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Abstract

We describe UMA (Unified Model of Arithmetic), a theory of children's arithmetic implemented as a computational model. UMA extends a theory of fraction arithmetic (Braithwaite et al., 2017) to include arithmetic with whole numbers and decimals. We evaluated UMA in the domain of decimal arithmetic by training the model on problems from a math textbook series, then testing it on decimal arithmetic problems that were solved by 6th and 8th graders in a previous study. UMA's test performance closely matched that of children, supporting three assumptions of the theory: (1) most errors reflect small deviations from standard procedures, (2) between-problem variations in error rates reflect the distribution of input that learners receive, and (3) individual differences in strategy use reflect underlying variation in learning parameters.

Keywords: mathematical cognition; fractions; decimals; arithmetic; computational cognitive model; strategy choice

Introduction

Arithmetic involves combining numbers by addition, subtraction, multiplication, and division. The simplicity of this description belies the complexity of arithmetic, which is apparent when one considers calculation with different types of numbers. For example, one may add single digit whole numbers (e.g., 4+2) by counting or retrieval from memory, whereas adding fractions (e.g., 3/5+1/4) may require conversion to a common denominator, and adding multidigit whole numbers (e.g., 123+56) or decimals (e.g., 2.46+4.1) may involve column addition algorithms based on place value. Yet other skills are required for other types of problems. Apparently, arithmetic is not one skill, but many.

Reflecting this complexity, previous models of arithmetic skill have focused on only one or a few aspects of it, such as arithmetic with small whole numbers (Aubin et al., 2017; Campbell & Graham, 1985; Shrager & Siegler, 1998; Verguts & Fias, 2005), multidigit whole numbers (Brown & VanLehn, 1980), fractions (Braithwaite et al., 2017), or decimals (Hiebert & Wearne, 1985). It is unclear how well these models can explain children's arithmetic outside the domains for which they were created.

To address this challenge, we created UMA—a Unified Model of Arithmetic. UMA is a theory of arithmetic learning and performance implemented as a computational cognitive model. UMA extends FARRA (Fraction Arithmetic Reflects Rules and Associations; Braithwaite et al., 2017), a model of children's fraction arithmetic. Unlike FARRA, UMA simulates arithmetic not only with fractions, but also with single digit and multidigit whole numbers and decimals.

UMA's viability as a unified model ultimately depends on its ability to explain performance in all of these domains of arithmetic. Here, we take a first step toward such a comprehensive test by applying the model to the domain of decimal arithmetic.

First, we describe our theoretical assumptions, which are shared by FARRA and UMA. These assumptions enabled FARRA to account for phenomena in children's fraction arithmetic and to generate novel predictions about children's decimal arithmetic that were subsequently confirmed, suggesting the feasibility of a unified theory including both fraction and decimal arithmetic. Next, we describe how UMA implements our theoretical assumptions in a computational model. Last, we evaluate UMA in the domain of decimal arithmetic by comparing its performance to that of children.

Theoretical Assumptions

We propose answers to three fundamental questions about children's arithmetic learning and performance: (1) Where do incorrect answers come from? (2) Why are errors more common for some problems than others? (3) What causes individual differences in strategy use?

Where do incorrect answers come from?

We assume that incorrect answers in arithmetic mostly reflect deviations from standard correct procedures. This assumption implies that correct procedures are a good starting point for understanding incorrect performance.

UMA posits two types of deviations: *overgeneralization* and *omission*. Overgeneralization (Table 1) refers to using a procedure that is not appropriate for the problem at hand but would be appropriate for a different type of problem. Overgeneralization encompasses errors that have previously been attributed to "whole number bias" (Ni & Zhou, 2005; Table 1, row 1) as well as many other types of errors (Table 1, other rows). Omission (Table 2) refers to executing some, but not all, of the steps required by a procedure.

UMA assumes that overgeneralization is the main source of error in both fraction and decimal arithmetic. The rationale for this assumption is that, in these domains, procedures for different arithmetic operations are easily confusable. Further, instruction often emphasizes how to execute each procedure but not how to choose which procedure should be used. This approach opens the door to overgeneralization errors in which procedures are used in inappropriate contexts.

In J. Culbertson, A. Perfors, H. Rabagliati & V. Ramenzoni (Eds.), *Proceedings of the 44th Annual Conference of the Cognitive Science Society.*

Table 1: Examples of Overgeneralization Errors.

Error	Procedure	Procedure is appropriate for ...
$\frac{3}{5}+\frac{1}{4}=\frac{4}{9}$	Apply operation to numerators and denominators	Multiplying fractions
$\frac{4}{5}\times\frac{3}{5}=\frac{12}{5}$	Apply operation to numerators, keep common denominator	Adding or subtracting fractions with equal denominators
$\begin{array}{r} 12.3 \\ \underline{+5.6} \\ 1.79 \end{array}$	Add decimal digits of operands to place decimal point in answer	Multiplying decimals
$\begin{array}{r} 2.4 \\ \underline{\times\ 1.2} \\ 4.8 \\ \underline{24.0} \\ 28.8 \end{array}$	Bring decimal point down from operands into answer	Adding or subtracting decimals

Table 2: Examples of Omission Errors.

Error	Omitted Step
$\frac{3}{5}+\frac{1}{4}=\frac{3}{20}+\frac{1}{20}=\frac{4}{20}$	Conversion of numerators when converting operands to a common denominator
$\begin{array}{r} 2.4 \\ \underline{\times\ 1.2} \\ 48 \\ \underline{24} \\ 0.72 \end{array}$	Shifting second partial product ("24") one column to the left (i.e., the "4" in "24" should be under the "4," rather than the "8," in "48")

Consistent with the above assumption, 91% of fraction arithmetic errors committed by sixth and eighth graders in (Siegler & Pyke, 2013) involved incorrect strategies. FARRA generated the large majority of these errors via overgeneralization (Braithwaite et al., 2017).

Also consistent with the assumption, 70% of sixth and eighth graders' decimal arithmetic errors involved using a strategy that would have been correct for a different type of problem, as in the last two rows of Table 1 (Braithwaite et al., 2021). The present study tests whether UMA can simulate this latter result, as FARRA did the former.

Why are errors more common for some problems?

Variation in error rates among problems in part reflects differences in the procedures required to solve the problems. For example, conversion to a common denominator is required for adding fractions with unequal denominators but not ones with equal denominators. Thus, errors resulting from incorrect conversion (e.g., Table 2, row 1) can only occur on the former type of problem.

However, some between-problem differences in error rates cannot be explained in this manner. For example, although the same procedure applies when multiplying fractions with equal or unequal denominators (e.g., 3/5×1/5 and 3/5×1/4), children err more on the former type of problem than the latter (e.g., 63% vs. 42% of trials in Siegler & Pyke, 2013).

Braithwaite et al. (2017) interpreted this phenomenon in terms of the distribution of input that children receive, a factor that has long been emphasized in research on language learning (e.g., Saffran et al., 1999) but that has received less attention in math cognition. Textbook analysis revealed that multiplication problems involving equal denominator fractions are very rare (Table 3), which Braithwaite et al. (2017) argued explains why children often err on them. When trained on problems from the textbooks, FARRA similarly erred more often on the rarer types of problems.

Table 3: Percentage of Fraction Arithmetic Problems in Textbooks with Different Operations and Operands (data from Braithwaite et al., 2017).

	Operand Denominators	
Operation	Equal	Unequal
Addition	12	13
Subtraction	13	12
Multiplication	1	29
Division	1	19

Braithwaite and colleagues (2021; Tian et al., 2021) predicted that decimal arithmetic would also reveal parallels between problem distributions and error rates. They found that in math textbooks, problems with a whole number and a decimal operand (WD) are rarer than those with two decimal operands (DD) for addition and subtraction, while the reverse is true for multiplication and division (Table 4). Similarly, children err more on addition with WD than DD operands, but err more often multiplication with DD than WD operands.

Table 4: Percentage of Decimal Arithmetic Problems in Textbooks Involving Different Operations and Operands (Tian et al., 2021).

	Operands	
Operation	Two decimals	One decimal, one whole number
Addition	14	1
Subtraction	14	1
Multiplication	15	21
Division	12	22

UMA, like FARRA, explains correspondences between problem distributions and error rates by assuming that the likelihood of using a procedure to solve a problem depends on how often one has used that procedure to solve similar problems in the past. This assumption is implemented via a reinforcement learning mechanism that is described in more detail below. The present study tests whether that mechanism can generate the variations in decimal arithmetic error rates described above.

What causes individual differences in strategy use?

As illustrated in Table 1, when doing arithmetic, children use a variety of strategies—some correct, others not. Individuals differ not only in speed or accuracy, but also in the strategies they use. UMA assumes that such individual differences reflect underlying parametric variation, which the theory characterizes in terms of two parameters: *decision determinism* (g) and *error discount* (d).

Decision determinism determines how strongly strategies that have received more reinforcement are preferred. High g reflects stronger preferences and therefore implies more consistent behavior. Low g implies more random behavior.

Error discount governs the reduction in reinforcement that occurs in response to negative feedback. Low d reflects indifference to such feedback, such that mere use of strategies leads to similar reinforcement regardless of the outcome. High d reflects greater sensitivity to negative feedback.

Braithwaite et al. (2019) showed that different values of g and d caused FARRA to generate different patterns of strategy use. High g and d led to consistent use of correct strategies. High g and low d led to consistent use of one strategy, both when it was appropriate and when not. Low g led to variable use of multiple strategies. These predicted patterns were all found in children's data and jointly accounted for the performance of over 90% of children.

If parametric variation among individuals affects decimal arithmetic strategy choices in a similar manner, then children should display patterns of strategy use in decimal arithmetic analogous to those found in fraction arithmetic. This prediction was confirmed by Braithwaite et al. (2021). The present study tests whether UMA can generate these patterns.

Computational Model

UMA is a production system model in the tradition of ACT-R[1] (Anderson, 2013). Its main components are its production rules, decision rule, and learning rule. The architecture is similar to FARRA (Braithwaite et al., 2017) except as noted.

Production Rules

Each production rule is a condition-action pair representing part of a procedure for solving problems. When presented a problem, UMA selects a production rule whose conditions are met and executes its action. Doing so causes conditions of other rules to be met. UMA then selects and executes another rule, continuing until an answer is obtained.

FARRA included rules representing algorithms for adding, subtracting, multiplying, and dividing fractions. UMA includes not only these rules, but also rules for adding, subtracting, and multiplying multidigit whole numbers and decimals[2]. Each algorithm is represented by multiple rules, each of which encapsulates a small part of the algorithm.

Some rules, called *strategy rules*, have conditions that enable UMA to select them when beginning a problem. Strategy rules create goals for other rules to achieve. UMA's strategy rule for adding or subtracting decimals creates goals to align the operands so their decimal points line up, optionally add zeroes to equalize decimal digits in the operands, operate (e.g., add) as with multidigit whole numbers, then bring the decimal from the operands into the answer. UMA's strategy rule for multiplying decimals creates goals to align the operands so their rightmost digits line up, operate as with multidigit whole numbers, then place the decimal in the answer according to the sum of the numbers of decimal digits in the operands.

Unlike FARRA, UMA's rules can generate sub-problems. For example, the two decimal arithmetic strategy rules just described each cause UMA to create a sub-problem involving multidigit whole numbers, which UMA would then solve using its rules for multidigit whole number arithmetic.

Like FARRA, UMA includes not only rules representing correct procedures but also *mal-rules*, which generate errors. Reflecting the assumption that errors reflect deviations from correct procedures, mal-rules were created by modifying correct rules via overgeneralization or omission.

Overgeneralization was implemented by removing part of a correct rule's condition, thus allowing the rule to be used in situations for which it was not appropriate. UMA includes an overgeneralized version of the decimal addition/subtraction strategy that can be used on multiplication problems, leading to errors like the one in Table 1, row 4. UMA also includes an overgeneralized version of the multiplication strategy that can be used when adding or subtracting, as in Table 1, row 3.

Omission was implemented by removing part of a correct rule's action, enabling UMA to skip a step of an otherwise correct procedure. For example, one rule specifies that when using a column algorithm to multiply multidigit whole numbers, each partial product after the first must be shifted one column to the left. Omission enables UMA to skip this step, resulting in errors like the one in Table 2, last row.

Decision Rule

When the conditions of multiple production rules are met, UMA chooses among them according to Equations 1 and 2:

$$A(r_j|C) = \sum_{x_i \in C} w_{ij} / \sum_{x_i \in C} 1 \quad (1)$$

$$P(r_j|C) = e^{gA(r_j|C)} / \sum_k e^{gA(r_k|C)} \quad (2)$$

Equation 1 states that the activation of rule r_j in the context of problem C is the average of the weights w_{ij} associating rule r_j with features x_i in C. The problem context C indicates the features that are present in the problem UMA is currently working on. For example, if UMA is working on 2.4×1.2, the problem context would include features "multiplication" and "DD" (i.e., "two decimal operands"). When UMA creates a sub-problem, the problem context changes to the sub-

[1] However, UMA does not use an existing cognitive architecture.

[2] The version of UMA tested here, like FARRA, performs single digit arithmetic using "expert" rules that never generate errors. The full version of UMA includes rules that solve single digit arithmetic problems by counting or retrieval and can generate errors.

problem until it is solved, then reverts to the original problem.

Equation 2 states that probability of selecting rule r_j given C is a softmax function of the activation in C of that rule and all other rules r_k whose conditions are met. The decision determinism parameter g determines how strongly the probability of selecting a rule depends on its activation.

Learning Rule

The weights in Equation 1 are initially set to 0. Each time UMA solves a problem while in learning mode, UMA adjusts the weights w_{ij} according to Equation 3:

$$\Delta w_{ij} = 1 - err * d \tag{3}$$

Here *err* represents error feedback (0 = correct, 1 = incorrect) and d is the error discount, which determines how much less rule weights are reinforced after errors than after correct answers. Reinforcement after errors is positive if $d < 1$ and negative if $d > 1$.

Equation 3 is used to adjust the weight w_{ij} connecting every rule r_j that was used to solve the problem with every feature x_i that was in the context at the time that r_j was used. These adjustments to rule weights will affect subsequent rule choices on other problems that have similar features. UMA tends to associate correct rules most strongly with the problem features that it encounters most often.

Simulations

Method

To evaluate UMA, we (1) created multiple instances of the model; (2) trained each instance on a learning set; (3) tested each instance on a test set, which differed from the learning set; and (4) compared UMA's performance on the test set to that of children.

Creating Multiple Instances of the Model. We simulated a cohort of students by creating 450 instances of UMA, including 10 instances for each combination of 9 values for g (.01, .02, .03, .04, .05, .06, .07, .08, .09) and 5 values for d (.5, .75, 1.0, 1.25, 1.5). These parameter ranges were chosen because they yielded reasonable results in initial testing, although systematic parameter fitting was not conducted.

Training the Model. The learning set was extracted from the grade 1-6 volumes of GO MATH! (Dixon et al., 2015), a math textbook series that was used in the schools where children's data (described below) were collected. The set consisted of 2510 whole number problems and 307 decimal problems involving addition, subtraction, or multiplication[3] with two operands. Such problems were included in the learning set if they were in symbolic form (not story problems) and required an exact answer (not an estimate) in open answer format (not multiple choice).

As in the textbooks analyzed by Tian et al. (2021), in the learning set, decimal addition and subtraction problems involved DD operands much more often than WD operands (118 vs. 4), whereas decimal multiplication problems involved WD operands more often than DD (107 vs 78).

Each instance of UMA was trained by running it once, in learning mode, on each problem in the learning set in the same order as in the textbook. The model received feedback (correct or incorrect) and updated rule weights after solving each problem before beginning the next problem.

Testing the Model. After training, the model instance was run once on each problem in a test set comprising six addition problems (24.45+0.34, 12.3+5.6, 2.46+4.1, 0.826+0.12, 5.61+23, 0.415+52) and six multiplication problems (0.41×0.31, 2.4×1.2, 2.3×0.13, 0.31×2.1, 31×3.2, 14×0.21). The problems for each arithmetic operation included four DD problems, of which equally many involved decimals with equal or unequal numbers of decimal digits, and two WD problems. The model did not receive feedback or update its rule weights during the test.

Comparison to Children's Data. The model's performance on the test set was compared to that of participants in a previous study of children's decimal arithmetic (Authors, 2021a). This sample consisted of 92 children, 57 sixth graders and 35 eighth graders, who solved the set of problems that served as UMA's test set. The problems were presented in one of four sequences, which were counterbalanced among participants. Children solved the problems in paper-and-pencil format. They were asked to show their work and to think aloud while working. Further details regarding children's data are provided by Braithwaite et al. (2021).

Results

Mean percent correct on the test set was 64% (addition: 79%, multiplication: 49%) for children and 74% (addition: 85%, multiplication: 62%) for UMA. Thus, UMA's accuracy was comparable to, though slightly higher than, that of children.

Errors Committed. Answers generated by children and UMA on a subset of test problems are shown in Table 5. On 56% of trials that children answered incorrectly, their answers were generated by UMA, and on 94% of trials that UMA answered incorrectly, its answers were generated by children. Children's errors that were not generated by UMA mostly reflected a "long tail" of strategies used by only a few children, such as multiplying decimals by multiplying each column of digits (e.g., 2.4×1.2 = 2.8) as in column addition.

[3] Whole number problems were included in the learning set, although the test did not include such problems, because practice with whole numbers affects subsequent performance with decimals for UMA and, presumably, children. Similarly, subtraction problems were included in the learning set, although the test set did not include such problems, because algorithms for subtraction partially overlap with those for addition, so subtraction practice affects addition performance. Division was excluded from the learning set because this version of UMA cannot perform long division.

Table 5: Answers Generated by Children and UMA.

Problem	Answer	% Trials on Which Answer Was Generated by Children	UMA
12.3+5.6	17.9 *	94	88
	1.79	2	12
2.46+4.1	6.56	89	91
	0.287	3	5
	2.87	1	4
5.61+23	28.61 *	70	79
	5.84	26	21
2.4×1.2	2.88 *	63	55
	28.8	14	34
	7.2	8	7
	2.8	5	0
	0.72	1	3
0.32×2.1	0.672 *	42	55
	6.72	18	31
	0.96	6	2
	67.2	6	3
	0.32	3	0
	1.05	0	4
31×3.2	99.2 *	61	74
	9.92	10	9
	15.5	5	6
	992	4	8

Note. * denotes correct answers. All answers generated on ≥3% of trials by either children or UMA are shown.

For all problems, the incorrect answer generated most often by children was also generated by UMA. UMA generated these common errors via overgeneralization, either by using a multiplication strategy on an addition problem (as in 12.3+5.6 = 1.79; see Table 1, row 3) or by using an addition strategy on a multiplication problem (as in 2.4×1.2 = 28.8; see Table 1, row 4). Many of children's less common errors were also generated by UMA, either via omission (as in 2.4×1.2 = 0.72; see Table 2, row 2) or a combination of overgeneralization and omission.

Error Rates. As shown in Figure 1, when adding, children erred more often on WD than DD addition problems (e.g., 5.61+23 vs. 2.46+4.1), but erred more often on DD than WD multiplication problems (e.g., 0.32×2.1 vs. 31×3.2). UMA displayed a very similar pattern of error rates. These patterns in children's and UMA's error rates paralleled the distributions of decimal arithmetic problems in math textbooks, including the textbook from which UMA's learning set was drawn, in which WD operands appeared less often than DD operands on addition problems, whereas DD operands appeared less often than WD operands on multiplication problems.

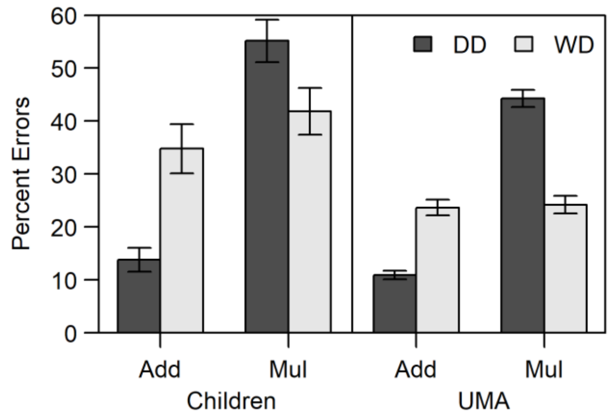

Figure 1: Percent Errors for Different Problem Types. "Add" = addition, "Mul" = multiplication, "DD" = two decimal operands, and "WD" = one whole number and one decimal operand. Error bars represent standard errors.

Individual Differences. Braithwaite et al. (2021) coded children's solution to each problem (based on written work) as consistent with an addition strategy, a multiplication strategy, both, or neither. Correct solutions were coded as displaying the strategy appropriate to the arithmetic operation on that trial. Incorrect solutions were coded as displaying an addition strategy if the operands were aligned at the decimal point and/or the decimal point was brought down from the operands into the answer (as in Table 1, row 4), or as displaying a multiplication strategy if the operands were aligned at the rightmost digit and/or the decimal point was placed in the answer according to the sum of numbers of decimal digits in the operands (as in Table 1, row 3).

Next, children were classified as consistently using correct strategies—addition strategies when adding and multiplication strategies when multiplying—if they did so on ≥75% of trials. If not, children were classified as relying on one flawed strategy if they displayed that strategy on ≥75% of trials. Remaining children were classified as using varied strategies if they displayed each strategy at least once on both addition and multiplication problems. Table 6 shows the percentage of children receiving each classification.

Table 6: Percentage of Children and Instances of UMA That Displayed Each Pattern of Strategy Use.

	Children	UMA
Consistent correct strategies	47	68
Reliance on one flawed strategy		
Addition	25	12
Multiplication	4	0.4
Using varied strategies	21	19
None of the above	3	0

The 450 instances of UMA were classified in the same way, except that UMA's strategies were determined by examining the production rules used on each the trial. As shown in Table 6, UMA generated all patterns of strategy use that were observed among children, and no others.

To understand the origins of differences in UMA's strategy use, we examined the values of UMA's parameters within each group (Table 7). Consistent correct strategies were associated with moderate *g* and high *d*. Persistent reliance on an addition strategy was associated with high *g* and low *d*. Persistent reliance on a multiplication strategy, and using varied strategies, were associated with low *g* and moderate *d*.

Table 7: Mean (SD) Values of UMA's Free Parameters Among Instances Classified Into Each Strategy Pattern.

	g	*d*
Consistent correct strategies	.05 (.03)	1.1 (0.3)
Reliance on one flawed strategy		
Addition	.07 (.02)	0.7 (0.2)
Multiplication	.02 (.01)	1.1 (0.5)
Using varied strategies	.03 (.02)	0.9 (0.3)

Discussion

Origins of Errors

The results demonstrate the explanatory power of UMA's error-generating mechanisms, that is, overgeneralization and omission. Using these mechanisms, UMA generated all of the most common errors, and over half of all errors, observed in children's decimal arithmetic. These results dovetail with Braithwaite et al.'s (2017) demonstration that the same mechanisms could account for a majority of children's errors in fraction arithmetic. Together, these findings support the assumption that most errors in rational number arithmetic reflect small deviations from standard correct procedures.

In fact, overgeneralization alone enabled UMA to generate all of children's most common decimal arithmetic errors (Table 5). Similarly, FARRA generated children's most common fraction arithmetic errors via overgeneralization alone (Siegler & Pyke, 2013). Although overgeneralization appears to be the main source of error in rational number arithmetic, the same may not be true in whole number arithmetic. For example, many single digit addition errors among young children are thought to reflect counting mistakes, resulting in answers slightly larger or smaller than the correct answer (e.g., Siegler & Shrager, 1984). We speculate that most such errors can be simulated via omission, but this remains to be tested.

Variation in Error Rates

A second theoretical assumption supported by the present results is that children are sensitive to distributional characteristics of the problems that they receive. UMA models this sensitivity using (1) a decision rule in which the likelihood of choosing a rule when solving a problem depends on the association between the rule and the problem's features, and (2) a learning rule by which that association depends largely on how often the rule has been used on problems with similar features in the past. Together, these mechanisms cause correct rules to be used more often on frequently encountered types of problems than on rare ones. These mechanisms thereby enabled UMA to simulate correspondences between error rates and textbook problem distributions in decimal arithmetic, and enabled FARRA to do the same for fraction arithmetic (Braithwaite et al., 2017).

However, the textbook analysis shown in Table 2 suggests that children receive more opportunities to practice decimal multiplication than decimal addition, whereas children are more accurate on the latter than the former. UMA suggests that the reason involves differences in the intrinsic difficulty of the procedures required to add and multiply decimals. Specifically, the possibility of strategy over-generalization exists for both addition and multiplication, but another common source of error for multiplication—failing to left-shift partial products (Table 2, last row) has no analogue in addition.

In sum, explaining between-problem variation in error rates requires considering both learning experience and intrinsic difficulty. By doing so, UMA goes beyond a prior theory of decimal arithmetic (Hiebert & Wearne, 1985), which considered only intrinsic difficulty.

Individual Differences in Strategy Use

UMA generated four patterns of strategy use, which matched those previously observed in children's decimal arithmetic (Braithwaite et al., 2021). The patterns were analogous to ones generated by FARRA and observed in children's fraction arithmetic (Braithwaite et al., 2019).

UMA's generation of these patterns depended on the assumption that learners vary along two dimensions: consistency of rule use and sensitivity to error feedback, represented by *g* and *d*. The results suggest that consistent use of correct strategies depends on both dimensions. Consistency of rule use without sensitivity to error feedback (high *g*, low *d*) may lead to the strategy that is studied first (i.e., addition) continuing to be used even when it yields errors (e.g., on multiplication problems). Learners with low consistency (low *g*) may select strategies randomly, as in the varied strategies pattern, or occasionally, may converge on persistently using the strategy that is appropriate for the largest number of practice problems (i.e., multiplication).

The fact that varying *g* and *d* generated differences among runs in strategy use matching differences among children provides preliminary evidence for the above assumption. Obtaining direct evidence for individual differences in *g* and *d*, and relating such differences to other measurable competencies, are important goals for future research.

Conclusion

Despite large differences in the specific procedures involved in fraction and decimal arithmetic, the same theoretical assumptions can explain empirical phenomena observed in both domains. Future research should test how well these assumptions can explain children's whole number arithmetic, which forms a foundation for fraction and decimal arithmetic.

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