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GROUND STATE SOLUTIONS FOR NON-AUTONOMOUS DYNAMICAL SYSTEMS

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Abstract

We study the existence of periodic solutions for a second order non-autonomous dynamical system. We allow both sublinear and superlinear problems. We obtain ground state solutions.

1 Introduction

We consider the following problem. One wishes to solve

$$(1) \quad -\ddot{x}(t) = B(t)x(t) + \nabla_x V(t, x(t)),$$

where

$$(2) \quad x(t) = (x_1(t), \dots, x_n(t))$$

is a map from $I = [0, T]$ to $\mathbb{R}^n$ such that each component $x_j(t)$ is a periodic function in H^1 with period T , and the function $V(t, x) = V(t, x_1, \dots, x_n)$ is continuous from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ with

$$(3) \quad \nabla_x V(t, x) = (\partial V / \partial x_1, \dots, \partial V / \partial x_n) \in C(\mathbb{R}^{n+1}, \mathbb{R}^n).$$

We shall study this problem under several sets of assumptions. Our assumption on $B(t)$ is

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(B1) The elements of the symmetric matrix $B(t)$ are real-valued functions $b_{jk}(t) = b_{kj}(t) \in L^1(I)$.

Although this assumption is very weak, it is sufficient to allow us to find an extension $\mathcal{D}$ of the operator

$$\mathcal{D}_0 x = -\ddot{x}(t) - B(t)x(t)$$

having a discrete, countable spectrum consisting of isolated eigenvalues of finite multiplicity with a finite lower bound -L

$$(4) \quad -\infty < -L \leq \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_l < \dots$$

.

Let λ_l be the first positive eigenvalue of $\mathcal{D}$. We allow $\lambda_{l-1} = 0$. We have

Theorem 1.1. *Assume*

1.

$$2V(t, x) \geq \lambda_{l-1}|x|^2, \quad t \in I, \quad x \in \mathbb{R}^n.$$

2. *There are positive constants $\mu < \lambda_l$ and m such that*

$$2V(t, x) \leq \mu|x|^2, \quad |x| \leq m, \quad x \in \mathbb{R}^n.$$

3. *There is a constant C such that*

$$V(t, x) \leq C(|x|^2 + 1), \quad t \in I, \quad x \in \mathbb{R}^n.$$

4. *The function given by*

$$(5) \quad H(t, x) = 2V(t, x) - \nabla_x V(t, x) \cdot x$$

satisfies

$$(6) \quad H(t, x) \leq W(t) \in L^1(I), \quad t \in I, \quad x \in \mathbb{R}^n,$$

and

$$(7) \quad H(t, x) \rightarrow -\infty, \quad |x| \rightarrow \infty, \quad t \in I, \quad x \in \mathbb{R}^n.$$

Then the system

$$(8) \quad \mathcal{D}x(t) = \nabla_x V(t, x(t))$$

has a solution.

Theorem 1.2. *If, in addition, we assume*

There are a constant $\gamma > \lambda_l$ and a function $W(t) \in L^1(I)$ such that

$$2V(t, x) \geq \gamma|x|^2 - W(t), \quad t \in I, x \in \mathbb{R}^n.$$

then the system (8) has a nontrivial solution.

Theorem 1.3. *The system (8) has a solution if we assume*

1.

$$2V(t, x) \geq \lambda_{l-1}|x|^2 - W(t), \quad t \in I, x \in \mathbb{R}^n.$$

2.

$$2V(t, x) \leq \lambda_l|x|^2 + W(t), \quad t \in I, x \in \mathbb{R}^n.$$

3. *There is a constant C such that*

$$V(t, x) \leq C(|x|^2 + 1), \quad t \in I, x \in \mathbb{R}^n.$$

4. *The function given by*

$$(9) \quad H(t, x) = 2V(t, x) - \nabla_x V(t, x) \cdot x$$

satisfies

$$(10) \quad H(t, x) \leq W(t) \in L^1(I), \quad t \in I, x \in \mathbb{R}^n,$$

and

$$(11) \quad H(t, x) \rightarrow -\infty, \quad |x| \rightarrow \infty, \quad t \in I, x \in \mathbb{R}^n,$$

where $W(t) \in L^1(I)$.

Theorem 1.4. *The conclusions of Theorems 1.1 - 1.3 hold if we replace (10) and (7) with*

$$(12) \quad H(t, x) \geq -W_1(t) \in L^1(I), \quad x \in \mathbb{R}^n, t \in I,$$

and

$$(13) \quad H(t, x) \rightarrow \infty, \quad |x| \rightarrow \infty, \quad t \in I, x \in \mathbb{R}^n.$$

Theorem 1.5. *The conclusions of Theorems 1.1 - 1.3 hold if in place of (10), (7) we assume*

$$(14) \quad H(t, x) \geq -W_1(t)(|x|^\alpha + 1), \quad x \in \mathbb{R}^n, t \in I,$$

and

$$(15) \quad \nu(t) := \liminf_{|x| \rightarrow \infty} H(t, x)/|x|^\alpha > 0 \quad \text{a.e.}$$

for some α satisfying $0 < \alpha \leq 2$, where $W_1(t) \in L^1(I)$.

Theorem 1.6. *The conclusions of Theorems 1.1 - 1.3 hold if in place of (10), (7) we assume that there is an α satisfying $0 < \alpha \leq 2$ such that*

$$(16) \quad H(t, x) \leq W_1(t)(|x|^\alpha + 1), \quad t \in I, \quad x \in \mathbb{R}^n,$$

and

$$(17) \quad \nu(t) := \limsup_{|x| \rightarrow \infty} H(t, x)/|x|^\alpha < 0 \quad \text{a.e.},$$

where $W_1(t) \in L^1(I)$.

Let $\mathcal{M}$ be the set of all solutions of

$$(18) \quad \mathcal{D}x(t) = \nabla_x V(t, x(t)).$$

A solution $\tilde{x}$ is called a “ground state solution” if it minimizes the functional

$$(19) \quad G(x) = d(x) - 2 \int_I V(t, x) dt$$

over the set $\mathcal{M}$.

We have

Theorem 1.7. *Under the hypotheses of any of the Theorems 1.1 - 1.6, system (18) has a ground state solution.*

The periodic non-autonomous problem

$$(20) \quad \ddot{x}(t) = \nabla_x V(t, x(t)),$$

has an extensive history in the case of singular systems (cf., e.g., Ambrosetti-Coti Zelati [1]). The first to consider it for potentials satisfying (3) were Berger and the author [5] in 1977. We proved the existence of solutions to (8) under the condition that

$$V(t, x) \rightarrow \infty \quad \text{as} \quad |x| \rightarrow \infty$$

uniformly for a.e. $t \in I$. Subsequently, Willem [54], Mawhin [25], Mawhin-Willem [27], Tang [47, 48], Tang-Wu [51, 52], Wu-Tang [55] and others proved existence under various conditions (cf. the references given in these publications).

Most previous work considered the case when $B(t) = 0$. Ding and Girardi [11] considered the case of (1) when the potential oscillates in magnitude and sign,

$$(21) \quad -\ddot{x}(t) = B(t)x(t) + b(t)\nabla W(x(t)),$$

and found conditions for solutions when the matrix $B(t)$ is symmetric and negative definite and the function $W(x)$ grows superquadratically and satisfies a homogeneity condition. Antonacci [3, 4] gave conditions for existence of solutions with stronger constraints on the potential but without the homogeneity condition, and without the negative definite condition on the matrix. Generalizations of the above results are given by Antonacci and Magrone [2], Barletta and Livrea [6], Guo and Xu [16], Li and Zou [24], Faraci and Livrea [15], Bonanno and Livrea [7, 8], Jiang [21, 22], Shilgba [39, 40], Faraci and Iannizzotto [14] and Tang and Xiao [53].

Some authors considered the second order system (1) where the potential function $V(t, x)$ is quadratically bounded as $|x| \rightarrow \infty$. Berger and Schechter [5] considered the case of (1) where $B(t)$ is a constant symmetric matrix that is positive definite, and showed existence of solutions when the magnitude of $\nabla_x V(t, x)$ is uniformly bounded, the potential is strictly convex, and if $y(t)$ is a T -periodic solutions of the linear system $-\ddot{y} = Ay$, then there exists a function $x(t)$ which is weakly differentiable with $\dot{x} \in L^2(I, \mathbb{R}^n)$ and satisfies

$$\int_0^T \langle \nabla_x V(t, x(t)), y(t) \rangle_{\mathbb{R}^n} dt = 0.$$

Han [17] gave conditions for existence of solutions when $B(t)$ was a multiple of the identity matrix, the system satisfies the resonance condition, and the potential has upper and lower subquadratic bounds. Li and Zou [24] considered the case where $B(t)$ is continuous and nonconstant and the system satisfies the resonance condition, and showed existence of solutions when the potential is even and grows no faster than linearly. Tang and Wu [49] required the function that satisfies the resonance condition to pass through the zero vector, and gave upper and lower conditions for subquadratic growth of the magnitude of $V(t, x)$ without the requirement that the potential be even. Faraci [13] considered the case where for each $t \in I$, $B(t)$ is negative definite with elements that are bounded but not necessarily continuous and the potential has an upper quadratic bound as $|x| \rightarrow \infty$, showing existence of a solution when the gradient of the potential is bounded near the origin and exceeds the matrix product in at least one direction.

We shall prove Theorems 1.1 – 1.7 in Section 3 after we introduce the operator $\mathcal{D}$ in the next section. We use linking and sandwich methods of critical point theory and then apply the monotonicity trick introduced by Struwe in [42, 43] for minimization problems. (This trick was also used by others to solve Landesman-Lazer type problems, for bifurcation problems, for Hamiltonian systems and Schrödinger equations.)

The theory of sandwich pairs began in [41] and [34, 35] and was developed in subsequent publications such as [36, 37].

2 The operator $\mathcal{D}$

In proving our theorems we shall make use of the following considerations.

We define a bilinear form $a(\cdot, \cdot)$ on the set $L^2(I, \mathbb{R}^n) \times L^2(I, \mathbb{R}^n)$,

$$(22) \quad a(u, v) = (\dot{u}, \dot{v}) + (u, v).$$

The domain of the bilinear form is the set $D(a) = H$, consisting of those periodic $x(t) = (x_1(t), \dots, x_n(t)) \in L^2(I, \mathbb{R}^n)$ having weak derivatives in $L^2(I, \mathbb{R}^n)$. H is a dense subset of $L^2(I, \mathbb{R}^n)$. Note that H is a Hilbert space. Thus we can define an operator $\mathcal{A}$ such that $u \in D(\mathcal{A})$ if and only if $u \in D(a)$ and there exists $g \in L^2(I, \mathbb{R}^n)$ such that

$$(23) \quad a(u, v) = (g, v), \quad v \in D(a).$$

If u and g satisfy this condition we say $\mathcal{A}u = g$.

Lemma 2.1. *The operator $\mathcal{A}$ is a self-adjoint Fredholm operator from $L^2(I, \mathbb{R}^n)$ to $L^2(I, \mathbb{R}^n)$. It is one-to-one and onto.*

Proof. Let $f \in L^2(I, \mathbb{R}^n)$. Then

$$(v, f) \leq \|v\| \cdot \|f\| \leq \|v\|_H \|f\|, \quad v \in H.$$

Thus (v, f) is a bounded linear functional on H . Since H is complete, there is a $u \in H$ such that

$$(u, v)_H = (f, v), \quad v \in H.$$

Consequently, $u \in D(\mathcal{A})$ and $\mathcal{A}u = f$. Moreover, if $\mathcal{A}u = 0$, then

$$(u, v)_H = 0, \quad v \in H.$$

Thus, $u = 0$. Hence, $\mathcal{A}$ is one-to-one and onto.

For any two functions $x, y \in D(\mathcal{A})$,

$$(24) \quad (\mathcal{A}x, y) = (\dot{x}, \dot{y}) + (x, y) = (x, \mathcal{A}y).$$

Thus, $\mathcal{A}$ is symmetric. It is now easy to show that $D(\mathcal{A}) \subset D(a)$ is also a dense subset of $L^2(I, \mathbb{R}^n)$. In fact, if $f \in L^2(I, \mathbb{R}^n)$ satisfies $(f, v) = 0 \forall v \in D(\mathcal{A})$, then $w = \mathcal{A}^{-1}f$ satisfies $(w, \mathcal{A}v) = (\mathcal{A}w, v) = 0 \forall v \in D(\mathcal{A})$. Since $\mathcal{A}$ is onto, $w = 0$. Hence, $f = \mathcal{A}w = 0$.

Next, we show that $\mathcal{A}$ is self-adjoint. Consider any $u, f \in L^2(I, \mathbb{R}^n)$, and suppose for any $v \in D(\mathcal{A})$,

$$(25) \quad (u, \mathcal{A}v) = (f, v).$$

Since $\mathcal{A}$ is onto and $f \in L^2(I, \mathbb{R}^n)$, there exists $w \in D(\mathcal{A})$ such that $\mathcal{A}w = f$. Then using (24),

$$(u - w, \mathcal{A}v) = (f, v) - (\mathcal{A}w, v) = 0.$$

Since $u - w \in L^2(I, \mathbb{R}^n)$, we can find a $v \in D(\mathcal{A})$ such that $\mathcal{A}v = u - w$, and

$$\|u - w\|^2 = 0.$$

This implies $u = w$ in the space $L^2(I, \mathbb{R}^n)$, and therefore $u \in D(\mathcal{A})$. Hence, $\mathcal{A}u = \mathcal{A}w = f$. $\square$

Lemma 2.2. *The essential spectrum of $\mathcal{A}$ is the null set.*

Proof. By (2.1), $\mathcal{A}$ is linear, self-adjoint, and onto $L^2(I, \mathbb{R}^n)$.

Next, we note that

$$\|\mathcal{A}^{-1}u\| \leq \|u\|.$$

To see this, let $f = \mathcal{A}u$. Then $u = \mathcal{A}^{-1}f$, and

$$(u, v)_H = (f, v), \quad v \in H.$$

Thus,

$$\|u\|_H^2 \leq \|f\| \cdot \|u\| \leq \|f\| \cdot \|u\|_H.$$

Hence, $\|u\| \leq \|f\|$.

Now we show that the inverse operator $\mathcal{A}^{-1}$ is compact on $L^2(I, \mathbb{R}^n)$. Let (u_k) be a bounded sequence in $L^2(I, \mathbb{R}^n)$, and let $C > 0$ satisfy for each k , $\|u_k\| \leq C$. By applying the inverse operator, let (x_k) be the sequence such that for each k , $\mathcal{A}x_k = u_k$. From the above statements, for each k , $\|x_k\| \leq C$. From the definition of the operator $\mathcal{A}$, for any $x \in D(\mathcal{A})$,

$$(\mathcal{A}x, x) = (\dot{x}, \dot{x}) + (x, x) = \|x\|_H^2 \geq 0.$$

Hence, $\mathcal{K} = \mathcal{A}^{-1}$ is a positive compact operator, and the eigenvalues μ_k of $\mathcal{K}$ are denumerable and have 0 as their only possible limit point. The eigenfunctions ϕ_k of $\mathcal{K}$ are also eigenfunctions of $\mathcal{K}^{-1} = \mathcal{A}$ and satisfy

$$\mathcal{A}\phi_k = \frac{1}{\mu_k} \phi_k.$$

Since the values μ_k are bounded and have no limit point except 0, there are no limit points of the set $(1/\mu_k)$ and the essential spectrum of $\mathcal{A}$ is the null set. $\square$

We will use two theorems of Schechter [30] on bilinear forms to prove Lemma 2.5.

Theorem 2.3. Let $a(\cdot, \cdot)$ be a closed Hermitian bilinear form with dense domain in $L^2(I, \mathbb{R}^n)$. If for some real number N ,

$$(26) \quad a(u, u) + N\|u\|^2 \geq 0,$$

then the operator $\mathcal{A}$ associated with $a(\cdot, \cdot)$ is self-adjoint and $\sigma(\mathcal{A}) \subset [-N, \infty)$.

Theorem 2.4. Suppose $a(\cdot, \cdot)$ is a bilinear form satisfying the hypotheses of Theorem 2.3. Let $b(\cdot, \cdot)$ be a Hermitian bilinear form such that $D(a) \subset D(b)$ and for some positive real number K , for any $u \in D(a)$,

$$(27) \quad |b(u, u)| \leq Ka(u, u).$$

Assume that every sequence $(u_k) \subset D(a)$ which satisfies

$$(28) \quad \|u_k\|^2 + a(u_k, u_k) \leq C$$

has a subsequence (v_j) such that

$$(29) \quad b(v_j - v_k, v_j - v_k) \rightarrow 0.$$

Assume also that if (28), (29) hold and $v_j \rightarrow 0$ in the $L^2(I, \mathbb{R}^n)$ norm, then $b(v_j, v_j) \rightarrow 0$. Set

$$(30) \quad c(u, v) = a(u, v) + b(u, v).$$

and let $\mathcal{A}, \mathcal{C}$ be the operators associated with a, c , respectively. Then

$$\sigma_e(\mathcal{A}) = \sigma_e(\mathcal{C}).$$

Let

$$(31) \quad b(u, v) = - \sum_{j=1}^n \sum_{k=1}^n \int_0^T (b_{jk}(t) + \delta_{jk}) u_k(t) v_j(t) dt$$

and

$$(32) \quad d(u, v) = a(u, v) + b(u, v).$$

We shall prove

Lemma 2.5. The operator $\mathcal{D}$ associated with the bilinear form $d(\cdot, \cdot)$ under assumption (B1) is self-adjoint. Its essential spectrum is the null set and there exists a finite real value L such that $\sigma(\mathcal{D}) \subset [-L, \infty)$. $\mathcal{D}$ has a discrete, countable spectrum consisting of isolated eigenvalues of finite multiplicity with a finite lower bound $-L$

$$(33) \quad -\infty < -L \leq \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_l < \dots$$

To show the bilinear form $b(\cdot, \cdot)$ is Hermitian, we can use the symmetry of the matrix $B(t) + I$ to rearrange the order of the finite summation,

$$\begin{aligned}
b(u, v) &= - \sum_{j=1}^n \sum_{k=1}^n \int_0^T (b_{jk}(t) + \delta_{jk}) u_k(t) v_j(t) dt \\
&= - \sum_{k=1}^n \sum_{j=1}^n \int_0^T (b_{jk}(t) + \delta_{jk}) v_j(t) u_k(t) dt \\
&= - \sum_{k=1}^n \sum_{j=1}^n \int_0^T (b_{kj}(t) + \delta_{kj}) v_j(t) u_k(t) dt \\
&= b(v, u).
\end{aligned}$$

Also the magnitude of $b(u) = b(u, u)$ is bounded by a multiple of the bilinear form $a(\cdot, \cdot)$ and satisfies (27),

$$\begin{aligned}
|b(u)| &\leq K_B \|u\|_{L^\infty(I, \mathbb{R}^n)}^2 \\
&\leq K_B (M \|u\|_H)^2 \\
(34) \quad &\leq K_B \cdot M^2 \|u\|_H^2 = K a(u).
\end{aligned}$$

Consider a sequence $(x_k) \subset D(\mathcal{A})$ which is bounded by a constant C in the H norm. Then each term of the sequence satisfies

$$\|x_k\|^2 + a(x_k) = 2(x_k, x_k) + (\dot{x}_k, \dot{x}_k) \leq 2\|x_k\|_H^2 \leq 4C^2.$$

Since,

$$(35) \quad \|u\|_{L^\infty(I, \mathbb{R}^n)} \leq C \|u\|_H, \quad u \in H,$$

we can find a subsequence $(x_{\bar{k}})$ which converges weakly in H and strongly in $L^\infty(I, \mathbb{R}^n)$ and $L^2(I, \mathbb{R}^n)$ to some function $x \in H$. Because the subsequence is convergent in $L^\infty(I, \mathbb{R}^n)$ it is also Cauchy under this norm. As $\bar{j}, \bar{k} \rightarrow \infty$ we can apply (34) to show this subsequence satisfies (29),

$$(36) \quad |b(x_{\bar{j}} - x_{\bar{k}})| \leq K_B \|x_{\bar{j}} - x_{\bar{k}}\|_{L^\infty(I, \mathbb{R}^n)}^2 \rightarrow 0.$$

If in addition the subsequence $(x_{\bar{k}})$ converges to zero in $L^2(I, \mathbb{R}^n)$, the subsequence must also converge in $L^\infty(I, \mathbb{R}^n)$ to the zero function, and

$$b(x_{\bar{k}}) \rightarrow 0.$$

Then the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ satisfy the conditions of Theorem 2.4. The bilinear form $d(\cdot, \cdot)$ is the sum of these two bilinear forms as in (30). By this theorem, the operator $\mathcal{D}$ associated with this bilinear form has the same essential spectrum as the operator $\mathcal{A}$ associated with the bilinear form $a(\cdot, \cdot)$. Now we show that for any constant $\epsilon > 0$ there exists a positive constant K_ϵ such that

$$(37) \quad |b(x)| \leq \epsilon \|\dot{x}\|^2 + K_\epsilon \|x\|^2 \quad x \in D(\mathcal{A}).$$

We can use (34) to find a constant K_B , and for any $\epsilon > 0$, let $\xi = \epsilon/K_B$. Then there is a constant C_ξ which satisfies

$$\begin{aligned} |b(x)| &\leq K_B \|x\|_{L^\infty(I, \mathbb{R}^n)}^2 \\ &\leq K_B \left(\frac{\epsilon}{K_B} \|\dot{x}\|^2 + C_\xi \|x\|^2 \right) \\ &\leq \epsilon \|\dot{x}\|^2 + (K_B \cdot C_\xi) \|x\|^2 . \end{aligned}$$

Setting $K_\epsilon = K_B \cdot C_\xi$ gives the stated inequality. To show $d(\cdot, \cdot)$ is closed, first apply (37) with $\epsilon = 1/2$. Thus there is a constant C_0 such that

$$(38) \quad |b(u)| \leq \frac{1}{2} a(u) + C_0 \|u\|^2 .$$

Now suppose a sequence $(u_k) \subset D(d)$ satisfies

$$(39) \quad d(u_j - u_k) \rightarrow 0 ,$$

and $(u_k) \rightarrow u$ in $L^2(I, \mathbb{R}^n)$. The sequence is Cauchy in $L^2(I, \mathbb{R}^n)$ and as j, k increase

$$\|u_j - u_k\|^2 \rightarrow 0 .$$

Suppose that $u \notin D(d)$. Because the domains of $d(\cdot, \cdot)$ and $a(\cdot, \cdot)$ are the same, $u \notin D(a)$. We have shown above that $a(\cdot, \cdot)$ is closed, so the sequence cannot be Cauchy and as j, k increase $a(u_j - u_k)$ does not approach zero. But by (39),

$$a(u_j - u_k) - b(u_j - u_k) \rightarrow 0 .$$

Applying the inequality in (38) bounds the magnitude of each $b(\cdot, \cdot)$ term, and since $a(u, u) \geq 0$, the following inequality is satisfied,

$$a(u_j - u_k) - b(u_j - u_k) \geq \frac{1}{2} a(u_j - u_k) - C_0 \|u_j - u_k\|^2 .$$

Adding the last term to both sides leaves only the positive bilinear form on the right side,

$$\begin{aligned} a(u_j - u_k) - b(u_j - u_k) + C_0 \|u_j - u_k\|^2 \\ \geq \frac{1}{2} a(u_j - u_k) \\ \geq 0 . \end{aligned}$$

As j, k increase the left side of this equation approaches zero so the center term must also approach zero, a contradiction to the statement above. Therefore, $u \in D(a) = D(d)$, and $d(\cdot, \cdot)$ is also a closed bilinear form.

Next we show that there exists a positive constant N such that for any $x \in D(a)$,

$$(40) \quad d(x) + N \|x\|^2 \geq 0 .$$

For any positive constant $\epsilon > 0$ we can find K_ϵ which satisfies (37) and thereby find a lower bound for $b(x, x)$,

$$a(x) + b(x) + N\|x\|^2 \geq a(x) - \epsilon\|\dot{x}\|^2 - K_\epsilon\|x\|.$$

We have shown that $d(\cdot, \cdot)$ is closed, and as the sum of two Hermitian bilinear forms, $d(\cdot, \cdot)$ is clearly Hermitian. Its domain is dense in $L^2(I, \mathbb{R}^n)$ and the N in (40) satisfies the conditions of Theorem 2.3, so the operator $\mathcal{D}$ associated with this bilinear form is self-adjoint and has its spectrum bounded below by $-N$. We have shown that the essential spectrum of this operator is the null set, so the spectrum is discrete and we can number the eigenvalues in increasing order, and each eigenvalue is of finite multiplicity.

3 Proofs

Proof of Theorem 1.1.

Define the subspaces M and N of H as

$$N = \bigoplus_{k < l} E(\lambda_k), \quad M = N^\perp, \quad H = M \oplus N.$$

Let

$$(41) \quad G(x) = d(x) - 2 \int_I V(t, x) dt.$$

We note that Hypothesis 1 implies

$$(42) \quad G(v) \leq 0, \quad v \in N.$$

In fact, we have

$$G(x) = d(x) - 2 \int_I V(t, x) dt \leq \int_I [\lambda_{l-1}|x|^2 - 2V(t, x)] dt \leq 0, \quad x \in N.$$

Note that there is a positive $\rho > 0$ such that

$$|x(t)| < m$$

when $\|x\|_H = \rho$. In fact, we have $|x(t)| \leq c_0\|x\|_H$. If $x \in M$, then

$$G(x) = d(x) - 2 \int_I V(t, x) dt \geq d(x) \left[1 - \frac{2\mu\|x\|^2}{d(x)}\right] > \varepsilon > 0.$$

Take

$$\begin{aligned} A &= \partial \mathbf{B}_\rho \cap M, \\ B &= N, \end{aligned}$$

where

$$\mathbf{B}_\rho = \{x \in H : \|x\|_H < \rho\}.$$

By Example 8, p. 22 of [37], A links B . Moreover,

$$(43) \quad \sup_A [-G] \leq 0 \leq \inf_B [-G].$$

Hence, we may apply Corollary 2.8.2 of [31] to conclude that there is a sequence $\{x^{(k)}\} \subset H$ such that

$$(44) \quad G(x^{(k)}) = d(x^{(k)}) - 2 \int_I V(t, x^{(k)}(t)) dt \rightarrow c \leq 0,$$

$$(45) \quad (G'(x^{(k)}), z)/2 = d(x^{(k)}, z) - \int_I \nabla_x V(t, x^{(k)}) \cdot z(t) dt \rightarrow 0, \quad z \in H$$

and

$$(46) \quad (G'(x^{(k)}), x^{(k)})/2 = d(x^{(k)}) - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)} dt \rightarrow 0.$$

Moreover, (44) and (46) imply that

$$(47) \quad \int_I H(t, x^{(k)}(t)) dt \rightarrow -c.$$

If

$$\rho_k = \|x^{(k)}\|_H \leq C,$$

there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in H and uniformly on I . From (45) we see that

$$(G'(x), z)/2 = d(x, z) - \int_I \nabla_x V(t, x(t)) \cdot z(t) dt = 0, \quad z \in H,$$

from which we conclude easily that x is a solution of (1). On the other hand, if

$$\rho_k = \|x^{(k)}\|_H \rightarrow \infty,$$

let $\tilde{x}^{(k)} = x^{(k)}/\rho_k$. Then there is a renamed subsequence such that $\tilde{x}^{(k)}$ converges to a limit $\tilde{x} \in H$ weakly in H and uniformly on I . Let

$$x^{(k)} = w^{(k)} + v^{(k)}, \quad w^{(k)} \in M, \quad v^{(k)} \in N.$$

Since

$$d(\tilde{x}^{(k)}) - 2 \int_I V(t, x^{(k)}(t)) dt / \rho_k^2 \rightarrow 0$$

and there is a constant $c_0 > 0$ such that

$$c_0 \|x^{(k)}\|_H^2 \leq d(w^{(k)}) - d(v^{(k)}) + \|g^{(k)}\|^2,$$

where $g^{(k)}$ is the component of $x^{(k)}$ in $N(\mathcal{D})$, we have

$$\begin{aligned} c_0 = c_0 \|\tilde{x}^{(k)}\|_H^2 &\leq -2d(\tilde{v}^{(k)}) + \|\tilde{g}^{(k)}\|^2 + 2 \int_I V(t, x^{(k)}(t)) dt / \rho_k^2 + o(1) \\ &\leq -2d(\tilde{v}^{(k)}) + \|\tilde{g}^{(k)}\|^2 + 2C \int_I (|\tilde{x}^{(k)}(t)|^2 + \rho_k^{-2}) dt + o(1) \\ &\rightarrow -2d(\tilde{v}) + \|\tilde{g}\|^2 + 2C \int_I |\tilde{x}(t)|^2 dt. \end{aligned}$$

Hence, $\tilde{x}(t) \not\equiv 0$. Let $\Omega_0 \subset I$ be the set on which $\tilde{x}(t) \neq 0$. The measure of Ω_0 is positive. Moreover, $|x^{(k)}(t)| \rightarrow \infty$ as $k \rightarrow \infty$ for $t \in \Omega_0$. Thus,

$$\int_I H(t, x^{(k)}(t)) dt \leq \int_{\Omega_0} H(t, x^{(k)}(t)) dt + \int_{I \setminus \Omega_0} W(t) dt \rightarrow -\infty$$

by hypothesis. But this contradicts (47). Hence, the ρ_k are bounded, and the proof is complete.

Proof of Theorem 1.3. We note that Hypothesis 1 implies

$$(48) \quad G(v) \leq Q, \quad v \in N$$

where

$$Q = \int_I W(t) dt.$$

In fact, we have

$$G(x) = d(x) - 2 \int_I V(t, x) dt \leq \int_I [\lambda_{l-1}|x|^2 - 2V(t, x)] dt \leq Q, \quad x \in N.$$

If $x \in M$, we have by Hypothesis 2 that

$$\begin{aligned} (49) \quad G(x) &\geq d(x) - \int \lambda_l |x(t)|^2 dt - Q \\ &\geq (\lambda_l - \lambda_l) \|x\|^2 - Q \geq -Q. \end{aligned}$$

By Theorem 3.17, p.26 of [37], M and N form a sandwich pair. We can now follow the proof of Theorem 1.1 to come to the same conclusion.

Proof of Theorem 1.2 Let

$$y(t) = v + sw_0,$$

where $v \in N$, $s \geq 0$, and $w_0 \in M$ is an eigenfunction of $\mathcal{D}$ corresponding to λ_l . Consequently,

$$\begin{aligned} G(y) &= s^2 d(w_0) + d(v) - 2 \int_I V(t, y(t)) dt \\ &\leq \lambda_l s^2 \|w_0\|^2 + \lambda_{l-1} \|v\|^2 - 2\gamma \int_I |y(t)|^2 dt + Q \\ &\leq (\lambda_{l-1} - \gamma) \|v\|^2 + (\lambda_l - \gamma) s^2 \|w_0\|^2 + Q \\ &\rightarrow -\infty \text{ as } s^2 + |v|^2 \rightarrow \infty, \end{aligned}$$

where

$$Q = \int_I W(t) dt.$$

Take

$$\begin{aligned} A &= \{v \in N : \|v\|_H \leq R\} \cup \{sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\|_H = R\}, \\ B &= \partial \mathbf{B}_\rho \cap M, \quad 0 < \rho < R. \end{aligned}$$

By Example 3, p.38, of [31], A links B . Moreover, if R is sufficiently large,

$$(50) \quad \sup_A G \leq 0 < \varepsilon \leq \inf_B G.$$

We may now apply Corollary 2.8.2 of [31] to conclude that there is a sequence $\{x^{(k)}\} \subset H$ such that

$$(51) \quad G(x^{(k)}) = d(x^{(k)}) - 2 \int_I V(t, x^{(k)}(t)) dt \rightarrow c \geq \varepsilon > 0,$$

$$(52) \quad (G'(x^{(k)}), z)/2 = d(x^{(k)}, z) - \int_I \nabla_x V(t, x^{(k)}) \cdot z(t) dt \rightarrow 0, \quad z \in H$$

and

$$(53) \quad (G'(x^{(k)}), x^{(k)})/2 = d(x^{(k)}) - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)} dt \rightarrow 0.$$

If

$$\rho_k = \|x^{(k)}\|_H \leq C,$$

there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in H and uniformly on I . From (52) we see that

$$(G'(x), z)/2 = d(x, z) - \int_I \nabla_x V(t, x(t)) \cdot z(t) dt = 0, \quad z \in H,$$

from which we conclude easily that x is a solution of (1). Moreover, (53) implies

$$(54) \quad d(x^{(k)}) \rightarrow \int_I \nabla_x V(t, x) \cdot x dt = d(x).$$

Consequently,

$$(55) \quad x^{(k)} \rightarrow x$$

strongly in H . This means that

$$(56) \quad G(x) = d(x) - 2 \int_I V(t, x) dt = c \geq \varepsilon > 0.$$

But

$$G(0) = -2 \int_I V(t, 0) dt \leq 0.$$

Hence, $x(t) \neq 0$. On the other hand, if

$$\rho_k = \|x^{(k)}\|_H \rightarrow \infty,$$

we can follow the proof of Theorem 1.1 to obtain a contradiction. Thus ρ_k is bounded, and the theorem is proved.

In proving Theorem 1.4 we merely replace $H(t, x)$ by $-H(t, x)$ in Theorems 1.1 and 1.2.

Proof of Theorem 1.5.

In this case we have

$$\liminf_{k \rightarrow \infty} \int_{\Omega_0} \frac{H(t, x^{(k)})}{\rho_k^\alpha} \geq \int_{\Omega_0} \liminf_{k \rightarrow \infty} \frac{H(t, x^{(k)})}{|x^{(k)}|^\alpha} |\tilde{x}^{(k)}|^\alpha \geq \int_{\Omega_0} \nu(t) |\tilde{x}|^\alpha \geq 0,$$

where Ω_0 is the set where $|x(t)^{(k)}| \rightarrow \infty$. Moreover, $\tilde{x} = 0$ on $\Omega_1 = I \setminus \Omega_0$. Since $\nu(t) > 0$ a.e., it follows that $\tilde{x} = 0$ a.e. The rest of the proof proceeds as before.

Proof of Theorem 1.6.

In this case we have

$$\limsup_{k \rightarrow \infty} \int_{\Omega_0} \frac{H(t, x^{(k)})}{\rho_k^\alpha} \leq \int_{\Omega_0} \limsup_{k \rightarrow \infty} \frac{H(t, x^{(k)})}{|x^{(k)}|^\alpha} |\tilde{x}^{(k)}|^\alpha \leq \int_{\Omega_0} \nu(t) |\tilde{x}|^\alpha \leq 0,$$

where Ω_0 is the set where $|x(t)^{(k)}| \rightarrow \infty$. Moreover, $\tilde{x} = 0$ on $\Omega_1 = I \setminus \Omega_0$. Since $\nu(t) < 0$ a.e., it follows that $\tilde{x} = 0$ a.e. The rest of the proof proceeds as before.

Proof of Theorem 1.7.

Let

$$\alpha = \inf_{\mathcal{M}} G(x).$$

There is a sequence $\{x^{(k)}\} \in \mathcal{M}$ such that

$$(57) \quad G(x^{(k)}) = d(x^{(k)}) - 2 \int_I V(t, x^{(k)}(t)) dt \rightarrow \alpha,$$

$$(58) \quad (G'(x^{(k)}), z)/2 = d(x^{(k)}, z) - \int_I \nabla_x V(t, x^{(k)}) \cdot z(t) dt = 0, \quad z \in H$$

and

$$(59) \quad (G'(x^{(k)}), x^{(k)})/2 = d(x^{(k)}) - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)} dt = 0.$$

Thus,

$$\int_I H(t, x^{(k)}(t)) dt = -G(x^{(k)}) \rightarrow -\alpha.$$

This implies that

$$\rho_k = \|x^{(k)}\|_H \leq C.$$

Hence, there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in H and uniformly on I . From (57) and (58) we see that

$$(60) \quad G(x) = d(x) - 2 \int_I V(t, x(t)) dt = \alpha,$$

and

$$(G'(x), z)/2 = d(x, z) - \int_I \nabla_x V(t, x(t)) \cdot z(t) dt = 0, \quad z \in H,$$

from which we conclude easily that x is a solution of (1). Hence, $x \in \mathcal{M}$ and $G(x) = \alpha$. This completes the proof.

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