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Probabilistic response of dynamical systems based on the global attractor with the compatible cell mapping method

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Abstract

A generalized compatible cell mapping (CCM) method is proposed in this paper to take advantages of the simple cell mapping (SCM) method, the generalized cell mapping (GCM) method together with a subdivision procedure. A coarse cell partition is first used to obtain a covering set of the global attractor. Then, a finer global attractor is obtained by the subdivision process. The probabilistic response of stochastic dynamic systems is obtained by the sparse matrix analysis algorithm applied to the covering set of the global attractor. Because the computational domain is the covering set of the global attractor rather than the whole state space, the numerical efficiency of the proposed method can be greatly improved as compared to the GCM. A three-dimensional and a four-dimensional dynamical system under Poisson white noise excitation are studied to demonstrate the effectiveness of the proposed method for the probabilistic response analysis. Monte Carlo simulations show a good agreement with the proposed method.

Keywords: global attractor, compatible cell mapping method, probabilistic response, Poisson white noise

1 Introduction

Stochastic nonlinear dynamical systems have important application in various areas of science and engineering, such as physics, seismology, economics, aerospace structures, civil engineering structures, ocean structures and so on [1-4]. To obtain the probability density function (PDF) of the response is the main goal in stochastic analysis. The PDF solution of the system under Gaussian white noise is governed by the Fokker-Planck-Kolmogorov (FPK) equation and by the Kolmogorov-Feller (KF) equation under Poisson white noise [5]. However, exact analytical solutions for the PDF of the response are only available for very restricted classes of nonlinear systems [6-11]. Various approximate and numerical procedures have been developed to solve PDF

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solutions for the FPK equation or the KF equation [12-25], such as stochastic averaging method, path integral method, cell mapping method and exponential-polynomial closure method. Among these methods, the cell mapping method has been demonstrated to be a very efficient tool due to its ability of global analysis of the strongly nonlinear systems. It can evaluate the PDF solutions of the transient and steady state responses of the system driven by Gaussian and Poisson white noise excitations [26-28].

The SCM method was first proposed by Hsu [29] to increase the computational efficiency of point mapping. Various improvements of the SCM method appeared and were applied to analyze the dynamical phenomena, such as bifurcation, stochastic response, first-passage problem [21, 30-41]. As the most important improvement, the GCM method can determine all the global properties including stable and unstable manifolds with the help of the digraph theory [36, 41]. The transient and steady-state PDFs of the stochastic response can be obtained by the GCM method [37]. The CCM method is proposed and applied to deterministic and stochastic dynamical systems by combining the advantages of the SCM and GCM methods [32]. The set-oriented method is another extension of the cell mapping method and just concentrates on the invariant sets. It can find fine details of the unstable manifold and the global attractor of dynamical systems by the subdivision technique [30]. The GCM method originally computed the one-step transition probability matrix of stochastic systems by the Monte Carlo simulations, which proved to be time consuming. The short-time Gaussian approximation is a procedure to increase the computational speed of transition probability matrix, and is effective when the moment equations with the Gaussian closure are derived.

As shown in Ref. [30], the global attractor contains all the invariant sets. Then the global properties related to the stochastic response of dynamical systems, such as attractors, saddles and unstable manifold should also be contained in the global attractor. It is, however, a formidable task to find all these information. This paper proposes a new approach to conduct the global analysis with a goal to discover all the above properties for stochastic dynamical systems by developing the CCM method and applying an algorithm to focus the computational effort on the covering set of the global attractor.

The example considered in this paper is a stochastic system subject to the Poisson white noise [42-44]. That is to say that the excitation consists of a sequence of independent, identically distributed pulses arriving at random times of a Poisson process and provides a more realistic

description of random discrete events than the Gaussian white noise. Many engineering and science problems, such as traffic loads, ground acceleration due to earthquakes, forces acting on railway vehicles traveling on imperfect tracks, can be modeled as a dynamical system under Poisson white noise. Thus, the probabilistic response of dynamical systems under the Poisson white noise excitation is of great importance. The Poisson white noise is given in the form [5, 14, 27, 45]

$$\eta(t) = \sum_{i=1}^{N(t)} Y_i \cdot \delta(t - t_i) \quad (1)$$

where $\delta(t)$ is the Dirac delta function. $N(t)$ is a Poisson counting process with mean arrival rate λ , which gives the total number of impulse occurrences in the time interval $[0, t)$. $\{Y_i, i \geq 1\}$ is a family of independent and identically distributed variables and is independent of the impulse arrive time t_k . The correlation function of the Poisson white noise is given by

$$K^{(k)}[\eta(t_1), \eta(t_2), \dots, \eta(t_k)] = \lambda E[Y^k] \delta(t_2 - t_1) \cdots \delta(t_k - t_1) \quad (k = 1, 2, \dots, \infty) \quad (2)$$

where $E[\cdot]$ is the mathematical expectation. In this paper, the random variable Y is assumed to the normal distribution with mean zero. $I = \lambda \cdot E[Y^2]$ is the intensity of the Poisson white noise.

The rest of this paper is arranged as follows. Section 2 reviews the SCM and GCM methods and introduces the procedure of determining the global attractor with a generalized CCM method. The detailed scheme of analyzing the probabilistic response with the proposed method is presented in Sec. 3. Section 4 demonstrates the proposed method with two examples of different types. Concluding remarks are made in Section 5.

2 The Cell Mapping Methods

2.1 SCM Method

Consider a dynamical system governed by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t), \quad \mathbf{x} \in \mathbf{D}. \quad (3)$$

where t is the time variable, \mathbf{x} is an N -vector and \mathbf{F} is a nonlinear vector-value function of \mathbf{x} . \mathbf{D} is a given bounded domain in the state space \mathbf{R}^N . By discretizing the state space into a cell state space, the domain \mathbf{D} is divided into finite small cells numbered from 1 to N_c . The region outside the domain \mathbf{D} constitutes one single cell, defined as the 0th cell. The image of each cell can be represented by the point mapping of the original system from a point within the cell.

For the SCM method, only one trajectory of system (3) is generated to obtain the image from each cell with the time duration τ , starting at $t = \tau_0$. The cell mapping equation can be created as

follows

$$\mathbf{Z}(n+1) = \mathbf{C}(\mathbf{Z}(n)) \quad (4)$$

where \mathbf{C} is the cell function, which is a mapping from an integer to an integer, n is the mapping step and $\mathbf{Z}(n)$ is an integer representing the cell where the system resides at the n th step. When the mapping \mathbf{C} is obtained, the global properties such as equilibrium cells, the periodic solutions consisting of periodic cells and basins of attraction can be studied in a systematic way. During the computational process, the group number array, the step number array and the periodicity number array are assigned to each cell for the global analysis. The original algorithms were developed by Hsu [29].

2.2 GCM Method

For the GCM method, V trajectories of system (3) are constructed out of each cell with the time duration τ , starting at $t=\tau_0$, corresponding to multiple image cells. For a cell j , if there are S_i trajectories falling in the image cell i , then the one-step transition probability from cell j to cell i is $p_{ij}=S_i/V$ such that

$$\sum_{i=1}^{N_c} p_{ij} = 1 \quad \text{and} \quad \sum_{i=1}^{N_c} V_i = V \quad (5)$$

The evolution of the dynamical system is described by Markov chains

$$\mathbf{p}(n+1) = \mathbf{P} \cdot \mathbf{p}(n) \quad \text{or} \quad \mathbf{p}(n) = \mathbf{P}^n \cdot \mathbf{p}(0) \quad (6)$$

where $\mathbf{P}=\{p_{ij}\}$ is the one-step transition probability matrix. $\mathbf{p}(n)=\{p_i(n)\}$ is the probability distribution of the system at the n th mapping step and $p_i(n)$ is the probability of finding the system in the i th cell. $\mathbf{p}(0)$ is the initial probability vector. With the GCM method, not only the probabilistic responses of the system but also the global properties such as basins of attraction, saddles and invariant manifolds can be obtained by analyzing the characteristics of the Markov chains with the digraph algorithms [28].

The transient and steady-state responses of dynamical systems are described by the vector $\mathbf{p}(n)$ [37]. We should note that most elements of the transition probability matrix \mathbf{P} are 0. Thus, from Eq. (6) it can be seen that the key point of computing the probabilistic response is the multiplication of a sparse matrix and a vector, which can be fast by using the sparse matrix computational methods.

2.3 Global Attractor with the CCM Method

The SCM and GCM methods are both used for the global analysis of dynamical systems. The SCM method can quickly find the periodic motions while the GCM method can determine the saddle-like properties. By combining their advantages, the CCM method was proposed to treat the nonlinear vibration problems, which may be deterministic or stochastic [32]. In this section we will investigate a generalized CCM method to obtain the global attractor.

Let $Q \subset \mathbf{R}^N$, the global attractor relative to Q can be defined by [30]

$$A_Q = \bigcap_{j \geq 0} f^j(Q) \quad (7)$$

where f is a discrete dynamical system $x_{j+1}=f(x_j)$, $j=0, 1, 2, \dots$ or a Poincare map of an ordinary differential equation. A global attractor A is an attracting set and contains all the invariant sets of the dynamical systems. The basin of attraction of A is the whole space \mathbf{R}^N . A_Q is a subset of the global attractor A . The SCM method can obtain the periodic solutions of dynamical systems. These solutions may be stable or unstable. The trajectories generated from the periodic cells forward in time will approach the sets that contain the global attractors. By the subdivision process, which will be introduced in detail in Sec. 3, the fine structure of the global attractor can be delineated.

The algorithm of the generalized CCM method for finding the global attractor is presented as follows:

- (1) Select a domain of interest D_1 with a coarse partition, in which the GCM is constructed with V sampling points in each cell. A SCM is selected from the GCM so that the two mappings are compatible.
- (2) The set of N_s periodic cells is extracted from the SCM. For each cell c_i in this set, denote the cells that c_i leads to as F_i . Then the covering set of the global attractor is given by

$$S_g = \bigcup_{i=1}^{N_s} [F_i \cup c_i] \quad (8)$$

- (3) Construct the SCM and the GCM subsequently on the set S_g and repeat steps 2 and 3 to obtain the new sets of the global attractor until we reach the desired accuracy of the solution in terms of the size of the cells.

In general, the maximum or minimum transition probability is used to extract the SCM from the GCM. Since we are interested in the unstable and stable periodic solutions, which are small probability events, we have found that the image of the center of a cell should be used for the compatible SCM. When one of the image cells is the same as the pre-image cell, the mapping from the cell to itself is taken as the compatible SCM.

3 Probabilistic Response Analysis with the CCM Method

3.1 Approximation of Global Attractor with Subdivision

From Sec. 2.3, we can find that under the coarse partitions the global attractor is contained in the set S_g and we can find the global attractor by the iteration forward in time from the periodic solutions. **The purpose of coarse partitions is to find the spread of the attractors in the state space so that we can have an estimate of the computational domain.** The more accurate results of the set S_g can be obtained by the subdivision process, which is described as follows.

(1) Numbering the Cells in the Subdivision Process

For a goal set S , create the cell state space in it. Because the set S is only a part of the whole space \mathbf{R}^N , the numbering rule is changed compared with the traditional GCM method. Supposed there are totally K_m cells in S after m iterations, where K_0 represents the number of cells in the original cell state space. If there are M_m small cells in each cell for the GCM in the m th iteration, then the cell numbers are assigned from 1 to $K_{m-1} \cdot M_m$. The cell numbers in the i th cell, which is the sequence number of output cells in the m th iteration, are from $(i-1) \cdot M_m + 1$ to $i \cdot M_m$.

(2) Creation of the Cell Mapping in the New State Space

The region out of the current cell state space is called the sink cell. The image of a cell may be located within the cell state space or within the sink cell. For a cell in the m th iteration, if it is located in the i th cell of K_{m-1} cells and in the j th cell of M_m cells, then the cell number is $(i-1) \cdot M_m + j$. During the m th iteration, the initial point P_0 with the cell c is mapped to the point P_1 after one-step mapping. The cell number of point P_1 can be determined just by computing its location in the cell sequence of K_{m-1} and M_m . Denote them as i' and j' , then the cell number of point P_1 is $c' = (i'-1) \cdot M_m + j'$. The cell mapping from cell c to cell c' is created.

(3) Correspondence Between Computed Cells and its Coordinates

After the creation of the GCM for $K_{m-1} \cdot M_m$ cells, the periodic solutions are calculated by the retrieved SCM and then the trajectories are generated from them to determine the global attractor. In order to output the computation results, we introduce an array of structures $AR_m[i]$ in the m th iteration step to determine the center coordinates of each cell, where $i=1,2,\dots,K_{m-1} \cdot M_m$. This structure has $N+1$ (N is the dimension of state space) elements, where the first N elements of AR_m represent the coordinate of the i th cell, and the $(N+1)$ th element represents the cell number of the i th cell in the cell state space of the $(m+1)$ th iteration. With the aid of this cell number, we can

determine its location by the correspondence rules of the GCM method between cells and points. For example, if the cell state space of the m th iteration step is $C_{m1} \times C_{m2} \times \cdots \times C_{mN}$, then the $(N+1)$ th element represents the cell number in the next cell state space with the cell structure of $(C_{m1} \times C_{m2} \times \cdots \times C_{mN}) \cdot M_m$, where M_m is usually taken as 2^N during the computation.

(4) Determination of Global Attractor

Once the cell mapping is created, the SCM method is used to find the periodic solution. The iteration process forward in time is carried out by the search algorithm of digraph from these periodic solutions to obtain the global attractor. The other global properties such as attractors and saddles can be found by analyzing the m th cell state space with digraph GCM method. Repeat all the above steps until the accuracy is met. Suppose that the initial cell state space is $K_0 = C_1 \times C_2 \times \cdots \times C_N$, then the number of cells in the cell state space at the m th iteration step becomes into $L_m = (2^{m-1}C_1) \times (2^{m-1}C_2) \times \cdots \times (2^{m-1}C_N)$ that is considered in the framework of the GCM. However, the actual number of cells at the m th iteration step is K_{m-1} , which is far less than L_{m-1} . The computation efficiency is improved by L_{m-1}/K_{m-1} times with the proposed method at the m th iteration step.

3.2 Probabilistic Response on the Global Attractor

The steady-state response of the dynamical system under stochastic excitation should be contained in its global attractor. Consider a nonlinear dynamical system under stochastic excitation, the probabilistic response analysis based on the covering set of the global attractor under given partition scales is shown as follows:

Firstly, find the covering set of the global attractor according to the process shown in Sec. 3.1. and take the set as the initial cell state space for computing the probabilistic response. The only difference is that the image cells are determined by the random trajectories from the center point of each cell. Under the stochastic condition, the global attractor is just used to cover the region of steady state response, so the cell state space is not required to be too refined. Suppose that the partition scale in the framework of the GCM is $H_0 = C_1 \times C_2 \times \cdots \times C_N$ and there are K_0 cells in the set of the global attractor. Then the computation efficiency under stochastic condition is improved by H_0/K_0 times.

Secondly, based on the numbering rules and correspondences between cells and points of Sec. 3.1, the set of the obtained global attractor is considered as the initial cell state space to create the one-step transition probability matrix. For a cell j , if there are V randomly sampled trajectories

generated from each cell, and V_i trajectories falling in the image cell i , then the one-step transition probability $p_{ij}=V_i/V$. We just consider the one-step transition probability of the cells within the global attractor.

Finally, the probabilistic response is obtained with the help of the matrix analysis algorithm [37]. Because most elements of the transition probability matrix are zero, it can be stored as a sparse matrix.

If we want to improve the precision of the probabilistic response, the initial cell state space should be further divided. Suppose that M_m small cells are divided from each cell and V random trajectories are generated from each small cell, then in this way, the final cell state space equivalent to GCM is changed into $H_0 \cdot M_m$.

Under stochastic excitations, the sampling points for one-step transition probability may be numerous, especially when the dimension of dynamical systems is high. The parallel technique can be used to further increase the computational efficiency [46]. The most time-consuming part is the creation of transition probability matrix. The generation process of image cells is independent with each other. Therefore, a simple and easy parallel strategy can be implemented [47].

4 Illustrative Examples

Two examples are presented in this section to demonstrate the ability of the proposed method in the probabilistic response analysis. The dimension of these two examples is respectively 3 and 4. The results by the generalized CCM method and the Monte Carlo simulation are both obtained on a laptop with Intel core i7 6820 4-core processor.

Example 1: In this example we will present the probabilistic responses of a nonlinear system with non-viscous exponential damping under stochastic excitation, the governing equation of which can be expressed as [48]

$$\ddot{x} + 2\zeta \int_0^t \frac{e^{-1/\beta(t-\tau)}}{\beta} \dot{x} d\tau + ax + bx^3 = f \cos(\omega t) + \eta(t) \quad (9)$$

By using the Leibnitz rule for differentiation of an integral [49], the equation can be rewritten as a three-dimensional system as follows

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2\zeta x_3 - ax - bx^3 + f \cos(\omega t) + \eta(t) \\ \dot{x}_3 &= \frac{x_2 - x_3}{\beta} \end{aligned} \quad (10)$$

The parameters are taken as $\zeta=0.1$, $\beta=0.2$, $a=0.5$, $b=0.5$, $f=1.5$, $\omega=2.0$, $\eta(t)$ is the Poisson white noise. The time duration for the CCM method is $\tau = T = 2\pi/\omega$, which is the period of harmonic excitation. The domain $\mathbf{D}_1=\{-6\leq x_1\leq 6, -8\leq x_2\leq 10, -6\leq x_3\leq 10\}$ is firstly divided into a coarse cell structure of $16\times 16\times 16$ (K_0) cells. $30\times 30\times 30$ (V) points are uniformly selected within each cell to determine the image cells with the CCM method. In the subdivision process, 1000 (V) random sampling points are generated. Under deterministic condition, Figs. 1(a), 1(b) and 1(c) show the coverings of the global attractor (unstable manifold) of the system (10) with different divisions. A fine structure of the global attractor can be obtained after 8 iteration steps, which is equivalent to the cell structure of $2048\times 2048\times 2048$ cells for the GCM method.

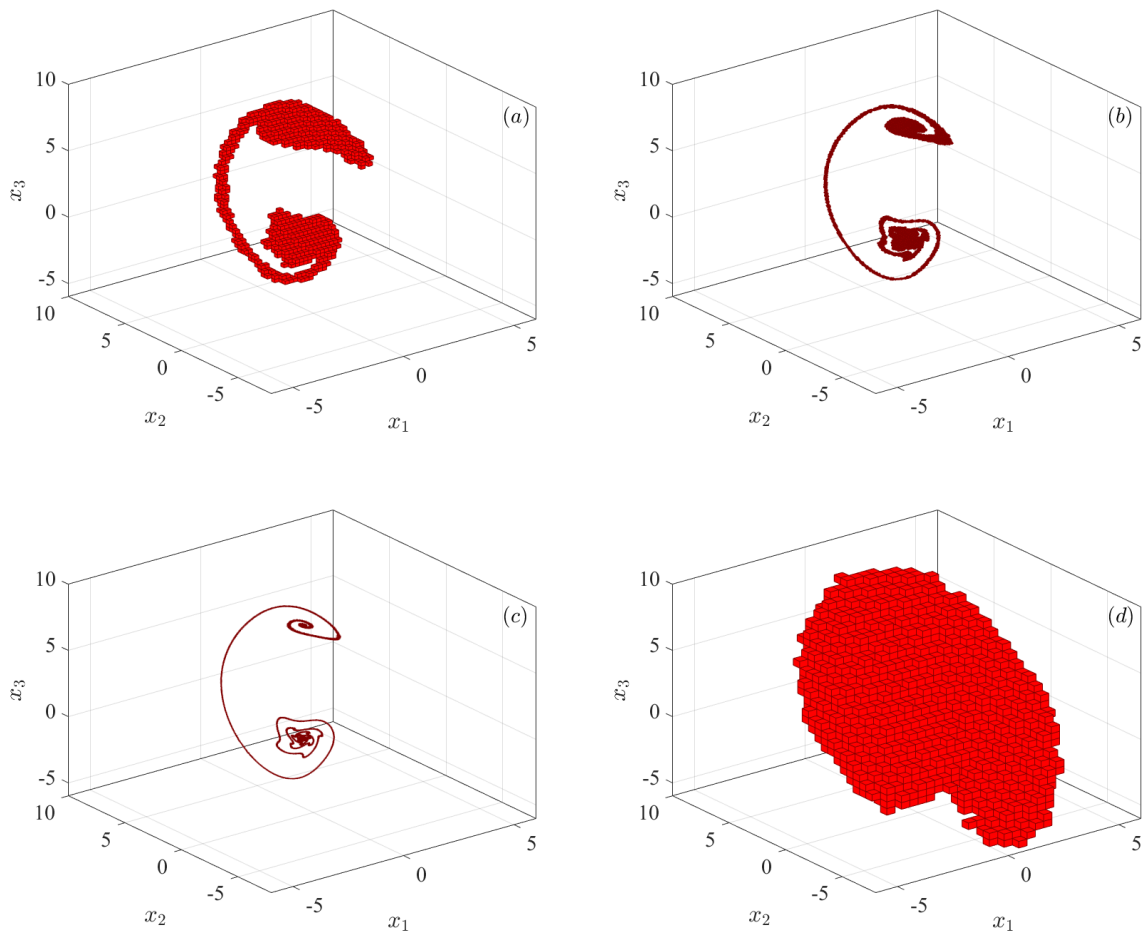


Fig. 1 The coverings of the global attractor for system (10) with different divisions. (a) $m=3$, $I=0$; (b) $m=5$, $I=0$; (c) $m=8$, $I=0$; (d) $32\times 32\times 32$, $\lambda=0.5$, $I=0.05$.

For the response analysis of stochastic dynamical systems, it is not needed to find the global attractor very fine and a proper cell structure is sufficient. Under the Poisson white noise excitation, the same initial coarse partition is carried out in the domain \mathbf{D}_1 . 2500 (V) random sampling points are generated to determine the image cells. After 2 iteration steps, the covering of the global

attractor of system (10) is obtained with the cell structure of $32 \times 32 \times 32$ for GCM method. Fig. 1(d) shows the covering of the global attractor with $\lambda=0.5$, $I=0.05$ for Poisson white noise, where $H_0=32 \times 32 \times 32$ and $K_0=1543$. Therefore, the computation efficiency is improved by around 21 times.

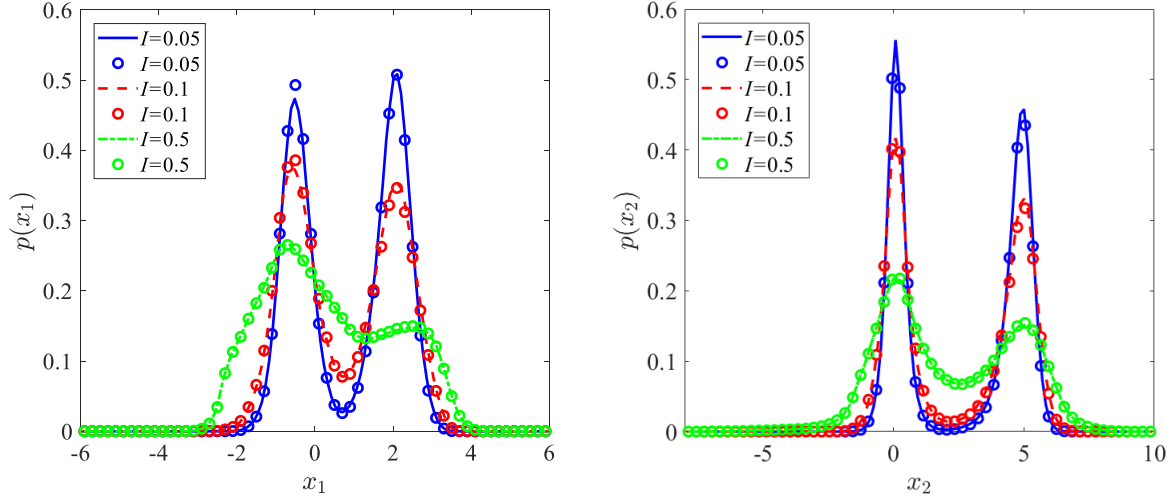


Fig. 2 The marginal PDFs of system (10) for x_1 and x_2 with different intensities of Poisson white noise with $\lambda=0.5$. Lines: the GCM method. Circles: the direct Monte Carlo simulations.

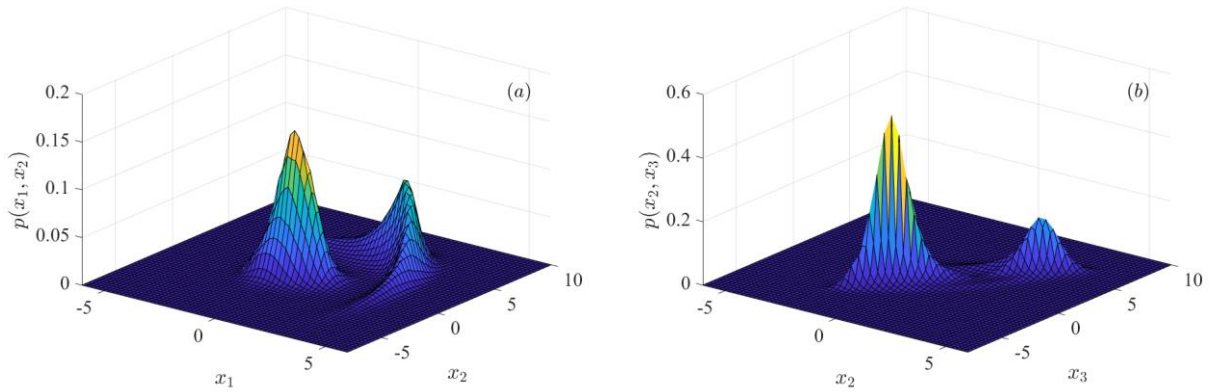


Fig. 3 The surface plots of joint PDFs of system (10) on different planes for $\lambda=0.5$ and $I=0.5$. (a) x_1 - x_2 plane; (b) x_2 - x_3 plane.

In the following we fix the mean arrive rate $\lambda=0.5$ and consider the influences of Poisson white noise intensity on the probabilistic response based on the obtained global attractor. **The response results are obtained when the time $t > 200T$.** For each cell, $M_m=2 \times 2 \times 2$ small cells are divided and **3200 (V) random** sampling trajectories are generated to determine the one-step transition probability. Figure 2 gives the marginal PDFs of system (10) for x_1 and x_2 when the intensities of Poisson white noise are respectively taken as $I=0.05$, $I=0.1$, $I=0.5$. The accuracy of the proposed CCM method is examined by the Monte Carlo simulation, where 5×10^6 sample points are used. When the noise intensity $I=0.5$, Figure 3 shows the surface plots of joint PDFs on different planes to exhibit the space structure of probabilistic responses.

Figure 4 shows the contour plots of joint PDFs of system (10) on x_1 - x_2 plane as the intensity of Poisson white noise is varied with $\lambda=0.5$. When $I=0.05$, shown in Fig. 4(a), the system (10) has two stable states and the probability distribution is concentrated around them. As the intensity I increases, the probability distribution of two stable states becomes broader and gradually merges. The probabilistic response becomes more random on a broader scale. We can also find that the evolutionary direction of steady-state probability distributions of response is in accordance with the unstable manifold shown in Fig. 1(c).

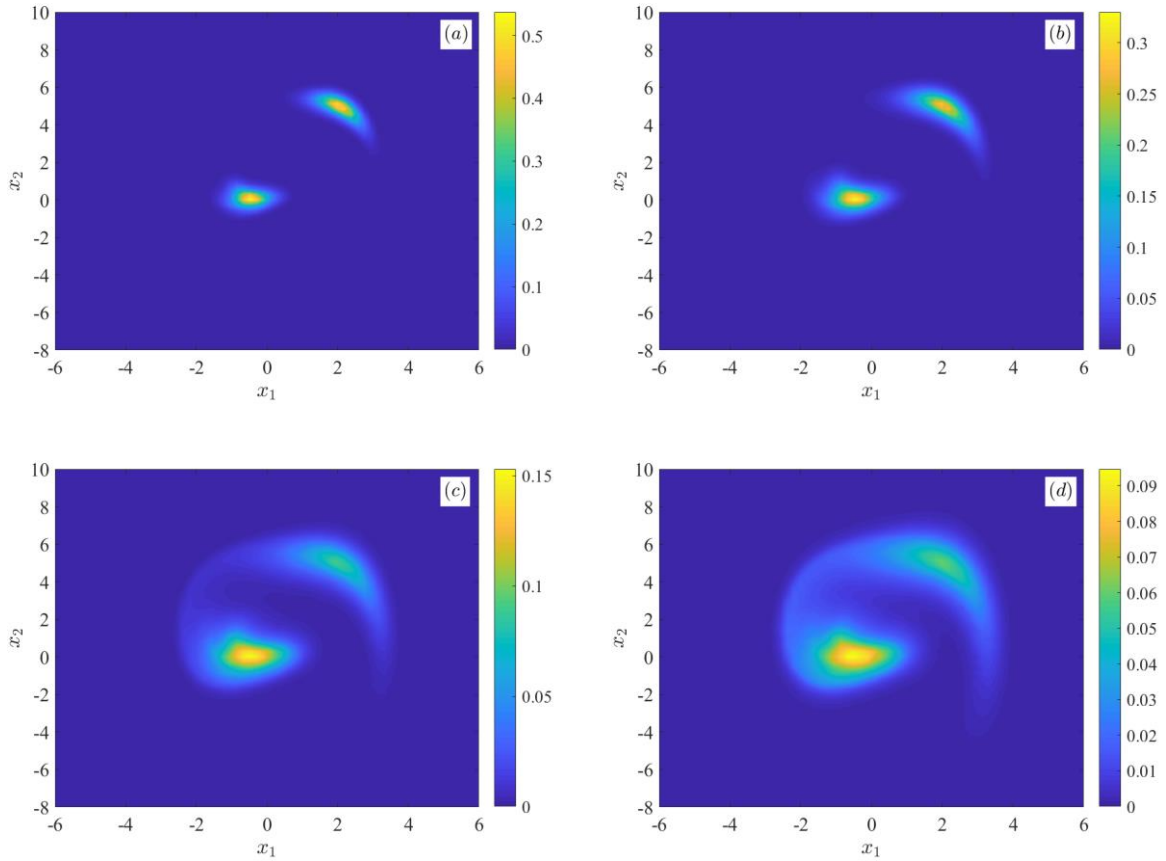


Fig. 4 The contour plots of joint PDFs of system (10) on x_1 - x_2 plane for different intensities of Poisson white noise with $\lambda=0.5$. (a) $I=0.05$; (b) $I=0.1$; (c) $I=0.3$; (d) $I=0.5$.

Example 2: Now, we consider a coupled nonlinear system, which describes the escape from a two-dimensional potential well [50]. The equation of motion under sinusoidal and stochastic excitations is given by

$$\begin{aligned} \dot{x} + \beta(\dot{x} - \dot{u}) + (x - u) - (x - u)^2 &= F \sin(\omega t) + \eta(t) \\ \ddot{u} + \gamma \dot{u} - \beta(\dot{x} - \dot{u}) + ku - (x - u) + (x - u)^2 &= 0 \end{aligned} \quad (11)$$

where $\eta(t)$ is the Poisson white noise and the parameters are taken as $\beta=0.05$, $\gamma=0.05$, $k=3.0$, $F=3.0$, $\omega=0.75$. We rewrite the equation as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\beta(x_2 - x_4) - (x_1 - x_3) + (x_1 - x_3)^2 + F \sin(\omega t) + \eta(t) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\gamma x_4 + \beta(x_2 - x_4) - kx_3 + (x_1 - x_3) - (x_1 - x_3)^2
\end{aligned} \tag{12}$$

The selected domain $\mathbf{D}_2 = \{-1.5 \leq x_1 \leq 1.5, -1.5 \leq x_2 \leq 1.5, -1.5 \leq x_3 \leq 1.5, -1.5 \leq x_4 \leq 1.5\}$ is divided into a coarse cell structure of $16 \times 16 \times 16 \times 16$ (K_0) for the CCM method with the time duration $\tau = T = 2\pi/\omega$. 625 (V) points are uniformly selected within each cell to determine the image cells in the following subdivision process. For the system without stochastic excitation, Figs. 5(a), 5(b) and 5(c) show the coverings of the global attractor of system (12) with different divisions in the three-dimensional space of (x_1, x_2, x_3) . It can be found that the structure of the global attractor is already very fine after 6 iteration steps, which is equivalent to the cell structure of $256 \times 256 \times 256 \times 256$ cells for the GCM method.

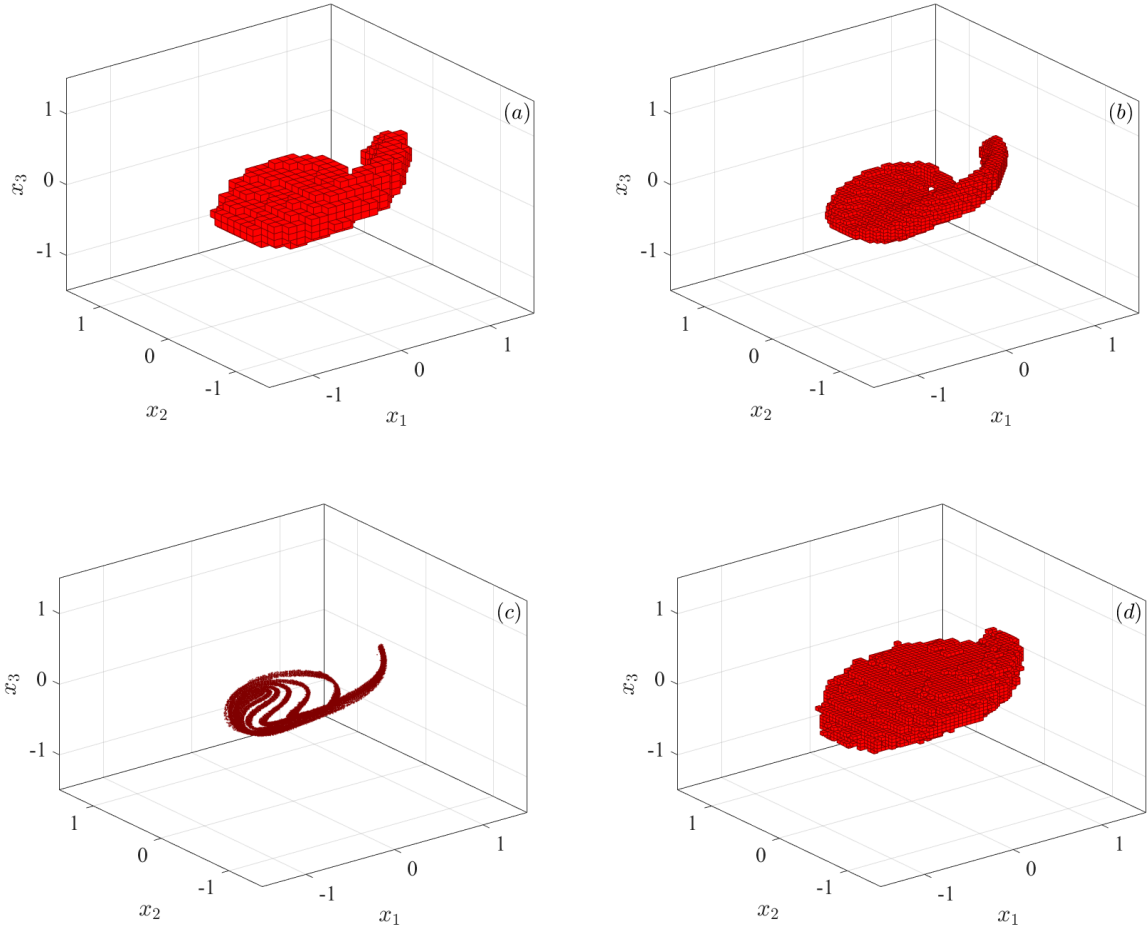


Fig. 5 The coverings of the global attractor of system (12) with different divisions in the three-dimensional space of (x_1, x_2, x_3) . (a) $m=2, I=0$; (b) $m=3, I=0$; (c) $m=5, I=0$; (d) $64 \times 64 \times 64 \times 64, \lambda=1.0, I=0.002$.

Under the Poisson white noise excitation, firstly a coarse cell structure of $16 \times 16 \times 16 \times 16$ (K_0) for

the CCM method is carried out in the domain \mathbf{D}_2 . And the image cells are generated by 800 (I) random sampling points. After 3 iteration steps, the covering of the global attractor is obtained equivalent to the cell structure of $64 \times 64 \times 64 \times 64$ for GCM method. For example, when the mean rate and intensity of Poisson white noise are taken as $\lambda=1.0$ and $I=0.002$, shown in Fig. 5(d), the number of cells in the initial cell state space for response analysis is $K_0=76126$ and $H_0=64 \times 64 \times 64 \times 64$. Therefore, the computation efficiency is improved by around 220 times.

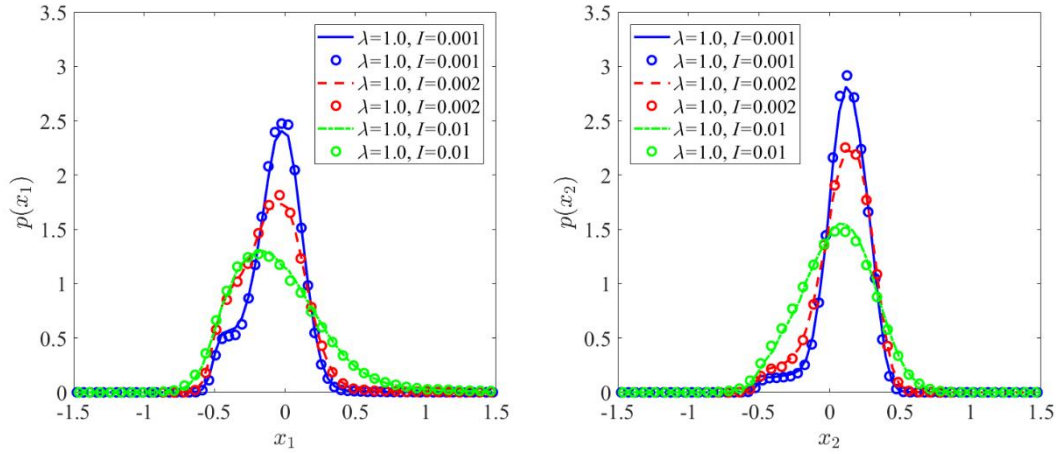


Fig. 6 The marginal PDFs of system (12) for x_1 and x_2 with different intensities of Poisson white noise with $\lambda=1.0$. Lines: the GCM method. Circles: the direct Monte Carlo simulations.

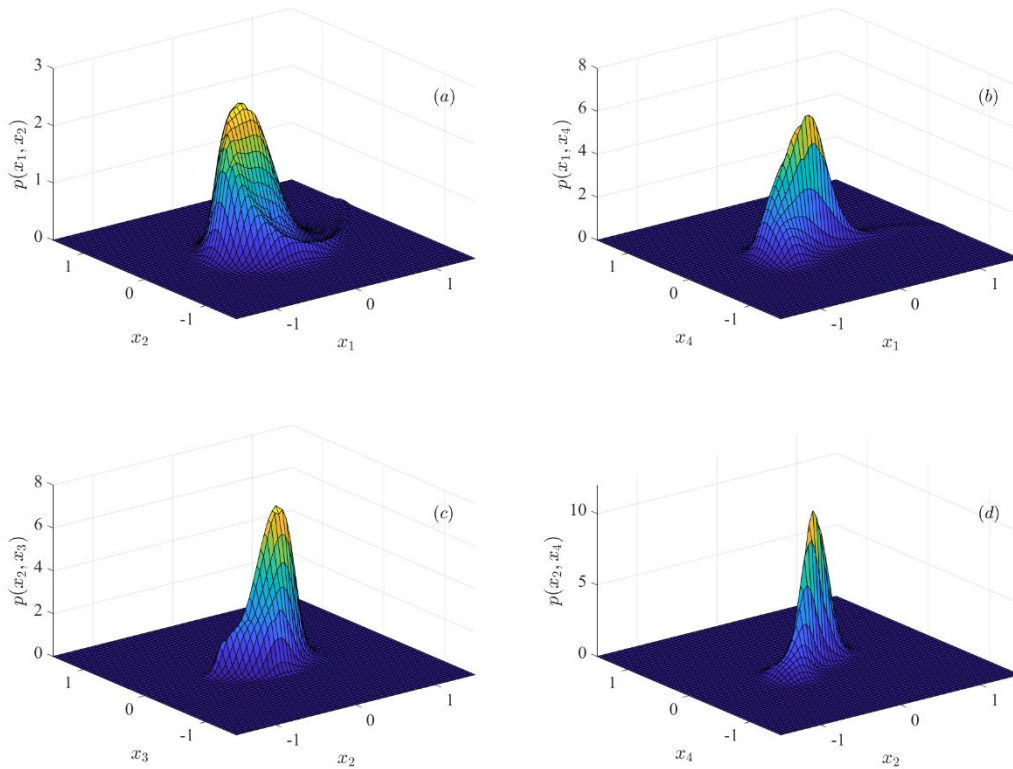


Fig. 7 The surface plots of joint PDFs of system (12) on different planes for $\lambda=1.0$ and $I=0.005$. (a) x_1 - x_2 plane; (b) x_1 - x_4 plane; (c) x_2 - x_3 plane; (d) x_2 - x_4 plane.

The influences of Poisson white noise intensity on the probabilistic response based on the obtained global attractor is then considered by fixing $\lambda=1.0$. 2560 (V) random sampling trajectories are selected within each cell to compute the one-step transition probability. Figure 6 gives the marginal PDFs of system (12) for x_1 and x_2 with the noise intensities $I=0.001$, $I=0.002$, $I=0.05$. The response results are obtained when the time $t > 200T$. The Monte Carlo simulation with 10^9 sampling points is used to demonstrate the efficiency of the proposed compatible cell mapping. In order to exhibit the space structure of probabilistic responses, Fig. 7 shows the surface plots of joint PDFs on different planes when the noise intensity $I=0.005$.

When the intensity of Poisson white noise is small, the probability distribution is concentrated around the attractor. The system is located in the safe basin and disturbed by a minor perturbation, shown in Fig. 8(a). As the intensity I increases, we can find from Fig. 8 that the probabilistic response becomes more and more random. The system will escape from the safe basin with a high probability. And also, the evolutionary direction of steady-state probability distributions of response is in accordance with the unstable manifold shown in Fig. 5(c).

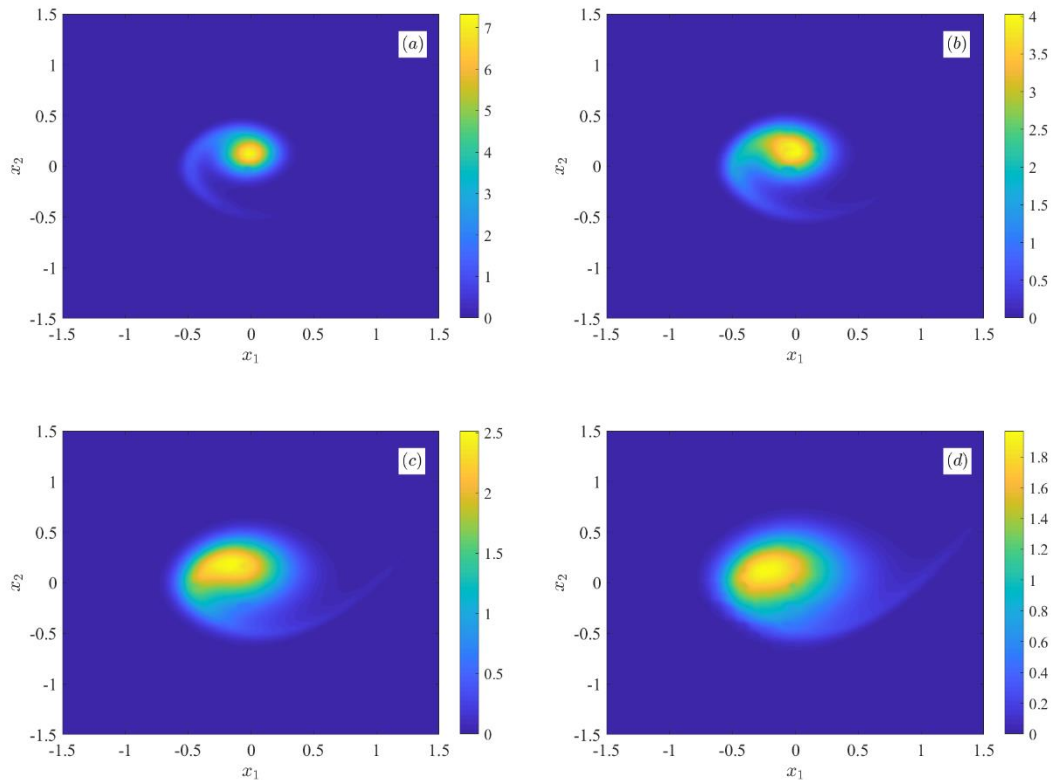


Fig. 8 The contour plots of joint PDFs of system (12) on x_1 - x_2 plane for different intensities of Poisson white noise with $\lambda=1.0$. (a) $I=0.001$; (b) $I=0.002$; (c) $I=0.005$; (d) $I=0.01$.

5 Conclusions

A generalized CCM method is presented in this paper. With the subdivision process, our proposed

method can find the fine global attractor of dynamical systems. Based on the coarse covering of the global attractor, the probabilistic response can be efficiently obtained. Two examples, a three-dimensional nonlinear stochastic dynamical system and a coupled nonlinear stochastic dynamical system, are presented to show the effectiveness of our proposed method. The fine global attractor is presented for the three-dimensional and four-dimensional systems, which is usually difficult for the traditional GCM method. For the stochastic response analysis, the covering set of the global attractor with a coarse partition is enough. The Monte Carlo results verify the accuracy of solving the PDFs of the stochastic response. Because the covering set of the global attractor is a small percentage of the whole state space, the computational efficiency of the proposed method can be greatly improved compared with the traditional GCM method. For the coupled nonlinear system, the computation efficiency is improved by around 220 times. With such efficiency, the solution can be computed on a laptop.

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