Title
Generalized Span Categories in Classical Mechanics and the Functoriality of the Legendre Transformation

Author
Yassine, Adam

Publication Date
2020

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
RIVERSIDE

Generalized Span Categories in Classical Mechanics and the Functoriality of the Legendre Transformation

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Adam Maher Yassine

June 2020

Dissertation Committee:

Dr. Michel L. Lapidus, Co-Chairperson
Dr. David Weisbart, Co-Chairperson
Dr. Ziv Ran
Dr. Bun Wong

The Dissertation of Adam Maher Yassine is approved:

Committee Co-Chairperson

Committee Co-Chairperson

University of California, Riverside

Acknowledgements

First and foremost, I would like to thank my PhD advisors Doctors Michel L. Lapidus and David Weisbart for their incredible guidance and patience throughout my graduate school experience. I am lucky to have two advisors who shared their knowledge, which shaped my viewpoint in appreciating the wonderful intricacies of mathematics. Words cannot express how much I appreciate their tutelage, and generosity. This thesis could not have been completed without their support. I also want to thank Doctor Edriss Titi for his endless support and counsel. His magnanimity and wisdom had a major impact on my overall development as a mathematician. Doctor Titi is a third mathematical father and I am grateful to have him in my life.

During my time at the University of California Riverside I have had the opportunity to learn from several exceptional instructors. First, I would like to thank my doctoral committee members Doctor Ziv Ran and Doctor Bun Wong for their time, help, and service. I also want to thank Doctor John Baez for introducing me to the fascinating world of mathematical physics as well as his guidance. His humor and teaching made my interactions with him some of the most enjoyable and memorable moments in graduate school. I want to thank Doctors Julie Bergner, Mei-Chu Chang, Vyjayanthi Chari, Po-Ning Chen, Yat Tin (Raymond) Chow, Gerhard Gierz, Jacob Greenstein, Mike Hartglass, Juhi Jang, Bangti Jin, James Kelliher, Yat Sun Poon, Yunied Puig De Dios, Reinhard Schultz, Christina Vasilakopoulou, Frederick Wilhelm, Qi Zhang, and Zhenghe Zhang for their support and instruction. I also want to thank my mathematical uncles Doctors Rahul Fernandez, and Boyan Kostadinov and mathematical aunt Doctor Rita Fioresi for their support and wisdom. I want to thank the wonderful staff members Deidra Kornfeld, Melissa Gomez, Margarita Roman and Gena Thompson.

The friends that I made in graduate school and the memories we have are truly some of the highlights of my time at UC Riverside. I would not have made it through graduate school without their support and companionship. Doctors Mikahl Banwarth-Kuhn, Jesse Benavides, Josh Buli, Bryan Carrillo, Daniel Cicala, Tim Cobler, Kenny Courser, Tim McEldowney, Ryan Moruzzi, James Ogaja, Priyanka Rajan, Alex Sherbetjian, John Simanyi, Kevin Tsai, Edward Voskanian, and Andrew Walker. I also want to thank Taylor Baldwin, Erin Davison, Linette Gharibi, Nick Lanni, Nick Salinas and Thomas Schellhous. I want to especially thank Megan Weisbart and Zuma for their support, and kindness throughout this entire process.

I was also very fortunate to study and learn from so many great instructors at California State University, Northridge where I obtained my Master's degree. I want to thank my Master's thesis advisor Doctor Chad Sprouse for his wonderful wit and tutelage. His guidance and knowledge prepared me to pursue a doctoral degree. I also want to thank Doctors Jorge Balbás, Alberto Candel, Rabia Djellouli, Terry Fuller, Werner Horn, Jing Li, Elena Marchisotto, Jerry Rosen and the late Ali Zakeri for their support and guidance. I thank Craig Euler, Gevork Demirchyan, Julio Fernando Guerrero-Gonzalez, his wife Denise Denulio, Arin Gregorian, Mike Hubbard, Greg Imhoff, Matthew Jaime, Theodora Jaime, their daughter Hypatia Jaime, Lorena Lopez, Doctor Shant Mahserejian, Doctor Patrick Medina, Doctor Alina Ostrander, Doctor Robert Ostrander, Sharon Paesachov, Eddie Tchertchian, Nancy Vusca, Doctor Gerardo Zelaya Eufemia, Katherine Zelaya and their children Rose, and Gerardo.

I want to take this opportunity to acknowledge the amazing people that I met at the MSRI workshop “From Symplectic Geometry to Chaos" in 2018. I thank Doctor Leonid Polterovich and his students Doctors Yaniv Ganor and Vukasin Stojisavljevic for their guidance and mentorship. I also want to thank my roommates and future doctors Will Clark, Thomas Melistas and Pedro Valentín de Jesús for their support and friendship, which made the experience unforgettable with many treasured memories.

In 2019, I had the great opportunity to be a teaching assistant for the FLEAP program at UC Riverside, which allowed my advisor and I to teach a probability theory course at University College London. I thank the coordinator Doctor Karolyn Andrews. I also had the opportunity to visit and give a talk at the American University of Beirut. I thank Doctors Abbas Alhakim, Rafael Andrist, Richard Aoun, Florian Bertrand, Giuseppe Della Sala, Kamal Khuri-Makdisi, Wissam Raji, Ahmad Sabra and Jihad Touma for their generosity and wonderful hospitality. In addition, I want to thank Doctor Joseph Malkoun at Notre Dame University-Louaize.

Of course the journey to pursuing a doctoral degree had to start somewhere and for me that was at the University of California Los Angeles. I was incredibly lucky to learn from many fantastic instructors in so many diverse fields. I want to thank Doctors Aravind Asok, the late Rodolfo De Sapio, David Gieseker, Nicolette Meshkat, Fred Park, Peter Petersen, and Owen Sizemore. I especially want to thank Doctor Robert Greene for his amazing instruction, which inspired me to go to graduate school. In fact, Doctor Greene has had a major impact on my development as a mathematician. He not only introduced me to the wonderful world of geometry but he also taught my advisors Doctors Chad Sprouse and David Weisbart. Finally, I want to thank my mathematical grandfather the late Doctor V.S. Varadarajan to whom this thesis is dedicated to. Even though we never met, his wisdom and knowledge that he imparted on Doctor David Weisbart has had a monumental impact on me. Although he is no longer with us, he continues to inspire me to be a better person through his dedication and love for his students. His memory will be eternal.

I thank my parents for raising me to be a person with humility and understanding the meaning of hard work. I want to thank my mother Amal for her endless support and encouragement. I will always be grateful for the sacrifices that she made so that I could achieve my dreams. I want to thank my sister Ryann for her support. Her delightful humor and kindness always brightens my day. I am truly thankful to have such an amazing sister. Even though I lost my father Hassan long ago, his memory lives on. My memories of his generosity, kindness, and humor is something that I truly treasure. In addition, his honesty and incredible work ethic is something that I try to emulate, which led to the completion of my doctorate. I miss him greatly. This doctoral thesis is dedicated to both my parents.

Finally, I want to acknowledge friends and family whose unconditional love and support gave me motivation to complete my doctoral degree. I thank the Sanaknaki family, which includes my aunt Hanan, her husband Mohammad and their children Jinane, Rawan, and Ali al-Hadi. I especially thank Rawan for all the lovely conversations we had in math and for knowing when I should relax when stuck on a problem. I want to thank several members of the Khawaja family along with their spouses and children. The first is my late uncle Fawzi who told me that if I did not get an education that he would not see me as his nephew. He always knew that I would one day get a doctorate and I thank him for his encouragement. He was a great man who I miss dearly. I want to thank his wonderful wife Ahlam, their first daughter Sara, her husband Haytham and children Celine and Joud. I thank their first son Ali, his wife Manal and children Hassan, Talia and Mohamad Ali. I thank their second daughter Najla, her husband Fouad, and children Lara, Hadi, and Leah. I thank their second son Zaher, his wife Yara and Yara's mother Mouna. I thank their third daughter Layal and her husband Ali. I thank my uncle Mohamad, his wife Maryam and children Loubna and Malak. I thank my uncle Safi, his wife Sana and children Hassan, Zaharat, and Maryam. I thank my aunt Asmahan, her husband Hassan and children Hanan, Akram, Ghinwa and Najwa. I thank my uncle Nassif. Words cannot express how much they all mean to me. I am truly blessed to have them all in my life.

I want to also thank my late aunt Layla Tayeh, and late uncle Moufid Tayeh along with their children Joseph and Rima. I was glad that I got to see my aunt Layla in 2019 and create new memories. I thank my cousin Mustafa Yassine and his amazing mother Hajja Fatima for their love and support. I was glad to see them and reconnect. I thank Ibrahim Kdouh, his wife Nadia and their children Sawsan and Lena who have been gracious to my mother. I also thank Mona Abbani, Mustafa "Jano" Marwani, Fatima Marwani, Jafar Marwani, and Ahmad Saleh for their lovely hospitality and for being so great to my mother over the years. I thank Saadiya, her late husband Abbas Misilmani and their children Ghada, Fatima, and Nariman for all the memories of my time in Lebanon. I also thank Nariman's children Lana, Joanna and Ali. I thank my father's late brother Hajj Ali Yassine, his wife Najat and children Mirvat, Fatima, Hussein, Mohamad, and Hassan. I thank the late Youssef Fakhreddine, his son Sam and daughter Marium. I thank Mustafa and Mahmoud Kasfy along with Mahmoud's wife Wadad and their children Rula, Ali, Mohammed, Maya and Ahmad. I especially want to thank Mohammed who is like a brother to me for all the great memories. I thank Mohammad Marwani, his wife Randa and their children Sarah, Hassan, Ali and Fatima. I thank Moustafa Fakhreddine, his wife Fatme and their children Moe, Youssef, Ranin and Ali. I thank my late uncle Hajj Mohamad Abbas Saleh and his wife Hajja Awatif for their caring support. I thank their daughter Hajja Amal, her husband Abdullah and their children Hussein, Mohammed and Ali. I also thank the Diab family Amneh, Nadia, Fatima, Faten and their late mother Hajja Hayat who was truly a wonderful person. I try to live by her example every day. I thank Faten's husband Ahmad Mohamad for his support and encouragement. I thank Fatima's husband Mike Saleh and their children Sally, Lama, and Zena for their support. I also thank Sally's husband Rabih Khawaja. I thank Hajj Abdullah and Hajja Mona Ramadan. I would also like to thank Soha Aoun, Mohamad Ayache, Ibrahim Jaber (Sheikh), Soha

Khalefe, the late Mahmoud Khawaja (Abu Salah), Mohamad Majid, Mariam Sabra, and Wafa Sabra for all the support and great memories. I want to thank Sayyid Murtadha al-Qazwini and Doctor Halima Shaikley for their mentorship and guidance.

I want to take the opportunity to also thank the Kobaissi family for their love and encouragement. I thank Hajj Abdullah, his wife Hajja Nasseemah who have been like second grandparents. I thank their first son Doctor Hassan Kobaissi, his wife Kawssar and their children Hadi, Laila, and Zane. I thank their second son Doctor Ali Kobaissi, his wife Lina and his children Mohammad and Abdullah. I thank their third son Nidal, his wife Zaineb and their children Kareem, Yasmeen and Aleeya. I thank their daughter Fatima, her husband Rick and their children Mia, Linnea and Richard. Finally, I want to thank Samira Elzayat-Messelmani for all the laughs and good times. I thank her husband Mohammed Messelmani and their children Nadine, Hassan, Maya, Jamil and Rami.

Lastly, I want to thank several friends who have supported me from the very beginning. I thank Omar Abouelnasr, Mishal Al-Ajmi, Doctor Bachar Ali, Mustafa Ali, Hussein Alshara, Doctor Jafar Badday, Doctor Alexander Gregath, Sarin Haroutounian, Talin Haroutounian, Ali Hassan, Jaryd Hochberger, Jaffer Jaffer, Cyrus Jenani, Casey Franklyn Proulx, Mahdi Qazwini, Mothafar Qazwini, Nancy Que, Ali Rahnama, Omid Rahnama, Haider Shaikley, and Sina Surouri. Thank you all for your camaraderie and laughs.

To my mother Amal.

In memory of my father Hassan and mathematical grandfather V.S. Varadarajan.

“In mathematics there is an empty canvas before you which can be filled without reference to external reality."

-Harish-Chandra

ABSTRACT OF THE DISSERTATION

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by

Adam Maher Yassine

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2020
Dr. Michel L. Lapidus, Co-Chairperson
Dr. David Weisbart, Co-Chairperson

Span categories provide an abstract framework for formalizing mathematical models of certain physical systems. The categories appearing in classical mechanics do not have pullbacks and this limits the utility of span categories in describing such systems. We introduce the notion of span tightness of a functor $\mathcal{F}$ from categories $\mathscr{C}$ to $\mathscr{C}'$ as well as the notion of an $\mathcal{F}$-pullback of a cospan in $\mathscr{C}$. If $\mathcal{F}$ is span tight, then we can form a generalized span category $\mathrm{Span}(\mathscr{C}, \mathcal{F})$ and circumvent the technical difficulty of $\mathscr{C}$ failing to have pullbacks. Composition in $\mathrm{Span}(\mathscr{C}, \mathcal{F})$ uses $\mathcal{F}$-pullbacks rather than pullbacks. We introduce the augmented generalized span categories LagSy and HamSy that respectively provide a categorical framework for the Lagrangian and Hamiltonian descriptions of certain classical mechanical systems. The morphisms of LagSy and HamSy contain all kinematical and dynamical information about these systems and composition of morphisms models the construction of systems from subsystems. A functor from LagSy to HamSy translates from the Lagrangian to the Hamiltonian perspective and is a categorical analog of the Legendre transformation.

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Chapter 1

Introduction

Category theory provides a formalism for unifying ideas across a wide spectrum of disciplines. The last few decades have seen rapid growth in the application of category theory to the study of systems and the emergence of applied category theory as a field of study. The recent book [31] is an introductory text for the general scientific community in which Spivak discusses some applications of category theory. Baez and Dolan apply category theory to study topological quantum field theory in [5]. Fuchs, Runkel, and Schweigert discuss categorification in the context of conformal field theory in [20] and give many references to work in this direction. Brunetti, Fredenhagen and Verch use category theory in [14] to study model-independent descriptions of quantum field theories. Thaule discusses open and closed strings in [32], building on the earlier work [4] of Baas, Cohen and Ramírez. Recently, Baez, Fritz, and Leinster gave a categorical interpretation of entropy in [7], demonstrating a connection between category theory and information theory.

A prominent program in applied category theory is to describe systems as the morphisms of an appropriate category, where the composition of morphisms describes the way in which systems compose to form more complicated systems. Category theory has found applications in the study of quantum theory and information theory, but there is a striking absence in the literature of its application in the study of classical mechanics. We introduce an abstract framework for classical mechanics that makes precise some physical heuristics and permits the Legendre transformation to be viewed as a functor from a category of Lagrangian systems to a category of Hamiltonian systems.

Since the study of classical systems involves solving differential equations that describe paths on general Riemannian and symplectic manifolds, it is in some ways more complicated than the study of the quantum counterparts, at least in the setting of flat spacetimes. This thesis investigates some previously unidentified structures that appear critical to the study of classical mechanics in an abstract setting and that promise more generally to significantly enlarge the scope of application of categories to the study of complicated systems.

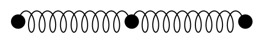

Figure 1.1: Three Masses

Figure 1.1 represents a system with three point masses attached by springs, where all motion is along the same line. Figure 1.2 represents the more complicated system formed by attaching additional point masses and springs in series. View a pair of point masses attached by a spring as a fundamental component, or subsystem, of one of these more complicated systems. The spring-mass subsystems are open systems in the sense that both forces internal to the subsystem and external forces of the larger system govern the dynamics of the subsystems. A study of the combined spring-mass system of Figure 1.2 motivates our current investigation. The system has a state space that is either the tangent space to a Riemannian manifold in the Lagrangian description or is a symplectic manifold in the Hamiltonian description [2].

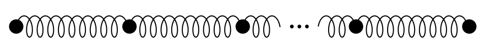

Figure 1.2: Many Masses

A path in the state space models the path of motion of each of the masses. Mappings from the state space of the combined spring mass system to the state spaces of the subsystems should permit the state spaces of the subsystems to be viewed locally as embedded Riemannian or symplectic submanifolds of the state space of the combined system, where the Riemannian or symplectic structures are consistent with that of the larger manifold. This restriction on the admissible mappings between the state spaces implies that a Lagrangian description involves objects and morphisms in a category of Riemannian manifolds with surjective Riemannian submersions and that a Hamiltonian description involves objects and morphisms in a category of symplectic manifolds with surjective Poisson maps.

Figure 1.3 depicts a linking of subsystems to form a larger system, where two spring-mass systems combine by identification of a center mass given by the right mass of the spring-mass system on the left and the left mass of the spring-mass system on the right. Figure 1.4 depicts the state spaces of the systems in Figure 1.3 from a Hamiltonian perspective. Each of the maps that Figure 1.4 depicts is a canonical projection. At the lowest level in Figure 1.3 are the three distinct masses. View each mass as moving along a line where the forces acting on each mass are external to the system. Each system has $T^*\mathbb{R}$, the cotangent bundle to $\mathbb{R}$, as its state space. At the middle level, view the system as two spring-mass systems, each with a state space given by $T^*\mathbb{R}^2$ and with an external force acting on one of the masses. The total system is a system with three masses interacting in series, where connecting springs mediate the interaction of the masses. The state space for this system is a fibered product of two copies of the symplectic manifold $T^*\mathbb{R}^2$ over the manifold $T^*\mathbb{R}$.

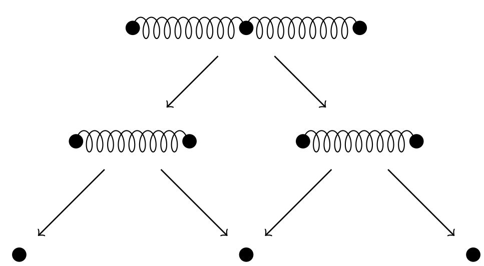

Figure 1.3: Three Point Masses

The fibered product is a six dimensional symplectic manifold, whereas the cartesian product of the state spaces is an eight dimensional symplectic manifold. While the fibered product is an embedded submanifold of the product, it will not be a symplectic submanifold when endowed with the symplectic structure that it requires to be the state space of the given physical system. The Lagrangian setting is similar, but uses tangent bundles rather than cotangent bundles as the state spaces. The fibered product together with its canonical projections appear to encapsulate the physical

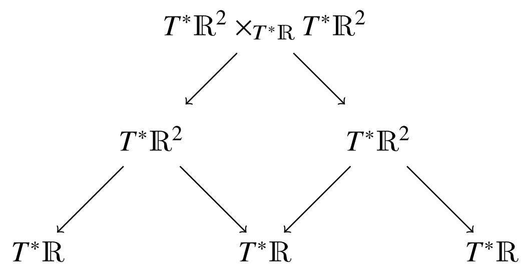

Figure 1.4: Three Mass Phase Space

meaning of identifying the right mass of the left spring-mass system with the left mass of the right spring-mass system. Both Dazord in [17] and Marle in [27] had similar insights with respect to studying constrained systems, which are similar to the systems given above in the sense that the masses that connect our systems can be thought of as a geometric constraint. In fact, Dazord explicitly uses fibered products to construct the configuration and state spaces for certain constrained systems.

Suppose that X, Y, and Z are sets and f and g are functions that respectively map X and Y to the set Z. Denote by ρ_X and ρ_Y the respective canonical projections

$$\rho_X\colon X \times Y \to X \quad \text{and} \quad \rho_Y\colon X \times Y \to Y.$$

Denote by π_X and π_Y the respective restrictions of ρ_X and ρ_Y to the fibered product $X \times_Z Y$, the subset of $X \times Y$ consisting of all elements on which f is equal to g. Maintain this notation henceforth. The fibered product in the category Set, whose objects are sets and whose morphisms are functions, has certain universal properties to be studied in Section 3.2. The connection between these universal properties and the construction of span categories for modeling classical mechanical systems is a central theme of the current investigation.

A span in the category Set is a pair of functions with the same source. The fibered product together with the span (π_X, π_Y) gives a prescription for composing certain spans in Set. Bénabou proved in [12] that if $\mathscr{C}$ is a category with pullbacks then there is a bicategory, Span($\mathscr{C}$), whose objects, morphisms, and 2-morphisms are the respective objects, spans, and maps of spans in $\mathscr{C}$. To avoid unnecessary complications, view this bicategory as a category, a *span category*, by ignoring the bicategory structure and taking isomorphism classes of spans in $\mathscr{C}$, to be defined in Section 3.1, as the morphisms. Fibered products define a composition of isomorphism classes of certain spans in Set that seems strikingly similar to the way in which classical mechanical systems appear to compose. Earlier works have used span and cospan categories to study the composition of physical systems. For example, Baez and Pollard used cospans in [9] to study reaction networks. Haugseng used spans to study classical topological field theories in [22]. In [19], Fong developed the notion of a decorated cospan, broadening the potential use of cospan categories in the modeling of physical systems.

Professor John Baez initiated the current line of research by proposing that the study of classical mechanics might have a foundation in category theory, in particular, that classical systems could be morphisms in an appropriate span category, where composition of morphisms using fibered products would describe the composition of physical systems. An abstract formalization of classical mechanics should deepen our understanding of the foundations of classical mechanics and may also offer a way to automate the modeling of classical mechanical systems. It also promises to provide model independent descriptions of classical mechanical systems. The current study requires substantial extensions of known tools in category theory. Modeling classical mechanical systems necessitates working with spans in categories other than Set, where the fibered product lacks the universal properties that it has in Set.

Chapter 5 defines an augmented span, a physical system, and an isomorphism class of augmented spans. The language and approach it employs is arguably nonstandard from a category theorist's perspective but we have found it both helpful for presenting the results to non-specialists in category theory and for use in practical applications. An isomorphism class of augmented spans that can describe a physical system from either a Lagrangian or Hamiltonian perspective encodes all observable information in a physical system. It is natural to view a physical system as an isomorphism class of spans in the category of Riemannian manifolds with surjective Riemannian submersions in the Lagrangian setting or as an isomorphism class of spans in the category of symplectic manifolds with surjective Poisson maps in the Hamiltonian setting. Section 5.3 makes use of Example 3.3.4 to demonstrate that neither of these categories has pullbacks, and so the work of Bénabou does not apply. For this same reason, it does not appear that the work of Fong can be modified from its cospan setting to a span setting that is useful to the present discussion. Denote by Diff the category whose objects are smooth manifolds and whose morphisms are smooth functions. Since two submanifolds of a given manifold may not intersect transversally, the fibered product of manifolds is not necessarily a manifold and so Diff does not have pullbacks. This technical difficulty that Spivak encounters in [30] parallels a central technical difficulty of the thesis. Spivak uses a homotopy pullback rather than a pullback because the fibered product in his setting is not necessarily a smooth manifold. The fibered products appearing in the thesis will necessarily be smooth manifolds, but the universality condition of a pullback fails. Spivak's approach does not seem applicable to the current setting because the categories that appear in classical mechanics have more structure than Diff and the study of classical mechanical systems requires some preservation of the additional structure.

Section 4.1 defines an $\mathcal{F}$-pullback of a cospan in $\mathscr{C}$ and the span tightness of the functor $\mathcal{F}$, as well as the composite of two spans along an $\mathcal{F}$-pullback. While the notion of an $\mathcal{F}$-pullback generalizes the notion of a pullback in a way that is sufficient for the current setting, without an additional condition on $\mathcal{F}$ it is not enough to provide a method for composing isomorphism classes of spans. Section 4.2 proves that if the functor $\mathcal{F}$ is span tight, then there exists a category $\mathrm{Span}(\mathscr{C}, \mathcal{F})$ whose objects are the objects of $\mathscr{C}$ and whose morphisms are isomorphism classes of spans in $\mathscr{C}$. Composition in this generalized span category is defined using $\mathcal{F}$-pullbacks and appears to depend on the functor $\mathcal{F}$. Generalized span categories determine the kinematical properties of a physical system in the Hamiltonian setting and the free systems in the Lagrangian setting. We use the notion of an augmentation of a span in order to construct, in Chapter 6, the augmented generalized span categories HamSy and LagSy. In the Hamiltonian setting, the augmentations determine the dynamical evolution of the system. In the Lagrangian setting, the augmentations determine the potentials of the physical systems, hence their dynamics as well. The categories LagSy and HamSy provide a framework for studying physical systems respectively from the Lagrangian and Hamiltonian perspectives. Section 6.2 introduces a functor $\mathscr{L}$ from LagSy to HamSy that translates from the Lagrangian to the Hamiltonian perspective, an analog of the Legendre transformation in a category theoretic setting. The augmentations we introduce greatly generalize certain aspects of Fong's work in [19]. Further generalization of augmentations should more completely generalize the decorations of [19]. These categories provide a precise framework for describing certain complicated physical systems as composite physical systems with open constituent parts that are each easier to model than the original system. While this section works out a basic example, future work will more thoroughly address applications to more complicated systems.

This thesis is based on and heavily borrows from [10] and [33].

Chapter 2

Background

2.1 Differential Geometry

Smooth Manifolds

Refer to [8] and [23] as standard references for smooth manifold theory. We present some well known definitions in order to explicitly establish language and notational conventions.

Definition 2.1.1. An *m-dimensional manifold* is a triple $(M, \mathcal{T}_M, \mathcal{A}_M)$ such that

(1) M is a set;

(2) $\mathcal{T}_M$ is a topology for M that is Hausdorff and second countable;

(3) $\mathcal{A}_M$ is an atlas, a collection of homeomorphisms such that the domain of each element of $\mathcal{A}_M$ is an open subset of M, the collection of domains of the elements of $\mathcal{A}_M$ form an open cover for M, and the range of each element of $\mathcal{A}_M$ is an open subset of $\mathbb{R}^m$.

If $\mathcal{A}_M$ is maximal with respect to the property that for any ϕ and ψ in $\mathcal{A}_M$ that have intersecting domains, the *transition function* $\phi \circ \psi^{-1}$ and its inverse are of class C^r (r-times continuously differentiable), then M is a C^r*– manifold*. Only the smooth case, when r is infinity, is relevant to the present work. Refer to the elements of $\mathcal{A}_M$ as *coordinates* and refer to their domains as *charts*.

It is customary to denote by M a manifold $(M, \mathcal{T}_M, \mathcal{A}_M)$ and we generally follow this convention, except when it is important to explicitly distinguish between the manifold, the topological space associated to the manifold, and the underlying set. Reference to the manifold M, the topological space M, and the underlying set M, will respectively be a reference to the triple $(M, \mathcal{T}_M, \mathcal{A}_M)$, the pair $(M, \mathcal{T}_M)$, and the set M. Unless stated otherwise, all manifolds in this thesis are smooth. Denote the set of smooth real-valued functions on M by $C^\infty(M)$.

Definition 2.1.2. A *derivation* D at the point x in M is a linear function from $C^\infty(M)$ to $C^\infty(M)$ that has the Leibniz property, meaning for all f and g in $C^\infty(M)$,

$$D(fg)(x) = (Df)g(x) + f(x)(Dg).$$

Definition 2.1.3. Let M be a manifold and p be in M. Define T_pM, the *tangent space of M at p*, to be the set of all derivations at the point p.

Definition 2.1.4. A *bundle* is a pair of manifolds E and B together with a map $\pi\colon E \to B$, a triple (E, B, π). The manifold B is the *base space*. The manifold E is the *total space*. The map π is called a *projection*. For any point x in B the set $\pi^{-1}(x)$ is the *fiber over x*.

Definition 2.1.5. A bundle with total space E, base space B and projection π is *locally trivializable* if there is a manifold F, the *standard fiber*, such that for any x in B there is an open subset U of B containing x and a homeomorhism $\phi\colon \pi^{-1}(U) \to U \times F$ such that for each z in $\pi^{-1}(U)$,

$$\pi(z) = \mathrm{proj}_1(\phi(z)),$$

where proj_1 is the projection onto the first coordinate.

Definition 2.1.6. A *fibre bundle* is a locally trivializable bundle (E, B, π) where the map π is a continuous surjection. A *smooth fibre bundle* is a fibre bundle in the category of smooth manifolds.

Definition 2.1.7. The *tangent bundle of a manifold* M is the triple (TM, M, ρ_M) where TM is the disjoint union

$$TM = \bigsqcup_{x \in M} T_x M \quad \text{and} \quad \rho_M(v) = x \quad \forall v \in T_x M.$$

Definition 2.1.8. Let M be a smooth manifold and suppose x is in M. The set $T^*_x M$ of all linear maps from $T_x M$ to $\mathbb{R}$ is the *cotangent space of* M *at* x.

Definition 2.1.9. The *cotangent bundle* is the triple (T^*M, M, π_M) where T^*M is the disjoint union

$$T^*M = \bigsqcup_{x \in M} T^*_x M \quad \text{and} \quad \pi_M(\theta) = x \quad \forall \theta \in T^*_x M.$$

As is customary, refer respectively to TM and T^*M as the tangent and cotangent bundles rather than the appropriate triple. If M is manifold of dimension m, then for each x in M, $T_x M$ and $T^*_x M$ are m-dimensional vector spaces and both TM and T^*M are $2m$-dimensional smooth manifolds.

Definition 2.1.10. A *section* of a bundle (E, B, π) is a map $\sigma\colon B \to E$ such that for any x in B, $\sigma(x)$ is in $\pi^{-1}(x)$.

Definition 2.1.11. A *smooth vector field* (henceforth just a vector field) on a manifold M is a smooth section of TM. A *smooth covector field* (henceforth just a vector field) or 1-*form* on a manifold M is a smooth section of T^*M.

Definition 2.1.12. Suppose that v is a vector field on M. An *integral curve of* v is a differentiable curve $\gamma\colon [0,1] \to M$ such that for any differentiable function f on M,

$$v\Big|_{\gamma(0)} f = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f \circ \gamma)(t).$$

Poisson Geometry

For further background and discussion on Poisson geometry refer to [25] and [15]. We provide some common definitions for the reader's convenience.

Definition 2.1.13. A *Poisson bracket* on a smooth manifold M is a bilinear function

$$\{\cdot,\cdot\}\colon C^\infty(M)\times C^\infty(M)\to C^\infty(M)$$

that satisfies the following:

(1) *Antisymmetry:* $\{f,g\} = -\{g,f\}$

(2) *Bilinearity:* $\{f, ag+bh\} = a\{f,g\} + b\{f,h\}$

(3) *Jacobi Identity:* $\{f,\{g,h\}\} + \{\{g,h\},f\} + \{h,\{f,g\}\} = 0$

(4) *Leibniz Law:* $\{fg,h\} = \{f,h\}g + f\{g,h\}$.

Definition 2.1.14. A *Poisson manifold* is the pair consisting of a smooth manifold M and a Poisson bracket on M.

Definition 2.1.15. Suppose that $(M,\{\cdot,\cdot\}_M)$ and $(N,\{\cdot,\cdot\}_N)$ are Poisson manifolds. For each f in $C^\infty(M)$, the *Poisson vector field associated to* f is the derivation v_f given by

$$v_f(\cdot) = \{\cdot, f\}_M .$$

Note that the fact that the Poisson bracket satisfies the Leibniz law implies that the Poisson vector field v_f associated to a function f is, indeed, a derivation. The fact that the Poisson bracket satisfies the Jacobi identity implies that v_f is a derivation on the Lie algebra $C^\infty(M)$, where the Poisson bracket gives $C^\infty(M)$ the structure of a Lie algebra.

Definition 2.1.16. A smooth map Φ from M to N is a *Poisson map* if for any f and g in $C^\infty(N)$,

$$\{f,g\}_N\circ\Phi \;=\; \{f\circ\Phi, g\circ\Phi\}_M .$$

The above equality can be alternatively written as

$$\Phi^*\{f,g\}_N \;= \{\Phi^* f, \Phi^* g\}_M .$$

Symplectic Geometry

Symplectic manifolds are the primary objects of study in Hamiltonian mechanics. For further background in symplectic geometry see [2], [23] and [28].

Definition 2.1.17. A *symplectic vector space* is a pair (V, ω_V) where V is a vector space and ω_V is a symplectic form on V, a function on $V \times V$ that for each u, v, and w in V and each a and b in $\mathbb{R}$ satisfies

(1) (Linearity): $\omega_V(au + bv, w) = a\omega_V(u, w) + b\omega_V(v, w)$;

(2) (Skew-symmetry): $\omega_V(v, w) = -\omega_V(w, v)$;

(3) (Nondegeneracy): if $\omega_V(v, y) = 0$ for all y in V, then v is the zero vector.

Definition 2.1.18. Let (V, ω_V) be a symplectic vector space and W be a linear subspace of V. Define the *symplectic complement* of W to be the set

$$W^\omega = \{v \in V : \quad \omega_V(v, w) = 0 \quad \text{for all} \quad w \in W\}.$$

Definition 2.1.19. A linear subspace W of a vector space V is *symplectic* if

$$W \cap W^\omega = \{0\}.$$

Definition 2.1.20. A linear subspace W of a vector space V is *Lagrangian* if $W = W^\omega$.

Definition 2.1.21. A *symplectic manifold* is a pair (M, ω_M), where M is an even dimensional smooth manifold and ω_M is a 2-form on M that is a symplectic form on each fiber of TM.

Example 2.1.22. The smooth even dimensional manifold $\mathbb{R}^{2n}$ paired with ω is a symplectic manifold, where $(q_i, p_i)_{i=1}^n$ are coordinate functions on $\mathbb{R}^{2n}$ and

$$\omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

The pair $(\mathbb{R}^{2n}, \omega)$ is a symplectic manifold.

Example 2.1.23. The projection π maps T^*M to M and so $\mathrm{d}\pi$ is a map from $T(T^*M)$ to TM. Define a 1-form λ in the following way. If v is in $T(T^*M)$, then there is an ℓ in T^*M so that v is in $T_\ell(T^*M)$, and so $\mathrm{d}\pi_\ell$ maps v to a tangent vector of M. Take

$$\lambda(v) = \ell(\mathrm{d}\pi(v)).$$

The form λ is the *tautological 1-form* on the cotangent bundle. If $(x_1, x_2, \dots, x_m)$ are smooth local coordinates on M and $(x_1, x_2, \dots, x_m, \ell_1, \ell_2, \dots, \ell_m)$ are smooth local coordinates on T^*M, then

$$\lambda = \sum_{i=1}^{m} \ell_i \mathrm{d}x_i.$$

The 2-form, $-\omega_{T^*M}$, is the exterior derivative of the tautological 1-form and is a symplectic form on T^*M, [2, p. 202]. Since ω_{T^*M} is exact, it will be closed. Write ω_{T^*M} in the above local coordinates to see that it is the standard symplectic form on $\mathbb{R}^{2m}$, implying that ω_{T^*M} is nondegenerate. The pair (T^*M, ω_{T^*M}) is a symplectic manifold and ω_{T^*M} is the *canonical symplectic form* on T^*M.

Definition 2.1.24. Suppose that (X, ω_X) is a symplectic manifold with an embedded submanifold N and suppose that p is a point in N. The submanifold N is *symplectic* (*Lagrangian*) if the linear subspace T_pN of T_pX is symplectic (Lagrangian).

Definition 2.1.25. Let (X, ω_X) and (Y, ω_Y) be symplectic manifolds. A smooth map Φ from X to Y is *symplectic* if

$$\Phi^*\omega_Y = \omega_M.$$

Definition 2.1.26. A diffeomorphism Φ from a symplectic manifold (X, ω_X) to a symplectic manifold (Y, ω_Y) that is symplectic is a *symplectomorphism*.

A basic argument shows that any symplectic vector space is necessarily even dimensional. If M is a symplectic manifold, then for any point x in M, the vector space T_xM is a symplectic vector space and so even dimensional, implying that M is even dimensional. The requirement that every symplectic manifold be even dimensional is discussed in [28, p.38-40]. The following theorem shows that symplectic manifolds have no local invariants and we refer the reader to the proof by

V.I. Arnol'd in [2, p.230-232]. The symplectic 2-form also naturally distinguishes position and momentum coordinates on M.

Theorem 2.1.27 (*Darboux*)**.** *Suppose that the dimension of M is $2m$. For each x in M, there is a chart U containing x such that the symplectic 2-form gives rise to* Darboux coordinates $(q_i, p_i)_{i=1}^m$ *on U, coordinates such that*

$$\omega_M = \sum_{i=1}^{m} \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

Proposition 2.1.28. *Let (X, ω_X) and (Y, ω_Y) be symplectic manifolds. Suppose that $\rho_X\colon X \times Y \to X$ and $\rho_Y\colon X \times Y \to Y$ are the standard projection maps. Then $(X \times Y, \omega_{X\times Y})$ is a symplectic manifold with $\omega_{X\times Y} = \rho_X^* \omega_X + \rho_Y^* \omega_Y$.*

Proof. Let X and Y be symplectic manifolds of respective dimensions $2m$ and $2n$. Since X and Y are smooth manifolds, $X \times Y$ is a smooth manifold of dimension $2m + 2n$. To show that the even dimensional manifold $X \times Y$ is symplectic, it suffices to show that the 2−form $\omega_{X\times Y}$ given in the statement of the lemma is closed and nondegenerate.

Since d commutes with ρ_X^* and ρ_Y^* and since ω_X and ω_Y are closed,

$$\mathrm{d}\omega_{X\times Y} = \mathrm{d}(\rho_X^* \omega_X + \rho_Y^* \omega_Y) = \mathrm{d}(\rho_X^* \omega_X) + \mathrm{d}(\rho_Y^* \omega_Y) = \rho_X^* \mathrm{d}\omega_X + \rho_Y^* \mathrm{d}\omega_Y = 0,$$

implying that $\omega_{X\times Y}$ is a closed 2−form.

Since X is symplectic, Darboux's theorem implies that for any x in X there exists an open neighborhood U of x and local coordinates $(x_i, p_i)_{i=1}^m$ on U such that

$$\omega_X = \sum_{i=1}^{m} \mathrm{d}x_i \wedge \mathrm{d}p_i.$$

Similarly, for any y in Y there exists an open neighborhood V of y and local coordinates $(y_i, q_i)_{j=1}^n$ on V such that

$$\omega_Y = \sum_{j=1}^{n} \mathrm{d}y_j \wedge \mathrm{d}q_j.$$

Let $(\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{p}_1, \ldots, \tilde{p}_m, \tilde{y}_1, \ldots, \tilde{y}_n, \tilde{q}_1, \ldots, \tilde{q}_n)$ be local coordinates on $U \times V$ with

$$\tilde{x}_i = x_i \circ \rho_X, \tilde{p}_i = p_i \circ \rho_X, \tilde{y}_j = y_j \circ \rho_Y \quad \text{and} \quad \tilde{\mathrm{q}}_{\mathrm{j}} = \mathrm{q}_{\mathrm{j}} \circ \rho_{\mathrm{Y}}$$

so that

$$\rho_X^*(\mathrm{d}x_i) = \mathrm{d}(x_i \circ \rho_X) = \mathrm{d}\tilde{x}_i.$$

Analogous equalities hold for the other coordinates, implying that $\omega_{X\times Y}$ can be written in local coordinates on $U \times V$ as

$$\omega_{X\times Y} = \sum_{i=1}^{m} \mathrm{d}\tilde{x}_i \wedge \mathrm{d}\tilde{p}_i + \sum_{j=1}^{n} \mathrm{d}\tilde{y}_j \wedge \mathrm{d}\tilde{q}_j.$$

For $\omega_{X\times Y}$ to be nondegenerate means that for any α in $X \times Y$ and any nonzero v in $T_\alpha(X \times Y)$ there exists u in $T_\alpha(X \times Y)$ such that $\omega_{X\times Y}(v, u)$ is nonzero. Suppose v is in $T_\alpha(X \times Y)$ and for any u in $T_\alpha(X \times Y)$, $\omega_{X\times Y}(v, u)$ is 0. There exists coefficients a^i, b^i, c^j, e^j such that

$$v = a^i \partial\tilde{x}_i + b^i \partial\tilde{p}_i + c^j \partial\tilde{y}_j + e^j \partial\tilde{q}_j.$$

If u is equal to $\partial\tilde{x}_i$ then

$$-\omega_{X\times Y}(v, u) = -\omega_{X\times Y}(a^i \partial\tilde{x}_i - b^i \partial\tilde{p}_i - c^j \partial\tilde{y}_j - e^j \partial\tilde{q}_j, \partial\tilde{x}_i) = b^i = 0.$$

By assumption,

$$\omega_{X\times Y}(\mathrm{v}, \partial\tilde{x}_i) = \omega_{X\times Y}(v, \partial\tilde{p}_i) = \omega_{X\times Y}(v, \partial\tilde{y}_j) = \omega_{X\times Y}(v, \partial\tilde{q}_j) = 0.$$

Follow the above calculation to obtain the equalities

$$a^i = c^j = e^j = 0, \quad \text{hence,} \quad \mathrm{v} = 0.$$

By contraposition, $\omega_{X\times Y}$ is nondegenerate. □

Every symplectic manifold has a Poisson structure that it inherits from its symplectic structure in the following way. The symplectic 2-form induces an isomorphism Ω_M between the tangent and cotangent bundles. Given tangent vectors v and w in the same fiber of TM, define by $\Omega_M(v)$ the covector

$$\Omega_M(v) = \omega_M(\cdot, v) \colon w \mapsto \omega_M(w, v).$$

Since ω_M is nondegenerate, the map Ω_M is invertible. For each function f in $C^\infty(M)$, denote by D_f the *symplectic gradient of* f, which is defined by

$$D_f = \Omega_M^{-1}(\mathrm{d}f).$$

Definition 2.1.29. For any symplectic manifold (M, ω_M), define a Poisson bracket $\{\cdot, \cdot\}_M$ on pairs (f, g) in $C^\infty(M) \times C^\infty(M)$ by

$$\{f, g\}_M = \omega_M\left(D_f, D_g\right).$$

The *symplectic gradient* D_f is the Poisson vector field v_f associated to f, implying that

$$\{f, g\}_M = \omega_M\left(v_f, v_g\right).$$

Definition 2.1.30. An *almost symplectic manifold* is a pair (M, ω_M), where ω_M is a nondegenerate 2-form that satisfies the Leibniz law, but may or may not satisfy the Jacobi identity.

An almost symplectic manifold has a bracket that is induced by its nondegenerate 2-form in the same way that the symplectic form on a symplectic manifold gives rise to a bracket. The statement of Theorem 2.1.31 can be found in [15, p.21].

Theorem 2.1.31. *The bracket* $\{\cdot, \cdot\}$ *on an almost symplectic manifold* (M, ω_M) *satisfies the Jacobi identity if and only if* $\mathrm{d}\omega_M = 0$.

The real valued function Π_M defined by

$$\Pi_M(\mathrm{d}f, \mathrm{d}g) = \{f, g\}_M$$

is a section of $(T^*M \wedge T^*M)^*$.

Definition 2.1.32. The *Poisson bivector of* $(M, \{\cdot,\cdot\}_M)$ is the image of the function Π_M under the canonical isomorphism that takes $(T^*M \wedge T^*M)^*$ to $\Lambda^2 TM$. To simplify notation, denote henceforth by Π_M the Poisson bivector of $(M, \{\cdot,\cdot\}_M)$.

Clairaut's theorem implies the following proposition.

Proposition 2.1.33. *The manifold* $\mathbb{R}^{2n}$ *with coordinate functions* $(q_i, p_i)_{i=1}^n$ *is a Poisson manifold with the bracket*

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}.$$

Refer to [15, p. 30] for Proposition 2.1.34 and [15, p. 44] for Proposition 2.1.35.

Proposition 2.1.34. *A smooth map* Φ *from* $(M, \{\cdot,\cdot\}_M)$ *to* $(N, \{\cdot,\cdot\}_N)$ *is a Poisson map if and only if*

$$\mathrm{d}\Phi(\Pi_M) = \Pi_N.$$

Proof. Suppose Φ is a Poisson map. For any functions f and g in $C^\infty(N)$ and and point x be a point in M,

$$(\mathrm{d}\Phi_x \Pi_M)(\mathrm{d}f, \mathrm{d}g) = \Pi_M\Big|_x(\Phi^*\mathrm{d}f, \Phi^*\mathrm{d}g) = \{\Phi^* f, \Phi^* g\}_M(x).$$

The map Φ is Poisson and so

$$\begin{aligned}\{\Phi^* f, \Phi^* g\}_M(x) &= \{f \circ \Phi, g \circ \Phi\}_M(x)\\ &= \{f, g\}_N \circ \Phi(x) = \Pi_N\Big|_{\Phi(x)}(\mathrm{d}f, \mathrm{d}g).\end{aligned}$$

If $\mathrm{d}\Phi(\Pi_M)$ is equal to Π_N, then

$$\begin{aligned}(\{f, g\}_N \circ \Phi)(x) &= \Pi_N\Big|_{\Phi(x)}(\mathrm{d}f, \mathrm{d}g)\\ &= \mathrm{d}\Phi_x \Pi_M(\mathrm{d}f, \mathrm{d}g)\\ &= \Pi_M\Big|_x(\mathrm{d}(f \circ \Phi), \mathrm{d}(g \circ \Phi)) = \{f \circ \Phi, g \circ \Phi\}_M.\end{aligned}$$

Therefore,

$$\{f, g\}_N \circ \Phi = \{f \circ \Phi, g \circ \Phi\}_M,$$

and so Φ is a Poisson map. □

The following proposition is stated and proved in [15, p. 44].

Proposition 2.1.35. *Suppose that $(M, \{\cdot, \cdot\}_M)$ is a Poisson manifold and (N, ω_N) symplectic manifold. Every Poisson map from M to N is a submersion.*

Icthyomorphisms and Symplectomorphisms

Definition 2.1.36. A diffeomorphism Φ from $(M, \{\cdot, \cdot\}_M)$ to $(N, \{\cdot, \cdot\}_N)$ that is a Poisson map is an *icthyomorphism*.

Proposition 2.1.37. *If $(M, \{\cdot, \cdot\}_M)$ and $(N, \{\cdot, \cdot\}_N)$ are Poisson manifolds and Φ is an icthyomorphism from M to N, then $\Phi^{-1} \colon N \to M$ is an icthyomorphism.*

Proof. Since Φ is a diffeomorphism, Φ^{-1} is a smooth bijection. It suffices to show that Φ^{-1} is a Poisson map. Suppose that h and k are in $C^\infty(M)$. Since Φ is Poisson,

$$\begin{aligned}\Phi^*\{(\Phi^{-1})^*h, (\Phi^{-1})^*k\}_N &= \{\Phi^*(h \circ \Phi^{-1}), \Phi^*(k \circ \Phi^{-1})\}_M \\ &= \{h \circ \Phi^{-1} \circ \Phi, k \circ \Phi^{-1} \circ \Phi\}_M = \{h, k\}_M.\end{aligned}$$

Therefore,

$$\Phi^*\{(\Phi^{-1})^*h, (\Phi^{-1})^*k\}_N = \{h, k\}_M$$

and so

$$(\Phi^{-1})^* \circ \Phi^*\{(\Phi^{-1})^*h, (\Phi^{-1})^*k\}_N = (\Phi^{-1})^*\{h, k\}_M,$$

hence,

$$(\Phi^{-1})^*\{h, k\}_M = \{(\Phi^{-1})^*h, (\Phi^{-1})^*k\}_N.$$

□

We now discuss the difference between an icthyomorphism and symplectomorphism. In general, symplectic maps between symplectic manifolds are immersions whereas Poisson maps between symplectic manifolds are submersions. An example in [15, p. 37] explains the difference, which we now present.

Example 2.1.38. Let $\mathbb{R}^2$ and $\mathbb{R}^4$ be symplectic manifolds and let ι be the inclusion map from $\mathbb{R}^2$ to $\mathbb{R}^4$ defined by mapping the coordinates $(q_1, p_1) \mapsto (q_1, p_1, 0, 0)$. The map ι will be symplectic but not Poisson because $\{q_2, p_2\}_{\mathbb{R}^4} = 1$, whereas the bracket on $\mathbb{R}^2$ of their pull-backs is zero. Now let π be the projection map from $\mathbb{R}^4$ to $\mathbb{R}^2$ defined by $(q_1, p_1, q_2, p_2) \mapsto (q_1, p_1)$. Then π is a Poisson map but not symplectic. This is because $\pi^*\omega_{\mathbb{R}^2} = dq_1 \wedge dp_1 \neq \omega_{\mathbb{R}^4}$.

The next proposition provides conditions that guarantee the equivalence of icthyomorphisms and symplectomorphism. The proof can be found in [1, p.195]

Proposition 2.1.39. *Let (X, ω_X) and (Y, ω_Y) be symplectic manifolds and let Φ be a diffeomorphism from X to Y. The diffeomorphism Φ is a symplectomorphism if and only if Φ is an icthyomorphism.*

Riemannian Geometry

We present here some basic ideas in Riemannian geometry. For further background see [24].

Definition 2.1.40. A *Riemannian manifold* is a pair (M, g_M) where g_M is a smooth $(0, 2)$-tensor field that is symmetric and positive definite, that is:

(1) (Symmetric) for all p in M and all (v, w) in T_pM,

$$g_M(v, w) = g_M(w, v)\,;$$

(2) (Positive-Definite) for all non-zero v in TM,

$$g_M(v, v) > 0.$$

Example 2.1.41. Take g to be the standard inner product on $\mathbb{R}^n$. The pair $(\mathbb{R}^n, g)$ is a Riemannian manifold.

Riemannian manifolds are the primary objects of study in Lagrangian mechanics. The metric on the tangent bundle of a Riemannian manifold gives a kinetic energy associated to a particle moving in the base manifold which is the configuration space for the system, [2, p.83-84].

Definition 2.1.42. A *Riemannian submersion* Φ from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N) is a smooth submersion with the property that if v and w are vector fields tangent to the horizontal space $(\ker(\mathrm{d}\Phi))^{\perp}$, then

$$g_M(v, w) = g_N(\mathrm{d}\Phi(v), \mathrm{d}\Phi(w)) .$$

Definition 2.1.43. Let (M, g_M) and (N, g_N) be Riemannian manifolds and let Φ be a diffeomorphism from M to N. If Φ is a Riemannian submersion, then Φ is an *isometry*.

2.2 Classical Mechanics

Refer to [2] and [11] as sources for further background material in classical mechanics.

Definition 2.2.1. Take M to be a symplectic manifold of dimension $2m$. The *Hamiltonian* is a smooth real valued function, H, on M.

The *Hamiltonian vector field* is the vector field v_H where

$$v_H(f) = \{f, H\}.$$

Equivalently, this is the vector field with

$$\omega(v_H, \cdot) = \mathrm{d}H.$$

Darboux's theorem implies that every point of M lies in a chart U with coordinates $(q_1, \ldots, q_m, p_1, \ldots p_m)$ so that

$$\omega_M = \sum_{i=1}^{m} \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

A curve γ is an integral curve of v_H if and only if

$$\frac{\mathrm{d}(q_i \circ \gamma)}{\mathrm{d}t}(t) = \frac{\partial H}{\partial p_i}(\gamma(t)) \quad \text{and} \quad \frac{\mathrm{d}(p_i \circ \gamma)}{dt}(t) = -\frac{\partial H}{\partial q_i}(\gamma(t)).$$

These equations are known as Hamilton's equations. For any such curve γ,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \gamma(t) = v_H(\gamma(t_0))$$

and so the hamiltonian function is constant along the integral curves of the hamiltonian vector field. The hamiltonian will describe the energy of the system, the integral curves of the hamiltonian vector fields will be paths of motion of the system, and the energy is conserved along the paths of motion.

For any smooth function

$$F \colon M \to \mathbb{R},$$

Hamilton's equations for a path of motion imply that if γ is a path of motion, then

$$\begin{aligned}\frac{d}{dt}F(\gamma(t)) &= \sum_{i=1}^{m}\left(\frac{\partial F}{\partial q_i}(\gamma(t))\frac{d(q_i \circ \gamma)}{dt}(t) + \frac{\partial F}{\partial p_i}(\gamma(t))\frac{d(p_i \circ \gamma)}{\mathrm{d}t}(t)\right) \\ &= \sum_{i=1}^{m}\left(\frac{\partial F}{\partial q_i}(\gamma(t))\frac{\partial H}{\partial p_i}(\gamma(t)) - \frac{\partial F}{\partial p_i}(\gamma(t))\frac{\partial H}{\partial q_i}\right) = \{H, F\}(\gamma(t)).\end{aligned}$$

Euler-Lagrange Equations on a Riemannian Manifold

Suppose that M is a Riemannian manifold, g_M is the Riemannian metric on M, and V_M is a potential associated to M. Define the *Lagrangian of* M on TM to be the function $\mathcal{L}_M$, where

$$\mathcal{L}_M(v) = \frac{1}{2}g_M(v, v) - V_M(\rho_M(v)) \quad \text{with} \quad v \in TM.$$

Definition 2.2.2. A path in the Riemannian manifold (M, g_M) is a *path of motion of M* if it is extremal for the action integral of $\mathcal{L}_M$ under smooth variations with fixed endpoints.

Define on each v in TM the function $\flat_M$ by

$$\flat_M(v) = g_M(v, \cdot).$$

The non-degeneracy of the metric g_M implies that the map $\flat_M$ is an invertible function from TM to T^*M. Define by $\sharp_M$ the inverse of $\flat_M$ with

$$\sharp_M : T^*M \to TM \quad \text{by} \quad \theta \mapsto v, \quad \text{where} \quad \theta = g_M(v, \cdot) \quad \text{and} \quad (\theta, v) \in T^*M \times TM.$$

Denote by $\mathrm{grad}_M(V_M)$ the vector field

$$\mathrm{grad}_M(V_M) = \sharp_M(\mathrm{d}V_M).$$

Denote by ∇^M the Levi-Civita connection on the Riemannian manifold (M, g_M). A standard calculation shows that γ is a path of motion of the Riemannian manifold M if and only if it satisfies

$$\nabla^M_{\gamma'} \gamma' + \mathrm{grad}_M(V_M)\big|_\gamma = 0, \tag{EL}$$

the Euler–Lagrange equations. See [16] for further details.

2.3 Category Theory

We introduce the notion of a category here. For further background, see [26].

Definition 2.3.1. A *category $\mathscr{C}$* consists of:

(1) a class $Ob(\mathscr{C})$ of *objects* in $\mathscr{C}$ and a class $\mathrm{Hom}(\mathscr{C})$ of *morphisms* in $\mathscr{C}$;

(2) for each morphism f in $\mathrm{Hom}(\mathscr{C})$, a pair (A, B) of objects, respectively called the *source* and *target* of f;

(3) for each triple of objects A, B, and C, a mapping called *composition*,

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C),$$

written as $(f, g) \mapsto g \circ f$. Composition satisfies the following axioms:

(i) *Associativity:* $(f \circ g) \circ h = f \circ (g \circ h)$;

(ii) *Existence of Identity Morphisms:* for any objects A and B, there exists *identity morphisms* Id_A and Id_B of $\mathrm{Hom}(A, A)$ such that for every morphism f in $\mathrm{Hom}(A, B)$,

$$Id_B \circ f = f = f \circ Id_A.$$

Example 2.3.2. The class Set, whose objects are sets, morphisms are functions, and where composition of functions defines composition is a category.

Example 2.3.3. The class Top, whose objects are topological spaces, morphisms are continuous functions, and where composition of continuous functions defines composition is a category.

Example 2.3.4. The class Diff, whose objects are smooth manifolds, morphisms are smooth functions, and where composition of smooth functions defines composition is a category.

Definition 2.3.5. A *functor* $\mathcal{F}$ between two categories $\mathscr{C}$ and $\mathscr{C}'$ is a mapping that

(1) associates every object A in $\mathscr{C}$ to an object $\mathcal{F}(A)$ in $\mathscr{C}'$;

(2) associates every morphism $f\colon A \to B$ in $\mathscr{C}$ to a morphism $\mathcal{F}(f)\colon \mathcal{F}(A) \to \mathcal{F}(B)$ in $\mathscr{C}'$ such that

   (i) $\mathcal{F}(Id_A) = Id_{\mathcal{F}(A)}$;

   (ii) for all morphisms f, g in $\mathscr{C}$,

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Example 2.3.6. The *forgetful functor* from Diff to Top maps $(M, \mathcal{T}_M, \mathcal{A}_M)$ to $(M, \mathcal{T}_M)$ and maps the smooth functions to the same functions on the underlying topological space.

Example 2.3.7. The *forgetful functor* from Diff to Set which maps $(M, \mathcal{T}_M, \mathcal{A}_M)$ to M and maps the smooth functions to the same functions on the underlying set.

Chapter 3

Pullbacks and Span Categories

3.1 Span Categories

Spans and their Isomorphism Classes

Definition 3.1.1. A *span* in a category $\mathscr{C}$ is a pair of morphisms in $\mathscr{C}$ with the same source and a *cospan* in $\mathscr{C}$ is a pair of morphisms in $\mathscr{C}$ with the same target. For any span S in $\mathscr{C}$, write

$$S = (s_L, s_R),$$

where S_L, S_R, and S_A are objects in $\mathscr{C}$,

$$s_L \colon S_A \to S_L, \quad \text{and} \quad s_R \colon S_A \to S_R.$$

Utilize the same notation if S is a cospan, but where s_L and s_R respectively map S_L and S_R to S_A. For any span or cospan S of $\mathscr{C}$, refer respectively to the objects S_A, S_L, and S_R in $\mathscr{C}$ as the *apex, left foot, and right foot of* S.

Spans and cospans have respective diagrammatical realizations given by Figure 3.1 and Figure 3.2.

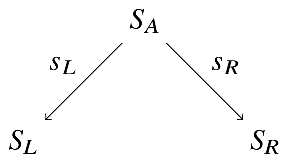

Figure 3.1: The Span S

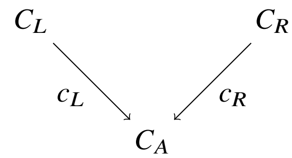

Figure 3.2: The Cospan C

Definition 3.1.2. A span S in $\mathscr{C}$ is *paired with* a cospan C in $\mathscr{C}$ if

$$C_L = S_L, \quad C_R = S_R, \quad \text{and} \quad c_L \circ s_L = c_R \circ s_R.$$

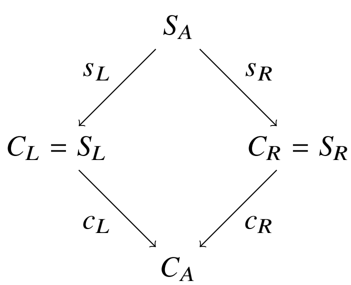

Figure 3.3: The Pairing of S with C

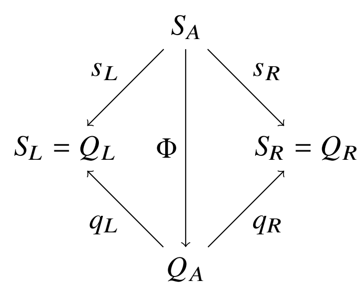

Figure 3.4: A Span Morphism from S to Q

View the pairing of a span S with a cospan C as a commutative square (Figure 3.3). Suppose that S and Q are spans in $\mathscr{C}$ with S_L equal to Q_L and S_R equal to Q_R.

Definition 3.1.3. A *span morphism in $\mathscr{C}$ from S to Q* is a morphism Φ (Figure 3.4) in $\mathscr{C}$ from S_A to Q_A with

$$s_L = q_L \circ \Phi \quad \text{and} \quad s_R = q_R \circ \Phi.$$

A *span isomorphism in $\mathscr{C}$ from S to Q* is a span morphism that is additionally an isomorphism.

Proposition 3.1.4. *For any span isomorphism Φ, the inverse Φ^{-1} is also a span isomorphism. Furthermore, a composite of span morphisms is a span morphism.*

Pullbacks in a Category $\mathscr{C}$

Composing isomorphism classes of spans in a span category requires the existence of a pullback. This subsection introduces the notion of a pullback of a cospan.

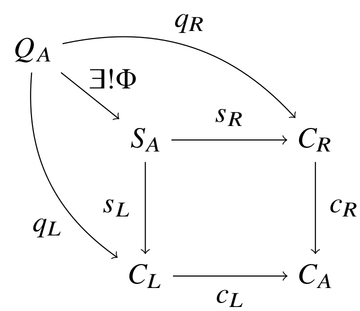

Figure 3.5: Pullback Diagram

Definition 3.1.5. A span S in $\mathscr{C}$ is a *pullback of a cospan C in $\mathscr{C}$* if it is paired with C and if for any other span Q in $\mathscr{C}$ that is also paired with C there exists a unique span morphism Φ in $\mathscr{C}$ from Q to S (Figure 3.5).

Definition 3.1.6. A category $\mathscr{C}$ *has pullbacks* if for any cospan C in $\mathscr{C}$ there is a span S in $\mathscr{C}$ that is a pullback of C and S is unique up to a span isomorphism in $\mathscr{C}$.

The pairing of a pullback S of a cospan C with C is a pullback square. We have found it useful to separately define the parts of a pullback square.

3.2 Examples of Categories that have Pullbacks

Denote by Top the category whose objects are topological spaces and whose morphisms are continuous functions. The categories Set and Top are examples of categories that have pullbacks, as S. MacLane discusses in [26] and S. Awodey discusses more specifically for Set in [3]. We provide a proof here for the convenience of the reader.

Let C be a cospan in Set and let ρ_L and ρ_R be the canonical projections

$$\rho_L\colon C_L \times C_R \to C_L \quad \text{and} \quad \rho_R\colon C_L \times C_R \to C_R.$$

Denote by S_A the fibered product

$$C_L \times_{C_A} C_R := \{(x, y) \in C_L \times C_R \colon (c_L \circ \rho_L)(x, y) = (c_R \circ \rho_R)(x, y)\}.$$

Take S_L and S_R to be respectively equal to C_L and C_R, and let s_L and s_R be the respective restrictions of ρ_L and ρ_R to the set S_A. Suppose that P is a span that is paired with C. Denote by Φ the function

$$\Phi \colon P_A \to C_L \times C_R \quad \text{by} \quad a \mapsto (p_L(a), p_R(a)) \quad (\forall a \in P_A),$$

the unique function from P_A to $C_L \times C_R$ such that

$$p_L = \rho_L \circ \Phi \quad \text{and} \quad p_R = \rho_R \circ \Phi. \tag{3.1}$$

The image of Φ is S_A and so Φ is a span morphism from P to S. Since any other span morphism from P to S defines a function from P to $C_L \times C_R$ with the property given by (3.1), the function Φ is the unique span morphism from P to S. Since P was arbitrarily chosen, the span S is a pullback of the cospan C.

Suppose that C is a cospan in Top and let ρ_L and ρ_R again be the canonical projections on $C_L \times C_R$. The product $C_L \times C_R$ with the product topology is a topological space. The fibered product S_A given above is a subset of $C_L \times C_R$ and is a topological space with the subspace topology. The projections s_L and s_R are continuous maps and so (s_L, s_R) is a pullback of C. The proof of this fact is nearly the same as the proof in the setting of Set, with the straightforward check that the mappings involved are continuous as the only modification of the proof.

The Category Span($\mathscr{C}$)

Suppose that $\mathscr{C}$ is a category with pullbacks. Suppose that $[S]$ and $[Q]$ are isomorphism classes of spans with respective representatives S and Q, and S_R is equal to Q_L. Since $\mathscr{C}$ has pullbacks, there is a span P that is a pullback of the cospan (s_R, q_L). Define by $[(s_L \circ p_L, q_R \circ p_R)]$ the composite $[S] \circ [Q]$. Take the objects in $\mathscr{C}$ to be the objects in Span($\mathscr{C}$), the isomorphism classes of spans in $\mathscr{C}$

to be the morphisms in Span($\mathscr{C}$), and S_R and S_L to respectively be the source and target of the span $[S]$. Given an object X in $\mathscr{C}$ and the identity morphism I taking X to X, define by $[(I, I)]$ the identity morphism in Span($\mathscr{C}$) with X as both source and target. It is well known that Span($\mathscr{C}$) is a category, [12]. Our treatment in Section 4.1 of generalized span categories specializes in the case when $\mathscr{C}$ has pullbacks to give a proof that Span($\mathscr{C}$) is a category. If $\mathscr{C}$ does not have pullbacks, then the existence of P is not guaranteed. The next section will demonstrate that some categories important in classical mechanics, and more generally in differential geometry, do not have pullbacks.

3.3 Some Categories that do not have Pullbacks

Some Functors that preserve Pullbacks

Denote by Diff the category whose objects are smooth manifolds and whose morphisms are smooth maps between smooth manifolds.

Suppose that $\mathscr{C}$ is a locally small category and that X is an object in $\mathscr{C}$. Denote by $\mathrm{Hom}(X, -)$ the hom functor that maps an object Y in $\mathscr{C}$ to the set $\mathrm{Hom}(X, Y)$. A functor $\mathcal{F}$ with

$$\mathcal{F} : \mathscr{C} \to \mathsf{Set}$$

is said to be *representable* if there is an object B in $\mathscr{C}$ so that $\mathcal{F}$ is naturally isomorphic to $\mathrm{Hom}(B, -)$.

The categories Diff, Top, and Set are locally small and there are forgetful functors, each to be ambiguously denoted by $\mathcal{F}$, from Diff to Top and from Top to Set given by

$$\mathcal{F}(M, \mathcal{T}_M, \mathcal{A}_M) = (M, \mathcal{T}_M) \quad \text{and} \quad \mathcal{F}(M, \mathcal{T}_M) = M.$$

The morphisms in Diff and Top are entirely determined by their action on the underlying sets and so the forgetful functor in each case maps a given source category to a subcategory of the target category. The functor obtained by composing the above forgetful functors is the forgetful functor, denoted again by $\mathcal{F}$, from Diff to Set.

We say that a functor $\mathcal{F}$ from a category $\mathscr{C}$ to a category $\mathscr{C}'$ *preserves pullbacks* if for any cospan C in $\mathscr{C}$, if S is a pullback of C, then $\mathcal{F}(S)$ is a pullback of $\mathcal{F}(C)$. The following lemma is a special case of a more general result that guarantees that representable functors preserve pullbacks [13, p. 64]. The proof of Lemma 3.3.1 is presented here for the convenience of the reader because we use a slightly different language in our definition of a pullback than does Borceux.

Lemma 3.3.1. *Suppose that $\mathscr{C}$ is a locally small category and B is an object in $\mathscr{C}$. The functor* $\mathrm{Hom}(B, -)$ *preserves pullbacks, where*

$$\mathrm{Hom}(B, -) \colon \mathscr{C} \to \mathsf{Set}.$$

Proof. Suppose that X and Y are objects in $\mathscr{C}$. For any morphism f in $\mathscr{C}$ from X to Y, denote by $\tilde{f}$ the morphism $\mathrm{Hom}(B, f)$, that is defined to act on any β in $\mathrm{Hom}(B, X)$ by

$$\tilde{f}(\beta) = f \circ \beta.$$

Suppose that C is a cospan in $\mathscr{C}$ and that S is a pullback of C. Since $\mathscr{C}$ is locally small, the functor $\mathrm{Hom}(B, -)$ maps the cospan C to a cospan $\mathrm{Hom}(B, C)$ in Set, taking the pair (c_L, c_R) to the pair $(\tilde{c_L}, \tilde{c_R})$. It similarly maps the span S to the span $\mathrm{Hom}(B, S)$. For any ψ in $\mathrm{Hom}(B, S_A)$, the fact that S is a pullback of C implies that

$$(\tilde{c_L} \circ \tilde{s_L})(\psi) = c_L \circ s_L \circ \psi = c_R \circ s_R \circ \psi = (\tilde{c_R} \circ \tilde{s_R})(\psi).$$

The span $\mathrm{Hom}(B, S)$ is therefore paired with the cospan $\mathrm{Hom}(B, C)$.

Denote respectively by ρ_L and ρ_R the canonical projections from $\mathrm{Hom}(B, C_L) \times \mathrm{Hom}(B, C_R)$ to $\mathrm{Hom}(B, C_L)$ and $\mathrm{Hom}(B, C_R)$, and by Q_A the set

$$\begin{aligned} &\mathrm{Hom}(B, C_L) \times_{\mathrm{Hom}(B, C_A)} \mathrm{Hom}(B, C_R) \\ &\qquad = \left\{\alpha \in \mathrm{Hom}(B, C_L) \times \mathrm{Hom}(B, C_R) \colon (\tilde{c_L} \circ \pi_L)(\alpha) = (\tilde{c_R} \circ \pi_R)(\alpha)\right\}. \end{aligned}$$

Let q_L and q_R be the respective restrictions of ρ_L and ρ_R to Q_A. Denote by Q the span (q_L, q_R) in Set, a pullback of the cospan $\mathrm{Hom}(B, C)$.

Suppose that α is in Q_A. In this case, there are morphisms α_L and α_R in $\mathscr{C}$ that map B to C_A, where α is equal to (α_L, α_R). Furthermore,

$$c_L \circ \alpha_L = \tilde{c_L}(\alpha_L) = (\tilde{c_L} \circ q_L)(\alpha) = (\tilde{c_R} \circ q_R)(\alpha) = \tilde{c_R}(\alpha_R) = c_R \circ \alpha_R.$$

The pair (α_L, α_R) is therefore a span in $\mathscr{C}$ that is paired with C and, since S is a pullback of C, there is a unique span morphism ϕ_α in $\mathscr{C}$ from (α_L, α_R) to S_A that maps B to S_A. Let Φ be the function from Q to $\mathrm{Hom}(B, S)$ that is defined for each α in Q_A by

$$\Phi(\alpha) = \phi_\alpha.$$

The morphism ϕ_α is a span morphism, implying that

$$s_L \circ \phi_\alpha = \alpha_L \quad \text{and} \quad s_R \circ \phi_\alpha = \alpha_R.$$

These equalities further imply that

$$(\tilde{s_L} \circ \Phi)(\alpha) = s_L \circ \phi_\alpha, \quad \alpha_L = q_L(\alpha), \quad (\tilde{s_R} \circ \Phi)(\alpha) = s_R \circ \phi_\alpha, \quad \text{and} \quad \alpha_R = q_R(\alpha),$$

and so

$$(\tilde{s_L} \circ \Phi)(\alpha) = q_L(\alpha) \quad \text{and} \quad (\tilde{s_R} \circ \Phi)(\alpha) = q_R(\alpha).$$

The morphism Φ in Set is, therefore, a span morphism and is unique since ϕ_α is uniquely determined. Since Q is a pullback of $\mathrm{Hom}(B, C)$, the span $\mathrm{Hom}(B, S)$ is as well and so $\mathrm{Hom}(B, -)$ maps pullbacks in $\mathscr{C}$ to pullbacks in Set. □

Suppose the $\mathbf{1}$ is the one point manifold in Diff. Lemma 3.3.1 and the fact that the forgetful functor $\mathcal{F}$ from Diff to Set is naturally isomorphic to the functor $\mathrm{Hom}(\mathbf{1}, -)$ together imply Propostion 3.3.2.

Proposition 3.3.2. *The forgetful functor $\mathcal{F}$ from* Diff *to* Set *preserves pullbacks.*

SurjSub does not have Pullbacks

Theorem 3.3.3. *SurjSub whose objects are smooth manifolds and morphisms are surjective submersions and composition of surjective submersions defines composition is a category.*

Proof. Let M, M', N, N' be smooth manifolds and $f\colon M \to M'$, $g\colon M' \to N$ and $h\colon N \to N'$ be surjective submersions. It suffices to show that the compositon of surjective submersions is again a surjective submersion. For any x in M

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

by the chain rule. If f and g are smooth surjective submersions then dg and df are surjective. The composition of smooth maps is smooth and the composition of surjective maps is surjective, therefore the composition of smooth surjective maps is smooth surjective. If the composition of smooth submersions is a smooth submersion, then

$$d((h \circ g) \circ f)_x = d(h \circ g)_{f(x)} \circ df_x = dh_{g \circ f(x)} \circ dg_{f(x)} \circ df_x.$$

This is a smooth submersion and doing a similar computation we get

$$d((h \circ g) \circ f)_x = d(h \circ (g \circ f))_x,$$

which verifies associativity. For the right unit law, let 1_x be the identity map on the point x. By the chain rule we have

$$d(f \circ 1_x)_x = df_x \circ d1_x = df_x \circ 1_{T_x M} = df_x.$$

Similarly, the left unit law holds. Hence, SurjSub is a category. □

This category is important in the study of classical mechanical systems because a map that takes the configuration space of a classical mechanical system to the configuration space of a subsystem should be a surjective submersion. The category SurjSub is an example of a category that does not have pullbacks.

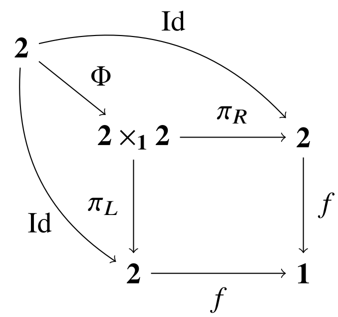

Figure 3.6: Two Point Manifold Contradiction

Example 3.3.4. Let **1** and **2** respectively denote the one and two point manifolds (Figure 3.6). Let f be the unique map from **2** to **1** and C be the cospan (f, f). Denote by Id the identity map from **2** to **2**. The span (Id, Id) is paired with C.

Suppose that π_L and π_R are the canonical projections from $\mathbf{2} \times_{\mathbf{1}} \mathbf{2}$ to **2**. Suppose that S is a pullback of the cospan C in SurjSub. Proposition 3.3.2 together with the discussion immediately following Definition 3.1.6 imply that the image of S under the forgetful functor from Diff to Set is the span (π_L, π_R). Since $\mathbf{2} \times_{\mathbf{1}} \mathbf{2}$ is isomorphic to $\mathbf{2} \times \mathbf{2}$, a set with four elements, there cannot be a span morphism in SurjSub from **2** to $\mathbf{2} \times_{\mathbf{1}} \mathbf{2}$, as such a map would necessarily be surjective and **2** has only two elements. Therefore, the cospan C does not have a pullback in SurjSub and so SurjSub does not have pullbacks.

3.4 Diff does not have Pullbacks

Suppose throughout this section that f and g are morphisms in Diff that have mutual target $(Z, \mathcal{T}_Z, \mathcal{A}_Z)$ and respective sources $(X, \mathcal{T}_X, \mathcal{A}_X)$ and $(Y, \mathcal{T}_Y, \mathcal{A}_Y)$. Recall that π_X and π_Y are the respective projections from the set $X \times_Z Y$ to X and Y. Let $\mathcal{T}_{X \times_Z Y}$ be the subspace topology on $X \times_Z Y$ that $X \times_Z Y$ inherits from the product topology on $X \times Y$ and with respect to which π_X and π_Y are both continuous. View the functions f and g as functions in Top that have the topological space $(Z, \mathcal{T}_Z)$ as their mutual target and the topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ as their respective sources. Suppose that $(W, \mathcal{T}_W, \mathcal{A}_W)$ is an embedded submanifold of $(Z, \mathcal{T}_Z, \mathcal{A}_Z)$. Refer to [23, p. 143-144] for further discussion of transversality and, in particular, for the proof of Proposition 3.4.3.

Definition 3.4.1. The smooth function f is *transverse to W* if for every x in $f^{-1}(W)$, the spaces $T_{f(x)}W$ and $\mathrm{d}f(T_x X)$ together span $T_{f(x)}Z$. The smooth functions f and g are *transverse* if for every point x in X and y in Y with $f(x)$ and $g(y)$ both equal to z,

$$\mathrm{d}f(T_x X) + \mathrm{d}g\left(T_y Y\right) = T_z Z.$$

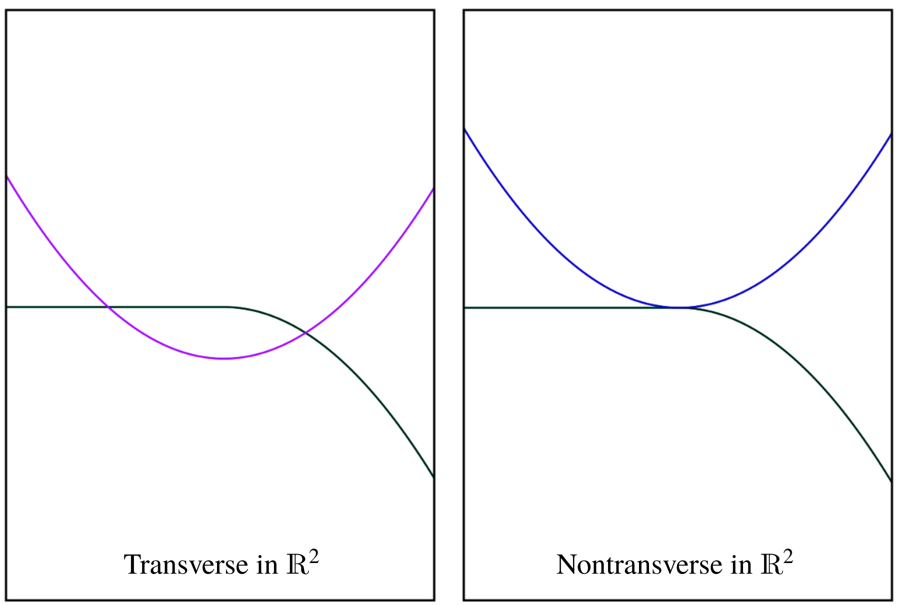

Figure 3.7: Transverse and Nontransverse Curves

Proposition 3.4.2. *If f is a surjective submersion from X to Z and g is a smooth map from Y to Z then f and g are transverse.*

Proof. If f is a surjective submersion then df is surjective. For any point z in Z and any tangent vector v in $T_z Z$ choose x in $f^{-1}(z)$, which is possible by surjectivity. But since f is a submersion, then there exists a tangent vector w in $T_{f^{-1}(z)}X$ such that $df(w) = v$. Therefore, $Im(df) = T_z Z$ and

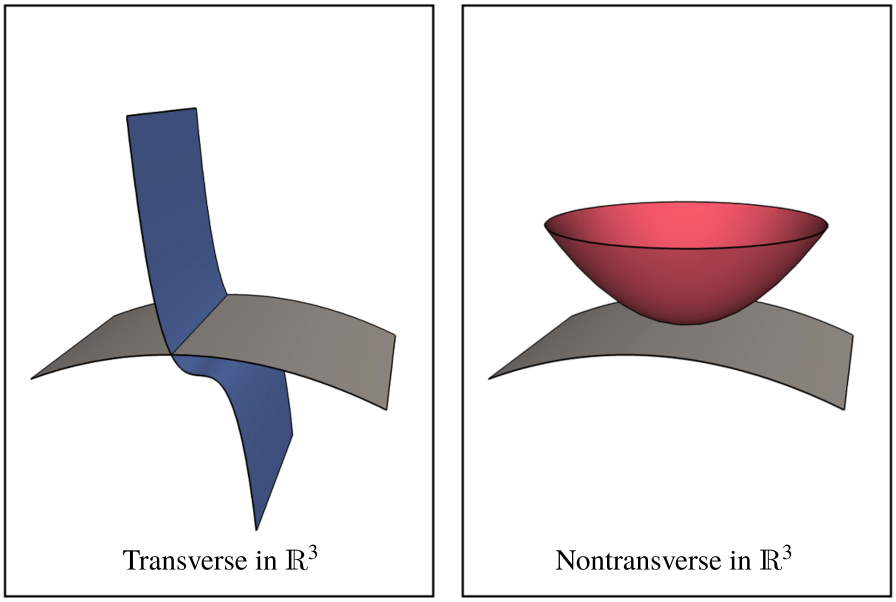

Figure 3.8: Transverse and Nontransverse Surfaces

hence

$$\mathrm{d}f(T_xX) + \mathrm{d}g\left(T_yY\right) = T_zZ.$$

□

Proposition 3.4.3. *Suppose that X and Z are smooth manifolds and W is an embedded submanifold of Z. If f is a smooth map from X to Z that is transverse to W, then $f^{-1}(W)$ is an embedded submanifold of X whose codimension is equal to the codimension of W in Z.*

Proposition 3.4.4. *If f and g are transverse, then the fibered product $X \times_Z Y$ is a smooth embedded submanifold of codimension equal to the dimension of Z. Furthermore, the span (π_X, π_Y) in* Diff *is a pullback of (f, g).*

Proof. Denote by Δ_Z the diagonal $\{(z, z) : z \in Z\}$ of $Z \times Z$, an embedded submanifold of $Z \times Z$. The function $f \times g$, with

$$f \times g \colon X \times Y \to Z \times Z \quad \text{by} \quad (x, y) \mapsto (f(x), g(y)),$$

is smooth and $(f \times g)^{-1}(\Delta_Z)$ is equal to $X \times_Z Y$. Since f and g are transverse, the function $f \times g$ is transverse to Δ_Z. Proposition 3.4.3 implies that $X \times_Z Y$ is a smooth manifold of codimension in $X \times Y$ equal to the dimension of Δ_Z. The dimension of Δ_Z is equal to that of Z, implying that $X \times_Z Y$ has codimension in $X \times Y$ equal to the dimension of Z.

To show that (π_X, π_Y) is a pullback of (f, g), suppose that S is a span in Diff that is paired with (f, g). Define for each s in S_A the span morphism Φ from S to (π_X, π_Y) by

$$\Phi(s) = (s_L(s), s_R(s)).$$

Suppose that Φ' is another span morphism from S to (π_X, π_Y). For any s in S_A,

$$\pi_X(\Phi'(s)) = s_L(s) \quad \text{and} \quad \pi_Y(\Phi'(s)) = s_R(s),$$

implying that $\Phi'(S)$ is equal to $\Phi(s)$. Since s was arbitrarily chosen, the morphism Φ' is equal to Φ and so Φ is unique, hence (π_X, π_Y) is a pullback. □

If f and g are in $\mathsf{SurjSub}$ with mutual target Z, then they are transverse and so Proposition 3.4.4 implies the following.

Proposition 3.4.5. *If (f, g) is a cospan in $\mathsf{SurjSub}$, then the fibered product $X \times_Z Y$ is a smooth embedded submanifold of $X \times Y$ of dimension* $\dim(X \times_Z Y)$*, where*

$$\dim(X \times_Z Y) = \dim(X) + \dim(Y) - \dim(Z)\,.$$

For the following proposition, take (f, g) to be a cospan in Diff but where the maps f and g are not assumed to be transverse.

Proposition 3.4.6. *If S is a span in Diff that is a pullback of (f, g), and if (π_X, π_Y) and (s_L, s_R) are span isomorphic as spans in Top, then $X \times_Z Y$ has a manifold structure.*

Proof. Let Φ be the unique span morphism in Top from S to (π_X, π_Y). The homeomorphism Φ transports the manifold structure of S_A to $X \times_Z Y$, giving it a manifold structure as well. □

If S is a span in Diff that is paired with (f, g), then the map Φ, that is defined for each s in S_A by

$$\Phi(s) = (s_L(s), s_R(s)),$$

is a smooth map from S_A to $X \times Y$. If $X \times_Z Y$ is an embedded submanifold of $X \times Y$, then Φ is a smooth map from S_A to $X \times_Z Y$ and is the unique such map, implying the following proposition.

Proposition 3.4.7. *If $X \times_Z Y$ is an embedded submanifold of $X \times Y$, then (π_X, π_Y) is a span in Diff and a pullback of (f, g).*

Propositions 3.4.4 and 3.4.7 together imply the following proposition.

Proposition 3.4.8. *If (f, g) is a cospan in Diff and f and g are transverse, then (π_X, π_Y) is a pullback of (f, g) in Diff.*

The following example demonstrates that $X \times_Z Y$ may be a manifold and the projections π_x and π_Y may be continuous, but $X \times_Z Y$ is not an embedded submanifold of $X \times Y$. In light of Proposition 3.4.4, such an example requires the functions f and g to be non-transverse.

Example 3.4.9. Let X and Z be $\mathbb{R}$ and Y be the one point manifold $\mathbf{1}$. Suppose that f is smooth, that (a_n) is a sequence in $\mathbb{R}$ that converges to a point a_0 that is not equal to a_n for any natural number n, and that the zero set of f is the set $\{a_0\} \cup \{a_n \colon n \in \mathbb{N}\}$. Suppose further that the range of g is $\{0\}$. The set $X \times_Z Y$ is the subset $\{a_0\} \cup \{a_n \colon n \in \mathbb{N}\}$ of $\mathbb{R}$.

In Top, if (π_X, π_Y) is a pullback, then $X \times_Z Y$ must be endowed with the subspace topology $\mathcal{T}_S$ that makes each set $\{a_n\}$ an open set, where n varies over $\mathbb{N}$. Any open set containing a_0 contains infinitely many points.

If $X \times_Z Y$ has a manifold structure, then each point must contain a neighborhood that is homeomorphic to a point, and so as a manifold $X \times_Z Y$ must be endowed with the discrete topology $\mathcal{T}_D$. In this case, the manifold $X \times_Z Y$ is not an embedded submanifold of $X \times Y$ since its topology is not the subspace topology. The span (π_X, π_Y) is, nevertheless in this case, a pullback of (f, g) in Diff.

The above example demonstrates that f and g may be non-transverse, but (f, g) nevertheless has a pullback that is a span in Diff. The forgetful functor $\mathcal{F}$ from Diff to Set preserves pullbacks and so if S is a span in Diff and a pullback of (f, g), then $\mathcal{F}(S)$ is a span in Set that is a pullback of (f, g) as a cospan in Set. Since Set has pullbacks, there is a span isomorphism in Set from $\mathcal{F}(S)$ to (π_X, π_Y). This span isomorphism is only a bijection and there should be no expectation that it preserves topological structure.

The category Top also has pullbacks and so if f and g are continuous, then the pullback of (f, g) will exist and, in fact, the span (π_X, π_Y) in Top is a pullback of (f, g) where the maps π_X and π_Y have $(X \times_Z Y, \mathcal{T}_S)$ as their common source. Since the forgetful functor from Diff to Top does not preserve pullbacks, there is no guarantee that S being a pullback of (f, g) implies that it is a pullback when mapped by a forgetful functor to Top. The topology on the image of the manifold $X \times_Z Y$ under the forgetful functor from Diff to Top is $\mathcal{T}_D$, which is a finer topology than $\mathcal{T}_S$. The identity map taking $(X \times_Z Y, \mathcal{T}_D)$ to $(X \times_Z Y, \mathcal{T}_S)$ is a continuous span morphism from (π_X, π_Y) to (π_X, π_Y), but the inverse is not continuous. So the forgetful functor $\mathcal{F}$ from Diff to Top maps the pullback (π_X, π_Y), where maps π_X and π_Y have the manifold $X \times_Z Y$ as their common source, to the span (π_X, π_Y), where the maps have $(X \times_Z Y, \mathcal{T}_D)$ as their common source. This demonstrates that the forgetful functor from Diff to Top does not preserve pullbacks.

The former discussion demonstrates that there is some subtlety involved in determining that Diff does not have pullbacks and such a determination requires a carefully selected counterexample. The proof of Proposition 3.4.11 presents such an example that is fortunately quite basic. Refer to Figure 3.9 to visualize the various mapping involved in the proof of Lemma 3.4.10.

Lemma 3.4.10. *If (f, g) is a cospan in Diff and S is a span in Diff that is a pullback of (f, g), then there is a bijective span morphism in Top from $\mathcal{F}(S)$ to (π_X, π_Y), where $\mathcal{F}$ is the forgetful functor from Diff to Top and $X \times_Z Y$ is endowed with the topology $\mathcal{T}_S$.*

Proof. Suppose that S is a span in Diff that is a pullback of (f, g). Define for each a in S_A the function Φ by

$$\Phi(a) = (s_L(a), s_R(a)).$$

The map Φ from S_A to $X \times Y$ is smooth because the functions s_L and s_R are smooth. The span S is paired with (f, g), implying that the range of Φ is $X \times_Z Y$, and so Φ is a continuous function from S_A

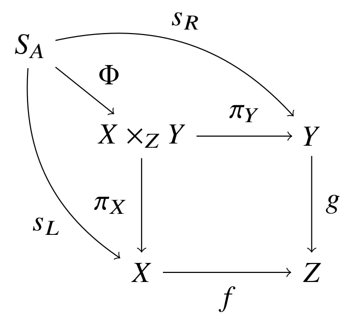

Figure 3.9: Φ is a Bijection

to $X \times_Z Y$. Proposition 3.3.2 implies that the forgetful functor $\mathcal{F}$ from Diff to Set preserves pullbacks, therefore $\mathcal{F}(\Phi)$ is a span morphism in Set from $\mathcal{F}(S)$ to (π_X, π_Y), where the pair of projections is viewed only as a pair of maps in Set. The span S is a pullback in Diff, hence $\mathcal{F}(S)$ is a span in Set that is a pullback of (f, g), and so the map $\mathcal{F}(\Phi)$ is a bijection. Maps between manifolds are determined by their behavior on the underlying sets, hence Φ is a continuous bijection. □

Although the fact that Diff does not have pullbacks is commonly cited in the literature, we found it difficult to locate a detailed proof of this fact and so present it here for the convenience of the reader.

Proposition 3.4.11. *The category Diff does not have pullbacks.*

Proof. Define f and g to be the functions from $\mathbb{R}$ to $\mathbb{R}$ given for each x in $\mathbb{R}$ by mapping x to x^2. Suppose that S is a span in Diff that is a pullback of (f, g). The fibered product $X \times_Z Y$ is the set

$$X \times_Z Y = \{(v, w) \colon |v| = |w|\}.$$

The restrictions of f and g to the open sets $(-\infty, 0)$ and $(0, \infty)$ are surjective submersions onto $(0, \infty)$. If the sets $s_L^{-1}(-\infty, 0) \cap s_R^{-1}(-\infty, 0)$, $s_L^{-1}(0, \infty) \cap s_R^{-1}(-\infty, 0)$, $s_L^{-1}(-\infty, 0) \cap s_R^{-1}(0, \infty)$, and $s_L^{-1}(0, \infty) \cap s_R^{-1}(0, \infty)$ are all empty, then the underlying set S_A is a single point. However, there is a bijection between the underlying set S_A and $X \times_Z Y$ since they are isomorphic in Set as the apices of pullbacks of the same cospan. Therefore, at least one of the above intersections is not empty.

Let U be of one of the four intersections given above that is not empty. The set U is an open subset of S_A as a non-empty intersection of open sets, hence a manifold. Proposition 3.4.5 implies that the dimension of U is equal to 1. The dimension of the manifold S_A is also 1 since S_A contains U as an open subset and is therefore homeomorphic to either a line, an open interval, a half-open interval, or a circle, [21]. The map Φ which maps S_A to $X \times Y$, defined for each a in S_A by

$$\Phi(a) = (s_L(a), s_R(a)),$$

is a smooth map that is a span morphism and maps S_A onto the subspace $X \times_Z Y$. Since S_A is a pullback and the forgetful functor from Diff to Set preserves pullbacks, the underlying set S_A is the apex of a span in Set that is a pullback of (f, g) and so there is a span isomorphism from S to (π_X, π_Y) in Set, a bijection between the set S_A and the set $X \times_Z Y$. Since the span morphism in Diff from S to (π_X, π_Y) that maps S_A onto $X \times_Z Y$ is also a morphism in Set of the underlying sets and is unique, the map Φ is a bijection. Therefore, the preimage $\Phi^{-1}(X \times_Z Y \setminus \{(0, 0)\})$ is the set S_A with one point removed and so has either one or two connected components. However, the subspace $X \times_Z Y \setminus \{(0, 0)\}$ of $X \times Y$ has four components and this contradicts the continuity of Φ, which must map connected components to connected components. □

Chapter 4

$\mathcal{F}$-Pullbacks, Span Tightness, and Generalized Span Categories

4.1 Composition by $\mathcal{F}$-Pullbacks and Span Tightness

Assume henceforth that $\mathscr{C}$ and $\mathscr{C}'$ are categories and that $\mathcal{F}$ is a functor from $\mathscr{C}$ to $\mathscr{C}'$. For any span S in $\mathscr{C}$, denote by $\mathcal{F}(S)$ the span $(\mathcal{F}(s_L), \mathcal{F}(s_R))$. For any cospan C in $\mathscr{C}$, denote by $\mathcal{F}(C)$ the cospan $(\mathcal{F}(c_L), \mathcal{F}(c_R))$ in $\mathscr{C}'$.

$\mathcal{F}$-Pullbacks and Span Tightness

Definition 4.1.1. The category $\mathscr{C}$ *has $\mathcal{F}$-pullbacks in $\mathscr{C}'$* if for any cospan C in $\mathscr{C}$, there is a span S in $\mathscr{C}$ that is paired with C and the span $\mathcal{F}(S)$ is a pullback of the cospan $\mathcal{F}(C)$ in $\mathscr{C}'$. In this case, the span S is an *$\mathcal{F}$-pullback* of C.

Note that if $\mathscr{C}'$ is equal to $\mathscr{C}$ and $\mathcal{F}$ is the identity functor, then an $\mathcal{F}$-pullback is simply a pullback.

Definition 4.1.2. Suppose that S and Q are spans in $\mathscr{C}$ such that:

(1) $S_R = Q_L$;

(2) there is a span P in $\mathscr{C}$ that is a pullback of the cospan (s_R, q_L).

The *composite of S and Q along P* is the span in $\mathscr{C}$ given by

$$S \circ_P Q = (s_L \circ p_L, q_R \circ p_R).$$

If P is an $\mathcal{F}$-pullback, then the span $S \circ_P Q$ is an *$\mathcal{F}$-pullback composite of S and Q along P*.

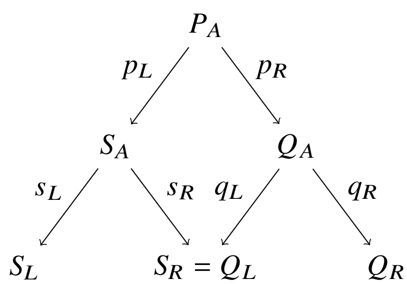

Figure 4.1: Composing S and Q along P

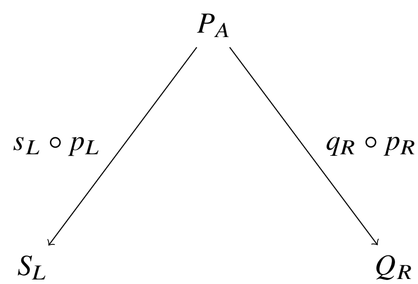

Figure 4.2: The Composite $S \circ_P Q$

Diagram 4.2 is a diagrammatical realization of the composite of S and Q along P and Diagram 4.1 depicts the construction of this composite by the $\mathcal{F}$-pullback P.

Definition 4.1.3. Suppose that $\mathscr{C}$ has $\mathcal{F}$-pullbacks in $\mathscr{C}'$. The functor $\mathcal{F}$ is *span tight* if for any $\mathcal{F}$-pullbacks S and Q of the same cospan, the unique span isomorphism Φ from $\mathcal{F}(S)$ to $\mathcal{F}(Q)$ is $\mathcal{F}(\Psi)$ for some span isomorphism Ψ from S to Q.

$\mathcal{F}$-Pullbacks of SurjSub

Suppose that X, Y, and Z are smooth manifolds. Suppose further that f is a smooth map from X to Z and that g is a smooth map from Y to Z. Again denote by ρ_X and ρ_Y the respective projections from $X \times Y$ to X and Y and let π_X and π_Y be their respective restrictions to the embedded submanifold $X \times_Z Y$.

Proposition 4.1.4. *The span (π_X, π_Y) is a pullback in Diff of the cospan (f, g).*

Proof. Suppose that Q is a span in Diff that is paired with the cospan (f, g). Define the map Ψ from Q_A to $X \times Y$ as the product of q_L and q_R, so that $\Psi(a)$ is equal to $(q_L(a), q_R(a))$. This map is smooth as a product of smooth maps and unique since Diff has categorical products. Furthermore, for any a in Q_A,

$$(f \circ \rho_X \circ \Psi)(a) = f(q_L(a)) \quad \text{and} \quad (g \circ \rho_Y \circ \Psi)(a) = g(q_R(a)) \,.$$

Since Q is paired with (f, g), $f(q_L(a))$ is equal to $g(q_R(a))$, and so $\Psi(a)$ is in $X \times_Z Y$. Since Q was an arbitrarily chosen span paired with (f, g), the span (π_X, π_Y) is a pullback in Diff. □

Note that while SurjSub is a subcategory of Diff, the category SurjSub does not have pullbacks. Let $\mathcal{F}$ be the inclusion functor from SurjSub to Diff. Suppose that (f, g) is a cospan in SurjSub, where f and g have respective sources X and Y and both maps have target Z. In this case, Proposition 4.1.4 implies that the span (π_X, π_Y) is an $\mathcal{F}$-pullback of the cospan (f, g) and this, together with the fact that every diffeomorphism is a surjective submersion, implies Theorem 4.1.5.

Theorem 4.1.5. *The inclusion functor from SurjSub to Diff is span tight.*

4.2 The Generalized Span Category

Identify the objects in Span$(\mathscr{C}, \mathcal{F})$ to be the objects in $\mathscr{C}$ and the isomorphism classes of spans in $\mathscr{C}$ to be the morphisms in Span$(\mathscr{C}, \mathcal{F})$. If $[S]$ is an isomorphism class of spans in Span$(\mathscr{C}, \mathcal{F})$, then identify S_R and S_L respectively to be the source and target of $[S]$. Define composition of isomorphism classes of spans by

$$\left[S^1\right] \circ \left[S^2\right] = \left[S^1 \circ_P S^2\right],$$

where $S^1 \circ_P S^2$ is an $\mathcal{F}$-pullback composite of S^1 and S^2. Theorem 4.2.1 is the main result of the section and the lemmata that follow simplify the proof of the theorem.

Theorem 4.2.1. *If $\mathcal{F}$ is a span tight functor from $\mathscr{C}$ to $\mathscr{C}'$, then* Span$(\mathscr{C}, \mathcal{F})$ *is a category.*

If the functor $\mathcal{F}$ from $\mathscr{C}$ to $\mathscr{C}'$ is span tight and S and Q are spans in $\mathscr{C}$ with S_R equal to Q_L, then there is an $\mathcal{F}$-pullback P of the cospan (s_R, q_L) and so there is an $\mathcal{F}$-pullback composite of S and Q along P. The $\mathcal{F}$-pullback P is, however, only defined up to a span isomorphism Φ. The following lemma shows that changing P up to an isomorphism changes the resulting composite span only up to a span isomorphism in $\mathscr{C}$.

Lemma 4.2.2. *Suppose that $\mathcal{F}$ is span tight, that S and Q are spans in $\mathscr{C}$, and that $S \circ_{P^i} Q$ is an $\mathcal{F}$-pullback composite, with i equal to* 1 *or* 2*. There is a span isomorphism Φ in $\mathscr{C}$ from $S \circ_{P^1} Q$ to $S \circ_{P^2} Q$.*

Proof. Since P^1 and P^2 are both $\mathcal{F}$-pullbacks of the cospan (s_R, q_L), there is a span isomorphism Φ in $\mathscr{C}'$ from $\mathcal{F}\left(P^1\right)$ to $\mathcal{F}\left(P^2\right)$. Since $\mathcal{F}$ is span tight, there is a span isomorphism Ψ in $\mathscr{C}$ from P^1 to P^2 with $\mathcal{F}(\Psi)$ equal to Φ, and so

$$p_L^1 = p_L^2 \circ \Psi \quad \text{and} \quad p_R^1 = p_R^2 \circ \Psi.$$

These equalities imply that

$$s_L \circ p_L^1 = s_L \circ p_L^2 \circ \Psi \quad \text{and} \quad q_R \circ p_R^1 = q_R \circ p_R^2 \circ \Psi,$$

establishing that Ψ is a span isomorphism from $S \circ_{P^1} Q$ to $S \circ_{P^2} Q$. □

Lemma 4.2.3. *Suppose that $\mathcal{F}$ is span tight, that S^i and Q^i are spans in $\mathscr{C}$, and that $S^i \circ_{P^i} Q^i$ is an $\mathcal{F}$-pullback composite, with i equal to* 1 *or* 2*. Suppose that S^1 and Q^1 are respectively span isomorphic to S^2 and Q^2. There is a span isomorphism in $\mathscr{C}$ between spans $S^1 \circ_{P^1} Q^1$ and $S^2 \circ_{P^2} Q^2$.*

Lemma 4.2.3 generalizes Lemma 4.2.2 and reduces to Lemma 4.2.2 when S^1 is equal to S^2, when C^1 is equal to C^2, and when P^1 and P^2 are pullbacks that are not necessarily equal to each other. Refer to Diagram 4.3 to visualize the mappings involved in the proof of Lemma 4.2.3.

Proof. Let α and β be span isomorphisms respectively from S^1 to S^2 and from Q^1 to Q^2. The span P^1 is an $\mathcal{F}$-pullback of $\left(s_R^1, q_L^1\right)$. Since α and β are span morphisms, the span $\left(\alpha \circ p_L^1, \beta \circ p_R^1\right)$ is paired with $\left(s_R^2, q_L^2\right)$. Since $\mathcal{F}\left(P^2\right)$ is a pullback of $\left(\mathcal{F}\left(s_R^2\right), \mathcal{F}\left(q_L^2\right)\right)$, there is a span morphism, Φ_1, in $\mathscr{C}'$ from $\left(\mathcal{F}\left(\alpha \circ p_L^1\right), \mathcal{F}\left(\beta \circ p_R^1\right)\right)$ to $\mathcal{F}\left(P^2\right)$.

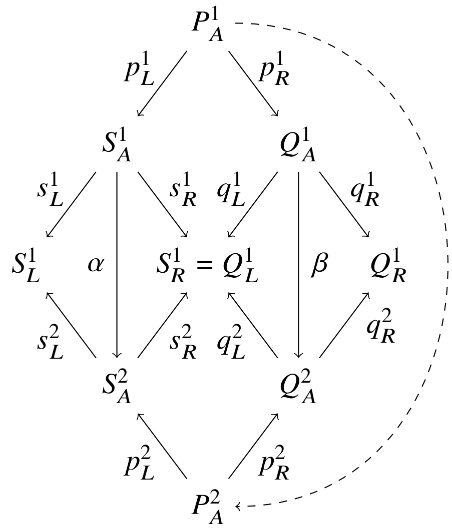

Figure 4.3: Isomorphic Compositions of Isomorphic Spans

If T is a span in $\mathscr{C}'$ paired with the $\mathcal{F}\left(s^1_R, q^1_L\right)$, then there is a span morphism Φ_2 in $\mathscr{C}'$ from T to $\mathcal{F}\left(P^1\right)$. The composite $\Phi_1 \circ \Phi_2$ maps T to $\mathcal{F}\left(\alpha^{-1} \circ p^2_L, \beta^{-1} \circ p^2_R\right)$, which is also paired with $\mathcal{F}\left(s^1_R, q^1_L\right)$. Uniqueness of the pullback of $\mathcal{F}\left(s^1_R, q^1_L\right)$ up to a span isomorphism implies that there is a span isomorphism Φ_3 in $\mathscr{C}'$ from $\mathcal{F}\left(\alpha^{-1} \circ p^2_L, \beta^{-1} \circ p^2_R\right)$ to $\mathcal{F}\left(P^1\right)$. Since $\mathcal{F}$ is span tight, there is a span isomorphism Ψ in $\mathscr{C}$ such that $\mathcal{F}(\Psi)$ is Φ_3. Use the fact that Ψ is a span isomorphism to obtain the equalities

$$\alpha^{-1} \circ p^2_L = p^1_L \circ \Psi \quad \text{and} \quad \beta^{-1} \circ p^2_R = p^1_R \circ \Psi.$$

The equalities

$$s^2_L = s^1_L \circ \alpha^{-1} \quad \text{and} \quad q^2_R = q^1_R \circ \beta^{-1}$$

imply that

$$s^2_L \circ p^2_L = s^1_L \circ \alpha^{-1} \circ p^2_L = s^1_L \circ p^1_L \circ \Psi$$

and similarly that

$$q^2_R \circ p^2_R = q^1_R \circ p^1_R \circ \Psi.$$

Therefore, the isomorphism Ψ is a span isomorphism from $S^2 \circ_{P^2} Q^2$ to $S^1 \circ_{P^1} Q^1$. □

Lemma 4.2.4. *Suppose that $\mathcal{F}$ is span tight and that S, Q, and T are spans in $\mathscr{C}$ with S_R equal to Q_L and Q_R equal to T_L. Suppose that $S \circ_{P^1} Q$ and $Q \circ_{P^4} T$ are $\mathcal{F}$-pullback composites and that $(S \circ_{P^1} Q) \circ_{P^2} T$ and $S \circ_{P^3} (Q \circ_{P^4} T)$ are also $\mathcal{F}$-pullback composites. There is a span isomorphism Φ in $\mathscr{C}$ from $(S \circ_{P^1} Q) \circ_{P^2} T$ to $S \circ_{P^4} (Q \circ_{P^3} T)$.*

Refer to Diagram 4.4 and Diagram 4.5 below to visualize the mappings involved in the proof of Lemma 4.2.4.

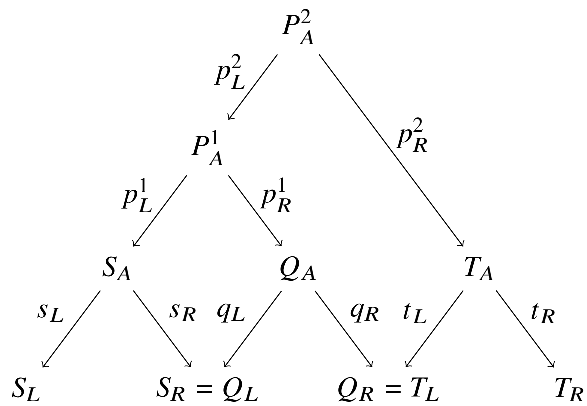

Figure 4.4: The Composite $(S \circ_{P^1} Q) \circ_{P^2} T$

Proof. Suppose that P^1 is an $\mathcal{F}$-pullback of the cospan (s_R, q_L), that P^3 is an $\mathcal{F}$-pullback of the cospan (q_R, t_L), and that P is an $\mathcal{F}$-pullback of the cospan (p_R^1, p_L^3) where

$$P_L = P_A^1 \quad \text{and} \quad P_R = P_A^3.$$

Suppose further that P^2 is an $\mathcal{F}$-pullback of the cospan $(q_R \circ p_R^1, t_L)$ and that P^4 is an $\mathcal{F}$-pullback of the cospan $(s_R, q_L \circ p_L^3)$.

Since P^2 is an $\mathcal{F}$-pullback of the cospan $(q_R \circ p_R^1, t_L)$, the span $(p_R^1 \circ p_L^2, p_R^2)$ is paired with the cospan (q_R, t_L) and so $\mathcal{F}(p_R^1 \circ p_L^2, p_R^2)$ is paired with the cospan $\mathcal{F}(q_R, t_L)$. The span P^3 is an $\mathcal{F}$-pullback, which implies the existence of a span morphism Φ_1 in $\mathscr{C}'$ from $\mathcal{F}(p_R^1 \circ p_L^2, p_R^2)$ to $\mathcal{F}(P^3)$. The span $(\mathcal{F}(p_L^2), \Phi_1)$ is paired with $\mathcal{F}(p_R^1, p_L^3)$ and so there is a span morphism Φ_2 in $\mathscr{C}'$ from $\mathcal{F}(p_L^2, \Phi_1)$ to $\mathcal{F}(P)$. If U is a span paired with $(q_R \circ p_R^1, t_L)$, then there is a span morphism

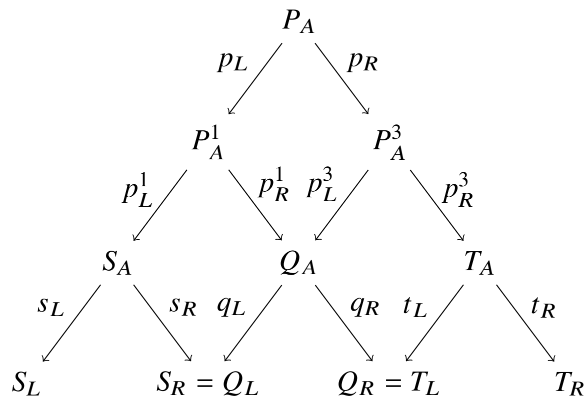

Figure 4.5: Comparator Span

Φ_3 in $\mathscr{C}'$ from $\mathcal{F}(U)$ to $\mathcal{F}(P^2)$. The composite $\Phi_2 \circ \Phi_3$ is a span morphism in $\mathscr{C}'$ from $\mathcal{F}(U)$ to $\mathcal{F}(p_L, p_R^3 \circ p_R)$ and so $\mathcal{F}(p_L, p_R^3 \circ p_R)$ is a pullback in $\mathscr{C}'$ of the cospan $\mathcal{F}(q_R \circ p_R^1, t_L)$. There is, therefore, a span isomorphism in $\mathscr{C}'$ from $\mathcal{F}(p_L, p_R^3 \circ p_R)$ to $\mathcal{F}(P^2)$. Span tightness of $\mathcal{F}$ implies that there is a span isomorphism Ψ_1 in $\mathscr{C}$ from $(p_L, p_R^3 \circ p_R)$ to P^2 with

$$p_R^2 \circ \Psi_1 = p_R^3 \circ p_R \quad \text{and so} \quad t_R \circ p_R^2 \circ \Psi_1 = t_R \circ p_R^3 \circ p_R.$$

The equality

$$p_L^2 \circ \Psi_1 = p_L \quad \text{implies that} \quad s_L \circ p_L^1 \circ p_L^2 \circ \Psi_1 = s_L \circ p_L^1 \circ p_L.$$

The isomorphism Ψ_1 in $\mathscr{C}$ is, therefore, a span isomorphism with

$$\Psi_1(s_L \circ p_L^1 \circ p_L, t_R \circ p_R^3 \circ p_R) = (s_L \circ p_L^1 \circ p_L^2, t_R \circ p_R^2), \tag{4.1}$$

where the second span is that given in Diagram 4.4.

A similar argument shows that there is a span isomorphism Ψ_2 in $\mathscr{C}$ with

$$\Psi_2(s_L \circ p_L^1 \circ p_L, t_R \circ p_R^3 \circ p_R) = S \circ_{P^4} (Q \circ_{P^3} T), \tag{4.2}$$

where P^4 is an $\mathcal{F}$-pullback of the cospan $\left(s_R, q_L \circ p_L^3\right)$. Together with the Proposition 3.1.4 and its corollary, (4.1) and (4.2) imply Lemma 4.2.4. □

Proof of Theorem 4.2.1. To prove the theorem, it suffices to show that the composition of morphisms in Span$(\mathscr{C}, \mathcal{F})$ is well defined, satisfies the left and right unit laws, and is associative.

If $[S^1]$ and $[S^2]$ are isomorphism classes of spans and the source of $\left[S^1\right]$ is the target of $\left[S^2\right]$, then for any representatives S^1 and S^2 respectively of $\left[S^1\right]$ and $\left[S^2\right]$, span tightness of $\mathcal{F}$ implies that there is an $\mathcal{F}$-pullback P of (s_R^1, s_L^2), hence there exists a composite $S^1 \circ_P S^2$. Lemma 4.2.2 implies that the equivalence class $\left[S^1 \circ_P S^2\right]$ is independent of P. Lemma 4.2.3 additionally implies that $\left[S^1 \circ_P S^2\right]$ is independent of choice of representatives S^1 and S^2. Furthermore, the objects S_R^2 and S_L^1 are the respective source and target of $\left[S^1\right] \circ \left[S^2\right]$, implying that the composition $\circ$ is well defined.

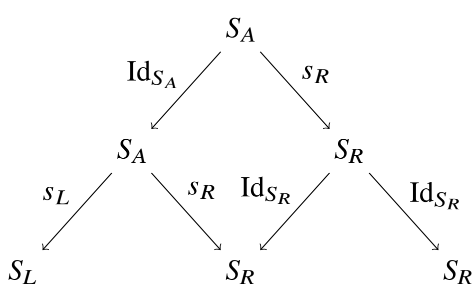

Figure 4.6: Composing S with Id_{S_R}

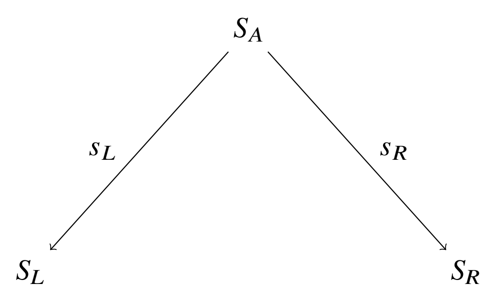

Figure 4.7: The Composite $S \circ_S \mathrm{Id}_{S_R}$

Suppose that $[S]$ is an isomorphism class of spans in $\mathscr{C}$ and that $[\mathrm{I}_{S_R}]$ is the isomorphism class of spans containing $(\mathrm{Id}_{S_R}, \mathrm{Id}_{S_R})$, where

$$\mathrm{Id}_{S_R} : S_R \to S_R$$

is the identity map from S_R to S_R.

Let P be the span (Id_{S_A}, s_R). For any span Q in $\mathscr{C}'$ that is paired with $(\mathcal{F}(s_R), \mathcal{F}(\mathrm{Id}_{S_R}))$,

$$\mathcal{F}(s_R) \circ q_L = \mathcal{F}(\mathrm{Id}_{S_R}) \circ q_R = q_R$$

and so the map q_L is a span morphism in $\mathscr{C}'$ from Q to $\mathcal{F}(P)$. Given any other span morphism Φ in $\mathscr{C}'$ from Q to $\mathcal{F}(P)$,

$$q_L = \mathcal{F}(\mathrm{Id}_{S_A}) \circ \Phi = \mathrm{Id}_{\mathcal{F}(S_A)} \circ \Phi = \Phi$$

and so the span morphism in $\mathscr{C}'$ from Q to $\mathcal{F}(P)$ is unique. Since Q was arbitrarily chosen, the span $\mathcal{F}(P)$ is a pullback in $\mathscr{C}'$ of the cospan $(\mathcal{F}(s_R), \mathcal{F}(\mathrm{Id}_{S_R}))$, and so P is an $\mathcal{F}$-pullback of the cospan (s_R, Id_{S_R}). Since composition is well defined and $S \circ_P \mathrm{I}_{S_R}$ is span isomorphic in $\mathscr{C}$ to S, the composite $[S] \circ [\mathrm{I}_{S_R}]$ is equal to $[S]$. Similar arguments will show that $[\mathrm{I}_{S_L}] \circ [S]$ is equal to $[S]$, and so Span$(\mathscr{C}, \mathcal{F})$ has both a right and left unit law.

Lemma 4.2.4 implies that $\circ$ is associative. □

4.3 Structures on the Fibered Product

Given Riemannian manifolds X, Y, and Z, we construct a metric tensor on $X \times_Z Y$ that makes $X \times_Z Y$ a Riemannian manifold and makes the projections from the fibered product surjective Riemannian submersions. Similarly, when X, Y, and Z are symplectic manifolds we construct a symplectic form on $X \times_Z Y$ that makes $X \times_Z Y$ a symplectic manifold and makes the projections from the fibered product surjective Poisson maps.

Figure 4.8 specifies the categories to be henceforth denoted by Diff, SurjSub, RiemSurj, and SympSurj.

Category Name	Objects	Morphisms
Diff	Smooth manifolds	Smooth maps
SurjSub	Smooth manifolds	Surjective submersions
RiemSurj	Riemannian manifolds	Surjective Riemannian submersions
SympSurj	Symplectic manifolds	Surjective Poisson maps

Figure 4.8: Table of Categories

Denote by π_Z the map

$$\pi_Z = f \circ \pi_X = g \circ \pi_Y,$$

where π_X and π_Y are the projections from $X \times_Z Y$ to Z. More generally, for any span Q that is paired with a cospan (f, g), define by q_M the map

$$q_M = f \circ q_L = g \circ q_R.$$

Suppose X is a symplectic manifold. The Poisson bivector Π_X of X induces a map $\widetilde{\Pi}_X$ from T^*X to TX that takes any η in T^*X to the vector field $\widetilde{\Pi}_X(\eta)$ with the property that for any ν in T^*X,

$$\nu(\widetilde{\Pi}_X(\eta)) = \Pi_X(\eta, \nu).$$

Since X is symplectic, the map $\widetilde{\Pi}_X$ is an isomorphism [15, p. 17]. This isomorphism gives a way to pull back vector fields by surjective Poisson maps, a fact that, along with Proposition 2.1.35, is critical to the proof of Theorem 4.3.1. Theorem 4.3.1 establishes the existence of a local splitting of the tangent space of a symplectic manifold by a local foliation given by the inverse image of a surjective Poisson map.

Theorem 4.3.1. *Suppose that X and Z are symplectic manifolds with respective dimensions 2ℓ and $2n$ and that f is a surjective Poisson map from X to Z. Given any z in Z and a choice of Darboux coordinates $(q_i^Z, p_i^Z)_{i=1}^n$ on a chart U containing z, and given any x in X with $f(x)$ equal to z, there exist Darboux coordinates $(q_i^X, p_i^X)_{i=1}^\ell$ on a chart V containing x such that for any i in $\{1, \ldots, n\}$,*

$$q_i^X = q_i^Z \circ f \quad \text{and} \quad p_i^X = p_i^Z \circ f.$$

Proof. Suppose that x_0 is in X, that U is a chart containing $f(x_0)$, and that $(q_i^Z, p_i^Z)_{i=1}^n$ is a Darboux coordinate system on U. Proposition 2.1.35 guarantees that f is a surjective submersion, hence it is an open map and so there is a chart V' containing x_0 with a Darboux coordinate system $(q_i^X, p_i^X)_{i=1}^\ell$ such that $f(V')$ is an open subset of U. Denote by $\mathcal{H}$ the set of all vector fields v on $f(V')$ for which there is some α in $C^\infty(f(V'))$ such that for any β in $C^\infty(f(V'))$,

$$v(\beta) = \{\beta, \alpha\}_Z.$$

Denote such a vector field by v_α. Denote by $f^*(\mathcal{H})$ the set of all vector fields w on V' for which there is an α in $C^\infty(f(V'))$ such that for any h in $C^\infty(V')$,

$$w = \{h, \alpha \circ f\}_X.$$

Denote such a vector field by w_α. For any x in V' and any z in $f(V')$, denote respectively by $f^*(\mathcal{H})(x)$ and $\mathcal{H}(z)$ the set of all vector fields in $f^*(\mathcal{H})$ evaluated at x and the set of all vector fields in $\mathcal{H}$ evaluated at z. The bilinearity of the bracket implies that $\mathcal{H}(z)$ and $f^*(\mathcal{H})(x)$ are vectors spaces. Since

$$v_{-q_i^Z} = \frac{\partial}{\partial p_i^Z} \quad \text{and} \quad v_{p_i^Z} = \frac{\partial}{\partial q_i^Z},$$

for any z in $f(V')$, the vector space $\mathcal{H}(z)$ spans $T_z(U)$.

Let F be the function

$$F\colon \mathcal{H} \to f^*(\mathcal{H}) \quad \text{by} \quad F(v_\alpha) = w_\alpha.$$

The fact that f is Poisson implies that

$$\begin{aligned} \mathrm{d}f(w_\alpha)(\beta) &= w_\alpha(\beta \circ f) \\ &= \{\beta \circ f, \alpha \circ f\}_X \\ &= \{\beta, \alpha\}_Z = v_\alpha(\beta), \end{aligned}$$

and so

$$\mathrm{d}f(F(v_\alpha)) = v_\alpha.$$

Similarly, for any w_α in $f^*(\mathcal{H})$,

$$F(\mathrm{d}f(w_\alpha)) = F(v_\alpha) = w_\alpha.$$

The maps F and $\mathrm{d}f|_{\mathcal{H}}$ are therefore inverses of each other and so for each x in V', the vector spaces $\mathcal{H}(f(x))$ and $f^*(\mathcal{H})(x)$ are isomorphic. Both of these vector spaces are of the same dimension as Z.

For any w_α and $w_{\alpha'}$ in $f^*(\mathcal{H})$, the Jacobi identity implies that

$$\begin{aligned}
[w_\alpha, w_{\alpha'}]_{TX} &= w_\alpha(w_{\alpha'}(\beta)) - w_{\alpha'}(w_\alpha(\beta)) \\
&= \{w_{\alpha'}(\beta), \alpha \circ f\}_X - \{w_\alpha(\beta), \alpha' \circ f\}_X \\
&= \{\{\beta \circ f, \alpha' \circ f\}_X, \alpha \circ f\}_X - \{\{\beta \circ f, \alpha \circ f\}_X, \alpha' \circ f\}_X \\
&= \{\beta, \{\alpha' \circ f, \alpha \circ f\}_X\}_X = w_{\{\alpha,\alpha'\}}(\beta),
\end{aligned}$$

and so the space of vector fields $f^*(\mathcal{H})$ is closed under the bracket $[\cdot,\cdot]_{TX}$ on TX. Frobenius' Theorem for involutive distributions implies that for any x in V' there is a submanifold W of V' such that $f^*(\mathcal{H})(x)$ is the tangent space T_xW. Since

$$f^*(\mathcal{H})(x) \cap \ker(\mathrm{d}f|_x) = \{0\},$$

the rank-nullity theorem implies that

$$T_xV' = f^*(\mathcal{H})(x) \oplus \ker(\mathrm{d}f|_x).$$

Define the function g from W to Z to be the restriction of f to the submanifold W. The form $g^*(\omega_Z)$ is a closed 2-form on W as the pullback of the closed 2-form ω_Z restricted to $f(V')$. Suppose that there is a v in TW such that for all w in TW, $g^*(\omega_Z)(v,w)$ is equal to zero. In this case,

$$0 = g^*(\omega_Z)(v,w) = \omega_Z(\mathrm{d}g(v), \mathrm{d}g(w)),$$

and so

$$\omega_Z(\mathrm{d}g(v), \cdot) = 0$$

since $\mathrm{d}g|_x$ is surjective at each point x of W. Nondegeneracy of ω_Z implies that $\mathrm{d}g(v)$ is equal to zero and the injectivity of $\mathrm{d}g$ further implies that v is equal to zero. The form $g^*(\omega_Z)$ is, therefore, a symplectic form on W.

For any (η, ζ) in $C^\infty(V') \times C^\infty(V')$,

$$\begin{aligned}(f^*(\omega_Z|_x))(w_\eta, w_\zeta) &= \omega_Z(\mathrm{d}f(w_\eta), \mathrm{d}f(w_\zeta))|_{f(x)} \\ &= \omega_Z(v_\eta, v_\zeta)|_{f(x)} \\ &= \{\eta, \zeta\}_Z|_{f(x)} \\ &= \{\eta \circ f, \zeta \circ f\}_X|_x = \omega_X(w_\eta, w_\zeta)|_x, \end{aligned} \tag{4.3}$$

where the assumption that f is Poisson implies the penultimate equality. The pullback $f^*(\omega_Z)$ is therefore the restriction of ω_X to $TW \times TW$. The manifold W is an embedded symplectic submanifold of V' and so [28, p.124, Exercise 3.38] implies that there is an open set V of V' that contains x_0 and a Darboux coordinate system $(q_i^X, p_i^X)_{i=1}^\ell$ on V such that for any x in V and i strictly larger than n,

$$q_i^X(x) = p_i^X(x) = 0.$$

Define

$$\omega_A = \sum_{i=1}^n \mathrm{d}q_i^X \wedge \mathrm{d}p_i^X \quad \text{and} \quad \omega_B = \sum_{i=n+1}^\ell \mathrm{d}q_i^X \wedge \mathrm{d}p_i^X,$$

so that in the open set V, ω_X is equal to the sum of ω_A and ω_B. The form ω_B is the restriction of ω_X to $(TW \times TW) \cap (TV \times TV)$ and so (4.3) implies that ω_B is equal to $f^*(\omega_X)$. Furthermore, for any θ in $C^\infty(U)$,

$$\begin{aligned}(f^*(\mathrm{d}q_i^Z))(w_\theta)|_x &= \mathrm{d}q_i^Z(\mathrm{d}f(w_\theta))|_x \\ &= \mathrm{d}q_i^Z(v_\theta)|_{f(x)} \\ &= v_\theta(q_i^Z)|_{f(x)} \\ &= \{q_i^Z, \theta\}_Z|_{f(x)}\end{aligned}$$

$$= \{q_i^Z \circ f, \theta \circ f\}_X\big|_x = \mathrm{d}(q_i^Z \circ f) w_\theta\big|_x.$$

Every element of TW is of the form w_θ for some θ in $C^\infty(U)$, implying that

$$f^*(\mathrm{d}q_i^Z) = \mathrm{d}(q_i^Z \circ f) \quad \text{and} \quad f^*(\mathrm{d}p_i^Z) = \mathrm{d}(p_i^Z \circ f). \tag{4.4}$$

Use (4.4) together with the coordinate representation of ω_Z to obtain the equality

$$f^*(\omega_Z) = \sum_{i=1}^{n} \mathrm{d}(q_i^Z \circ f) \wedge \mathrm{d}(p_i^Z \circ f),$$

that implies that in the chart V,

$$\omega_X = \sum_{i=1}^{n} \mathrm{d}(q_i^Z \circ f) \wedge \mathrm{d}(p_i^Z \circ f) + \sum_{i=n+1}^{\ell} \mathrm{d}q_i^X \wedge \mathrm{d}p_i^X.$$

The coordinate system ϕ on V given by

$$\phi = (q_1^Z \circ f, p_1^Z \circ f, \dots, q_n^Z \circ f, p_n^Z \circ f, q_{n+1}^X, p_{n+1}^X, \dots, q_\ell^X, p_\ell^X)$$

is, therefore, a Darboux coordinate system on V. □

Despite having a local splitting of the tangent space by a local foliation given Poisson maps, it is not always true that the image of a symplectic manifold under a Poisson map is symplectic as the next example demonstrates.

The following example was inspired by a conversation with L. Polterovich [29].

Example 4.3.2. Let Φ be the Poisson map from $\mathbb{R}^4$ to $\mathbb{R}^2$ defined by $(p_1, q_1, p_2, q_2) \mapsto (p_1, q_1)$. The manifold $\mathbb{R}^2$ is an embedded submanifold of $\mathbb{R}^4$ with basis vectors e_1 and e_2 for its tangent space and with $\omega_{\mathbb{R}^4}(e_1, e_2) > 0$, hence $\mathbb{R}^2$ is a symplectic submanifold of $\mathbb{R}^4$. Let e_1' be the vector $e_1 + e_2$ so that $\omega_{\mathbb{R}^4}(e_1', e_2) > 0$. Take A to be the $\mathrm{Span}\big(\mathrm{e}_1', \mathrm{e}_2\big)$ such that e_2 is in $\ker(\Phi|_A)$ and e_1' is not in $\ker(\Phi|_A)$. The submanifold A is a symplectic submanifold of $\mathbb{R}^4$ and $\Phi(A)$ is a line in $\mathbb{R}^2$, which is a

Lagrangian submanifold. Therefore, the image of a symplectic submanifold under a Poisson map need not be symplectic.

We now look at a particular manifold, the diagonal submanifold of the product of a symplectic manifold with itself and see that changing the 2-form on the diagonal can make the diagonal symplectic or Lagrangian. Let X be a symplectic manifold with symplectic form ω_X. Let π_1 and π_2 be the projections that map the product $X \times X$ onto X by

$$\pi_1(x, y) = x \quad \text{and} \quad \pi_2(x, y) = y \quad \text{with} \quad x, y \in X.$$

Take c to be a non-zero real number. The form $\omega_{X\times X}$, given by

$$\omega_{X\times X} = c\pi_1^*\omega + c\pi_2^*\omega,$$

is closed and nondegenerate. Therefore, the manifold $X \times X$ with this form is a symplectic manifold. Denote by D the diagonal submanifold of $X \times X$ and by ι the inclusion map

$$\iota \colon D \to X \times X.$$

Denote by ω_D the form given by

$$\omega_D = c\iota^*\pi_1^*\omega_X + c\iota^*\pi_2^*\omega_X.$$

We will show that (D, ω_D) is a symplectic submanifold of $(X \times X, \omega_{X\times X})$ and that the map ϕ with

$$\phi \colon X \to D \subset X \times X \quad \text{by} \quad x \mapsto (x, x)$$

is a Poisson map onto its image.

Suppose that V and W are sections of TD that are defined at a point P in D. There are sections v and w of TX and a point p in X such that

$$V = (v, v), \quad W = (w, w), \quad \text{and} \quad P = (p, p)$$

and both v and w are defined at p. In Lemma 4.3.3 and in Proposition 4.3.4 below, we will use the notational convention that the uppercase letters V and W denote sections of TD that are respectively the pairs (v, v) and (w, w) where v and w are sections of TX.

Lemma 4.3.3. *If V and W are sections of TD defined at the same point P in D, then*

$$\left(\iota^*\pi_1^*\omega_X|_P\right)(V, W) = \left(\iota^*\pi_2^*\omega_X|_P\right)(V, W) = \omega_X|_p(v, w).$$

Proof. The definition of the pull back functions on forms gives us the equalities

$$\begin{aligned} \iota^*\pi_1^*\omega_X|_P\,(V, W) &= \iota^*\pi_1^*\omega_X|_{(p,p)}\,((v, v), (w, w)) \\ &= \omega_X|_p\,(\pi_{1*}\,((v, v), (w, w))) \\ &= \omega_X|_p\,((\pi_{1*}(v, v), \pi_{1*}(w, w))) = \omega_X|_p(v, w). \end{aligned}$$

On replacing π_1^* with π_2^* in the above calculation, we obtain the equality

$$\iota^*\pi_2^*\omega_X|_P\,(V, W) = \omega_X|_p(v, w),$$

hence,

$$\iota^*\pi_1^*\omega_X|_P(V, W) = \iota^*\pi_2^*\omega_X|_P(V, W).$$

□

Proposition 4.3.4. *The form ω_D is closed and nondegenerate on D. Therefore, D is a symplectic submanifold of $X \times X$.*

Proof. The form ω_X is closed, therefore

$$d\iota^*\pi_i^*\omega_X = \iota^*\pi_i^* d\omega_X = 0,$$

and so

$$d\omega_D = c d\iota^*\pi_i^*\omega_X + c d\iota^*\pi_i^*\omega_X = 0.$$

Lemma 4.3.3 implies that

$$\omega_D|_P (V, W) = 2c\omega_X|_p (v, w) .$$

Therefore,

$$\omega_D|_P (V, W) = 0$$

for any section W of TD defined at P if and only if

$$\omega_X|_p (v, w) = 0$$

for any section w of TX defined at p. Since ω_X is nondegenerate, v_p is the zero vector and so V_P is the zero vector as well. Therefore, ω_D is nondegenerate. □

We now set c equal to $\frac{1}{2}$. Let $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ be Darboux coordinates in an open neighborhood U of a point a in X and denote these coordinates by (q_i, p_i) to compress notation. Since (q_i, p_i) are Darboux, Lemma 4.3.3 implies $\left(\iota^*\pi_1^* q_i, \iota^*\pi_1^* p_i\right)$ is a Darboux coordinate system on D in an open neighborhood of the point (a, a). For clarity, we rename the Darboux coordinates so that

$$\left(\iota^*\pi_1^* q_i, \iota^*\pi_1^* p_i\right) = (\alpha^* q_i, \alpha^* p_i) .$$

Denote respectively by $\{\cdot,\cdot\}_D$ and $\{\cdot,\cdot\}_X$ the Poisson brackets on (D, ω_D) and (X, ω_X).

Lemma 4.3.5. *Suppose that z_i is equal to either q_i or p_i. If f and g are in $C^\infty(X \times X)$, then*

$$\frac{\partial f}{\partial \alpha^* z_i} \circ \phi^* = \frac{\partial \phi^* f}{\partial z_i}.$$

Proof. We will assume that z_i is equal to q_1 since the proofs for the other cases are all similar. Denote by ψ the homeomorphism

$$\psi\colon U \to \mathbb{R}^{2n} \quad \text{by} \quad u \mapsto (q_i(u), p_i(u)).$$

Suppose that a is in U, then $\phi(a)$ equals (a, a), an element D. Let γ the curve in $\psi(U)$ given by

$$\gamma(t) = (q_1(a) + t, q_2(a), \ldots, q_n(a), p(a))$$

where t varies in an open interval containing zero that is small enough so that the curve remains in $\psi(U)$. We have the equalities

$$\begin{aligned}
\left(\frac{\partial f}{\partial \alpha^* q_1} \circ \phi\right)(a) &= \frac{\partial f}{\partial \alpha^* q_1}(a, a) \\
&= \frac{d}{dt}\Big|_{t=0} f(\iota \circ \phi \circ \psi^{-1}(\gamma(t))) \\
&= \frac{d}{dt}\Big|_{t=0} f(\psi^{-1}(x_t), \psi^{-1}(x_t)) \\
&= \frac{d}{dt}\Big|_{t=0} f \circ \phi(\psi^{-1}(x_t)) \\
&= \frac{d}{dt}\Big|_{t=0} \phi^* f(\psi^{-1}(x_t)) = \frac{\partial \phi^* f}{\partial q_1}.
\end{aligned}$$

□

Proposition 4.3.6. *The map ϕ is a Poisson map onto its image D.*

Proof. If f and g are in $C^\infty(X \times X)$, then

$$\phi^*\{f, g\}_D(a) = \phi^* \sum_i \left(\frac{\partial f}{\partial \alpha^* q_i} \cdot \frac{\partial g}{\partial \alpha^* p_i} - \frac{\partial f}{\partial \alpha^* p_i} \cdot \frac{\partial g}{\partial \alpha^* q_i}\right)(a)$$

$$= \sum_i \left(\left(\frac{\partial f}{\partial \alpha^* q_i} \circ \phi \right) (a) \left(\frac{\partial g}{\partial \alpha^* p_i} \circ \phi \right) (a) - \left(\frac{\partial f}{\partial \alpha^* p_i} \circ \phi \right) (a) \left(\frac{\partial g}{\partial \alpha^* q_i} \circ \phi \right) \right).$$

Furthermore,

$$\{\phi^* f, \phi^* g\}_X(a) = \sum_i \left(\frac{\partial \phi^* f}{\partial q_i}(a) \cdot \frac{\partial \phi^* g}{\partial p_i}(a) - \frac{\partial \phi^* f}{\partial p_i}(a) \cdot \frac{\partial \phi^* g}{\partial q_i}(a) \right).$$

Therefore, Lemma 4.3.5 implies that

$$\phi^*\{f, g\}_D(a) = \phi^*\{f, g\}_D(a),$$

hence ϕ is Poisson onto D. □

Notice that the symplectic form on $X \times X$ that is induced by the symplectic form on X and the projections π_1 and π_2 is in no way unique. In fact, so long as a and b are nonzero real numbers, the form ω on $X \times X$ given by

$$\omega = a\pi_1^*\omega_X + b\pi_2^*\omega_X$$

is symplectic. Different choices of a and b can profoundly affect the properties of D. In our setting, D is a symplectic submanifold. However, if we take an $\omega'_{X\times X}$ defined as

$$\omega'_{X\times X} = \pi_1^*\omega_X - \pi_2^*\omega_X,$$

then D will no longer be a symplectic leaf but a Lagrangian submanifold.

Proposition 4.3.7. *The diagonal submanifold D is a Lagrangian submanifold of* $(X \times X, \omega'_{X\times X})$.

Proof. If (v, w) is an element of $T_{(a,a)}(X \times X)$, then

$$\begin{aligned}
(\pi_1^*\omega_X - \pi_2^*\omega_X)\left((v, w), \cdot\right) &= \pi_1^*\omega_X\left((v, w), \cdot\right) - \pi_2^*\omega_X\left((v, w), \cdot\right) \\
&= \omega_X\left(\pi_{1*}(v, w), \cdot\right) - \omega_X\left(\pi_{2*}(v, w), \cdot\right) \\
&= \omega_X\left(v, \cdot\right) - \omega_X\left(w, \cdot\right) \\
&= \omega_X\left(v - w, \cdot\right).
\end{aligned}$$

Since ω is nondegenerate, $\omega_X(v-w,\cdot)$ is identically zero if and only if $v-w$ is the zero vector. Therefore $\omega'_{X\times X}((v,w),\cdot)$ is identically zero if and only if (v,w) is in $T_{(a,a)}D$, which proves that D is a Lagrangian submanifold of $X\times X$. □

As we have seen with the diagonal, we can change the symplectic form and end up with different symplectic structures. There are many symplectic forms possible on the fibered product, $X\times_Z Y$ but not all will make the projection maps from $X\times_Z Y$ Poisson. For instance, one could pair $X\times_Z Y$ with the symplectic form induced by the product manifold $X\times Y$. Example 4.3.8 shows that such structure leads to a double-counting of coordinate functions when studying Hamiltonian systems on the the fibered product.

Example 4.3.8. Consider three point masses attached by springs as shown in Figure 1.1 with the left spring having a spring constant k_1 and the right spring having spring constant k_2. Let f be a surjective Poisson map from X to Z, g be a surjective Poisson map from Y to Z, ρ_X be the projection map from $X\times Y$ to X and ρ_Y be the projection map from $X\times Y$ to Y. The phase space of the left mass, m_X, is X and has position and momentum coordinate (q_X,p_X). The phase space of the middle mass, m_Z, is Z with position and momentum coordinates (q_Z,p_Z). The phase space of the right mass, m_Y, is Y with position and momentum coordinates (q_Y,p_Y). The Hamiltonian for the system is

$$H=\frac{1}{m_X}p_X^2+\frac{1}{m_Z}p_Z^2+\frac{1}{m_Y}p_Y^2+\frac{1}{2}k_1(q_Z-q_X)^2+\frac{1}{2}k_2(q_Y-q_Z)^2.$$

Hamilton's equations are as follows:

$$\dot{p_X}=-\frac{\partial H}{\partial q_X}=k_1(q_Z-q_X),\quad \dot{q_X}=\frac{\partial H}{\partial p_X}=\frac{1}{m_X}p_X,$$

$$\dot{p_Y}=-\frac{\partial H}{\partial q_Y}=k_2(q_Y-q_Z),\quad \dot{q_Y}=\frac{\partial H}{\partial p_Y}=\frac{1}{m_Y}p_Y,$$

and

$$\dot{q_Z}=\frac{\partial H}{\partial p_Z}=\frac{1}{m_Z}p_Z,\quad \dot{p_Z}=-\frac{\partial H}{\partial q_Z}=k_1(q_Z-q_X)+k_2(q_Y-q_Z).$$

We can view the system as two subsystems, namely the left mass, middle mass, left spring and the middle mass, right mass and right spring. Each mass will have a symplectic manifold for its phase space. We can compose the two subsystems by “gluing" along the middle mass to build the larger system, which means taking a pullback along Z. The pullback $X \times_Z Y$ will be the phase space for the composite system. Now suppose the symplectic form on $X \times_Z Y$ is the induced symplectic form on $X \times Y$. The symplectic form will be

$$\widetilde{\omega} = \omega_{b,\beta} + \widetilde{\omega}_X + \widetilde{\omega}_Y,$$

where

$$q_Z \circ f \circ \rho_X = \widetilde{q_Z}^X, \quad p_Z \circ f \circ \rho_X = \widetilde{p_Z}^X, \quad q_Z \circ g \circ \rho_Y = \widetilde{q_Z}^Y, \quad p_Z \circ g \circ \rho_Y = \widetilde{p_Z}^Y,$$

$$q_X \circ \rho_X = \widetilde{q_X}, \quad p_X \circ \rho_X = \widetilde{p_X}, \quad q_Y \circ \rho_Y = \widetilde{q_Y}, \quad p_Y \circ \rho_Y = \widetilde{p_Y},$$

$$\beta = \widetilde{p_Z}^X + \widetilde{p_Z}^Y \quad \text{and} \quad b = \widetilde{q_Z}^X + \widetilde{q_Z}^Y.$$

Rewrite the symplectic form as

$$\widetilde{\omega} = db \wedge d\beta + d\widetilde{q_X} \wedge \widetilde{p_X} + d\widetilde{q_Y} \wedge d\widetilde{p_Y}.$$

This construction gives rise to the Poisson bracket

$$\begin{aligned}\{\cdot,\cdot\} : (\phi,\psi) \mapsto & \frac{\partial \phi}{\partial(\widetilde{q_Z}^X + \widetilde{q_Z}^Y)} \frac{\partial \psi}{\partial(\widetilde{p_Z}^X + \widetilde{p_Z}^Y)} - \frac{\partial \phi}{\partial(\widetilde{p_Z}^X + \widetilde{p_Z}^Y)} \frac{\partial \psi}{\partial(\widetilde{q_Z}^X + \widetilde{q_Z}^Y)} + \frac{\partial \phi}{\partial \widetilde{q_X}} \frac{\partial \psi}{\partial \widetilde{p_X}} \\ & - \frac{\partial \phi}{\partial \widetilde{p_X}} \frac{\partial \psi}{\partial \widetilde{q_X}} + \frac{\partial \phi}{\partial \widetilde{q_Y}} \frac{\partial \psi}{\partial \widetilde{p_Y}} - \frac{\partial \phi}{\partial \widetilde{q_Y}} \frac{\partial \psi}{\partial \widetilde{p_Y}}.\end{aligned}$$

The Hamiltonian is

$$\begin{aligned}H = & \frac{1}{2m_X}\widetilde{p_X}^2 + \frac{1}{2m_Z}(\widetilde{p_Z}^X + \widetilde{p_Z}^Y)^2 + \frac{1}{2m_Y}\widetilde{p_Y}^2 \\ & + \frac{1}{2}k_1((\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_X})^2 + \frac{1}{2}k_2((\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_Y})^2\end{aligned}$$

and the Hamiltonian vector field is

$$
\begin{aligned}
\{\cdot, H\} = & \frac{\partial}{\partial(\widetilde{q_Z}^X + \widetilde{q_Z}^Y)} \frac{\partial H}{\partial(\widetilde{p_Z}^X + \widetilde{p_Z}^Y)} \\
& - \frac{\partial}{\partial(\widetilde{p_Z}^X + \widetilde{p_Z}^Y)} \frac{\partial H}{\partial(\widetilde{q_Z}^X + \widetilde{q_Z}^Y)} + \frac{\partial}{\partial\widetilde{q_X}} \frac{\partial H}{\partial\widetilde{p_X}} \\
& - \frac{\partial}{\partial\widetilde{p_X}} \frac{\partial H}{\partial\widetilde{q_X}} + \frac{\partial}{\partial\widetilde{q_Y}} \frac{\partial H}{\partial\widetilde{p_Y}} - \frac{\partial}{\partial\widetilde{p_Y}} \frac{\partial H}{\partial\widetilde{q_Y}}.
\end{aligned}
$$

Denote by v_H this vector field to obtain

$$
\begin{aligned}
v_H = & \frac{1}{m_Z}(\widetilde{p_Z}^X + \widetilde{p_Z}^Y) \frac{\partial}{\partial(\widetilde{q_Z}^X + \widetilde{q_Y}^Y)} - (k_1((\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_X}) \\
& + k_2((\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_Y})) \frac{\partial}{\partial(\widetilde{p_Z}^X + \widetilde{p_Z}^Y)} \\
& + \frac{1}{m_X}\widetilde{p_X} \frac{\partial}{\partial\widetilde{q_X}} - k_1((\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_X}) \frac{\partial}{\partial\widetilde{p_X}} + \frac{1}{m_Y}\widetilde{p_Y} \frac{\partial}{\partial\widetilde{q_Y}} - k_2((\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_Y}) \frac{\partial}{\partial\widetilde{p_Y}}.
\end{aligned}
$$

Hamilton's equations for this system are

$$
\dot{\widetilde{p_X}} = -\frac{\partial H}{\partial\widetilde{q_X}} = k_1((\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_X}), \quad \dot{\widetilde{q_X}} = \frac{\partial H}{\partial\widetilde{p_X}} = \frac{1}{m_X}\widetilde{p_X},
$$

$$
\dot{\widetilde{p_Y}} = -\frac{\partial H}{\partial\widetilde{q_Y}} = k_2((\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_Y}), \quad \dot{\widetilde{q_Y}} = \frac{\partial H}{\partial\widetilde{p_Y}} = \frac{1}{m_Y}\widetilde{p_Y},
$$

$$
(\widetilde{p_Z}^X + \widetilde{p_Z}^Y)^{\cdot} = \frac{\partial H}{\partial(\widetilde{q_Z}^X + \widetilde{q_Z}^Y)} = -(k_1((\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_X}) + k_2(\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_Y}),
$$

and

$$
(\widetilde{q_Z}^X + \widetilde{q_Z}^Y)^{\cdot} = -\frac{\partial H}{\partial(\widetilde{p_Z}^X + \widetilde{p_Z}^Y)} = \frac{1}{m_Z}(\widetilde{p_Z}^X + \widetilde{p_Z}^Y).
$$

On the pullback we have $\widetilde{p_Z}^X = \widetilde{p_Z}^Y$ and $\widetilde{q_Z}^X = \widetilde{q_Z}^Y$. Hence,

$$
2(\widetilde{q_Z}^X)^{\cdot} = -\frac{\partial H}{\partial(\widetilde{p_Z}^X + \widetilde{p_Z}^Y)} = -\frac{\partial H}{2\partial(\widetilde{p_Z}^X)} = \frac{2}{m_Z}(\widetilde{p_Z}^X)
$$

or $(\widetilde{q_Z}^X)^\cdot = \frac{2}{m_Z}(\widetilde{p_Z}^X)$. Similarly,

$$(\widetilde{p_Z}^X + \widetilde{p_Z}^Y)^\cdot = \frac{\partial H}{\partial(\widetilde{q_Z}^X + \widetilde{q_Z}^Y)} = -(k_1((\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_X}) + k_2(\widetilde{q_Z}^X + \widetilde{q_Z}^Y) - \widetilde{q_Y})$$
$$= -(k_1(2\widetilde{q_Z}^X - \widetilde{q_X}) + k_2(2\widetilde{q_Z}^X - \widetilde{q_Y})).$$

This shows illustrates the double counting due to the incorrect form on the fibered product.

Example 4.3.8 shows that we have chosen the incorrect form on the pullback and do not retrieve from the calculation the paths of motion. In Theorem 4.3.9 we construct the correct symplectic form on $X \times_Z Y$ where the projection maps from $X \times_Z Y$ will be Poisson.

Theorem 4.3.9. *Suppose that (f, g) is a cospan in* SympSurj *with*

$$f : X \to Z \quad \text{and} \quad g : Y \to Z,$$

with 2ℓ, $2m$, $2n$ the respective dimensions of X, Y, and Z, and suppose that ω_X, ω_Y, and ω_Z are the respective symplectic forms on X, Y, and Z. Suppose that Q is a span in SympSurj *that is paired with (f, g) and suppose that Q_A has dimension $2(\ell + m - n)$. The 2-form ω_{Q_A}, given by*

$$\omega_{Q_A} = q_L^*(\omega_X) + q_R^*(\omega_Y) - q_M^*(\omega_Z),$$

is the symplectic form on Q_A. Moreover, the 2-form ω, given by

$$\omega = \pi_X^*(\omega_X) + \pi_Y^*(\omega_Y) - \pi_Z^*(\omega_Z)$$

is the unique symplectic form on $X \times_Z Y$ with the property that (π_X, π_Y) is paired with (f, g).

Proof. Suppose that a is in Q_A. Since Z is a symplectic manifold, there is on some chart U_Z containing $q_M(a)$ a Darboux coordinate system Ψ^Z with

$$\Psi^Z = \left(q_k^Z, p_k^Z\right)_{k \in \{1,\ldots,n\}} : U_Z \to \mathbb{R}^{2n}.$$

Since $q_M(a)$ is equal to $f(q_L(a))$, Theorem 4.3.1 implies that there is a chart U_X containing $q_L(a)$ and a Darboux coordinate system Ψ^X on U_X with

$$\Psi^X = \left(q_i^X, p_i^X, q_k^Z \circ f, p_k^Z \circ f\right)_{\substack{i \in \{1,\ldots,\ell-n\} \\ k \in \{1,\ldots,n\}}} : U_X \to \mathbb{R}^{2\ell}.$$

Similarly, there is a chart U_Y containing $q_R(a)$ and a Darboux coordinate system Ψ^Y on U_Y with

$$\Psi^Y = \left(q_j^Y, p_j^Y, q_k^Z \circ g, p_k^Z \circ g\right)_{\substack{j \in \{1,\ldots,m-n\} \\ k \in \{1,\ldots,n\}}} : U_Y \to \mathbb{R}^{2m}.$$

For each k in $\{1, \ldots, n\}$, the equality of $f \circ q_L$ and $g \circ q_R$ implies that

$$q_k^Z \circ f \circ q_L = q_k^Z \circ g \circ q_R = q_k^Z \circ q_M \quad \text{and} \quad p_k^Z \circ f \circ q_L = p_k^Z \circ g \circ q_R = p_k^Z \circ q_M.$$

Furthermore, there is a chart U containing a with the property that $q_L(U)$ and $q_R(U)$ are, respectively, subsets of U_X and U_Y. Denote respectively by $\tilde{q}_i^X, \tilde{p}_i^X, \tilde{q}_j^Y, \tilde{p}_j^Y, \tilde{q}_k^Z, \tilde{p}_k^Z$ the functions $q_i^X \circ q_L$, $p_i^X \circ q_L$, $q_j^Y \circ q_R$, $p_j^Y \circ q_R$, $q_k^Z \circ q_M$, and $p_k^Z \circ q_M$ acting on Q_A. The map Ψ given by

$$\Psi = \left(\tilde{q}_i^X, \tilde{p}_i^X, \tilde{q}_j^Y, \tilde{p}_j^Y, \tilde{q}_k^Z, \tilde{p}_k^Z\right)_{\substack{i \in \{1,\ldots,\ell-n\} \\ j \in \{1,\ldots,m-n\} \\ k \in \{1,\ldots,n\}}} : U \to \mathbb{R}^{2(\ell+m-n)}$$

is a homeomorphism from U to an open subset of $\mathbb{R}^{2(\ell+m-n)}$ and hence a coordinate system on U that is a Darboux coordinate system. The 2-form ω_{Q_A} is therefore the form

$$\omega_{Q_A} = \sum_{i=1}^{\ell-n} \mathrm{d}\tilde{q}_i^X \wedge \mathrm{d}\tilde{p}_i^X + \sum_{j=1}^{m-n} \mathrm{d}\tilde{q}_j^Y \wedge \mathrm{d}\tilde{p}_j^Y + \sum_{k=1}^{n} \mathrm{d}\tilde{q}_k^Z \wedge \mathrm{d}\tilde{p}_k^Z,$$

proving that if there is a span Q with the given properties, then the symplectic form on Q_A is determined by the cospan (f, g). It does not, however, prove that there is such a span.

Proposition 3.4.4 implies that $X \times_Z Y$ is a smooth manifold of dimension $2(\ell + m - n)$. Suppose v is in $T_a(X \times_Z Y)$ and for any w in $T_a(X \times_Z Y)$, $\omega(v, w)$ is zero. There are coefficients $a^i, b^i, c^j, e^j, s^k, t^k$

such that, using Einstein summation convention,

$$v = a^i \partial \tilde{q}_i^X + b^i \partial \tilde{p}_i^X + c^j \partial \tilde{q}_j^Y + e^j \partial \tilde{p}_j^Y + s^k \partial \tilde{q}_k^Z + t^k \partial \tilde{p}_k^Z.$$

For a fixed i,

$$-\omega(v, \partial \tilde{q}_i^X) = b^i = 0.$$

A similar calculation shows that all of the given coefficients are zero, implying that v is equal to zero and so ω is nondegenerate. The form ω is the sum of pullbacks of smooth closed forms, and so smooth and closed itself, hence symplectic. The construction of ω ensures that the smooth surjections π_X and π_Y are Poisson maps on the symplectic manifold $(X \times_Z Y, \omega)$, hence (π_X, π_Y) is paired with (f, g). □

Theorem 4.3.10. *Suppose that (f, g) is a cospan in* RiemSurj *with*

$$f \colon X \to Z \quad \text{and} \quad g \colon Y \to Z$$

and that g_X, g_Y, and g_Z are the metric tensors on X, Y, and Z, respectively. The tensor $g_{X \times_Z Y}$, given by

$$g_{X \times_Z Y} = \pi_X^*(g_X) + \pi_Y^*(g_Y) - \pi_Z^*(g_Z),$$

is the unique metric tensor on $X \times_Z Y$ such that the span (π_X, π_Y) is paired with (f, g).

Proof. Since every surjective Riemannian submersion is a surjective submersion, the fibered product $X \times_Z Y$ is a smooth manifold. If $g_{X \times_Z Y}$ is positive definite, then $(X \times_Z Y, g_{X \times_Z Y})$ is a Riemannian manifold since $g_{X \times_Z Y}$ is a symmetric tensor as a sum of pullbacks of symmetric tensors. It suffices to show that $g_{X \times_Z Y}$ is nondegenerate.

Follow the proof of Theorem 4.3.9, using the splitting of the tangent spaces

$$TX = (\ker(\mathrm{d}f))^\perp \oplus (\ker(\mathrm{d}f)) \quad \text{and} \quad TY = (\ker(\mathrm{d}g))^\perp \oplus (\ker(\mathrm{d}g))$$

rather than the previous appeal to Theorem 4.3.1 to obtain an expression for $g_{X\times_Z Y}$ in local coordinates. Together with this local coordinate representation of $g_{X\times_Z Y}$, the fact that the maps π_X, π_Y and π_Z are surjective Riemannian submersions imply that $g_{X\times_Z Y}$ is nondegenerate. The proof is similar to the proof of Theorem 4.3.9 and so the details are left to the reader to verify. □

Note that the symplectic form on $X \times_Z Y$ in Theorem 4.3.9 is not the pullback by the inclusion map of the symplectic form on $X \times Y$ to the manifold $X \times_Z Y$. While the pullback form is symplectic, the span (π_X, π_Y) will no longer be a span in SympSurj when $X \times_Z Y$ is endowed instead with the pullback form. The analogous statements about the potential choices for the metric tensor are true in the Riemannian setting.

4.4 Examples

Below are some first examples of generalized span categories. We will develop more examples in the next chapter that involve looking at categories of Riemannian and symplectic manifolds.

Example 4.4.1. (*Categories that have Pullbacks*) Suppose that $\mathscr{C}$ is a category that has pullbacks and let $\mathcal{F}$ be the identity functor from $\mathscr{C}$ to $\mathscr{C}$. The functor $\mathcal{F}$ is span tight and so Span$(\mathscr{C}, F)$ is a category. Since every $\mathcal{F}$-pullback of a cospan is a pullback of a cospan, the category Span$(\mathscr{C}, \mathcal{F})$ is the category Span$(\mathscr{C})$. In this way, the concept of a generalized span category Span$(\mathscr{C}, \mathcal{F})$ generalizes the notion of a span category and reduces to it when $\mathscr{C}$ has pullbacks and $\mathcal{F}$ is the identity functor.

Example 4.4.2. (*Smooth Manifolds and Surjective Submersions*) Suppose that $\mathcal{F}$ is the inclusion functor from SurjSub to Diff. Theorems 4.1.5 and 4.2.1 together imply that Span(SurjSub, $\mathcal{F}$) is a category.

Example 4.4.3. (*Classical Mechanics*) We work in the categories RiemSurj, whose objects are Riemannian manifolds and whose morphisms are surjective Riemannian submersions, and SympSurj, whose objects are symplectic manifolds and whose morphisms are surjective Poisson maps. Unlike SurjSub, these categories are not subcategories of Diff. However, the forgetful functors from these categories into Diff are still span tight and so it is possible to construct generalized span categories in these settings which are critical to the study of classical mechanics.

In the next chapter, we will in a limited setting extend the work of Fong in [19] by introducing the notion of an augmented generalized span category. Such categories are critical to the categorification of classical mechanics and the study of the functoriality of the Legendre transformation.

Chapter 5

Lagrangian and Hamiltonian Systems

5.1 Systems as Isomorphism Classes of Augmented Spans

We now introduce the notion of an augmentation of a span and cospan in the restricted settings that are significant to the current discussion. The description of a Lagrangian or Hamiltonian system respectively requires not only the identification of a Riemannian or Poisson span, but the additional information of a potential or a Hamiltonian, both of which are augmentations.

Definition 5.1.1. An *augmented manifold* is a pair (M, F_M), where M is a smooth manifold and F_M is a smooth real valued function defined on M. The pair given by (M, F_M) is an *augmented Riemannian* (*symplectic*) *manifold* if M is a Riemannian (symplectic) manifold. Refer to F_M as a *potential* (*or Hamiltonian*), denoting it by V_M (or H_M) if M is respectively a Riemannian (or symplectic) manifold.

For sake of concision, denote by $\mathfrak{M}$ any of the categories listed in Figure 4.8.

Definition 5.1.2. An *augmented* (*co*)*span* in $\mathfrak{M}$ is a pair (S, F_S), where S is a (co) span in $\mathfrak{M}$ and F_S is a triple $(F_{S_A}, F_{S_L}, F_{S_R})$ of smooth real valued functions defined respectively on S_A, S_L, and S_R. If $\mathfrak{M}$ is RiemSurj (or SympSurj), then the given augmented span is an *augmented Riemannian* (*co*)*span* (or *augmented Poisson* (*co*)*span*). A *physical* (*co*)*span* is an augmented (co)span that is either Riemannian or Poisson. If (S, F_S) is an augmented Riemannian (Poisson) span, then refer to F_S as a *potential* (*or Hamiltonian*) and denote it by V_S (or H_S).

The apex of a Poisson span determines the kinematical properties of the system and the mapping of the apex to its feet determines the way in which the span composes with other spans and, therefore, how components of systems compose to form more complicated systems. The apex of a Riemannian span determines a free system and the augmentation will be a potential that determines the interactions in the system. The fundamental object of our study should be an isomorphism class of augmented spans rather than an augmented span because composition using $\mathcal{F}$-pullbacks is only determined up to isomorphism.

Definition 5.1.3. Suppose that physical spans (S, F_S) and $\left(Q, F_Q\right)$ are either both Riemannian or both Poisson and that

$$(S_L, F_{S_L}) = \left(Q_L, F_{Q_L}\right) \quad \text{and} \quad (S_R, F_{S_R}) = \left(Q_R, F_{Q_R}\right).$$

A span morphism Φ from S_A to Q_A is *compatible with* F_S and F_Q if F_{S_A} is equal to $F_{Q_A} \circ \Phi$ and is, in this case, a *morphism of physical spans*. If Φ is additionally an isomorphism, then Φ is an *isomorphism of physical spans* and (S, F_S) and $\left(Q, F_Q\right)$ are *isomorphic physical spans*.

The inverse of an isometry is again an isometry. The inverse of an icthyomorphism is again an icthyomorphism, [18, p. 10]. Proposition 5.1.4 follows from these facts.

Proposition 5.1.4. *The inverse of any Riemannian* (*or Poisson*) *span isomorphism from S to Q is a Riemannian* (*or Poisson*) *span isomorphism from Q to S.*

Denote by $[S, F_S]$ the set of all physical spans that are isomorphic to a physical span (S, F_S). Together with the fact that the composition of physical span isomorphisms is again a physical span isomorphism, Proposition 5.1.4 implies that isomorphism of physical spans is an equivalence relation, hence the set $[S, F_S]$ is an equivalence class.

Definition 5.1.5. A *Lagrangian* (or *Hamiltonian*) *system* is an isomorphism class of Riemannian or Poisson) spans. If $[S, F_S]$ is either a Hamiltonian system or a Lagrangian system, then $[S, F_S]$ is a *physical system*. Physical systems $[S, F_S]$ and $[Q, F_Q]$ are *of the same type* if they are both Hamiltonian systems or both Lagrangian systems.

5.2 Paths of Motion

Refer to Section 2.2 for review of the Euler-Lagrange equations on a Riemannian manifold.

Definition 5.2.1. Suppose that S is a Poisson span. Denote by $\{\cdot, \cdot\}_{S_A}$ the Poisson bracket associated to the symplectic form ω_{S_A} on the symplectic manifold S_A. A path γ in S_A is a *path of motion of S* if it is an integral curve of the the vector field v where

$$v = \{\cdot, H_{S_A}\}_{S_A} .$$

Proposition 5.2.2. *Suppose that (S, F_S) and (Q, F_Q) are physical spans of the same type and Φ is an isomorphism of physical spans taking (S, F_S) to (Q, F_Q). If γ is a path of motion of (S, F_S), then $\Phi \circ \gamma$ is a path of motion of (Q, F_Q). Furthermore, every path of motion of (Q, F_Q) is the image of a path of motion of (S, F_S).*

Proof. If S and Q are Riemannian spans and Φ is an isomorphism from S to Q, then Φ is an isometry from S_A to Q_A and V_{S_A} is equal to $V_{Q_A} \circ \Phi$. Denote by ∇^{S_A} and ∇^{Q_A} the respective Levi-Civita connections on S_A and Q_A. Suppose that p is an element of S_A and that X and Y are tangent vector fields on S_A. The map Φ is an isometry and so

$$d\Phi_p\left(\left(\nabla^{S_A}_X Y\right)(p)\right) = \nabla^{Q_A}_{d\Phi(X)} d\Phi(Y)(\Phi(p)) \quad \text{and} \quad d\Phi\left(\operatorname{grad}_{S_A}\left(V_{Q_A} \circ \Phi\right)\right) = \operatorname{grad}_{Q_A}\left(V_{Q_A}\right).$$

If γ is a path of motion of (S, F_S), then $\Phi \circ \gamma$ is a curve in Q_A and

$$\begin{aligned}
\nabla^{Q_A}_{(\Phi\circ\gamma)'}(\Phi \circ \gamma)' + \operatorname{grad}_{Q_A}\left(V_{Q_A}\right)|_{\Phi\circ\gamma} &= \nabla^{Q_A}_{d\Phi(\gamma')}(d\Phi(\gamma')) + \operatorname{grad}_{Q_A}\left(V_{Q_A}\right)|_{\Phi\circ\gamma} \\
&= d\left(\nabla^{S_A}_{\gamma'}(\gamma') + \operatorname{grad}_{S_A}(V_{S_A})|_{\gamma}\right) \\
&= d(0) = 0,
\end{aligned}$$

where the fact that γ satisfies (EL) in S_A implies the penultimate equality. The path $\Phi \circ \gamma$ is therefore a path of motion of (Q, F_Q).

If S and Q are Poisson spans and Φ is an isomorphism from S to Q, then Φ is an icthyomorphism from S_A to Q_A and H_{S_A} is equal to $H_{Q_A} \circ \Phi$. The curve γ is path of motion of (S, F_S) if and only if it is an integral curve of the vector field $\{\cdot, H_{S_A}\}$. Suppose that α and β are smooth functions on Q_A. Since Φ is Poisson,

$$\mathrm{d}\Phi\Big(\{\cdot, \alpha \circ \Phi\}_{S_A}\Big)(\beta) = \{\cdot, \alpha \circ \Phi\}_{S_A}(\beta \circ \Phi) = \Big(\{\beta \circ \Phi, \alpha \circ \Phi\}_{S_A}\Big) = \{\beta, \alpha\}_{Q_A}$$

and so

$$\begin{aligned}(\Phi \circ \gamma)' &= \mathrm{d}\Phi|_\gamma\Big(\{\cdot, H_{S_A}\}_{S_A}\Big) \\ &= \mathrm{d}\Phi|_\gamma\Big(\Big\{\cdot, H_{Q_A} \circ \Phi\Big\}_{S_A}\Big) = \Big\{\cdot, H_{Q_A}\Big\}_{Q_A}|_{\Phi\circ\gamma}.\end{aligned}$$

The curve $\Phi \circ \gamma$ is, therefore, a path of motion of (Q, F_Q).

In both the Riemannian and Poisson settings, the map Φ^{-1} is also an isomorphism of physical spans and so every path of motion of (Q, F_Q) is the image of a path of motion of (S, F_S). □

5.3 $\mathcal{F}$-Pullbacks of SympSurj and RiemSurj in Diff

Recall Example 3.3.4, which demonstrated that SurjSub does not have pullbacks. This same example can be adopted in the Riemannian or symplectic setting because any discrete manifold can be endowed with the trivial Riemannian metric or symplectic form. Therefore, RiemSurj and SympSurj do not have pullbacks. Proposition 5.2.2 implies that an isomorphism class of physical spans determines the dynamics of a physical system. Composing such isomorphism classes requires both the existence of $\mathcal{F}$-pullbacks in these categories, where $\mathcal{F}$ is an appropriate forgetful functor into Diff, as well as the span tightness of the functor $\mathcal{F}$.

Theorem 5.3.1. *The forgetful functors from SympSurj to Diff and from RiemSurj to Diff are span tight.*

Proof. Suppose that $\mathcal{F}$ is the forgetful functor from SympSurj to Diff. Since every morphism in SympSurj is a surjective submersion, the functor $\mathcal{F}$ maps SympSurj to the subcategory SurjSub of Diff. If (f, g) is a cospan in SympSurj, and π_X and π_Y are, as defined above, the respective projections from $X \times_Z Y$ to X and Y, then Proposition 4.1.4 implies that $(\mathcal{F}(\pi_X), \mathcal{F}(\pi_Y))$ is a span in Diff that is a pullback of the cospan $(\mathcal{F}(f), \mathcal{F}(g))$. Therefore, SympSurj has $\mathcal{F}$-pullbacks in Diff. Suppose now that Q is a span in SympSurj that is also an $\mathcal{F}$-pullback of (f, g). In this case, the span $\mathcal{F}(Q)$ is a span in Diff that is a pullback of $(\mathcal{F}(f), \mathcal{F}(g))$ and so there is a span diffeomorphism Φ from $\mathcal{F}(Q)$ to $\mathcal{F}(X \times_Z Y)$. Since Φ is a span morphism,

$$\mathcal{F}(q_L) \circ \Phi^{-1} = \mathcal{F}(\pi_X), \quad \mathcal{F}(q_R) \circ \Phi^{-1} = \mathcal{F}(\pi_Y), \quad \text{and} \quad \mathcal{F}(f) \circ \mathcal{F}(q_L) \circ \Phi^{-1} = \mathcal{F}(\pi_Z). \tag{5.1}$$

Denote respectively by ω, ω_X, ω_Y, and ω_Z the symplectic forms on $X \times_Z Y$, X, Y, and Z. The equalities of (5.1) imply that

$$\begin{aligned}
\omega &= \mathcal{F}(\pi_X)^*(\omega_X) + \mathcal{F}(\pi_Y)^*(\omega_Y) - \mathcal{F}(\pi_Z)^*(\omega_Z) \\
&= \left(\mathcal{F}(q_L) \circ \Phi^{-1}\right)^*(\omega_X) + \left(\mathcal{F}(q_R) \circ \Phi^{-1}\right)^*(\omega_Y) - \left(\mathcal{F}(f) \circ \mathcal{F}(q_L) \circ \Phi^{-1}\right)^*(\omega_Z) \\
&= \left(\Phi^{-1}\right)^* \left(\mathcal{F}(q_L)^*(\omega_X) + \mathcal{F}(q_R)^*(\omega_Y) - (\mathcal{F}(f) \circ \mathcal{F}(q_L))^*(\omega_Z)\right) \\
&= \left(\Phi^{-1}\right)^* \left(\omega_{Q_A}\right),
\end{aligned}$$

where ω_{Q_A} is the unique 2-form on Q_A such that Q is paired with (f, g). Let Ψ be the map from (Q_A, ω_{Q_A}) to $(X \times_Z Y, \omega)$ that acts as Φ on the underlying manifolds. The map Ψ is, therefore, a diffeomorphism and Ψ^{-1} is a symplectic map, hence Ψ is a symplectomorphism. Since every symplectomorphism is an icthyomorphism, Ψ isomorphism in the category SympSurj with $\mathcal{F}(\Psi)$ equal to Φ, [1, p. 195].

A similar argument proves the theorem in the case of RiemSurj. □

Corollary. *If $\mathcal{F}$ is the forgetful functor from SympSurj to Diff (resp. RiemSurj to Diff), then* Span(*SympSurj*, $\mathcal{F}$) (*resp.* Span(*RiemSurj*, $\mathcal{F}$)) *is a category.*

While Theorems 4.2.1 and 5.3.1 imply that $\mathrm{Span}(\mathsf{SympSurj}, \mathcal{F})$ and $\mathrm{Span}(\mathsf{RiemSurj}, \mathcal{F})$ are categories, where $\mathcal{F}$ is the appropriate forgetful functor into Diff, to show that physical systems are morphisms of a category requires additional verifications. The next section provides the necessary verifications.

Chapter 6

Physical Systems as Morphisms

6.1 The Categories HamSy and LagSy

This section constructs the categories LagSy and HamSy, whose objects are respectively augmented Riemannian manifolds or augmented symplectic manifolds and whose morphisms are isomorphism classes of the physical spans appropriate to the given category.

Definition 6.1.1. The physical system $[S, F_S]$ is *composable* with the physical system $[Q, F_Q]$ if:

(1) both are physical systems of the same type;

(2) if (S, F_S) and $\left(Q, F_Q\right)$ are respective representatives of the equivalence classes $[S, F_S]$ and $[Q, F_Q]$, then (S_R, F_{S_R}) is equal to $\left(Q_L, F_{Q_L}\right)$.

Assume below that the physical system $[S, F_S]$ is composable with $\left[Q, F_Q\right]$, and (S, F_S) and $\left(Q, F_Q\right)$ are, respectively, representatives of $[S, F_S]$ and $\left[Q, F_Q\right]$. To simplify notation, let

$$S_A = X,\ S_L = V,\ S_R = Q_L = Z,\ Q_A = Y,\ \text{and}\ Q_R = W.$$

Again denote by $X \times_Z Y$ the fibered product and by π_X, π_Y, and π_Z the respective projections to X, Y, and Z. Define by $[S, F_S] \circ [Q, F_Q]$ the augmented span given by

$$[S, F_S] \circ [Q, F_Q] = \left[(s_L \circ \pi_X, q_R \circ \pi_Y), F_{S \circ Q}\right],$$

where

$$F_{S\circ Q} = (F_X \circ \pi_X + F_Y \circ \pi_Y - F_Z \circ \pi_Z, F_V, F_W)\,.$$

Theorem 6.1.2. *The Hamiltonian systems are the morphisms in a category,* HamSy*, whose objects are augmented symplectic manifolds. The Lagrangian systems are the morphisms in a category,* LagSy*, whose objects are augmented Riemannian manifolds.*

Proof. To prove the theorem, it suffices to show that: (1) composition of morphisms in HamSy and in LagSy is well defined; (2) both HamSy and LagSy have left and right unit laws; and (3) composition of morphisms in HamSy and in LagSy is associative. Since Span(RiemSurj, $\mathcal{F}$) and Span(SympSurj, $\mathcal{F}$) are categories, to show that HamSy and LagSy are categories, it suffices to show that the augmentations are compatible with the various span isomorphisms that arise in defining the categories Span(RiemSurj, $\mathcal{F}$) and Span(SympSurj, $\mathcal{F}$). Suppose that $[S, F_S]$ and $\left[Q, F_Q\right]$ are both morphisms in HamSy and denote by $\mathcal{F}$ the forgetful functor from SympSurj to Diff.

(1) Suppose that $[S', F_{S'}]$ is equal to $[S, F_S]$ and that α is an isomorphism of augmented spans with

$$\alpha\colon X = S_A \to S'_A.$$

Suppose that $\left[Q', F_{Q'}\right]$ is equal to $\left[Q, F_Q\right]$ and that β is an isomorphism of augmented spans with

$$\beta\colon Y = Q_A \to Q'_A.$$

Since (Z, F_Z) is the right foot of (S, F_S) and the left foot of $\left(Q, F_Q\right)$,

$$\left(S'_R, F_{S'_R}\right) = \left(Q'_L, F_{Q'_L}\right) = (Z, F_Z)\,.$$

If P is an $\mathcal{F}$-pullback of $\left(s'_R, q'_L\right)$, then there is a span isomorphism Φ in SympSurj with

$$\Phi\colon X \times_Z Y \to P_A.$$

The augmented span $(S', F_{S'}) \circ_P \left(Q', F_{Q'}\right)$ is given by

$$(S', F_{S'}) \circ_P \left(Q', F_{Q'}\right) = \left(\left(s'_L \circ p_L, q'_R \circ p_R\right), F_{S' \circ_P Q'}\right),$$

where

$$F_{S' \circ_P Q'} = \left(F_{S'_A} \circ p_L + F_{Q'_A} \circ p_R - F_Z \circ s'_R \circ p_L, F_V, F_W\right).$$

Since α and β are isomorphisms of augmented spans,

$$F_{S'_A} \circ \alpha = F_X \quad \text{and} \quad F_{Q'_A} \circ \beta = F_Y.$$

The function Φ is a span isomorphism and so

$$p_L \circ \Phi = \alpha \circ \pi_X \quad \text{and} \quad p_R \circ \Phi = \beta \circ \pi_Y,$$

hence

$$F_{S'_A} \circ p_L \circ \Phi = F_{S'_A} \circ \alpha \circ \pi_X = F_X \circ \pi_X.$$

Similar arguments show that

$$F_{Q'_A} \circ p_R \circ \Phi = F_Y \circ \pi_Y \quad \text{and} \quad F_Z \circ s'_R \circ p_L \circ \Phi = F_Z \circ \pi_Z,$$

and so

$$F_{S \circ Q} = \left(F_{S' \circ_P Q'}\right) \circ \Phi. \tag{6.1}$$

Equality (6.1) implies that Φ is an augmented span isomorphism, hence the composition of $[S, F_S]$ and $\left[Q, F_Q\right]$ is independent of representative. The composite $[S, F_S] \circ \left[Q, F_Q\right]$ is, therefore, a well defined morphism from (Q_R, F_{Q_R}) to (S_L, F_{S_L}).

(2) Let $[S, F_S]$ be a morphism with source (S_R, F_{S_R}) and target (S_L, F_{S_L}). Let $(\mathrm{I}_{S_R}, F_{\mathrm{I}_{S_R}})$ be a representative of the identity augmented span with source (S_R, F_{S_R}) and target (S_R, F_{S_R}). The

equality

$$[S] \circ [\mathrm{I}_{S_R}] = [S]$$

follows from the fact that Span(SympSurj, $\mathcal{F}$) is a category. Let the span P be an $\mathcal{F}$-pullback of (s_R, I_{S_R}), where

$$P_L = P_A = S_A,\ P_R = S_R,\ p_L = \mathrm{Id}_X,\ \text{and } p_R = s_R.$$

The equalities

$$\begin{aligned} F_{P_A} &= F_{S_L} \circ p_L + F_{S_R} \circ s_R - F_{S_R} \circ s_R \circ p_L \\ &= F_{S_L} \circ \mathrm{Id}_\mathrm{X} + F_{S_R} \circ s_R - F_{S_R} \circ s_R \circ \mathrm{Id}_X = F_{S_L} \end{aligned}$$

imply that there is an augmented span isomorphism from $(S, F_S) \circ (\mathrm{I}_{S_R}, F_{S_R})$ to (S, F_S), and so

$$[S, F_S] \circ [\mathrm{I}_{S_R}, F_{S_R}] = [S, F_S]\,.$$

A similar argument shows that

$$[\mathrm{I}_{S_L}, F_{S_L}] \circ [S, F_S] = [S, F_S]\,.$$

Therefore, HamSy has left and right unit laws.

(3) Refer to Figure 6.1 for the naming of the maps below, where all spans paired with a given cospan are augmented $\mathcal{F}$-pullbacks of the given cospan and the diagram is commutative. Let (P^3, F_{P^3}) be an $\mathcal{F}$-pullback of (p_R^1, p_L^2) and let (P^4, F_{P^4}) be an $\mathcal{F}$-pullback of $(q_R \circ p_R^1, t_L)$.

To prove (3), show first that there is an augmented span isomorphism from the augmented span $((S, F_S) \circ_{(P^1, F_{P^1})} (Q, F_Q)) \circ_{(P^4, F_{P^4})} (T, F_T)$ to the augmented span (P, F_P) that is given by the composite $((S, F_S) \circ_{(P^1, F_{P^1})} (Q, F_Q)) \circ_{(P^3, F_{P^3})} ((Q, F_Q) \circ_{(P^2, F_{P^2})} (T, F_T))$. A similar argument will show that there is an augmented span isomorphism from the augmented span $(S, F_S) \circ ((Q, F_Q) \circ (T, F_T))$ to (P, F_P) and the result follows by the fact that inverses and compositions of augmented span isomorphisms are augmented span isomorphisms. Since Lemma 4.2.2 proves the existence of a

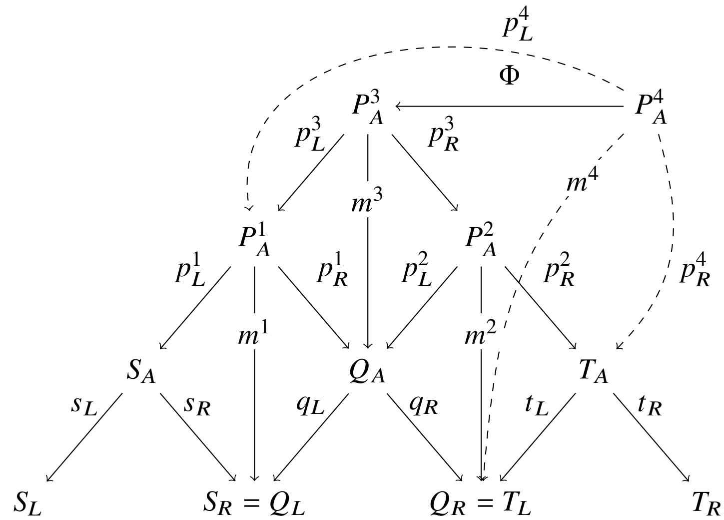

Figure 6.1: Associativity of Augmented Span Composition

span isomorphism between the non-augmented spans, it suffices to show that this span isomorphism is compatible with the augmentations for the two composite spans.

The commutativity of the diagram in Figure 6.1 and the definition of the composition of augmented spans together imply that

$$
\begin{aligned}
F_{P_A^4} &= F_{P_A^1} \circ p_L^4 + F_{T_A} \circ p_R^4 - F_{Q_R} \circ m^4 \\
&= F_{P_A^1} \circ p_L^3 \circ \Phi + F_{T_A} \circ p_R^2 \circ p_R^3 \circ \Phi - F_{Q_R} \circ m^2 \circ p_R^3 \circ \Phi. \\
&= \left(F_{P_A^1} \circ p_L^3 + F_{T_A} \circ p_R^2 \circ p_R^3 - F_{Q_R} \circ m^2 \circ p_R^3\right) \circ \Phi \\
&= \left(F_{P_A^1} \circ p_L^3 + \left(F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\right) \circ p_R^3\right) \circ \Phi \\
&= \left(F_{P_A^1} \circ p_L^3 + \left(F_{Q_A} \circ p_L^2 - F_{Q_A} \circ p_L^2 + F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\right) \circ p_R^3\right) \circ \Phi \\
&= \left(F_{P_A^1} \circ p_L^3 + \left(F_{Q_A} \circ p_L^2 + F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\right) \circ p_R^3 - F_{Q_A} \circ p_L^2 \circ p_R^3\right) \circ \Phi \\
&= \left(F_{P_A^1} \circ p_L^3 + \left(F_{Q_A} \circ p_L^2 + F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\right) \circ p_R^3 - F_{Q_A} \circ m^3\right) \circ \Phi \\
&= \left(F_{P_A^1} \circ p_L^3 + F_{P^2} \circ p_R^3 - F_{Q_A} \circ m^3\right) \circ \Phi \\
&= F_{P_A^3} \circ \Phi.
\end{aligned}
$$

Therefore, the span isomorphism Φ is compatible with the augmentations F_{P^4} and F_{P^3}.

The above arguments are independent of the morphisms being in HamSy. Repeat the arguments above in the setting of LagSy to complete the proof of the theorem. □

6.2 The Legendre Functor

This section constructs a functor $\mathscr{L}$ from LagSy to HamSy, the Legendre functor, that preserves the paths of motion.

Suppose that (M, g_M) is a Riemannian manifold of dimension m. Denote respectively by π_M and ρ_M the canonical projections from T^*M to M and from TM to M. Suppose a is a point of M. There is a chart U of M containing a that is the domain of coordinates $(x_i)_{i\in\{1,\dots,m\}}$. The set of 1-forms $\{\mathrm{d}x_i : i \in \{1, \dots, m\}\}$ trivializes the subbundle T^*U. Define for each i the real valued functions p_i^M on T^*U with the property that for all θ in T^*M,

$$\theta = \sum_{i=1}^{m} p_i^M(\theta) \frac{\partial}{\partial x_i}\bigg|_{\pi_M(\theta)}.$$

The p_i^M are the *momenta* associated with the x_i coordinates. For each i, the function p_i^M is the evaluation map $\mathrm{ev}_{\frac{\partial}{\partial x_i}\big|_{\pi_M(\theta)}}$ that is defined by the equality

$$\mathrm{ev}_{\frac{\partial}{\partial x_i}\big|_{\pi(\theta)}}(\theta) = \theta\left(\frac{\partial}{\partial x_i}\bigg|_{\pi_M(\theta)}\right).$$

For each i, define q_i^M by

$$q_i^M = x_i \circ \pi_M.$$

The function given by $\left(q_i^M, p_i^M\right)_{i\in\{1,\dots,m\}}$ on $\pi_M^{-1}(U)$ is a Darboux coordinate system, that is

$$\omega_{T^*M} = \sum_{i=1}^{m} \mathrm{d}q_i^M \wedge \mathrm{d}p_i^M.$$

Define for each i the real valued function $\hat{q}_i^M$ on TM with the property that if v is in $\rho_M^{-1}(U)$, then

$$v = \sum_{i=1}^{m} \hat{q}_i^M(v) \left.\frac{\partial}{\partial x_i}\right|_{\rho_M(v)}.$$

Note that $\hat{q}_i^M$ is the function defined for each v in TU by

$$\hat{q}_i^M(v) = \mathrm{d}x_i|_{\rho_M(v)}(v).$$

Denote ambiguously by q_i^M the function

$$q_i^M = x_i \circ \rho_M$$

on TU. The coordinate system $\left(q_i^M, \hat{q}_i^M\right)$ is a coordinate system on $\rho_M^{-1}(\pi_M(U))$.

The Riemannian metric g_M on TM induces a Riemannian metric on the cotangent bundle T^*M, to be denoted g_M^* and for each a in U defined on the pair (θ_1, θ_2) in $T_a^*M \times T_a^*M$ by

$$g_M^*(\theta_1, \theta_2) = g_M(\sharp_M(\theta_1), \sharp_M(\theta_2)) = \sum_{i,j=1}^{m} g_M^{ij}(a) p_i^M(\theta_1) p_j^M(\theta_2),$$

where g_M^{ij} denotes the (i, j) entry of the inverse of the matrix given by g_M in the $(q_i^M, \hat{q}_i^M)$ coordinates. For all v in TM and θ in T^*M, denote respectively by $g_M(\cdot)$ and $g_M^*(\cdot)$ the quadratic forms

$$g_M(v) = g_M(v, v) \quad \text{and} \quad g_M^*(\theta) = g_M^*(\theta, \theta). \tag{6.2}$$

Define $\mathscr{K}$ as a map from Riemannian manifolds to symplectic manifolds by

$$\mathscr{K}(M, g_M) = (T^*M, \omega_{T^*M}).$$

For any surjective Riemannian submersion f from M to N, define (see Figure 6.2) $\mathscr{K}(f)$ by

$$\mathscr{K}(f) = \flat_N \circ \mathrm{d}f \circ \sharp_M.$$

To simplify the notation, denote by F the function $\mathscr{K}(f)$.

Suppose that M and N are smooth manifolds of respective dimensions m and n and suppose further that f is a surjective Riemannian submersion from M to N. For any point p in M there is a coordinate system $(x_1, \ldots, x_m)$ of $\mathcal{A}_M$ on a chart containing p and a coordinate system $(y_1, \ldots, y_n)$ of $\mathcal{A}_N$ on a chart containing $f(p)$ such that for all i in $\{1, \ldots, n\}$ and k in $\{n+1, \ldots, m\}$,

$$x_i = y_i \circ f \quad \text{and} \quad \frac{\partial}{\partial x_k} \in \ker(\mathrm{d}f).$$

Let j be an index varying in the set $\{1, \ldots, n\}$. For each i and each j, denote respectively by q_i^M and q_j^N the functions $x_i \circ \pi_M$ and $y_j \circ \pi_N$ and denote by p_i^M and p_j^N the momenta associated with the coordinate functions x_i and y_j. Use the above notation for the following lemma, as well as for the rest of the section.

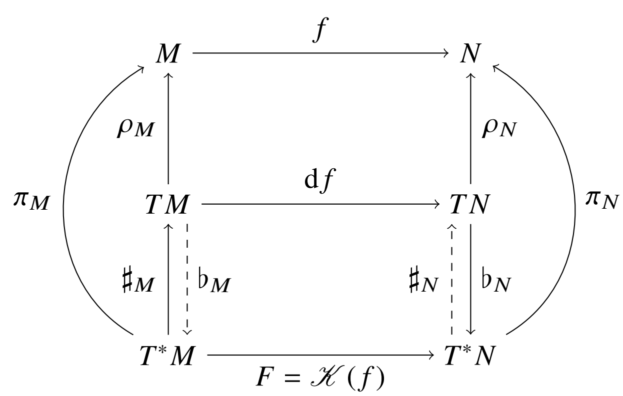

Figure 6.2: Composition of $\mathrm{d}f$ with the Musical Isomorphisms

Lemma 6.2.1. *For all p_j^M, p_j^N, and F defined as above,*

$$p_j^M = p_j^N \circ F.$$

Proof. For all j in $\{1, \ldots, n\}$,

$$\mathrm{d}f\left(\left.\frac{\partial}{\partial x_j}\right|_a\right) = \mathrm{d}f\left(\left.\frac{\partial}{\partial (y_j \circ f)}\right|_a\right) = \left.\frac{\partial}{\partial y_j}\right|_{f(a)}.$$

For all θ in T^*U, there is an element X of TU with θ equal to $g_M(X, \cdot)$. In this case, the form $F(\theta)$ is equal to $g_N(\mathrm{d}f(X), \cdot)$, and so

$$p_j^M(\theta) = \mathrm{ev}_{\frac{\partial}{\partial x_j}\big|_{\pi_M(\theta)}}(\theta) = g_M\left(X, \frac{\partial}{\partial x_j}\bigg|_{\pi_M(\theta)}\right).$$

The function f is Riemannian, implying that

$$g_M\left(X, \frac{\partial}{\partial (y_j \circ f)}\bigg|_{\pi_M(\theta)}\right) = g_N\left(\mathrm{d}f(X), \mathrm{d}f\left(\frac{\partial}{\partial (y_j \circ f)}\bigg|_{\pi_M(\theta)}\right)\right)$$

and so

$$\begin{aligned} p_j^M(\theta) &= g_N\left(\mathrm{d}f(X), \frac{\partial}{\partial y_j}\bigg|_{f(\pi_M(\theta))}\right) \\ &= g_N\left(\mathrm{d}f(X), \frac{\partial}{\partial y_j}\bigg|_{\pi_N(F(\theta))}\right) \\ &= F(\theta)\left(\frac{\partial}{\partial y_j}\bigg|_{\pi_N(F(\theta))}\right) \\ &= \mathrm{ev}_{\frac{\partial}{\partial y_j}\big|_{\pi_M(\theta)}}(F(\theta)) = (\pi_N \circ F)(\theta), \end{aligned}$$

which proves the desired equality. □

Proposition 6.2.2. *For any surjective Riemannian submersion f from a Riemannian manifold M to a Riemannian manifold N, the function $\mathscr{K}(f)$ is a surjective Poisson map.*

Proof. Suppose M and N have respective dimensions m and n. The map $\mathscr{K}$ maps Riemannian manifolds to symplectic manifolds. Once again denote by F the map $\mathscr{K}(f)$. Suppose that Π_{T^*M} and Π_{T^*N} respectively denote the Poisson bivectors for T^*M and T^*N. For any α and β in $C^\infty(N)$ and any a in M,

$$\begin{aligned} \mathrm{d}F_a(\Pi_{T^*M})(\alpha, \beta) &= \Pi_{T^*M}(\alpha \circ F, \beta \circ F)\Big|_a \\ &= \sum_{i=1}^{m}\left(\frac{\partial(\alpha \circ F)}{\partial q_i^M}\frac{\partial(\beta \circ F)}{\partial p_i^M} - \frac{\partial(\beta \circ F)}{\partial q_i^M}\frac{\partial(\alpha \circ F)}{\partial p_i^M}\right)\bigg|_a \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \left(\frac{\partial(\alpha \circ F)}{\partial q_i^M} \frac{\partial(\beta \circ F)}{\partial p_i^M} - \frac{\partial(\beta \circ F)}{\partial q_i^M} \frac{\partial(\alpha \circ F)}{\partial p_i^M} \right)\Bigg|_a \\
&= \sum_{i=1}^{n} \left(\frac{\partial(\alpha \circ F)}{\partial(q_i^N \circ F)} \frac{\partial(\beta \circ F)}{\partial(p_i^N \circ F)} - \frac{\partial(\beta \circ F)}{\partial(q_i^N \circ F)} \frac{\partial(\alpha \circ F)}{\partial(p_i^N \circ F)} \right)\Bigg|_a \qquad (6.3) \\
&= \sum_{i=1}^{n} \left(\frac{\partial(\alpha)}{\partial q_i^N} \frac{\partial(\beta)}{\partial p_i^N} - \frac{\partial(\beta)}{\partial q_i^N} \frac{\partial(\alpha)}{\partial p_i^N} \right)\Bigg|_{F(a)} = \Pi_{T^*N}(\alpha, \beta)\Big|_{F(a)},
\end{aligned}
$$

where Lemma 6.2.1 implies the equality in (6.3). Therefore, $\mathrm{d}F(\Pi_{T^*M})$ is equal to Π_{T^*N}, which implies that F is a Poisson map. The map f is a surjective submersion, therefore $\mathrm{d}f$ is surjective. The nondegeneracy of g implies that F is also surjective and so $\mathscr{K}$ maps the morphisms in RiemSurj to morphisms in SympSurj. □

Lemma 6.2.3. *For any Riemannian spans S and Q and any span isomorphism Φ from S to Q, the function $\mathscr{K}(\Phi)$ is a span isomorphism from $\mathscr{K}(S)$ to $\mathscr{K}(Q)$.*

Proof. Suppose that Φ is a span isomorphism from S and Q. In this case, $\mathscr{K}(\Phi)$ is Poisson. Since $\mathscr{K}(\Phi)$ is an icthyomorphism, it is an isomorphism in the category SympSurj. Recall that the isomorphisms in SympSurj are icthyomorphisms, which are symplectomorphisms since the objects in SympSurj are symplectic manifolds, [1, p. 195]. Since Φ is a span morphism,

$$s_L = q_L \circ \Phi \quad \text{and} \quad s_R = q_R \circ \Phi,$$

implying that

$$
\begin{aligned}
\mathscr{K}(s_L) &= \mathscr{K}(q_L \circ \Phi) \\
&= \flat_{Q_L} \circ \mathrm{d}(q_L \circ \Phi) \circ \sharp_{S_A} \\
&= \flat_{Q_L} \circ \mathrm{d}q_L \circ \mathrm{d}\Phi \circ \sharp_{S_A} \\
&= \flat_{Q_L} \circ \mathrm{d}q_L \circ \left(\sharp_{Q_A} \circ \flat_{Q_A}\right) \circ \mathrm{d}\Phi \circ \sharp_{S_A} \\
&= \left(\flat_{Q_L} \mathrm{d}q_L \circ \sharp_{Q_A}\right) \circ \left(\flat_{Q_L} \circ \mathrm{d}\Phi \circ \sharp_{S_A}\right) = \mathscr{K}(q_L) \circ \mathscr{K}(\Phi).
\end{aligned}
$$

A similar argument shows that

$$\mathscr{K}(s_R) = \mathscr{K}(q_R) \circ \mathscr{K}(\Phi),$$

proving that $\mathscr{K}(\Phi)$ is a span morphism. Therefore, for any spans S and Q in RiemSurj that are span isomorphic, the spans $\mathscr{K}(S)$ and $\mathscr{K}(Q)$ are also span isomorphic. □

Lemma 6.2.4. *For any Riemannian submersion f that is compatible with a Riemannian augmentation, the function $\mathscr{K}(f)$ is a Poisson map that is compatible with the Hamiltonian augmentation that is the image under $\mathscr{K}$ of the Riemannian augmentation.*

Proof. For any span isomorphism Φ from S to Q that is compatible with F_S and F_Q,

$$V_{S_A} = V_{Q_A} \circ \Phi.$$

The isomorphism Φ is Riemannian, hence an isometry. Therefore,

$$g^*_{S_A} = g^*_{Q_A} \circ \mathscr{K}(\Phi),$$

and so

$$\begin{aligned} H_{S_A} &= \frac{1}{2} g^*_{S_A} + V_{S_A} \circ \pi_{S_A} \\ &= \frac{1}{2} g^*_{Q_A} \circ \mathscr{K}(\Phi) + V_{Q_A} \circ \pi_{Q_A} \circ \mathscr{K}(\Phi) = H_{Q_A} \circ \mathscr{K}(\Phi). \end{aligned}$$

□

Suppose that S is a Riemannian span and let $\star$ denote either of the letters A, L, or R. Define $\mathscr{K}(S_\star, V_\star)$ by

$$\mathscr{K}(S_\star, V_\star) = (\mathscr{K}(S_\star), H_\star)$$

where for all η in $S_\star$,

$$H_{S_\star}(\eta) = \frac{1}{2} g^*_{S_\star}(\eta) + (V_\star \circ \pi_{S_\star})(\eta).$$

Each object of LagSy is an augmented Riemannian manifold and so $\mathscr{K}$ maps the objects of LagSy to the objects of HamSy. Define $\mathscr{L}$ to be $\mathscr{K}$ on the objects of LagSy and for each morphism $[S]$ in LagSy, define $\mathscr{L}([S])$ by

$$\mathscr{L}([S]) = [\mathscr{K}(S)].$$

Theorem 6.2.5. *The map $\mathscr{L}$ is a functor from LagSy to HamSy. Suppose that π_{S_A} is the canonical projection from T^*S_A to S_A. Suppose that the Lagrangian system $[S]$ has a path of motion γ on the manifold S_A that is specified by the representative S of $[S]$ and suppose that γ intersects a point x of S_A at time zero. In this case, the path $\mathscr{K} \circ \gamma$ is a path determined by $\mathscr{L}([S])$, valued in the symplectic manifold $\mathscr{K}(S_A)$, and $\pi_{S_A} \circ \mathscr{K} \circ \gamma$ also intersects x at time zero.*

Proof. The map $\mathscr{L}$ maps Riemannian manifolds to symplectic manifolds and potentials to Hamiltonians, and therefore maps the objects of LagSy to the objects of HamSy. Proposition 6.2.2 implies that $\mathscr{L}$ maps surjective Riemannian submersions to surjective Poisson maps, and so if S is a Riemannian span, then $\mathscr{K}(S)$ is a Poisson span. Lemma 6.2.4 implies that if (S, F_S) and (Q, F_Q) are isomorphic as augmented Riemannian spans, then $\mathscr{K}(S, F_S)$ and $\mathscr{K}(Q, F_Q)$ are also isomorphic as augmented Poisson spans and so $\mathscr{L}$ is well defined on Lagrangian systems, mapping them to Hamiltonian systems.

Suppose that M is a Riemannian manifold. Denote by $\mathcal{L}_M$ the Lagrangian on TM, where for each v in TM,

$$\mathcal{L}_M(v) = \frac{1}{2} g_M(v, v) - V_M(\rho_M(v)) \, .$$

Denote by H_M the Hamiltonian associated to V_M and by $\{\cdot, \cdot\}_{T^*M}$ the Poisson bracket as given above in the construction of $\mathscr{L}$. It is a standard result in classical mechanics that a path γ on M is a solution to (EL) if and only if it is an integral curve of $\{\cdot, H_M\}_M$, [16, p.25, Theorem 3.13]. This proves the last two statements of the theorem. To prove that $\mathscr{L}$ is a functor, it suffices to show further that: (1) $\mathscr{L}$ commutes with composition and (2) $\mathscr{L}$ maps identity morphisms to identity morphisms.

To show (1), suppose that $[S, F_S]$ and $[Q, F_Q]$ are augmented Riemannian spans and that $[S, F_S]$ is composable with $[Q, F_Q]$. Suppose that P is an $\mathcal{F}$-pullback of (s_R, q_L), where P_A is the fibered product $S_A \times_{S_R} Q_A$ and p_R and p_L are the respective restrictions of the projections on $S_A \times Q_A$ to S_A

and Q_A. The map $\mathscr{K}$ maps $S_A \times_{S_R} Q_A$ to its cotangent bundle $T^*(S_A \times_{S_R} Q_A)$, which is isomorphic in SympSurj to the manifold $(T^*S_A) \times_{(T^*S_R)} (T^*Q_A)$. The symplectic form on $T^*(S_A \times_{S_R} Q_A)$ is given by the canonical 2-form and the symplectic form ω on $(T^*S_A) \times_{(T^*S_R)} (T^*Q_A)$ is given by

$$\omega = \mathscr{K}(p_L)^*(\omega_{T^*S_A}) + \mathscr{K}(p_R)^*(\omega_{T^*Q_A}) - \mathscr{K}(p_L)^*(\mathscr{K}(s_R)^*(\omega_{T^*S_R})).$$

The symplectomorphism Φ from $T^*(S_A \times_{S_R} Q_A)$ to $(T^*S_A) \times_{(T^*S_R)} (T^*Q_A)$ is consistent with the augmentations. Lemma 6.2.4 implies that

$$\begin{aligned} \mathscr{L}([S, F_S] \circ [Q, F_Q]) &= \mathscr{L}([(S, F_S) \circ_P (Q, F_Q)]) \\ &= [\mathscr{K}((S, F_S) \circ_P (Q, F_Q))] \\ &= [\mathscr{K}(S, F_S) \circ_{\mathscr{K}(P)} \mathscr{K}(Q, F_Q)] \\ &= [\mathscr{K}(S, F_S)] \circ [\mathscr{K}(Q, F_Q)] = \mathscr{L}([S, F_S]) \circ \mathscr{L}([Q, F_Q]), \end{aligned}$$

where the penultimate equality holds because $\mathscr{K}(P)$ is an $\mathcal{F}$-pullback.

To show (2), suppose that (X, V_X) is an augmented Riemannian manifold and that Id_X is the identity map from X to X. Denote by I_X the span $(\mathrm{Id}_X, \mathrm{Id}_X)$. The span $\mathscr{K}(\mathrm{I}_X)$ is the pair $(\mathscr{K}(\mathrm{Id}_X), \mathscr{K}(\mathrm{Id}_X))$ where $\mathscr{K}(\mathrm{Id}_X)$ is the identity map Id_{T^*X} from T^*X to T^*X. Furthermore, $\mathscr{K}$ maps the augmentation V_X to the augmentation H_{T^*X} where

$$H_{T^*X} = \frac{1}{2} g_X^* + V_X \circ \pi_X.$$

Suppose that S is an augmented Hamiltonian span with (S_L, H_{S_L}) equal to (T^*X, H_{T^*X}). Let Q be the $\mathcal{F}$-pullback of the cospan $(\mathscr{K}(\mathrm{Id}_X), s_L)$ with the property that Q_A is the symplectic manifold $T^*X \times_{T^*X} S_A$. The maps q_L and q_R are the respective restrictions to the manifold $T^*X \times_{T^*X} S_A$ of the canonical projections of the manifold $T^*X \times S_A$ to T^*X and S_A and are symplectomorphisms. The definition of the augmentation on a pullback implies that

$$H_{Q_A} = \left(\frac{1}{2} g_X^* + V_X \circ \pi_X\right) \circ q_L + \left(\frac{1}{2} g_{S_A}^* + V_{S_A} \circ \pi_{S_A}\right) \circ q_R$$

$$
\begin{aligned}
&\quad - \left(\frac{1}{2}g_X^* + V_X \circ \pi_X\right) \circ q_L \circ \mathrm{Id}_{T^*X} \\
&= \left(\frac{1}{2}g_X^* + V_X \circ \pi_X\right) \circ q_L + \left(\frac{1}{2}g_{S_A}^* + V_{S_A} \circ \pi_{S_A}\right) \circ q_R - \left(\frac{1}{2}g_X^* + V_X \circ \pi_X\right) \circ q_L \\
&= \left(\frac{1}{2}g_{S_A}^* + V_{S_A} \circ \pi_{S_A}\right) \circ q_R = H_{S_A} \circ q_R,
\end{aligned}
$$

hence

$$H_{Q_A} = H_{S_A} \circ q_R.$$

The map q_R is, therefore, compatible with the augmentations. Since Q is paired with $(\mathscr{K}(\mathrm{Id}_X), s_L)$,

$$s_L \circ q_R = \mathrm{Id}_X \circ q_L = q_L,$$

and so q_R is a span isomorphism mapping the composite $(\mathscr{K}(\mathrm{I}_X) \circ q_L, s_R \circ q_R)$ to the span S that is compatible with the augmentations. This compatibility implies that

$$\mathscr{L}([\mathrm{I}_X, V_{\mathrm{I}_X}]) \circ [S, H_S] = [\mathscr{K}(\mathrm{I}_X, V_X) \circ (S, H_S)] = [S, H_S].$$

Similar arguments show that for any augmented Hamiltonian span $(S', H_{S'})$ such that $(S'_R, H_{S'_R})$ is equal to (T^*X, H_{T^*X}),

$$[S', H_{S'}] \circ \mathscr{L}([\mathrm{I}_X, V_X]) = [S', H_{S'}],$$

and so $\mathscr{L}([\mathrm{I}_X, V_X])$ is the identity map with source and target (T^*X, H_{T^*X}).

□

Refer to the functor $\mathcal{L}$ from LagSy to HamSy as the *Legendre functor*. It is an analog of the Legendre transformation and translates from Lagrangian to Hamiltonian descriptions of a physical system.

6.3 Motivating Example

Suppose that the spring-mass system with three masses given in Figure 1.3 has masses m_1, m_2, and m_3 respectively as the left, middle, and right masses of the system. Suppose further that the spring constants of the left and right springs are respectively k_1 and k_2. The spring-mass system with three masses is a composite of two spring-mass systems with two masses each. We now discuss a category theoretic construction of a model for the composite system with its subsystems.

Let $[S, V_S]$ be a Lagrangian system describing the left-spring mass system and $[Q, V_Q]$ be a Lagrangian systems describing the right spring-mass system. Denote both S_R and Q_L by Z, since S_R is equal to Q_L, and by V_Z the augmentation on Z. Take a representative (S, V_S) of the Langrangian system $[S, V_S]$ to be the augmented Riemannian span with the manifold S_A equal to $\mathbb{R}^2$ and the manifolds S_L and Z equal to $\mathbb{R}$. Let g_1 be the standard Riemannian metric on $\mathbb{R}$. Let ρ_L and ρ_R be the canonical projections on $\mathbb{R}^2$ with

$$\rho_L(q_1, q_2) = q_1 \quad \text{and} \quad \rho_R(q_1, q_2) = q_2.$$

Denote by g_2 the standard Riemannian metric on $\mathbb{R}^2$. Endow S_L with the Riemannian metric g_{S_L} and Z with the Riemannian metric g_Z, where g_{S_L} and g_Z are given by

$$g_{S_L} = m_1 g_1 \quad \text{and} \quad g_Z = m_2 g_1.$$

Define by g_{S_A} the metric on $\mathbb{R}^2$ given for all v and w in $T_{(q_1,q_2)}\mathbb{R}^2$ by

$$g_{S_A}(v, w) = g_{S_L}(\mathrm{d}\rho_L(v), \mathrm{d}\rho_L(w)) + g_Z(\mathrm{d}\rho_R(v), \mathrm{d}\rho_R(w)).$$

Denote respectively by s_L and s_R the functions from S_A to S_L and from S_A to Z that act on underlying manifolds as the projections ρ_L and ρ_R. The augmentation V_S is the triple of maps

$$V_S = (V_{S_A}, V_{S_L}, V_Z) \quad \text{with} \quad V_{S_A}(q_1, q_2) = \frac{k_1}{2}(q_1 - q_2)^2,\ V_{S_L} \equiv 0,\ \text{and}\ V_Z \equiv 0.$$

Define similarly the Riemannian span (Q, V_Q), but with the Riemannian metric g_{Q_R} on Q_R and the augmentations V_{Q_A} and V_{Q_R} given by

$$g_{Q_R} = m_3 g_1, \; V_{Q_A}(q_2, q_3) = \frac{k_2}{2}(q_2 - q_3)^2, \text{ and } V_{Q_R} \equiv 0.$$

Define by g_{Q_A} the metric on $\mathbb{R}^2$ given for all v and w in $T_{(q_2,q_3)}\mathbb{R}^2$ by

$$g_{Q_A}(v, w) = g_Z(\mathrm{d}\rho_L(v), \mathrm{d}\rho_L(w)) + g_{Q_R}(\mathrm{d}\rho_R(v), \mathrm{d}\rho_R(w)).$$

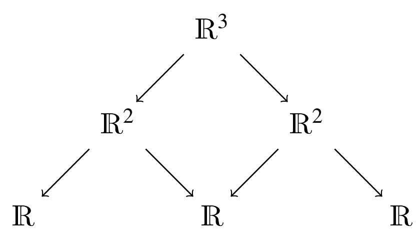

Figure 6.3: Configuration Spaces for Three Point Masses

Denote by π_L and π_R the respective projections from $S_A \times_Z Q_A$ to S_A and to Q_A and by π_M the map $s_R \circ \pi_L$, which is also the map $q_R \circ \pi_R$. Denote by $g_{S_A \times_Z Q_A}$ the Riemannian metric on $S_A \times_Z Q_A$ given by

$$g_{S_A \times_Z Q_A} = \pi_L^*(g_{S_A}) + \pi_R^*(Q_A) - \pi_Z^*(g_Z).$$

The augmentation $V_{S_A \times_Z Q_A}$ is then given by

$$V_{S_A \times_Z Q_A} = \pi_L^*(V_{S_A}) + \pi_R^*(V_{Q_A}) - \pi_M^*(V_Z).$$

Let Φ be the diffeomorphism from $S_A \times_Z Q_A$ to $\mathbb{R}^3$ given by

$$\Phi(q_1, q_2, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_2, \dot{q}_3) = (q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3).$$

Denote by P_A the Riemannian manifold $\mathbb{R}^3$, and by p_L and p_R the maps

$$p_L = s_L \circ \pi_L \circ \Phi^{-1} \quad \text{and} \quad p_R = s_R \circ \pi_R \circ \Phi^{-1}.$$

Denote similarly by V_{P_A} the potential

$$V_{P_A} = V_{S_A \times_Z Q_A} \circ \Phi^{-1}.$$

Define a Riemannian metric g_{P_A} on P_A by

$$g_{P_A} = (\Phi^{-1})^*(g_{S_A \times_Z Q_A}),$$

making Φ an isometry. The Lagrangian for the composite system is $\mathcal{L}_{P_A}$ where for every v in TP_A,

$$\mathcal{L}_{P_A}(v) = \frac{1}{2} g_{P_A}(v, v) - V_{P_A}\left(\rho_{P_A}(v)\right).$$

The Lagrangian $\mathcal{L}$ of the system with configuration space given by $\mathbb{R}^3$ is given with respect to coordinate system (q_1, q_2, q_3) by

$$\begin{aligned}
\mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) &= \frac{m_1}{2}(\dot{q}_1)^2 + \frac{m_2}{2}(\dot{q}_2)^2 + \frac{m_2}{2}(\dot{q}_2)^2 + \frac{m_3}{2}(\dot{q}_3)^2 - \frac{m_2}{2}(\dot{q}_2)^2 \\
&\quad - \frac{k_1}{2}(q_1 - q_2)^2 - \frac{k_1}{2}(q_2 - q_3)^2 + 0 \quad (\text{since } V_Z \equiv 0) \\
&= \frac{m_1}{2}(\dot{q}_1)^2 + \frac{m_2}{2}(\dot{q}_2)^2 + \frac{m_3}{2}(\dot{q}_3)^2 - \frac{k_1}{2}(q_1 - q_2)^2 - \frac{k_1}{2}(q_2 - q_3)^2.
\end{aligned}$$

The Riemannian span (P, F_P) is a representative of the Lagrangian system $[S, F_S] \circ [Q, F_Q]$. The Lagrangian $\mathcal{L}$ on P_A is the Lagrangian for the given system of three masses and two springs with configuration space equal to $\mathbb{R}^3$. We leave the determination of the Hamiltonian system to the reader as it is a straightforward exercise given the previous discussion and the result of the next section.

In general, a description of a composite system requires a prior description of the subsystems. The subsystems need not themselves have descriptions as composite systems and it remains an open problem to determine the simplest subsystems that are required to construct from them any other system as a composite. If two subsystems that share a common component form a complicated system, and if we know how to map the subsystems into two pieces, one of which is the common component, then we can view the complicated system as a composite system in our formalism.

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