Title

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Journal

Mathematische Annalen, 342(2)

ISSN

0025-5831

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Publication Date

2008-10-01

DOI

10.1007/s00208-008-0240-5

Peer reviewed

L-functions of Symmetric Products of the Kloosterman Sheaf over $\mathbf{Z}$

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Abstract

The classical n-variable Kloosterman sums over the finite field $\mathbf{F}_p$ give rise to a lisse $\overline{\mathbf{Q}}_l$-sheaf Kl_{n+1} on $\mathbf{G}_{m,\mathbf{F}_p} = \mathbf{P}^1_{\mathbf{F}_p} - \{0,\infty\}$, which we call the Kloosterman sheaf. Let $L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}, s)$ be the L-function of the k-fold symmetric product of Kl_{n+1}. We construct an explicit virtual scheme X of finite type over Spec $\mathbf{Z}$ such that the p-Euler factor of the zeta function of X coincides with $L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}, s)$. We also prove similar results for $\otimes^k\mathrm{Kl}_{n+1}$ and $\bigwedge^k \mathrm{Kl}_{n+1}$.

0. Introduction

For each prime number p, let $\mathbf{F}_p$ be a finite field with p elements. Fix an algebraic closure $\overline{\mathbf{F}}_p$ of $\mathbf{F}_p$. For any power q of p, let $\mathbf{F}_q$ be the subfield of $\overline{\mathbf{F}}_p$ with q elements. Let l be a prime number distinct from p. Fix a nontrivial additive character $\psi : \mathbf{F}_p \to \overline{\mathbf{Q}}_l^*$. Thus, $\psi(1)$ is a primitive p-th root of unity, which is denoted by ζ_p. For any nonzero $x \in \mathbf{F}_q$, we define the n-variable Kloosterman sum by

$$\mathrm{Kl}_{n+1}(\mathbf{F}_q, x) = \sum_{x_1,\ldots,x_{n+1}\in\mathbf{F}_q^*, x_1\cdots x_{n+1}=x} \psi(\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x_1 + \cdots + x_{n+1})) \in \mathbf{Z}[\zeta_p].$$

In [SGA $4\frac{1}{2}$] [Sommes trig.] §7, Deligne constructs a lisse $\overline{\mathbf{Q}}_l$-sheaf Kl_{n+1} on $\mathbf{G}_{m,\mathbf{F}_p} = \mathbf{P}^1_{\mathbf{F}_p} - \{0,\infty\}$ such that for any $x \in \mathbf{G}_m(\mathbf{F}_q) = \mathbf{F}_q^*$, we have

$$\mathrm{Tr}(F_x, \mathrm{Kl}_{n+1,\bar{x}}) = (-1)^n \mathrm{Kl}_{n+1}(\mathbf{F}_q, x),$$

Mathematics Subject Classification: 14F20, 11L05.

*The research of Lei Fu is supported by the NSFC (10525107).

where F_x is the geometric Frobenius element at the point x. For any natural number k, consider the L-function

$$L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}, s) = \prod_{x\in|\mathbf{G}_{m,\mathbf{F}_p}|} \det(1 - F_x p^{-s\deg(x)}, (\mathrm{Sym}^k\mathrm{Kl}_{n+1})_{\bar{x}})^{-1}$$

of the k-fold symmetric product $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$ of Kl_{n+1}, where $|\mathbf{G}_{m,\mathbf{F}_p}|$ is the set of Zariski closed points in $\mathbf{G}_{m,\mathbf{F}_p}$. This L-function in s has two parameters k and p. It was first studied by Robba [Ro] in the case $n = 1$ via p-adic methods. More recently, its basic properties and p-adic variation as k varies p-adicaly have been studied extensively in connection with Dwork's unit root conjecture. See [W1], [GK], [FW1] and [FW2].

In this paper, we fix k and study the variation of this L-function as p varies. It was observed in [FW1] Lemma 2.2 that for each p, the L-function $L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}, s)$ is a polynomial in p^{-s} with coefficients in $\mathbf{Z}$. This naturally leads to the conjecture that the infinite product

$$\zeta_{k,n}(s) := \prod_p L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}, s)$$

is automorphic and thus extends to a meromorphic function in $s \in \mathbf{C}$. This is easy to prove if $n = 1$ and $k \leq 4$. If $n = 1$ and $k = 5$, the series $\zeta_{5,1}(s)$ is essentially the L-function of an elliptic curve with complex multiplication and thus meromprhic in $s \in \mathbf{C}$, see [PTV]. If $n = 1$ and $k = 6$, the modularity of $\zeta_{6,1}(s)$ follows from [HS] and the references listed there. In this case, one obtains a rigid Calabi-Yau threefold. In the case $n = 1$ and $k = 7$, the series $\zeta_{7,1}(s)$ is conjectured by Evans [Ev] to be given by the L-function associated to an explicit modular form of weight 3 and level 525. With the recent progress on the modularity problem due to Taylor and Harris, it may be possible to prove the meromorphic continuation of $\zeta_{k,n}(s)$ for some larger k and n.

To prove the meromorphic continuation of $\zeta_{k,n}(s)$, the first step would be to prove that $\zeta_{k,n}(s)$ is motivic (or geometric) in nature, i.e., it arises as the zeta function of a motive over $\mathrm{Spec}\mathbf{Z}$. This question was raised in [FW1] and is solved in this paper. We will construct a virtual $\overline{\mathbf{Q}}_l$-sheaf $\mathcal{G}$ of geometric origin on $\mathrm{Spec}\,\mathbf{Z}$ so that the Euler factor

$$L_p(\mathrm{Spec}\,\mathbf{Z}, \mathcal{G}, s) = \det(1 - F_p p^{-s}, \mathcal{G}_{\bar{p}})^{-1}$$

of the L-function of $\mathcal{G}$ coincides with $L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}, s)$ for each prime number p, where F_p is the geometric Frobenius element at p. We also prove similar results for $\otimes^k\mathrm{Kl}_{n+1}$ and $\bigwedge^k \mathrm{Kl}_{n+1}$.

To describe our results, we introduce the following schemes over $\mathbf{Z}$.

Definition 0.1. Denote the homogeneous coordinates of $\mathbf{P}^{kn-1}$ by $[x_{ij}]$ $(i = 1, \ldots, n,\ j = 1, \ldots, k)$. Let Y_k be the subscheme of $\mathbf{P}^{kn-1}$ defined by

$$x_{ij} \neq 0,$$

let Y_{k0} be the subscheme defined by

$$x_{ij} \neq 0, \ \sum_{i,j} x_{ij} = 0,$$

let Z_k be the subscheme defined by the conditions

$$x_{ij} \neq 0, \ \sum_{j=1}^{k} \frac{1}{\prod_{i=1}^{n} x_{ij}} = 0,$$

and let Z_{k0} be the subscheme defined by

$$x_{ij} \neq 0, \ \sum_{j=1}^{k} \frac{1}{\prod_{i=1}^{n} x_{ij}} = 0, \ \sum_{i,j} x_{ij} = 0.$$

These are schemes of finite type over $\mathbf{Z}$. Let $\mathfrak{S}_k$ be the group of permutations of the set $\{1, \ldots, k\}$. It acts on $\mathbf{P}^{kn-1}$ by permuting the homogenous coordinates $x_{i1}, \ldots, x_{ik}$ for each i. Similarly $\mathfrak{S}_k$ acts on Z_{k0}, Z_k, Y_{k0} and Y_k. The notations $Y_k/\mathfrak{S}_k$, $Y_{k0}/\mathfrak{S}_k$, $Z_k/\mathfrak{S}_k$, and $Z_{k0}/\mathfrak{S}_k$ denote the quotient scheme of Y_k, Y_{k0}, Z_k, and Z_{k0} by $\mathfrak{S}_k$, respectively. Our main result is the following theorem.

Theorem 0.2. For a scheme X of finite type over $\mathbf{Z}$, let $\zeta_X(s)$ denote its zeta function. We have

$$L_p(\mathbf{G}_{m,\mathbf{F}_p}, \otimes^k \mathrm{Kl}_{n+1}, s) = \left(\frac{\zeta_{Z_{k0,\mathbf{F}_p}}(s-2)\zeta_{Y_{k,\mathbf{F}_p}}(s)}{\zeta_{Z_{k,\mathbf{F}_p}}(s-1)\zeta_{Y_{k0,\mathbf{F}_p}}(s-1)} \right)^{(-1)^{kn}},$$

$$L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}, s) = \left(\frac{\zeta_{Z_{k0,\mathbf{F}_p}/\mathfrak{S}_k}(s-2)\zeta_{Y_{k,\mathbf{F}_p}/\mathfrak{S}_k}(s)}{\zeta_{Z_{k,\mathbf{F}_p}/\mathfrak{S}_k}(s-1)\zeta_{Y_{k0,\mathbf{F}_p}/\mathfrak{S}_k}(s-1)} \right)^{(-1)^{kn}}.$$

Thus,

$$\prod_p L_p(\mathbf{G}_{m,\mathbf{F}_p}, \otimes^k \mathrm{Kl}_{n+1}, s) = \left(\frac{\zeta_{Z_{k0}}(s-2)\zeta_{Y_k}(s)}{\zeta_{Z_k}(s-1)\zeta_{Y_{k0}}(s-1)} \right)^{(-1)^{kn}},$$

$$\prod_p L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}, s) = \left(\frac{\zeta_{Z_{k0}/\mathfrak{S}_k}(s-2)\zeta_{Y_k/\mathfrak{S}_k}(s)}{\zeta_{Z_k/\mathfrak{S}_k}(s-1)\zeta_{Y_{k0}/\mathfrak{S}_k}(s-1)} \right)^{(-1)^{kn}}.$$

The above formulas can be simplified significantly. This is done in §4. To prove the above results, we need to relate Kloosterman sheaves by the l-adic Fourier transformation. This is done in §1. We prove Theorem 0.2 in §2 and §3.

Remark 0.3. For any partition λ of k, let $S_\lambda(\mathrm{Kl}_{n+1})$ be the Weyl construction applied to Kl_{n+1}. (Confer [FH] §6.1.) The method developed in this paper can also be used to show that $L_p(\mathbf{G}_{m,\mathbf{F}_p}, S_\lambda(\mathrm{Kl}_{n+1}), s)$ is the Euler factor at p of the L-function of a virtual $\overline{\mathbf{Q}}_l$-sheaf on $\mathrm{Spec}\,\mathbf{Z}$

of geometric origin for each prime number p. An example is given in Theorem 3.2 for the k-th exterior product.

1. Kloosterman Sheaves and the Fourier Transformation

In this section, we give an inductive construction of Kloosterman sheaves using the l-adic Fourier transformation. We refer the reader to [L] for the definition and properties of the Fourier transformation.

The morphism

$$\mathcal{P} : \mathbf{A}^1_{\mathbf{F}_p} \to \mathbf{A}^1_{\mathbf{F}_p}$$

corresponding to the $\mathbf{F}_p$-algebra homomorphism

$$\mathbf{F}_p[t] \to \mathbf{F}_p[t],\ t \mapsto t^p - t$$

is a finite galois étale covering space, and it defines an $\mathbf{F}_p$-torsor

$$0 \to \mathbf{F}_p \to \mathbf{A}^1_{\mathbf{F}_p} \stackrel{\mathcal{P}}{\to} \mathbf{A}^1_{\mathbf{F}_p} \to 0.$$

Pushing-forward this torsor by $\psi^{-1} : \mathbf{F}_p \to \overline{\mathbf{Q}}_l$, we get a lisse $\overline{\mathbf{Q}}_l$-sheaf $\mathcal{L}_\psi$ of rank 1 on $\mathbf{A}^1_{\mathbf{F}_p}$, which we call the Artin-Schreier sheaf. Let X be a scheme over $\mathbf{F}_p$ and let f be an element in the ring of global sections $\Gamma(X, \mathcal{O}_X)$ of the structure sheaf of X. Then f defines an $\mathbf{F}_p$-morphism $X \to \mathbf{A}^1_{\mathbf{F}_p}$ so that the induced $\mathbf{F}_p$-algebra homomorphism $\mathbf{F}_p[t] \to \Gamma(X, \mathcal{O}_X)$ maps t to f. We often denote this canonical morphism also by f, and denote by $\mathcal{L}_\psi(f)$ the inverse image of $\mathcal{L}_\psi$ under this morphism.

The main result of this section is the following.

Proposition 1.1. Let $i : \mathbf{G}_{m,\mathbf{F}_p} \to \mathbf{G}_{m,\mathbf{F}_p}$ be the morphism $x \mapsto \frac{1}{x}$, and let $j : \mathbf{G}_{m,\mathbf{F}_p} \to \mathbf{A}^1_{\mathbf{F}_p}$ be the canonical open immersion. For each integer $n \geq 1$, define Kl_n inductively as follows:

$$\begin{aligned} \mathrm{Kl}_1 &= \mathcal{L}_\psi|_{\mathbf{G}_{m,\mathbf{F}_p}}, \\ \mathrm{Kl}_{n+1} &= (\mathcal{F}(j_! i^* \mathrm{Kl}_n))|_{\mathbf{G}_{m,\mathbf{F}_p}}, \end{aligned}$$

where $\mathcal{F}(-) = Rp_{2!}(p_1^*(-) \otimes^L \mathcal{L}_\psi(tt'))[1]$ denotes the Fourier transformation. Here

$$p_1, p_2 : \mathbf{A}^1_{\mathbf{F}_p} \times_{\mathbf{F}_p} \mathbf{A}^1_{\mathbf{F}_p} \to \mathbf{A}^1_{\mathbf{F}_p}$$

are the projections, and tt' is regarded as an element in

$$\Gamma(\mathbf{A}^1_{\mathbf{F}_p} \times_{\mathbf{F}_p} \mathbf{A}^1_{\mathbf{F}_p}, \mathcal{O}_{\mathbf{A}^1_{\mathbf{F}_p} \times_{\mathbf{F}_p} \mathbf{A}^1_{\mathbf{F}_p}}) \cong \mathbf{F}_p[t, t'].$$

(i) For any $t \in \mathbf{G}_m(\mathbf{F}_q)$, we have

$$\mathrm{Tr}(F_t, \mathrm{Kl}_{n,\bar{t}}) = (-1)^{n-1} \sum_{x_1, \ldots, x_n \in \mathbf{F}_q^*,\ x_1 \cdots x_n = t} \psi(\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x_1 + \cdots + x_n)).$$

(ii) Kl_n is a lisse $\overline{\mathbf{Q}}_l$-sheaf on $\mathbf{G}_{m,\mathbf{F}_p}$ of rank n. It is tame at 0, and its Swan conductor at ∞ is 1.

It follows from the proposition that the sheaf Kl_n defined inductively using the Fourier transformation as above coincides with the Kloosterman sheaf constructed by Deligne.

Proof. We use induction on n. When $n = 1$, the assertions are clear. Suppose the assertions hold for Kl_n. We have

$$\begin{aligned}
& \mathrm{Tr}(F_t, \mathrm{Kl}_{n+1,\bar{t}}) \\
= \; & \mathrm{Tr}(F_t, (\mathcal{F}(j_! i^* \mathrm{Kl}_n))_{\bar{t}}) \\
= \; & -\sum_{s\in\mathbf{F}_q} \psi(\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(st))\mathrm{Tr}(F_s, (j_! i^*\mathrm{Kl}_n)_{\bar{s}}) \\
= \; & (-1)^n \sum_{s\in\mathbf{F}_q^*} \psi(\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(st)) \sum_{x_1,\ldots,x_n\in\mathbf{F}_q^*,\, x_1\cdots x_n=\frac{1}{s}} \psi(\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x_1+\cdots+x_n)) \\
= \; & (-1)^n \sum_{s,x_1,\ldots,x_n\in\mathbf{F}_q^*,\, x_1\cdots x_n=\frac{1}{s}} \psi(\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x_1+\cdots+x_n+st)) \\
= \; & (-1)^n \sum_{x_1,\ldots,x_{n+1}\in\mathbf{F}_q^*,\, x_1\cdots x_{n+1}=t} \psi(\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x_1+\cdots+x_{n+1})),
\end{aligned}$$

where the second equality follows from the definition of the Fourier transformation, and the third equality follows from the induction hypothesis. This proves (i) holds for Kl_{n+1}. Let η_0 (resp. η_∞) be the generic point of the strict henselization of $\mathbf{P}^1_{\mathbf{F}_p}$ at 0 (resp. ∞). By the induction hypothesis, Kl_n is tame at 0. Hence $(i^*\mathrm{Kl}_n))|\eta_\infty$ is tame. By [L] 2.3.1.3 (i), $\mathrm{Kl}_{n+1} = (\mathcal{F}(j_! i^*\mathrm{Kl}_n))|_{\mathbf{G}_{m,\mathbf{F}_p}}$ is a lisse sheaf on $\mathbf{G}_{m,\mathbf{F}_p}$. Moreover, by [L] 2.5.3.1, $\mathcal{F}^{(\infty,0')}((i^*\mathrm{Kl}_n)|\eta_\infty)$ is tame. It follows that Kl_{n+1} is tame at 0. By the stationary phase principle [L] 2.3.3.1 (iii), we have

$$\mathrm{Kl}_{n+1}|\eta_{\infty'} = \mathcal{F}^{(0,\infty')}((i^*\mathrm{Kl}_n)|\eta_0) \oplus \mathcal{F}^{(\infty,\infty')}((i^*\mathrm{Kl}_n)|\eta_\infty).$$

Since $(i^*\mathrm{Kl}_n))|\eta_\infty$ is tame, we have $\mathcal{F}^{(\infty,\infty')}((i^*\mathrm{Kl}_n))|\eta_\infty) = 0$ by [L] 2.4.3 (iii) b). By the induction hypothesis, the Swan conductor of $(i^*\mathrm{Kl}_n)|\eta_0$ is 1 and its rank is n. By [L] 2.4.3 (i) b), the Swan conductor of $\mathcal{F}^{(0,\infty')}((i^*\mathrm{Kl}_n)|\eta_0)$ is 1, and its rank is $n+1$. Hence the Swan conductor of Kl_{n+1} at ∞ is 1, and the rank of Kl_{n+1} is $n+1$. This proves (ii) holds for Kl_{n+1}.

2. The L-function of $\otimes^k \mathrm{Kl}_{n+1}$

Let

$$\tilde{\mathbf{A}}^{n+1}_{\mathbf{F}_p} = \{(x,y) \in \mathbf{A}^{n+1}_{\mathbf{F}_p} \times_{\mathbf{F}_p} \mathbf{P}^n_{\mathbf{F}_p} | x \text{ lies on the line determined by } y\}$$

be the blowing-up of $\mathbf{A}^{n+1}_{\mathbf{F}_p}$ at the origin, let

$$\pi_1 : \tilde{\mathbf{A}}^{n+1}_{\mathbf{F}_p} \to \mathbf{A}^{n+1}_{\mathbf{F}_p},\ \pi_2 : \tilde{\mathbf{A}}^{n+1}_{\mathbf{F}_p} \to \mathbf{P}^n_{\mathbf{F}_p}$$

be the projections, let

$$H = \{[x_0 : \ldots : x_n] \in \mathbf{P}^n | \sum x_i = 0\},$$

and let

$$\kappa : H \to \mathbf{P}^n_{\mathbf{F}_p}$$

be the canonical closed immersion. Consider the morphism

$$s : \mathbf{A}^{n+1}_{\mathbf{F}_p} \to \mathbf{A}^1_{\mathbf{F}_p},\ s(x_0, \ldots, x_n) = x_0 + \cdots + x_n.$$

We have

$$R\pi_{2!}\pi_1^* s^* \mathcal{L}_\psi = \kappa_! \overline{\mathbf{Q}}_l(-1)[-2].$$

This follows from the fact that $\tilde{\mathbf{A}}^{n+1}_{\mathbf{F}_p}$ is a line bundle over $\mathbf{P}^n_{\mathbf{F}_p}$, and that for any point $a = [a_0 : \ldots : a_n]$ in $\mathbf{P}^n_{\mathbf{F}_p}$, we have

$$R\Gamma_c(\pi_2^{-1}(a) \otimes \overline{\mathbf{F}}_p, \pi_1^* s^* \mathcal{L}_\psi) \cong R\Gamma_c(\mathbf{A}^1_{\overline{\mathbf{F}}_p}, \mathcal{L}_\psi(t \sum a_i)) = \begin{cases} 0 & \text{if } \sum a_i \neq 0, \\ \overline{\mathbf{Q}}_l(-1)[-2] & \text{otherwise.} \end{cases}$$

Lemma 2.1. For a subscheme Z of $\mathbf{P}^n_{\mathbf{F}_p}$, let

$$Z_0 = Z \cap H,\ \tilde{X} = \pi_2^{-1}(Z),\ X = \pi_1(\tilde{X}).$$

We have a natural distinguished triangle

$$R\Gamma_c((X - \{0\}) \otimes \overline{\mathbf{F}}_p, s^* \mathcal{L}_\psi) \to R\Gamma_c(Z_0 \otimes \overline{\mathbf{F}}_p, \overline{\mathbf{Q}}_l(-1)[-2]) \to R\Gamma_c(Z \otimes \overline{\mathbf{F}}_p, \overline{\mathbf{Q}}_l) \to .$$

Proof. Let $\pi_1' : \tilde{X} \to X$ be the restriction of π_1 to $\tilde{X}$. We have a distinguished triangle

$$R\Gamma_c(\pi_1'^{-1}(X - \{0\}) \otimes \overline{\mathbf{F}}_p, \pi_1^* s^* \mathcal{L}_\psi) \to R\Gamma_c(\tilde{X} \otimes \overline{\mathbf{F}}_p, \pi_1^* s^* \mathcal{L}_\psi) \to R\Gamma_c(\pi_1'^{-1}(\{0\}) \otimes \overline{\mathbf{F}}_p, \pi_1^* s^* \mathcal{L}_\psi) \to .$$

On the other hand, we have

$$X - \{0\} \cong \pi_1'^{-1}(X - \{0\}),\ \pi_1'^{-1}(X) = \tilde{X},\ \pi_1'^{-1}(\{0\}) \cong Z,$$

and hence

$$\begin{aligned} R\Gamma_c(\pi_1'^{-1}(X - \{0\}) \otimes \overline{\mathbf{F}}_p, \pi_1^* s^* \mathcal{L}_\psi) &\cong R\Gamma_c((X - \{0\}) \otimes \overline{\mathbf{F}}_p, s^* \mathcal{L}_\psi), \\ R\Gamma_c(\tilde{X} \otimes \overline{\mathbf{F}}_p, \pi_1^* s^* \mathcal{L}_\psi) &\cong R\Gamma_c(Z \otimes \overline{\mathbf{F}}_p, R\pi_{2!}\pi_1^* s^* \mathcal{L}_\psi) \\ &\cong R\Gamma_c(Z \otimes \overline{\mathbf{F}}_p, \kappa_! \overline{\mathbf{Q}}_l(-1)[-2]) \\ &\cong R\Gamma_c(Z_0 \otimes \overline{\mathbf{F}}_p, \overline{\mathbf{Q}}_l(-1)[-2]), \\ R\Gamma_c(\pi_1'^{-1}(\{0\}) \otimes \overline{\mathbf{F}}_p, \pi_1^* s^* \mathcal{L}_\psi) &\cong R\Gamma_c(Z \otimes \overline{\mathbf{F}}_p, \overline{\mathbf{Q}}_l). \end{aligned}$$

Our assertion follows.

By Proposition 1.1, we have

$$\mathcal{F}(j_! i^* \mathrm{Kl}_n)|_{\mathbf{G}_{m,\mathbf{F}_p}} \cong \mathrm{Kl}_{n+1}.$$

By [L] 1.2.2.7, we have

$$\mathcal{F}(*^k (j_! i^* \mathrm{Kl}_n))|_{\mathbf{G}_{m,\mathbf{F}_p}} \cong \otimes^k \mathrm{Kl}_{n+1}[1-k], \tag{1}$$

where $*^k$ denotes the k-fold convolution product. Let

$$\begin{aligned} s_n : \mathbf{G}_m^n &\to \mathbf{A}^1, \\ p_n : \mathbf{G}_m^n &\to \mathbf{G}_m \end{aligned}$$

be the morphisms

$$s_n(x_1, \ldots, x_n) = x_1 + \cdots + x_n,$$
$$p_n(x_1, \ldots, x_n) = x_1 \cdots x_n,$$

respectively. By [SGA $4\frac{1}{2}$] [Sommes trig.] §7, we have

$$\mathrm{Kl}_n \cong R p_{n!} s_n^* \mathcal{L}_\psi [n-1]. \tag{2}$$

Denote the coordinates of $\mathbf{G}_m^{kn}$ by x_{ij} $(i = 1, \ldots, n,\ j = 1, \ldots, k)$. Let

$$\begin{aligned} s_{kn} : \mathbf{G}_m^{kn} &\to \mathbf{A}^1, \\ f_{kn} : \mathbf{G}_m^{kn} &\to \mathbf{A}^1 \end{aligned}$$

be the morphisms

$$s_{kn}((x_{ij})) = \sum_{i,j} x_{ij},$$
$$f_{kn}((x_{ij})) = \sum_{j=1}^{k} \frac{1}{\prod_{i=1}^{n} x_{ij}},$$

respectively. By the Künneth formula, the definition of the convolution product [L] 1.2.2.6, and the isomorphism (2), we have

$$*^k (j_! i^* \mathrm{Kl}_n) \cong R f_{kn,!} s_{kn}^* \mathcal{L}_\psi [k(n-1)].$$

Combined with the isomorphism (1), we get

$$(\mathcal{F}(R f_{kn,!} s_{kn}^* \mathcal{L}_\psi)[kn-1])|_{\mathbf{G}_{m,\mathbf{F}_p}} \cong \otimes^k \mathrm{Kl}_{n+1}. \tag{3}$$

By Grothendieck's formula for L-functions, we have

$$L_p(\mathbf{G}_{m,\mathbf{F}_p}, \otimes^k \mathrm{Kl}_{n+1}, s) = \det(1 - Fp^{-s}, R\Gamma_c(\mathbf{G}_{m,\overline{\mathbf{F}}_p}, \otimes^k \mathrm{Kl}_{n+1}))^{-1}.$$

Taking into account of the isomorphism (3), we get

$$\begin{aligned} L_p(\mathbf{G}_{m,\mathbf{F}_p}, \otimes^k \mathrm{Kl}_{n+1}, s) &= \det(1 - Fp^{-s}, R\Gamma_c(\mathbf{G}_{m,\overline{\mathbf{F}}_p}, (\mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1])|_{\mathbf{G}_{m,\overline{\mathbf{F}}_p}}))^{-1} \\ &= \frac{\det(1 - Fp^{-s}, R\Gamma_c(\mathbf{A}^1_{\overline{\mathbf{F}}_p}, \mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1]))^{-1}}{\det(1 - Fp^{-s}, (\mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1])|_{\bar{0}})^{-1}} \end{aligned}$$

By the definition of the Fourier transformation, we have

$$(\mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi))|_{\bar{0}} \cong R\Gamma_c(\mathbf{A}^1_{\overline{\mathbf{F}}_p}, Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[1] \cong R\Gamma_c(\mathbf{G}^{kn}_{m,\overline{\mathbf{F}}_p}, s_{kn}^*\mathcal{L}_\psi)[1].$$

Hence

$$\det(1 - Fp^{-s}, (\mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1])|_{\bar{0}})^{-1} = \det(1 - Fp^{-s}, R\Gamma_c(\mathbf{G}^{kn}_{m,\overline{\mathbf{F}}_p}, s_{kn}^*\mathcal{L}_\psi)[kn])^{-1}.$$

By the inversion formula for the Fourier transformation [L] 1.2.2.1, we have

$$\begin{aligned} R\Gamma_c(\mathbf{A}^1_{\overline{\mathbf{F}}_p}, \mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)) &\cong (\mathcal{F}(\mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)))_{\bar{0}}[-1] \\ &\cong (Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)_{\bar{0}}(-1)[-1] \\ &\cong R\Gamma_c(X_{k,\overline{\mathbf{F}}_p}, s_{kn}^*\mathcal{L}_\psi)(-1)[-1], \end{aligned}$$

where X_k is the subscheme of $\mathbf{G}_m^{kn}$ over $\mathbf{Z}$ defined by the equation

$$\sum_{j=1}^k \frac{1}{\prod_{i=1}^n x_{ij}} = 0.$$

Hence

$$\begin{aligned} &\det(1 - Fp^{-s}, R\Gamma_c(\mathbf{A}^1_{\overline{\mathbf{F}}_p}, \mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1]))^{-1} \\ = \; &\det(1 - Fp^{-s}, R\Gamma_c(X_{k,\overline{\mathbf{F}}_p}, s_{kn}^*\mathcal{L}_\psi)(-1)[kn-2])^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} L_p(\mathbf{G}_{m,\mathbf{F}_p}, \otimes^k \mathrm{Kl}_{n+1}, s) &= \frac{\det(1 - Fp^{-s}, R\Gamma_c(\mathbf{A}^1_{\overline{\mathbf{F}}_p}, \mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1]))^{-1}}{\det(1 - Fp^{-s}, (\mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1])|_{\bar{0}})^{-1}} \\ &= \frac{\det(1 - Fp^{-s}, R\Gamma_c(X_{k,\overline{\mathbf{F}}_p}, s_{kn}^*\mathcal{L}_\psi)(-1)[kn-2])^{-1}}{\det(1 - Fp^{-s}, R\Gamma_c(\mathbf{G}^{kn}_{m,\overline{\mathbf{F}}_p}, s_{kn}^*\mathcal{L}_\psi)[kn])^{-1}}. \end{aligned}$$

Let Z_k be the subscheme of $\mathbf{P}^{kn-1}$ over $\mathbf{Z}$ defined by the conditions

$$x_{ij} \neq 0, \ \sum_{j=1}^k \frac{1}{\prod_{i=1}^n x_{ij}} = 0,$$

and let Z_{k0} be the subscheme defined by

$$x_{ij} \neq 0, \sum_{j=1}^{k} \frac{1}{\prod_{i=1}^{n} x_{ij}} = 0, \sum_{i,j} x_{ij} = 0.$$

By Lemma 2.1, we have

$$\begin{aligned}
\det(1 - Fp^{-s}, R\Gamma_c(X_{k,\overline{\mathbf{F}}_p}, s_{kn}^*\mathcal{L}_\psi)(-1)[kn-2])^{-1} &= \frac{\det(1 - Fp^{-s}, R\Gamma_c(Z_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-2)[kn-4])^{-1}}{\det(1 - Fp^{-s}, R\Gamma_c(Z_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-1)[kn-2])^{-1}} \\
&= \frac{\zeta_{Z_{k0,\mathbf{F}_p}}(s-2)^{(-1)^{kn}}}{\zeta_{Z_{k,\mathbf{F}_p}}(s-1)^{(-1)^{kn}}}.
\end{aligned}$$

Let Y_k be the subscheme of $\mathbf{P}^{kn-1}$ over $\mathbf{Z}$ defined by the condition

$$x_{ij} \neq 0,$$

and let Y_{k0} be the subscheme defined by

$$x_{ij} \neq 0, \sum_{i,j} x_{ij} = 0.$$

By Lemma 2.1 again, we have

$$\begin{aligned}
\det(1 - Fp^{-s}, R\Gamma_c(\mathbf{G}_{m,\overline{\mathbf{F}}_p}^{kn}, s_{kn}^*\mathcal{L}_\psi)[kn])^{-1} &= \frac{\det(1 - Fp^{-s}, R\Gamma_c(Y_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-1)[kn-2])^{-1}}{\det(1 - Fp^{-s}, R\Gamma_c(Y_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)[kn])^{-1}} \\
&= \frac{\zeta_{Y_{k0,\mathbf{F}_p}}(s-1)^{(-1)^{kn}}}{\zeta_{Y_{k,\mathbf{F}_p}}(s)^{(-1)^{kn}}}.
\end{aligned}$$

So we finally get

$$\begin{aligned}
L_p(\mathbf{G}_{m,\mathbf{F}_p}, \otimes^k \mathrm{Kl}_{n+1}, s) &= \frac{\det(1 - Fp^{-s}, R\Gamma_c(X_{k,\overline{\mathbf{F}}_p}, s_{kn}^*\mathcal{L}_\psi)(-1)[kn-2])^{-1}}{\det(1 - Fp^{-s}, R\Gamma_c(\mathbf{G}_{m,\overline{\mathbf{F}}_p}^{kn}, s_{kn}^*\mathcal{L}_\psi)[kn])^{-1}} \\
&= \left(\frac{\zeta_{Z_{k0,\mathbf{F}_p}}(s-2)\zeta_{Y_{k,\mathbf{F}_p}}(s)}{\zeta_{Z_{k,\mathbf{F}_p}}(s-1)\zeta_{Y_{k0,\mathbf{F}_p}}(s-1)} \right)^{(-1)^{kn}}.
\end{aligned}$$

Hence

$$\prod_p L_p(\mathbf{G}_{m,\mathbf{F}_p}, \otimes^k \mathrm{Kl}_{n+1}, s) = \left(\frac{\zeta_{Z_{k0}}(s-2)\zeta_{Y_k}(s)}{\zeta_{Z_k}(s-1)\zeta_{Y_k}(s-1)} \right)^{(-1)^{kn}}.$$

This proves the assertions about the L-functions of $\otimes^k \mathrm{Kl}_{n+1}$ in Theorem 0.1.

3. The L-function of $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$

Lemma 3.1. Let V be a $\overline{\mathbf{Q}}_l$-vector space, let $\pi : V \to V$ and $F : V \to V$ be two linear maps such that $\pi^2 = \pi$ and $F\pi = \pi F$. Then we have

$$\det(1 - Ft, \mathrm{im}(\pi)) = \det(1 - F\pi t, V).$$

Proof. Since $\pi^2=\pi$, we have

$$V=\ker(\pi)\oplus \mathrm{im}(\pi),$$

and

$$\pi|_{\ker(\pi)}=0,\ \pi|_{\mathrm{im}(\pi)}=\mathrm{id}.$$

Since $F\pi=\pi F$, the subspaces $\ker(\pi)$ and $\mathrm{im}(\pi)$ are stable under F. It follows that

$$\begin{aligned}\det(1-F\pi t,V) &= \det(1-F\pi t,\mathrm{im}(\pi))\det(1-F\pi t,\ker(\pi))\\ &= \det(1-Ft,\mathrm{im}(\pi)).\end{aligned}$$

Denote the coordinates of $\mathbf{G}_m^{kn}$ by x_{ij} $(i=1,\ldots,n,\ j=1,\ldots,k)$. Let

$$\begin{aligned}s_{kn}:\mathbf{G}_m^{kn} &\to \mathbf{A}^1,\\ f_{kn}:\mathbf{G}_m^{kn} &\to \mathbf{A}^1\end{aligned}$$

be the morphisms

$$s_{kn}((x_{ij}))=\sum_{i,j}x_{ij},$$

$$f_{kn}((x_{ij}))=\sum_{j=1}^{k}\frac{1}{\prod_{i=1}^{n}x_{ij}},$$

respectively. Recall that in the previous section, we obtain the isomorphisms (1) and (3):

$$\otimes^k\mathrm{Kl}_{n+1}\cong\Big(\mathcal{F}(*^k(j_!i^*\mathrm{Kl}_n))[k-1]\Big)|_{\mathbf{G}_m}\cong\Big(\mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1]\Big)|_{\mathbf{G}_m}.$$

The group $\mathfrak{S}_k$ acts on $\otimes^k\mathrm{Kl}_{n+1}$ and on $*^k(j_!i^*\mathrm{Kl}_n)$ by permuting the factors, and it acts on $Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi$ by permuting the coordinates $x_{i1},\ldots,x_{ik}$ of $\mathbf{G}_m^{kn}$ for each i. These actions are compatible with the above isomorphisms. By Grothendieck's formula for L-functions, we have

$$L_p(\mathbf{G}_{m,\mathbf{F}_p},\mathrm{Sym}^k\mathrm{Kl}_{n+1},s)=\det(1-Fp^{-s},R\Gamma_c(\mathbf{G}_{m,\overline{\mathbf{F}}_p},\mathrm{Sym}^k\mathrm{Kl}_{n+1}))^{-1}.$$

Let

$$\pi=\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\sigma.$$

We have $\pi^2=\pi$, and π induces the projection of $\otimes^k\mathrm{Kl}_{n+1}$ to its direct factor $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$. It follows that

$$H_c^m(\mathbf{G}_{m,\overline{\mathbf{F}}_p},\mathrm{Sym}^k\mathrm{Kl}_{n+1})\ \cong\ \mathrm{im}(H_c^m(\mathbf{G}_{m,\overline{\mathbf{F}}_p},\otimes^k\mathrm{Kl}_{n+1})\xrightarrow{\pi}H_c^m(\mathbf{G}_{m,\overline{\mathbf{F}}_p},\otimes^k\mathrm{Kl}_{n+1}))$$

for all m. Applying Lemma 3.1 to

$$\pi:H_c^m(\mathbf{G}_{m,\overline{\mathbf{F}}_p},\otimes^k\mathrm{Kl}_{n+1})\to H_c^m(\mathbf{G}_{m,\overline{\mathbf{F}}_p},\otimes^k\mathrm{Kl}_{n+1}),$$

we get

$$\det(1 - Fp^{-s}, H_c^m(\mathbf{G}_{m,\overline{\mathbf{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})) = \det(1 - F\pi p^{-s}, H_c^m(\mathbf{G}_{m,\overline{\mathbf{F}}_p}, \otimes^k\mathrm{Kl}_{n+1})).$$

It follows that

$$\begin{aligned} L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}, s) &= \det(1 - F\pi p^{-s}, R\Gamma_c(\mathbf{G}_{m,\overline{\mathbf{F}}_p}, \otimes^k\mathrm{Kl}_{n+1}))^{-1} \\ &= \det(1 - F\pi p^{-s}, R\Gamma_c(\mathbf{G}_{m,\overline{\mathbf{F}}_p}, (\mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1])|_{\mathbf{G}_{m,\overline{\mathbf{F}}_p}}))^{-1} \\ &= \frac{\det(1 - F\pi p^{-s}, R\Gamma_c(\mathbf{A}^1_{\overline{\mathbf{F}}_p}, \mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1]))^{-1}}{\det(1 - F\pi p^{-s}, (\mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1])|_{\bar{0}})^{-1}}. \end{aligned}$$

The same argument as in §2 shows that

$$\begin{aligned} &\det(1 - F\pi p^{-s}, R\Gamma_c(\mathbf{A}^1_{\overline{\mathbf{F}}_p}, \mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1]))^{-1} \\ = \; &\det(1 - F\pi p^{-s}, R\Gamma_c(X_{k,\overline{\mathbf{F}}_p}, s_{kn}^*\mathcal{L}_\psi)(-1)[kn-2])^{-1} \\ = \; &\frac{\det(1 - F\pi p^{-s}, R\Gamma_c(Z_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-2)[kn-4])^{-1}}{\det(1 - F\pi p^{-s}, R\Gamma_c(Z_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-1)[kn-2])^{-1}}, \end{aligned}$$

where Z_k is the subscheme of $\mathbf{P}^{kn-1}$ over $\mathbf{Z}$ defined by the condition

$$x_{ij} \neq 0,\ \sum_{j=1}^k \frac{1}{\prod_{i=1}^n x_{ij}} = 0,$$

Z_{k0} is the subscheme defined by

$$x_{ij} \neq 0,\ \sum_{j=1}^k \frac{1}{\prod_{i=1}^n x_{ij}} = 0,\ \sum_{i,j} x_{ij} = 0,$$

and the group $\mathfrak{S}_k$ acts on Z_k and on Z_{k0} by permuting the homogeneous coordinates $x_{i1}, \ldots, x_{ik}$ for each i. The same argument as in §2 also shows that

$$\begin{aligned} &\det(1 - F\pi p^{-s}, (\mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1])|_{\bar{0}})^{-1} \\ = \; &\det(1 - F\pi p^{-s}, R\Gamma_c(\mathbf{G}^{kn}_{m,\overline{\mathbf{F}}_p}, s_{kn}^*\mathcal{L}_\psi)[kn])^{-1} \\ = \; &\frac{\det(1 - F\pi p^{-s}, R\Gamma_c(Y_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-1)[kn-2])^{-1}}{\det(1 - F\pi p^{-s}, R\Gamma_c(Y_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)[kn])^{-1}}, \end{aligned}$$

where Y_k is the subscheme of $\mathbf{P}^{kn-1}$ defined by

$$x_{ij} \neq 0,$$

Y_{k0} is the subscheme defined by

$$x_{ij} \neq 0,\ \sum_{i,j} x_{ij} = 0,$$

and the group $\mathfrak{S}_k$ acts on Y_k and on Y_{k0} by permuting the homogeneous coordinates $x_{i1}, \ldots, x_{ik}$ for each i. So we have

$$\begin{aligned} & L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}, s) \\ = & \frac{\det(1-F\pi p^{-s}, R\Gamma_c(\mathbf{A}^1_{\overline{\mathbf{F}}_p}, \mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1]))^{-1}}{\det(1-F\pi p^{-s}, (\mathcal{F}(Rf_{kn,!}s_{kn}^*\mathcal{L}_\psi)[kn-1])|_{\bar{0}})^{-1}} \\ = & \frac{\det(1-F\pi p^{-s}, R\Gamma_c(Z_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-2)[kn-4])^{-1}\det(1-F\pi p^{-s}, R\Gamma_c(Y_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)[kn])^{-1}}{\det(1-F\pi p^{-s}, R\Gamma_c(Z_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-1)[kn-2])^{-1}\det(1-F\pi p^{-s}, R\Gamma_c(Y_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-1)[kn-2])^{-1}}. \end{aligned}$$

Let

$$a: Z_{k0} \to \mathrm{Spec}\,\mathbf{Z},\ b: Z_k \to \mathrm{Spec}\,\mathbf{Z},\ c: Y_{k0} \to \mathrm{Spec}\,\mathbf{Z},\ d: Y_k \to \mathrm{Spec}\,\mathbf{Z}$$

be the structure morphisms of Z_{k0}, Z_k, Y_{k0} and Y_k, respectively. By Lemma 3.1, we have

$$\begin{aligned} \det(1-F\pi p^{-s}, H_c^m(Z_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-2)) &= \det(1-Fp^{-(s-2)}, \mathrm{im}(H_c^m(Z_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l) \xrightarrow{\pi} H_c^m(Z_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l))) \\ &= \det(1-F_p p^{-(s-2)}, \mathrm{im}(R^m a_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m a_! \overline{\mathbf{Q}}_l)) \\ \det(1-F\pi p^{-s}, H_c^m(Z_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-1)) &= \det(1-Fp^{-(s-1)}, \mathrm{im}(H_c^m(Z_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l) \xrightarrow{\pi} H_c^m(Z_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l))) \\ &= \det(1-F_p p^{-(s-1)}, \mathrm{im}(R^m b_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m b_! \overline{\mathbf{Q}}_l)), \\ \det(1-F\pi p^{-s}, H_c^m(Y_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-1)) &= \det(1-Fp^{-(s-1)}, \mathrm{im}(H_c^m(Y_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l) \xrightarrow{\pi} H_c^m(Y_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l))) \\ &= \det(1-F_p p^{-(s-1)}, \mathrm{im}(R^m c_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m c_! \overline{\mathbf{Q}}_l)), \\ \det(1-F\pi p^{-s}, H_c^m(Y_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)) &= \det(1-Fp^{-s}, \mathrm{im}(H_c^m(Y_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l) \xrightarrow{\pi} H_c^m(Y_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l))) \\ &= \det(1-F_p p^{-s}, \mathrm{im}(R^m d_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m d_! \overline{\mathbf{Q}}_l)). \end{aligned}$$

So we have

$$\begin{aligned} & L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}, s) \\ = & \frac{\det(1-F\pi p^{-s}, R\Gamma_c(Z_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-2)[kn-4])^{-1}\det(1-F\pi p^{-s}, R\Gamma_c(Y_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)[kn])^{-1}}{\det(1-F\pi p^{-s}, R\Gamma_c(Z_{k,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-1)[kn-2])^{-1}\det(1-F\pi p^{-s}, R\Gamma_c(Y_{k0,\overline{\mathbf{F}}_p}, \overline{\mathbf{Q}}_l)(-1)[kn-2])^{-1}} \\ = & \prod_m \left(\frac{\det(1-F_p p^{-(s-2)}, \mathrm{im}(R^m a_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m a_! \overline{\mathbf{Q}}_l))\det(1-F_p p^{-s}, \mathrm{im}(R^m d_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m d_! \overline{\mathbf{Q}}_l))}{\det(1-F_p p^{-(s-1)}, \mathrm{im}(R^m b_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m b_! \overline{\mathbf{Q}}_l))\det(1-F_p p^{-(s-1)}, \mathrm{im}(R^m c_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m c_! \overline{\mathbf{Q}}_l))} \right)^{(-1)^{kn+m+1}}, \end{aligned}$$

and

$$\begin{aligned} & \prod_p L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}, s) \\ = & \prod_m \left(\frac{L(\mathrm{Spec}\,\mathbf{Z}, \mathrm{im}(R^m a_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m a_! \overline{\mathbf{Q}}_l), s-2) L(\mathrm{Spec}\,\mathbf{Z}, \mathrm{im}(R^m d_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m d_! \overline{\mathbf{Q}}_l), s)}{L(\mathrm{Spec}\,\mathbf{Z}, \mathrm{im}(R^m b_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m b_! \overline{\mathbf{Q}}_l), s-1) L(\mathrm{Spec}\,\mathbf{Z}, \mathrm{im}(R^m c_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m c_! \overline{\mathbf{Q}}_l), s-1)} \right)^{(-1)^{kn+m}}. \end{aligned}$$

The above sheaf $\mathrm{im}(R^m a_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m a_! \overline{\mathbf{Q}}_l)$ and the similar sheaves for the morphisms b, c and d can be made more explicit. The group $\mathfrak{S}_k$ acts on $R^m a_! \overline{\mathbf{Q}}_l$. We have

$$(R^m a_! \overline{\mathbf{Q}}_l)^{\mathfrak{S}_k} \cong \mathrm{im}(R^m a_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m a_! \overline{\mathbf{Q}}_l).$$

Let $a' : Z_{k0}/\mathfrak{S}_k \to \mathrm{Spec}\,\mathbf{Z}$ be the structure morphism of the quotient of Z_{k0} by $\mathfrak{S}_k$. Then we have

$$(R^m a_! \overline{\mathbf{Q}}_l)^{\mathfrak{S}_k} \cong R^m a'_! \overline{\mathbf{Q}}_l.$$

To prove this, we use the Hochschild-Serre type spectral sequences in [G] 5.2.1. These spectral sequences are constructed by Grothendieck for the cohomology of sheaves of abelian groups on topological spaces. We can construct similar spectral sequences for the cohomology of étale sheaves of torsion abelian groups on schemes. We then use the fact that $H^i(\mathfrak{S}_k, -)$ are annihilated by $k!$ for all $i > 0$ to conclude that similar spectral sequences degenerate for cohomology of $\overline{\mathbf{Q}}_l$-sheaves. So we have

$$R^m a'_! \overline{\mathbf{Q}}_l \cong \mathrm{im}(R^m a_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m a_! \overline{\mathbf{Q}}_l).$$

Therefore we have

$$\begin{aligned}
& L_p(\mathbf{G}_{m,\mathbf{F}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}, s) \\
= \ & \prod_m \left(\frac{\det(1 - F_p p^{-(s-2)}, \mathrm{im}(R^m a_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m a_! \overline{\mathbf{Q}}_l)) \det(1 - F_p p^{-s}, \mathrm{im}(R^m d_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m d_! \overline{\mathbf{Q}}_l))}{\det(1 - F_p p^{-(s-1)}, \mathrm{im}(R^m b_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m b_! \overline{\mathbf{Q}}_l)) \det(1 - F_p p^{-(s-1)}, \mathrm{im}(R^m c_! \overline{\mathbf{Q}}_l \xrightarrow{\pi} R^m c_! \overline{\mathbf{Q}}_l))} \right)^{(-1)^{kn+m+1}} \\
= \ & \prod_m \left(\frac{\det(1 - F_p p^{-(s-2)}, R^m a'_! \overline{\mathbf{Q}}_l) \det(1 - F_p p^{-s}, R^m d'_! \overline{\mathbf{Q}}_l)}{\det(1 - F_p p^{-(s-1)}, R^m b'_! \overline{\mathbf{Q}}_l) \det(1 - F_p p^{-(s-1)}, R^m c'_! \overline{\mathbf{Q}}_l)} \right)^{(-1)^{kn+m+1}} \\
= \ & \left(\frac{\zeta_{Z_{k0,\mathbf{F}_p}/\mathfrak{S}_k}(s-2) \zeta_{Y_{k,\mathbf{F}_p}/\mathfrak{S}_k}(s)}{\zeta_{Z_{k,\mathbf{F}_p}/\mathfrak{S}_k}(s-1) \zeta_{Y_{k0,\mathbf{F}_p}/\mathfrak{S}_k}(s-1)} \right)^{(-1)^{kn}}.
\end{aligned}$$

This proves the assertions about the L-functions of $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ in Theorem 0.2.

Similarly, by working with

$$\pi' = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) \sigma$$

instead of π, we can prove the following result for the k-th exterior power.

Theorem 3.2. Notation as above. We have

$$\begin{aligned}
& L_p(\mathbf{G}_{m,\mathbf{F}_p}, \bigwedge^k \mathrm{Kl}_{n+1}, s) \\
= \ & \prod_m \left(\frac{\det(1 - F_p p^{-(s-2)}, \mathrm{im}(R^m a_! \overline{\mathbf{Q}}_l \xrightarrow{\pi'} R^m a_! \overline{\mathbf{Q}}_l)) \det(1 - F_p p^{-s}, \mathrm{im}(R^m d_! \overline{\mathbf{Q}}_l \xrightarrow{\pi'} R^m d_! \overline{\mathbf{Q}}_l))}{\det(1 - F_p p^{-(s-1)}, \mathrm{im}(R^m b_! \overline{\mathbf{Q}}_l \xrightarrow{\pi'} R^m b_! \overline{\mathbf{Q}}_l)) \det(1 - F_p p^{-(s-1)}, \mathrm{im}(R^m c_! \overline{\mathbf{Q}}_l \xrightarrow{\pi'} R^m c_! \overline{\mathbf{Q}}_l))} \right)^{(-1)^{kn+m+1}},
\end{aligned}$$

and

$$\begin{aligned}
& \prod_p L_p(\mathbf{G}_{m,\mathbf{F}_p}, \bigwedge^k \mathrm{Kl}_{n+1}, s) \\
= \ & \prod_m \left(\frac{L(\mathrm{Spec}\,\mathbf{Z}, \mathrm{im}(R^m a_! \overline{\mathbf{Q}}_l \xrightarrow{\pi'} R^m a_! \overline{\mathbf{Q}}_l), s-2) L(\mathrm{Spec}\,\mathbf{Z}, \mathrm{im}(R^m d_! \overline{\mathbf{Q}}_l \xrightarrow{\pi'} R^m d_! \overline{\mathbf{Q}}_l), s)}{L(\mathrm{Spec}\,\mathbf{Z}, \mathrm{im}(R^m b_! \overline{\mathbf{Q}}_l \xrightarrow{\pi'} R^m b_! \overline{\mathbf{Q}}_l), s-1) L(\mathrm{Spec}\,\mathbf{Z}, \mathrm{im}(R^m c_! \overline{\mathbf{Q}}_l \xrightarrow{\pi'} R^m c_! \overline{\mathbf{Q}}_l), s-1)} \right)^{(-1)^{kn+m}}.
\end{aligned}$$

The sheaf $\mathrm{im}(R^m a_! \overline{\mathbf{Q}}_l \xrightarrow{\pi'} R^m a_! \overline{\mathbf{Q}}_l)$ and the similar sheaves for the morphisms b, c and d can again be made more explicit. Let $\mathcal{S}$ be the constant sheaf $\overline{\mathbf{Q}}_l$ on $Z_{k0}/\mathfrak{S}_k$ provided with an action of $\mathfrak{S}_k$ so that $\sigma \in \mathfrak{S}_k$ acts as multiplication by $\mathrm{Sgn}(\sigma)$. Let $p_{Z_{k0}} : Z_{k0} \to Z_{k0}/\mathfrak{S}_k$ be the projection. Using Hochschild-Serre type spectral sequences, one can show that

$$R^m a'_! \left((p_{Z_{k0},*} \overline{\mathbf{Q}}_l \otimes \mathcal{S})^{\mathfrak{S}_k} \right) \cong \mathrm{im}(R^m a_! \overline{\mathbf{Q}}_l \xrightarrow{\pi'} R^m a_! \overline{\mathbf{Q}}_l).$$

4. Simplified formulas

The formula for $\prod_p L_p(\mathbf{G}_{m,\mathbf{F}_p}, \otimes^k \mathrm{Kl}_{n+1}, s)$ in Theorem 0.2 can be significantly simplified. Since Y_k is isomorphic to $\mathbf{G}_m^{kn-1}$, we have

$$\# Y_k(\mathbf{F}_q) = (q-1)^{kn-1}.$$

A simple inclusion-exclusion argument shows that

$$\# Y_{k0}(\mathbf{F}_q) = \frac{1}{q} \left((q-1)^{kn-1} + (-1)^{kn} \right).$$

This gives the relation

$$\frac{\zeta_{Y_k}(s)}{\zeta_{Y_{k0}}(s-1)} = \zeta(s)^{(-1)^{kn}},$$

where $\zeta(s)$ is the Riemann zeta function. Similarly, one checks that

$$\# Z_k(\mathbf{F}_q) = (q-1)^{k(n-1)} \frac{1}{q} \left((q-1)^{k-1} + (-1)^k \right) = \frac{1}{q} \left((q-1)^{kn-1} + (-1)^k (q-1)^{k(n-1)} \right).$$

Thus $\zeta_{Z_k}(s)$ is also determined explicitly by the Riemann zeta function. The only non-trivial factor in the formula for $\prod_p L_p(\mathbf{G}_{m,\mathbf{F}_p}, \otimes^k \mathrm{Kl}_{n+1}, s)$ is the zeta function $\zeta_{Z_{k0}}(s)$. From the last equation defining Z_{k0}, we get

$$x_{nk} = -\left(\sum_{i=1}^{n-1} x_{ik} + \sum_{i=1}^{n} \sum_{j=1}^{k-1} x_{ij} \right).$$

Substituting this into the second equation defining Z_{k0}, we see that Z_{k0} is isomorphic to the toric hypersurface W_k in

$$\{[x_{ij}] \in \mathbf{P}^{kn-1} | x_{11} = 1, x_{ij} \neq 0\} \cong \mathbf{G}_m^{kn-1}$$

defined by

$$x_{11} = 1, \ \sum_{j=1}^{k-1} \frac{1}{\prod_{i=1}^{n} x_{ij}} \left(\sum_{i=1}^{n-1} x_{ik} + \sum_{i=1}^{n} \sum_{j=1}^{k-1} x_{ij} \right) - \frac{1}{\prod_{i=1}^{n-1} x_{ik}} = 0.$$

Thus, we obtain the simplified formula

$$\prod_p L_p(\mathbf{G}_{m,\mathbf{F}_p}, \otimes^k \mathrm{Kl}_{n+1}, s) = \zeta(s) \left(\frac{\zeta_{W_k}(s-2)}{\zeta_{Z_k}(s-1)} \right)^{(-1)^{kn}}.$$

The formula for the L-function of $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$ is more complicated. The scheme $Y_k/\mathfrak{S}_k$ can be explicitly described as follows. Let $S=k[x_{ij}]$ be the polynomial ring with the canonical grading by the degrees of polynomials. The group $\mathfrak{S}_k$ acts on S by permuting the indeterminates $x_{i1},\ldots,x_{ik}$. Let $f=\prod_{i,j}x_{ij}$. Then $Y_k=\mathrm{Spec}\,S_{(f)}$. Let s_{ij} be the j-th elementary symmetric polynomial of $x_{i1},\ldots,x_{ik}$. Then the subring of S fixed by $\mathfrak{S}_k$ is

$$S^{\mathfrak{S}_k}=k[s_{ij}].$$

Let $S'=k[s_{ij}]$. It is isomorphic to a polynomial ring. Introduce a grading on S' by setting $\deg(s_{ij})=j$. Then we have

$$(S_{(f)})^{\mathfrak{S}_k}=S'_{(f)}$$

and hence

$$Y_k/\mathfrak{S}_k=\mathrm{Spec}\,S'_{(f)}.$$

Let $\mathbf{Q}^{kn-1}=\mathrm{Proj}\,S'$ which is a weighted projective space. Then $Y_k/\mathfrak{S}_k$ is the complement of the hypersurface $f=0$ in $\mathbf{Q}^{kn-1}$.

References

[CE] T. Choi and R. Evans, *Congruences for powers of Kloosterman sums*, Intern. J. Number Theory, 3 (2007), 105-117.

[Ev] R. Evans, *Seventh power moments of Kloosterman sums*, to appear.

[FW1] L. Fu and D. Wan, *L-functions for symmetric products of Kloosterman sums*, J. Reine Angew. Math., 589 (2005), 79-103.

[FW2] L. Fu and D. Wan, *Trivial factors for L-functions of symmetric products of Kloosterman sheaves*, Finite Fields and Their Applications, to appear.

[FH] W. Fulton and J. Harris, *Representation theory, a first course*, Springer-Verlag (1991).

[G] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. 9 (1957), 119-221.

[GK] E. Grosse-Klönne, *On families of pure slope L-functions*, Doc. Math., 8 (2003), 1-42.

[HS] K. Hulek, J. Spandaw, B. van Geemen, D. van Straten, *The modularity of the Barth-Nieto quintic and its relatives*, Adv. Geom. 1 (2001), no. 3, 263-289.

[L] G. Laumon, *Transformation de Fourier, constantes d'équations fontionnelles, et conjecture de Weil*, Publ. Math. IHES 65 (1987), 131-210.

[PTV] C. Peters, J. Top and M. van der Vlugt, *The Hasse zeta function of a K3 surface related to the number of words of weight* 5 *in the Melas codes*, J. Reine Angew. Math., 432 (1992), 151-176.

[Ro] P. Robba, *Symmetric powers of p-adic Bessel equation*, J. Reine Angew. Math., 366 (1986), 194-220.

[SGA $4\frac{1}{2}$] P. Deligne et al, *Cohomologie Étale*, Lecture Notes in Math., 569, Springer-Verlag (1977).

[W1] D. Wan, *Dwork's conjecture on unit root zeta functions*, Ann. Math., 150 (1999), 867-927.