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A Riemann—Hilbert approach to asymptotic analysis of Toeplitz+Hankel determinants II

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Journal

Nonlinearity, 39(8)

ISSN

0951-7715

Authors

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Publication Date

2026-08-31

DOI

10.1088/1361-6544/ae8d79

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A Riemann–Hilbert approach to asymptotic analysis of Toeplitz+Hankel determinants II

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12 November 2025REVISED
2 June 2026ACCEPTED FOR PUBLICATION
20 July 2026PUBLISHED
3 August 2026Original content from
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citation and DOI.A Riemann–Hilbert approach to asymptotic analysis of
Toeplitz+Hankel determinants IIRoozbeh Gharakhloo^{1,*} and Alexander Its²¹ Mathematics Department, University of California Santa Cruz, Santa Cruz, CA, United States of America² Department of Mathematical Sciences, Indiana University Indianapolis, Indianapolis, IN, United States of America

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E-mail: roozbeh@ucsc.edu and aits@iu.edu**Keywords:** Riemann–Hilbert problems, Toeplitz+Hankel determinants, asymptotic analysis**Mathematics subject classification numbers:** 15B05, 30E15, 30E25, 41A60, 42C05, 82B20

Recommended by Dr Kuo-Chang Chen

Abstract

In this article, we further develop the Riemann–Hilbert formalism introduced in our earlier work [Gharakhloo and Its, SIGMA 16 (2020), 100] for the asymptotic analysis of Toeplitz+Hankel determinants with distinct Toeplitz and Hankel symbols. In that work, we showed that the associated Riemann–Hilbert problem can be analyzed by the Deift–Zhou nonlinear steepest descent method under a special restriction on the winding numbers of the symbols. This restriction excludes several natural cases, including the zero–winding configuration for both ϕ and w . The main goal of the present paper is to extend the asymptotic analysis of Toeplitz+Hankel determinants to a broader class of winding–number configurations. As an application, we study the case associated with the Ising model on the zig–zag layered half–plane with critical boundary magnetic field, for which the winding numbers of the Toeplitz and Hankel symbols are 0 and -1 , respectively. In this setting, we compute the asymptotics of the norms of the corresponding system of orthogonal polynomials.

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1. Introduction

Structured matrices and their determinants, such as Toeplitz, Hankel, and their various extensions, play a central role in diverse areas of mathematics and physics, especially in random matrix theory and statistical mechanics. These include classical Toeplitz and Hankel matrices, combinations like Toeplitz+Hankel, bordered and framed variants, and other generalizations such as slant Toeplitz matrices. There is a vast literature on the study of these structured determinants; see, e.g. [5–13, 16, 24, 26, 27, 29, 30] and references therein.

The $n \times n$ Toeplitz and Hankel matrices associated respectively with the symbols ϕ and w on the unit circle $\mathbb{T}$, traversed once counterclockwise, are defined by

$$T_n[\phi; r] := \{\phi_{j-k+r}\}_{j,k=0}^{n-1}, \quad \phi_k = \int_{\mathbb{T}} z^{-k} \phi(z) \frac{dz}{2\pi iz}, \quad (1.1)$$

and

$$H_n[w; s] := \{w_{j+k+s}\}_{j,k=0}^{n-1}, \quad w_k = \int_{\mathbb{T}} z^{-k} w(z) \frac{dz}{2\pi iz}, \quad (1.2)$$

where $r, s \in \mathbb{Z}$ are fixed offset values. If the Hankel symbol w is supported on a subset I of the real line, then w_k in (1.2) are instead given by

$$w_k = \int_I x^k w(x) dx. \quad (1.3)$$

The Toeplitz+Hankel determinant associated with symbols ϕ and w , respectively generating Toeplitz and Hankel components with the offset pair $(r, s) \in \mathbb{Z} \times \mathbb{Z}$, is of the form

$$D_n[\phi, w; r, s] := \det \{\phi_{j-k+r} + w_{j+k+s}\}_{j,k=0}^{n-1},$$

i.e.

$$D_n[\phi, w; r, s] = \det \begin{pmatrix} \phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} \\ \phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} \end{pmatrix}. \quad (1.4)$$

Our ultimate goal is to obtain the large- n asymptotics of $D_n[\phi, w; r, s]$ for arbitrary offset pairs and to determine how the analytical properties of the symbols ϕ and w affect this asymptotics.

This paper is a sequel to our first paper [25] where we studied the offset pair $(r, s) = (1, 1)$ for the Szegő-type symbols—that is, nonvanishing smooth symbols on the unit circle with zero winding number which admit an analytic continuation to a neighborhood of the circle. In [25] we computed the asymptotics of $D_n[\phi, w; 1, 1]$, as $n \rightarrow \infty$, up to the constant. At the end of [25] we formulated a number of open questions. Among those open questions was the extension of analysis to other offset pairs (r, s) . In this paper we provide a Riemann–Hilbert framework for this extension for any offset pair. We explicitly work out the details for the offset pairs $(r, s) \in \{(0, 0), (0, 1), (0, 2)\}$ which are of more interest due to their appearance in applications. The motivation for these specific offset pairs is discussed in the [appendix](#). Additionally, for the offset pair $(0, 1)$, we asymptotically solve the associated Riemann–Hilbert problem and obtain the asymptotics for the norms of the associated orthogonal polynomials.

1.1. Background

Suppose that ϕ has zero winding number with Fourier coefficients ϕ_j given by (1.1), and w_j 's be either:

- the Fourier coefficients of a symbol w with zero winding number, given by (1.2), or
- the moments

$$w_j = \int_I x^j w(x) dx, \quad (1.5)$$

of a weight w supported on some subset I of the real line. Similar to the cases of pure Toeplitz and pure Hankel determinants, in addition to operator-theoretic techniques (see [5–8] and references therein), an alternative approach to analyzing the large- n asymptotics of $D_n[\phi, w; r, s]$ —or of the associated system of

(bi)orthogonal polynomials— is provided by the Riemann–Hilbert formulation. In what follows we shall describe this formulation in detail.

It was observed in [25] that the determinant (1.4) is related to the system $\{\mathcal{P}_n(z; r, s)\}_{n \in \mathbb{Z}_{\geq 0}}$, $\deg \mathcal{P}_n(z; r, s) = n$, of monic polynomials determined by the orthogonality relations³

$$\int_{\mathbb{T}} \mathcal{P}_n(z; r, s) z^{-k-r} \phi(z) \frac{dz}{2\pi i z} + \int_{\mathbb{T}} \mathcal{P}_n(z; r, s) z^{k+s} \tilde{w}(z) \frac{dz}{2\pi i z} = h_n^{(r,s)} \delta_{n,k}, \quad k = 0, 1, \dots, n. \quad (1.6)$$

These polynomials exist and are unique if the Toeplitz+Hankel determinants (1.4) are non-zero. Indeed, if $D_n[\phi, w; r, s] \neq 0$, the polynomials $\mathcal{P}_n$ can be explicitly written as

$$\mathcal{P}_n(z; r, s) := \frac{1}{D_n[\phi, w; r, s]} \det \begin{pmatrix} \phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n} + w_{s+n} \\ \phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+1} + w_{s+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_{r-1} + w_{s+2n-1} \\ 1 & z & \cdots & z^n \end{pmatrix}, \quad (1.7)$$

while the uniqueness follows from the fact that the linear system which determines the vector of coefficients $\mathbf{a} := (a_0, \dots, a_{n-1})^T$ of the polynomials $\mathcal{P}_n(z; r, s) = z^n + \sum_{k=0}^{n-1} a_k z^k$ is of the form $\{\phi_{j-k+r} + w_{j+k+s}\}_{j,k=0}^{n-1} \mathbf{a} = \mathbf{b}$, and thus can be inverted if $D_n[\phi, w; r, s] \neq 0$. Using (1.6) and (1.7) we find

$$h_n^{(r,s)} = \frac{D_{n+1}(\phi, w; r, s)}{D_n(\phi, w; r, s)}. \quad (1.8)$$

Define now the 2×2 matrix valued function,

$$\mathcal{Y}(z; n, r, s) = \begin{pmatrix} \mathcal{P}_n(z; r, s) & \int_{\mathbb{T}} \frac{\xi^s \tilde{w}(\xi) \mathcal{P}_n(\xi; r, s) + \xi^r \tilde{\phi}(\xi) \tilde{\mathcal{P}}_n(\xi; r, s)}{\xi - z} \frac{d\xi}{2\pi i \xi} \\ -\frac{1}{h_{n-1}^{(r,s)}} \mathcal{P}_{n-1}(z; r, s) & -\frac{1}{h_{n-1}^{(r,s)}} \int_{\mathbb{T}} \frac{\xi^s \tilde{w}(\xi) \mathcal{P}_{n-1}(\xi; r, s) + \xi^r \tilde{\phi}(\xi) \tilde{\mathcal{P}}_{n-1}(\xi; r, s)}{\xi - z} \frac{d\xi}{2\pi i \xi} \end{pmatrix}. \quad (1.9)$$

In [25], it was shown that $\mathcal{Y}$ satisfies the following Riemann–Hilbert type analytical problem:

- **RH- $\mathcal{Y}1$** $\mathcal{Y}$ is holomorphic in $\mathbb{C} \setminus \mathbb{T}$.
- **RH- $\mathcal{Y}2$** For a given function f , and an oriented contour Γ , we write $f_+(z)$ (resp. $f_-(z)$) to denote the limiting value of $f(\zeta)$, as ζ approaches $z \in \Gamma$ from the left (resp. right) hand side of the oriented contour Γ with respect to its orientation. For $z \in \mathbb{T}$ we have

$$\mathcal{Y}_+^{(1)}(z; n, r, s) = \mathcal{Y}_-^{(1)}(z; n, r, s), \quad (1.10)$$

and

$$\mathcal{Y}_+^{(2)}(z; n, r, s) = \mathcal{Y}_-^{(2)}(z; n, r, s) + z^{-1+s} \tilde{w}(z) \mathcal{Y}_-^{(1)}(z; n, r, s) + z^{-1+r} \tilde{\phi}(z) \mathcal{Y}_-^{(1)}(z^{-1}; n, r, s), \quad (1.11)$$

where $\mathbb{T}$ is positively oriented in the counter-clockwise direction.

- **RH- $\mathcal{Y}3$** As $z \rightarrow \infty$, $\mathcal{Y}$ satisfies

$$\mathcal{Y}(z; n, r, s) = (I + O(z^{-1})) z^{n\sigma_3} = \begin{pmatrix} z^n + O(z^{n-1}) & O(z^{-n-1}) \\ O(z^{n-1}) & z^{-n} + O(z^{-n-1}) \end{pmatrix}, \quad (1.12)$$

where $\mathcal{Y}^{(1)}$ and $\mathcal{Y}^{(2)}$ are the first and second columns of $\mathcal{Y}$ respectively. The relation of this problem to the Toeplitz + Hankel determinants with offset $r, s \in \mathbb{Z}$ is given by the following theorem.

Theorem 1.1 [25, theorem 2.1]. *The following statements are true.*

- (1) Suppose that $D_n[\phi, w; r, s], D_{n-1}[\phi, w; r, s] \neq 0$. Then, the Riemann–Hilbert problem **RH- $\mathcal{Y}1$** through **RH- $\mathcal{Y}3$** is uniquely solvable and its solution $\mathcal{Y}$ is defined by (1.9). Moreover,

$$h_{n-1}^{(r,s)} = - \lim_{z \rightarrow \infty} z^{n-1} / \mathcal{Y}_{21}(z; n, r, s). \quad (1.13)$$

³ **Notation.** Throughout the paper we will frequently use the notation $\tilde{f}(z)$, to denote $f(z^{-1})$.

- (2) Suppose that the Riemann–Hilbert problem **RH- $\mathcal{Y}1$** through **RH- $\mathcal{Y}3$** has a unique solution. Then $D_n[\phi, w; r, s] \neq 0$, $\text{rank}(T_{n-1}[\phi; r] + H_{n-1}[w; s]) \geq n-2$, and $\mathcal{P}_n(z; r, s) = \mathcal{Y}_{11}(z; n, r, s)$.
- (3) Suppose that the Riemann–Hilbert problem **RH- $\mathcal{Y}1$** through **RH- $\mathcal{Y}3$** has a unique solution. Suppose also that

$$\lim_{z \rightarrow \infty} \mathcal{Y}_{21}(z; n, r, s) z^{-n+1} \neq 0.$$

Then, as before, $D_n[\phi, w; r, s] \neq 0$, $\mathcal{P}_n(z; r, s) = \mathcal{Y}_{11}(z; n, r, s)$, and, in addition,

$$D_{n-1}[\phi, w; r, s] \neq 0, \quad h_{n-1}^{(r,s)} = - \lim_{z \rightarrow \infty} \mathcal{Y}_{21}^{-1}(z; n, r, s) z^{n-1}, \quad \mathcal{P}_{n-1}(z; r, s) = -h_{n-1}^{(r,s)} \mathcal{Y}_{21}(z; n, r, s).$$

Corollary 1.1.1 [25, corollary 2.2]. Suppose that the $\mathcal{Y}$ -RH problem has a unique solution for n and $n-1$. Then

$$D_n[\phi, w; r, s] \neq 0, \quad D_{n-1}[\phi, w; r, s] \neq 0, \quad \text{and} \quad h_{n-1}^{(r,s)} \neq 0,$$

where $h_{n-1}^{(r,s)}$ can be reconstructed from the RHP data as

$$h_{n-1}^{(r,s)} = - \lim_{z \rightarrow \infty} z^{n-1} / \mathcal{Y}_{21}(z; n, r, s). \quad (1.14)$$

The above ‘Riemann–Hilbert type’ problem can be transformed to a genuine Riemann–Hilbert problem, i.e. to the problem whose jump conditions can be written in the usual matrix-multiplication form. To this end, in [25], the following enlargement of the $\mathcal{Y}$ -RHP was considered

$$\widehat{X}(z; n, r, s) := \left(\mathcal{Y}^{(1)}(z; n, r, s), \widetilde{\mathcal{Y}}^{(1)}(z; n, r, s), \mathcal{Y}^{(2)}(z; n, r, s), \widetilde{\mathcal{Y}}^{(2)}(z; n, r, s) \right). \quad (1.15)$$

From (1.10)–(1.12) we obtain the following 2×4 Riemann–Hilbert problem for $\widehat{X}$:

- **RH- $\widehat{X}1$** $\widehat{X}$ is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \{0\})$.
- **RH- $\widehat{X}2$** For $z \in \mathbb{T}$, $\widehat{X}$ satisfies

$$\widehat{X}_+(z; n, r, s) = \widehat{X}_-(z; n, r, s) \begin{pmatrix} 1 & 0 & z^{s-1} \widetilde{w}(z) & -z^{-r+1} \phi(z) \\ 0 & 1 & z^{r-1} \widetilde{\phi}(z) & -z^{-s+1} w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.16)$$

- **RH- $\widehat{X}3$** As $z \rightarrow \infty$ we have

$$\begin{aligned} \widehat{X}(z; n, r, s) &= \begin{pmatrix} 1 + O(z^{-1}) & C_1(n, r, s) + O(z^{-1}) & O(z^{-1}) & C_3(n, r, s) + O(z^{-1}) \\ O(z^{-1}) & C_2(n, r, s) + O(z^{-1}) & 1 + O(z^{-1}) & C_4(n, r, s) + O(z^{-1}) \end{pmatrix} \\ &\quad \times \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (1.17)$$

- **RH- $\widehat{X}4$** As $z \rightarrow 0$ we have

$$\widehat{X}(z; n, r, s) = \begin{pmatrix} C_1(n, r, s) + O(z) & 1 + O(z) & C_3(n, r, s) + O(z) & O(z) \\ C_2(n, r, s) + O(z) & O(z) & C_4(n, r, s) + O(z) & 1 + O(z) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}, \quad (1.18)$$

where

$$\begin{aligned} C_1(n, r, s) &= \mathcal{Y}_{11}(0; n, r, s), & C_3(n, r, s) &= \mathcal{Y}_{12}(0; n, r, s), \\ C_2(n, r, s) &= \mathcal{Y}_{21}(0; n, r, s), & C_4(n, r, s) &= \mathcal{Y}_{22}(0; n, r, s). \end{aligned}$$

It is a natural next step to associate with the above formulated 2×4 Riemann–Hilbert problem the following, canonically normalized at $z = \infty$, 4×4 square Riemann–Hilbert problem:

- **RH-X1** $X(\cdot; n, r, s) : \mathbb{C} \setminus (\mathbb{T} \cup \{0\}) \rightarrow \mathbb{C}^{4 \times 4}$ is analytic.
- **RH-X2** For $z \in \mathbb{T}$, we have $X_+(z; n, r, s) = X_-(z; n, r, s)J_X(z; r, s)$, where

$$J_X(z; r, s) = \begin{pmatrix} 1 & 0 & z^{s-1}\tilde{w}(z) & -z^{1-r}\phi(z) \\ 0 & 1 & z^{r-1}\tilde{\phi}(z) & -z^{1-s}w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1.19)$$

- **RH-X3** As $z \rightarrow \infty$

$$X(z; n, r, s) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

- **RH-X4** As $z \rightarrow 0$

$$X(z; n, r, s) = P(n, r, s) (I + O(z)) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}.$$

Note that the matrix $P(n, r, s)$ is not *a priori* prescribed, and thus by the standard Liouville theorem-based arguments one can show that:

Lemma 1.2. *The solution of the Riemann–Hilbert problem RH-X1 through RH-X4 is unique, if it exists.*

Remark 1.3. The expansions at $z = \infty$ and $z = 0$ are to be understood as convergent local expansions of the corresponding normalized matrix functions. For instance, set

$$Y_\infty(z; n, r, s) := X(z; n, r, s) \begin{pmatrix} z^{-n} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then Y_∞ is analytic in a punctured neighborhood of infinity, and the normalization gives

$$Y_\infty(z; n, r, s) = I + O(z^{-1}), \quad z \rightarrow \infty.$$

Hence Y_∞ is bounded in a punctured neighborhood of infinity, and therefore $z = \infty$ is a removable singularity for Y_∞ . It follows that Y_∞ admits a convergent expansion in powers of z^{-1} , namely

$$Y_\infty(z; n, r, s) = I + \frac{\tilde{X}_1(n, r, s)}{z} + \frac{\tilde{X}_2(n, r, s)}{z^2} + O(z^{-3}), \quad z \rightarrow \infty.$$

The expansion at $z = 0$ is interpreted in the same way.

The connection between $\hat{X}$ and X —Riemann Hilbert problems is given by the following lemma whose proof is given in section 2.2 of [25].

Lemma 1.4. *The solution to the $\hat{X}$ -RHP can be reconstructed from the solution of the X -RHP using*

$$\hat{X}(z; n, r, s) = \mathfrak{R}(z; n, r, s)X(z; n, r, s), \quad (1.20)$$

where

$$\mathfrak{R}(z; n, r, s) = \begin{pmatrix} 1 & C_1(n, r, s) & 0 & C_3(n, r, s) \\ 0 & C_2(n, r, s) & 1 & C_4(n, r, s) \end{pmatrix}. \quad (1.21)$$

Moreover, the following linear system for solving $C_j(n, r, s)$ in terms of $P_{jk}(n, r, s)$

$$\begin{pmatrix} 1 & C_1(n, r, s) & 0 & C_3(n, r, s) \\ 0 & C_2(n, r, s) & 1 & C_4(n, r, s) \end{pmatrix} = \begin{pmatrix} C_1(n, r, s) & 1 & C_3(n, r, s) & 0 \\ C_2(n, r, s) & 0 & C_4(n, r, s) & 1 \end{pmatrix} P^{-1}(n, r, s), \quad (1.22)$$

is well defined and is uniquely solvable if at least one of the following inequalities is true:

$$P_{22}(n, r, s)P_{44}(n, r, s) - P_{42}(n, r, s)P_{24}(n, r, s) \neq 0, \quad (1.23)$$

$$(1 - P_{21}(n, r, s))P_{42}(n, r, s) + P_{22}(n, r, s)P_{41}(n, r, s) \neq 0, \quad (1.24)$$

$$(1 - P_{43}(n, r, s))P_{22}(n, r, s) + P_{23}(n, r, s)P_{42}(n, r, s) \neq 0, \quad (1.25)$$

$$(1 - P_{21}(n, r, s))P_{44}(n, r, s) + P_{41}(n, r, s)P_{24}(n, r, s) \neq 0, \quad (1.26)$$

$$(1 - P_{21}(n, r, s))(P_{43}(n, r, s) - 1) + P_{41}(n, r, s)P_{23}(n, r, s) \neq 0, \quad (1.27)$$

$$(1 - P_{43}(n, r, s))P_{24}(n, r, s) + P_{23}(n, r, s)P_{44}(n, r, s) \neq 0. \quad (1.28)$$

Lemma 1.5 [25, lemma 2.9]. Suppose that the solution of the X-RHP exists. Then, if at least one of the conditions (1.23) through (1.28) holds, one can uniquely reconstruct the solution of the $\mathcal{Y}$ -RHP.

Remark 1.6. The reconstruction goes through equations (1.20) and (1.15). The crucial issue is to prove the uniqueness of the solution of the $\tilde{X}$ -RHP (C_j are not a priori prescribed).

Corollary 1.6.1 [25, corollary 2.10]. Suppose that the solution of the X-RHP exists for n and $n - 1$, then if at least one of the conditions (1.23) through (1.28) holds also for n and $n - 1$, then we have

$$D_n[\phi, w; r, s] \neq 0, \quad D_{n-1}[\phi, w; r, s] \neq 0, \quad \text{and} \quad h_{n-1}^{(r,s)} \neq 0.$$

Moreover,

$$h_{n-1}^{(r,s)} = - \lim_{z \rightarrow \infty} z^{n-1} / \mathcal{Y}_{21}(z; n, r, s) \quad (1.29)$$

where in the present context,

$$\mathcal{Y}_{21}(z; n, r, s) = C_2(n, r, s)X_{21}(z; n, r, s) + X_{31}(z; n, r, s) + C_4(n, r, s)X_{41}(z; n, r, s).$$

In a similar way, as detailed in [25], the analogue of the Riemann–Hilbert problem **RH-X1** through **RH-X4** for the Toeplitz+Hankel determinants with offset $r, s \in \mathbb{Z}$, when ϕ_j is still given by (1.1) but w_j 's are instead of (1.2) given by

$$w_j = \int_a^b x^j w(x) dx, \quad 0 < a < b < 1, \quad (1.30)$$

is the problem of finding the 4×4 matrix-valued function $Z(z; n, r, s)$ satisfying

- **RH-Z1** $Z(\cdot; n, r, s) : \mathbb{C} \setminus (\mathbb{T} \cup [a, b] \cup [b^{-1}, a^{-1}] \cup \{0\}) \rightarrow \mathbb{C}^{4 \times 4}$ is analytic,
- **RH-Z2** For $z \in \Sigma := \mathbb{T} \cup [a, b] \cup [b^{-1}, a^{-1}]$, we have $Z_+(z; n, r, s) = Z_-(z; n, r, s)J_Z(z; r, s)$, where

$$J_Z(z; r, s) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 & -z^{1-r}\phi(z) \\ 0 & 1 & z^{r-1}\tilde{\phi}(z) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & z \in \mathbb{T}, \\ \begin{pmatrix} 1 & 0 & 2\pi i x^s w(x) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & z \equiv x \in (a, b), \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2\pi i x^{-s}\tilde{w}(x) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & z \equiv x \in (b^{-1}, a^{-1}). \end{cases}$$

- **RH-Z3** As $z \rightarrow \infty$

$$Z(z; n, r, s) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH-Z4** As $z \rightarrow 0$

$$Z(z; n, r, s) = Q(n, r, s) (I + O(z)) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}.$$

In this case the associated system of orthogonal polynomials are characterized by

$$\int_{\mathbb{T}} Q_n(z; r, s) z^{-k-r} \phi(z) \frac{dz}{2\pi iz} + \int_a^b Q_n(x; r, s) x^{k+s} w(x) dx = h_n \delta_{n,k}, \quad k = 0, 1, \dots, n.$$

The analogues of theorem 1.1, corollary 1.1.1, lemma 1.5 and corollary 1.6.1, for **RH-Z1** through **RH-Z4** are detailed in [25] in theorem 3.1, corollary 3.2, lemma 3.5, lemma 3.6 and corollary 3.7. The analog of lemma 1.4 is proven in section 3.1.

Lemma 1.5 describes the (asymptotic) solvability of **RH-X1** through **RH-X4** as the sufficient condition for the existence and uniqueness of the orthogonal polynomials $\{\mathcal{P}_n(z; r, s)\}$. For particular choices of the offset values, the Riemann–Hilbert problems for **X** and **Z** are amenable to the Deift–Zhou nonlinear steepest descent analysis. Such requirements on the offset values are important in the Deift–Zhou method, when one does the so-called *lens-opening* transformation to construct the *global parametrix*. To that end, the Deift–Zhou nonlinear steepest descent analysis of **RH-X1** through **RH-X4** and that of **RH-Z1** through **RH-Z4** were worked out in [25] respectively when $r = s = 1$ and $(r, s) \in \{1\} \times \mathbb{Z}$.

2. Main results

Our main results establish direct algebraic relations between the solutions $X(z; n, r, s)$ of **RH-X1–RH-X4**, for $(r, s) \in \{(0, 0), (0, 1), (0, 2)\}$, and $X(z; n, 1, 1)$, which, as shown in [25], is amenable to the Deift–Zhou nonlinear steepest descent method. These algebraic relations allow one to obtain large- n asymptotics for $X(z; n, 0, 0)$, $X(z; n, 0, 1)$, and $X(z; n, 0, 2)$ from the Deift–Zhou asymptotic analysis of $X(z; n, 1, 1)$ carried out in [25]. Before presenting our main results, we introduce some notation and suitable transformations. Although we do not carry out the detailed calculations, we also describe the corresponding framework for deriving direct algebraic relations between the solutions $Z(z; n, r, s)$ of **RH-Z1–RH-Z4** and $Z(z; n, 1, s)$, which also admits a successful Deift–Zhou analysis [25].

Let us particularly denote the functions $X(z; n, 1, 1)$ and $Z(z; n, 1, s)$, which are amenable for the Deift–Zhou nonlinear steepest descent analysis [25] respectively by $\mathcal{X}(z; n)$ and $\mathcal{Z}(z; n, s)$, and thus, these functions respectively satisfy the following Riemann–Hilbert problems

- **RH- $\mathcal{X}1$** $\mathcal{X}$ is holomorphic in the complement of $\mathbb{T} \cup \{0\}$.
- **RH- $\mathcal{X}2$** For $z \in \mathbb{T}$, $\mathcal{X}$ satisfies

$$\mathcal{X}_+(z; n) = \mathcal{X}_-(z; n) \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{X}3$** As $z \rightarrow \infty$, we have

$$\mathcal{X}(z; n) = \left(I + \frac{\tilde{\mathcal{X}}_1}{z} + \frac{\tilde{\mathcal{X}}_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{X}4$** As $z \rightarrow 0$, we have

$$\mathcal{X}(z; n) = P(n) \left(I + \mathring{\mathcal{X}}_1 z + \mathring{\mathcal{X}}_2 z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix},$$

and

- **RH-Z1** $\mathcal{Z}(\cdot; n, s) : \mathbb{C} \setminus (\mathbb{T} \cup [a, b] \cup [b^{-1}, a^{-1}] \cup \{0\}) \rightarrow \mathbb{C}^{4 \times 4}$ is analytic.
- **RH-Z2** For $z \in \Sigma := \mathbb{T} \cup [a, b] \cup [b^{-1}, a^{-1}]$, we have $\mathcal{Z}_+(z; n, s) = \mathcal{Z}_-(z; n, s)J_{\mathcal{Z}}(z, s)$, where

$$J_{\mathcal{Z}}(z; s) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & z \in \mathbb{T}, \\ \begin{pmatrix} 1 & 0 & 2\pi i x^s w(x) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & z \equiv x \in (a, b), \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2\pi i x^{-s} \tilde{w}(x) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & z \equiv x \in (b^{-1}, a^{-1}). \end{cases}$$

- **RH-Z3** As $z \rightarrow \infty$

$$\mathcal{Z}(z; n, s) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH-Z4** As $z \rightarrow 0$

$$\mathcal{Z}(z; n, s) = Q(n, 1, s) (I + O(z)) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}.$$

Our principal methodological idea which we propose to use for the asymptotic analysis of the solutions of X and Z —RHPs with arbitrary r, s is to algebraically connect these RHPs to $\mathcal{X}$ and $\mathcal{Z}$ —RHPs, respectively. In more details, in the case of the X -RHP, we suggest to consider the following transformations,

$$\mathcal{W}(z; n, r, s) := X(z; n, r, s) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{r-s} & 0 & 0 \\ 0 & 0 & z^{1-s} & 0 \\ 0 & 0 & 0 & z^{r-1} \end{pmatrix}, \quad (2.1)$$

and

$$\mathcal{V}(z; n, r, s) := X(z; n, r, s) \begin{pmatrix} z^{s-r} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{1-r} & 0 \\ 0 & 0 & 0 & z^{s-1} \end{pmatrix}, \quad (2.2)$$

which coincide when $r = s$. The key observation is that these functions, both, satisfy the same jump condition on the unit circle as $\mathcal{X}$, indeed

$$\begin{aligned} \mathcal{W}_-^{-1}(z; n, r, s) \mathcal{W}_+(z; n, r, s) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{s-r} & 0 & 0 \\ 0 & 0 & z^{s-1} & 0 \\ 0 & 0 & 0 & z^{1-r} \end{pmatrix} \begin{pmatrix} 1 & 0 & z^{s-1} \tilde{w}(z) & -z^{1-r} \phi(z) \\ 0 & 1 & z^{r-1} \tilde{\phi}(z) & -z^{1-s} w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{r-s} & 0 & 0 \\ 0 & 0 & z^{1-s} & 0 \\ 0 & 0 & 0 & z^{r-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} \mathcal{V}_-^{-1}(z; n, r, s) \mathcal{V}_+(z; n, r, s) &= \begin{pmatrix} z^{r-s} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{r-1} & 0 \\ 0 & 0 & 0 & z^{1-s} \end{pmatrix} \begin{pmatrix} 1 & 0 & z^{s-1} \tilde{w}(z) & -z^{1-r} \phi(z) \\ 0 & 1 & z^{r-1} \tilde{\phi}(z) & -z^{1-s} w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\times \begin{pmatrix} z^{s-r} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{1-r} & 0 \\ 0 & 0 & 0 & z^{s-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (2.4)$$

This observation means that the functions

$$\mathcal{R}(z; n, r, s) := \mathcal{W}(z; n, r, s) \mathcal{X}^{-1}(z; n) \quad (2.5)$$

$$R(z; n, r, s) := \mathcal{V}(z; n, r, s) \mathcal{X}^{-1}(z; n) \quad (2.6)$$

are both rational functions in z , with singular behavior only at zero and infinity. If one of the rational functions $\mathcal{R}$ or R is completely determined using the data from the $\mathcal{X}$ -RHP, then accordingly, either

$$X(z; n, r, s) = \mathcal{R}(z; n, r, s) \mathcal{X}(z; n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{s-r} & 0 & 0 \\ 0 & 0 & z^{s-1} & 0 \\ 0 & 0 & 0 & z^{1-r} \end{pmatrix}, \quad (2.7)$$

or

$$X(z; n, r, s) = R(z; n, r, s) \mathcal{X}(z; n) \begin{pmatrix} z^{r-s} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{r-1} & 0 \\ 0 & 0 & 0 & z^{1-s} \end{pmatrix} \quad (2.8)$$

directly relates the solution of the X-RHP to the solution of the $\mathcal{X}$ -RHP which admits a successful non-linear steepest descent analysis [25], and thus paves the path for an asymptotic analysis of the X-RHP.

Similarly, we could also consider

$$\mathcal{K}(z; n, r, s) := Z(z; n, r, s) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{r-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{r-1} \end{pmatrix}, \quad (2.9)$$

and

$$\mathcal{N}(z; n, r, s) := Z(z; n, r, s) \begin{pmatrix} z^{1-r} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{1-r} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.10)$$

It can be simply checked that both $\mathcal{K}$ and $\mathcal{N}$ satisfy the same jump conditions as $\mathcal{Z}$ does (see **RH-Z2**), and therefore the functions

$$\mathcal{S}(z; n, r, s) := \mathcal{K}(z; n, r, s) \mathcal{Z}^{-1}(z; n, s) \quad (2.11)$$

$$S(z; n, r, s) := \mathcal{N}(z; n, r, s) \mathcal{Z}^{-1}(z; n, s) \quad (2.12)$$

are both rational functions in z , with singular behavior only at zero and infinity. Provided that either $\mathcal{S}$ or S are given explicitly by the data from the solution of the Riemann–Hilbert problem **RH-Z1** through **RH-Z4**, then one of

$$Z(z; n, r, s) = \mathcal{S}(z; n, r, s) \mathcal{Z}(z; n, s) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{1-r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{1-r} \end{pmatrix}, \quad (2.13)$$

or

$$Z(z; n, r, s) = S(z; n, r, s) \mathcal{Z}(z; n, s) \begin{pmatrix} z^{r-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{r-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.14)$$

expresses the solution of the Z -RHP in terms of the solution of the $\mathcal{Z}$ -RHP which admits a successful nonlinear steepest descent analysis [25], and thus provides a pathway for asymptotic analysis of the Z -RHP.

In this paper we show how to effectively use this idea by considering a few examples. In our exposition we focus on determining one of $\mathcal{R}$ or R (depending on the particular choice of $(r, s) \in \mathbb{Z}^2$) explicitly in terms of the data from the solution of the $\mathcal{X}$ -RHP. It will be evident that the determination of one of $\mathcal{S}$ or S (depending on the choice of $r \in \mathbb{Z}$) can be achieved in an identical way.

Our first choice of the offsets will be $r = 0$ and $s = 1$. This choice is of special interest, both because it is used in the operator-theoretic approach of [5] and because it appears in the statistical-mechanics application discussed in section A.2.1. We obtain a connection between the solution of the X -RHP with $r = 0$ and $s = 1$, which we denote by $\mathcal{U}(z; n)$, and the solution of the $\mathcal{X}$ -RHP which is amenable for the Deift–Zhou nonlinear steepest descent analysis. This will pave the way for the asymptotic analysis of $\mathcal{U}$ and eventually $h_n^{(0,1)}$, details of which are presented in section 7. By its definition, the matrix valued function $\mathcal{U}(z; n)$ is the solution of the following RHP

- **RH- $\mathcal{U}1$** $\mathcal{U}$ is holomorphic in the complement of $\mathbb{T} \cup \{0\}$.
- **RH- $\mathcal{U}2$** For $z \in \mathbb{T}$, $\mathcal{U}$ satisfies

$$\mathcal{U}_+(z; n) = \mathcal{U}_-(z; n) \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -z\phi(z) \\ 0 & 1 & z^{-1}\tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{U}3$** As $z \rightarrow \infty$ we have

$$\mathcal{U}(z; n) = \left(I + \frac{\tilde{\mathcal{U}}_1}{z} + \frac{\tilde{\mathcal{U}}_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{U}4$** As $z \rightarrow 0$ we have

$$\mathcal{U}(z; n) = \hat{\mathcal{U}} \left(I + \overset{\circ}{\mathcal{U}}_1 z + \overset{\circ}{\mathcal{U}}_2 z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}.$$

In section 3 we prove the following theorem which represents the solution of the $\mathcal{U}$ -RHP in terms of the $\mathcal{X}$ -RHP.

Theorem 2.1. *The solution $\mathcal{U}$ to the Riemann–Hilbert problem **RH- $\mathcal{U}1$** through **RH- $\mathcal{U}4$** can be expressed in terms of the data extracted from the solution $\mathcal{X}$ of the Riemann–Hilbert problem **RH- $\mathcal{X}1$** through **RH- $\mathcal{X}4$** as*

$$\mathcal{U}(z) = R(z) \mathcal{X}(z) \begin{pmatrix} z^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.15)$$

where

$$R(z) = \begin{pmatrix} \overset{\infty}{\mathcal{U}}_{1,11} - \overset{\infty}{\mathcal{X}}_{1,11} & -\overset{\infty}{\mathcal{X}}_{1,12} & \overset{\infty}{\mathcal{U}}_{1,13} - \overset{\infty}{\mathcal{X}}_{1,13} & -\overset{\infty}{\mathcal{X}}_{1,14} \\ \overset{\infty}{\mathcal{U}}_{1,21} & 1 & \overset{\infty}{\mathcal{U}}_{1,23} & 0 \\ \overset{\infty}{\mathcal{U}}_{1,31} - \overset{\infty}{\mathcal{X}}_{1,31} & -\overset{\infty}{\mathcal{X}}_{1,32} & \overset{\infty}{\mathcal{U}}_{1,33} - \overset{\infty}{\mathcal{X}}_{1,33} & -\overset{\infty}{\mathcal{X}}_{1,34} \\ \overset{\infty}{\mathcal{U}}_{1,41} & 0 & \overset{\infty}{\mathcal{U}}_{1,43} & 1 \end{pmatrix} + z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.16)$$

and $\{\tilde{\mathcal{U}}_{1,j1}^\infty, \tilde{\mathcal{U}}_{1,j3}^\infty\}_{j=1}^4$ are explicitly given in terms of the following data from the $\mathcal{X}$ -RHP

$$\begin{aligned}\tilde{\mathcal{U}}_{1,11}^\infty &= \tilde{\mathcal{X}}_{1,11}^\infty + \frac{P_{33} \sum_{j \in \{2,4\}} \tilde{\mathcal{X}}_{1,1j}^\infty P_{j1} - P_{31} \sum_{j \in \{2,4\}} \tilde{\mathcal{X}}_{1,1j}^\infty P_{j3}}{P_{11}P_{33} - P_{13}P_{31}}, & \tilde{\mathcal{U}}_{1,21}^\infty &= \frac{P_{31}P_{23} - P_{33}P_{21}}{P_{11}P_{33} - P_{13}P_{31}}, \\ \tilde{\mathcal{U}}_{1,13}^\infty &= \tilde{\mathcal{X}}_{1,13}^\infty + \frac{P_{11} \sum_{j \in \{2,4\}} \tilde{\mathcal{X}}_{1,1j}^\infty P_{j3} - P_{13} \sum_{j \in \{2,4\}} \tilde{\mathcal{X}}_{1,1j}^\infty P_{j1}}{P_{11}P_{33} - P_{13}P_{31}}, & \tilde{\mathcal{U}}_{1,23}^\infty &= \frac{P_{13}P_{21} - P_{11}P_{23}}{P_{11}P_{33} - P_{13}P_{31}}, \\ \tilde{\mathcal{U}}_{1,31}^\infty &= \tilde{\mathcal{X}}_{1,31}^\infty + \frac{P_{33} \sum_{j \in \{2,4\}} \tilde{\mathcal{X}}_{1,3j}^\infty P_{j1} - P_{31} \sum_{j \in \{2,4\}} \tilde{\mathcal{X}}_{1,3j}^\infty P_{j3}}{P_{11}P_{33} - P_{13}P_{31}}, & \tilde{\mathcal{U}}_{1,41}^\infty &= \frac{P_{43}P_{31} - P_{33}P_{41}}{P_{11}P_{33} - P_{13}P_{31}}, \\ \tilde{\mathcal{U}}_{1,33}^\infty &= \tilde{\mathcal{X}}_{1,33}^\infty + \frac{P_{11} \sum_{j \in \{2,4\}} \tilde{\mathcal{X}}_{1,3j}^\infty P_{j3} - P_{13} \sum_{j \in \{2,4\}} \tilde{\mathcal{X}}_{1,3j}^\infty P_{j1}}{P_{11}P_{33} - P_{13}P_{31}}, & \tilde{\mathcal{U}}_{1,43}^\infty &= \frac{P_{13}P_{41} - P_{11}P_{43}}{P_{11}P_{33} - P_{13}P_{31}},\end{aligned}$$

where we have assumed a generic condition,

$$P_{11}P_{33} - P_{13}P_{31} \neq 0,$$

and in all objects we have suppressed the dependence on n .

Our next example is the X-RHP with $r = s = 0$, and whose solution we shall denote by $\mathcal{Y}(z; n)$. By its definition, $\mathcal{Y}$ satisfies the following RHP,

- **RH- $\mathcal{Y}1$** $\mathcal{Y}$ is holomorphic in the complement of $\mathbb{T} \cup \{0\}$.
- **RH- $\mathcal{Y}2$** For $z \in \mathbb{T}$, $\mathcal{Y}$ satisfies

$$\mathcal{Y}_+(z; n) = \mathcal{Y}_-(z; n) \begin{pmatrix} 1 & 0 & z^{-1}\tilde{w}(z) & -z\phi(z) \\ 0 & 1 & z^{-1}\tilde{\phi}(z) & -zw(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{Y}3$** As $z \rightarrow \infty$ we have

$$\mathcal{Y}(z; n) = \left(I + \frac{\tilde{\mathcal{Y}}_1^\infty}{z} + \frac{\tilde{\mathcal{Y}}_2^\infty}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{Y}4$** As $z \rightarrow 0$ we have

$$\mathcal{Y}(z; n) = \hat{\mathcal{Y}} \left(I + \hat{\mathcal{Y}}_1 z + \hat{\mathcal{Y}}_2 z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}.$$

In section 4 we prove:

Theorem 2.2. The solution $\mathcal{Y}$ to the Riemann–Hilbert problem **RH- $\mathcal{Y}1$** through **RH- $\mathcal{Y}4$** can be expressed in terms of the data extracted from the solution $\mathcal{X}$ of the Riemann–Hilbert problem **RH- $\mathcal{X}1$** through **RH- $\mathcal{X}4$** as

$$\mathcal{Y}(z) = \mathcal{R}(z)\mathcal{X}(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z \end{pmatrix}, \quad (2.17)$$

where

$$\mathcal{R}(z) = \frac{1}{z} \begin{pmatrix} \hat{\mathcal{Y}}_{14}P_{32} & \hat{\mathcal{Y}}_{14}P_{31} & \hat{\mathcal{Y}}_{14}P_{34} & \hat{\mathcal{Y}}_{14}P_{33} \\ \hat{\mathcal{Y}}_{24}P_{32} & \hat{\mathcal{Y}}_{24}P_{31} & \hat{\mathcal{Y}}_{24}P_{34} & \hat{\mathcal{Y}}_{24}P_{33} \\ \hat{\mathcal{Y}}_{34}P_{32} & \hat{\mathcal{Y}}_{34}P_{31} & \hat{\mathcal{Y}}_{34}P_{34} & \hat{\mathcal{Y}}_{34}P_{33} \\ \hat{\mathcal{Y}}_{44}P_{32} & \hat{\mathcal{Y}}_{44}P_{31} & \hat{\mathcal{Y}}_{44}P_{34} & \hat{\mathcal{Y}}_{44}P_{33} \end{pmatrix} + \begin{pmatrix} 1 & 0 & \tilde{\mathcal{Y}}_{1,13}^\infty & 0 \\ 0 & 1 & \tilde{\mathcal{Y}}_{1,23}^\infty & 0 \\ -\tilde{\mathcal{X}}_{1,31}^\infty & -\tilde{\mathcal{X}}_{1,32}^\infty & \tilde{\mathcal{Y}}_{1,33}^\infty - \tilde{\mathcal{X}}_{1,33}^\infty & -\tilde{\mathcal{X}}_{1,34}^\infty \\ 0 & 0 & \tilde{\mathcal{Y}}_{1,43}^\infty & 0 \end{pmatrix}$$

$$+ z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.18)$$

and $\left\{ \hat{\mathcal{Y}}_{j4}, \check{\mathcal{Y}}_{1,j3} \right\}_{j=1}^4$ are explicitly given in terms of the following data from the $\mathcal{X}$ -RHP

$$\begin{aligned} \hat{\mathcal{Y}}_{14} &= \frac{P_{13} \check{\mathcal{X}}_{1,34} - P_{33} \check{\mathcal{X}}_{1,14}}{P_{33}^2 - \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,43}}, & \check{\mathcal{Y}}_{1,13} &= \frac{\check{\mathcal{X}}_{1,43} \check{\mathcal{X}}_{1,14} - P_{33} P_{13}}{P_{33}^2 - \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,43}}, \\ \hat{\mathcal{Y}}_{24} &= \frac{P_{23} \check{\mathcal{X}}_{1,34} - P_{33} \check{\mathcal{X}}_{1,24}}{P_{33}^2 - \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,43}}, & \check{\mathcal{Y}}_{1,23} &= \frac{\check{\mathcal{X}}_{1,43} \check{\mathcal{X}}_{1,24} - P_{33} P_{23}}{P_{33}^2 - \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,43}}, \\ \hat{\mathcal{Y}}_{44} &= \frac{P_{33}}{P_{33}^2 - \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,43}}, & \check{\mathcal{Y}}_{1,43} &= \frac{-\check{\mathcal{X}}_{1,43}}{P_{33}^2 - \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,43}}, \\ \hat{\mathcal{Y}}_{34} &= \frac{\Delta}{P_{33}^2 - \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,43}}, & \check{\mathcal{Y}}_{1,33} &= \frac{\Lambda}{P_{33}^2 - \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,43}}, \end{aligned}$$

with

$$\begin{aligned} \Delta &= P_{33} \left(-\check{\mathcal{X}}_{2,34} + \check{\mathcal{X}}_{1,31} \check{\mathcal{X}}_{1,14} + \check{\mathcal{X}}_{1,32} \check{\mathcal{X}}_{1,24} + \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,44} \right) \\ &\quad - \check{\mathcal{X}}_{1,34} \left(\check{\mathcal{X}}_{1,31} P_{13} + \check{\mathcal{X}}_{1,32} P_{23} + \check{\mathcal{X}}_{1,34} P_{43} \right), \end{aligned}$$

and

$$\begin{aligned} \Lambda &= -\check{\mathcal{X}}_{1,43} \left(-\check{\mathcal{X}}_{2,34} + \check{\mathcal{X}}_{1,31} \check{\mathcal{X}}_{1,14} + \check{\mathcal{X}}_{1,32} \check{\mathcal{X}}_{1,24} + \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,33} + \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,44} \right) \\ &\quad + P_{33} \left(\check{\mathcal{X}}_{1,31} P_{13} + \check{\mathcal{X}}_{1,32} P_{23} + \check{\mathcal{X}}_{1,33} P_{33} + \check{\mathcal{X}}_{1,34} P_{43} \right). \end{aligned}$$

Here, we have assumed a generic condition,

$$P_{33}^2 - \check{\mathcal{X}}_{1,34} \check{\mathcal{X}}_{1,43} \neq 0$$

and in all objects the dependence on n is suppressed.

Our third choice will be $r = 0$ and $s = 2$. As we have already explained in the introduction, these offsets appear in the theory of Ising model on the zig-zag layered half-plane. Denoting $X(z; n, 0, 2)$ as $\mathcal{T}(z; n)$ we shall arrive this time to the following RHP.

- **RH- $\mathcal{T}1$** $\mathcal{T}$ is holomorphic in the complement of $\mathbb{T} \cup \{0\}$.
- **RH- $\mathcal{T}2$** For $z \in \mathbb{T}$, $\mathcal{T}$ satisfies

$$\mathcal{T}_+(z; n) = \mathcal{T}_-(z; n) \begin{pmatrix} 1 & 0 & z\tilde{w}(z) & -z\phi(z) \\ 0 & 1 & z^{-1}\tilde{\phi}(z) & -z^{-1}w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{T}3$** As $z \rightarrow \infty$ we have

$$\mathcal{T}(z; n) = \left(I + \frac{\check{\mathcal{T}}_1}{z} + \frac{\check{\mathcal{T}}_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{T}4$** As $z \rightarrow 0$ we have

$$\mathcal{T}(z; n) = \widehat{\mathcal{T}} \left(I + \overset{\circ}{\mathcal{T}}_1 z + \overset{\circ}{\mathcal{T}}_2 z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}.$$

In section 5 we prove:

Theorem 2.3. *The solution $\mathcal{T}$ to the Riemann–Hilbert problem **RH- $\mathcal{T}1$** through **RH- $\mathcal{T}4$** can be expressed in terms of the data extracted from the solution $\mathcal{X}$ of the Riemann–Hilbert problem **RH- $\mathcal{X}1$** through **RH- $\mathcal{X}4$** as*

$$\mathcal{T}(z; n) = R(z) \mathcal{X}(z; n) \begin{pmatrix} z^{-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix}, \quad (2.19)$$

with

$$R(z) = z^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + zE + B, \quad (2.20)$$

where

$$E = \begin{pmatrix} \overset{\infty}{\mathcal{T}}_{1,11} - \overset{\infty}{\mathcal{X}}_{1,11} & -\overset{\infty}{\mathcal{X}}_{1,12} & -\overset{\infty}{\mathcal{X}}_{1,13} & -\overset{\infty}{\mathcal{X}}_{1,14} \\ \overset{\infty}{\mathcal{T}}_{1,21} & 0 & 0 & 0 \\ \overset{\infty}{\mathcal{T}}_{1,31} & 0 & 1 & 0 \\ \overset{\infty}{\mathcal{T}}_{1,41} & 0 & 0 & 1 \end{pmatrix}, \quad (2.21)$$

and

$$B = \begin{pmatrix} \overset{\infty}{\mathcal{T}}_{2,11} + \left[\overset{\infty}{\mathcal{X}}_1 \right]_{11} - \overset{\infty}{\mathcal{X}}_{2,11} & \left[\overset{\infty}{\mathcal{X}}_1 \right]_{12} - \overset{\infty}{\mathcal{X}}_{2,12} & \overset{\infty}{\mathcal{T}}_{1,13} + \left[\overset{\infty}{\mathcal{X}}_1 \right]_{13} - \overset{\infty}{\mathcal{X}}_{2,13} & \overset{\infty}{\mathcal{T}}_{1,14} + \left[\overset{\infty}{\mathcal{X}}_1 \right]_{14} - \overset{\infty}{\mathcal{X}}_{2,14} \\ \overset{\infty}{\mathcal{T}}_{2,21} & 1 & \overset{\infty}{\mathcal{T}}_{1,23} & \overset{\infty}{\mathcal{T}}_{1,24} \\ \overset{\infty}{\mathcal{T}}_{2,31} - \overset{\infty}{\mathcal{X}}_{1,31} & -\overset{\infty}{\mathcal{X}}_{1,32} & \overset{\infty}{\mathcal{T}}_{1,33} - \overset{\infty}{\mathcal{X}}_{1,33} & \overset{\infty}{\mathcal{T}}_{1,34} - \overset{\infty}{\mathcal{X}}_{1,34} \\ \overset{\infty}{\mathcal{T}}_{2,41} - \overset{\infty}{\mathcal{X}}_{1,41} & -\overset{\infty}{\mathcal{X}}_{1,42} & \overset{\infty}{\mathcal{T}}_{1,43} - \overset{\infty}{\mathcal{X}}_{1,43} & \overset{\infty}{\mathcal{T}}_{1,44} - \overset{\infty}{\mathcal{X}}_{1,44} \end{pmatrix} - \begin{pmatrix} \overset{\infty}{\mathcal{T}}_{1,11} \overset{\infty}{\mathcal{X}}_{1,11} & \overset{\infty}{\mathcal{T}}_{1,11} \overset{\infty}{\mathcal{X}}_{1,12} & \overset{\infty}{\mathcal{T}}_{1,11} \overset{\infty}{\mathcal{X}}_{1,13} & \overset{\infty}{\mathcal{T}}_{1,11} \overset{\infty}{\mathcal{X}}_{1,14} \\ \overset{\infty}{\mathcal{T}}_{1,21} \overset{\infty}{\mathcal{X}}_{1,11} & \overset{\infty}{\mathcal{T}}_{1,21} \overset{\infty}{\mathcal{X}}_{1,12} & \overset{\infty}{\mathcal{T}}_{1,21} \overset{\infty}{\mathcal{X}}_{1,13} & \overset{\infty}{\mathcal{T}}_{1,21} \overset{\infty}{\mathcal{X}}_{1,14} \\ \overset{\infty}{\mathcal{T}}_{1,31} \overset{\infty}{\mathcal{X}}_{1,11} & \overset{\infty}{\mathcal{T}}_{1,31} \overset{\infty}{\mathcal{X}}_{1,12} & \overset{\infty}{\mathcal{T}}_{1,31} \overset{\infty}{\mathcal{X}}_{1,13} & \overset{\infty}{\mathcal{T}}_{1,31} \overset{\infty}{\mathcal{X}}_{1,14} \\ \overset{\infty}{\mathcal{T}}_{1,41} \overset{\infty}{\mathcal{X}}_{1,11} & \overset{\infty}{\mathcal{T}}_{1,41} \overset{\infty}{\mathcal{X}}_{1,12} & \overset{\infty}{\mathcal{T}}_{1,41} \overset{\infty}{\mathcal{X}}_{1,13} & \overset{\infty}{\mathcal{T}}_{1,41} \overset{\infty}{\mathcal{X}}_{1,14} \end{pmatrix}. \quad (2.22)$$

In the above formula, the sixteen unknowns $\left\{ \overset{\infty}{\mathcal{T}}_{2,j1}, \overset{\infty}{\mathcal{T}}_{1,j1}, \overset{\infty}{\mathcal{T}}_{1,j3}, \overset{\infty}{\mathcal{T}}_{1,j4} \right\}_{j=1}^4$ are explicitly given in terms of the $\mathcal{X}$ -RHP data as described below. Consider

$$\mathcal{A} := P \overset{\circ}{\mathcal{X}}_1, \quad \mathcal{B} := \overset{\infty}{\mathcal{X}}_1 P, \quad \mathcal{C} := \overset{\infty}{\mathcal{X}}_2 - \overset{\infty}{\mathcal{X}}_1^2, \quad \mathcal{D} := P - \overset{\infty}{\mathcal{X}}_1 \mathcal{A}, \quad (2.23)$$

and the objects

$$\alpha := \left(\frac{\mathcal{A}_{11} \mathcal{B}_{11}}{P_{11}} + \mathcal{D}_{11} \right)^{-1}, \quad \theta := \frac{\mathcal{A}_{11}}{P_{11}}, \quad \rho_j := \mathcal{A}_{j1} - \frac{\mathcal{A}_{11} P_{j1}}{P_{11}},$$

$$\omega_{jk} := \frac{P_{1j} P_{k1}}{P_{11}}, \quad \eta_j := \frac{P_{1j}}{P_{11}}, \quad \nu_j := \frac{\mathcal{B}_{11} P_{1j}}{P_{11}} - \mathcal{B}_{1j},$$

assuming that they are well defined. Using these objects define

$$\mathcal{M}_{jk} := -\alpha \rho_k \nu_j - \omega_{jk} + P_{kj}, f_j(x, y, z) := \alpha \nu_j (z - \theta x) + \eta_j x - y,$$

and the following four functions

$$\begin{aligned} \mathcal{F}_1(x, y, w, z) = & \frac{x}{P_{11}} + \left(\left(\frac{P_{41} + \alpha \mathcal{B}_{11} \rho_4}{P_{11} \Delta} \right) \mathcal{M}_{43} - \left(\frac{P_{31} + \alpha \mathcal{B}_{11} \rho_3}{P_{11} \Delta} \right) \mathcal{M}_{44} \right) f_3(x, y, z) \\ & + \left(\left(\frac{P_{31} + \alpha \mathcal{B}_{11} \rho_3}{P_{11} \Delta} \right) \mathcal{M}_{34} - \left(\frac{P_{41} + \alpha \mathcal{B}_{11} \rho_4}{P_{11} \Delta} \right) \mathcal{M}_{33} \right) f_4(x, w, z) + \frac{\alpha \mathcal{B}_{11}}{P_{11}} (z - \theta x), \end{aligned} \quad (2.24)$$

$$\mathcal{F}_2(x, y, w, z) = \alpha \left(z - \theta x + \left(\frac{\rho_4 \mathcal{M}_{43} - \rho_3 \mathcal{M}_{44}}{\Delta} \right) f_3(x, y, z) + \left(\frac{\rho_3 \mathcal{M}_{34} - \rho_4 \mathcal{M}_{33}}{\Delta} \right) f_4(x, w, z) \right), \quad (2.25)$$

$$\mathcal{F}_3(x, y, w, z) = \frac{1}{\Delta} \left(\frac{1}{\mathcal{M}_{43}} f_3(x, y, z) - \mathcal{M}_{34} f_4(x, w, z) \right), \quad (2.26)$$

$$\mathcal{F}_4(x, y, w, z) = \frac{\mathcal{M}_{33} f_4(x, w, z) - f_3(x, y, z)}{\Delta}, \quad (2.27)$$

where

$$\Delta := \mathcal{M}_{34} \mathcal{M}_{43} - \mathcal{M}_{33} \mathcal{M}_{44}, \quad (2.28)$$

and it is assumed that $\Delta \neq 0$. Finally, the $\mathcal{T}$ -RHP data $\left\{ \tilde{\mathcal{T}}_{2,j1}^\infty, \tilde{\mathcal{T}}_{1,j1}^\infty, \tilde{\mathcal{T}}_{1,j3}^\infty, \tilde{\mathcal{T}}_{1,j4}^\infty \right\}_{j=1}^4$ describing the constant matrices B and E in (2.20) are determined by the $\mathcal{X}$ -RHP data as

$$\begin{aligned} \tilde{\mathcal{T}}_{2,11}^\infty &= \mathcal{F}_1([CP]_{11}, [CP]_{13}, [CP]_{14}, [CA + B]_{11}), & \tilde{\mathcal{T}}_{1,13}^\infty &= \mathcal{F}_3([CP]_{11}, [CP]_{13}, [CP]_{14}, [CA + B]_{11}), \\ \tilde{\mathcal{T}}_{1,11}^\infty &= \mathcal{F}_2([CP]_{11}, [CP]_{13}, [CP]_{14}, [CA + B]_{11}), & \tilde{\mathcal{T}}_{1,14}^\infty &= \mathcal{F}_4([CP]_{11}, [CP]_{13}, [CP]_{14}, [CA + B]_{11}), \\ \tilde{\mathcal{T}}_{2,21}^\infty &= \mathcal{F}_1(-P_{21}, -P_{23}, -P_{24}, -\mathcal{A}_{21}), & \tilde{\mathcal{T}}_{1,23}^\infty &= \mathcal{F}_3(-P_{21}, -P_{23}, -P_{24}, -\mathcal{A}_{21}), \\ \tilde{\mathcal{T}}_{1,21}^\infty &= \mathcal{F}_2(-P_{21}, -P_{23}, -P_{24}, -\mathcal{A}_{21}), & \tilde{\mathcal{T}}_{1,24}^\infty &= \mathcal{F}_4(-P_{21}, -P_{23}, -P_{24}, -\mathcal{A}_{21}), \\ \tilde{\mathcal{T}}_{2,31}^\infty &= \mathcal{F}_1(\mathcal{B}_{31}, \mathcal{B}_{33}, \mathcal{B}_{34}, -\mathcal{D}_{31}), & \tilde{\mathcal{T}}_{1,33}^\infty &= \mathcal{F}_3(\mathcal{B}_{31}, \mathcal{B}_{33}, \mathcal{B}_{34}, -\mathcal{D}_{31}), \\ \tilde{\mathcal{T}}_{1,31}^\infty &= \mathcal{F}_2(\mathcal{B}_{31}, \mathcal{B}_{33}, \mathcal{B}_{34}, -\mathcal{D}_{31}), & \tilde{\mathcal{T}}_{1,34}^\infty &= \mathcal{F}_4(\mathcal{B}_{31}, \mathcal{B}_{33}, \mathcal{B}_{34}, -\mathcal{D}_{31}), \\ \tilde{\mathcal{T}}_{2,41}^\infty &= \mathcal{F}_1(\mathcal{B}_{41}, \mathcal{B}_{43}, \mathcal{B}_{44}, -\mathcal{D}_{41}), & \tilde{\mathcal{T}}_{1,43}^\infty &= \mathcal{F}_3(\mathcal{B}_{41}, \mathcal{B}_{43}, \mathcal{B}_{44}, -\mathcal{D}_{41}), \\ \tilde{\mathcal{T}}_{1,41}^\infty &= \mathcal{F}_2(\mathcal{B}_{41}, \mathcal{B}_{43}, \mathcal{B}_{44}, -\mathcal{D}_{41}), & \tilde{\mathcal{T}}_{1,44}^\infty &= \mathcal{F}_4(\mathcal{B}_{41}, \mathcal{B}_{43}, \mathcal{B}_{44}, -\mathcal{D}_{41}). \end{aligned}$$

Again, the dependence of all objects on n is suppressed.

In section 7 we will illustrate a general procedure of how theorems 2.1–2.3 can be used to find the asymptotics of the norms $h_n^{(r,s)}$ for $(r, s) \in \{(0, 0), (0, 1), (0, 2)\}$. In addition to these theorems, we will need two more ingredients.

First, we will need the following general fact concerning the X-RHP⁴:

Theorem 2.4. For any choice of $(r, s) \in \mathbb{Z} \times \mathbb{Z}$, it holds that

- (a) $J_X^{-1}(z; r, s) = W J_X(z^{-1}; r, s) W$,
- (b) $P^{-1}(n, r, s) = W P(n; r, s) W$, and
- (c) $\overset{\circ}{X}_1(n, r, s) = W \overset{\circ}{X}_1^\infty(n, r, s) W$, where W is the following permutation matrix:

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (2.29)$$

We give a proof of this general theorem in section 6.

The second ingredient which we will use in section 7 is the asymptotic results concerning the X-RHP with $(r, s) = (1, 1)$, i.e. the $\mathcal{X}$ -RHP, obtained in [25]. Here are the details.

⁴ Part (b) of this theorem was used in the case of $(r, s) = (1, 1)$ in [25].

Given the Szegő-type symbols $\phi(z)$ and $w(z) = d(z)\phi(z)$, let

$$U_0 := \{z : r_i < |z| < r_o : 0 < r_i < 1 < r_o\}, \quad (2.30)$$

be the neighborhood of the unit circle where both functions, $\phi(z)$ and $d(z)$ are analytic and denote

$$r_0 := \max\{r_i, r_o^{-1}\}. \quad (2.31)$$

Define

$$\alpha(z) = \exp \left[\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\ln(\phi(\tau))}{\tau - z} d\tau \right], \quad \beta(z) = \exp \left[\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\ln(d(\tau))}{\tau - z} d\tau \right], \quad (2.32)$$

$$C_\rho(z) = -\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{1}{\beta_-(\tau)\beta_+(\tau)\tilde{\alpha}_-(\tau)\alpha_+(\tau)(\tau - z)} d\tau, \quad (2.33)$$

$$g_{23}(z) = -\frac{\alpha(0)\tilde{d}(z)\beta(z)}{\tilde{\alpha}(z)}, \quad g_{43}(z) = -\alpha^2(0)\beta(z) \left(\frac{\alpha(z)}{\tilde{\phi}(z)} + \frac{\tilde{d}(z)C_\rho(z)}{\tilde{\alpha}(z)} \right), \quad (2.34)$$

$$R_{1,23}(z; n) = \frac{1}{2\pi i} \int_{\Gamma'_i} \frac{\mu^n g_{23}(\mu)}{\mu - z} d\mu, \quad R_{1,43}(z; n) = \frac{1}{2\pi i} \int_{\Gamma'_i} \frac{\mu^n g_{43}(\mu)}{\mu - z} d\mu, \quad (2.35)$$

and finally

$$\mathcal{E}(n) = \frac{2}{\alpha(0)} R_{1,43}(0; n) - C_\rho(0) R_{1,23}(0; n). \quad (2.36)$$

In (2.35), the contour Γ'_i is a circle, oriented counter-clockwise, with radius $r' \in (r_0, 1)$.

Theorem 2.5 [25, theorem 1.1]. Suppose that $\phi(e^{i\theta})$ is smooth and nonzero on the unit circle with zero winding number, which admits an analytic continuation in a neighborhood of the unit circle. Let $w = d\phi$, where d satisfies all the properties of ϕ in addition to $d(e^{i\theta})d(e^{-i\theta}) = 1$, for all $\theta \in [0, 2\pi)$. Suppose that there exists $C > 0$ such that for sufficiently large n ,

$$|\mathcal{E}(n)| \geq C\tau^n, \quad \text{for some } \tau : r_0 \leq \tau < 1, \quad (2.37)$$

where $\mathcal{E}(n)$ is the functional of the weights ϕ and w defined in (2.36). Then, for sufficiently large n the determinant $D_n(\phi, w; 1, 1) \neq 0$ and the asymptotics of

$$h_{n-1}^{(1,1)} \equiv \frac{D_n(\phi, w; 1, 1)}{D_{n-1}(\phi, w; 1, 1)},$$

is given by

$$h_{n-1}^{(1,1)} = -\alpha(0) \frac{\mathcal{E}(n)}{\mathcal{E}(n-1)} (1 + O(e^{-c_1 n})), \quad n \rightarrow \infty, \quad (2.38)$$

where $c_1 = -\log\left(\frac{r_1^2}{\tau}\right) > 0$, and r_1 is any number satisfying the conditions: $\tau < r_1 < 1$ and $r_1^2 < \tau^5$.

We conclude this section by presenting the asymptotics of $h^{(0,1)}$ which will be obtained in section 7 based on theorem 2.1. We treat this as a case study: the same procedure can be used in view of theorems 2.2 and 2.3 to find asymptotic expressions for $h_n^{(0,0)}$ and $h_n^{(0,2)}$.

Theorem 2.6. Suppose that $\phi(e^{i\theta})$ is smooth and nonzero on the unit circle with zero winding number, which admits an analytic continuation in a neighborhood of the unit circle. Let $w = d\phi$, where d satisfies all the properties of ϕ in addition to $d(e^{i\theta})d(e^{-i\theta}) = 1$, for all $\theta \in [0, 2\pi)$. Let also $\alpha(z)$, $\beta(z)$ and $C_\rho(z)$ be defined via (2.32) and (2.33), and fix r_* to be any number satisfying the condition $r_0 < r_* < 1$, where r_0 is defined in (2.31). Define

$$\begin{aligned} \mathcal{R}_{1,jk}(z; n) &:= \frac{1}{2\pi i} \int_{\Gamma'_i} \frac{\mu^n g_{jk}(\mu)}{\mu - z} d\mu, & jk = 12, 14, \\ \mathcal{R}_{1,jk}(z; n) &:= \frac{1}{2\pi i} \int_{\Gamma'_o} \frac{\mu^{-n} g_{jk}(\mu)}{\mu - z} d\mu, & jk = 32, 34, \end{aligned} \quad (2.39)$$

⁵ See section 4.2 of [25] for the requirements on τ and r_1

where the contours Γ'_i and Γ'_o are circles, oriented counter-clockwise, with radii $r_* \in (r_0, 1)$ and $1/r_*$ respectively, and

$$\begin{aligned} g_{12}(z) &= -\frac{\alpha(z)}{\phi(z)\beta(z)} - \frac{\tilde{w}(z)C_\rho(z)}{\phi(z)\beta(z)\tilde{\alpha}(z)}, & g_{14}(z) &= \frac{\tilde{w}(z)}{\phi(z)\beta(z)\tilde{\alpha}(z)\alpha(0)}, \\ g_{32}(z) &= -\frac{1}{\alpha(0)\tilde{\phi}(z)} \left(\frac{\tilde{\alpha}(z)}{\beta(z)} - w(z)\tilde{\alpha}^2(z)\beta(z)\alpha(z)C_\rho(z) \right), & g_{34}(z) &= \frac{w(z)\tilde{\alpha}^2(z)\beta(z)\alpha(z)}{\tilde{\phi}(z)\alpha^2(0)}. \end{aligned}$$

Assume further that ϕ and d be symbols for which there exist $\hat{n} \in \mathbb{N}$, constants $r_1, r_2 \in (r_*^3, r_*^2)$, $r_3, r_4 \in (r_*^2, r_*)$, and $C_j > 0$, $j = 1, \dots, 4$, such that

$$|\mathcal{R}_{1,32}(0; n)\mathcal{R}_{1,14}(0; n)| \geq C_1 r_1^n, \quad (2.40)$$

$$|\mathcal{R}_{1,32}(0; n)\mathcal{R}_{1,14}(0; n) - \mathcal{R}_{1,12}(0; n)\mathcal{R}_{1,34}(0; n)| \geq C_2 r_2^n, \quad (2.41)$$

$$\left| \frac{\mathcal{R}_{1,12}(0; n)}{\alpha(0)} - \mathcal{R}_{1,32}(0; n-1) \right| \geq C_3 r_3^n, \quad (2.42)$$

$$|-C_\rho(0)\alpha(0)\mathcal{R}_{1,34}(0; n) - \mathcal{R}_{1,32}(0; n) + \mathcal{R}_{1,32}(0; n-1)| \geq C_4 r_4^n, \quad (2.43)$$

for all $n > \hat{n}$. Let $\mathfrak{c} := \min\{c_1, c_2, c_3, c_4\}$ where

$$2c_1 := -\log\left(\frac{r_*^3}{r_1}\right) > 0, \quad c_2 := -\log\left(\frac{r_*^3}{r_2}\right) > 0,$$

$$2c_3 := -\log\left(\frac{r_*^2}{r_3}\right) > 0, \quad c_4 := -\log\left(\frac{r_*^2}{r_4}\right) > 0.$$

Then, for sufficiently large n the determinant $D_n(\phi, w; 0, 1) \neq 0$ and the asymptotics of

$$h_{n-1}^{(0,1)} = \frac{D_n(\phi, w; 0, 1)}{D_{n-1}(\phi, w; 0, 1)},$$

is given by

$$\begin{aligned} h_{n-1}^{(0,1)} &= \frac{(\alpha(0)\mathcal{R}_{1,32}(0; n-1) - \mathcal{R}_{1,12}(0; n))(\mathcal{R}_{1,32}(0; n)\mathcal{R}_{1,14}(0; n) - \mathcal{R}_{1,12}(0; n)\mathcal{R}_{1,34}(0; n))}{\mathcal{R}_{1,32}(0; n)\mathcal{R}_{1,14}(0; n)(C_\rho(0)\alpha(0)\mathcal{R}_{1,34}(0; n) + \mathcal{R}_{1,32}(0; n) - \mathcal{R}_{1,32}(0; n-1))} \\ &\quad \times (1 + O(e^{-cn})), \end{aligned} \quad (2.44)$$

as $n \rightarrow \infty$.

Remark 2.7. As detailed in section 7.4, the conditions (2.41) and (2.42) ensure that the condition (1.24) and thus the statement of corollary 1.6.1 is in place. This justifies the statement in theorem 2.6 that for sufficiently large n the determinant $D_n(\phi, w; 0, 1) \neq 0$.

Remark 2.8. Being applied to the Ising Model in the half plane studied in [14] and described in section A.2.1, equation (2.44) yields the estimate

$$h_{n-1}^{(0,1)} = 1 + O(e^{-cn}).$$

This in turn would mean that the corresponding magnetization M_n is approaching a constant as $n \rightarrow \infty$,

$$M_n = D_n(\phi, d\phi; 0, 1) \sim \text{constant}, \quad n \rightarrow \infty. \quad (2.45)$$

We shall present the details in the forthcoming publication where we also hope to produce the exact value of the constant in (2.45).

3. Proof of theorem 2.1

Let us recall the Riemann–Hilbert problem satisfied by $\mathcal{U}$:

- **RH- $\mathcal{U}1$** $\mathcal{U}$ is holomorphic in the complement of $\mathbb{T} \cup \{0\}$.

- **RH- $\mathcal{U}2$** For $z \in \mathbb{T}$, $\mathcal{U}$ satisfies

$$\mathcal{U}_+(z; n) = \mathcal{U}_-(z; n) \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -z\phi(z) \\ 0 & 1 & z^{-1}\tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{U}3$** As $z \rightarrow \infty$ we have

$$\mathcal{U}(z; n) = \left(I + \frac{\tilde{\mathcal{U}}_1}{z} + \frac{\tilde{\mathcal{U}}_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{U}4$** As $z \rightarrow 0$ we have

$$\mathcal{U}(z; n) = \hat{\mathcal{U}} \left(I + \overset{\circ}{\mathcal{U}}_1 z + \overset{\circ}{\mathcal{U}}_2 z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}.$$

Recall also the function

$$\mathcal{V}(z; n) := \mathcal{U}(z; n) \begin{pmatrix} z & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.1)$$

This function satisfies the following Riemann–Hilbert problem

- **RH- $\mathcal{V}1$** $\mathcal{V}$ is holomorphic in the complement of $\mathbb{T} \cup \{0\}$.
- **RH- $\mathcal{V}2$** For $z \in \mathbb{T}$, $\mathcal{V}$ satisfies

$$\mathcal{V}_+(z; n) = \mathcal{V}_-(z; n) \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{V}3$** As $z \rightarrow \infty$ we have

$$\mathcal{V}(z; n) = \left(I + \frac{\tilde{\mathcal{U}}_1}{z} + \frac{\tilde{\mathcal{U}}_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{V}4$** As $z \rightarrow 0$ we have

$$\mathcal{V}(z; n) = \hat{\mathcal{U}} \left(I + \overset{\circ}{\mathcal{U}}_1 z + \overset{\circ}{\mathcal{U}}_2 z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix} \begin{pmatrix} z & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\mathcal{V}$ and $\mathcal{X}$ have the same jump matrices on the unit circle, their ratio

$$R(z; n) = \mathcal{V}(z; n) \mathcal{X}^{-1}(z; n) \quad (3.2)$$

must be a rational function with singular behavior only at zero and ∞ . From **RH- $\mathcal{X}4$** and **RH- $\mathcal{V}4$** we readily observe that $R(z; n)$ is holomorphic at zero and thus is an entire function. Let us consider its behavior at ∞ using **RH- $\mathcal{X}3$** and **RH- $\mathcal{V}3$** ; we indeed have

$$R(z; n) = \left(I + \frac{\tilde{\mathcal{U}}_1}{z} + \frac{\tilde{\mathcal{U}}_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(I - \frac{\tilde{\mathcal{X}}_1}{z} + \frac{\tilde{\mathcal{X}}_1^2 - \tilde{\mathcal{X}}_2}{z^2} + O(z^{-3}) \right)$$

$$= z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \tilde{\mathcal{U}}_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tilde{\mathcal{X}}_1 + O(z^{-1}), \quad (3.3)$$

and thus

$$R(z; n) = z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \tilde{\mathcal{U}}_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tilde{\mathcal{X}}_1. \quad (3.4)$$

It follows that to determine $R(z; n)$, and, as a consequence, $\mathcal{U}(z; n)$, in terms of the $\mathcal{X}$ -RHP data, we only need to find the eight unknowns in the first and the third columns of $\tilde{\mathcal{U}}_1$ in terms of the $\mathcal{X}$ -RHP data. To that end, we use the above expression for $R(z; n)$ in

$$\mathcal{U}(z; n) = R(z; n) \mathcal{X}(z; n) \begin{pmatrix} z^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.5)$$

which is a combination of (3.1) and (3.2), and try to match the behavior with **RH- $\mathcal{U}3$** and **RH- $\mathcal{U}4$** . As far as the behavior **RH- $\mathcal{U}3$** at infinity is concerned, it holds automatically—because of the indicated in (3.4) structure of matrix function $R(z; n)$. Hence we have to hope that the matching the behavior with **RH- $\mathcal{U}4$** will give us all the eight unknowns in the first and third columns of $\tilde{\mathcal{U}}_1$. It turns out to be indeed the case. To see this, let us rewrite (3.4) as

$$R(z; n) = z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A, \quad (3.6)$$

where

$$A := \begin{pmatrix} \tilde{\mathcal{U}}_{1,11} - \tilde{\mathcal{X}}_{1,11} & -\tilde{\mathcal{X}}_{1,12} & \tilde{\mathcal{U}}_{1,13} - \tilde{\mathcal{X}}_{1,13} & -\tilde{\mathcal{X}}_{1,14} \\ \tilde{\mathcal{U}}_{1,21} & 1 & \tilde{\mathcal{U}}_{1,23} & 0 \\ \tilde{\mathcal{U}}_{1,31} - \tilde{\mathcal{X}}_{1,31} & -\tilde{\mathcal{X}}_{1,32} & \tilde{\mathcal{U}}_{1,33} - \tilde{\mathcal{X}}_{1,33} & -\tilde{\mathcal{X}}_{1,34} \\ \tilde{\mathcal{U}}_{1,41} & 0 & \tilde{\mathcal{U}}_{1,43} & 1 \end{pmatrix}. \quad (3.7)$$

Using **RH- $\mathcal{X}4$** , the behavior of (3.5) as $z \rightarrow 0$ reads

$$\begin{aligned} \mathcal{U}(z; n) &= \left[z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A \right] P(n) \left(I + \dot{\mathcal{X}}_1 z + \dot{\mathcal{X}}_2 z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix} \begin{pmatrix} z^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \left[z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A \right] P(n) \left(I + \dot{\mathcal{X}}_1 z + \dot{\mathcal{X}}_2 z^2 + O(z^3) \right) \begin{pmatrix} z^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}. \end{aligned} \quad (3.8)$$

Comparing this with **RH- $\mathcal{U}4$** suggests that

$$\begin{aligned} &\left[z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A \right] P(n) \left(I + \dot{\mathcal{X}}_1 z + \dot{\mathcal{X}}_2 z^2 + O(z^3) \right) \begin{pmatrix} z^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \hat{\mathcal{U}} \left(I + \dot{\mathcal{U}}_1 z + \dot{\mathcal{U}}_2 z^2 + O(z^3) \right). \end{aligned} \quad (3.9)$$

Since the right hand side has no $\frac{1}{z}$ behavior, the left hand side does not as well. This condition gives the needed eight equations to determine the eight unknowns in A . Indeed, we need to have

$$AP(n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0, \quad (3.10)$$

which means that all eight entries in the first and third columns of $AP(n)$ must vanish. Thus we have

$$\begin{aligned} (AP(n))_{11} &= \sum_{j=1}^4 A_{1j}P_{j1}(n) = \left(\tilde{\mathcal{U}}_{1,11}^\infty - \tilde{\mathcal{X}}_{1,11}^\infty \right) P_{11}(n) - \tilde{\mathcal{X}}_{1,12}^\infty P_{21}(n) + \left(\tilde{\mathcal{U}}_{1,13}^\infty - \tilde{\mathcal{X}}_{1,13}^\infty \right) P_{31}(n) \\ &\quad - \tilde{\mathcal{X}}_{1,14}^\infty P_{41}(n) = 0, \end{aligned} \quad (3.11)$$

$$\begin{aligned} (AP(n))_{13} &= \sum_{j=1}^4 A_{1j}P_{j3}(n) = \left(\tilde{\mathcal{U}}_{1,11}^\infty - \tilde{\mathcal{X}}_{1,11}^\infty \right) P_{13}(n) - \tilde{\mathcal{X}}_{1,12}^\infty P_{23}(n) + \left(\tilde{\mathcal{U}}_{1,13}^\infty - \tilde{\mathcal{X}}_{1,13}^\infty \right) P_{33}(n) \\ &\quad - \tilde{\mathcal{X}}_{1,14}^\infty P_{43}(n) = 0, \end{aligned} \quad (3.12)$$

$$(AP(n))_{21} = \sum_{j=1}^4 A_{2j}P_{j1}(n) = \tilde{\mathcal{U}}_{1,21}^\infty P_{11}(n) + P_{21}(n) + \tilde{\mathcal{U}}_{1,23}^\infty P_{31}(n) = 0, \quad (3.13)$$

$$(AP(n))_{23} = \sum_{j=1}^4 A_{2j}P_{j3}(n) = \tilde{\mathcal{U}}_{1,21}^\infty P_{13}(n) + P_{23}(n) + \tilde{\mathcal{U}}_{1,23}^\infty P_{33}(n) = 0, \quad (3.14)$$

$$\begin{aligned} (AP(n))_{31} &= \sum_{j=1}^4 A_{3j}P_{j1}(n) = \left(\tilde{\mathcal{U}}_{1,31}^\infty - \tilde{\mathcal{X}}_{1,31}^\infty \right) P_{11}(n) - \tilde{\mathcal{X}}_{1,32}^\infty P_{21}(n) + \left(\tilde{\mathcal{U}}_{1,33}^\infty - \tilde{\mathcal{X}}_{1,33}^\infty \right) P_{31}(n) \\ &\quad - \tilde{\mathcal{X}}_{1,34}^\infty P_{41}(n) = 0, \end{aligned} \quad (3.15)$$

$$\begin{aligned} (AP(n))_{33} &= \sum_{j=1}^4 A_{3j}P_{j3}(n) = \left(\tilde{\mathcal{U}}_{1,31}^\infty - \tilde{\mathcal{X}}_{1,31}^\infty \right) P_{13}(n) - \tilde{\mathcal{X}}_{1,32}^\infty P_{23}(n) + \left(\tilde{\mathcal{U}}_{1,33}^\infty - \tilde{\mathcal{X}}_{1,33}^\infty \right) P_{33}(n) \\ &\quad - \tilde{\mathcal{X}}_{1,34}^\infty P_{43}(n) = 0, \end{aligned} \quad (3.16)$$

$$(AP(n))_{41} = \sum_{j=1}^4 A_{4j}P_{j1}(n) = \tilde{\mathcal{U}}_{1,41}^\infty P_{11}(n) + \tilde{\mathcal{U}}_{1,43}^\infty P_{31}(n) + P_{41}(n) = 0, \quad (3.17)$$

$$(AP(n))_{43} = \sum_{j=1}^4 A_{4j}P_{j3}(n) = \tilde{\mathcal{U}}_{1,41}^\infty P_{13}(n) + \tilde{\mathcal{U}}_{1,43}^\infty P_{33}(n) + P_{43}(n) = 0. \quad (3.18)$$

We view these eight equations as four mutually decoupled systems for determining the unknowns, for example $\tilde{\mathcal{U}}_{1,11}^\infty$ and $\tilde{\mathcal{U}}_{1,13}^\infty$ can be found by solving the system (3.11) and (3.12), $\tilde{\mathcal{U}}_{1,21}^\infty$ and $\tilde{\mathcal{U}}_{1,23}^\infty$ can be found by solving the system (3.13) and (3.14), and so on. We will skip these quite routine calculations whose result is the formulae for $\tilde{\mathcal{U}}_{1,jk}^\infty$ given in theorem 2.1. We have thus proven the theorem.

4. Proof of theorem 2.2

Let us recall the Riemann–Hilbert problem **RH- $\mathcal{Y}1$** through **RH- $\mathcal{Y}4$** below, being the specialization $r = s = 0$ of the Riemann–Hilbert problem **RH- $\mathbf{X}1$** through **RH- $\mathbf{X}4$** :

- **RH- $\mathcal{Y}1$** $\mathcal{Y}$ is holomorphic in the complement of $\mathbb{T} \cup \{0\}$.
- **RH- $\mathcal{Y}2$** For $z \in \mathbb{T}$, $\mathcal{Y}$ satisfies

$$\mathcal{Y}_+(z; n) = \mathcal{Y}_-(z; n) \begin{pmatrix} 1 & 0 & z^{-1}\tilde{w}(z) & -z\phi(z) \\ 0 & 1 & z^{-1}\tilde{\phi}(z) & -zw(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{Y}3$** As $z \rightarrow \infty$ we have

$$\mathcal{Y}(z; n) = \left(I + \frac{\mathring{\mathcal{Y}}_1}{z} + \frac{\mathring{\mathcal{Y}}_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{Y}4$** As $z \rightarrow 0$ we have

$$\mathcal{Y}(z; n) = \hat{\mathcal{Y}} \left(I + \mathring{\mathcal{Y}}_1 z + \mathring{\mathcal{Y}}_2 z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}.$$

Consider the transformation $\mathcal{Y} \mapsto \mathcal{W}$ defined by

$$\mathcal{W}(z; n) := \mathcal{Y}(z; n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix}. \quad (4.1)$$

This is the $r = s = 0$ specification of the general transformation (2.1), and it satisfies

- **RH- $\mathcal{W}1$** $\mathcal{W}$ is holomorphic in the complement of $\mathbb{T} \cup \{0\}$.
- **RH- $\mathcal{W}2$** For $z \in \mathbb{T}$, $\mathcal{W}$ satisfies

$$\mathcal{W}_+(z; n) = \mathcal{W}_-(z; n) \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{W}3$** As $z \rightarrow \infty$ we have

$$\mathcal{W}(z; n) = \left(I + \frac{\mathring{\mathcal{Y}}_1}{z} + \frac{\mathring{\mathcal{Y}}_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix}.$$

- **RH- $\mathcal{W}4$** As $z \rightarrow 0$ we have

$$\mathcal{W}(z; n) = \hat{\mathcal{Y}} \left(I + \mathring{\mathcal{Y}}_1 z + \mathring{\mathcal{Y}}_2 z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix}.$$

Since $\mathcal{W}$ and $\mathcal{X}$ have the same jump matrices on the unit circle, their ratio

$$\mathcal{R}(z; n) = \mathcal{W}(z; n) \mathcal{X}^{-1}(z; n) \quad (4.2)$$

must be a rational function with singular behavior only at zero and ∞ . Let us consider the behavior at ∞ using **RH- $\mathcal{X}3$** and **RH- $\mathcal{W}3$**

$$\begin{aligned} \mathcal{R}(z; n) &= \left(I + \frac{\mathring{\mathcal{Y}}_1}{z} + \frac{\mathring{\mathcal{Y}}_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix} \left(I - \frac{\mathring{\mathcal{X}}_1}{z} + \frac{\mathring{\mathcal{X}}_1^2 - \mathring{\mathcal{X}}_2}{z^2} + O(z^{-3}) \right) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathring{\mathcal{Y}}_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathring{\mathcal{X}}_1 + O(z^{-1}). \end{aligned} \quad (4.3)$$

Considering the behavior of $\mathcal{R}$ near zero using **RH- $\mathcal{X}4$** and **RH- $\mathcal{W}4$** we find

$$\begin{aligned}\mathcal{R}(z; n) &= \hat{\mathcal{Y}} \left(I + \hat{\mathcal{Y}}_1 z + O(z^2) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix} \left(I - \hat{\mathcal{X}}_1 z + O(z^2) \right) P^{-1}(n) \\ &= \frac{1}{z} \hat{\mathcal{Y}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^{-1}(n) + O(1).\end{aligned}\quad (4.4)$$

Therefore by the Liouville's theorem we have the following formula for $\mathcal{R}$ in terms of the *unknown* matrices $\hat{\mathcal{Y}}$ and $\hat{\mathcal{Y}}_1$:

$$\mathcal{R}(z; n) = \frac{1}{z} \hat{\mathcal{Y}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^{-1}(n) + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \hat{\mathcal{Y}}_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \hat{\mathcal{X}}_1. \quad (4.5)$$

Since the matrices are sparse in the above formula, to determine $\mathcal{R}$, we only need to determine four entries from each one of $\hat{\mathcal{Y}}$ and $\hat{\mathcal{Y}}_1$. To this end, let us introduce the relevant entries of these matrices by the formulae,

$$\hat{\mathcal{Y}} \equiv \begin{pmatrix} * & * & * & \hat{\mathcal{Y}}_{14} \\ * & * & * & \hat{\mathcal{Y}}_{24} \\ * & * & * & \delta \\ * & * & * & \gamma \end{pmatrix}, \quad \text{and} \quad \hat{\mathcal{Y}}_1 \equiv \begin{pmatrix} * & * & \hat{\mathcal{Y}}_{1,13}^{\infty} & * \\ * & * & \hat{\mathcal{Y}}_{1,23}^{\infty} & * \\ * & * & \hat{\mathcal{Y}}_{1,33}^{\infty} & * \\ * & * & \hat{\mathcal{Y}}_{1,43}^{\infty} & * \end{pmatrix}. \quad (4.6)$$

For simplicity of notations, below we suppress the dependence of quantities on n . In the notations above, $\mathcal{R}$ can be written as

$$\begin{aligned}\mathcal{R}(z) &= \frac{1}{z} \begin{pmatrix} 0 & 0 & 0 & \hat{\mathcal{Y}}_{14} \\ 0 & 0 & 0 & \hat{\mathcal{Y}}_{24} \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \gamma \end{pmatrix} P^{-1} + \begin{pmatrix} 1 & 0 & \hat{\mathcal{Y}}_{1,13}^{\infty} & 0 \\ 0 & 1 & \hat{\mathcal{Y}}_{1,23}^{\infty} & 0 \\ -\hat{\mathcal{X}}_{1,31}^{\infty} & -\hat{\mathcal{X}}_{1,32}^{\infty} & \hat{\mathcal{Y}}_{1,33}^{\infty} - \hat{\mathcal{X}}_{1,33}^{\infty} & -\hat{\mathcal{X}}_{1,34}^{\infty} \\ 0 & 0 & \hat{\mathcal{Y}}_{1,43}^{\infty} & 0 \end{pmatrix} \\ &\quad + z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}\quad (4.7)$$

We will try to find these unknowns using the above form for $\mathcal{R}$ and trying to satisfy **RH- $\mathcal{Y}3$** and **RH- $\mathcal{Y}4$** . Using (4.1) and (4.2) we have

$$\mathcal{Y}(z) = \mathcal{R}(z) \mathcal{X}(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z \end{pmatrix}, \quad (4.8)$$

so

$$\mathcal{Y}(z) = \left(\frac{1}{z} \begin{pmatrix} 0 & 0 & 0 & \hat{\mathcal{Y}}_{14} \\ 0 & 0 & 0 & \hat{\mathcal{Y}}_{24} \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \gamma \end{pmatrix} P^{-1} + \begin{pmatrix} 1 & 0 & \hat{\mathcal{Y}}_{1,13}^{\infty} & 0 \\ 0 & 1 & \hat{\mathcal{Y}}_{1,23}^{\infty} & 0 \\ -\hat{\mathcal{X}}_{1,31}^{\infty} & -\hat{\mathcal{X}}_{1,32}^{\infty} & \hat{\mathcal{Y}}_{1,33}^{\infty} - \hat{\mathcal{X}}_{1,33}^{\infty} & -\hat{\mathcal{X}}_{1,34}^{\infty} \\ 0 & 0 & \hat{\mathcal{Y}}_{1,43}^{\infty} & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

$$\times \mathcal{X}(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z \end{pmatrix}. \quad (4.9)$$

Using **RH- $\mathcal{X}3$** we have as $z \rightarrow \infty$

$$\begin{aligned} \mathcal{Y}(z) = & \left(\frac{1}{z} \begin{pmatrix} 0 & 0 & 0 & \widehat{\mathcal{Y}}_{14} \\ 0 & 0 & 0 & \widehat{\mathcal{Y}}_{24} \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \gamma \end{pmatrix} P^{-1} + \begin{pmatrix} 1 & 0 & \mathcal{Y}_{1,13}^\infty & 0 \\ 0 & 1 & \mathcal{Y}_{1,23}^\infty & 0 \\ -\mathcal{X}_{1,31}^\infty & -\mathcal{X}_{1,32}^\infty & \mathcal{Y}_{1,33}^\infty - \mathcal{X}_{1,33}^\infty & -\mathcal{X}_{1,34}^\infty \\ 0 & 0 & \mathcal{Y}_{1,43}^\infty & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \\ & \times \left(I + \frac{\mathcal{X}_1^\infty}{z} + \frac{\mathcal{X}_2^\infty}{z^2} + O(z^{-3}) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z \end{pmatrix} \begin{pmatrix} z'' & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (4.10)$$

Now, we compare this with **RH- $\mathcal{Y}3$** , thus we must have

$$\begin{aligned} & \left(\frac{1}{z} \begin{pmatrix} 0 & 0 & 0 & \widehat{\mathcal{Y}}_{14} \\ 0 & 0 & 0 & \widehat{\mathcal{Y}}_{24} \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \gamma \end{pmatrix} P^{-1} + \begin{pmatrix} 1 & 0 & \mathcal{Y}_{1,13}^\infty & 0 \\ 0 & 1 & \mathcal{Y}_{1,23}^\infty & 0 \\ -\mathcal{X}_{1,31}^\infty & -\mathcal{X}_{1,32}^\infty & \mathcal{Y}_{1,33}^\infty - \mathcal{X}_{1,33}^\infty & -\mathcal{X}_{1,34}^\infty \\ 0 & 0 & \mathcal{Y}_{1,43}^\infty & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \\ & \times \left(I + \frac{\mathcal{X}_1^\infty}{z} + \frac{\mathcal{X}_2^\infty}{z^2} + O(z^{-3}) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z \end{pmatrix} \equiv I + \frac{\mathcal{Y}_1^\infty}{z} + \frac{\mathcal{Y}_2^\infty}{z^2} + O(z^{-3}). \end{aligned} \quad (4.11)$$

Therefore we have

$$\begin{aligned} & \begin{pmatrix} 0 & 0 & 0 & \widehat{\mathcal{Y}}_{14} \\ 0 & 0 & 0 & \widehat{\mathcal{Y}}_{24} \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \gamma \end{pmatrix} P^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & \mathcal{Y}_{1,13}^\infty & 0 \\ 0 & 1 & \mathcal{Y}_{1,23}^\infty & 0 \\ -\mathcal{X}_{1,31}^\infty & -\mathcal{X}_{1,32}^\infty & \mathcal{Y}_{1,33}^\infty - \mathcal{X}_{1,33}^\infty & -\mathcal{X}_{1,34}^\infty \\ 0 & 0 & \mathcal{Y}_{1,43}^\infty & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ & + \begin{pmatrix} 1 & 0 & \mathcal{Y}_{1,13}^\infty & 0 \\ 0 & 1 & \mathcal{Y}_{1,23}^\infty & 0 \\ -\mathcal{X}_{1,31}^\infty & -\mathcal{X}_{1,32}^\infty & \mathcal{Y}_{1,33}^\infty - \mathcal{X}_{1,33}^\infty & -\mathcal{X}_{1,34}^\infty \\ 0 & 0 & \mathcal{Y}_{1,43}^\infty & 0 \end{pmatrix} \mathcal{X}_1^\infty \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathcal{X}_1^\infty \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathcal{X}_2^\infty \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv I. \end{aligned} \quad (4.12)$$

Simplifying (4.12) we get:

$$\begin{aligned} & \begin{pmatrix} 0 & 0 & 0 & \widehat{\mathcal{Y}}_{14} P_{33} \\ 0 & 0 & 0 & \widehat{\mathcal{Y}}_{24} P_{33} \\ 0 & 0 & 0 & \delta P_{33} \\ 0 & 0 & 0 & \gamma P_{33} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\mathcal{X}_{1,31}^\infty & -\mathcal{X}_{1,32}^\infty & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{X}_{1,31}^\infty & \mathcal{X}_{1,32}^\infty & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ & + \begin{pmatrix} 1 & 0 & \mathcal{Y}_{1,13}^\infty & 0 \\ 0 & 1 & \mathcal{Y}_{1,23}^\infty & 0 \\ -\mathcal{X}_{1,31}^\infty & -\mathcal{X}_{1,32}^\infty & \mathcal{Y}_{1,33}^\infty - \mathcal{X}_{1,33}^\infty & -\mathcal{X}_{1,34}^\infty \\ 0 & 0 & \mathcal{Y}_{1,43}^\infty & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \mathcal{X}_{1,14}^\infty \\ 0 & 0 & 0 & \mathcal{X}_{1,24}^\infty \\ 0 & 0 & 0 & \mathcal{X}_{1,34}^\infty \\ 0 & 0 & 0 & \mathcal{X}_{1,44}^\infty \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{X}_{2,34}^\infty \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv I, \end{aligned} \quad (4.13)$$

where we have used $(P^{-1})_{44} = P_{33}$, which is due to the representation of P^{-1} as *WPW* with

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

given by part (b) of theorem 2.4, also see [25]. Combining terms and simplifying (4.13) we find

$$\begin{pmatrix} 1 & 0 & 0 & P_{33}\widehat{\mathcal{Y}}_{14} + \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{Y}}_{1,13} + \overset{\infty}{\mathcal{X}}_{1,14} \\ 0 & 1 & 0 & P_{33}\widehat{\mathcal{Y}}_{24} + \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{Y}}_{1,23} + \overset{\infty}{\mathcal{X}}_{1,24} \\ 0 & 0 & 1 & P_{33}\delta + \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{Y}}_{1,33} + \overset{\infty}{\mathcal{X}}_{2,34} - \overset{\infty}{\mathcal{X}}_{1,31}\overset{\infty}{\mathcal{X}}_{1,14} - \overset{\infty}{\mathcal{X}}_{1,32}\overset{\infty}{\mathcal{X}}_{1,24} - \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{X}}_{1,33} - \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{X}}_{1,44} \\ 0 & 0 & 0 & P_{33}\gamma + \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{Y}}_{1,43} \end{pmatrix} \equiv I. \quad (4.14)$$

This gives four linear equations in the eight unknowns $\{\widehat{\mathcal{Y}}_{j4}, \overset{\infty}{\mathcal{Y}}_{1,j3}\}_{j=1}^4$:

$$P_{33}\widehat{\mathcal{Y}}_{14} + \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{Y}}_{1,13} + \overset{\infty}{\mathcal{X}}_{1,14} = 0, \quad (4.15)$$

$$P_{33}\widehat{\mathcal{Y}}_{24} + \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{Y}}_{1,23} + \overset{\infty}{\mathcal{X}}_{1,24} = 0, \quad (4.16)$$

$$P_{33}\delta + \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{Y}}_{1,33} + \overset{\infty}{\mathcal{X}}_{2,34} - \overset{\infty}{\mathcal{X}}_{1,31}\overset{\infty}{\mathcal{X}}_{1,14} - \overset{\infty}{\mathcal{X}}_{1,32}\overset{\infty}{\mathcal{X}}_{1,24} - \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{X}}_{1,33} - \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{X}}_{1,44} = 0, \quad (4.17)$$

$$P_{33}\gamma + \overset{\infty}{\mathcal{X}}_{1,34}\overset{\infty}{\mathcal{Y}}_{1,43} = 1. \quad (4.18)$$

To find the complementary equations we consider the behavior of (4.9) near zero. Using **RH- $\mathcal{X}4$** we have as $z \rightarrow 0$

$$\begin{aligned} \mathcal{Y}(z; n) &= \left(\frac{1}{z} \begin{pmatrix} 0 & 0 & 0 & \widehat{\mathcal{Y}}_{14} \\ 0 & 0 & 0 & \widehat{\mathcal{Y}}_{24} \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \gamma \end{pmatrix} P^{-1}(n) + \begin{pmatrix} 1 & 0 & \overset{\infty}{\mathcal{Y}}_{1,13} & 0 \\ 0 & 1 & \overset{\infty}{\mathcal{Y}}_{1,23} & 0 \\ -\overset{\infty}{\mathcal{X}}_{1,31} & -\overset{\infty}{\mathcal{X}}_{1,32} & \overset{\infty}{\mathcal{Y}}_{1,33} - \overset{\infty}{\mathcal{X}}_{1,33} & -\overset{\infty}{\mathcal{X}}_{1,34} \\ 0 & 0 & \overset{\infty}{\mathcal{Y}}_{1,43} & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \\ &\quad \times P(n) \left(I + \overset{\circ}{\mathcal{X}}_{1z} + \overset{\circ}{\mathcal{X}}_{2z^2} + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix} \\ &= \left(\frac{1}{z} \begin{pmatrix} 0 & 0 & 0 & \widehat{\mathcal{Y}}_{14} \\ 0 & 0 & 0 & \widehat{\mathcal{Y}}_{24} \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \gamma \end{pmatrix} + \begin{pmatrix} 1 & 0 & \overset{\infty}{\mathcal{Y}}_{1,13} & 0 \\ 0 & 1 & \overset{\infty}{\mathcal{Y}}_{1,23} & 0 \\ -\overset{\infty}{\mathcal{X}}_{1,31} & -\overset{\infty}{\mathcal{X}}_{1,32} & \overset{\infty}{\mathcal{Y}}_{1,33} - \overset{\infty}{\mathcal{X}}_{1,33} & -\overset{\infty}{\mathcal{X}}_{1,34} \\ 0 & 0 & \overset{\infty}{\mathcal{Y}}_{1,43} & 0 \end{pmatrix} P(n) + z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P(n) \right) \\ &\quad \times \left(I + \overset{\circ}{\mathcal{X}}_{1z} + \overset{\circ}{\mathcal{X}}_{2z^2} + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}. \quad (4.19) \end{aligned}$$

The coefficient of z^{-2} in the above expression is

$$\begin{pmatrix} 0 & 0 & 0 & \widehat{\mathcal{Y}}_{14} \\ 0 & 0 & 0 & \widehat{\mathcal{Y}}_{24} \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$$

The coefficient of z^{-1} in the (4.19) is

$$\begin{pmatrix} 0 & 0 & \overset{\circ}{\mathcal{X}}_{1,43}\widehat{\mathcal{Y}}_{14} + P_{33}\overset{\infty}{\mathcal{Y}}_{1,13} + P_{13} & 0 \\ 0 & 0 & \overset{\circ}{\mathcal{X}}_{1,43}\widehat{\mathcal{Y}}_{24} + P_{33}\overset{\infty}{\mathcal{Y}}_{1,23} + P_{23} & 0 \\ 0 & 0 & \overset{\circ}{\mathcal{X}}_{1,43}\delta + P_{33}\overset{\infty}{\mathcal{Y}}_{1,33} - \overset{\infty}{\mathcal{X}}_{1,31}P_{13} - \overset{\infty}{\mathcal{X}}_{1,32}P_{23} - \overset{\infty}{\mathcal{X}}_{1,33}P_{33} - \overset{\infty}{\mathcal{X}}_{1,34}P_{43} & 0 \\ 0 & 0 & \overset{\circ}{\mathcal{X}}_{1,43}\widehat{\mathcal{Y}}_{44} + P_{33}\overset{\infty}{\mathcal{Y}}_{1,43} & 0 \end{pmatrix},$$

which we need to set equal to zero according to **RH- $\mathcal{Y}4$** :

$$\overset{\circ}{\mathcal{X}}_{1,43}\widehat{\mathcal{Y}}_{14} + P_{33}\overset{\infty}{\mathcal{Y}}_{1,13} + P_{13} = 0, \quad (4.20)$$

$$\overset{\circ}{\mathcal{X}}_{1,43}\widehat{\mathcal{Y}}_{24} + P_{33}\overset{\infty}{\mathcal{Y}}_{1,23} + P_{23} = 0, \quad (4.21)$$

$$\overset{\circ}{\mathcal{X}}_{1,43}\delta + P_{33}\overset{\infty}{\mathcal{Y}}_{1,33} - \overset{\infty}{\mathcal{X}}_{1,31}P_{13} - \overset{\infty}{\mathcal{X}}_{1,32}P_{23} - \overset{\infty}{\mathcal{X}}_{1,33}P_{33} - \overset{\infty}{\mathcal{X}}_{1,34}P_{43} = 0, \quad (4.22)$$

$$\overset{\circ}{\mathcal{X}}_{1,43}\widehat{\mathcal{Y}}_{44} + P_{33}\overset{\infty}{\mathcal{Y}}_{1,43} = 0. \quad (4.23)$$

Notice that the 8 equations (4.15)–(4.18) and (4.20)–(4.23) *decouples* into four sets of two equations in two unknowns. Solving these equations we will find formulae for $\widehat{\mathcal{Y}}_{jk}$ as given in theorem 2.2.

Finally, using (4.7) and $P^{-1} = WPW$ with

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

we arrived at the desired explicit representation of $\mathcal{R}$ in terms of the data from the solution of the Riemann–Hilbert problem **RH- $\mathcal{X}1$** through **RH- $\mathcal{X}4$** :

$$\begin{aligned} \mathcal{R}(z) = & \frac{1}{z} \begin{pmatrix} \widehat{\mathcal{Y}}_{14}P_{32} & \widehat{\mathcal{Y}}_{14}P_{31} & \widehat{\mathcal{Y}}_{14}P_{34} & \widehat{\mathcal{Y}}_{14}P_{33} \\ \widehat{\mathcal{Y}}_{24}P_{32} & \widehat{\mathcal{Y}}_{24}P_{31} & \widehat{\mathcal{Y}}_{24}P_{34} & \widehat{\mathcal{Y}}_{24}P_{33} \\ \widehat{\mathcal{Y}}_{34}P_{32} & \widehat{\mathcal{Y}}_{34}P_{31} & \widehat{\mathcal{Y}}_{34}P_{34} & \widehat{\mathcal{Y}}_{34}P_{33} \\ \widehat{\mathcal{Y}}_{44}P_{32} & \widehat{\mathcal{Y}}_{44}P_{31} & \widehat{\mathcal{Y}}_{44}P_{34} & \widehat{\mathcal{Y}}_{44}P_{33} \end{pmatrix} + \begin{pmatrix} 1 & 0 & \overset{\infty}{\mathcal{Y}}_{1,13} & 0 \\ 0 & 1 & \overset{\infty}{\mathcal{Y}}_{1,23} & 0 \\ -\overset{\infty}{\mathcal{X}}_{1,31} & -\overset{\infty}{\mathcal{X}}_{1,32} & \overset{\infty}{\mathcal{Y}}_{1,33} - \overset{\infty}{\mathcal{X}}_{1,33} & -\overset{\infty}{\mathcal{X}}_{1,34} \\ 0 & 0 & \overset{\infty}{\mathcal{Y}}_{1,43} & 0 \end{pmatrix} \\ & + z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (4.24)$$

5. Proof of theorem 2.3

For $r = 0$ and $s = 2$, recall that we denote $X(z; n, 0, 2)$ by $\mathcal{T}(z; n)$, which satisfies

- **RH- $\mathcal{T}1$** $\mathcal{T}(\cdot; n)$ is holomorphic in the complement of $\mathbb{T} \cup \{0\}$.
- **RH- $\mathcal{T}2$** For $z \in \mathbb{T}$, $\mathcal{T}$ satisfies

$$\mathcal{T}_+(z; n) = \mathcal{T}_-(z; n) \begin{pmatrix} 1 & 0 & z\tilde{w}(z) & -z\phi(z) \\ 0 & 1 & z^{-1}\tilde{\phi}(z) & -z^{-1}w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{T}3$** As $z \rightarrow \infty$ we have

$$\mathcal{T}(z; n) = \left(I + \frac{\overset{\infty}{\mathcal{T}}_1}{z} + \frac{\overset{\infty}{\mathcal{T}}_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH- $\mathcal{T}4$** As $z \rightarrow 0$ we have

$$\mathcal{T}(z; n) = \widehat{\mathcal{T}} \left(I + \overset{\circ}{\mathcal{T}}_1 z + \overset{\circ}{\mathcal{T}}_2 z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}.$$

Let us recall (2.8) in this case

$$\mathcal{T}(z) = R(z)\mathcal{X}(z) \begin{pmatrix} z^{-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix}. \quad (5.1)$$

Let us also recall (2.2) in this case

$$\mathcal{V}(z) = \mathcal{T}(z) \begin{pmatrix} z^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix}. \quad (5.2)$$

Behavior of $\mathcal{V}$ as $z \rightarrow \infty$

$$\mathcal{V}(z) = \left(I + \frac{\overset{\infty}{\mathcal{T}}_1}{z} + \frac{\overset{\infty}{\mathcal{T}}_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix}. \quad (5.3)$$

Behavior of $\mathcal{V}$ as $z \rightarrow 0$

$$\mathcal{V}(z) = \widehat{\mathcal{T}} \left(I + \overset{\circ}{\mathcal{T}}_1 z + \overset{\circ}{\mathcal{T}}_2 z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix} \begin{pmatrix} z^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix}. \quad (5.4)$$

Notice that $R = \mathcal{V}\mathcal{X}^{-1}$ is an entire function, due to (5.4) and **RH- $\mathcal{X}4$** . Behavior of $R = \mathcal{V}\mathcal{X}^{-1}$ as $z \rightarrow \infty$ is given by

$$\begin{aligned} R(z) = & z^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + z \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \overset{\infty}{\mathcal{T}}_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{\infty}{\mathcal{X}}_1 \right\} \\ & + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{\infty}{\mathcal{X}}_1 + \overset{\infty}{\mathcal{T}}_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \overset{\infty}{\mathcal{T}}_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{\infty}{\mathcal{X}}_1 \\ & + \overset{\infty}{\mathcal{T}}_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \left(\overset{\infty}{\mathcal{X}}_1^2 - \overset{\infty}{\mathcal{X}}_2 \right). \end{aligned} \quad (5.5)$$

Further simplification gives

$$\begin{aligned}
 R(z) = & z^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + z \begin{pmatrix} \mathcal{T}_{1,11} - \mathcal{X}_{1,11} & -\mathcal{X}_{1,12} & -\mathcal{X}_{1,13} & -\mathcal{X}_{1,14} \\ \mathcal{T}_{1,21} & 0 & 0 & 0 \\ \mathcal{T}_{1,31} & 0 & 1 & 0 \\ \mathcal{T}_{1,41} & 0 & 0 & 1 \end{pmatrix} \\
 & + \begin{pmatrix} \mathcal{T}_{2,11} + [\mathcal{X}_1^2]_{11} - \mathcal{X}_{2,11} & [\mathcal{X}_1^2]_{12} - \mathcal{X}_{2,12} & \mathcal{T}_{1,13} + [\mathcal{X}_1^2]_{13} - \mathcal{X}_{2,13} & \mathcal{T}_{1,14} + [\mathcal{X}_1^2]_{14} - \mathcal{X}_{2,14} \\ \mathcal{T}_{2,21} & 1 & \mathcal{T}_{1,23} & \mathcal{T}_{1,24} \\ \mathcal{T}_{2,31} - \mathcal{X}_{1,31} & -\mathcal{X}_{1,32} & \mathcal{T}_{1,33} - \mathcal{X}_{1,33} & \mathcal{T}_{1,34} - \mathcal{X}_{1,34} \\ \mathcal{T}_{2,41} - \mathcal{X}_{1,41} & -\mathcal{X}_{1,42} & \mathcal{T}_{1,43} - \mathcal{X}_{1,43} & \mathcal{T}_{1,44} - \mathcal{X}_{1,44} \end{pmatrix} \\
 & - \begin{pmatrix} \mathcal{T}_{1,11} \mathcal{X}_{1,11} & \mathcal{T}_{1,11} \mathcal{X}_{1,12} & \mathcal{T}_{1,11} \mathcal{X}_{1,13} & \mathcal{T}_{1,11} \mathcal{X}_{1,14} \\ \mathcal{T}_{1,21} \mathcal{X}_{1,11} & \mathcal{T}_{1,21} \mathcal{X}_{1,12} & \mathcal{T}_{1,21} \mathcal{X}_{1,13} & \mathcal{T}_{1,21} \mathcal{X}_{1,14} \\ \mathcal{T}_{1,31} \mathcal{X}_{1,11} & \mathcal{T}_{1,31} \mathcal{X}_{1,12} & \mathcal{T}_{1,31} \mathcal{X}_{1,13} & \mathcal{T}_{1,31} \mathcal{X}_{1,14} \\ \mathcal{T}_{1,41} \mathcal{X}_{1,11} & \mathcal{T}_{1,41} \mathcal{X}_{1,12} & \mathcal{T}_{1,41} \mathcal{X}_{1,13} & \mathcal{T}_{1,41} \mathcal{X}_{1,14} \end{pmatrix} \equiv z^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + zE + B.
 \end{aligned} \tag{5.6}$$

To completely determine $R(z)$, in this case we need to find the sixteen unknowns $\{\mathcal{T}_{1,j1}, \mathcal{T}_{1,j3}, \mathcal{T}_{1,j4}, \mathcal{T}_{2,j1}\}_{j=1}^4$ in terms of the data from the solution of the Riemann–Hilbert problem **RH- $\mathcal{X}1$** through **RH- $\mathcal{X}4$** . To this end, we substitute the above expression for R into (5.1) and compare the resulting asymptotics of $\mathcal{T}$, as $z \rightarrow \infty$ and as $z \rightarrow 0$, respectively with **RH- $\mathcal{T}3$** and **RH- $\mathcal{T}4$** . Similar to the proof of theorem 2.1 we conclude that **RH- $\mathcal{T}3$** is satisfied automatically because of the structure of $R(z)$, and we only have to take care of **RH- $\mathcal{T}4$** . It turns out, as shown below, that all sixteen equations to determine the sixteen unknowns come from the facts that there are no terms in

$$\mathcal{T}(z; n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix}$$

with z^{-1} and z^{-2} as required by **RH- $\mathcal{T}4$** . Indeed,

$$\begin{aligned}
 \mathcal{T}(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix} &= R(z) \mathcal{X}(z) \begin{pmatrix} z^{-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix} \\
 &= \left(z^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + zE + B \right) P \left(I + \mathcal{X}_1 z + \mathcal{X}_2 z^2 + O(z^3) \right) \begin{pmatrix} z^{-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix} \\
 &= \left(z^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + zE + B \right) \left(P + P \mathcal{X}_1 z + P \mathcal{X}_2 z^2 + O(z^3) \right) \\
 &\quad \times \left(z^{-2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + z^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right).
 \end{aligned} \tag{5.7}$$

The equations corresponding to the coefficients of z^{-2} and z^{-1} , in the above expression in view of **RH- $\mathcal{T}4$** are

$$BP \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0, \tag{5.8}$$

and

$$BP \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \left(BP \overset{\circ}{\mathcal{X}}_1 + EP \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0, \quad (5.9)$$

respectively. These equations imply that the first, third and fourth columns of BP are zero. So we have the following twelve equations:

$$\begin{aligned}
(BP)_{11} = & \left(\bar{\mathcal{T}}_{2,11} + \left[\bar{\mathcal{X}}_1^2 \right]_{11} - \bar{\mathcal{X}}_{2,11} - \bar{\mathcal{T}}_{1,11} \bar{\mathcal{X}}_{1,11} \right) P_{11} + \left(\left[\bar{\mathcal{X}}_1^2 \right]_{12} - \bar{\mathcal{X}}_{2,12} - \bar{\mathcal{T}}_{1,11} \bar{\mathcal{X}}_{1,12} \right) P_{21} \\
& + \left(\bar{\mathcal{T}}_{1,13} + \left[\bar{\mathcal{X}}_1^2 \right]_{13} - \bar{\mathcal{X}}_{2,13} - \bar{\mathcal{T}}_{1,11} \bar{\mathcal{X}}_{1,13} \right) P_{31} \\
& + \left(\bar{\mathcal{T}}_{1,14} + \left[\bar{\mathcal{X}}_1^2 \right]_{14} - \bar{\mathcal{X}}_{2,14} - \bar{\mathcal{T}}_{1,11} \bar{\mathcal{X}}_{1,14} \right) P_{41} = 0,
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
(BP)_{13} = & \left(\bar{\mathcal{T}}_{2,11} + \left[\bar{\mathcal{X}}_1^2 \right]_{11} - \bar{\mathcal{X}}_{2,11} - \bar{\mathcal{T}}_{1,11} \bar{\mathcal{X}}_{1,11} \right) P_{13} + \left(\left[\bar{\mathcal{X}}_1^2 \right]_{12} - \bar{\mathcal{X}}_{2,12} - \bar{\mathcal{T}}_{1,11} \bar{\mathcal{X}}_{1,12} \right) P_{23} \\
& + \left(\bar{\mathcal{T}}_{1,13} + \left[\bar{\mathcal{X}}_1^2 \right]_{13} - \bar{\mathcal{X}}_{2,13} - \bar{\mathcal{T}}_{1,11} \bar{\mathcal{X}}_{1,13} \right) P_{33} \\
& + \left(\bar{\mathcal{T}}_{1,14} + \left[\bar{\mathcal{X}}_1^2 \right]_{14} - \bar{\mathcal{X}}_{2,14} - \bar{\mathcal{T}}_{1,11} \bar{\mathcal{X}}_{1,14} \right) P_{43} = 0,
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
(BP)_{14} = & \left(\overset{\infty}{\mathcal{T}}_{2,11} + \left[\overset{\infty}{\mathcal{X}}_1^2 \right]_{11} - \overset{\infty}{\mathcal{X}}_{2,11} - \overset{\infty}{\mathcal{T}}_{1,11} \overset{\infty}{\mathcal{X}}_{1,11} \right) P_{14} + \left(\left[\overset{\infty}{\mathcal{X}}_1^2 \right]_{12} - \overset{\infty}{\mathcal{X}}_{2,12} - \overset{\infty}{\mathcal{T}}_{1,11} \overset{\infty}{\mathcal{X}}_{1,12} \right) P_{24} \\
& + \left(\overset{\infty}{\mathcal{T}}_{1,13} + \left[\overset{\infty}{\mathcal{X}}_1^2 \right]_{13} - \overset{\infty}{\mathcal{X}}_{2,13} - \overset{\infty}{\mathcal{T}}_{1,11} \overset{\infty}{\mathcal{X}}_{1,13} \right) P_{34} + \left(\overset{\infty}{\mathcal{T}}_{1,14} + \left[\overset{\infty}{\mathcal{X}}_1^2 \right]_{14} - \overset{\infty}{\mathcal{X}}_{2,14} - \overset{\infty}{\mathcal{T}}_{1,11} \overset{\infty}{\mathcal{X}}_{1,14} \right) P_{44} = 0, \quad (5.12)
\end{aligned}$$

$$\begin{aligned} (BP)_{21} = & \left(\bar{\mathcal{T}}_{2,21} - \bar{\mathcal{T}}_{1,21} \bar{\mathcal{X}}_{1,11} \right) P_{11} + \left(1 - \bar{\mathcal{T}}_{1,21} \bar{\mathcal{X}}_{1,12} \right) P_{21} + \left(\bar{\mathcal{T}}_{1,23} - \bar{\mathcal{T}}_{1,21} \bar{\mathcal{X}}_{1,13} \right) P_{31} \\ & + \left(\bar{\mathcal{T}}_{1,24} - \bar{\mathcal{T}}_{1,21} \bar{\mathcal{X}}_{1,14} \right) P_{41} = 0, \end{aligned} \quad (5.13)$$

$$\begin{aligned}
(BP)_{23} = & \left(\bar{\mathcal{T}}_{2,21} - \bar{\mathcal{T}}_{1,21} \bar{\mathcal{X}}_{1,11} \right) P_{13} + \left(1 - \bar{\mathcal{T}}_{1,21} \bar{\mathcal{X}}_{1,12} \right) P_{23} + \left(\bar{\mathcal{T}}_{1,23} - \bar{\mathcal{T}}_{1,21} \bar{\mathcal{X}}_{1,13} \right) P_{33} \\
& + \left(\bar{\mathcal{T}}_{1,24} - \bar{\mathcal{T}}_{1,21} \bar{\mathcal{X}}_{1,14} \right) P_{43} = 0,
\end{aligned} \tag{5.14}$$

$$\begin{aligned} (BP)_{24} = & \left(\tilde{\mathcal{T}}_{2,21} - \tilde{\mathcal{T}}_{1,21} \tilde{\mathcal{X}}_{1,11} \right) P_{14} + \left(1 - \tilde{\mathcal{T}}_{1,21} \tilde{\mathcal{X}}_{1,12} \right) P_{24} + \left(\tilde{\mathcal{T}}_{1,23} - \tilde{\mathcal{T}}_{1,21} \tilde{\mathcal{X}}_{1,13} \right) P_{34} \\ & + \left(\tilde{\mathcal{T}}_{1,24} - \tilde{\mathcal{T}}_{1,21} \tilde{\mathcal{X}}_{1,14} \right) P_{44} = 0, \end{aligned} \quad (5.15)$$

$$\begin{aligned}
(BP)_{31} = & \left(\bar{\mathcal{T}}_{2,31} - \bar{\mathcal{X}}_{1,31} - \bar{\mathcal{T}}_{1,31} \bar{\mathcal{X}}_{1,11} \right) P_{11} + \left(-\bar{\mathcal{X}}_{1,32} - \bar{\mathcal{T}}_{1,31} \bar{\mathcal{X}}_{1,12} \right) P_{21} \\
& + \left(\bar{\mathcal{T}}_{1,33} - \bar{\mathcal{X}}_{1,33} - \bar{\mathcal{T}}_{1,31} \bar{\mathcal{X}}_{1,13} \right) P_{31} + \left(\bar{\mathcal{T}}_{1,34} - \bar{\mathcal{X}}_{1,34} - \bar{\mathcal{T}}_{1,31} \bar{\mathcal{X}}_{1,14} \right) P_{41} = 0,
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
(BP)_{33} = & \left(\bar{\mathcal{T}}_{2,31} - \bar{\mathcal{X}}_{1,31} - \bar{\mathcal{T}}_{1,31} \bar{\mathcal{X}}_{1,11} \right) P_{13} + \left(-\bar{\mathcal{X}}_{1,32} - \bar{\mathcal{T}}_{1,31} \bar{\mathcal{X}}_{1,12} \right) P_{23} \\
& + \left(\bar{\mathcal{T}}_{1,33} - \bar{\mathcal{X}}_{1,33} - \bar{\mathcal{T}}_{1,31} \bar{\mathcal{X}}_{1,13} \right) P_{33} + \left(\bar{\mathcal{T}}_{1,34} - \bar{\mathcal{X}}_{1,34} - \bar{\mathcal{T}}_{1,31} \bar{\mathcal{X}}_{1,14} \right) P_{43} = 0,
\end{aligned} \tag{5.17}$$

$$(BP)_{34} = \left(\tilde{\mathcal{T}}_{2,31}^{\infty} - \tilde{\mathcal{X}}_{1,31}^{\infty} - \tilde{\mathcal{T}}_{1,31}^{\infty} \tilde{\mathcal{X}}_{1,11}^{\infty} \right) P_{14} + \left(-\tilde{\mathcal{X}}_{1,32}^{\infty} - \tilde{\mathcal{T}}_{1,31}^{\infty} \tilde{\mathcal{X}}_{1,12}^{\infty} \right) P_{24} \\ + \left(\tilde{\mathcal{T}}_{1,33}^{\infty} - \tilde{\mathcal{X}}_{1,33}^{\infty} - \tilde{\mathcal{T}}_{1,31}^{\infty} \tilde{\mathcal{X}}_{1,13}^{\infty} \right) P_{34} + \left(\tilde{\mathcal{T}}_{1,34}^{\infty} - \tilde{\mathcal{X}}_{1,34}^{\infty} - \tilde{\mathcal{T}}_{1,31}^{\infty} \tilde{\mathcal{X}}_{1,14}^{\infty} \right) P_{44} = 0, \quad (5.18)$$

$$(BP)_{41} = \left(\tilde{\mathcal{T}}_{2,41}^{\infty} - \tilde{\mathcal{X}}_{1,41}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,11}^{\infty} \right) P_{11} + \left(-\tilde{\mathcal{X}}_{1,42}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,12}^{\infty} \right) P_{21} \\ + \left(\tilde{\mathcal{T}}_{1,43}^{\infty} - \tilde{\mathcal{X}}_{1,43}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,13}^{\infty} \right) P_{31} + \left(\tilde{\mathcal{T}}_{1,44}^{\infty} - \tilde{\mathcal{X}}_{1,44}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,14}^{\infty} \right) P_{41} = 0, \quad (5.19)$$

$$(BP)_{43} = \left(\tilde{\mathcal{T}}_{2,41}^{\infty} - \tilde{\mathcal{X}}_{1,41}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,11}^{\infty} \right) P_{13} + \left(-\tilde{\mathcal{X}}_{1,42}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,12}^{\infty} \right) P_{23} \\ + \left(\tilde{\mathcal{T}}_{1,43}^{\infty} - \tilde{\mathcal{X}}_{1,43}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,13}^{\infty} \right) P_{33} + \left(\tilde{\mathcal{T}}_{1,44}^{\infty} - \tilde{\mathcal{X}}_{1,44}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,14}^{\infty} \right) P_{43} = 0, \quad (5.20)$$

$$(BP)_{44} = \left(\tilde{\mathcal{T}}_{2,41}^{\infty} - \tilde{\mathcal{X}}_{1,41}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,11}^{\infty} \right) P_{14} + \left(-\tilde{\mathcal{X}}_{1,42}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,12}^{\infty} \right) P_{24} \\ + \left(\tilde{\mathcal{T}}_{1,43}^{\infty} - \tilde{\mathcal{X}}_{1,43}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,13}^{\infty} \right) P_{34} + \left(\tilde{\mathcal{T}}_{1,44}^{\infty} - \tilde{\mathcal{X}}_{1,44}^{\infty} - \tilde{\mathcal{T}}_{1,41}^{\infty} \tilde{\mathcal{X}}_{1,14}^{\infty} \right) P_{44} = 0. \quad (5.21)$$

The complementary equations come from setting the first column of (5.9) equal to zero, which are

$$\mathcal{A}_{11}B_{11} + \mathcal{A}_{21}B_{12} + \mathcal{A}_{31}B_{13} + \mathcal{A}_{41}B_{14} + P_{11} \left(\tilde{\mathcal{T}}_{1,11}^{\infty} - \tilde{\mathcal{X}}_{1,11}^{\infty} \right) - \tilde{\mathcal{X}}_{1,12}^{\infty}P_{21} - \tilde{\mathcal{X}}_{1,13}^{\infty}P_{31} - \tilde{\mathcal{X}}_{1,14}^{\infty}P_{41} = 0, \quad (5.22)$$

$$\mathcal{A}_{11}B_{21} + \mathcal{A}_{21}B_{22} + \mathcal{A}_{31}B_{23} + \mathcal{A}_{41}B_{24} + P_{11}\tilde{\mathcal{T}}_{1,21}^{\infty} = 0, \quad (5.23)$$

$$\mathcal{A}_{11}B_{31} + \mathcal{A}_{21}B_{32} + \mathcal{A}_{31}B_{33} + \mathcal{A}_{41}B_{34} + P_{11}\tilde{\mathcal{T}}_{1,31}^{\infty} + P_{31} = 0, \quad (5.24)$$

and

$$\mathcal{A}_{11}B_{41} + \mathcal{A}_{21}B_{42} + \mathcal{A}_{31}B_{43} + \mathcal{A}_{41}B_{44} + P_{11}\tilde{\mathcal{T}}_{1,41}^{\infty} + P_{41} = 0, \quad (5.25)$$

where $\mathcal{A} \equiv \mathring{P}\tilde{\mathcal{X}}_1$. Using the explicit expression for B_{jk} in (5.6) we can write the last four equations as

$$\mathcal{A}_{11} \left(\tilde{\mathcal{T}}_{2,11}^{\infty} + \left[\tilde{\mathcal{X}}_1^2 \right]_{11}^{\infty} - \tilde{\mathcal{X}}_{2,11}^{\infty} - \tilde{\mathcal{T}}_{1,11}^{\infty} \tilde{\mathcal{X}}_{1,11}^{\infty} \right) + \mathcal{A}_{21} \left(\left[\tilde{\mathcal{X}}_1^2 \right]_{12}^{\infty} - \tilde{\mathcal{X}}_{2,12}^{\infty} - \tilde{\mathcal{T}}_{1,11}^{\infty} \tilde{\mathcal{X}}_{1,12}^{\infty} \right) \\ + \mathcal{A}_{31} \left(\tilde{\mathcal{T}}_{1,13}^{\infty} + \left[\tilde{\mathcal{X}}_1^2 \right]_{13}^{\infty} - \tilde{\mathcal{X}}_{2,13}^{\infty} - \tilde{\mathcal{T}}_{1,11}^{\infty} \tilde{\mathcal{X}}_{1,13}^{\infty} \right) + \mathcal{A}_{41} \left(\tilde{\mathcal{T}}_{1,14}^{\infty} + \left[\tilde{\mathcal{X}}_1^2 \right]_{14}^{\infty} - \tilde{\mathcal{X}}_{2,14}^{\infty} - \tilde{\mathcal{T}}_{1,11}^{\infty} \tilde{\mathcal{X}}_{1,14}^{\infty} \right) \\ + P_{11} \left(\tilde{\mathcal{T}}_{1,11}^{\infty} - \tilde{\mathcal{X}}_{1,11}^{\infty} \right) - \tilde{\mathcal{X}}_{1,12}^{\infty}P_{21} - \tilde{\mathcal{X}}_{1,13}^{\infty}P_{31} - \tilde{\mathcal{X}}_{1,14}^{\infty}P_{41} = 0, \quad (5.26)$$

$$\mathcal{A}_{11} \left(\tilde{\mathcal{T}}_{2,21}^{\infty} - \tilde{\mathcal{T}}_{1,21}^{\infty} \tilde{\mathcal{X}}_{1,11}^{\infty} \right) + \mathcal{A}_{21} \left(1 - \tilde{\mathcal{T}}_{1,21}^{\infty} \tilde{\mathcal{X}}_{1,12}^{\infty} \right) + \mathcal{A}_{31} \left(\tilde{\mathcal{T}}_{1,23}^{\infty} - \tilde{\mathcal{T}}_{1,21}^{\infty} \tilde{\mathcal{X}}_{1,13}^{\infty} \right) \\ + \mathcal{A}_{41} \left(\tilde{\mathcal{T}}_{1,24}^{\infty} - \tilde{\mathcal{T}}_{1,21}^{\infty} \tilde{\mathcal{X}}_{1,14}^{\infty} \right) + P_{11}\tilde{\mathcal{T}}_{1,21}^{\infty} = 0, \quad (5.27)$$

$$\mathcal{A}_{11} \left(\tilde{\mathcal{T}}_{2,31}^{\infty} - \tilde{\mathcal{X}}_{1,31}^{\infty} - \tilde{\mathcal{T}}_{1,31}^{\infty} \tilde{\mathcal{X}}_{1,11}^{\infty} \right) + \mathcal{A}_{21} \left(-\tilde{\mathcal{X}}_{1,32}^{\infty} - \tilde{\mathcal{T}}_{1,31}^{\infty} \tilde{\mathcal{X}}_{1,12}^{\infty} \right) \\ + \mathcal{A}_{31} \left(\tilde{\mathcal{T}}_{1,33}^{\infty} - \tilde{\mathcal{X}}_{1,33}^{\infty} - \tilde{\mathcal{T}}_{1,31}^{\infty} \tilde{\mathcal{X}}_{1,13}^{\infty} \right) + \mathcal{A}_{41} \left(\tilde{\mathcal{T}}_{1,34}^{\infty} - \tilde{\mathcal{X}}_{1,34}^{\infty} - \tilde{\mathcal{T}}_{1,31}^{\infty} \tilde{\mathcal{X}}_{1,14}^{\infty} \right) + P_{11}\tilde{\mathcal{T}}_{1,31}^{\infty} + P_{31} = 0, \quad (5.28)$$

$$\begin{aligned} & \mathcal{A}_{11} \left(\bar{\mathcal{T}}_{2,41} - \bar{\mathcal{X}}_{1,41} - \bar{\mathcal{T}}_{1,41} \bar{\mathcal{X}}_{1,11} \right) + \mathcal{A}_{21} \left(-\bar{\mathcal{X}}_{1,42} - \bar{\mathcal{T}}_{1,41} \bar{\mathcal{X}}_{1,12} \right) + \\ & \mathcal{A}_{31} \left(\bar{\mathcal{T}}_{1,43} - \bar{\mathcal{X}}_{1,43} - \bar{\mathcal{T}}_{1,41} \bar{\mathcal{X}}_{1,13} \right) + \mathcal{A}_{41} \left(\bar{\mathcal{T}}_{1,44} - \bar{\mathcal{X}}_{1,44} - \bar{\mathcal{T}}_{1,41} \bar{\mathcal{X}}_{1,14} \right) + P_{11} \bar{\mathcal{T}}_{1,41} + P_{41} = 0. \end{aligned} \quad (5.29)$$

Now we observe that the sixteen equations (5.10)–(5.21) and (5.26)–(5.29) can be split into four sets of mutually decoupled equations, where solving each set of equations gives four unknowns in

$$\left\{ \bar{\mathcal{T}}_{1,j1}, \bar{\mathcal{T}}_{1,j3}, \bar{\mathcal{T}}_{1,j4}, \bar{\mathcal{T}}_{2,j1} \right\}_{j=1}^4.$$

To this end, we categorize these equations below.

- The four equations (5.10)–(5.12) and (5.26) are in four unknowns $\bar{\mathcal{T}}_{2,11}, \bar{\mathcal{T}}_{1,11}, \bar{\mathcal{T}}_{1,13}, \bar{\mathcal{T}}_{1,14}$, which can be written as

$$\begin{pmatrix} P_{11} & -\mathcal{B}_{11} & P_{31} & P_{41} \\ P_{13} & -\mathcal{B}_{13} & P_{33} & P_{43} \\ P_{14} & -\mathcal{B}_{14} & P_{34} & P_{44} \\ \mathcal{A}_{11} & \mathcal{D}_{11} & \mathcal{A}_{31} & \mathcal{A}_{41} \end{pmatrix} \begin{pmatrix} \bar{\mathcal{T}}_{2,11} \\ \bar{\mathcal{T}}_{1,11} \\ \bar{\mathcal{T}}_{1,13} \\ \bar{\mathcal{T}}_{1,14} \end{pmatrix} = \begin{pmatrix} [\mathcal{CP}]_{11} \\ [\mathcal{CP}]_{13} \\ [\mathcal{CP}]_{14} \\ [\mathcal{CA} + \mathcal{B}]_{11} \end{pmatrix}, \quad (5.30)$$

where for simplicity of notations we have introduced

$$\mathcal{A} \equiv P\bar{\mathcal{X}}_1, \quad \mathcal{B} \equiv \bar{\mathcal{X}}_1 P, \quad \mathcal{C} \equiv \bar{\mathcal{X}}_2 - \bar{\mathcal{X}}_1^2, \quad \text{and} \quad \mathcal{D} \equiv P - \bar{\mathcal{X}}_1 \mathcal{A}. \quad (5.31)$$

- The four equations (5.13)–(5.15) and (5.27) are in four unknowns $\bar{\mathcal{T}}_{2,21}, \bar{\mathcal{T}}_{1,21}, \bar{\mathcal{T}}_{1,23}, \bar{\mathcal{T}}_{1,24}$, which can be written as

$$\begin{pmatrix} P_{11} & -\mathcal{B}_{11} & P_{31} & P_{41} \\ P_{13} & -\mathcal{B}_{13} & P_{33} & P_{43} \\ P_{14} & -\mathcal{B}_{14} & P_{34} & P_{44} \\ \mathcal{A}_{11} & \mathcal{D}_{11} & \mathcal{A}_{31} & \mathcal{A}_{41} \end{pmatrix} \begin{pmatrix} \bar{\mathcal{T}}_{2,21} \\ \bar{\mathcal{T}}_{1,21} \\ \bar{\mathcal{T}}_{1,23} \\ \bar{\mathcal{T}}_{1,24} \end{pmatrix} = \begin{pmatrix} -P_{21} \\ -P_{23} \\ -P_{24} \\ -\mathcal{A}_{21} \end{pmatrix}. \quad (5.32)$$

- The four equations (5.16)–(5.18) and (5.28) are in four unknowns $\bar{\mathcal{T}}_{2,31}, \bar{\mathcal{T}}_{1,31}, \bar{\mathcal{T}}_{1,33}, \bar{\mathcal{T}}_{1,34}$, which can be written as

$$\begin{pmatrix} P_{11} & -\mathcal{B}_{11} & P_{31} & P_{41} \\ P_{13} & -\mathcal{B}_{13} & P_{33} & P_{43} \\ P_{14} & -\mathcal{B}_{14} & P_{34} & P_{44} \\ \mathcal{A}_{11} & \mathcal{D}_{11} & \mathcal{A}_{31} & \mathcal{A}_{41} \end{pmatrix} \begin{pmatrix} \bar{\mathcal{T}}_{2,31} \\ \bar{\mathcal{T}}_{1,31} \\ \bar{\mathcal{T}}_{1,33} \\ \bar{\mathcal{T}}_{1,34} \end{pmatrix} = \begin{pmatrix} \mathcal{B}_{31} \\ \mathcal{B}_{33} \\ \mathcal{B}_{34} \\ -\mathcal{D}_{31} \end{pmatrix}. \quad (5.33)$$

- The four equations (5.19)–(5.21) and (5.29) are in four unknowns $\bar{\mathcal{T}}_{2,41}, \bar{\mathcal{T}}_{1,41}, \bar{\mathcal{T}}_{1,43}, \bar{\mathcal{T}}_{1,44}$, which can be written as

$$\begin{pmatrix} P_{11} & -\mathcal{B}_{11} & P_{31} & P_{41} \\ P_{13} & -\mathcal{B}_{13} & P_{33} & P_{43} \\ P_{14} & -\mathcal{B}_{14} & P_{34} & P_{44} \\ \mathcal{A}_{11} & \mathcal{D}_{11} & \mathcal{A}_{31} & \mathcal{A}_{41} \end{pmatrix} \begin{pmatrix} \bar{\mathcal{T}}_{2,41} \\ \bar{\mathcal{T}}_{1,41} \\ \bar{\mathcal{T}}_{1,43} \\ \bar{\mathcal{T}}_{1,44} \end{pmatrix} = \begin{pmatrix} \mathcal{B}_{41} \\ \mathcal{B}_{43} \\ \mathcal{B}_{44} \\ -\mathcal{D}_{41} \end{pmatrix}. \quad (5.34)$$

We see that all four linear systems above have the same matrix coefficient. So we try to invert the following linear system

$$\begin{pmatrix} P_{11} & -\mathcal{B}_{11} & P_{31} & P_{41} \\ P_{13} & -\mathcal{B}_{13} & P_{33} & P_{43} \\ P_{14} & -\mathcal{B}_{14} & P_{34} & P_{44} \\ \mathcal{A}_{11} & \mathcal{D}_{11} & \mathcal{A}_{31} & \mathcal{A}_{41} \end{pmatrix} \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \\ \mathcal{F}_4 \end{pmatrix} = \begin{pmatrix} x \\ y \\ w \\ z \end{pmatrix}, \quad (5.35)$$

and then find the desired unknowns by replacing $(x \ y \ w \ z)^T$ by the appropriate right hand side in (5.30), (5.32) – (5.34). Now, we introduce the objects (see the formulation of theorem 2.3)

$$\begin{aligned} \alpha &:= \left(\frac{\mathcal{A}_{11}\mathcal{B}_{11}}{P_{11}} + \mathcal{D}_{11} \right)^{-1}, & \theta &:= \frac{\mathcal{A}_{11}}{P_{11}}, & \rho_j &:= \mathcal{A}_{j1} - \frac{\mathcal{A}_{11}P_{j1}}{P_{11}}, \\ \omega_{jk} &:= \frac{P_{1j}P_{k1}}{P_{11}}, & \eta_j &:= \frac{P_{1j}}{P_{11}}, & \nu_j &:= \frac{\mathcal{B}_{11}P_{1j}}{P_{11}} - \mathcal{B}_{1j}, \end{aligned}$$

assuming that they are well defined. Using these objects define

$$\mathcal{M}_{jk} = -\alpha\rho_k\nu_j - \omega_{jk} + P_{kj}, f_j(x, y, z) = \alpha\nu_j(z - \theta x) + \eta_j x - y,$$

and

$$\Delta := \mathcal{M}_{34}\mathcal{M}_{43} - \mathcal{M}_{33}\mathcal{M}_{44}, \quad (5.36)$$

and it is assumed that $\Delta \neq 0$. Then, inverting (5.35) yields

$$\begin{aligned} \mathcal{F}_1(x, y, w, z) &= \frac{x}{P_{11}} + \left(\left(\frac{P_{41} + \alpha\mathcal{B}_{11}\rho_4}{P_{11}\Delta} \right) \mathcal{M}_{43} - \left(\frac{P_{31} + \alpha\mathcal{B}_{11}\rho_3}{P_{11}\Delta} \right) \mathcal{M}_{44} \right) f_3(x, y, z) \\ &\quad + \left(\left(\frac{P_{31} + \alpha\mathcal{B}_{11}\rho_3}{P_{11}\Delta} \right) \mathcal{M}_{34} - \left(\frac{P_{41} + \alpha\mathcal{B}_{11}\rho_4}{P_{11}\Delta} \right) \mathcal{M}_{33} \right) f_4(x, w, z) + \frac{\alpha\mathcal{B}_{11}}{P_{11}}(z - \theta x), \end{aligned} \quad (5.37)$$

$$\mathcal{F}_2(x, y, w, z) = \alpha \left(z - \theta x + \left(\frac{\rho_4\mathcal{M}_{43} - \rho_3\mathcal{M}_{44}}{\Delta} \right) f_3(x, y, z) + \left(\frac{\rho_3\mathcal{M}_{34} - \rho_4\mathcal{M}_{33}}{\Delta} \right) f_4(x, w, z) \right), \quad (5.38)$$

$$\mathcal{F}_3(x, y, w, z) = \frac{1}{\Delta} \left(\frac{1}{\mathcal{M}_{43}} f_3(x, y, z) - \mathcal{M}_{34} f_4(x, w, z) \right), \quad (5.39)$$

$$\mathcal{F}_4(x, y, w, z) = \frac{\mathcal{M}_{33}f_4(x, w, z) - f_3(x, y, z)}{\Delta}. \quad (5.40)$$

These are exactly the four functions that appear in theorem 2.3. This finishes the proof of this theorem.

Proof of theorem 2.4

From (1.19) and (2.29) one can directly check that

$$WJ_X^{-1}(z^{-1}; r, s)W = J_X(z; r, s), \quad (6.1)$$

which is part (a) of theorem 2.4. To prove the other two parts, let us consider the function

$$B(z; n, r, s) := WP^{-1}(n, r, s)X(z^{-1}; n, r, s)W. \quad (6.2)$$

We have

$$B_-^{-1}(z; n, r, s)B_+(z; n, r, s) = W(X(z^{-1}; n, r, s))_-^{-1}(X(z^{-1}; n, r, s))_+ W. \quad (6.3)$$

So

$$\begin{aligned} B_-^{-1}(z; n, r, s)B_+(z; n, r, s) &= W \left(\lim_{\zeta \rightarrow z} X^{-1}(\zeta^{-1}; n, r, s) \right) \left(\lim_{\zeta \rightarrow z} X(\zeta^{-1}; n, r, s) \right) W \\ &= W \left(\lim_{\zeta^{-1} \rightarrow z^{-1}} X^{-1}(\zeta^{-1}; n, r, s) \right) \left(\lim_{\zeta^{-1} \rightarrow z^{-1}} X(\zeta^{-1}; n, r, s) \right) W \\ &= W \left(\lim_{\tau \rightarrow z^{-1}} X^{-1}(\tau; n, r, s) \right) \left(\lim_{\tau \rightarrow z^{-1}} X(\tau; n, r, s) \right) W \\ &= WX_+^{-1}(z^{-1}; n, r, s)X_-(z^{-1}; n, r, s)W \\ &= WJ_X^{-1}(z^{-1}; r, s)W. \end{aligned} \quad (6.4)$$

Therefore, from (6.1) we have

$$B_+(z; n, r, s) = B_-(z; n, r, s)J_X(z; r, s). \quad (6.5)$$

Let us now remind the asymptotic behavior of $X(z; n, r, s)$ near zero and infinity with the following notations for the subleading terms:

- **RH-X3** As $z \rightarrow \infty$

$$X(z; n, r, s) = \left(I + \frac{\tilde{X}_1(n, r, s)}{z} + \frac{\tilde{X}_2(n, r, s)}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH-X4** As $z \rightarrow 0$

$$X(z; n, r, s) = P(n, r, s) \left(I + \tilde{X}_1(n, r, s)z + \tilde{X}_2(n, r, s)z^2 + O(z^3) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}.$$

Let us consider the asymptotic behavior of B as $z \rightarrow \infty$:

$$\begin{aligned} B(z; n, r, s) &= WP^{-1}(n, r, s)X(z^{-1}; n, r, s)W \\ &= W \left(I + \tilde{X}_1(n, r, s)z^{-1} + O(z^{-2}) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix} W \\ &= W \left(I + \tilde{X}_1(n, r, s)z^{-1} + O(z^{-2}) \right) WW \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix} W \\ &= \left(I + W\tilde{X}_1(n, r, s)Wz^{-1} + O(z^{-2}) \right) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \quad (6.6)$$

where we have used $W^2 = I$. Next, consider the asymptotic behavior of B as $z \rightarrow 0$:

$$\begin{aligned} B(z; n, r, s) &= WP^{-1}(n, r, s)X(z^{-1}; n, r, s)W \\ &= WP^{-1}(n, r, s) \left(I + \tilde{X}_1(n, r, s)z + O(z) \right) \begin{pmatrix} z^{-n} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} W \\ &= WP^{-1}(n, r, s)WW \left(I + \tilde{X}_1(n, r, s)z + O(z) \right) WW \begin{pmatrix} z^{-n} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} W \\ &= WP^{-1}(n, r, s)W \left(I + W\tilde{X}_1(n, r, s)Wz + O(z) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}. \end{aligned} \quad (6.7)$$

By definition (6.2), B is analytic in $\mathbb{C} \setminus (\mathbb{T} \cup \{0\})$. This, together with (6.5)–(6.7) and lemma 1.2 gives

$$B(z; n, r, s) \equiv X(z; n, r, s), \quad (6.8)$$

and as a result we immediately confirm parts (b) and (c) of theorem 2.4:

$$P(n, r, s) = WP^{-1}(n, r, s)W, \quad (6.9)$$

and

$$\overset{\circ}{X}_1(n, r, s) = W \overset{\circ}{X}_1^\infty(n, r, s) W. \quad (6.10)$$

7. Asymptotics of the norms of the orthogonal polynomials $\mathcal{P}_n(z; 0, 1)$

In this section we illustrate how one can use theorem 2.4 and the expression of $\mathcal{U}$ in terms of $\mathcal{X}$, as given in theorem 2.1, to find the large- n asymptotics of the norm $h_{n-1}^{(0,1)}$. This is a case study as we focus on $(r, s) = (0, 1)$, but we would like to highlight that the same procedure can be followed for other choices of (r, s) once we have expressions for $X(z; n, r, s)$ in terms of $\mathcal{X}(z; n)$, such as those in theorems 2.2 and 2.3. What is presented below partially follows section 4.2 of [25]. Let us start with formula (1.14) for the norm $h_{n-1}^{(0,1)}$ which can be rewritten as

$$\frac{-1}{h_{n-1}^{(0,1)}} = \lim_{z \rightarrow 0} z^{n-1} \mathcal{Y}_{21}(z^{-1}; n, 0, 1). \quad (7.1)$$

In view of RH- $\mathcal{U}4$ let us define

$$\mathcal{H}(z; n) := \widehat{\mathcal{U}}(n)^{-1} \mathcal{U}(z; n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix}, \quad (7.2)$$

which implies

$$\mathcal{H}(z; n) = I + \overset{\circ}{\mathcal{U}}_1(n)z + \overset{\circ}{\mathcal{U}}_2(n)z^2 + O(z^3), \quad \text{as } z \rightarrow 0. \quad (7.3)$$

From (7.2) and (1.20)–(1.22) and remembering that $\mathcal{U}(z; n) = X(z; n, 0, 1)$ we have

$$\widehat{X}(z; n, 0, 1) = \begin{pmatrix} C_1(n, 0, 1) & 1 & C_3(n, 0, 1) & 0 \\ C_2(n, 0, 1) & 0 & C_4(n, 0, 1) & 1 \end{pmatrix} \mathcal{H}(z; n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}, \quad (7.4)$$

recalling that, by definition, $\widehat{\mathcal{U}}(n) = P(n, 0, 1)$. From (1.15) it follows that

$$\mathcal{Y}_{21}(z^{-1}; n, 0, 1) = \widehat{X}_{22}(z; n, 0, 1), \quad (7.5)$$

hence,

$$z^{n-1} \mathcal{Y}_{21}(z^{-1}; n, 0, 1) = C_2(n, 0, 1) z^{-1} \mathcal{H}_{12}(z; n) + C_4(n, 0, 1) z^{-1} \mathcal{H}_{32}(z; n) + z^{-1} \mathcal{H}_{42}(z; n). \quad (7.6)$$

From (7.3) we conclude that

$$z^{-1} \mathcal{H}_{j2}(z; n) = \overset{\circ}{\mathcal{U}}_{1,j2}(n) + O(z), \quad j = 1, 3, 4. \quad (7.7)$$

Therefore, (7.1) yields the formula,

$$\frac{-1}{h_{n-1}^{(0,1)}} = C_2(n, 0, 1) \overset{\circ}{\mathcal{U}}_{1,12}(n) + C_4(n, 0, 1) \overset{\circ}{\mathcal{U}}_{1,32}(n) + \overset{\circ}{\mathcal{U}}_{1,42}(n). \quad (7.8)$$

7.1. $h_{n-1}^{(0,1)}$ and the $\mathcal{X}$ -RHP data

In order to find the asymptotics of $h_{n-1}^{(0,1)}$, we need to find the expressions for the objects on the right hand side of (7.8) in terms of the $\mathcal{X}$ -RHP data. From part (c) of theorem 2.4 we have $\overset{\circ}{\mathcal{U}}_1(n) = W \overset{\circ}{\mathcal{U}}_1(n) W$, which can be written as

$$\begin{pmatrix} \overset{\circ}{\mathcal{U}}_{1,11} & \overset{\circ}{\mathcal{U}}_{1,12} & \overset{\circ}{\mathcal{U}}_{1,13} & \overset{\circ}{\mathcal{U}}_{1,14} \\ \overset{\circ}{\mathcal{U}}_{1,21} & \overset{\circ}{\mathcal{U}}_{1,22} & \overset{\circ}{\mathcal{U}}_{1,23} & \overset{\circ}{\mathcal{U}}_{1,24} \\ \overset{\circ}{\mathcal{U}}_{1,31} & \overset{\circ}{\mathcal{U}}_{1,32} & \overset{\circ}{\mathcal{U}}_{1,33} & \overset{\circ}{\mathcal{U}}_{1,34} \\ \overset{\circ}{\mathcal{U}}_{1,41} & \overset{\circ}{\mathcal{U}}_{1,42} & \overset{\circ}{\mathcal{U}}_{1,43} & \overset{\circ}{\mathcal{U}}_{1,44} \end{pmatrix} = \begin{pmatrix} \overset{\circ}{\mathcal{U}}_{1,22} & \overset{\circ}{\mathcal{U}}_{1,21} & \overset{\circ}{\mathcal{U}}_{1,24} & \overset{\circ}{\mathcal{U}}_{1,23} \\ \overset{\circ}{\mathcal{U}}_{1,12} & \overset{\circ}{\mathcal{U}}_{1,11} & \overset{\circ}{\mathcal{U}}_{1,14} & \overset{\circ}{\mathcal{U}}_{1,13} \\ \overset{\circ}{\mathcal{U}}_{1,42} & \overset{\circ}{\mathcal{U}}_{1,41} & \overset{\circ}{\mathcal{U}}_{1,44} & \overset{\circ}{\mathcal{U}}_{1,43} \\ \overset{\circ}{\mathcal{U}}_{1,32} & \overset{\circ}{\mathcal{U}}_{1,31} & \overset{\circ}{\mathcal{U}}_{1,34} & \overset{\circ}{\mathcal{U}}_{1,33} \end{pmatrix}. \quad (7.9)$$

So we have

$$\overset{\circ}{\mathcal{U}}_{1,12} = \overset{\infty}{\mathcal{U}}_{1,21} = \frac{P_{31}P_{23} - P_{33}P_{21}}{P_{11}P_{33} - P_{13}P_{31}}, \quad (7.10)$$

$$\overset{\circ}{\mathcal{U}}_{1,32} = \overset{\infty}{\mathcal{U}}_{1,41} = \frac{P_{43}P_{31} - P_{33}P_{41}}{P_{11}P_{33} - P_{13}P_{31}}, \quad (7.11)$$

$$\overset{\circ}{\mathcal{U}}_{1,42} = \overset{\infty}{\mathcal{U}}_{1,31} = \overset{\infty}{\mathcal{X}}_{1,31} + \frac{P_{33} \sum_{j \in \{2,4\}} \overset{\infty}{\mathcal{X}}_{1,3j}P_{j1} - P_{31} \sum_{j \in \{2,4\}} \overset{\infty}{\mathcal{X}}_{1,3j}P_{j3}}{P_{11}P_{33} - P_{13}P_{31}}, \quad (7.12)$$

where we have used theorem 2.1.

Now move on to find C_2 and C_4 in terms of the $\mathcal{X}$ -RHP data. We are not going to study each and every one of conditions (1.23) through (1.28) of lemma 1.4, rather like what is presented in [25] we consider⁶ the condition (1.24):

$$(1 - \widehat{\mathcal{U}}_{21}(n))\widehat{\mathcal{U}}_{42}(n) + \widehat{\mathcal{U}}_{22}(n)\widehat{\mathcal{U}}_{41}(n) \neq 0. \quad (7.13)$$

Then, from (1.22) we can find:

$$C_2(n, 0, 1) = \frac{\widehat{\mathcal{U}}_{42}(n)\widehat{\mathcal{U}}_{31}(n) - \widehat{\mathcal{U}}_{41}(n)\widehat{\mathcal{U}}_{32}(n)}{(1 - \widehat{\mathcal{U}}_{21}(n))\widehat{\mathcal{U}}_{42}(n) + \widehat{\mathcal{U}}_{41}(n)\widehat{\mathcal{U}}_{22}(n)}, \quad (7.14)$$

and

$$C_4(n, 0, 1) = -\frac{\widehat{\mathcal{U}}_{22}(n)\widehat{\mathcal{U}}_{31}(n) + [1 - \widehat{\mathcal{U}}_{21}(n)]\widehat{\mathcal{U}}_{32}(n)}{(1 - \widehat{\mathcal{U}}_{21}(n))\widehat{\mathcal{U}}_{42}(n) + \widehat{\mathcal{U}}_{41}(n)\widehat{\mathcal{U}}_{22}(n)}. \quad (7.15)$$

Let us recall from theorem 2.1 that

$$\mathcal{U}(z) = R(z)\mathcal{X}(z) \begin{pmatrix} z^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (7.16)$$

where $R(z) = A + z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ with

$$A = \begin{pmatrix} \overset{\infty}{\mathcal{U}}_{1,11} - \overset{\infty}{\mathcal{X}}_{1,11} & -\overset{\infty}{\mathcal{X}}_{1,12} & \overset{\infty}{\mathcal{U}}_{1,13} - \overset{\infty}{\mathcal{X}}_{1,13} & -\overset{\infty}{\mathcal{X}}_{1,14} \\ \overset{\infty}{\mathcal{U}}_{1,21} & 1 & \overset{\infty}{\mathcal{U}}_{1,23} & 0 \\ \overset{\infty}{\mathcal{U}}_{1,31} - \overset{\infty}{\mathcal{X}}_{1,31} & -\overset{\infty}{\mathcal{X}}_{1,32} & \overset{\infty}{\mathcal{U}}_{1,33} - \overset{\infty}{\mathcal{X}}_{1,33} & -\overset{\infty}{\mathcal{X}}_{1,34} \\ \overset{\infty}{\mathcal{U}}_{1,41} & 0 & \overset{\infty}{\mathcal{U}}_{1,43} & 1 \end{pmatrix}. \quad (7.17)$$

From (7.16) and **RH- $\mathcal{X}4$** we have

$$\mathcal{U}(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix} = \alpha_n z^{-1} + \beta_n + O(z), \quad \text{as } z \rightarrow 0, \quad (7.18)$$

where

$$\alpha_n = AP(n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (7.19)$$

⁶ In section 7.4 we impose conditions on the symbols to ensure this; see (7.35) and (7.95)–(7.98), as well as the assumptions in theorem 2.6.

and

$$\beta_n = AP(n) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + AP(n) \overset{\circ}{\mathcal{X}}_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P(n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (7.20)$$

Notice that by (3.10) we have

$$\alpha_n = 0. \quad (7.21)$$

Comparing (7.18) with **RH-U4** gives

$$\beta_n \equiv \widehat{\mathcal{U}}. \quad (7.22)$$

Using (7.21) we can simplify the second term on the right hand side of (7.20), indeed

$$\begin{aligned} AP(n) \overset{\circ}{\mathcal{X}}_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &= AP(n) \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] \overset{\circ}{\mathcal{X}}_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= AP(n) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{\circ}{\mathcal{X}}_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= AP(n) \begin{pmatrix} 0 & 0 & 0 & 0 \\ \overset{\circ}{\mathcal{X}}_{1,21} & 0 & \overset{\circ}{\mathcal{X}}_{1,23} & 0 \\ 0 & 0 & 0 & 0 \\ \overset{\circ}{\mathcal{X}}_{1,41} & 0 & \overset{\circ}{\mathcal{X}}_{1,43} & 0 \end{pmatrix}. \end{aligned} \quad (7.23)$$

Combining this, with (7.20) and (7.22) and after some simplifications we find

$$\widehat{\mathcal{U}} = AP(n) \begin{pmatrix} 0 & 0 & 0 & 0 \\ \overset{\circ}{\mathcal{X}}_{1,21} & 1 & \overset{\circ}{\mathcal{X}}_{1,23} & 0 \\ 0 & 0 & 0 & 0 \\ \overset{\circ}{\mathcal{X}}_{1,41} & 0 & \overset{\circ}{\mathcal{X}}_{1,43} & 1 \end{pmatrix} + \begin{pmatrix} P_{11} & 0 & P_{13} & 0 \\ 0 & 0 & 0 & 0 \\ P_{31} & 0 & P_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (7.24)$$

We therefore find

$$\widehat{\mathcal{U}}_{21} = \overset{\infty}{\mathcal{X}}_{1,12} (A_{21}P_{12} + A_{23}P_{32} + P_{22}) + \overset{\infty}{\mathcal{X}}_{1,32} (A_{21}P_{14} + A_{23}P_{34} + P_{24}), \quad (7.25)$$

$$\widehat{\mathcal{U}}_{42} = A_{41}P_{12} + A_{43}P_{32} + P_{42}, \quad (7.26)$$

$$\widehat{\mathcal{U}}_{41} = \overset{\infty}{\mathcal{X}}_{1,12} (A_{41}P_{12} + A_{43}P_{32} + P_{42}) + \overset{\infty}{\mathcal{X}}_{1,32} (A_{41}P_{14} + A_{43}P_{34} + P_{44}), \quad (7.27)$$

$$\widehat{\mathcal{U}}_{22} = A_{21}P_{12} + A_{23}P_{32} + P_{22}, \quad (7.28)$$

$$\widehat{\mathcal{U}}_{32} = A_{31}P_{12} + A_{32}P_{22} + A_{33}P_{32} + A_{34}P_{42}, \quad (7.29)$$

$$\begin{aligned} \widehat{\mathcal{U}}_{31} &= \overset{\infty}{\mathcal{X}}_{1,12} (A_{31}P_{12} + A_{32}P_{22} + A_{33}P_{32} + A_{34}P_{42}) \\ &\quad + \overset{\infty}{\mathcal{X}}_{1,32} (A_{31}P_{14} + A_{32}P_{24} + A_{33}P_{34} + A_{34}P_{44}) + P_{31}, \end{aligned} \quad (7.30)$$

where we have used that $\overset{\circ}{\mathcal{X}}_{1,21} = \overset{\infty}{\mathcal{X}}_{1,12}$ and $\overset{\circ}{\mathcal{X}}_{1,41} = \overset{\infty}{\mathcal{X}}_{1,32}$ using part (c) of theorem 2.4. Below, we recall the expressions for the relevant entries of the matrix A in terms of the $\mathcal{X}$ -RHP data, using notations that simplify the resulting formulas.

To streamline the expressions, we define the determinant of 2×2 minors of $P(n)$ as follows:

$$D_{jk}^{rs} := P_{jk}P_{rs} - P_{js}P_{rk}, \quad 1 \leq j < r \leq 4, \quad 1 \leq k < s \leq 4. \quad (7.31)$$

Assuming the generic condition, $D_{11}^{33} \neq 0$ (see note that this is the condition we assumed in theorem 2.1), we have

$$\begin{aligned} A_{21} &= -\frac{D_{21}^{33}}{D_{11}^{33}}, & A_{41} &= \frac{D_{31}^{43}}{D_{11}^{33}}, & A_{32} &= -\mathcal{X}_{1,32}^\infty, & A_{31} &= \frac{\mathcal{X}_{1,32}^\infty D_{21}^{33} - \mathcal{X}_{1,34}^\infty D_{31}^{43}}{D_{11}^{33}}, \\ A_{23} &= -\frac{D_{11}^{23}}{D_{11}^{33}}, & A_{43} &= -\frac{D_{11}^{43}}{D_{11}^{33}}, & A_{34} &= -\mathcal{X}_{1,34}^\infty, & A_{33} &= \frac{\mathcal{X}_{1,32}^\infty D_{11}^{23} + \mathcal{X}_{1,34}^\infty D_{11}^{43}}{D_{11}^{33}}. \end{aligned}$$

Using (7.17) and theorem 2.1 we find the following expression for the numerator of $C_2(n)$:

$$\begin{aligned} \widehat{\mathcal{U}}_{42}(n)\widehat{\mathcal{U}}_{31}(n) - \widehat{\mathcal{U}}_{41}(n)\widehat{\mathcal{U}}_{32}(n) &= \frac{-1}{D_{11}^{33}} [P_{31} (P_{41}D_{12}^{33} - P_{42}D_{11}^{33} + P_{43}D_{11}^{32})] - \frac{\left(\mathcal{X}_{1,32}^\infty\right)^2}{D_{11}^{33}} \left[D_{11}^{32}D_{23}^{44} \right. \\ &\quad \left. - D_{23}^{34}D_{11}^{42} - D_{11}^{22}D_{33}^{44} - D_{21}^{32}D_{13}^{44} + P_{14}P_{41}D_{22}^{33} - P_{14}P_{42}D_{21}^{33} - P_{13}P_{41}D_{22}^{34} + P_{13}P_{42}D_{21}^{34} \right]. \end{aligned} \quad (7.32)$$

Similarly, for the numerator of $C_4(n)$ we have

$$\begin{aligned} -\widehat{\mathcal{U}}_{22}(n)\widehat{\mathcal{U}}_{31}(n) - [1 - \widehat{\mathcal{U}}_{21}(n)]\widehat{\mathcal{U}}_{32}(n) &= -\frac{1}{D_{11}^{33}} \left[(\mathcal{X}_{1,32}^\infty - P_{31}) (P_{32}D_{11}^{23} - P_{31}D_{12}^{23} - P_{33}D_{11}^{22}) \right. \\ &\quad \left. + \mathcal{X}_{1,34}^\infty (P_{41}D_{12}^{33} - P_{42}D_{11}^{33} + P_{43}D_{11}^{32}) \right] - \frac{\mathcal{X}_{1,32}^\infty \mathcal{X}_{1,34}^\infty}{D_{11}^{33}} \left[D_{11}^{32}D_{23}^{44} - D_{23}^{34}D_{11}^{42} - D_{11}^{22}D_{33}^{44} - D_{21}^{32}D_{13}^{44} \right. \\ &\quad \left. + P_{14}P_{41}D_{22}^{33} - P_{14}P_{42}D_{21}^{33} - P_{13}P_{41}D_{22}^{34} + P_{13}P_{42}D_{21}^{34} \right], \end{aligned} \quad (7.33)$$

while for the shared denominator of $C_2(n)$ and $C_4(n)$ we have

$$\begin{aligned} (1 - \widehat{\mathcal{U}}_{21}(n))\widehat{\mathcal{U}}_{42}(n) + \widehat{\mathcal{U}}_{41}(n)\widehat{\mathcal{U}}_{22}(n) &= \frac{-1}{D_{11}^{33}} (P_{41}D_{12}^{33} - P_{42}D_{11}^{33} + P_{43}D_{11}^{32}) \\ &\quad - \frac{\mathcal{X}_{1,32}^\infty}{D_{11}^{33}} \left(D_{11}^{32}D_{23}^{44} - D_{23}^{34}D_{11}^{42} - D_{11}^{22}D_{33}^{44} - D_{21}^{32}D_{13}^{44} \right. \\ &\quad \left. + P_{14}P_{41}D_{22}^{33} - P_{14}P_{42}D_{21}^{33} - P_{13}P_{41}D_{22}^{34} + P_{13}P_{42}D_{21}^{34} \right). \end{aligned} \quad (7.34)$$

Let the quantity

$$E(n) := -D_{11}^{33} \left((1 - \widehat{\mathcal{U}}_{21}(n))\widehat{\mathcal{U}}_{42}(n) + \widehat{\mathcal{U}}_{41}(n)\widehat{\mathcal{U}}_{22}(n) \right) \quad (7.35)$$

denote the remaining expression after clearing out the denominator of (7.34). Noticing the common terms among the expressions (7.32)–(7.34), we obtain (generically, $E(n) \neq 0$)

$$C_2(n) = \mathcal{X}_{1,32}^\infty + \frac{\left(P_{31} - \mathcal{X}_{1,32}^\infty \right) (P_{41}D_{12}^{33} - P_{42}D_{11}^{33} + P_{43}D_{11}^{32})}{E(n)}, \quad (7.36)$$

and

$$C_4(n) = \mathcal{X}_{1,34}^\infty + \frac{(P_{31} - \mathcal{X}_{1,32}^\infty) (P_{33}D_{11}^{22} + P_{31}D_{12}^{23} - P_{32}D_{11}^{23})}{E(n)}. \quad (7.37)$$

Combining these with equations (7.8) and (7.10)–(7.12) and straightforward simplifications we obtain the following exact formula for $h_{n-1}^{(0,1)}$:

$$\begin{aligned} \frac{-1}{h_{n-1}^{(0,1)}} &= \mathcal{X}_{1,31}^\infty + \frac{P_{31} - \mathcal{X}_{1,32}^\infty}{E(n)D_{11}^{33}} \left[D_{31}^{43} (P_{33}D_{11}^{22} + P_{31}D_{12}^{23} - P_{32}D_{11}^{23}) \right. \\ &\quad \left. - D_{21}^{33} (P_{41}D_{12}^{33} - P_{42}D_{11}^{33} + P_{43}D_{11}^{32}) \right]. \end{aligned} \quad (7.38)$$

Now we focus on finding the large n asymptotics of the right hand side of the above equation by recalling equation (4.17) of [25] which gives the asymptotic formula for $P(n)$,

$$P(n) = \begin{pmatrix} -C_\rho(0)\alpha(0)\mathcal{R}_{1,14}(0;n) - \mathcal{R}_{1,12}(0;n) & 0 & \mathcal{R}_{1,14}(0;n) & -\alpha(0) \\ -1 & -\frac{\mathcal{R}_{1,23}(0;n)}{\alpha(0)} & 0 & -\alpha(0)\mathcal{R}_{1,21}(0;n) \\ -C_\rho(0)\alpha(0)\mathcal{R}_{1,34}(0;n) - \mathcal{R}_{1,32}(0;n) & -\frac{1}{\alpha(0)} & \mathcal{R}_{1,34}(0;n) & 0 \\ -C_\rho(0)\alpha(0) & -\frac{\mathcal{R}_{1,43}(0;n)}{\alpha(0)} & 1 & -\alpha(0)\mathcal{R}_{1,41}(0;n) \end{pmatrix} + O(e^{-2cn}), \quad (7.39)$$

as $n \rightarrow \infty$.

Notice that

$$D_{11}^{33} = P_{11}P_{33} - P_{13}P_{31} = \mathcal{R}_{1,32}(0;n)\mathcal{R}_{1,14}(0;n) - \mathcal{R}_{1,12}(0;n)\mathcal{R}_{1,34}(0;n) + O(e^{-3cn}), \quad (7.40)$$

$$D_{31}^{43} = P_{31}P_{43} - P_{33}P_{41} = -\mathcal{R}_{1,32}(0;n) + O(e^{-2cn}), \quad (7.41)$$

$$D_{11}^{22} = P_{11}P_{22} - P_{12}P_{21} = \left(\frac{\mathcal{R}_{1,23}(0;n)}{\alpha(0)} \right) (C_\rho(0)\alpha(0)\mathcal{R}_{1,14}(0;n) + \mathcal{R}_{1,12}(0;n)) + O(e^{-3cn}), \quad (7.42)$$

$$D_{12}^{23} = P_{12}P_{23} - P_{13}P_{22} = \frac{\mathcal{R}_{1,14}(0;n)\mathcal{R}_{1,23}(0;n)}{\alpha(0)} + O(e^{-3cn}), \quad (7.43)$$

$$D_{11}^{23} = P_{11}P_{23} - P_{13}P_{21} = \mathcal{R}_{1,14}(0;n) + O(e^{-2cn}), \quad (7.44)$$

$$D_{21}^{33} = P_{21}P_{33} - P_{31}P_{23} = -\mathcal{R}_{1,34}(0;n) + O(e^{-2cn}), \quad (7.45)$$

$$D_{12}^{33} = P_{12}P_{33} - P_{13}P_{32} = \frac{\mathcal{R}_{1,14}(0;n)}{\alpha(0)} + O(e^{-2cn}), \quad (7.46)$$

$$D_{11}^{32} = P_{11}P_{32} - P_{12}P_{31} = C_\rho(0)\mathcal{R}_{1,14}(0;n) + \frac{\mathcal{R}_{1,12}(0;n)}{\alpha(0)} + O(e^{-2cn}). \quad (7.47)$$

Notice also that

$$P_{33}D_{31}^{43}D_{11}^{22} = -\frac{\mathcal{R}_{1,34}(0;n)\mathcal{R}_{1,23}(0;n)\mathcal{R}_{1,32}(0;n)}{\alpha(0)} (C_\rho(0)\alpha(0)\mathcal{R}_{1,14}(0;n) + \mathcal{R}_{1,12}(0;n)) + O(e^{-5cn}), \quad (7.48)$$

$$P_{31}D_{31}^{43}D_{12}^{23} = (C_\rho(0)\alpha(0)\mathcal{R}_{1,34}(0;n) + \mathcal{R}_{1,32}(0;n)) \frac{\mathcal{R}_{1,32}(0;n)\mathcal{R}_{1,14}(0;n)\mathcal{R}_{1,23}(0;n)}{\alpha(0)} + O(e^{-5cn}), \quad (7.49)$$

$$-P_{32}D_{31}^{43}D_{11}^{23} = -\frac{\mathcal{R}_{1,32}(0;n)\mathcal{R}_{1,14}(0;n)}{\alpha(0)} + O(e^{-3cn}), \quad (7.50)$$

$$-P_{41}D_{21}^{33}D_{12}^{33} = -C_\rho(0)\mathcal{R}_{1,34}(0;n)\mathcal{R}_{1,14}(0;n) + O(e^{-3cn}), \quad (7.51)$$

$$P_{42}D_{21}^{33}D_{11}^{33} = \frac{\mathcal{R}_{1,43}(0;n)}{\alpha(0)} \left(\mathcal{R}_{1,34}(0;n)\mathcal{R}_{1,32}(0;n)\mathcal{R}_{1,14}(0;n) - (\mathcal{R}_{1,34}(0;n))^2\mathcal{R}_{1,12}(0;n) \right) + O(e^{-5cn}), \quad (7.52)$$

$$-P_{43}D_{21}^{33}D_{11}^{32} = C_\rho(0)\mathcal{R}_{1,34}(0;n)\mathcal{R}_{1,14}(0;n) + \frac{\mathcal{R}_{1,34}(0;n)\mathcal{R}_{1,12}(0;n)}{\alpha(0)} + O(e^{-3cn}). \quad (7.53)$$

Therefore we observe that

$$D_{31}^{43} (P_{33}D_{11}^{22} + P_{31}D_{12}^{23} - P_{32}D_{11}^{23}) - D_{21}^{33} (P_{41}D_{12}^{33} - P_{42}D_{11}^{33} + P_{43}D_{11}^{32}) = -\frac{\mathcal{R}_{1,32}(0;n)\mathcal{R}_{1,14}(0;n)}{\alpha(0)} + O(e^{-3cn}), \quad (7.54)$$

where the left hand side appears in (7.38).

7.2. Asymptotics of relevant entries in $\tilde{\mathcal{X}}_1^\infty(n)$

In view of equations (7.34), (7.35) and (7.38) we need large n asymptotics for $\tilde{\mathcal{X}}_{1,31}^\infty$ and $\tilde{\mathcal{X}}_{1,32}^\infty$. To this end, let us recall some relevant information from [25]. First of all let us start with *the model Riemann–Hilbert problem for the pair (ϕ, w)* :

- **RH- Λ 1** Λ is holomorphic in $\mathbb{C} \setminus \mathbb{T}$.
- **RH- Λ 2** $\Lambda_+(z) = \Lambda_-(z)J_\Lambda(z)$, for $z \in \mathbb{T}$, where

$$J_\Lambda(z) = \begin{pmatrix} 0 & 0 & 0 & -\phi(z) \\ -\frac{w(z)}{\phi(z)} & 0 & \tilde{\phi}(z) - \frac{w(z)\tilde{w}(z)}{\phi(z)} & 0 \\ 0 & -\frac{1}{\tilde{\phi}(z)} & 0 & 0 \\ \frac{1}{\phi(z)} & 0 & \frac{\tilde{w}(z)}{\phi(z)} & 0 \end{pmatrix}.$$

- **RH- Λ 3** As $z \rightarrow \infty$, we have $\Lambda(z) = I + \frac{\tilde{\Lambda}_1}{z} + \frac{\tilde{\Lambda}_2}{z^2} + O(z^{-3})$.

In [25] it was shown that if we consider $D_n(\phi, d\phi; 1, 1)$, where d is of Szegő-type and further satisfies the condition $d(z)\tilde{d}(z) = 1$ on the unit circle⁷, then this model problem is explicitly solvable and its solution can be written as

$$\Lambda(z) = \Lambda_\infty^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ C_\rho(z) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{cases} \begin{pmatrix} -\beta(z) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\tilde{\alpha}(z)\beta(z)\alpha(z)} & 0 \\ 0 & -\tilde{\alpha}(z) & 0 & 0 \\ 0 & 0 & 0 & -\alpha(z) \end{pmatrix}, & |z| < 1, \\ \begin{pmatrix} 0 & \beta(z) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta(z)\tilde{\alpha}(z)\alpha(z)} \\ 0 & 0 & \tilde{\alpha}(z) & 0 \\ \alpha(z) & 0 & 0 & 0 \end{pmatrix}, & |z| > 1, \end{cases} \quad (7.55)$$

where

$$\Lambda_\infty^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha(0)} & 0 \\ 0 & \alpha(0) & 0 & 0 \end{pmatrix}, \quad (7.56)$$

and the functions α , β , and C_ρ are defined in (2.32) and (2.33).

In what follows, we connect the desired quantities $\tilde{\mathcal{X}}_{1,31}^\infty$ and $\tilde{\mathcal{X}}_{1,32}^\infty$ to data from the Λ -RHP as well as $\mathcal{R}$, the solution of the small-norm Riemann–Hilbert problem:

- **RH- $\mathcal{R}$ 1** $\mathcal{R}$ is holomorphic in $\mathbb{C} \setminus \Gamma_{\mathcal{R}}$.
- **RH- $\mathcal{R}$ 2** $\mathcal{R}_+(z; n) = \mathcal{R}_-(z; n)J_{\mathcal{R}}(z; n)$, for $z \in \Gamma_{\mathcal{R}}$.
- **RH- $\mathcal{R}$ 3** As $z \rightarrow \infty$,

$$\mathcal{R}(z; n) = I + \frac{\tilde{\mathcal{R}}_1(n)}{z} + \frac{\tilde{\mathcal{R}}_2(n)}{z^2} + O(z^{-3}).$$

⁷ which makes $J_{\Lambda,23} \equiv 0$.

The contour $\Gamma_{\mathcal{R}} := \Gamma'_i \cup \Gamma'_o$ consists of two counter-clockwise oriented circles: Γ'_i has radius $r_* \in (r_0, 1)$ and Γ'_o has radius $1/r_*$. Here r_* is any number satisfying $r_0 < r_* < 1$ (see (2.30) and (2.31) for the definition and meaning of r_0). The jump matrix $J_{\mathcal{R}}$ is given by

$$J_{\mathcal{R}}(z; n) - I = \begin{cases} z^n \cdot \begin{pmatrix} 0 & g_{12}(z) & 0 & g_{14}(z) \\ 0 & 0 & g_{23}(z) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{43}(z) & 0 \end{pmatrix}, & z \in \Gamma'_i, \\ z^{-n} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ g_{21}(z) & 0 & 0 & 0 \\ 0 & g_{32}(z) & 0 & g_{34}(z) \\ g_{41}(z) & 0 & 0 & 0 \end{pmatrix}, & z \in \Gamma'_o, \end{cases} \quad (7.57)$$

with

$$\begin{aligned} g_{12}(z) &= -\frac{\alpha(z)}{\phi(z)\beta(z)} - \frac{\tilde{w}(z)C_{\rho}(z)}{\phi(z)\beta(z)\tilde{\alpha}(z)}, & g_{14}(z) &= \frac{\tilde{w}(z)}{\phi(z)\beta(z)\tilde{\alpha}(z)\alpha(0)}, \\ g_{23}(z) &= -\frac{\alpha(0)\tilde{w}(z)\beta(z)}{\tilde{\phi}(z)\tilde{\alpha}(z)}, & g_{43}(z) &= -\alpha^2(0) \left(\frac{\alpha(z)\beta(z)}{\tilde{\phi}(z)} + \frac{\beta(z)\tilde{w}(z)C_{\rho}(z)}{\tilde{\alpha}(z)\tilde{\phi}(z)} \right), \\ g_{21}(z) &= \frac{w(z)\beta(z)}{\phi(z)\alpha(z)}, & g_{32}(z) &= -\frac{1}{\alpha(0)\tilde{\phi}(z)} \left(\frac{\tilde{\alpha}(z)}{\beta(z)} - w(z)\tilde{\alpha}^2(z)\beta(z)\alpha(z)C_{\rho}(z) \right), \\ g_{34}(z) &= \frac{w(z)\tilde{\alpha}^2(z)\beta(z)\alpha(z)}{\tilde{\phi}(z)\alpha^2(0)}, & g_{41}(z) &= -\frac{\alpha(0)}{\phi(z)} \left(\frac{1}{\tilde{\alpha}(z)\beta(z)\alpha^2(z)} - \frac{w(z)\beta(z)C_{\rho}(z)}{\alpha(z)} \right). \end{aligned}$$

From (7.57) it follows that the jump matrix $J_{\mathcal{R}}$ satisfies on $\Gamma_{\mathcal{R}}$ the small-norm estimate,

$$\|J_{\mathcal{R}} - I\|_{L_2 \cap L_{\infty}} \leq C e^{-cn}, \quad (7.58)$$

for some positive C and $c = -\log r_*$. Therefore, by standard theory of small-norm Riemann–Hilbert problems [18, 19], there exists n_* such that for all $n > n_*$ the $\mathcal{R}$ —RH problem is solvable and

$$\mathcal{R}(z; n) = I + \mathcal{R}_1(z; n) + \mathcal{R}_2(z; n) + \mathcal{R}_3(z; n) + \cdots, \quad z \in \mathbb{C} \setminus \Gamma_{\mathcal{R}}, \quad n \geq n_*, \quad (7.59)$$

where each $\mathcal{R}_k$ is of order $O(e^{-kn})$ and they can be found recursively from

$$\mathcal{R}_k(z; n) = \frac{1}{2\pi i} \int_{\Gamma_{\mathcal{R}}} \frac{[\mathcal{R}_{k-1}(\mu; n)]_- (J_{\mathcal{R}}(\mu; n) - I)}{\mu - z} d\mu, \quad z \in \mathbb{C} \setminus \Gamma_{\mathcal{R}}, \quad k \geq 1. \quad (7.60)$$

Note that this recurrence also means that

$$\mathcal{R}_{k+1}(z; n) = o(\mathcal{R}_k(z; n)), \quad n \rightarrow \infty, \quad z \in \mathbb{C} \setminus \Gamma_{\mathcal{R}}, \quad k \geq 1.$$

More precisely we have

$$\mathcal{R}_{k,ij}(z; n) = \frac{O(e^{-kn})}{|z| + 1}, \quad n \rightarrow \infty, \quad k \geq 1, \quad (7.61)$$

uniformly for $z \in \mathbb{C} \setminus \Gamma_{\mathcal{R}}$, and the positive constant c is the same as in (7.58). From (7.60) we have

$$\mathcal{R}_1(z; n) = \frac{1}{2\pi i} \int_{\Gamma_{\mathcal{R}}} \frac{J_{\mathcal{R}}(\mu; n) - I}{\mu - z} d\mu = \begin{pmatrix} 0 & \mathcal{R}_{1,12}(z; n) & 0 & \mathcal{R}_{1,14}(z; n) \\ \mathcal{R}_{1,21}(z; n) & 0 & \mathcal{R}_{1,23}(z; n) & 0 \\ 0 & \mathcal{R}_{1,32}(z; n) & 0 & \mathcal{R}_{1,34}(z; n) \\ \mathcal{R}_{1,41}(z; n) & 0 & \mathcal{R}_{1,43}(z; n) & 0 \end{pmatrix}, \quad (7.62)$$

where

$$\begin{aligned} \mathcal{R}_{1,jk}(z; n) &= \frac{1}{2\pi i} \int_{\Gamma'_i} \frac{\mu^n g_{jk}(\mu)}{\mu - z} d\mu, & jk &= 12, 14, 23, 43, \\ \mathcal{R}_{1,jk}(z; n) &= \frac{1}{2\pi i} \int_{\Gamma'_o} \frac{\mu^{-n} g_{jk}(\mu)}{\mu - z} d\mu, & jk &= 21, 32, 34, 41. \end{aligned} \quad (7.63)$$

So, in view of (7.58) and (7.60), there exists a positive constant C_* such that

$$|\mathcal{R}_{1,jk}(0;n)| \leq C_* r_*^n, \quad jk = 12, 14, 23, 43, 21, 32, 34, 41, \quad n \geq n_*. \quad (7.64)$$

Recalling **RH- $\mathcal{X}3$** , let us consider

$$\mathcal{G}(z;n) := \mathcal{X}(z;n) \begin{pmatrix} z^{-n} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (7.65)$$

whose asymptotic behavior as $z \rightarrow \infty$ reads

$$\mathcal{G}(z;n) = I + \frac{\mathcal{X}_1^\infty}{z} + \frac{\mathcal{X}_2^\infty}{z^2} + O(z^{-3}). \quad (7.66)$$

As it is shown in [25], for $z \in \Omega_\infty$, we can write

$$\mathcal{G}(z;n) \equiv \mathcal{R}(z;n)\Lambda(z), \quad z \in \Omega_\infty, \quad (7.67)$$

and thus from **RH- $\mathcal{R}3$** and **RH- $\Lambda3$** which it is clear that

$$\mathcal{X}_1^\infty(n) = \tilde{\mathcal{R}}_1(n) + \tilde{\Lambda}_1. \quad (7.68)$$

From (7.59)–(7.61) we obtain that

$$\tilde{\mathcal{R}}_1(n) = \begin{pmatrix} 0 & \tilde{\mathcal{R}}_{1,12}(n) & 0 & \tilde{\mathcal{R}}_{1,14}(n) \\ \tilde{\mathcal{R}}_{1,21}(n) & 0 & \tilde{\mathcal{R}}_{1,23}(n) & 0 \\ 0 & \tilde{\mathcal{R}}_{1,32}(n) & 0 & \tilde{\mathcal{R}}_{1,34}(n) \\ \tilde{\mathcal{R}}_{1,41}(n) & 0 & \tilde{\mathcal{R}}_{1,43}(n) & 0 \end{pmatrix} = -\frac{1}{2\pi i} \int_{\Gamma_{\mathcal{R}}} (J_{\mathcal{R}}(\mu;n) - I) d\mu + O(e^{-2cn}), \quad (7.69)$$

so

$$\begin{aligned} \tilde{\mathcal{R}}_{1,jk}(n) &= -\frac{1}{2\pi i} \int_{\Gamma'_i} \mu^n g_{jk}(\mu) d\mu + O(e^{-2cn}), & jk &= 12, 14, 23, 43, \\ \tilde{\mathcal{R}}_{1,jk}(n) &= -\frac{1}{2\pi i} \int_{\Gamma'_o} \mu^{-n} g_{jk}(\mu) d\mu + O(e^{-2cn}), & jk &= 21, 32, 34, 41. \end{aligned} \quad (7.70)$$

Recalling (7.63), the equations (7.70) can be written as

$$\begin{aligned} \tilde{\mathcal{R}}_{1,jk}(n) &= -\mathcal{R}_{1,jk}(0;n+1) + O(e^{-2cn}), & jk &= 12, 14, 23, 43, \\ \tilde{\mathcal{R}}_{1,jk}(n) &= -\mathcal{R}_{1,jk}(0;n-1) + O(e^{-2cn}), & jk &= 21, 32, 34, 41. \end{aligned} \quad (7.71)$$

Now let us turn our attention to $\tilde{\Lambda}_1$. From (2.35), as $z \rightarrow \infty$ we have

$$\alpha(z) = 1 - \frac{1}{2\pi i z} \int_{\mathbb{T}} \ln(\phi(\tau)) d\tau + O(z^{-2}), \quad \beta(z) = 1 - \frac{1}{2\pi i z} \int_{\mathbb{T}} \ln(d(\tau)) d\tau + O(z^{-2}), \quad (7.72)$$

$$\tilde{\alpha}(z) = \alpha(0) \left(1 + \frac{1}{2\pi i z} \int_{\mathbb{T}} \ln(\phi(\tau)) \frac{d\tau}{\tau^2} + O(z^{-2}) \right), \quad C_\rho(z) = -\frac{1}{2\pi i z} \int_{\mathbb{T}} \rho(\tau) d\tau + O(z^{-2}), \quad (7.73)$$

where

$$\rho(\tau) = -\frac{1}{\beta_-(\tau)\beta_+(\tau)\tilde{\alpha}_-(\tau)\alpha_+(\tau)}, \quad \tau \in \mathbb{T}. \quad (7.74)$$

For $|z| > 1$, we can write (7.55) as

$$\Lambda(z) = \begin{pmatrix} \alpha(z) & 0 & 0 & 0 \\ 0 & \beta(z) & 0 & 0 \\ 0 & 0 & \frac{\tilde{\alpha}(z)}{\alpha(0)} & 0 \\ 0 & \alpha(0)\beta(z)C_\rho(z) & 0 & \frac{\alpha(0)}{\tilde{\alpha}(z)\alpha(z)\beta(z)} \end{pmatrix}. \quad (7.75)$$

Therefore in view of (7.68) and (7.75) we find

$$\mathcal{X}_{1,jk}^{\infty}(n) = \mathcal{R}_{1,jk}(n), \quad jk \in \{12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 43\}. \quad (7.76)$$

Therefore, in particular, we obtain

$$\mathcal{X}_{1,31}^{\infty}(n) = O(e^{-2cn}), \quad \mathcal{X}_{1,32}^{\infty}(n) = -\mathcal{R}_{1,32}(0; n-1) + O(e^{-2cn}), \quad (7.77)$$

in view of (7.69) and (7.71). Recalling (7.39) we have

$$P_{31} - \mathcal{X}_{1,32}^{\infty} = -C_{\rho}(0)\alpha(0)\mathcal{R}_{1,34}(0; n) - \mathcal{R}_{1,32}(0; n) + \mathcal{R}_{1,32}(0; n-1) + O(e^{-2cn}). \quad (7.78)$$

7.3. Asymptotics of $E(n)$

Recalling (7.34) and (7.35) we have

$$\begin{aligned} E(n) = & P_{41}D_{12}^{33} - P_{42}D_{11}^{33} + P_{43}D_{11}^{32} + \mathcal{X}_{1,32}^{\infty} \left(D_{11}^{32}D_{23}^{44} - D_{23}^{34}D_{11}^{42} - D_{11}^{22}D_{33}^{44} - D_{21}^{32}D_{13}^{44} \right. \\ & \left. + P_{14}P_{41}D_{22}^{33} - P_{14}P_{42}D_{21}^{33} - P_{13}P_{41}D_{22}^{34} + P_{13}P_{42}D_{21}^{34} \right). \end{aligned} \quad (7.79)$$

The asymptotics of the D_{jk}^{rs} terms in the above expression that are not among those already considered in equations (7.40)–(7.47) are given below:

$$D_{23}^{44} = P_{23}P_{44} - P_{24}P_{43} = \alpha(0)\mathcal{R}_{1,21}(0; n) + O(e^{-2cn}), \quad (7.80)$$

$$D_{23}^{34} = P_{23}P_{34} - P_{24}P_{33} = \alpha(0)\mathcal{R}_{1,21}(0; n)\mathcal{R}_{1,34}(0; n) + O(e^{-3cn}), \quad (7.81)$$

$$D_{33}^{44} = P_{33}P_{44} - P_{34}P_{43} = -\alpha(0)\mathcal{R}_{1,41}(0; n)\mathcal{R}_{1,34}(0; n) + O(e^{-3cn}), \quad (7.82)$$

$$D_{21}^{32} = P_{21}P_{32} - P_{22}P_{31} = \frac{1}{\alpha_0} + O(e^{-2cn}), \quad (7.83)$$

$$D_{13}^{44} = P_{13}P_{44} - P_{14}P_{43} = \alpha(0) + O(e^{-2cn}), \quad (7.84)$$

$$D_{22}^{33} = P_{22}P_{33} - P_{23}P_{32} = -\frac{1}{\alpha(0)}\mathcal{R}_{1,23}(0; n)\mathcal{R}_{1,34}(0; n) + O(e^{-3cn}), \quad (7.85)$$

$$D_{22}^{34} = P_{22}P_{34} - P_{24}P_{32} = -\mathcal{R}_{1,21}(0; n) + O(e^{-2cn}), \quad (7.86)$$

$$D_{21}^{34} = P_{21}P_{34} - P_{24}P_{31} = -\alpha(0)\mathcal{R}_{1,21}(0; n)(C_{\rho}(0)\alpha(0)\mathcal{R}_{1,34}(0; n) + \mathcal{R}_{1,32}(0; n)) + O(e^{-3cn}). \quad (7.87)$$

Using these we observe that the leading order asymptotics of (7.79) is given by the following three terms:

$$P_{41}D_{12}^{33} = -C_{\rho}(0)\mathcal{R}_{1,14}(0; n) + O(e^{-2cn}), \quad (7.88)$$

$$P_{43}D_{11}^{32} = C_{\rho}(0)\mathcal{R}_{1,14}(0; n) + \frac{\mathcal{R}_{1,12}(0; n)}{\alpha(0)} + O(e^{-2cn}), \quad (7.89)$$

$$\mathcal{X}_{1,32}^{\infty}D_{21}^{32}D_{13}^{44} = -\mathcal{R}_{1,32}(0; n-1)(1 + O(e^{-2cn})), \quad (7.90)$$

where in the last equation we have used (7.77). We thus have

$$E(n) = \frac{\mathcal{R}_{1,12}(0; n)}{\alpha(0)} - \mathcal{R}_{1,32}(0; n-1) + O(e^{-2cn}). \quad (7.91)$$

7.4. Asymptotics of $h_{n-1}^{(0,1)}$

In this subsection we combine the equations (7.38), (7.40), (7.54), (7.77), (7.78) and (7.91) to obtain the asymptotics of $h_{n-1}^{(0,1)}$. To that end, we consider generic symbols ϕ and d for which the following four properties hold

(1) There exists $n_1 \geq n_*$, $r_1 \in (r_*^3, r_*^2)$ ⁸ and a constant $C_1 > 0$ such that

$$|\mathcal{R}_{1,32}(0; n)\mathcal{R}_{1,14}(0; n)| \geq C_1 r_1^n. \quad (7.92)$$

⁸ Recall that the choice of r_* was fixed right below equation (7.58).

Using $r_1 > r_*^3$, we can rewrite the r.h.s. of (7.54) as

$$-\frac{1}{\alpha(0)}\mathcal{R}_{1,32}(0;n)\mathcal{R}_{1,14}(0;n)(1+O(e^{-c_1n})), \quad \text{with} \quad c_1 = -\log\left(\frac{r_*^3}{r_1}\right) > 0. \quad (7.93)$$

Notice that we need $r_1 < r_*^2$ in order to make the estimate (7.92) compatible with the following upper bound obtained from (7.64)

$$|\mathcal{R}_{1,32}(0;n)\mathcal{R}_{1,14}(0;n)| < C_0^2 r_*^{2n}. \quad (7.94)$$

(2) There exists $n_2 \geq n_1$, $r_2 \in (r_*^3, r_*^2)$ and a constant $C_2 > 0$ such that

$$|\mathcal{R}_{1,32}(0;n)\mathcal{R}_{1,14}(0;n) - \mathcal{R}_{1,12}(0;n)\mathcal{R}_{1,34}(0;n)| \geq C_2 r_2^n, \quad \text{for all} \quad n > n_2. \quad (7.95)$$

Using $r_2 > r_*^3$, we can now rewrite (7.40) as

$$D_{11}^{33} = (\mathcal{R}_{1,32}(0;n)\mathcal{R}_{1,14}(0;n) - \mathcal{R}_{1,12}(0;n)\mathcal{R}_{1,34}(0;n))(1+O(e^{-c_2n})), \quad (7.96)$$

with $c_2 = -\log\left(\frac{r_*^3}{r_2}\right) > 0$.

(3) There exists $n_3 \geq n_2$, $r_3 \in (r_*^2, r_*)$ and a constant $C_3 > 0$ such that

$$\left| \frac{\mathcal{R}_{1,12}(0;n)}{\alpha(0)} - \mathcal{R}_{1,32}(0;n-1) \right| \geq C_3 r_3^n, \quad \text{for all} \quad n > n_3. \quad (7.97)$$

Using $r_3 > r_*^2$, we can now rewrite (7.91) as

$$E(n) = \left(\frac{\mathcal{R}_{1,12}(0;n)}{\alpha(0)} - \mathcal{R}_{1,32}(0;n-1) \right) (1+O(e^{-c_3n})), \quad (7.98)$$

with $c_3 = -\log\left(\frac{r_*^2}{r_3}\right) > 0$.

(4) There exists $n_4 \geq n_3$, $r_4 \in (r_*^2, r_*)$ and a constant $C_4 > 0$ such that

$$|-C_\rho(0)\alpha(0)\mathcal{R}_{1,34}(0;n) - \mathcal{R}_{1,32}(0;n) + \mathcal{R}_{1,32}(0;n-1)| \geq C_4 r_4^n, \quad \text{for all} \quad n > n_4. \quad (7.99)$$

Using $r_4 > r_*^2$, we can now rewrite (7.78) as

$$P_{31} - \mathcal{X}_{1,32}^\infty = (-C_\rho(0)\alpha(0)\mathcal{R}_{1,34}(0;n) - \mathcal{R}_{1,32}(0;n) + \mathcal{R}_{1,32}(0;n-1))(1+O(e^{-c_4n})), \quad (7.100)$$

with $c_4 = -\log\left(\frac{r_*^2}{r_4}\right) > 0$.

Let

$$F(n) := \frac{\mathcal{R}_{1,32}(0;n)\mathcal{R}_{1,14}(0;n)(C_\rho(0)\alpha(0)\mathcal{R}_{1,34}(0;n) + \mathcal{R}_{1,32}(0;n) - \mathcal{R}_{1,32}(0;n-1))}{(\mathcal{R}_{1,12}(0;n) - \alpha(0)\mathcal{R}_{1,32}(0;n-1))(\mathcal{R}_{1,32}(0;n)\mathcal{R}_{1,14}(0;n) - \mathcal{R}_{1,12}(0;n)\mathcal{R}_{1,34}(0;n))}, \quad (7.101)$$

and

$$c := \min\{c_1, c_2, c_3, c_4\}. \quad (7.102)$$

Recalling (7.38), (7.40), (7.54), (7.78), (7.91) we have

$$\frac{-1}{h_{n-1}^{(0,1)}} - \mathcal{X}_{1,31}^\infty = F(n)(1+O(e^{-cn})). \quad (7.103)$$

Recalling (7.77) and the above assumptions we observe that there exists $\widehat{C}$ such that

$$\left| \frac{\mathcal{X}_{1,31}^\infty}{F_n} \right| \leq \widehat{C} \left(\frac{r_*^5}{r_4 r_1} \right)^n = \widehat{C} e^{-(c_1+c_4)n}. \quad (7.104)$$

So we can rewrite (7.103) as

$$\frac{-1}{h_{n-1}^{(0,1)}} = F(n) (1 + O(e^{-cn})), \quad (7.105)$$

and therefore

$$h_{n-1}^{(0,1)} = -F^{-1}(n) (1 + O(e^{-cn})). \quad (7.106)$$

We have just concluded the proof of theorem 2.6.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix. Offset pairs arising in applications

In this appendix, we explain why the offset pairs

$$(r, s) \in \{(0, 0), (0, 1), (0, 2)\}$$

arise naturally in the applications considered in this paper. We first review several settings in which the objects of interest are expressed as Toeplitz+Hankel determinants with these offset pairs, treating separately the cases $w = \phi$ and $w \neq \phi$. Although the main results of this work concern the latter case, we also include the case $w = \phi$ for completeness and to place the discussion in a broader context.

A.1. Identical Toeplitz and Hankel symbols

When the Toeplitz and Hankel symbols are identical, that is when $\phi = w$, the applications and asymptotic properties of Toeplitz+Hankel determinants have been studied extensively by several authors. In particular, E. Basor and T. Ehrhardt have made significant contributions to understanding various aspects of these determinants in a series of papers [6–9] using operator-theoretic methods. In most of these works, they focus on the offset pair $(0, 1)$, except in [8], where they establish Szegő-type limit theorems for $D_n[\phi, \phi; 0, s]$ with $s \geq 1$.

We now briefly outline several key applications of Toeplitz+Hankel determinants $D_n[\phi, \phi; 0, s]$. For a compact group G , and for any Haar-integrable function f on G , define

$$\mathbb{E}_{U \in G} f(U)$$

as the integral of $f(U)$ with respect to the normalized Haar measure on G ⁹. Matrix integrals over the classical groups $G(N)$ are connected to a wide range of fields, including combinatorics [4], quantum field theory [23], number theory [28], and integrable systems [1].

In their seminal work [4] on increasing subsequences of permutations under certain symmetry constraints, Baik and Rains established precise connections between Toeplitz+Hankel determinants $D_n[\phi, \phi; 0, s]$ and integrals over classical groups:

Theorem A.1 [4]. *Let $g(z)$ be any function on the unit circle such that the integrals*

$$\iota_j = \frac{1}{2\pi} \int_{[0, 2\pi]} g(e^{i\theta}) g(e^{-i\theta}) e^{ij\theta} d\theta$$

are well defined. Then

$$\mathbb{E}_{U \in O^+(2l)} \det(g(U)) = \frac{1}{2} \det(\iota_{j-k} + \iota_{j+k})_{0 \leq j, k < l}, \quad (A.1)$$

$$\mathbb{E}_{U \in O^-(2l)} \det(g(U)) = g(1)g(-1) \det(\iota_{j-k} - \iota_{j+k+2})_{0 \leq j, k < l-1}, \quad (A.2)$$

$$\mathbb{E}_{U \in O^+(2l+1)} \det(g(U)) = g(1) \det(\iota_{j-k} - \iota_{j+k+1})_{0 \leq j, k < l}, \quad (A.3)$$

⁹ When G is the orthogonal group, $\mathbb{E}_{U \in O^\pm(l)} f(U)$ denotes the integral of f over the coset of $O(l)$ with determinant ± 1 .

$$\mathbb{E}_{U \in O^-(2l+1)} \det(g(U)) = g(-1) \det(\iota_{j-k} + \iota_{j+k+1})_{0 \leq j, k < l}, \quad (\text{A.4})$$

$$\mathbb{E}_{U \in Sp(2l)} \det(g(U)) = \det(\iota_{j-k} - \iota_{j+k+2})_{0 \leq j, k < l}, \quad (\text{A.5})$$

except that $\mathbb{E}_{U \in O^+(0)} \det(g(U)) = 1$.

We would like to highlight that different choices of the offset pair $(0, s)$ correspond to averages over different classical groups. Through theorems 1.2 and 2.5 of [4], these integrals are related to the probability that the longest increasing subsequences of involutions with specific symmetry properties have length at most $2l$ or $2l + 1$.

Averages over classical groups, and thus Toeplitz+Hankel determinants, also arise in the description of the ground state density matrix $\rho_{N+1}(x, y)$ of the impenetrable Bose gas [21, 22]. In particular, $\rho_{N+1}(x, y)$ can be represented as

- a $U(N)$ average (and thus a pure Toeplitz determinant) for periodic boundary conditions,
- a $Sp(N)$ average for Dirichlet boundary conditions,
- an $O^+(2N)$ average for Neumann boundary conditions, and
- an $O^+(2N + 1)$ average for mixed Dirichlet–Neumann boundary conditions.

Forrester and Frankel combined these characterizations with the Baik–Rains theorem [4] and the asymptotic results of Basor and Ehrhardt for Toeplitz+Hankel determinants with Fisher–Hartwig singularities [7] to obtain large- N asymptotic formulas for $\rho_{N+1}(x, y)$ under Dirichlet, Neumann, and mixed Dirichlet–Neumann boundary conditions.

Toeplitz+Hankel determinants $D_n[\phi, \phi; 0, s]$ for various values of the offset parameter s appear in other contexts as well. For instance, the characterizations of theorem A.1 were applied in [2] to compute *moments of moments* of characteristic polynomials of orthogonal and symplectic groups, with connections to analytic number theory. In another application, [23] evaluated the corresponding Toeplitz+Hankel determinants for symplectic and orthogonal matrix integrals, obtaining explicit expressions for the partition functions, Wilson loops, and Hopf links of Chern–Simons theory on S^3 . Moreover, in [31, theorem 7.1], certain $k \times k$ Toeplitz+Hankel determinants were shown to characterize generating functions for enumerating column-strict tableaux with at most $2k$ and $2k + 1$ rows, corresponding respectively to offset pairs $(0, -1)$ and $(0, 0)$.

Finally, we note that for the large-size asymptotics of Toeplitz+Hankel determinants with identical symbols, in addition to the operator-theoretic approaches developed by E. Basor and T. Ehrhardt [6–8], a Riemann–Hilbert approach has also been developed and successfully applied in [16]. In particular, [16] derives precise large- n asymptotic formulae for $D_n[\phi, \phi; r, s]$ with a Fisher–Hartwig symbol ϕ and offset pairs $(r, s) \in \{(0, 0), (0, 1), (0, 2)\}$, using a 2×2 Riemann–Hilbert framework.

A.2. Distinct Toeplitz and Hankel symbols

Besides the inherent and natural motivation to obtain the asymptotics of $D_n[\phi, w; r, s]$ in the absence of the condition $\phi = w$, such asymptotic results for Toeplitz+Hankel determinants with distinct symbols are further justified due to their appearance in important applications. Below, we highlight two such applications that have significantly motivated this project. Regarding the asymptotics of $D_n[\phi, w; r, s]$ with $w \neq \phi$, there are two works [5, 25], respectively studying determinants

$$D_n[\phi, d\phi; 0, 1], \quad \text{and} \quad D_n[\phi, d\phi; 1, 1]$$

with Szegő-type functions ϕ and d , where the following extra condition is satisfied by d :

$$d(z)d(z^{-1}) \equiv 1, \quad z \in \mathbb{T}. \quad (\text{A.6})$$

The application from Statistical Mechanics highlighted below in section A.2.1 involves symbols ϕ and d which enjoy these properties, where the offset pairs of interest are $(r, s) \in \{(0, 1), (0, 2)\}$.

A.2.1. Ising model on the Zig–Zag layered half-plane

In [14], Chelkak, Hongler, and Mahfouf considered the Ising model on the so-called *zig–zag layered half-plane* $H_+^\diamond$, which is the left half-plane on the 45° -rotated square grid (see figure 3 of [14]) with $+$ boundary conditions along the rightmost column and at infinity. The *one-point function*

$$M_n = \mathbb{E}_{H^\diamond}^+ [\sigma_{(-2n-\frac{1}{2}, 0)}] \quad (\text{A.7})$$

is the *magnetization* in the $(2n)$ th column. Let the parameters θ and $q < 1$ be defined by

$$\tan \frac{\theta}{2} = \exp \left[-\frac{2J}{KT} \right], q = \tan \theta,$$

where J is the nearest neighbor coupling constant, K is the Boltzmann constant, and T is the temperature. Define the functions τ , ϕ and d as

$$\tau(\theta) := 1 - \frac{\cos^2 \theta_1}{\cos^2 \theta} \in (-q^2, 1),$$

$$\phi(z; q) = |1 - q^2 z|,$$

and

$$d(z; q) := -\frac{(\tau z - q^2)(q^2 z - 1)}{(z - q^2)(q^2 z - \tau)}, \quad (\text{A.8})$$

where in the last expression we think of τ being a function of the independent variable q , and θ_1 is interpreted as the *boundary magnetic field* [14] and is considered as a fixed parameter. Recalling the condition (A.6), it can be checked that $d(z; q)d(z^{-1}; q) \equiv 1$. Define the symbol w to be

$$w(z; q) := d(z; q)\phi(z; q), \quad (\text{A.9})$$

(see [5, 25]). If $\tau \neq 0$, we define

$$a := \frac{q^2}{\tau}. \quad (\text{A.10})$$

The critical value of external field h is specified by the condition that $a = 1$:

$$h = h_{\text{cr}}(q) \iff a = 1,$$

see remark 4.6 of [14]. Let

$$\gamma(z; q) := \begin{cases} \frac{c(q)}{1 - az}, & a < 1, \\ 0, & a \geq 1, \end{cases} \quad (\text{A.11})$$

where

$$c(q) = (\tau^2 - q^4)\tau^{-3/2}(\tau - q^4)^{-1/2}. \quad (\text{A.12})$$

Also, let us define

$$v(z; q) := w(z; q) + (1 - \tau)^{3/2}\gamma(z; q). \quad (\text{A.13})$$

Now we can recall the relevant result from [14].

Theorem A.2 [14]. *It holds that the magnetization in the $(2m)$ th column has the following Toeplitz+Hankel determinant representation*

$$M_n = (1 - \tau)^{-3/2} \det \left[\phi_{k-j} + w_{k+j} + (1 - \tau)^{3/2} \gamma_{k+j} \right]_{k,j=0}^{n-1}, \quad (\text{A.14})$$

where

$$f_m = \int_{\mathbb{T}} f(z) z^{-m} \frac{dz}{2\pi i z}, \quad f \in \{\phi, w, \gamma\}, \quad (\text{A.15})$$

is the m th Fourier coefficient of the symbol f .

Notice that when $a \geq 1$ we have the simpler Toeplitz+Hankel determinant

$$M_n = (1 - \tau)^{-3/2} \det [\phi_{k-j} + w_{k+j}]_{k,j=0}^{n-1} = (1 - r)^{-3/2} D_n[\phi, w; 0, 0], \quad (\text{A.16})$$

while when $a < 1$ we have

$$M_n = (1 - \tau)^{-3/2} \det [\phi_{k-j} + v_{k+j}]_{k,j=0}^{n-1} = (1 - r)^{-3/2} D_n[\phi, v; 0, 0]. \quad (\text{A.17})$$

For $z \in \mathbb{T}$, let us write ϕ as

$$\phi(z; q) = (1 - q^2 z)^{1/2} \overline{(1 - q^2 z)^{1/2}} = (1 - q^2 z)^{1/2} (1 - q^2 z^{-1})^{1/2} = -iq \sqrt{\frac{(z - q^{-2})(z - q^2)}{z}},$$

where, for $\alpha \in \{0, q^2, q^{-2}\}$, the branch cuts of $\sqrt{z - \alpha}$ are chosen to be $[\alpha, +\infty)$, and the branches are fixed by $0 < \arg(z - \alpha) < 2\pi$. Also, notice that since $0 < q < 1$, ϕ has no winding number.

First, let us focus on the case $a \geq 1$ and the determinant $D_n[\phi, w; 0, 0]$. Consider two cases: $\tau \neq 0$, and $\tau = 0$.

- (1) **The case $\tau \neq 0$.** If $\tau \neq 0$, that is if $\tau \in (-q^2, 0) \cup (0, 1)$, we define a by (A.10) and $d(z)$ can be written as

$$d(z; q) = -\frac{(z - a)(q^2 z - 1)}{(z - q^2)(az - 1)}. \quad (\text{A.18})$$

Let q_{cr} be such that $a(q_{\text{cr}}) = 1$. In this case $d(z; q)$ reduces to

$$d(z; q_{\text{cr}}) = -\frac{q_{\text{cr}}^2 z - 1}{z - q_{\text{cr}}^2} \quad (\text{A.19})$$

and thus

$$\text{wind } d(z; q) = \begin{cases} -1, & a = 1, \\ -2, & a > 1. \end{cases} \quad (\text{A.20})$$

- (1A) For the case $\tau \neq 0$ and $a > 1$, consider the function

$$\mathcal{d}(z; q) := z^2 d(z; q) \quad (\text{A.21})$$

which has zero winding number and also satisfies $\mathcal{d}(z; q)\mathcal{d}(1/z; q) \equiv 1$. So the Toeplitz+Hankel determinant to study is

$$M_n = \det(T_n[\phi] + H_n[z^{-2}\mathcal{d}\phi]) = D_n(\phi, \mathcal{d}\phi; 0, 2), \quad (\text{A.22})$$

since $(z^{-2}\mathcal{d}\phi)_j = (\mathcal{d}\phi)_{j+2}$. Therefore, when $\tau \neq 0$ and $a > 1$, the resulting Toeplitz+Hankel determinant has symbol pair $(\phi, \mathcal{d}\phi)$, with offsets $r = 0$ and $s = 2$, where both ϕ and $\mathcal{d}$ are of Szegő-type.

- (1B) For the case $\tau \neq 0$ and $a = 1$, consider the function

$$\mathfrak{d}(z) := z d(z; q_{\text{cr}}). \quad (\text{A.23})$$

which has zero winding number and also satisfies $\mathfrak{d}(z)\mathfrak{d}(1/z) \equiv 1$. So the Toeplitz+Hankel determinant to study is

$$M_n = \det(T_n[\phi] + H_n[z^{-1}\mathfrak{d}\phi]) = D_n(\phi, \mathfrak{d}\phi; 0, 1). \quad (\text{A.24})$$

So when $\tau \neq 0$ and $a = 1$, the associated Toeplitz+Hankel determinant has symbol pair $(\phi, \mathfrak{d}\phi)$ with offsets $r = 0$ and $s = 1$, where both ϕ and $\mathfrak{d}$ are of Szegő-type.

(2) **The case $\tau = 0$.** Let q_0 be such that $\tau(q_0) = 0$. In this case we have

$$d(z; q_0) := \frac{q_0^2 z - 1}{z(z - q_0^2)}, \quad (\text{A.25})$$

in which case, the winding number is still -2 . Similar to the case (1A), consider the function

$$\mathfrak{d}(z) := z^2 d(z; q_0) \quad (\text{A.26})$$

which has zero winding number and also satisfies $\mathfrak{d}(z)\mathfrak{d}(1/z) \equiv 1$. So in this case one needs to study

$$M_n = \det(T_n[\phi] + H_n[z^{-2}\mathfrak{d}\phi]) = D_n(\phi, \mathfrak{d}\phi; 0, 2), \quad (\text{A.27})$$

which is a Toeplitz+Hankel determinant with symbol pair $(\phi, \mathfrak{d}\phi)$ and offsets $r = 0$ and $s = 2$, where both ϕ and $\mathfrak{d}$ are of Szegő-type.

A.2.2. Spectral analysis of (pure) Hankel Matrices and offset pairs $(0, s), s \in \mathbb{N}_0$

The asymptotics of eigenvalues of a class of pure Toeplitz determinants were studied in [17], relying on the fact that the characteristic polynomial

$$\det[\lambda I - T_n[\phi; r]]$$

retains a Toeplitz structure. A natural problem, then, is to investigate the large-size asymptotic behavior of the characteristic polynomial

$$\det[\lambda I - H_n[w; s]]$$

of the Hankel matrix $H_n[w; s]$. Unlike in the Toeplitz case, this characteristic polynomial no longer inherits the structure of H_n ; rather, it is represented as a Toeplitz+Hankel determinant:

$$\det[\lambda I - H_n[w; s]] \equiv D_n(\lambda, w; 0, s),$$

with the Toeplitz symbol being the *constant* function λ .

In fact, this observation was the original motivation behind the Toeplitz+Hankel program initiated in [25], continued in the present work, and to be developed further in upcoming papers. This line of inquiry led us to study Toeplitz+Hankel determinants with non-coinciding symbols $\phi \neq w$. The main challenge in this direction is not related to offset pairs, but to the factorization of the model Riemann–Hilbert problem introduced in [25], where the symbols do *not* satisfy the relation $w = d\phi$, unlike the Ising model application discussed earlier. Should such a factorization be achieved, the issue of offset pairs can then be successfully addressed using the methods developed in this paper.

Two cases are particularly of interest:

- **Fourier case:** If the matrix $H_n[w; s]$ is generated by Fourier coefficients of $w \in L^1(\mathbb{T})$, then each offset pair satisfies $(0, s) \neq (1, 1)$, calling for further analysis beyond what was carried out in [25, section 2].
- **Moment case:** If the matrix $H_n[w; s]$ is generated by the moments of $w \in L^1(I)$ for some subset $I \subset \mathbb{R}$, then each offset pair satisfies $(0, s) \neq (1, s)$, and hence requires further investigation compared to the treatment in [25, section 3].

We will return to the first challenge outlined above in a forthcoming publication. In the present work, our focus has been on developing ideas to address the various offset arrangements.

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