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Bergman kernel asymptotics and exponential weights on phase space

A dissertation submitted in partial satisfaction of the

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in Mathematics

by

Matthew Ian Stone

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ABSTRACT OF THE DISSERTATION

Bergman kernel asymptotics and exponential weights on phase space

by

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Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2022
Professor Mikhail Khitrik, Chair

This dissertation has two parts. In the first part, we extend the direct approach to the semiclassical Bergman kernel asymptotics, developed recently in [12] for real analytic exponential weights, to the smooth case. Similar to [12], our approach avoids the use of the Kuranishi trick and it allows us to construct the amplitude of the asymptotic Bergman projection by means of an asymptotic inversion of an explicit Fourier integral operator. In the second part, we carry out a construction of a globally defined weight function on the phase space, associated to a class of non-self-adjoint semiclassical pseudodifferential operators with double characteristics.

This dissertation of Matthew Ian Stone is approved.

Wilfrid Dossou Gangbo

Ko Honda

Monica Vişan

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University of California, Los Angeles

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To my family, and the giants on whose shoulders I've stood

Contents

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VITA

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Chapter 1

Introduction

1.1 Overview

The purpose of this dissertation is to address some problems of semiclassical analysis in the complex domain. The new results established here are concerned with the following two topics, discussed in Chapters 2 and 3 of the thesis, respectively:

- Semiclassical asymptotics for Bergman kernels with smooth weights
- Operators with double characteristics and exponential weights on phase space

While these works can be read independently of each other, there is a common thread present throughout the thesis, namely that of pursuing microlocal analysis in exponentially weighted spaces of holomorphic functions.

We shall now proceed to give some background and motivation for each of the two works, providing also informal statements of the main results.

1.2 Semiclassical asymptotics for Bergman kernels with smooth weights

Let $\Omega \subset \mathbf{C}^n$ be open pseudoconvex and let $\Phi \in C^\infty(\Omega; \mathbf{R})$ be a strictly plurisubharmonic function, $i\partial\overline{\partial}\Phi > 0$. Associated to Φ is the exponentially weighted L^2 space

$$L^2_\Phi(\Omega) = \left\{ f : \Omega \to \mathbf{C};\ \int_\Omega |f(x)|^2 \, e^{-2\Phi(x)/h} \, L(dx) < \infty \right\}, \tag{1.2.1}$$

along with the subspace of holomorphic functions,

$$H_\Phi(\Omega) = \{ f \in L^2_\Phi(\Omega);\ \partial_{\overline{x}_j} f \equiv 0,\ j = 1, \ldots, n \}. \tag{1.2.2}$$

Here $L(dx)$ is the Lebesgue measure on $\mathbf{C}^n$ and $0 < h \ll 1$ is the semiclassical parameter. In the special case when $\Omega = \mathbf{C}^n$ and Φ is quadratic, the space $H_\Phi(\mathbf{C}^n)$ has been considered by V. Bargmann [1], as a natural setting for quantum mechanics expressed directly in the phase space $T^*\mathbf{R}^n \simeq \mathbf{C}^n$. More generally, a wealth of problems in analysis, physics, and geometry leads to questions involving exponentially weighted spaces of holomorphic functions of the form $H_\Phi(\Omega)$, from complex geometry and Toeplitz quantization [2], [8], to analytic microlocal analysis and the theory of FBI transforms [39], with applications to spectral theory [19] and inverse problems [32].

Since the subspace $H_\Phi(\Omega) \subset L^2_\Phi(\Omega)$ is closed, an orthogonal projection onto it,

$$\Pi_\Phi : L^2_\Phi(\Omega) \to H_\Phi(\Omega), \tag{1.2.3}$$

which we shall refer to as the Bergman projection, makes a natural appearance. Introducing the Schwartz kernel of Π_Φ, let us write

$$\Pi_\Phi u(x) = \int K(x,\overline{y})u(y)e^{-2\Phi(y)/h}\, L(dy), \tag{1.2.4}$$

where $K(x,\widetilde{y}) \in \mathrm{Hol}(\Omega \times \overline{\Omega})$, $\overline{\Omega} = \{x \in \mathbf{C}^n; \overline{x} \in \Omega\}$. The existence of a complete asymptotic expansion for the Bergman kernel $K(x,\overline{y})$ close to the diagonal in the semiclassical limit $h \to 0^+$ has been established in [6], [42], in the context of high powers of a holomorphic line bundle over a compact complex manifold. The original proofs in [6], [42] rely on the fundamental work [5], which in turn depends, in particular, on the full fledged machinery of Fourier integral operators with complex phase functions due to [35], and thus are not quite self-contained and explicit. A more explicit direct approach to the Bergman kernel asymptotics has subsequently been developed in [2], relying on a suitable change of variables, usually referred to as the *Kuranishi trick* in microlocal analysis. Now there exist interesting and natural situations where the Kuranishi trick becomes somewhat complicated to execute, e.g. when the Levi form $i\partial\overline{\partial}\Phi \geq 0$ develops some almost degenerate directions. Such nearly degenerate weights occur naturally, in particular, in the work in progress [25], devoted to a heat evolution approach to second microlocalization with respect to a real analytic hypersurface. Having an alternative direct approach to the semiclassical asymptotics for Bergman projections, not relying upon any changes of variables, becomes then valuable and significant, and such an approach has been developed in [12] recently in the non-degenerate case, assuming that the exponential weight Φ is real analytic. This is not sufficient for many applications, however, where the weight may be merely C^∞ or even of finite regularity. The principal achievement of our work in Chapter 2, see also [26], consists precisely of developing a direct approach to the semiclassical asymptotics for the kernel of Π_Φ in (1.2.3), avoiding the use of the Kuranishi trick, for general

C^∞ exponential weights.

Let us now state the main result established in Chapter 2.

Theorem 1.2.1. Let $\Omega \subset \mathbf{C}^n$ be open and let $\Phi \in C^\infty(\Omega; \mathbf{R})$ be strictly plurisubharmonic in Ω. Let $x_0 \in \Omega$. There exist a classical elliptic symbol $a(x, \widetilde{y}; h)$ in a neighborhood of $(x_0, \overline{x_0}) \in \mathbf{C}^{2n}$ of the form

$$a(x, \widetilde{y}; h) \sim \sum_{j=0}^{\infty} h^j a_j(x, \widetilde{y}),$$

in C^∞, with $a_j \in C^\infty(\mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}))$, holomorphic to ∞–order along the anti-diagonal $\widetilde{y} = \overline{x}$, satisfying

$$(Aa)(x, \overline{x}; h) = 1 + \mathcal{O}(h^\infty), \quad x \in \mathrm{neigh}(x_0, \mathbf{C}^n), \tag{1.2.5}$$

where A is an explicit elliptic Fourier integral operator, and small open neighborhoods $U \Subset V \Subset \Omega$ of x_0, such that the operator

$$\widetilde{\Pi}_V u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h}\Psi(x,\overline{y})} a(x, \overline{y}; h) u(y) e^{-\frac{2}{h}\Phi(y)}\, dy\, d\overline{y} \tag{1.2.6}$$

satisfies

$$\widetilde{\Pi}_V - 1 = \mathcal{O}(h^\infty) : H_\Phi(V) \to L^2(U, e^{-2\Phi/h}\, L(dx)). \tag{1.2.7}$$

Here in (1.2.6), the C^∞ function Ψ is holomorphic to ∞–order along the anti-diagonal, $\Psi(x, \overline{x}) = \Phi(x)$, and $L(dx)$ is the Lebesgue measure on $\mathbf{C}^n$.

We would like to emphasize here that adapting the arguments of [12] from the real analytic case to the smooth setting is far from routine, involving, in addition to the subtle techniques of almost holomorphic extensions, some essential new ideas of the $\overline{\partial}$

method. It also seems that the approach to computing the coefficients in the amplitude of the asymptotic Bergman projection, consisting of inverting the Fourier integral operator A in (1.2.5) asymptotically, is quite efficient and direct.

1.3 Operators with double characteristics and exponential weights on phase space

The study of (pseudo)differential operators with double characteristics has a long tradition in PDE [38], [4], and recent advances in our understanding of spectral asymptotics for Kramers-Fokker-Planck operators in the semiclassical limit [19] have given new impetus to the subject. In Chapter 3 of this thesis we are concerned with the spectral analysis for a broad class of semiclassical operators with double characteristics on $\mathbf{R}^n$, which is a natural generalization of that considered in [22], [23]. This class comprises, in particular, some natural non-self-adjoint accretive semiclassical Schrödinger operators, with discrete spectrum in a neighborhood of the imaginary axis.

Let $P = P(x,\xi;h) \in S(1)$ in the sense that $P(x,\xi;h) \in C^\infty(\mathbf{R}^{2n})$ is such that $\partial^\alpha_{x,\xi} P \in L^\infty(\mathbf{R}^{2n})$ for all $\alpha \in \mathbf{N}^{2n}$, and assume that

$$P(x,\xi;h) \sim \sum_{j=0}^{\infty} h^j p_j(x,\xi), \tag{1.3.8}$$

as $h \to 0^+$, in the space of such functions. We assume that the semiclassical leading symbol p_0 of P satisfies $\operatorname{Re} p_0 \geq 0$, as well as

$$\operatorname{Re} p_0(x,\xi) \geq \frac{1}{C}, \quad |(x,\xi)| \geq C, \tag{1.3.9}$$

for some $C > 0$. Associated to the symbol P is its semiclassical Weyl quantization,

$$P^w(x, hD_x; h)u(x) = \frac{1}{(2\pi h)^n} \iint_{\mathbf{R}^{2n}} e^{i\frac{(x-y)\cdot\xi}{h}} P\left(\frac{x+y}{2}, \xi; h\right) u(y) dy\, d\xi. \tag{1.3.10}$$

Broadly speaking, our goal is to determine the eigenvalues of the operator P^w, modulo $\mathcal{O}(h^\infty)$, in the semiclassical limit $h \to 0^+$, in an h–dependent region of the complex spectral plane of the form $\{z \in \mathbf{C}; \operatorname{Re} z = \mathcal{O}(h)\}$. This region is close to the boundary of the range of the leading symbol p_0, which we can roughly view as the "semiclassical pseudospectrum" of $P^w(x, hD_x; h)$, and the eigenvalues in this region can therefore be viewed as being physically the most relevant — indeed, the eigenvalues z such that $\operatorname{Re} z \gg h$ are too deep in the right half plane to have useful dynamical interpretation.

When carrying out the spectral analysis of $P^w(x, hD_x; h)$, we shall assume that the set

$$\mathcal{C} := \{X \in \mathbf{R}^{2n}; \operatorname{Re} p_0(X) = 0,\ H_{\operatorname{Im} p_0}(X) = 0\} \tag{1.3.11}$$

is finite, $\mathcal{C} = \{X_1, \ldots, X_N\}$. Here $H_{\operatorname{Im} p_0} = \partial_\xi \operatorname{Im} p_0 \cdot \partial_x - \partial_x \operatorname{Im} p_0 \cdot \partial_\xi$ is the Hamilton vector field of $\operatorname{Im} p_0$. Introducing the time averages

$$\langle \operatorname{Re} p_0 \rangle_{T, \operatorname{Im} p_0}(X) = \frac{1}{T} \int_0^T \operatorname{Re} p_0 \left(\exp\left(t H_{\operatorname{Im} p_0}\right)(X)\right)\, dt, \quad T > 0, \tag{1.3.12}$$

of $\operatorname{Re} p_0$ along the $H_{\operatorname{Im} p_0}$–trajectories, we can state our main hypotheses: first, we assume that for each $T > 0$ fixed, the following dynamical condition holds for $1 \leq j \leq N$,

$$\langle \operatorname{Re} p_0 \rangle_{T, \operatorname{Im} p_0}(X) \asymp \left|X - X_j\right|^2, \quad X \in \operatorname{neigh}(X_j, \mathbf{R}^{2n}). \tag{1.3.13}$$

We furthermore assume that in any set of the form

$$|X| \leq C, \quad \mathrm{dist}\,(X, \mathcal{C}) \geq \frac{1}{C}, \tag{1.3.14}$$

we have for each $T > 0$ fixed,

$$\langle \mathrm{Re}\, p_0 \rangle_{T,p_0}(X) \geq \frac{1}{\widetilde{C}(C)}, \quad \widetilde{C}(C) > 0. \tag{1.3.15}$$

The assumptions (1.3.11), (1.3.13), (1.3.15) are more general and natural than those made in [22], [23], and apply to a broad class of semiclassical Schrödinger operators with complex potentials, not covered by the analysis of those works. Roughly speaking, in the context of Schrödinger operators, the principal novelty is to allow the real part of the potential to vanish on a large open set, with the localization of the eigenvalues in the $\mathcal{O}(h)$–neighborhood of the imaginary axis being due to critical points of the imaginary part of the potential.

Following the general ideas and techniques of [19], [22], [23], the spectral analysis of the operator $P^w(x, hD_x; h)$ is to be carried out in a suitable microlocally exponentially weighted L^2-space, and Chapter 3 establishes the groundwork for this project, by constructing a suitable globally defined exponential phase space weight. Roughly speaking, given a suitable weight function $G \in C_0^\infty(\mathbf{R}^{2n}; \mathbf{R})$, one considers the formally conjugated operator,

$$e^{-\varepsilon G^w(x,hD_x)/h} \circ P^w(x, hD_x; h) \circ e^{\varepsilon G^w(x,hD_x)/h}, \tag{1.3.16}$$

acting on $L^2(\mathbf{R}^n)$ — a rigorous definition is obtained by modifying an exponential weight on the FBI-Bargmann transform side, see [19], [22], and the references given there. Here we view $e^{\varepsilon G^w(x,hD_x)/h}$ as a Fourier integral operator with the associated canonical transformation $\exp(i\varepsilon H_G)$, approximately equal to $(x,\xi) \mapsto (x,\xi) + i\varepsilon H_G(x,\xi)$, since $0 < \varepsilon$ will be small. Indeed, following the general philosophy of the method of "bounded exponential weights", we shall take $\varepsilon = \mathcal{O}(h)$. By Egorov's theorem we expect the conjugated operator in (1.3.16) to be an h-pseudodifferential operator with the leading symbol

$$p_0\left(\exp\left(i\varepsilon H_G(x,\xi)\right)\right) \approx p_0\left((x,\xi) + i\varepsilon H_G(x,\xi)\right) \approx p_0(x,\xi) - i\varepsilon H_{p_0}(G). \tag{1.3.17}$$

For a suitable weight G, the new leading symbol in (1.3.17) will have an increased real part, which is of crucial importance for the spectral analysis of $P^w(x, hD_x; h)$. The following is the main result of Chapter 3.

Theorem 1.3.1. Let $P(x,\xi;h) \in S(1)$ be such that the assumptions (1.3.9), (1.3.11), (1.3.13), (1.3.15) hold, and let $\widetilde{p}_0 \in C^\infty(\mathbf{C}^{2n})$ be an almost holomorphic extension of the semiclassical leading symbol p_0 of P, supported in a tubular neighborhood of $\mathbf{R}^{2n}$, bounded together with all derivatives. There exist constants $\widetilde{C} > 1$, $0 < \delta_0 \leq 1$, $0 < \varepsilon_0 \leq 1$, and a function $\widetilde{G}_\varepsilon \in C_0^\infty(\mathbf{R}^{2n}; \mathbf{R})$ depending on the parameter $0 < \varepsilon \leq \varepsilon_0$, satisfying

$$\partial^\alpha G_\varepsilon = \mathcal{O}(\varepsilon^{1-\frac{|\alpha|}{2}}), \quad |\alpha| \leq 2,$$

uniformly, such that we have for all $0 < \varepsilon \leq \varepsilon_0$, $0 < \delta \leq \delta_0$,

$$\operatorname{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))\right) \geq \frac{\delta \operatorname{dist}(X,\mathcal{C})^2}{\widetilde{C}}, \quad \operatorname{dist}(X,\mathcal{C}) \leq \varepsilon^{1/2},$$

$$\operatorname{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))\right) \geq \frac{\delta\,\varepsilon}{\widetilde{C}}, \quad \operatorname{dist}(X,\mathcal{C}) \geq \varepsilon^{1/2}.$$

A precise spectral analysis for the operator $P^w(x, hD_x; h)$, relying crucially on Theorem

1.3.1, remains to be carried out, and will be pursued after the completion of the Ph.D. thesis.

Chapter 2

Asymptotics for Bergman projections with smooth weights: a direct approach.

2.1 Introduction

Let $\Omega \subset \mathbf{C}^n$ be an open pseudoconvex domain and let $\Phi \in C^\infty(\Omega)$ be a strictly plurisubharmonic function. Exponentially weighted spaces of holomorphic functions of the form $H_\Phi(\Omega) = \mathrm{Hol}(\Omega) \cap L^2(\Omega, e^{-2\Phi/h})$ occur naturally in analytic microlocal analysis [39], [17], [40], [36], [24], among other areas, when passing from the real to the complex domain by means of an FBI transform [39], [40]. Associated to the space $H_\Phi(\Omega)$ is the orthogonal (Bergman) projection

$$\Pi : L^2(\Omega, e^{-2\Phi/h}) \to H_\Phi(\Omega), \tag{2.1.1}$$

and complex microlocal techniques have long been known to be useful in the study of the asymptotic behavior of Π in the semiclassical limit $h \to 0^+$. The existence of a complete asymptotic expansion for the Schwartz kernel of Π close to the diagonal has been established in the pioneering contributions [6], [42], in the context of high powers of a holomorphic line bundle with positive curvature, over a complex compact manifold. See also [7], [10], [27], [33], as well as [15], [16] for the case of domains in $\mathbf{C}^n$. The original proofs in [6], [42] relied on the description of singularities of the Szegő kernel on the boundary of a strictly pseudoconvex smooth domain given by the Boutet de Monvel – Sjöstrand parametrix in the seminal work [5], which in turn depended, in particular, on the theory of Fourier integral operators with complex phase functions developed in [35]. More self-contained explicit approaches to the Bergman kernel asymptotics have subsequently been developed in [36, Section 3] and [2], with the former work starting with the approximate projection property $\widetilde{\Pi}^2 = \widetilde{\Pi} + \mathcal{O}(h^\infty)$, while the approach of [2] focuses more directly on the reproducing property of the Bergman projection on the space $H_\Phi(\Omega)$.

To motivate a bit further, let us recall that the basic idea of the approach of [2] consists, roughly speaking, of expressing the identity operator on $H_\Phi(\Omega)$ in such a way that it automatically becomes the asymptotic Bergman projection, at least locally. This is accomplished, essentially, by first representing the identity as an h–pseudodifferential operator in the complex domain, and then passing to a non-standard phase via a suitable change of variables, usually referred to as the Kuranishi trick. See [39, Chapter 4], [2, Section 2.2], and also [18, Chapter 3] for the standard application of the Kuranishi trick to changes of variables for pseudodifferential operators in the real domain. Now the Kuranishi trick becomes somewhat complicated to execute in situations when the Levi form $i\partial\overline{\partial}\Phi \geq 0$ of the weight Φ becomes almost degenerate in some directions. Such nearly degenerate weights occur naturally, in particular, in the work in progress [25], devoted to a heat evolution approach to second microlocalization with respect to a real analytic hypersurface. A direct approach to the semiclassical asymptotics for Bergman projections, not relying upon any changes of variables, is therefore desirable, and it has recently been developed in [12] in the non-degenerate case, assuming that the weight Φ is real analytic. The approach of [12] has allowed, in particular, to give a quick proof of a result of [11], [37], stating that in the analytic case, the amplitude of the asymptotic Bergman projection is given by a classical analytic symbol. Our purpose here is to extend the approach of [12] to the case of weights that are merely C^∞ but not necessarily analytic. The following is the main result of this work.

Theorem 2.1.1. Let $\Omega \subset \mathbf{C}^n$ be open and let $\Phi \in C^\infty(\Omega; \mathbf{R})$ be strictly plurisubharmonic in Ω. Let $x_0 \in \Omega$. There exist a classical elliptic symbol $a(x, \widetilde{y}; h) \in S^0_{\rm cl}({\rm neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}))$ of the form

$$a(x, \widetilde{y}; h) \sim \sum_{j=0}^{\infty} h^j a_j(x, \widetilde{y}),$$

in C^∞, with $a_j \in C^\infty({\rm neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}))$, holomorphic to ∞–order along the anti-diagonal $\widetilde{y} = \overline{x}$, satisfying

$$(Aa)(x, \overline{x}; h) = 1 + \mathcal{O}(h^\infty), \quad x \in {\rm neigh}(x_0, \mathbf{C}^n), \tag{2.1.2}$$

where A is an elliptic Fourier integral operator, and small open neighborhoods $U \Subset V \Subset \Omega$ of x_0, with C^∞–boundaries, such that the operator

$$\widetilde{\Pi}_V u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h}\Psi(x,\overline{y})} a(x, \overline{y}; h) u(y) e^{-\frac{2}{h}\Phi(y)}\, dy\, d\overline{y} \tag{2.1.3}$$

satisfies

$$\widetilde{\Pi}_V - 1 = \mathcal{O}(h^\infty) : H_\Phi(V) \to L^2(U, e^{-2\Phi/h}\, L(dx)), \tag{2.1.4}$$

in the sense that the operator norm of $\widetilde{\Pi}_V - 1$ as a linear continuous map from $H_\Phi(V)$ to $L^2(U, e^{-2\Phi/h}\, L(dx))$ is $\mathcal{O}(h^\infty)$. Here in (2.1.3), the C^∞ function Ψ is holomorphic to ∞–order along the anti-diagonal, $\Psi(x, \overline{x}) = \Phi(x)$, and $L(dx)$ is the Lebesgue measure on $\mathbf{C}^n$.

When proving Theorem 2.1.1, we proceed largely along the general lines of [12], and the essential new ingredient in the proofs is the use of the techniques of almost holomorphic extensions [29], [34],[35]. The lack of holomorphy causes some of the estimates and asymptotic constructions in the proofs to become a bit more explicit and refined, demanding a greater technical investment overall. Compared to [12], we also have to rely on the L^2 estimates for the $\overline{\partial}$ operator [28] even more, to account for the fact that functions in the range of the operator $\widetilde{\Pi}_V$ in (2.1.3) are not quite holomorphic. The plan of the paper is as follows: In Section 2.2, we introduce an explicit elliptic Fourier integral operator A in the complex domain, with the phase defined via an almost holomorphic extension of the weight Φ, and obtain a C^∞ symbol a, holomorphic to ∞–order along the anti-diagonal, as a solution of (2.1.2). Let us observe that while the corresponding discussion in [12] in the analytic case makes use of the existence of a microlocal inverse of the corresponding analytic Fourier integral operator, here the asymptotic inversion of A, with $\mathcal{O}(h^\infty)$ errors, proceeds more directly by equipping the operator A with a suitable explicit contour of integration and by considering the stationary phase expansion for Aa. Section 2.3 is devoted to showing the approximate reproducing property for the operator $\widetilde{\Pi}_V$ in (2.1.3), on the level of scalar products, $(\widetilde{\Pi}_V u, v)_{L^2_\Phi(V)} = (u, v)_{H_\Phi(V)} + \mathcal{O}(h^\infty)$, for $u, v \in H_\Phi(V)$, with v concentrated in a small neighborhood of x_0. Similar to [12], the proof depends on a resolution of the identity and a contour deformation argument, with some additional care required due to the lack of holomorphy in (2.1.3). The proof of Theorem 2.1.1 is then concluded in Section 2.4, making use of the $\overline{\partial}$ techniques. Finally, in Section 2.5, we recall, following [2], the link between the operator $\widetilde{\Pi}_V$ in Theorem 2.1.1 and the orthogonal projection (2.1.1), showing that the kernels of (2.1.1) and (2.1.3) are close pointwise, locally.

2.2 Asymptotic inversion of a Fourier integral operator

The discussion in this section can be viewed as a natural analog in the C^∞–setting of [12, Section 3], working systematically with almost holomorphic extensions of the weights, [29], [35]. Let $\Omega \subset \mathbf{C}^n$ be open, and let $\Phi \in C^\infty(\Omega; \mathbf{R})$ be strictly plurisubharmonic in Ω,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial x_j \partial \overline{x}_k}(x)\xi_j \overline{\xi}_k \geq c(x)\left|\xi\right|^2, \quad x \in \Omega, \quad \xi \in \mathbf{C}^n, \tag{2.2.1}$$

where $0 < c \in C(\Omega)$. Let $x_0 \in \Omega$. Identifying $\mathbf{C}^n_x$ with the anti-diagonal $\{(x,y) \in \mathbf{C}^{2n}; y = \overline{x}\}$, we see that exists $\Psi \in C^\infty(\mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}_{x,y}))$ such that

$$\Psi(x, \overline{x}) = \Phi(x), \quad x \in \mathrm{neigh}(x_0, \mathbf{C}^n), \tag{2.2.2}$$

and for every N,

$$\left(\partial_{\overline{x}_j}\Psi\right)(x,y) = \mathcal{O}_N(|y - \overline{x}|^N), \quad \left(\partial_{\overline{y}_j}\Psi\right)(x,y) = \mathcal{O}_N(|y - \overline{x}|^N), \quad 1 \leq j \leq n, \tag{2.2.3}$$

locally uniformly, see [29], [34], [35], [14, Chapter 8]. Here we work with the usual operators,

$$\partial_{\overline{x}_j} = \frac{1}{2}\left(\partial_{\mathrm{Re}\, x_j} + i\partial_{\mathrm{Im}\, x_j}\right), \quad \partial_{\overline{y}_j} = \frac{1}{2}\left(\partial_{\mathrm{Re}\, y_j} + i\partial_{\mathrm{Im}\, y_j}\right), \quad 1 \leq j \leq n. \tag{2.2.4}$$

$$\partial_{x_j} = \frac{1}{2}\left(\partial_{\mathrm{Re}\, x_j} - i\partial_{\mathrm{Im}\, x_j}\right), \quad \partial_{y_j} = \frac{1}{2}\left(\partial_{\mathrm{Re}\, y_j} - i\partial_{\mathrm{Im}\, y_j}\right), \quad 1 \leq j \leq n. \tag{2.2.5}$$

We notice that (2.2.3) and [41, Lemma X.2.2] imply that for all α, β, γ, $\delta \in \mathbf{N}^n$ we have

$$\left(D_x^\alpha D_{\overline{x}}^\beta D_y^\gamma D_{\overline{y}}^\delta \partial_{\overline{x}_j}\Psi\right)(x,y) = \mathcal{O}_{\alpha,\beta,\gamma,\delta,N}(|y-\overline{x}|^N), \quad 1 \leq j \leq n, \tag{2.2.6}$$

$$\left(D_x^\alpha D_{\overline{x}}^\beta D_y^\gamma D_{\overline{y}}^\delta \partial_{\overline{y}_j}\Psi\right)(x,y) = \mathcal{O}_{\alpha,\beta,\gamma,\delta,N}(|y-\overline{x}|^N), \quad 1 \leq j \leq n, \tag{2.2.7}$$

locally uniformly.

Our starting point is the following classical estimate, see for instance [2]. Since related computations based on Taylor expansions will appear below, it will be natural to recall the proof.

Proposition 2.2.1. We have

$$\Phi(x) + \Phi(y) - 2\mathrm{Re}\,\Psi(x,\overline{y}) \asymp |x-y|^2, \quad x, y \in \mathrm{neigh}(x_0, \mathbf{C}^n). \tag{2.2.8}$$

Proof: Using (2.2.2), (2.2.6), (2.2.7) we get

$$(\partial_x \Psi)(x,\overline{x}) = \partial_x \Phi(x), \;\; (\partial_y \Psi)(x,\overline{x}) = \partial_{\overline{x}}\Phi(x), \tag{2.2.9}$$

$$\Psi''_{xx}(x,\overline{x}) = \Phi''_{xx}(x), \quad \Psi''_{xy}(x,\overline{x}) = \Phi''_{x\overline{x}}(x), \quad \Psi''_{yy}(x,\overline{x}) = \Phi''_{\overline{x}\,\overline{x}}(x). \tag{2.2.10}$$

By a Taylor expansion, we obtain then, using (2.2.2), (2.2.6), (2.2.7), (2.2.9), (2.2.10),

$$\Psi(x+z,\overline{x}+w) = \Phi(x) + \partial_x\Phi(x)\cdot z + \partial_{\overline{x}}\Phi(x)\cdot w \\ + \frac{1}{2}\left(\Phi''_{xx}(x)z\cdot z + 2\Phi''_{x\overline{x}}(x)w\cdot z + \Phi''_{\overline{x}\,\overline{x}}(x)w\cdot w\right) + \mathcal{O}(|(z,w)|^3),$$

and therefore,

$$2\mathrm{Re}\,\Psi(x+z,\overline{x}+w) = 2\Phi(x) + \partial_x\Phi(x)\cdot(z+\overline{w}) + \overline{\partial_x\Phi(x)}\cdot(\overline{z}+w) \\ + \frac{1}{2}\left(2\mathrm{Re}\,\left(\Phi''_{xx}(x)z\cdot z + \Phi''_{xx}(x)\overline{w}\cdot\overline{w}\right) + 2\Phi''_{x\overline{x}}(x)w\cdot z + 2\Phi''_{\overline{x}x}(x)\overline{w}\cdot\overline{z}\right) + \mathcal{O}(|(z,w)|^3). \tag{2.2.11}$$

Here we have used that $\partial_{\overline{x}}\Phi(x) = \overline{\partial_x\Phi(x)}$. Using also the Taylor expansions

$$\Phi(x+z) = \Phi(x) + \partial_x\Phi(x)\cdot z + \overline{\partial_x\Phi}\cdot\overline{z} \\ + \frac{1}{2}\left(\Phi''_{xx}(x)z\cdot z + 2\Phi''_{x\overline{x}}(x)\overline{z}\cdot z + \Phi''_{\overline{x}\,\overline{x}}(x)\overline{z}\cdot\overline{z}\right) + \mathcal{O}(|z|^3), \tag{2.2.12}$$

$$\Phi(x+\overline{w}) = \Phi(x) + \partial_x\Phi(x)\cdot\overline{w} + \overline{\partial_x\Phi}(x)\cdot w \\ + \frac{1}{2}\left(\Phi''_{xx}(x)\overline{w}\cdot\overline{w} + 2\Phi''_{x\overline{x}}(x)w\cdot\overline{w} + \Phi''_{\overline{x}\,\overline{x}}(x)w\cdot w\right) + \mathcal{O}(|w|^3), \tag{2.2.13}$$

we get from (2.2.11), (2.2.12), and (2.2.13),

$$\Phi(x+z)+\Phi(x+\overline{w})-2\mathrm{Re}\,\Psi(x+z,\overline{x}+w) = \Phi''_{x\overline{x}}(x)(w-\overline{z})\cdot(\overline{w}-z)+\mathcal{O}(|(z,w)|^3). \tag{2.2.14}$$

Hence we get for $x, y \in \mathrm{neigh}(x_0, \mathbf{C}^n)$,

$$\Phi(x) + \Phi(y) - 2\mathrm{Re}\,\Psi(x,\overline{y}) = \Phi''_{x\overline{x}}(x_0)(\overline{y}-\overline{x})\cdot(y-x) + \mathcal{O}(|y-x|^3), \tag{2.2.15}$$

and using the strict plurisubharmonicity of Φ given in (2.2.1), we infer (2.2.8). $\square$

Let us set, following [12, Section 3],

$$\varphi(y,\widetilde{x};x,\widetilde{y}) = \Psi(x,\widetilde{y}) - \Psi(x,\widetilde{x}) - \Psi(y,\widetilde{y}) + \Psi(y,\widetilde{x}). \tag{2.2.16}$$

We have $\varphi \in C^\infty(\mathrm{neigh}((x_0,\overline{x_0};x_0,\overline{x_0}),\mathbf{C}^{4n}))$. Of particular interest for us here are critical points of $(x,\widetilde{y}) \mapsto \varphi(y,\widetilde{x};x,\widetilde{y})$.

Proposition 2.2.2. For each $(y,\widetilde{x}) \in \mathrm{neigh}((x_0,\overline{x_0}),\mathbf{C}^{2n})$, the complex valued C^∞ function

$$\mathrm{neigh}((x_0,\overline{x_0}),\mathbf{C}^{2n}) \ni (x,\widetilde{y}) \mapsto \varphi(y,\widetilde{x};x,\widetilde{y})$$

has a unique critical point given by $(x,\widetilde{y}) = (y,\widetilde{x})$. The corresponding critical value is equal to 0, and when $\widetilde{x} = \overline{y}$, the quadratic part of the Taylor expansion of the C^∞ function $\mathbf{C}^{2n} \ni (z,w) \mapsto \varphi(y,\widetilde{x};y+z,\widetilde{x}+w)$ at $(z,w) = (0,0)$ is the non-degenerate holomorphic quadratic form on $\mathbf{C}^{2n}$ given by $(z,w) \mapsto \Phi''_{x\overline{x}}(y)w\cdot z$.

Proof: We shall consider the Taylor expansion of the C^∞ function $(z,w) \mapsto \varphi(y,\widetilde{x};y+z,\widetilde{x}+w)$ at $(z,w) = (0,0) \in \mathbf{C}^{2n}$. Here $(y,\widetilde{x}) \in \mathrm{neigh}((x_0,\overline{x_0}),\mathbf{C}^{2n})$. Let us write, using Taylor's formula and the almost holomorphy of Ψ along the anti-diagonal, as in (2.2.3), (2.2.6), (2.2.7),

$$\begin{aligned}\Psi(y+z,\widetilde{x}+w) = \Psi(y,\widetilde{x}) + \Psi'_x(y,\widetilde{x})\cdot z + \Psi'_{\overline{x}}(y,\widetilde{x})\cdot\overline{z} + \Psi'_y(y,\widetilde{x})\cdot w + \Psi'_{\overline{y}}(y,\widetilde{x})\cdot\overline{w}\\ + \frac{1}{2}\left(\Psi''_{xx}(y,\widetilde{x})z\cdot z + 2\Psi''_{xy}(y,\widetilde{x})w\cdot z + \Psi''_{yy}(y,\widetilde{x})w\cdot w\right)\\ + \mathcal{O}_N(|\overline{y}-\widetilde{x}|^N)(|(z,w)|^2) + \mathcal{O}(|(z,w)|^3),\end{aligned} \tag{2.2.17}$$

$$\Psi(y,\widetilde{x}+w)=\Psi(y,\widetilde{x})+\Psi'_y(y,\widetilde{x})\cdot w+\Psi'_{\overline{y}}(y,\widetilde{x})\cdot\overline{w}$$
$$+\frac{1}{2}\Psi''_{yy}(y,\widetilde{x})w\cdot w+\mathcal{O}_N(|\overline{y}-\widetilde{x}|^N)\,|w|^2+\mathcal{O}(|w|^3). \quad (2.2.18)$$

$$\Psi(y+z,\widetilde{x})=\Psi(y,\widetilde{x})+\Psi'_x(y,\widetilde{x})\cdot z+\Psi'_{\overline{x}}(y,\widetilde{x})\cdot\overline{z}$$
$$+\frac{1}{2}\Psi''_{xx}(y,\widetilde{x})z\cdot z+\mathcal{O}_N(|\overline{y}-\widetilde{x}|^N)\,|z|^2+\mathcal{O}(|z|^3). \quad (2.2.19)$$

Here $N\in\mathbf{N}$ is arbitrary. We get, using (2.2.16), (2.2.17), (2.2.18), (2.2.19),

$$\varphi(y,\widetilde{x};y+z,\widetilde{x}+w)=\Psi''_{xy}(y,\widetilde{x})w\cdot z+\mathcal{O}_N(|\overline{y}-\widetilde{x}|^N)(|(z,w)|^2)+\mathcal{O}(|(z,w)|^3). \quad (2.2.20)$$

This shows in particular that $(z,w)=(0,0)$ is a critical point of $(z,w)\mapsto\varphi(y,\widetilde{x};y+z,\widetilde{x}+w)$, with the corresponding critical value being equal to 0. We notice also that the matrix $\Psi''_{xy}(y,\widetilde{x})$ is invertible for $(y,\widetilde{x})\in\operatorname{neigh}((x_0,\overline{x_0}),\mathbf{C}^{2n})$. Restricting $(y,\widetilde{x})$ in (2.2.20) to the anti-diagonal in $\mathbf{C}^{2n}$, i.e. letting $\widetilde{x}=\overline{y}$, and using (2.2.10), we get,

$$\varphi(y,\overline{y};y+z,\overline{y}+w)=\Phi''_{x\overline{x}}(y)w\cdot z+\mathcal{O}(|(z,w)|^3). \quad (2.2.21)$$

Here, in view of (2.2.1), the holomorphic quadratic form $(z,w)\mapsto\Phi''_{x\overline{x}}(y)w\cdot z$ is non-degenerate on $\mathbf{C}^{2n}_{z,w}$, for $y\in\operatorname{neigh}(x_0,\mathbf{C}^n)$, and this completes the proof. $\square$

It follows from the last observation in the proof of Proposition 2.2.2 that the pluri-harmonic quadratic form $q(z,w):=\operatorname{Re}(\Phi''_{x\overline{x}}(x_0)w\cdot z)$ is non-degenerate on $\mathbf{C}^{2n}_{z,w}$, and hence necessarily of signature $(2n,2n)$. Explicitly, we have

$$q(z,\overline{z})=\operatorname{Re}(\Phi''_{x\overline{x}}(x_0)\overline{z}\cdot z)\asymp|z|^2,\quad z\in\mathbf{C}^n, \quad (2.2.22)$$

and therefore

$$q(iz, i\overline{z}) = -q(z, \overline{z}) \asymp -|z|^2, \quad z \in \mathbf{C}^n. \tag{2.2.23}$$

Using the continuity of $\Psi''_{xy}(y, \widetilde{x})$ we conclude that for all $(y, \widetilde{x}) \in \mathbf{C}^{2n}$ sufficiently close to $(x_0, \overline{x_0})$, we have

$$\mathrm{Re}\, \left(\Psi''_{xy}(y, \widetilde{x})\overline{z} \cdot z\right) \asymp |z|^2, \quad z \in \mathbf{C}^n, \tag{2.2.24}$$

$$\mathrm{Re}\, \left(\Psi''_{xy}(y, \widetilde{x})i\overline{z} \cdot iz\right) \asymp -|z|^2, \quad z \in \mathbf{C}^n. \tag{2.2.25}$$

Combining (2.2.20) with (2.2.24), (2.2.25), we get for all $(y, \widetilde{x}) \in \mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n})$,

$$\begin{aligned} \mathrm{Re}\, \varphi(y, \widetilde{x}; y + z, \widetilde{x} + \overline{z}) &= \mathrm{Re}\, \left(\Psi''_{xy}(y, \widetilde{x})\overline{z} \cdot z\right) + \mathcal{O}_N(|\overline{y} - \widetilde{x}|^N)\, |z|^2 + \mathcal{O}(|z|^3) \\ &\geq \frac{1}{C} |z|^2, \quad z \in \mathrm{neigh}(0, \mathbf{C}^n), \end{aligned} \tag{2.2.26}$$

$$\begin{aligned} \mathrm{Re}\, \varphi(y, \widetilde{x}; y + iz, \widetilde{x} + i\overline{z}) &= \mathrm{Re}\, \left(\Psi''_{xy}(y, \widetilde{x})i\overline{z} \cdot iz\right) + \mathcal{O}_N(|\overline{y} - \widetilde{x}|^N)\, |z|^2 + \mathcal{O}(|z|^3) \\ &\leq -\frac{1}{C} |z|^2, \quad z \in \mathrm{neigh}(0, \mathbf{C}^n), \end{aligned} \tag{2.2.27}$$

It follows from (2.2.20), (2.2.26), (2.2.27) that for each $(y, \widetilde{x}) \in \mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n})$, the critical point $(x, \widetilde{y}) = (y, \widetilde{x})$ of the real-valued C^∞ function $\mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}) \ni (x, \widetilde{y}) \mapsto \mathrm{Re}\, \varphi(y, \widetilde{x}; x, \widetilde{y})$ is non-degenerate of signature $(2n, 2n)$.

Let $\Gamma(y, \widetilde{x}) \subset \mathbf{C}^{2n}_{x,\widetilde{y}}$ be a smooth $2n$-dimensional contour of integration passing through the critical point $(x, \widetilde{y}) = (y, \widetilde{x})$ and depending smoothly on $(y, \widetilde{x}) \in \mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n})$, such that along $\Gamma(y, \widetilde{x})$, we have

$$\mathrm{Re}\, \varphi(y, \widetilde{x}; x, \widetilde{y}) \leq -\frac{1}{C} \mathrm{dist}\left((x, \widetilde{y}), (y, \widetilde{x})\right)^2. \tag{2.2.28}$$

Following [39, Chapter 3], we shall say that $\Gamma(y,\widetilde{x})$ is a good contour for the C^∞ real-valued function $(x,\widetilde{y}) \mapsto \operatorname{Re}\varphi(y,\widetilde{x};x,\widetilde{y})$. In particular, it follows from (2.2.27) that the $2n$-dimensional affine contour

$$\Gamma(y,\widetilde{x}) : \operatorname{neigh}(0,\mathbf{C}^n) \ni z \mapsto (y+z, \widetilde{x}-\overline{z}) \in \mathbf{C}^{2n}_{x,\widetilde{y}} \tag{2.2.29}$$

is good.

Proceeding similarly to [12, Section 3], we shall now introduce a suitable Fourier integral operator in the complex domain, with the function φ in (2.2.16) playing the role of the phase function, with no fiber variables present. To this end, let us first specify a suitable class of amplitudes. Let $a \in S^0_{\mathrm{cl}}(\operatorname{neigh}((x_0,\overline{x_0}),\mathbf{C}^{2n}))$,

$$a(x,\widetilde{y};h) \sim \sum_{j=0}^\infty a_j(x,\widetilde{y})h^j, \quad h \to 0^+, \tag{2.2.30}$$

be a classical C^∞ symbol, with the asymptotic expansion (2.2.30) in the space of smooth functions $C^\infty(\operatorname{neigh}((x_0,\overline{x_0}),\mathbf{C}^{2n}))$, such that $a_j \in C^\infty(\operatorname{neigh}((x_0,\overline{x_0}),\mathbf{C}^{2n}))$ satisfy

$$(\partial_{\overline{x}} a_j)(x,\widetilde{y}) = \mathcal{O}(|\widetilde{y}-\overline{x}|^\infty), \quad \left(\partial_{\overline{\widetilde{y}}} a_j\right)(x,\widetilde{y}) = \mathcal{O}(|\widetilde{y}-\overline{x}|^\infty), \quad j = 0,1,2,\ldots. \tag{2.2.31}$$

Given a good contour $\Gamma(y,\widetilde{x}) \subset \mathbf{C}^{2n}_{x,\widetilde{y}}$ for the C^∞ function $(x,\widetilde{y}) \mapsto \operatorname{Re}\varphi(y,\widetilde{x};x,\widetilde{y})$ and an amplitude $a(x,\widetilde{y};h)$ satisfying (2.2.30), (2.2.31), we set

$$(A_\Gamma a)(y,\widetilde{x};h) = \frac{1}{h^n} \iint_{\Gamma(y,\widetilde{x})} e^{\frac{2}{h}\varphi(y,\widetilde{x};x,\widetilde{y})} a(x,\widetilde{y};h)\, dx\, d\widetilde{y}. \tag{2.2.32}$$

We have $A_\Gamma a \in C^\infty(\operatorname{neigh}((x_0,\overline{x_0}),\mathbf{C}^{2n})$, and let us first check that the definition of

$A_\Gamma a$ is essentially independent of the choice of a good contour, up to a rapidly vanishing error as $h \to 0^+$, provided that $(y, \widetilde{x})$ is confined to the anti-diagonal.

Proposition 2.2.3. There exists an open neighborhood $V_0 \Subset \Omega \subset \mathbf{C}^n$ of x_0 such that for any two good contours $\Gamma(y, \overline{y})$, $\Gamma_0(y, \overline{y})$ for the function $(x, \widetilde{y}) \mapsto \operatorname{Re} \varphi(y, \overline{y}; x, \widetilde{y})$, for $y \in V_0$, and any amplitude a satisfying (2.2.30), (2.2.31), we have for $y \in V_0$,

$$(A_\Gamma a)(y, \overline{y}; h) - (A_{\Gamma_0} a)(y, \overline{y}; h) = \mathcal{O}(h^\infty), \tag{2.2.33}$$

in the $C^\infty(V_0)$ sense.

Proof: Let us start by making some general remarks concerning good contours when parameters are present, closely related to the discussion in [39, Chapter 3], [13, Chapter 1]. Let $f(x, y)$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}^{2N}$, be a real-valued C^∞ function in a neighborhood of $(0, 0) \in \mathbf{R}^n_x \times \mathbf{R}^{2N}_y$. Assume that $f'_y(0, 0) = 0$ and that $f''_{yy}(0, 0)$ is non-degenerate of signature (N, N). The implicit function theorem gives that the equation $f'_y(x, y) = 0$ uniquely defines a C^∞ function $y = y(x)$ in a neighborhood of 0, with $y(0) = 0$, and applying the Morse lemma with parameters [30, Appendix C], we obtain that there exist new C^∞ coordinates $z = z(y) = z(x, y)$ for $\mathbf{R}^{2N}_y$ near $y = 0$ when $x \in \mathbf{R}^n$ is small, such that writing $z = (t, s)$ with $t, s \in \mathbf{R}^N$, we have

$$f(x, y) = f(x, y(x)) + \frac{1}{2}(t^2 - s^2). \tag{2.2.34}$$

We may also recall from [30, Appendix C] that the new coordinates, which also depend on x, are centered at the critical point $y(x)$ and are of the form

$$z = Q(x, y)(y - y(x)), \tag{2.2.35}$$

where the matrix $Q(0,0)$ is invertible. Assume next that $\Gamma(x) \subset \mathbf{R}_y^{2N}$ is a good contour for $y \mapsto f(x,y)$, depending smoothly on x, so that $\Gamma(x)$ passes through $y(x)$ and along $\Gamma(x)$, we have for all $x \in \mathbf{R}^n$ small enough,

$$f(x,y) \leq f(x,y(x)) - \frac{1}{C}\left|y - y(x)\right|^2, \quad y \in \Gamma(x), \tag{2.2.36}$$

for some $C > 0$. It follows from (2.2.34), (2.2.35), and (2.2.36) that along $\Gamma(x)$, we have

$$\frac{1}{2}(t^2 - s^2) \leq -\frac{1}{\mathcal{O}(1)}\left(t^2 + s^2\right), \tag{2.2.37}$$

and by the implicit function theorem, we obtain therefore that in the Morse coordinates $z = (t,s)$, the contour $\Gamma(x)$ takes the form

$$t = g(s,x), \quad s \in \mathrm{neigh}(0, \mathbf{R}^N), \tag{2.2.38}$$

where

$$|g(s,x)| \leq \alpha\,|s|, \quad \alpha < 1. \tag{2.2.39}$$

See also [39, Chapter 3]. Throughout the discussion above, $x \in \mathbf{R}^n$ varies in a sufficiently small neighborhood of the origin. Letting $z \mapsto y(z)$ be the inverse of the map $y \mapsto z(y)$, well defined for x small, we obtain the following parametrization of the good contour $\Gamma(x)$,

$$\mathrm{neigh}(0, \mathbf{R}^N) \ni s \mapsto y(g(s,x), s) \in \mathbf{R}_y^{2N}. \tag{2.2.40}$$

A consequence of this discussion is that if $\Gamma_0(x) \subset \mathbf{R}_y^{2N}$ is another good contour for $y \mapsto f(x,y)$, depending smoothly on x, then its representation in the Morse coordinates

$z = (t, s)$ takes the form

$$t = g_0(s, x), \quad s \in \mathrm{neigh}(0, \mathbf{R}^N), \tag{2.2.41}$$

$$|g_0(s, x)| \le \alpha |s|, \quad \alpha < 1, \tag{2.2.42}$$

and the contours $\Gamma(x)$, $\Gamma_0(x)$ are therefore homotopic via the deformation

$$\mathrm{neigh}(0, \mathbf{R}^N) \times [0,1] \ni (s, \theta) \mapsto y(\theta g(s, x) + (1-\theta) g_0(s, x), s), \tag{2.2.43}$$

well defined for all x small enough, with the C^∞ dependence on x. It follows also from (2.2.34), (2.2.38), (2.2.39), (2.2.41), (2.2.42) that when $\theta \in [0,1]$, the contour

$$\Gamma_\theta(x) : \mathrm{neigh}(0, \mathbf{R}^N) \ni s \mapsto y(\theta g(s, x) + (1-\theta) g_0(s, x), s), \tag{2.2.44}$$

is good for $y \mapsto f(x, y)$, uniformly in $\theta \in [0,1]$ (and $x \in \mathbf{R}^n$ small enough).

We shall now apply the discussion above to the C^∞ function

$$\mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}) \ni (x, \widetilde{y}) \mapsto \mathrm{Re}\, \varphi(y, \overline{y}; x, \widetilde{y}), \tag{2.2.45}$$

with $y \in \mathrm{neigh}(x_0, \mathbf{C}^n)$ playing the role of the parameters. Let $\Gamma = \Gamma(y)$, $\Gamma_0 = \Gamma_0(y)$ be two good contours for the function in (2.2.45), and let $\Gamma_\theta(y) \subset \mathbf{C}^{2n}_{x,\widetilde{y}}$ be the "intermediate" contour defined as in (2.2.44), for $\theta \in [0,1]$, where $N = 2n$. Letting $G_{[0,1]}(y) \subset \mathbf{C}^{2n}_{x,\widetilde{y}}$ be the $(2n+1)$–dimensional contour formed by the union of the contours $\Gamma_\theta(y)$, for $\theta \in [0,1]$, parametrized as in (2.2.43), we may write by an application of

Stokes' formula to the $(2n,0)$–differential form $\omega = e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}a(x,\widetilde{y};h)\,dx\wedge d\widetilde{y}$,

$$\begin{aligned}\iint_{\partial G_{[0,1]}(y)}\omega &= \iiint_{G_{[0,1]}(y)} d\omega = \iiint_{G_{[0,1]}(y)} d\left(e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}a(x,\widetilde{y};h)\right)\wedge dx\wedge d\widetilde{y}\\ &= \iiint_{G_{[0,1]}(y)} \partial_{\overline{x}}\left(e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}a(x,\widetilde{y};h)\right)\wedge dx\wedge d\widetilde{y}\\ &\quad + \iiint_{G_{[0,1]}(y)} \partial_{\overline{\widetilde{y}}}\left(e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}a(x,\widetilde{y};h)\right)\wedge dx\wedge d\widetilde{y}. \end{aligned}\tag{2.2.46}$$

Using (2.2.16) we compute, for $1\leq j\leq n$,

$$\begin{aligned}\partial_{\overline{x}_j}\left(e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}a(x,\widetilde{y};h)\right) &= e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}\left(\frac{2}{h}\partial_{\overline{x}_j}\varphi(y,\overline{y};x,\widetilde{y})a(x,\widetilde{y};h)+\partial_{\overline{x}_j}a(x,\widetilde{y};h)\right)\\ &= e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}\left(\frac{2}{h}\partial_{\overline{x}_j}\left(\Psi(x,\widetilde{y})-\Psi(x,\overline{y})\right)a(x,\widetilde{y};h)+\partial_{\overline{x}_j}a(x,\widetilde{y};h)\right), \end{aligned}\tag{2.2.47}$$

and here we have in view of (2.2.3), for all N,

$$\begin{aligned}\left|\partial_{\overline{x}_j}\left(\Psi(x,\widetilde{y})-\Psi(x,\overline{y})\right)\right| &\leq \mathcal{O}_N(1)\left(|\overline{x}-\widetilde{y}|^N+|x-y|^N\right)\\ &\leq \mathcal{O}_N(1)\left(|\overline{x}-\overline{y}|^N+|\overline{y}-\widetilde{y}|^N+|x-y|^N\right)\leq \mathcal{O}_N(1)\left(|\overline{y}-\widetilde{y}|^N+|x-y|^N\right)\\ &\leq \mathcal{O}_N(1)\mathrm{dist}\left((x,\widetilde{y}),(y,\overline{y})\right)^N. \end{aligned}\tag{2.2.48}$$

Furthermore, using (2.2.30) and (2.2.31) we get

$$\left|\partial_{\overline{x}_j}a(x,\widetilde{y};h)\right|\leq \mathcal{O}_N(1)\left(|\overline{x}-\widetilde{y}|^N+h^N\right)\leq \mathcal{O}_N(1)\left(\mathrm{dist}\left((x,\widetilde{y}),(y,\overline{y})\right)^N+h^N\right).\tag{2.2.49}$$

We get therefore using (2.2.28), (2.2.47), (2.2.48), (2.2.49), uniformly along $G_{[0,1]}(y)$,

$$\begin{aligned}\left|\partial_{\overline{x}_j}\left(e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}a(x,\widetilde{y};h)\right)\right| &\le \mathcal{O}_N(1)e^{\frac{2}{h}\mathrm{Re}\,\varphi(y,\overline{y};x,\widetilde{y})}\left(\frac{1}{h}\mathrm{dist}\left((x,\widetilde{y}),(y,\overline{y})\right)^N+h^N\right)\\ &\mathcal{O}_N(1)e^{-\frac{1}{Ch}\mathrm{dist}\,((x,\widetilde{y}),(y,\overline{y}))^2}\left(\frac{1}{h}\mathrm{dist}\left((x,\widetilde{y}),(y,\overline{y})\right)^N+h^N\right). \quad (2.2.50)\end{aligned}$$

Using that

$$e^{-t^2/Ch}t^N \le \mathcal{O}_N(1)h^{N/2}, \quad t\ge 0,\ N=1,2,\ldots,$$

we conclude that we have uniformly along $G_{[0,1]}(y)$, for all N,

$$\left|\partial_{\overline{x}}\left(e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}a(x,\widetilde{y};h)\right)\right| \le \mathcal{O}_N(1)h^N. \quad (2.2.51)$$

This bound is uniform in $y\in \mathrm{neigh}(x_0,\mathbf{C}^n)$. Next, when considering

$$\begin{aligned}\partial_{\overline{\widetilde{y}}}\left(e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}a(x,\widetilde{y};h)\right) &= e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}\left(\frac{2}{h}\partial_{\overline{\widetilde{y}}}\varphi(y,\overline{y};x,\widetilde{y})a(x,\widetilde{y};h)+\partial_{\overline{\widetilde{y}}}a(x,\widetilde{y};h)\right)\\ &= e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}\left(\frac{2}{h}\partial_{\overline{\widetilde{y}}}\left(\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y})\right)a(x,\widetilde{y};h)+\partial_{\overline{\widetilde{y}}}a(x,\widetilde{y};h)\right), \quad (2.2.52)\end{aligned}$$

we write similarly to (2.2.48), for $N=1,2,\ldots$,

$$\begin{aligned}\left|\partial_{\overline{\widetilde{y}}}\left(\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y})\right)\right| &\le \mathcal{O}_N(1)\left(|\overline{x}-\widetilde{y}|^N+|\overline{y}-\widetilde{y}|^N\right)\\ &\le \mathcal{O}_N(1)\left(|\overline{x}-\overline{y}|^N+|\overline{y}-\widetilde{y}|^N+|\overline{y}-\widetilde{y}|^N\right)\le \mathcal{O}_N(1)\left(|x-y|^N+|\overline{y}-\widetilde{y}|^N\right)\\ &\le \mathcal{O}_N(1)\mathrm{dist}\left((x,\widetilde{y}),(y,\overline{y})\right)^N. \quad (2.2.53)\end{aligned}$$

Using also (2.2.30), (2.2.31), we conclude that

$$\left|\partial_{\widetilde{\overline{y}}}\left(e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}a(x,\widetilde{y};h)\right)\right| \leq \\ \mathcal{O}_N(1)e^{-\frac{1}{Ch}\mathrm{dist}\,((x,\widetilde{y}),(y,\overline{y}))^2}\left(\frac{1}{h}\mathrm{dist}\,((x,\widetilde{y}),(y,\overline{y}))^N + h^N\right), \quad (2.2.54)$$

and therefore, similar to (2.2.51), we get uniformly along $G_{[0,1]}(y)$, for all N,

$$\left|\partial_{\widetilde{\overline{y}}}\left(e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}a(x,\widetilde{y};h)\right)\right| \leq \mathcal{O}_N(1)h^N. \quad (2.2.55)$$

We get, combining (2.2.46), (2.2.51), and (2.2.55),

$$\iint_{\partial G_{[0,1]}(y)} e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})}a(x,\widetilde{y};h)\,dx\wedge d\widetilde{y} = \mathcal{O}(h^\infty), \quad (2.2.56)$$

uniformly for $y \in \mathrm{neigh}(x_0,\mathbf{C}^n)$. Here we may write, with a suitable orientation,

$$\partial G_{[0,1]}(y) = \Gamma(y) - \Gamma_0(y) + \Gamma_1(y),$$

where

$$\mathrm{Re}\,\varphi(y,\overline{y};x,\widetilde{y}) \leq -\frac{1}{C}, \quad (x,\widetilde{y}) \in \Gamma_1(y), \quad (2.2.57)$$

for some $C > 0$, when $y \in \mathrm{neigh}(x_0,\mathbf{C}^n)$. It follows that

$$(A_\Gamma a)(y,\overline{y};h) - (A_{\Gamma_0}a)(y,\overline{y};h) = \mathcal{O}(h^\infty), \quad (2.2.58)$$

uniformly for $y \in \mathrm{neigh}(x_0,\mathbf{C}^n)$. Let us see, finally, that the relation (2.2.58) holds in the C^∞ sense, i.e. also for the derivatives of $A_\Gamma a - A_{\Gamma_0}a$. To this end, we observe first that for all $\alpha, \beta \in \mathbf{N}^n$ there exists $M_{\alpha\beta} \geq 0$ such that

$$\partial_y^\alpha \partial_{\overline{y}}^\beta \left(A_\Gamma a - (A_{\Gamma_0} a) \right) (y, \overline{y}; h) = \mathcal{O}_\alpha(1) h^{-M_{\alpha\beta}}. \tag{2.2.59}$$

Combining (2.2.58), (2.2.59) with the convexity estimates for the derivatives of a smooth function [18, Chapter 1], we conclude that

$$\partial_y^\alpha \partial_{\overline{y}}^\beta \left(A_\Gamma a - (A_{\Gamma_0} a) \right) (y, \overline{y}; h) = \mathcal{O}(h^\infty), \tag{2.2.60}$$

uniformly, after an arbitrarily small decrease of the neighborhood of $x_0 \in \mathbf{C}^n$ where (2.2.58) holds. The proof is complete. $\square$

Remark. The proof of Proposition 2.2.3 shows that in order for the function

$$(A_\Gamma a)(y, \widetilde{x}; h)$$

to be independent of the choice of a good contour Γ, up to $\mathcal{O}(h^\infty)$, it is essential that $(y, \widetilde{x})$ should be confined to the anti-diagonal, so that $\widetilde{x} = \overline{y}$.

We shall next proceed to establish the existence of a complete asymptotic expansion for $(A_\Gamma a)(y, \overline{y}; h)$, as $h \to 0^+$. When doing so, thanks to Proposition 2.2.3, it will be convenient to work with the particular choice of the good contour $\Gamma(y, \overline{y})$ given in (2.2.29). Using the parametrization of $\Gamma(y, \overline{y})$ given in (2.2.29) and (2.2.32), we get

$$A_\Gamma a(y, \overline{y}; h) = \frac{C_n}{h^n} \int_U e^{\frac{i}{h} f(y,z)} b(y, z; h) \, L(dz) \tag{2.2.61}$$

Here $f(y, z) = -2i\varphi(y, \overline{y}; y + z, \overline{y} - \overline{z})$, $b(y, z; h) = a(y + z, \overline{y} - \overline{z}; h)$, and $L(dz)$ is the Lebesgue measure on $\mathbf{C}^n$. Furthermore, the constant $C_n \neq 0$ in (2.2.61) depends on the dimension n only and the region of integration $U \subset \mathbf{C}^n$ is a small neighborhood of the origin. Using (2.2.21) we see that

$$f(y,z) = 2i\Phi''_{x\overline{x}}(y)\overline{z}\cdot z + \mathcal{O}(|z|^3), \tag{2.2.62}$$

so that in particular

$$\operatorname{Im} f(y,z) \geq \frac{1}{C}\,|z|^2\,, \quad z \in U, \tag{2.2.63}$$

for some $C > 0$ and all $y \in \mathbf{C}^n$ close enough to x_0. For future reference, we shall now proceed to compute det $(\nabla_z^2 f(y,0)/i)$, where the Hessian ∇_z^2 is taken in the real sense of $\mathbf{R}^{2n} \simeq \mathbf{C}^n$, so that $\nabla_z^2 f(y,0)/i$ is a real symmetric $2n \times 2n$ matrix. Writing $\mathbf{C}^n \ni z = t + is$, $t, s \in \mathbf{R}^n$, we see that the quadratic part of the Taylor expansion of $z \mapsto f(y,z)/i$ at the origin, given in (2.2.62), is of the form

$$\begin{aligned} 2\Phi''_{x\overline{x}}(y)\overline{z}\cdot z &= 2\left(\Phi''_{x\overline{x}}(y)t\cdot t + \Phi''_{x\overline{x}}(y)s\cdot s + i\Phi''_{x\overline{x}}(y)t\cdot s - i\Phi''_{x\overline{x}}(y)s\cdot t\right) \\ &= 2\left(A_1 t\cdot t + A_1 s\cdot s - 2A_2 t\cdot s\right). \end{aligned} \tag{2.2.64}$$

Here we have written $\Phi''_{x\overline{x}}(y) = A_1 + iA_2$, with A_1, A_2 being $n \times n$ real matrices and observed that since $\Phi''_{x\overline{x}}(y)$ is Hermitian, we have $A_1^t = A_1$, $A_2^t = -A_2$. We get

$$2\Phi''_{x\overline{x}}(y)\overline{z}\cdot z = 2\begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}\begin{pmatrix} t \\ s \end{pmatrix}\cdot\begin{pmatrix} t \\ s \end{pmatrix}, \tag{2.2.65}$$

and writing

$$\Phi''_{x\overline{x}}(y)\overline{z}\cdot z = \frac{1}{2}\begin{pmatrix} \Phi''_{x\overline{x}}(y) & 0 \\ 0 & \Phi''_{\overline{x}x}(y) \end{pmatrix}\begin{pmatrix} \overline{z} \\ z \end{pmatrix}\cdot\begin{pmatrix} z \\ \overline{z} \end{pmatrix}, \tag{2.2.66}$$

$$\begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix}, \quad \begin{pmatrix} \overline{z} \\ z \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix}, \tag{2.2.67}$$

we obtain the factorization

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \Phi''_{x\overline{x}}(y) & 0 \\ 0 & \Phi''_{\overline{x}x}(y) \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}. \tag{2.2.68}$$

It follows from (2.2.62), (2.2.65), and (2.2.68) that

$$\det \left(\frac{\nabla_z^2 f(y,0)}{i} \right) = 2^{4n} \det \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix} = 2^{4n} \left(\det \left(\Phi''_{x\overline{x}}(y) \right) \right)^2, \tag{2.2.69}$$

which is a smooth strictly positive function near $y = x_0$. For future reference, let us also compute the quadratic form

$$\left(\nabla_z^2 f(y,0) \right)^{-1} \begin{pmatrix} t \\ s \end{pmatrix} \cdot \begin{pmatrix} t \\ s \end{pmatrix} = \frac{1}{2^2 i} \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}^{-1} \begin{pmatrix} t \\ s \end{pmatrix} \cdot \begin{pmatrix} t \\ s \end{pmatrix}, \tag{2.2.70}$$

dual to $\nabla_z^2 f(y,0)$. To this end, a simple computation using (2.2.68) shows that

$$\begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}^{-1} = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \left(\Phi''_{x\overline{x}}(y) \right)^{-1} & 0 \\ 0 & \left(\Phi''_{\overline{x}x}(y) \right)^{-1} \end{pmatrix} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}, \tag{2.2.71}$$

and therefore we get

$$\begin{aligned} \left(\nabla_z^2 f(y,0) \right)^{-1} \begin{pmatrix} t \\ s \end{pmatrix} \cdot \begin{pmatrix} t \\ s \end{pmatrix} &= \frac{1}{2^2 i} \begin{pmatrix} \left(\Phi''_{x\overline{x}}(y) \right)^{-1} & 0 \\ 0 & \left(\Phi''_{\overline{x}x}(y) \right)^{-1} \end{pmatrix} \begin{pmatrix} \overline{z} \\ z \end{pmatrix} \cdot \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \\ &= \frac{1}{2i} \left(\Phi''_{x\overline{x}}(y) \right)^{-1} \overline{z} \cdot z. \end{aligned} \tag{2.2.72}$$

Here the quadratic form in (2.2.72) can be regarded as the symbol of the second order constant coefficient differential operator on $\mathbf{C}_z^n$ given by

$$\frac{2}{i}\left(\Phi''_{x\overline{x}}(y)\right)^{-1} D_z \cdot D_{\overline{z}} = 2i\left(\Phi''_{x\overline{x}}(y)\right)^{-1} \partial_z \cdot \partial_{\overline{z}}, \tag{2.2.73}$$

where $D_z = \frac{1}{i}\partial_z = \frac{1}{2}(D_t - iD_s)$, $D_{\overline{z}} = \frac{1}{i}\partial_{\overline{z}} = \frac{1}{2}(D_t + iD_s)$.

It follows from (2.2.62), (2.2.63), and (2.2.69) that we are in the position to apply complex stationary phase in the form given in [30, Theorem 7.7.5] to derive a complete asymptotic expansion for $A_\Gamma a(y, \overline{y}; h)$ given in (2.2.61), as $h \to 0^+$. We obtain therefore that there exist differential operators $L_{j,y}(D)$ in (t, s) of order $2j$, which are C^∞ functions of $y \in \mathrm{neigh}(x_0, \mathbf{C}^n)$, such that for each N we have uniformly for $y \in \mathrm{neigh}(x_0, \mathbf{C}^n)$,

$$\left(A_\Gamma a\right)(y, \overline{y}; h) = \sum_{j=0}^{N-1} h^j \left(L_{j,y}(D) b\right)(y, 0) + \mathcal{O}_N(h^N). \tag{2.2.74}$$

Let us also recall from [30, Theorem 7.7.5], using also (2.2.73), the following explicit expressions for the operators L_j,

$$(L_{j,y}(D)b)(y,0) = \frac{C_n \pi^n}{2^n \det\left(\Phi''_{x\overline{x}}(y)\right)} \sum_{\nu-\mu=j} \sum_{2\nu \geq 3\mu} \frac{i^{\nu-j}}{\mu!\nu!} \left(\left(\Phi''_{x\overline{x}}(y)\right)^{-1} \partial_z \cdot \partial_{\overline{z}}\right)^\nu \left(g^\mu b\right)(y, 0) \tag{2.2.75}$$

where

$$g(y, z) = f(y, z) - 2i\Phi''_{x\overline{x}}(y)\overline{z} \cdot z = \mathcal{O}(|z|^3).$$

In particular,

$$L_{0,y} = \frac{C_n \pi^n}{2^n \det\left(\Phi''_{x\overline{x}}(y)\right)} \tag{2.2.76}$$

satisfies

$$\frac{1}{C} \leq |L_{0,y}| \leq C, \quad y \in \text{neigh}(x_0, \mathbf{C}^n). \tag{2.2.77}$$

The expansion (2.2.74) can be differentiated any number of times with respect to y, $\overline{y}$. Following [12, Section 3], our purpose is now to show that there exists an amplitude $a(x, \widetilde{y}; h) \in S^0_{\text{cl}}(\text{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}))$ satisfying (2.2.30), (2.2.31) such that

$$(A_\Gamma a)(y, \overline{y}; h) = 1 + \mathcal{O}(h^\infty), \tag{2.2.78}$$

for $y \in \text{neigh}(x_0, \mathbf{C}^n)$. Looking for a in the form (2.2.30), we may write in view of (2.2.74),

$$(A_\Gamma a)\,(y, \overline{y}; h) \sim \sum_{\ell=0}^{\infty} h^\ell c_\ell(y), \quad c_\ell(y) = \sum_{j+k=\ell} \left(L_{k,y}(D) b_j\right)(y, 0), \tag{2.2.79}$$

where $b_j(y, z) = a_j(y + z, \overline{y} - \overline{z})$. Using the expansion (2.2.79), we shall determine successively $a_0, a_1, \ldots$ satisfying (2.2.31), so that

$$c_0(y) = L_{0,y} a_0(y, \overline{y}) = 1, \tag{2.2.80}$$

$$c_\ell(y) = \sum_{j+k=\ell} (L_{k,y}(D) b_j)(y, 0) = 0, \quad \ell \geq 1. \tag{2.2.81}$$

First, (2.2.80) determines the C^∞ function $a_0(y, \overline{y})$ uniquely, in view of (2.2.76), (2.2.77), and taking an almost holomorphic extension from the anti-diagonal, we obtain

$$a_0(x, \widetilde{y}) \in C^\infty(\text{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n})),$$

satisfying (2.2.31) for $j = 0$. Assume next that $a_0, a_1, \ldots, a_{M-1}$, satisfying (2.2.31), have been determined so that (2.2.81) holds for $\ell \leq M - 1$. To determine a_M, we consider the equation (2.2.81) with $\ell = M$, writing

$$c_M(y) = L_{0,y} a_M(y,\overline{y}) + \sum_{\substack{j+k=M \\ j<M}} (L_{k,y}(D) b_j)(y,0) = 0. \tag{2.2.82}$$

Here we may notice that the expression in the sum in (2.2.82) only depends on the values of the a_j's along the anti-diagonal, for $j \leq M-1$. Indeed, for each $\alpha \in \mathbf{N}^n$, we have in view of the almost holomorphy of $a_j = a_j(x,\widetilde{y})$ along the anti-diagonal given in (2.2.31), for $j \leq M-1$,

$$(D_z^\alpha b_j)(y,0) = D_z^\alpha \left(a_j(y+z,\overline{y}-\overline{z})\right)|_{z=0} = \left(D_x^\alpha a_j\right)(y,\overline{y}) = D_y^\alpha \left(a_j(y,\overline{y})\right), \tag{2.2.83}$$

$$(D_{\overline{z}}^\alpha b_j)(y,0) = D_{\overline{z}}^\alpha \left(a_j(y+z,\overline{y}-\overline{z})\right)|_{z=0} = (-1)^\alpha \left(D_{\widetilde{y}}^\alpha a_j\right)(y,\overline{y}) = (-1)^\alpha D_{\overline{y}}^\alpha \left(a_j(y,\overline{y})\right). \tag{2.2.84}$$

It follows that (2.2.82) has a unique C^∞ solution $a_M(y,\overline{y})$, for $y \in \mathrm{neigh}(x_0,\mathbf{C}^n)$, and we may then take an almost holomorphic extension. Modifying the choice of an almost holomorphic extension of $a_M(y,\overline{y})$ will only affect $(A_\Gamma a)(y,\overline{y};h)$ by a term which is $\mathcal{O}(h^\infty)$.

The discussion above may be summarized in the following theorem, which is the main result of this section.

Theorem 2.2.4. There exists an elliptic symbol $a(x,\widetilde{y};h) \in S^0_{\mathrm{cl}}(\mathrm{neigh}((x_0,\overline{x_0}),\mathbf{C}^{2n}))$ of the form

$$a(x,\widetilde{y};h) \sim \sum_{j=0}^\infty a_j(x,\widetilde{y}) h^j, \tag{2.2.85}$$

in C^∞, with $a_j \in C^\infty(\mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}))$ satisfying

$$\left(\partial_{\overline{x}} a_j\right)(x, \widetilde{y}) = \mathcal{O}(|\widetilde{y} - \overline{x}|^\infty), \quad \left(\partial_{\overline{\widetilde{y}}} a_j\right)(x, \widetilde{y}) = \mathcal{O}(|\widetilde{y} - \overline{x}|^\infty), \quad j = 0, 1, 2, \ldots, \tag{2.2.86}$$

such that we have

$$(A_\Gamma a)(y, \overline{y}; h) = \frac{1}{h^n} \iint_{\Gamma(y,\overline{y})} e^{\frac{2}{h}\varphi(y,\overline{y};x,\widetilde{y})} a(x, \widetilde{y}; h)\, dx\, d\widetilde{y}$$
$$= 1 + \mathcal{O}(h^\infty), \quad y \in \mathrm{neigh}(x_0, \mathbf{C}^n), \tag{2.2.87}$$

in the C^∞ sense. Here $\Gamma(y, \overline{y})$ is a good contour for the function $(x, \widetilde{y}) \mapsto \mathrm{Re}\, \varphi(y, \overline{y}; x, \widetilde{y})$. The restrictions of the a_j's to the anti-diagonal $\widetilde{y} = \overline{x}$ are uniquely determined, for $j \geq 0$, and we have

$$a_0(x, \overline{x}) = A_n \det\left(\Phi''_{x\overline{x}}(x)\right), \quad x \in \mathrm{neigh}(x_0, \mathbf{C}^n), \tag{2.2.88}$$

with $A_n \neq 0$ depending on n only.

Remark. The elliptic symbol $a(x, \widetilde{y}; h)$ constructed in Theorem 2.2.4 is unique in the following sense: assume that $b(x, \widetilde{y}; h) \in S^0_{\mathrm{cl}}(\mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}))$ of the form

$$b(x, \widetilde{y}; h) \sim \sum_{j=0}^{\infty} b_j(x, \widetilde{y}) h^j, \tag{2.2.89}$$

such that

$$\left(\partial_{\overline{x}} b_j\right)(x, \widetilde{y}) = \mathcal{O}(|\widetilde{y} - \overline{x}|^\infty), \quad \left(\partial_{\overline{\widetilde{y}}} b_j\right)(x, \widetilde{y}) = \mathcal{O}(|\widetilde{y} - \overline{x}|^\infty), \quad j = 0, 1, 2, \ldots, \tag{2.2.90}$$

satisfies $(A_\Gamma b)(y,\overline{y};h) = 1 + \mathcal{O}(h^\infty)$, for a good contour Γ. We have then

$$a(x,\widetilde{y};h) - b(x,\widetilde{y};h) = \mathcal{O}\left(|\widetilde{y} - \overline{x}|^\infty + h^\infty\right). \tag{2.2.91}$$

2.3 Approximate reproducing property in the weak sense

Let $V \Subset \Omega$ be a small open neighborhood of $x_0 \in \Omega$, with C^∞–boundary. Let $\Psi \in C^\infty(\mathrm{neigh}((x_0,\overline{x_0}),\mathbf{C}^{2n}))$ be an almost holomorphic extension of the C^∞ strictly plurisubharmonic weight function Φ, so that (2.2.2), (2.2.3) hold. We may assume that Ψ, as well as the classical C^∞ symbol a, introduced in Theorem 2.2.4 and satisfying (2.2.87), are defined in a neighborhood of the closure of $V \times \rho(V)$. Here $\rho(x) = \overline{x}$ is the map of complex conjugation.

Let us set for $u \in L^2_\Phi(V) := L^2(V, e^{-2\Phi/h}L(dx))$,

$$\widetilde{\Pi}_V u(x) = \frac{1}{h^n}\int_V e^{\frac{2}{h}\Psi(x,\overline{y})} a(x,\overline{y};h) u(y) e^{-\frac{2}{h}\Phi(y)}\, dy\, d\overline{y}. \tag{2.3.1}$$

It follows from Proposition 2.2.1 and the Schur test that

$$\widetilde{\Pi}_V = \mathcal{O}(1) : L^2_\Phi(V) \to L^2_\Phi(V). \tag{2.3.2}$$

Furthermore, combining (2.2.3), (2.2.85), (2.2.86) with the Schur test, we obtain

$$\overline{\partial} \circ \widetilde{\Pi}_V = \mathcal{O}(h^\infty) : L^2_\Phi(V) \to L^2_{\Phi,(0,1)}(V). \tag{2.3.3}$$

Here the target space is a space of $(0,1)$–forms on V. Letting $u \in L^2_\Phi(V)$ be holomorphic, we can express $\widetilde{\Pi}_V u$ in the polarized form, as a contour integral in $\mathbf{C}^{2n}_{y,\widetilde{y}}$ of a $(2n,0)$–form,

$$\widetilde{\Pi}_V u(x) = \frac{1}{h^n} \iint_{\Gamma_V} e^{\frac{2}{h}(\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y}))} a(x,\widetilde{y};h) u(y)\, dy\, d\widetilde{y}, \quad u \in H_\Phi(V) := \mathrm{Hol}(V) \cap L^2_\Phi(V). \tag{2.3.4}$$

Here the contour of integration $\Gamma_V \subset V \times \rho(V)$ is given by

$$\Gamma_V = \{\widetilde{y} = \overline{y},\ y \in V\}, \tag{2.3.5}$$

and we observe that the restriction of the closed $(2n,0)$–form $dy \wedge d\widetilde{y}$ on $\mathbf{C}^{2n}_{y,\widetilde{y}}$ to the anti-diagonal $\widetilde{y} = \overline{y}$ agrees with $dy \wedge d\overline{y}$, a non-vanishing multiple of the Lebesgue volume form on $\mathbf{C}^n_y$.

The purpose of this section is to show that the operator $\widetilde{\Pi}_V$ in (2.3.4) satisfies an approximate reproducing property, in the weak formulation. Specifically, we shall prove that for a suitable class of $u, v \in H_\Phi(V)$, the sesquilinear form

$$H_\Phi(V) \times H_\Phi(V) \ni (u,v) \mapsto (\widetilde{\Pi}_V u, v)_{L^2_\Phi(V)} \tag{2.3.6}$$

agrees, up to an $\mathcal{O}(h^\infty)$–error, with the scalar product $(u,v)_{H_\Phi(V)}$. In [12], we have observed that this result cannot be expected to hold if u, v are general elements of $H_\Phi(V)$, and similar to [12], we shall demand that v should belong to an exponentially weighted space of holomorphic functions of the form $H_{\Phi_1}(V)$, where $\Phi_1 \leq \Phi$, with strict inequality away from a small neighborhood of x_0. The following theorem is the main result of this section.

Theorem 2.3.1. There exists a small open neighborhood $W \Subset V$ of x_0 with C^∞–

boundary such that for every $\Phi_1 \in C(\Omega; \mathbf{R})$, $\Phi_1 \leq \Phi$, with $\Phi_1 < \Phi$ on $\Omega \backslash \overline{W}$, and every $N \in \mathbf{N}$ there exists C_N such that for all $u \in H_\Phi(V)$, $v \in H_{\Phi_1}(V)$, we have

$$\left|(\widetilde{\Pi}_V u, v)_{L^2_\Phi(V)} - (u, v)_{H_\Phi(V)}\right| \leq C_N h^N ||\, u\, ||_{H_\Phi(V)} ||\, v\, ||_{H_{\Phi_1}(V)}. \tag{2.3.7}$$

Following [12, Section 4], the proof of Theorem 2.3.1 will proceed by a contour deformation argument. Compared with the analytic case treated in [12], here, when justifying the contour deformation, we shall have to take into account the lack of holomorphy in the integrand in (2.3.4), giving rise to an additional correction term, to be estimated.

Let $W \Subset V_1 \Subset V_2 \Subset V$ be open neighborhoods of x_0 with C^∞–boundaries, with W to be chosen small enough, and let $\Phi_1 \in C(\Omega; \mathbf{R})$ be such that

$$\Phi_1 \leq \Phi \text{ in } \Omega, \quad \Phi_1 < \Phi \text{ on } \Omega \backslash \overline{W}. \tag{2.3.8}$$

Arguing as in [12, Section 4], we find that the scalar product

$$(\widetilde{\Pi}_V u, v)_{L^2_\Phi(V)} = \int_V \widetilde{\Pi}_V u(x) \overline{v(x)} e^{-2\Phi(x)/h}\, L(dx), \quad u \in H_\Phi(V),\ v \in H_{\Phi_1}(V), \tag{2.3.9}$$

takes the form

$$(\widetilde{\Pi}_V u, v)_{L^2_\Phi(V)} = \int_{V_1} \widetilde{\Pi}_{V_2} u(x) \overline{v(x)} e^{-2\Phi(x)/h}\, L(dx) + \mathcal{O}(1) e^{-\frac{1}{Ch}} ||\, u\, ||_{H_\Phi(V)} ||\, v\, ||_{H_{\Phi_1}(V)}. \tag{2.3.10}$$

Here and below $C > 0$ is independent of u, v, and similar to (2.3.4), we have written

$$\widetilde{\Pi}_{V_2} u(x) = \frac{1}{h^n} \iint_{\Gamma_{V_2}} e^{\frac{2}{h}(\Psi(x,\widetilde{y}) - \Psi(y,\widetilde{y}))} a(x, \widetilde{y}; h) u(y)\, dy\, d\widetilde{y}. \tag{2.3.11}$$

Next, an application of [12, Proposition 2.2] gives that there exists $\eta > 0$ such that

$$v(x) = \int_V v_z(x) dz\, d\overline{z} + \mathcal{O}(1) || \, v \, ||_{H_\Phi(V)} e^{\frac{1}{h}(\Phi(x)-\eta)}, \quad x \in V_1. \tag{2.3.12}$$

Here

$$v_z(x) = \frac{1}{(2\pi h)^n} e^{\frac{i}{h}(x-z)\cdot\theta(x,z)} v(z)\chi(z) \det\left(\partial_{\overline{z}}\theta(x,z)\right) \in \mathrm{Hol}(V), \tag{2.3.13}$$

and $\theta(x,z)$ depends holomorphically on $x \in V$ with

$$-\operatorname{Im}\left((x-z)\cdot\theta(x,z)\right) + \Phi(z) \le \Phi(x) - \delta\left|x-z\right|^2, \quad x, z \in V, \tag{2.3.14}$$

for some $\delta > 0$. The function $\chi \in C_0^\infty(V;[0,1])$ in (2.3.13) satisfies $\chi = 1$ in V_2.

The resolution of the identity (2.3.12), (2.3.13), (2.3.14) is valid for an arbitrary element of $H_\Phi(V)$, and restricting the attention to $v \in H_{\Phi_1}(V)$, we get, letting $W \Subset W_1 \Subset V_1$,

$$v(x) = \int_{W_1} v_z(x)\, dz\, d\overline{z} + \mathcal{O}(1) || \, v \, ||_{H_{\Phi_1}(V)} e^{\frac{1}{h}(\Phi(x)-\frac{1}{C})}, \quad x \in V_1. \tag{2.3.15}$$

We get, combining (2.3.10), (2.3.15), and (2.3.2),

$$\begin{aligned}
&(\widetilde{\Pi}_V u, v)_{L^2_\Phi(V)} \\
&\quad = \int_{W_1}\int_{V_1} \widetilde{\Pi}_{V_2} u(x) \overline{v_z(x)} e^{-2\Phi(x)/h}\, L(dx)\, dz\, d\overline{z} + \mathcal{O}(1) e^{-\frac{1}{Ch}} || \, u \, ||_{H_\Phi(V)} || \, v \, ||_{H_{\Phi_1}(V)} \\
&\qquad\qquad = \int_{W_1} (\widetilde{\Pi}_{V_2} u, v_z)_{L^2_\Phi(V_1)} dz\, d\overline{z} + \mathcal{O}(1) e^{-\frac{1}{Ch}} || \, u \, ||_{H_\Phi(V)} || \, v \, ||_{H_{\Phi_1}(V)}.
\end{aligned} \tag{2.3.16}$$

Similar to [12], the advantage of the representation (2.3.16) lies in the good localization properties of the holomorphic functions v_z, for $z \in W_1$, in view of (2.3.14).

The following key observation is analogous to [12, Proposition 4.2].

Proposition 2.3.2. Let $\delta > 0$ be small and let us set for $z \in V$, $(x, \widetilde{x}, y, \widetilde{y}) \in V \times \rho(V) \times V \times \rho(V) \subset \mathbf{C}^{4n}$,

$$
\begin{aligned}
G_z(x, \widetilde{x}, y, \widetilde{y}) &= 2\mathrm{Re}\,\Psi(x, \widetilde{y}) - 2\mathrm{Re}\,\Psi(y, \widetilde{y}) + \Phi(y) + F_z(\widetilde{x}) - 2\mathrm{Re}\,\Psi(x, \widetilde{x}) \\
&= 2\mathrm{Re}\,\varphi(y, \widetilde{x}; x, \widetilde{y}) - 2\mathrm{Re}\,\Psi(y, \widetilde{x}) + \Phi(y) + F_z(\widetilde{x}), \quad (2.3.17)
\end{aligned}
$$

where

$$
F_z(\widetilde{x}) = \Phi(\overline{\widetilde{x}}) - \delta\left|\widetilde{x} - \overline{z}\right|^2. \quad (2.3.18)
$$

The C^∞ function G_z has a non-degenerate critical point at $(z, \overline{z}, z, \overline{z})$ of signature $(4n, 4n)$, with the critical value 0. The following submanifolds of $\mathbf{C}^{4n}$ are good contours for G_z in a neighbourhood of $(z, \overline{z}, z, \overline{z})$, i.e. they are both of dimension $4n$, pass through the critical point, and are such that the Hessian of G_z along the contours is negative definite:

1. The product contour

$$
\Gamma_V \times \Gamma_V = \{(x, \widetilde{x}, y, \widetilde{y});\, \widetilde{x} = \overline{x},\ \widetilde{y} = \overline{y},\ x \in V,\ y \in V\}. \quad (2.3.19)
$$

2. The composed contour

$$
\{(x, \widetilde{x}, y, \widetilde{y});\, (y, \widetilde{x}) \in \Gamma_V,\, (x, \widetilde{y}) \in \Gamma(y, \widetilde{x})\}. \quad (2.3.20)
$$

   Here $\Gamma(y, \widetilde{x}) \subset \mathbf{C}^{2n}_{x,\widetilde{y}}$ is a good contour for the C^∞ function $(x, \widetilde{y}) \mapsto \mathrm{Re}\,\varphi(y, \widetilde{x}; x, \widetilde{y})$ described in (2.2.28), see also Proposition 2.2.2.

Proof: We have the Taylor expansions at $(x,\widetilde{x},y,\widetilde{y})=(z,\overline{z},z,\overline{z})\in\mathbf{C}^{4n}$,

$$\begin{aligned}2\mathrm{Re}\,\Psi(x,\widetilde{y})-2\mathrm{Re}\,\Psi(y,\widetilde{y})&=2\Phi(z)+2\mathrm{Re}\,(\Phi'_x(z)\cdot(x-z)+\Phi'_{\overline{x}}(z)\cdot(\widetilde{y}-\overline{z}))\\ &-2\Phi(z)-2\mathrm{Re}\,(\Phi'_x(z)\cdot(y-z)+\Phi'_{\overline{x}}(z)\cdot(\widetilde{y}-\overline{z}))+\mathcal{O}((x-z,\widetilde{y}-\overline{z},y-z)^2)\\ &=2\mathrm{Re}\,(\Phi'_x(z)\cdot(x-z)-\Phi'_x(z)\cdot(y-z))+\mathcal{O}((x-z,\widetilde{y}-\overline{z},y-z)^2),\end{aligned}\tag{2.3.21}$$

$$\Phi(y)+F_z(\widetilde{x})=2\Phi(z)+2\mathrm{Re}\,\Big(\Phi'_x(z)\cdot(y-z)+\Phi'_x(z)\cdot(\overline{\widetilde{x}}-z)\Big)+\mathcal{O}((y-z,\widetilde{x}-\overline{z})^2),\tag{2.3.22}$$

$$2\mathrm{Re}\,\Psi(x,\widetilde{x})=2\Phi(z)+2\mathrm{Re}\,(\Phi'_x(z)\cdot(x-z)+\Phi'_{\overline{x}}(z)\cdot(\widetilde{x}-\overline{z}))+\mathcal{O}((x-z,\widetilde{x}-\overline{z})^2).\tag{2.3.23}$$

Here we have also used the almost holomorphy of Ψ. We get, combining (2.3.21), (2.3.22), (2.3.23) and using (2.3.17),

$$G_z(x,\widetilde{x},y,\widetilde{y})=\mathcal{O}\left(\mathrm{dist}\,((x,\widetilde{x},y,\widetilde{y}),(z,\overline{z},z,\overline{z}))^2\right).\tag{2.3.24}$$

The point $(z,\overline{z},z,\overline{z})$ is therefore a critical point of G_z with the critical value 0. When showing that it is non-degenerate of signature $(4n,4n)$, we observe that in view of Proposition 2.2.1, we have, using the expression for G_z on the first line of (2.3.17),

$$G_z(x,\overline{x},y,\overline{y})\leq-\frac{1}{C}\left|y-x\right|^2-\delta\left|x-z\right|^2\leq-\frac{1}{C}\left|x-z\right|^2-\frac{1}{C}\left|y-z\right|^2.\tag{2.3.25}$$

The contour (2.3.19) is therefore good for G_z, and using also the fact that the quadratic part of the Taylor expansion of G_z at the point $(z,\overline{z},z,\overline{z})$ is a plurisubharmonic quadratic form on $\mathbf{C}^{4n}$, we conclude that $(z,\overline{z},z,\overline{z})$ is a non-degenerate critical point of G_z, of signature $(4n,4n)$. The verification of the fact that the contour (2.3.20) is also good for G_z is performed exactly as in the proof of Proposition 4.2 of [12], using the expression for G_z given on the second line of (2.3.17). The proof is complete. $\square$

We can now carry out the contour deformation argument for the scalar product

$$(\widetilde{\Pi}_{V_2} u, v_z),$$

alluded to above.

Proposition 2.3.3. Let v_z be of the form (2.3.13), (2.3.14). There exists an open neighborhood $W_1 \Subset V_1$ of x_0 such that for all $u \in H_\Phi(V)$, we have,

$$(\widetilde{\Pi}_{V_2} u, v_z)_{L^2_\Phi(V_1)} = (u, v_z)_{H_\Phi(V_1)} + \mathcal{O}(h^\infty)||\, u\, ||_{H_\Phi(V)}\, |v(z)|\, e^{-\Phi(z)/h}, \tag{2.3.26}$$

uniformly in $z \in W_1$.

Proof: Writing the Lebesgue measure on $\mathbf{C}^n$ in the form $L(dx) = C_n dx\, d\overline{x}$, let us express the scalar product in the space $H_\Phi(V_1)$ in the polarized form,

$$(f, g)_{H_\Phi(V_1)} = \int_{V_1} f(x)\overline{g(x)} e^{-\frac{2\Phi(x)}{h}}\, L(dx) = C_n \iint_{\Gamma_{V_1}} f(x) g^*(\widetilde{x}) e^{-\frac{2}{h}\Psi(x,\widetilde{x})}\, dx\, d\widetilde{x}. \tag{2.3.27}$$

Here the contour Γ_{V_1} is defined as in (2.3.5) and we have also set

$$g^*(\widetilde{x}) = \overline{g(\overline{\widetilde{x}})} \in H_{\widehat{\Phi}}(\rho(V_1)), \quad \widehat{\Phi}(\widetilde{x}) = \Phi(\overline{\widetilde{x}}). \tag{2.3.28}$$

In view of (2.3.11) and (2.3.27), we may write

$$(\widetilde{\Pi}_{V_2} u, v_z)_{L^2_\Phi(V_1)} = \frac{C_n}{h^n} \iint_{\Gamma_{V_1}} \left(\iint_{\Gamma_{V_2}} e^{\frac{2}{h}(\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y}))} a(x,\widetilde{y};h) u(y)\, dy\, d\widetilde{y} \right) v_z^*(\widetilde{x}) e^{-\frac{2}{h}\Psi(x,\widetilde{x})}\, dx\, d\widetilde{x} = \iiiint_{\Gamma_{V_1}\times\Gamma_{V_2}} \omega. \quad (2.3.29)$$

Here ω is the $(4n,0)$–differential form on $\mathbf{C}^{4n}$ of the form

$$\omega = f(x,\widetilde{x},y,\widetilde{y})\, dx \wedge\, d\widetilde{x} \wedge\, dy \wedge\, d\widetilde{y}, \quad (2.3.30)$$

where $f \in C^\infty(V \times \rho(V) \times V \times \rho(V))$ is given by

$$f(x,\widetilde{x},y,\widetilde{y}) = \frac{C_n}{h^n} e^{\frac{2}{h}(\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y}))} a(x,\widetilde{y};h) u(y) v_z^*(\widetilde{x}) e^{-\frac{2}{h}\Psi(x,\widetilde{x})}. \quad (2.3.31)$$

When estimating f, we notice first, in view of (2.3.13), (2.3.14),

$$|v_z^*(\widetilde{x})| \leq \frac{\mathcal{O}(1)}{h^n} |v(z)|\, e^{-\Phi(z)/h} e^{F_z(\widetilde{x})/h}, \quad \widetilde{x} \in \rho(V_1), \quad (2.3.32)$$

where F_z is the strictly plurisubharmonic function in $\rho(V_1)$ given in (2.3.18). Combining (2.3.32) with [12, Proposition 2.3], we get

$$|f(x,\widetilde{x},y,\widetilde{y})| \leq \frac{\mathcal{O}(1)}{h^{3n}} ||\, u\, ||_{H_\Phi(V)}\, |v(z)|\, e^{-\Phi(z)/h} e^{G_z(x,\widetilde{x},y,\widetilde{y})/h}, \quad (2.3.33)$$

for $(x,\widetilde{x},y,\widetilde{y}) \in V_1 \times \rho(V_1) \times V_2 \times \rho(V_2)$, with $G_z(x,\widetilde{x},y,\widetilde{y})$ given in (2.3.17). Proposition 2.3.2 tells us that the contour $\Gamma_1 := \Gamma_{V_1} \times \Gamma_{V_2}$ and the composed contour Γ_2 defined in (2.3.20) are both good for G_z, and as reviewed in the proof of Proposition 2.2.3, there exists therefore a C^∞ homotopy between the contours Γ_1, Γ_2, well defined for all z in a small neighborhood of x_0, passing through good contours only, uniformly for z close enough to x_0. Let $\Sigma \subset \mathbf{C}^{4n}$ be the $(4n+1)$–dimensional contour of integration naturally associated to the homotopy above. We have, with a suitable orientation,

$$\partial\Sigma - (\Gamma_1 - \Gamma_2) \subset \left\{ (x, \widetilde{x}, y, \widetilde{y});\, G_z(x, \widetilde{x}, y, \widetilde{y}) \leq -\frac{1}{\mathcal{O}(1)} \right\}, \tag{2.3.34}$$

uniformly for all z in a small neighborhood of x_0. We may write therefore, using Stokes' formula, (2.3.33), and (2.3.34), for all z in a small neighborhood of x_0,

$$\begin{aligned} &\iiiint_{\Gamma_1} \omega - \iiiint_{\Gamma_2} \omega = \int\!\!\iiiint_{\Sigma} d\omega + \mathcal{O}(1) ||\, u \,||_{H_\Phi(V)} \, |v(z)| \, e^{-\Phi(z)/h} e^{-1/Ch} \\ &= \int\!\!\iiiint_{\Sigma} \overline{\partial} f \wedge dx \wedge d\widetilde{x} \wedge dy \wedge d\widetilde{y} + \mathcal{O}(1) ||\, u \,||_{H_\Phi(V)} \, |v(z)| \, e^{-\Phi(z)/h} e^{-1/Ch}. \end{aligned} \tag{2.3.35}$$

Using (2.3.31), we compute

$$\begin{aligned} &\partial_{\overline{x}} f(x, \widetilde{x}, y, \widetilde{y}) \\ &= \frac{C_n}{h^n} e^{\frac{2}{h}(\Psi(x,\widetilde{y}) - \Psi(y,\widetilde{y}) - \Psi(x,\widetilde{x}))} u(y) v_z^*(\widetilde{x}) \left(\frac{2}{h} \left(\partial_{\overline{x}} \Psi(x, \widetilde{y}) - \partial_{\overline{x}} \Psi(x, \widetilde{x}) \right) a + \partial_{\overline{x}} a \right), \end{aligned} \tag{2.3.36}$$

and recalling (2.2.3), (2.2.86), we infer for all $N \in \mathbf{N}$,

$$\begin{aligned} &|\partial_{\overline{x}} f| \leq \frac{\mathcal{O}_N(1)}{h^{3n+1}} ||\, u \,||_{H_\Phi(V)} \, |v(z)| \, e^{-\Phi(z)/h} e^{G_z(x,\widetilde{x},y,\widetilde{y})/h} \left(|\overline{x} - \widetilde{y}|^N + |\overline{x} - \widetilde{x}|^N + h^N \right) \\ &\leq \frac{\mathcal{O}_N(1)}{h^{3n+1}} ||\, u \,||_{H_\Phi(V)} \, |v(z)| \, e^{-\frac{\Phi(z)}{h}} e^{-\mathrm{dist}((x,\widetilde{x},y,\widetilde{y}),(z,\overline{z},z,\overline{z}))^2/Ch} \left(|\overline{x} - \widetilde{y}|^N + |\overline{x} - \widetilde{x}|^N + h^N \right). \end{aligned} \tag{2.3.37}$$

Here we write

$$|\overline{x}-\widetilde{y}|^N \leq (|x-z|+|\widetilde{y}-\overline{z}|)^N \leq \mathcal{O}_N(1)\left(\operatorname{dist}\left((x,\widetilde{x},y,\widetilde{y}),(z,\overline{z},z,\overline{z})\right)\right)^N, \tag{2.3.38}$$

and similarly,

$$|\overline{x}-\widetilde{x}|^N \leq \mathcal{O}_N(1)\left(\operatorname{dist}\left((x,\widetilde{x},y,\widetilde{y}),(z,\overline{z},z,\overline{z})\right)\right)^N. \tag{2.3.39}$$

We get therefore, combining (2.3.37), (2.3.38), (2.3.39),

$$|\partial_{\overline{x}} f(x,\widetilde{x},y,\widetilde{y})| \leq \mathcal{O}(h^\infty)||\, u\, ||_{H_\Phi(V)}\, |v(z)|\, e^{-\frac{\Phi(z)}{h}}. \tag{2.3.40}$$

Similar computations and estimates show that the bound (2.3.40) is also valid for $\partial_{\overline{\widetilde{x}}} f$, $\partial_{\overline{y}} f$, and $\partial_{\overline{\widetilde{y}}} f$. We conclude that there exists a small open neighborhood $W_1 \Subset V_1$ of x_0 such that for all $z \in W_1$, the right hand side of (2.3.35) is of the form

$$\mathcal{O}(h^\infty)||\, u\, ||_{H_\Phi(V)}\, |v(z)|\, e^{-\frac{\Phi(z)}{h}}. \tag{2.3.41}$$

It follows therefore from (2.3.29), (2.3.35), and (2.3.41) that for all $z \in W_1$, the scalar product $(\widetilde{\Pi}_{V_2} u, v_z)_{L^2_\Phi(V_1)}$ is equal to

$$C_n \iint_{\Gamma_{V_1}} \left(\frac{1}{h^n} \iint_{\Gamma(y,\widetilde{x})\cap(V_1\times\rho(V_1))} e^{\frac{2}{h}\varphi(y,\widetilde{x};x,\widetilde{y})} a(x,\widetilde{y};h)\,dx\,d\widetilde{y}\right) u(y) v_z^*(\widetilde{x}) e^{-\frac{2}{h}\Psi(y,\widetilde{x})}\,dy\,d\widetilde{x}$$
$$+ \mathcal{O}(h^\infty)||\, u\, ||_{H_\Phi(V)}\, |v(z)|\, e^{-\frac{\Phi(z)}{h}}. \tag{2.3.42}$$

Here in the contour of integration in the inner integral we have $(y,\widetilde{x}) \in \Gamma_{V_1} \iff \widetilde{x} = \overline{y}$, $y \in V_1$, and by Theorem 2.2.4 we obtain therefore that the inner integral is equal to $1 + \mathcal{O}(h^\infty)$, provided that the neighborhood V_1 is small enough. The integral (2.3.42) is therefore equal to

$$(u, v_z)_{H_\Phi(V_1)} + \mathcal{O}(h^\infty)||\, u\, ||_{H_\Phi(V)}\, |v(z)|\, e^{-\frac{\Phi(z)}{h}}, \tag{2.3.43}$$

uniformly for $z \in W_1$. The proof is complete. $\square$

We can now finish the proof of Theorem 2.3.1 as in [12, Section 4], letting $W \Subset W_1$, where W_1 is as in Proposition 2.3.3. We get, in view of (2.3.16) and (2.3.26),

$$(\widetilde{\Pi}_V u, v)_{L^2_\Phi(V)} = \int_{W_1} (u, v_z)_{H_\Phi(V_1)}\, dz\, d\overline{z} + \mathcal{O}(h^\infty)||\, u\, ||_{H_\Phi(V)}||\, v\, ||_{H_{\Phi_1}(V)}. \tag{2.3.44}$$

Using (2.3.15), we can also write

$$(u, v)_{H_\Phi(V)} = \int_{W_1} (u, v_z)_{H_\Phi(V_1)}\, dz\, d\overline{z} + \mathcal{O}(1)e^{-\frac{1}{Ch}}||\, u\, ||_{H_\Phi(V)}||\, v\, ||_{H_{\Phi_1}(V)}. \tag{2.3.45}$$

The proof of Theorem 2.3.1 is complete.

2.4 Completing the proof of Theorem 2.1.1

Let us first pass from the scalar products in Theorem 2.3.1 to weighted L^2 norm estimates. This will be done similarly to [12, Section 5], with appropriate modifications to accommodate the fact that the operator $\widetilde{\Pi}_V$ in (2.3.1) does not quite produce holomorphic functions. To this end, let $0 \leq \chi_1 \in C^\infty(\Omega; \mathbf{R})$ be such that $\chi_1 > 0$ on $\Omega \backslash \overline{W}$, where $W \Subset V$ is as in Theorem 2.3.1, and let us set

$$\Phi_1(x) = \Phi(x) - \delta\chi_1(x), \tag{2.4.1}$$

for $\delta > 0$ small enough. In particular, Φ_1 is strictly plurisubharmonic in V,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \Phi_1}{\partial x_j \partial \overline{x}_k}(x)\xi_j \overline{\xi}_k \geq \frac{|\xi|^2}{\mathcal{O}(1)}, \quad x \in V, \quad \xi \in \mathbf{C}^n. \tag{2.4.2}$$

In what follows, without loss of generality, we shall assume that the bounded open set V is convex, and we may even take it to be an open ball centered at x_0.

Using Proposition 2.2.1 together with the Schur test, we obtain

$$\widetilde{\Pi}_V = \mathcal{O}(1) : L^2_{\Phi_1}(V) \to L^2_{\Phi_2}(V). \tag{2.4.3}$$

Here $\Phi_2 = \Phi - \chi_2$, where $0 \leq \chi_2 \in C(\Omega; \mathbf{R})$ is given by

$$\chi_2(x) = \inf_{y \in V_0} \left(\frac{|x-y|^2}{2C} + \delta \chi_1(y) \right), \tag{2.4.4}$$

with $C > 0$ and V_0 being an open ball centered at x_0 such that $V \Subset V_0 \Subset \Omega$. In particular, we see that

$$\Phi_2 \leq \Phi \text{ in } \Omega, \quad \Phi_2 < \Phi \text{ on } \Omega \setminus \overline{W}, \tag{2.4.5}$$

and

$$\Phi_1 \leq \Phi_2 \text{ in } V. \tag{2.4.6}$$

Let us now take a closer look at the function χ_2 in (2.4.4). When doing so, let us observe first that the infimum in (2.4.4) is attained at a unique point in $\overline{V_0}$, in view of the strict convexity of the function $\overline{V_0} \ni y \mapsto F_x(y) := \dfrac{|x-y|^2}{2C} + \delta\chi_1(y)$, for $\delta > 0$ small enough. On the other hand, when $x \in V$, the function $F_x(y)$ has a unique critical point $y_c(x)$ in V_0, for $\delta > 0$ small enough, which is a non-degenerate local minimum.

Indeed, we have

$$F'_x(y) = 0 \Leftrightarrow CF'_0(y) = x,$$

and the map $V_0 \ni y \mapsto CF'_0(y) = y + C\delta\chi'_1(y)$ is a C^∞ diffeomorphism from V_0 onto its image, which contains the open ball V as a relatively compact subset, for $\delta > 0$ small enough. We have

$$y_c(x) = x - C\delta\chi'_1(x) + \mathcal{O}(\delta^2), \quad x \in V, \tag{2.4.7}$$

and we conclude therefore that

$$\chi_2(x) = F_x(y_c(x)) = \delta\chi_1(x) - \frac{C}{2}\delta^2 \left|\chi'_1(x)\right|^2 + \mathcal{O}(\delta^3), \quad x \in V. \tag{2.4.8}$$

In particular, we have $|| \Phi - \Phi_2 ||_{C^2(\overline{V})} = \mathcal{O}(\delta)$ is small enough, so that the function Φ_2 is strictly plurisubharmonic in V,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \Phi_2}{\partial x_j \partial \overline{x}_k}(x)\xi_j\overline{\xi}_k \geq \frac{|\xi|^2}{\mathcal{O}(1)}, \quad x \in V, \quad \xi \in \mathbf{C}^n. \tag{2.4.9}$$

Next, we let

$$\Pi_{\Phi_2} : L^2_{\Phi_2}(V) \to H_{\Phi_2}(V) \tag{2.4.10}$$

be the orthogonal projection. We shall make use of the following observation.

Proposition 2.4.1. We have

$$\Pi_{\Phi_2}\widetilde{\Pi}_V - \widetilde{\Pi}_V = \mathcal{O}(h^\infty) : L^2_{\Phi_1}(V) \to L^2_{\Phi_2}(V). \tag{2.4.11}$$

Proof: Using (2.2.3), (2.2.30), (2.2.31), and Proposition 2.2.1 together with the Schur test, we observe that

$$\overline{\partial} \circ \widetilde{\Pi}_V = \mathcal{O}(h^\infty) : L^2_{\Phi_1}(V) \to L^2_{\Phi_2,(0,1)}(V), \tag{2.4.12}$$

where the target space in (2.4.12) is a space of $(0,1)$–forms. Given $f \in L^2_{\Phi_1}(V)$, the solution of the equation

$$\overline{\partial} u = \overline{\partial} \widetilde{\Pi}_V f \tag{2.4.13}$$

of the minimal $L^2_{\Phi_2}(V)$–norm is given by $(1 - \Pi_{\Phi_2})\widetilde{\Pi}_V f$, and an application of Hörmander's L^2–estimates for the $\overline{\partial}$–operator in the open convex set V and the strictly plurisubharmonic weight Φ_2, [31, Proposition 4.2.5], gives that

$$|| \, (1 - \Pi_{\Phi_2})\widetilde{\Pi}_V f \, ||_{L^2_{\Phi_2}(V)} \leq \mathcal{O}(h^{1/2}) || \, \overline{\partial} \widetilde{\Pi}_V f \, ||_{L^2_{\Phi_2}(V)} \leq \mathcal{O}(h^\infty) || \, f \, ||_{L^2_{\Phi_1}(V)}. \tag{2.4.14}$$

Here we have also used (2.4.12). The proof is complete. $\Box$

It is now easy to derive a suitable estimate for the operator $\widetilde{\Pi}_V - 1$, proceeding as in [12]. Let $u \in H_{\Phi_1}(V)$, where $\Phi_1 \in C^\infty(\Omega; \mathbf{R})$ is given by (2.4.1), and let us apply Theorem 2.3.1, with

$$v = \Pi_{\Phi_2} \left((\widetilde{\Pi}_V - 1) u \right) = \Pi_{\Phi_2} \widetilde{\Pi}_V u - u \in H_{\Phi_2}(V) \tag{2.4.15}$$

and with Φ_2 in place of Φ_1. Here in the second equality in (2.4.15) we have also used (2.4.6). We obtain, using also (2.4.3),

$$\begin{aligned} \left| \left((\widetilde{\Pi}_V - 1) u, \Pi_{\Phi_2} \left((\widetilde{\Pi}_V - 1) u \right) \right)_{L^2_\Phi(V)} \right| &\leq \mathcal{O}(h^\infty) || \, u \, ||_{H_\Phi(V)} || \, \Pi_{\Phi_2} \widetilde{\Pi}_V u - u \, ||_{H_{\Phi_2}(V)} \\ &\leq \mathcal{O}(h^\infty) || \, u \, ||^2_{H_{\Phi_1}(V)}. \end{aligned} \tag{2.4.16}$$

Next, we write

$$\Pi_{\Phi_2}\left((\widetilde{\Pi}_V - 1)u\right) = (\widetilde{\Pi}_V - 1)u + Ru, \tag{2.4.17}$$

where $R = \Pi_{\Phi_2}\widetilde{\Pi}_V - \widetilde{\Pi}_V$. We get, combining (2.4.16), (2.4.17), and using that $\Phi_j \leq \Phi$, for $j = 1, 2$, together with (2.3.2),

$$\begin{aligned} ||\,(\widetilde{\Pi}_V - 1)u\,||^2_{L^2_\Phi(V)} &\leq \mathcal{O}(h^\infty)||\,u\,||^2_{H_{\Phi_1}(V)} + \mathcal{O}(1)||\,u\,||_{H_\Phi(V)}||\,Ru\,||_{L^2_\Phi(V)} \\ &\leq \mathcal{O}(h^\infty)||\,u\,||^2_{H_{\Phi_1}(V)} + \mathcal{O}(1)||\,u\,||_{H_\Phi(V)}||\,Ru\,||_{L^2_{\Phi_2}(V)} \\ \leq \mathcal{O}(h^\infty)||\,u\,||^2_{H_{\Phi_1}(V)} &+ \mathcal{O}(h^\infty)||\,u\,||_{H_\Phi(V)}||\,u\,||_{H_{\Phi_1}(V)} \leq \mathcal{O}(h^\infty)||\,u\,||^2_{H_{\Phi_1}(V)}. \end{aligned} \tag{2.4.18}$$

Here in the penultimate inequality we have also used Proposition 2.4.1. We obtain therefore that

$$||\,(\widetilde{\Pi}_V - 1)u\,||_{L^2_\Phi(V)} \leq \mathcal{O}(h^\infty)||\,u\,||_{H_{\Phi_1}(V)}, \quad u \in H_{\Phi_1}(V), \tag{2.4.19}$$

where $\Phi_1 \in C^\infty(\Omega; \mathbf{R})$ is of the form (2.4.1). The bound (2.4.19) is completely analogous to the estimate (5.5) in [12, Section 5], and we may therefore conclude the proof of Theorem 2.1.1 by repeating the arguments of [12, Section 5], which are based on the $\overline{\partial}$–techniques, exactly as they stand. Letting $U \Subset W \Subset V$ be an open neighborhood of x_0 with C^∞–boundary, we get therefore,

$$||\,(\widetilde{\Pi}_V - 1)u\,||_{L^2_\Phi(U)} \leq \mathcal{O}(h^\infty)||\,u\,||_{H_\Phi(V)}, \quad u \in H_\Phi(V). \tag{2.4.20}$$

The proof of Theorem 2.1.1 is complete.

2.5 From asymptotic local to global Bergman kernels

The purpose of this section is to establish a link between the operator $\widetilde{\Pi}_V$ in (2.1.3), enjoying the local approximate reproducing property (2.1.4), and the orthogonal projection

$$\Pi : L^2(\Omega, e^{-2\Phi/h} L(dx)) \to H_\Phi(\Omega). \tag{2.5.1}$$

When doing so, we shall follow [2] closely, where the Bergman projection was considered in the context of high powers of a holomorphic line bundle over a complex compact manifold. The discussion below is therefore essentially well known, and is given here mainly for the completeness and convenience of the reader. See also [15], [16].

We shall assume in what follows that the open set $\Omega \subset \mathbf{C}^n$ is pseudoconvex. It will also be convenient for us to choose a local polarization Ψ of $\Phi \in C^\infty(\Omega)$ satisfying (2.2.2), (2.2.3), such that the Hermitian property

$$\Psi(x,y) = \overline{\Psi(\overline{y}, \overline{x})}, \quad (x,y) \in \mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}_{x,y}) \tag{2.5.2}$$

holds. Indeed, if $\Psi(x,y)$ satisfies (2.2.2), (2.2.3), then so does $\overline{\Psi(\overline{y}, \overline{x})}$, and replacing $\Psi(x,y)$ by $(\Psi(x,y) + \overline{\Psi(\overline{y}, \overline{x})})/2$, we obtain (2.5.2).

Our starting point is the following well known result, allowing us to pass to pointwise estimates from the weighted L^2 estimates in Theorem 2.1.1.

Proposition 2.5.1. Let $V_1 \Subset V_2 \Subset \Omega$ be open. There exists $C > 0$ such that for all $f \in L^2_\Phi(V_2)$ satisfying $h\overline{\partial} f \in L^\infty(V_2)$ and all $h > 0$ small enough, we have

$$|f(x)| \leq C \left(\sup_{y \in V_2} \left| h\overline{\partial} f(y) \right| e^{-\Phi(y)/h} + h^{-n} || f ||_{L^2_\Phi(V_2)} \right) e^{\Phi(x)/h}, \quad x \in V_1. \tag{2.5.3}$$

Proof: Let $h_0 > 0$ be such that $B(x, h_0) = \{y \in \mathbf{C}^n, |y - x| < h_0\} \subset V_2$, for all $x \in V_1$. An application of [30, Lemma 15.1.8] gives for all $h \in (0, h_0]$,

$$|f(x)| \leq C \left(\sup_{B(x,h)} \left| h\overline{\partial} f(y) \right| + h^{-n} || \, f \, ||_{L^2(B(x,h))} \right), \quad x \in V_1. \tag{2.5.4}$$

Using that

$$e^{\Phi(y)/h} \leq \mathcal{O}(1) e^{\Phi(x)/h}, \quad y \in B(x, h),$$

for $x \in V_1$, we obtain (2.5.3). $\square$

We shall apply Proposition 2.5.1 to a function of the form

$$f = \widetilde{\Pi}_V u - u, \quad u \in H_\Phi(V), \tag{2.5.5}$$

satisfying

$$|| \, f \, ||_{L^2_\Phi(U)} \leq \mathcal{O}(h^\infty) || \, u \, ||_{H_\Phi(V)}, \tag{2.5.6}$$

in view of (2.1.4). As for the control of $h\overline{\partial} f = h\overline{\partial}\widetilde{\Pi}_V u$ in $L^\infty(U)$, we have in view of (2.2.3), (2.2.85), (2.2.86), for $N = 1, 2, \ldots$,

$$\begin{aligned} \left| h\overline{\partial} f(x) \right| &\leq \mathcal{O}_N(1) h^{-n} \int_V e^{\frac{2}{h} \operatorname{Re} \Psi(x, \overline{y})} \left| x - y \right|^N \left| u(y) \right| e^{-\frac{2}{h}\Phi(y)} \, L(dy) \\ &\leq \mathcal{O}_N(1) e^{\Phi(x)/h} h^{-n} \int_V e^{-|x-y|^2/Ch} \left| x - y \right|^N \left| u(y) \right| e^{-\Phi(y)/h} \, L(dy) \\ &\leq \mathcal{O}(h^{\frac{N}{2} - n}) e^{\Phi(x)/h} || \, u \, ||_{H_\Phi(V)}, \quad x \in U. \end{aligned} \tag{2.5.7}$$

Here we have also used Proposition 2.2.1 and the Cauchy-Schwarz inequality. Letting $\widetilde{U} \Subset U$ be an open neighborhood of x_0, and combining Proposition 2.5.1 with (2.5.6), (2.5.7), we get

$$\left|\widetilde{\Pi}_V u(x) - u(x)\right| \leq \mathcal{O}(h^\infty)\, e^{\Phi(x)/h} || \, u \, ||_{H_\Phi(V)}, \quad x \in \widetilde{U}. \tag{2.5.8}$$

We obtain therefore the following approximate local reproducing property,

$$u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h}\Psi(x,\overline{y})} a(x, \overline{y}; h) u(y) e^{-\frac{2}{h}\Phi(y)} \, L(dy) + \mathcal{O}(h^\infty) e^{\frac{\Phi(x)}{h}} || \, u \, ||_{H_\Phi(V)}, \quad x \in \widetilde{U}, \tag{2.5.9}$$

valid for all $u \in H_\Phi(V)$. In particular, we can apply (2.5.9) to $u \in H_\Phi(\Omega)$, and when doing so, let us recall that

$$|u(x)| \leq \mathcal{O}(1) h^{-n} e^{\frac{\Phi(x)}{h}} || \, u \, ||_{H_\Phi(\Omega)}, \quad x \in V, \tag{2.5.10}$$

in view of [12, Proposition 2.3]. Let $\chi \in C_0^\infty(V; [0,1])$ be such that $\chi = 1$ near $\overline{U}$. Using (2.5.10) and Proposition 2.2.1, we see that

$$\left| \frac{1}{h^n} \int_V e^{\frac{2}{h}\Psi(x,\overline{y})} (1 - \chi(y)) a(x, \overline{y}; h) u(y) e^{-\frac{2}{h}\Phi(y)} \, L(dy) \right| \\ \leq \mathcal{O}(1) e^{-\frac{1}{Ch}} e^{\frac{\Phi(x)}{h}} || \, u \, ||_{H_\Phi(\Omega)}, \quad x \in \widetilde{U}. \tag{2.5.11}$$

We get, combining (2.5.9) and (2.5.11), when $u \in H_\Phi(\Omega)$,

$$u(y) = \int_\Omega \widetilde{K}(y, \overline{z}) \chi(z) u(z) e^{-\frac{2}{h}\Phi(z)} \, L(dz) + \mathcal{O}(h^\infty) e^{\frac{\Phi(y)}{h}} || \, u \, ||_{H_\Phi(\Omega)}, \quad y \in \widetilde{U}, \tag{2.5.12}$$

where

$$\widetilde{K}(y, \overline{z}) = \frac{1}{h^n} e^{\frac{2}{h}\Psi(y,\overline{z})} a(y, \overline{z}; h), \quad (y, z) \in V \times V. \tag{2.5.13}$$

Next, arguing as in [37, Section 5], [9, Appendix A], we see that the Schwartz kernel of the orthogonal projection in (2.5.1) is of the form $K(x,\overline{y})e^{-2\Phi(y)/h}$, where $K(x,z) \in \mathrm{Hol}(\Omega \times \overline{\Omega})$ satisfies

$$y \mapsto \overline{K(x,\overline{y})} \in H_\Phi(\Omega), \quad x \mapsto K(x,\overline{y}) \in H_\Phi(\Omega). \tag{2.5.14}$$

Here we have written $\overline{\Omega} = \{x \in \mathbf{C}^n;\, \overline{x} \in \Omega\}$. Following [2] and applying (2.5.12) to the function $y \mapsto K(y,\overline{x}) \in H_\Phi(\Omega)$, we get

$$\begin{aligned} K(y,\overline{x}) = \int_\Omega \widetilde{K}(y,\overline{z})\chi(z)K(z,\overline{x})e^{-\frac{2}{h}\Phi(z)}\, L(dz) \\ + \mathcal{O}(h^\infty)e^{\frac{\Phi(y)}{h}} \|\, K(\cdot,\overline{x})\,\|_{H_\Phi(\Omega)}, \quad y \in \widetilde{U}. \end{aligned} \tag{2.5.15}$$

Here we have

$$\|\, K(\cdot,\overline{x})\,\|_{H_\Phi(\Omega)} \leq \mathcal{O}(1)h^{-n/2}e^{\frac{\Phi(x)}{h}}, \quad x \in \widetilde{U}, \tag{2.5.16}$$

see [3, Chapter 4], and combining (2.5.15) and (2.5.16), we infer that

$$K(y,\overline{x}) = \int_\Omega \widetilde{K}(y,\overline{z})\chi(z)K(z,\overline{x})e^{-\frac{2}{h}\Phi(z)}\, L(dz) + \mathcal{O}(h^\infty)e^{\frac{\Phi(x)+\Phi(y)}{h}}, \quad x,y \in \widetilde{U}. \tag{2.5.17}$$

Taking the complex conjugates in (2.5.17) and using the Hermitian property $K(x,\overline{y}) = \overline{K(y,\overline{x})}$, we get

$$K(x,\overline{y}) = \int_\Omega K(x,\overline{z})\chi(z)\widehat{K}(z,\overline{y})e^{-\frac{2}{h}\Phi(z)}\, L(dz) + \mathcal{O}(h^\infty)e^{\frac{\Phi(x)+\Phi(y)}{h}}, \quad x,y \in \widetilde{U}, \tag{2.5.18}$$

where, in view of (2.5.13) and (2.5.2),

$$\widehat{K}(z,\overline{y}) = \overline{\widetilde{K}(y,\overline{z})} = \frac{1}{h^n}e^{\frac{2}{h}\Psi(z,\overline{y})}b(z,\overline{y};h), \quad b(z,\overline{y};h) = \overline{a(y,\overline{z};h)}. \tag{2.5.19}$$

Writing

$$\Pi u(x) = \int_\Omega K(x,\overline{y}) u(y) e^{-2\Phi(y)/h}\, L(dy), \quad u \in L^2(\Omega, e^{-2\Phi/h} L(dx)), \tag{2.5.20}$$

we may express (2.5.18) as follows,

$$K(x,\overline{y}) = \Pi\left(\widehat{K}(\cdot,\overline{y})\chi\right)(x) + \mathcal{O}(h^\infty) e^{\frac{\Phi(x)+\Phi(y)}{h}}, \quad x, y \in \widetilde{U}. \tag{2.5.21}$$

Here we would like to show that $\Pi\left(\widehat{K}(\cdot,\overline{y})\chi\right)(x)$ is close to $\widehat{K}(x,\overline{y})\chi(x) = \widehat{K}(x,\overline{y})$ for $x \in \widetilde{U}$, and to this end we follow an argument in [2], see also Proposition 2.4.1. The function

$$\Omega \ni x \mapsto u_y(x) = \widehat{K}(x,\overline{y})\chi(x) - \Pi\left(\widehat{K}(\cdot,\overline{y})\chi\right)(x) \tag{2.5.22}$$

is the solution of the $\overline{\partial}$–equation

$$\overline{\partial} u_y = \overline{\partial}\left(\widehat{K}(\cdot,\overline{y})\chi\right) = \widehat{K}(\cdot,\overline{y})\overline{\partial}\chi + \chi\overline{\partial}\widehat{K}(\cdot,\overline{y}), \tag{2.5.23}$$

in Ω of the minimal $L^2(\Omega, e^{-2\Phi/h} L(dx))$ norm, and therefore, by Hörmander's L^2–estimates for the $\overline{\partial}$ operator, see [31, Proposition 4.2.5]), we get for any $y \in \widetilde{U}$,

$$\begin{aligned} &\int_\Omega |u_y(x)|^2 e^{-2\Phi(x)/h}\, L(dx) \le \mathcal{O}(h) \int_\Omega \frac{1}{c(x)} \left|\overline{\partial}_x \left(\widehat{K}(x,\overline{y})\chi(x)\right)\right|^2 e^{-2\Phi(x)/h}\, L(dx) \\ &\le \mathcal{O}(h)\left(\int_\Omega |\nabla\chi(x)|^2 \left|\widehat{K}(x,\overline{y})\right|^2 e^{-2\Phi(x)/h}\, L(dx) + \int_\Omega \chi(x)\left|\overline{\partial}_x \widehat{K}(x,\overline{y})\right|^2 e^{-2\Phi(x)/h}\, L(dx)\right) \end{aligned} \tag{2.5.24}$$

Here we get, in view of (2.5.19) and Proposition 2.2.1,

$$\int_{\Omega} |\nabla\chi(x)|^2 \left|\widehat{K}(x,\overline{y})\right|^2 e^{-2\Phi(x)/h}\, L(dx) \leq \mathcal{O}(1) \int_{V\setminus U} \left|\widehat{K}(x,\overline{y})\right|^2 e^{-2\Phi(x)/h}\, L(dx)$$
$$= \mathcal{O}(1)\, e^{2\Phi(y)/h} e^{-1/Ch}, \quad y \in \widetilde{U}. \quad (2.5.25)$$

Due to the almost holomorphy of Ψ and the symbol $b(x, \widetilde{y}; h)$ in (2.5.19) near the anti-diagonal,

$$\left|\overline{\partial}_x \widehat{K}(x,\overline{y})\right| \leq \frac{\mathcal{O}_N(1)}{h^{n+1}} e^{\frac{2}{h}\operatorname{Re}\Psi(x,\overline{y})} \left|x-y\right|^N, \quad N = 1, 2, \ldots, \quad (2.5.26)$$

and therefore, by another application of Proposition 2.2.1,

$$\int_{\Omega} \chi(x) \left|\overline{\partial}_x \widehat{K}(x,\overline{y})\right|^2 e^{-2\Phi(x)/h}\, L(dx) \leq \mathcal{O}(h^\infty)\, e^{2\Phi(y)/h}, \quad y \in \widetilde{U}. \quad (2.5.27)$$

Combining (2.5.24), (2.5.25), and (2.5.27), we get

$$||\, u_y \,||_{L^2_\Phi(\Omega)} \leq \mathcal{O}(h^\infty) e^{\Phi(y)/h}, \quad y \in \widetilde{U}. \quad (2.5.28)$$

It finally remains for us to pass from the weighted L^2–bound (2.5.28) on u_y to a pointwise estimate. To this end, we shall apply Proposition 2.5.1 to u_y, with $V_2 = U$, $V_1 = \widetilde{U}$. Recalling (2.5.22) and using that $\chi = 1$ on U, we see that we only have to estimate $h\overline{\partial}_z u_y(z) = h\overline{\partial}_z \widehat{K}(z,\overline{y})$ for $z \in U$, $y \in \widetilde{U}$. Using (2.5.26) and Proposition 2.2.1, we obtain that

$$\left|h\overline{\partial} u_y(z)\right| \leq \mathcal{O}(h^\infty) e^{(\Phi(z)+\Phi(y))/h}, \quad y \in \widetilde{U},\ z \in U. \quad (2.5.29)$$

Combining (2.5.3), (2.5.28), and (2.5.29), we get

$$|u_y(x)| \le \mathcal{O}(h^\infty)\, e^{(\Phi(x)+\Phi(y))/h}, \quad x, y \in \widetilde{U}, \tag{2.5.30}$$

and therefore, using (2.5.21), (2.5.22), and (2.5.30), we infer that

$$K(x, \overline{y}) = \widehat{K}(x, \overline{y}) + \mathcal{O}(h^\infty) e^{(\Phi(x)+\Phi(y))/h}, \quad x, y \in \widetilde{U}. \tag{2.5.31}$$

Recalling also (2.5.19), we obtain

$$K(x, \overline{y}) = \overline{\widetilde{K}(y, \overline{x})} + \mathcal{O}(h^\infty) e^{(\Phi(x)+\Phi(y))/h}, \quad x, y \in \widetilde{U}, \tag{2.5.32}$$

and taking the complex conjugates and using the Hermitian symmetry of K, we get

$$K(y, \overline{x}) = \widetilde{K}(y, \overline{x}) + \mathcal{O}(h^\infty) e^{(\Phi(x)+\Phi(y))/h}, \tag{2.5.33}$$

uniformly for $x, y \in \widetilde{U}$. Switching the variables x and y in (2.5.33), we may therefore summarize the discussion in this section in the following well known result, see [42], [6], [7], [10], [16], [2], [27].

Proposition 2.5.2. Let $\Omega \subset \mathbf{C}^n$ be open pseudoconvex, let $\Phi \in C^\infty(\Omega)$ be strictly plurisubharmonic so that (2.2.1) holds, and let $K(x, \overline{y}) e^{-2\Phi(y)/h}$ be the Schwartz kernel of the orthogonal projection (2.5.1). Let $x_0 \in \Omega$ and let $\widetilde{U} \Subset U \Subset V \Subset \Omega$ be small open neighborhoods of x_0, where U, V are as in Theorem 2.1.1. We have

$$e^{-\Phi(x)/h} \left(K(x, \overline{y}) - \frac{1}{h^n} e^{\frac{2}{h}\Psi(x, \overline{y})} a(x, \overline{y}; h) \right) e^{-\Phi(y))/h} = \mathcal{O}(h^\infty), \tag{2.5.34}$$

uniformly for $x, y \in \widetilde{U}$. Here $\Psi \in C^\infty(\mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}))$ is a polarization of Φ and the classical symbol $a \in S^0_{\mathrm{cl}}(\mathrm{neigh}((x_0, \overline{x_0}), \mathbf{C}^{2n}))$ has been introduced in (2.1.2).

Chapter 3

Operators with double characteristics and bounded exponential weights on phase space

3.1 Introduction and statement of results

Let us introduce the following standard symbol class of C^∞ functions on $\mathbf{R}^{2n}$, that are bounded together with all of their derivatives,

$$S(1) = \{a \in C^\infty(\mathbf{R}^{2n}); \partial^\alpha a \in L^\infty(\mathbf{R}^{2n}), \quad \forall \alpha \in \mathbf{N}^{2n}\}. \tag{3.1.1}$$

Let $P(x, \xi; h) \in S(1)$ and assume that $P(x, \xi; h)$ has a complete asymptotic expansion in the symbol space $S(1)$,

$$P(x, \xi; h) \sim \sum_{j=0}^{\infty} h^j p_j(x, \xi), \tag{3.1.2}$$

for some $p_j \in S(1)$, $j \in \mathbf{N}$, as $h \to 0^+$. We assume that the semiclassical principal symbol p_0 of P satisfies

$$\operatorname{Re} p_0(x, \xi) \geq 0, \quad (x, \xi) \in \mathbf{R}^{2n}, \tag{3.1.3}$$

and we make an important ellipticity assumption near infinity,

$$\operatorname{Re} p_0(x, \xi) \geq \frac{1}{C}, \quad |(x, \xi)| \geq C, \tag{3.1.4}$$

for some $C > 1$. It follows that the semiclassical Weyl quantization of P,

$$P = P^w(x, hD_x; h) \tag{3.1.5}$$

given by

$$P^w(x, hD_x; h)u(x) = \frac{1}{(2\pi h)^n} \iint_{\mathbf{R}^{2n}} e^{i\frac{(x-y)\cdot\xi}{h}} P\left(\frac{x+y}{2}, \xi; h\right) u(y) dy\, d\xi \tag{3.1.6}$$

is bounded on $L^2(\mathbf{R}^n)$, uniformly as $h \to 0^+$, and the analytic family of operators

$$P - z : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n) \tag{3.1.7}$$

is Fredholm of index 0, for all $h > 0$ small enough, provided that the spectral parameter $z \in \mathbf{C}$ is confined to a region of the form

$$\operatorname{Re} z \leq \frac{1}{\mathcal{O}(1)}. \tag{3.1.8}$$

Furthermore, an application of the sharp Gårding inequality allows us to conclude that

$$\operatorname{Re}(Pu, u)_{L^2} \geq -C_0 h || u ||_{L^2}^2, \quad u \in L^2(\mathbf{R}^n), \tag{3.1.9}$$

for some $C_0 > 0$, and we get for $z \in \mathbf{C}$ such that $\operatorname{Re} z < -C_0 h$,

$$|| u ||_{L^2} \leq \frac{1}{(-\operatorname{Re} z - C_0 h)} || (P - z) u ||_{L^2}. \tag{3.1.10}$$

It is therefore clear, in view of the analytic Fredholm theory, that the spectrum of P in a region of the form (3.1.8) is discrete for all $h > 0$ small enough, consisting of eigenvalues of finite algebraic multiplicity.

We shall make the basic assumption that the set

$$\mathcal{C} := \{X \in \mathbf{R}^{2n}; \operatorname{Re} p_0(X) = 0,\ H_{\operatorname{Im} p_0}(X) = 0\} \tag{3.1.11}$$

is finite, $\mathcal{C} = \{X_1, \ldots, X_N\}$. Here $H_{\operatorname{Im} p_0} = \partial_\xi \operatorname{Im} p_0 \cdot \partial_x - \partial_x \operatorname{Im} p_0 \cdot \partial_\xi$ is the Hamilton vector field of $\operatorname{Im} p_0$. It follows from (3.1.3) that we have

$$dp(X_j) = 0, \quad 1 \leq j \leq N,$$

and we may write therefore, Taylor expanding at the critical point X_j,

$$p_0(X_j+Y)=p_0(X_j)+q_j(Y)+\mathcal{O}(Y^3),\quad Y\to 0. \tag{3.1.12}$$

Here $p_0(X_j)\in i\mathbf{R}$ and q_j is quadratic such that $\operatorname{Re} q_j\geq 0$. Let us introduce the time averages

$$\langle \operatorname{Re} p_0\rangle_{T,\operatorname{Im} p_0}(X)=\frac{1}{T}\int_0^T \operatorname{Re} p_0\left(\exp\left(tH_{\operatorname{Im} p_0}\right)(X)\right)\,dt,\quad T>0, \tag{3.1.13}$$

of $\operatorname{Re} p_0$ along the $H_{\operatorname{Im} p_0}$–trajectories. We shall assume that for each $T>0$ fixed, the following dynamical condition holds for $1\leq j\leq N$,

$$\langle \operatorname{Re} p_0\rangle_{T,\operatorname{Im} p_0}(X)\asymp \left|X-X_j\right|^2,\quad X\in \operatorname{neigh}(X_j,\mathbf{R}^{2n}). \tag{3.1.14}$$

We may recall from [22], [23] that the assumption (3.1.14) is equivalent to the assumption that the quadratic form q_j in (3.1.12) satisfies

$$\{X\in\mathbf{R}^{2n};H^k_{\operatorname{Im} q_j}\operatorname{Re} q_j(X)=0,\quad \forall k\in\mathbf{N}\}=\{0\},\quad 1\leq j\leq N, \tag{3.1.15}$$

see also Proposition 3.2.1 below. We shall furthermore assume that in any set of the form

$$|X|\leq C,\quad \operatorname{dist}\left(X,\mathcal{C}\right)\geq \frac{1}{C}, \tag{3.1.16}$$

we have for each $T>0$ fixed,

$$\langle \operatorname{Re} p_0\rangle_{T,p_0}(X)\geq \frac{1}{\widetilde{C}(C)},\quad \widetilde{C}(C)>0. \tag{3.1.17}$$

Our purpose in this preliminary work is to carry out a construction of a suitable globally defined compactly supported weight function, assuming that the assumptions (3.1.3), (3.1.4), (3.1.14), (3.1.17) hold. The main result of this work is stated in Theorem 3.2.2 below. Working in the corresponding exponentially weighted space of holomorphic functions on the FBI-Bargmann transform side should then lead to some precise results concerning the eigenvalues and resolvent estimates for P in a region of the form $\{z \in \mathbf{C}; \operatorname{Re} z \leq \mathcal{O}(h)\}$, and this analysis will be pursued after the completion of the Ph.D. thesis.

3.2 Bounded weight function

Our starting point is the following proposition, which is a natural analog of [22, Proposition 2].

Proposition 3.2.1. For each fixed $T > 0$ and each j, $1 \leq j \leq N$, we have

$$\langle \operatorname{Re} p_0 \rangle_{T,\operatorname{Im} p_0}(X) = \langle \operatorname{Re} q_j \rangle_{T,\operatorname{Im} q_j}(X - X_j) + \mathcal{O}(|X - X_j|^3), \quad X \to X_j, \tag{3.2.1}$$

where

$$\langle \operatorname{Re} q_j \rangle_{T,\operatorname{Im} q_j}(Y) = \frac{1}{T} \int_0^T \operatorname{Re} q_j \left(\exp\left(tH_{\operatorname{Im} q_j}\right)(Y)\right)\, dt.$$

Proof: We begin by noticing that there exists $C > 0$ such that for all $X \in \mathbf{R}^{2n}$ and all $1 \leq j \leq N$, we have, since $p_0 \in S(1)$ and $H_{\operatorname{Im} p_0}(X_j) = 0$,

$$|H_{\operatorname{Im} p_0}(X)| \leq C\,|X - X_j|\,, \quad \left|H_{\operatorname{Im} q_j}(X)\right| \leq C\,|X|\,. \tag{3.2.2}$$

Writing

$$\exp(tH_{\operatorname{Im} p_0})(X) = X + \int_0^t H_{\operatorname{Im} p_0}(\exp(sH_{\operatorname{Im} p_0}X))\, ds, \tag{3.2.3}$$

we infer that, using (3.2.2),

$$\begin{aligned} |\exp(tH_{\operatorname{Im} p_0})(X) - X_j| &\leq |\exp(tH_{\operatorname{Im} p_0})(X) - X| + |X - X_j| \\ &\leq |X - X_j| + C\int_0^t |\exp(sH_{\operatorname{Im} p_0})(X) - X_j|\, ds, \end{aligned} \tag{3.2.4}$$

and therefore by Gronwall's Lemma,

$$|\exp(tH_{\operatorname{Im} p_0})(X) - X_j| \leq e^{Ct}\,|X - X_j|, \quad t \geq 0,\ X \in \mathbf{R}^{2n}. \tag{3.2.5}$$

Then by using (3.2.2), (3.2.3), and (3.2.5), we obtain

$$|\exp(tH_{\operatorname{Im} p_0})(X) - X| \leq \int_0^t Ce^{Cs}\,|X - X_j|\, ds = (e^{Ct} - 1)\,|X - X_j|.$$

In particular, for each fixed $T > 0$, there exists $c > 0$ such that for all $0 \leq t \leq T$ and $X \in \mathbf{R}^{2n}$, we have

$$|\exp(tH_{\operatorname{Im} p_0})(X) - X| \leq ct\,|X - X_j|. \tag{3.2.6}$$

Similar arguments applied to the quadratic form q_j give

$$\left|\exp(tH_{\operatorname{Im} q_j})X - X\right| \leq ct\,|X|. \tag{3.2.7}$$

Then it directly follows from (3.1.12) and (3.2.5) that

$$\langle \operatorname{Re} p_0\rangle_{T,\operatorname{Im} p_0}(X) = \frac{1}{T}\int_0^T \operatorname{Re} q_j(\exp(tH_{\operatorname{Im} p_0})X - X_j)\, dt + \mathcal{O}(|X - X_j|^3). \tag{3.2.8}$$

Let us now make the following observation,

$$\left|(\exp(tH_{\operatorname{Im} p_0})X - X_j) - \exp(tH_{\operatorname{Im} q_j})(X - X_j)\right| \leq \widetilde{c}t\,|X - X_j|^2, \quad 0 \leq t \leq T, \tag{3.2.9}$$

for some $\widetilde{c} > 0$. When verifying (3.2.9), let us set $r(X) = \operatorname{Im} p_0(X) - \operatorname{Im} q_j(X - X_j)$ and note that $H_r(X) = \mathcal{O}(1)\,|X - X_j|^2$ uniformly on $\mathbf{R}^{2n}$. Write next

$$\begin{aligned}
&(e^{tH_{\operatorname{Im} p_0}}X - X_j) - e^{tH_{\operatorname{Im} q_j}}(X - X_j) \\
&\qquad = \int_0^t \left[H_{\operatorname{Im} p_0}(e^{sH_{\operatorname{Im} p_0}}X) - H_{\operatorname{Im} q_j}(e^{sH_{\operatorname{Im} q_j}}(X - X_j))\right] ds \\
&= \int_0^t H_r(e^{sH_{\operatorname{Im} p_0}}X)\,ds + \int_0^t H_{\operatorname{Im} q_j}(e^{sH_{\operatorname{Im} p_0}}X - X_j) - H_{\operatorname{Im} q_j}(e^{sH_{\operatorname{Im} q_j}}(X - X_j))\,ds \\
&= \int_0^t H_r(e^{sH_{\operatorname{Im} p_0}}X)\,ds + \int_0^t H_{\operatorname{Im} q_j}((e^{sH_{\operatorname{Im} p_0}}X - X_j) - e^{sH_{\operatorname{Im} q_j}}(X - X_j))\,ds.
\end{aligned} \tag{3.2.10}$$

Here we have used the fact that $H_{\operatorname{Im} q}$ is a linear map. Setting

$$f(t) = (e^{tH_{\operatorname{Im} p_0}}X - X_j) - e^{tH_{\operatorname{Im} q_j}}(X - X_j),$$

we may rewrite (3.2.10) as follows,

$$f(t) = \int_0^t H_r(e^{sH_{\operatorname{Im} p_0}}X)\,ds + \int_0^t H_{\operatorname{Im} q_j}(f(s))\,ds, \tag{3.2.11}$$

which combined with (3.2.2), (3.2.5), and the fact that $H_r(X) = \mathcal{O}(|X - X_j|^2)$, gives

$$|f(t)| \leq \mathcal{O}(1)t\,|X - X_j|^2 + C\int_0^t |f(s)|\,ds, \quad 0 \leq t \leq T. \tag{3.2.12}$$

Then, by Gronwall's Lemma, we get

$$\int_0^t |f(s)|\, ds \leq \mathcal{O}(1) t^2 \left|X - X_j\right|^2, \tag{3.2.13}$$

and combining (3.2.13) with (3.2.12), we conclude the verification of the observation (3.2.9),

$$|f(t)| \leq \mathcal{O}(1) t |X - X_j|^2, \quad 0 \leq t \leq T. \tag{3.2.14}$$

It is now easy to complete the proof of the proposition. Using the fact that q_j is quadratic together with (3.2.9) we get that

$$\operatorname{Re} q_j(e^{tH_{\operatorname{Im} p_0}} X - X_j) = \operatorname{Re} q_j(e^{tH_{\operatorname{Im} q_j}}(X - X_j)) + \mathcal{O}(1) \left|X - X_j\right|^3, \quad 0 \leq t \leq T. \tag{3.2.15}$$

This together with (3.2.8) gives

$$\langle \operatorname{Re} p_0 \rangle_{T, \operatorname{Im} p_0}(X) = \frac{1}{T} \int_0^T \operatorname{Re} q_j(e^{tH_{\operatorname{Im} q_j}}(X - X_j))\, dt + \mathcal{O}(1) \left|X - X_j\right|^3, \tag{3.2.16}$$

which concludes the proof of Proposition 3.2.1. □

For the sake of convenience, we define $\widetilde{q}_j(X) = \langle \operatorname{Re} q_j \rangle_{T, \operatorname{Im} q_j}(X - X_j)$, thus giving in view of (3.2.1),

$$\langle \operatorname{Re} p_0 \rangle_{T, \operatorname{Im} p_0}(X) = \widetilde{q}_j(X) + \mathcal{O}(|X - X_j|^3). \tag{3.2.17}$$

Note that the assumption (3.1.14) is equivalent to the statement that $X \mapsto \widetilde{q}_j(X + X_j)$ is a positive definite quadratic form on $\mathbf{R}^{2n}$, $1 \leq j \leq N$.

We shall now proceed to construct a suitable bounded weight function on $\mathbf{R}^{2n}$. When doing so, we shall rely on some ideas and techniques of [20], [22]. Let $g \in C^\infty([0, \infty); [0, 1])$

be decreasing such that

$$g(t)=1,\ t\in[0,1],\ \text{and}\ g(t)=t^{-1},\ t\geq 2. \tag{3.2.18}$$

It follows that for each $k\in\mathbf{N}$, we have

$$g^{(k)}(t)=\mathcal{O}(\langle t\rangle^{-1-k}), \tag{3.2.19}$$

as $t\to+\infty$, where $\langle t\rangle=(1+t^2)^{1/2}$. Let $\Omega_j\subset\mathbf{R}^{2n}$ be small open sets, $1\leq j\leq N$, such that $X_j\in\Omega_j$ and $\overline{\Omega_j}\cap\overline{\Omega_k}=\emptyset$, for $j\neq k$. Letting $\chi_j\in C_0^\infty(\Omega_j;[0,1])$ be such that $\chi_j=1$ near X_j, we set for $0<\varepsilon\leq 1$ small enough,

$$(\operatorname{Re}p_0)_\varepsilon(X)=\sum_{j=1}^N\chi_j(X)g\left(\frac{|X-X_j|^2}{\varepsilon}\right)\operatorname{Re}p_0(X)+\varepsilon\left(1-\sum_{j=1}^N\chi_j(X)\right)\operatorname{Re}p_0(X). \tag{3.2.20}$$

In that way, for each $X\in\mathbf{R}^{2n}$, we have: either $X\notin\bigcup_{j=1}^N\Omega_j$, and thus $(\operatorname{Re}p_0)_\varepsilon(X)=\varepsilon\operatorname{Re}p_0(X)$, or $X\in\Omega_j$ for some unique j, and in this case we have,

$$(\operatorname{Re}p_0)_\varepsilon(X)=\chi_j(X)g\left(\frac{|X-X_j|^2}{\varepsilon}\right)\operatorname{Re}p_0(X)+\varepsilon\left(1-\chi_j(X)\right)\operatorname{Re}p_0(X).$$

In particular we obtain that

$$(\operatorname{Re}p_0)_\varepsilon(X)=\operatorname{Re}p_0(X),\ \text{for}\ |X-X_j|\leq\varepsilon^{1/2}, \tag{3.2.21}$$

$$(\operatorname{Re}p_0)_\varepsilon(X)\asymp\frac{\varepsilon\operatorname{Re}p_0(X)}{|X-X_j|^2},\quad\text{for}\ \varepsilon^{1/2}\leq|X-X_j|\leq\frac{1}{\mathcal{O}(1)}. \tag{3.2.22}$$

$$
\begin{aligned}
(\operatorname{Re} p_0)_\varepsilon(X) &= \chi_j(X)\frac{\varepsilon}{|X-X_j|^2}\operatorname{Re} p_0(X) + \varepsilon\,(1-\chi_j(X))\operatorname{Re} p_0(X) \\
&\geq \frac{\varepsilon \operatorname{Re} p_0(X)}{\mathcal{O}(1)}, \quad |X-X_j| \geq \frac{1}{\mathcal{O}(1)},\ X \in \Omega_j. \quad (3.2.23)
\end{aligned}
$$

Recalling also (3.1.12) we conclude that

$$
(\operatorname{Re} p_0)_\varepsilon(X) = \mathcal{O}(\varepsilon), \tag{3.2.24}
$$

uniformly on $\mathbf{R}^{2n}$. Straightforward estimates and computations making use of (3.1.12) and (3.2.19) show also that

$$
\partial_X^\alpha (\operatorname{Re} p_0)_\varepsilon(X) = \mathcal{O}(\varepsilon^{1-\frac{\alpha}{2}}), \quad |\alpha| \leq 2, \tag{3.2.25}
$$

uniformly on $\mathbf{R}^{2n}$.

We shall now introduce our bounded weight function. Let $T>0$, and let us set

$$
G_\varepsilon(X) = -\int J\left(-\frac{t}{T}\right)(\operatorname{Re} p_0)_\varepsilon(\exp{(tH_{\operatorname{Im} p_0}X)})\,dt. \tag{3.2.26}
$$

Here J is a real valued compactly supported piecewise affine function on $\mathbf{R}$ such that

$$
J'(t) = \delta(t) - 1_{[-1,0]}(t),
$$

with $1_{[-1,0]}$ being the characteristic function of the interval $[-1,0]$. Arguing as in [22], using an integration by parts, we see that

$$
H_{\operatorname{Im} p_0} G_\varepsilon = \langle (\operatorname{Re} p_0)_\varepsilon \rangle_{T,\operatorname{Im},p_0} - (\operatorname{Re} p_0)_\varepsilon, \tag{3.2.27}
$$

where similarly to (3.1.13), we have

$$\langle (\operatorname{Re} p_0)_\varepsilon \rangle_{T,\operatorname{Im} p_0}(X) := \frac{1}{T}\int_0^T (\operatorname{Re} p_0)_\varepsilon(\exp(tH_{\operatorname{Im} p_0})X)\, dt.$$

It follows from (3.2.25) and (3.2.26) that the function $G \in C^\infty(\mathbf{R}^{2n};\mathbf{R})$ satisfies

$$\partial_X^\alpha G_\varepsilon(X) = \mathcal{O}(\varepsilon^{1-\frac{\alpha}{2}}), \quad |\alpha| \leq 2, \tag{3.2.28}$$

uniformly on $\mathbf{R}^{2n}$.

Using (3.2.5), (3.2.20), and (3.2.26) we infer that

$$G_\varepsilon(X) = -\int J\left(-\frac{t}{T}\right) \operatorname{Re} p_0(\exp(tH_{\operatorname{Im} p_0})X)\, dt, \quad |X - X_j|^2 \leq \frac{\varepsilon}{2}, \tag{3.2.29}$$

provided that $T > 0$ is small enough fixed, independent of ε. We have therefore in this region,

$$H_{\operatorname{Im} p_0} G_\varepsilon(X) = \langle \operatorname{Re} p_0 \rangle_{T,\operatorname{Im} p_0}(X) - \operatorname{Re} p_0(X). \tag{3.2.30}$$

Recalling (3.1.14), we conclude in particular that

$$\langle (\operatorname{Re} p_0)_\varepsilon \rangle_{T,\operatorname{Im},p_0}(X) \asymp |X - X_j|^2, \quad |X - X_j|^2 \leq \frac{\varepsilon}{2}, \tag{3.2.31}$$

for all $1 \leq j \leq N$, or in other words,

$$\langle (\operatorname{Re} p_0)_\varepsilon \rangle_{T,\operatorname{Im},p_0}(X) \asymp \operatorname{dist}(X,\mathcal{C})^2, \quad \operatorname{dist}(X,\mathcal{C})^2 \leq \frac{\varepsilon}{2}. \tag{3.2.32}$$

Remark. It follows from the proof of Proposition 3.2.1 and (3.2.29) that we have for

$1 \leq j \leq N$,

$$G_\varepsilon(X) = G_j^0(X - X_j) + \mathcal{O}(|X - X_j|^3), \quad |X - X_j|^2 \leq \frac{\varepsilon}{2}, \tag{3.2.33}$$

where

$$G_j^0(X) = -\int J\left(-\frac{t}{T}\right) \operatorname{Re} q_j(e^{tH_{\operatorname{Im} q_j}} X)\, dt \tag{3.2.34}$$

is quadratic.

Let $\widetilde{p}_0 \in C^\infty(\mathbf{C}^{2n})$ be an almost holomorphic extension of p_0, supported in a fixed tubular neighborhood of $\mathbf{R}^{2n}$, such that $\partial^\alpha \widetilde{p}_0 \in L^\infty(\mathbf{C}^{2n})$ for each α. With $0 < \delta \leq 1$ to be chosen small enough, we get by a Taylor expansion that

$$\widetilde{p}_0\left(X + i\delta H_{G_\varepsilon}(X)\right) = p_0(X) + i\delta H_{G_\varepsilon} p_0(X) + \mathcal{O}(\delta^2 \left|\nabla G_\varepsilon(X)\right|^2), \quad X \in \mathbf{R}^{2n}. \tag{3.2.35}$$

It follows that

$$\operatorname{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\epsilon}(X))\right) = \operatorname{Re} p_0(X) + \delta H_{\operatorname{Im} p_0} G_\varepsilon(X) + \mathcal{O}(\delta^2 \left|\nabla G_\varepsilon(X)\right|^2), \quad X \in \mathbf{R}^{2n}. \tag{3.2.36}$$

We would like to get some improved positivity estimates for $\operatorname{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\epsilon}(X))\right)$, and to this end we shall first consider the region where $|X - X_j|^2 \leq \frac{\varepsilon}{2}$, for some $1 \leq j \leq N$. We get, using (3.2.17), (3.2.30),

$$\begin{aligned}\operatorname{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))\right) &= \operatorname{Re} p_0 + \delta(\langle \operatorname{Re} p_0\rangle_{T, \operatorname{Im} p_0}(X) - \operatorname{Re} p_0(X)) + \mathcal{O}(\delta^2 \left|X - X_j\right|^2) \\ &= (1-\delta)\operatorname{Re} p_0(X) + \delta \widetilde{q}_j(X) + \mathcal{O}(1)\left(\delta \left|X - X_j\right|^3 + \delta^2 \left|X - X_j\right|^2\right).\end{aligned} \tag{3.2.37}$$

Here we have also used the estimate

$$\nabla G_\varepsilon(X) = \mathcal{O}(|X - X_j|), \quad |X - X_j|^2 \leq \frac{\varepsilon}{2}, \tag{3.2.38}$$

which follows from (3.2.33), (3.2.34). Using that $(1 - \delta)\mathrm{Re}\, p_0(X) \geq 0$ for $\delta \in (0, 1]$, together with the fact that

$$\widetilde{q}_j(X) \asymp |X - X_j|^2, \quad X \in \mathbf{R}^{2n}, \tag{3.2.39}$$

we get from (3.2.37) that there exists a constant $\widehat{C} > 0$ such that

$$\begin{aligned}\mathrm{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))\right) &\geq \frac{\delta|X - X_j|^2}{\widehat{C}} - \mathcal{O}\left(\delta|X - X_j|^3\right) - \mathcal{O}\left(\delta^2|X - X_j|^2\right) \\ &\geq \frac{\delta|X - X_j|^2}{2\widehat{C}}, \quad |X - X_j|^2 \leq \frac{\varepsilon}{2},\end{aligned} \tag{3.2.40}$$

provided that $\delta > 0$, $\varepsilon > 0$ are chosen sufficiently small.

We shall next consider the region of the phase space where $8\varepsilon \leq |X - X_j|^2 \leq \dfrac{1}{\mathcal{O}(1)}$, for some $1 \leq j \leq N$, where the implicit constant is large enough so that $\chi_j = 1$ here. It follows from (3.2.36), (3.2.27), (3.2.20), and (3.2.28), that we have in this region,

$$\begin{aligned}\mathrm{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))\right) &= \mathrm{Re}\, p_0(X) + \delta\left(\langle(\mathrm{Re}\, p_0)_\varepsilon\rangle_{T,\mathrm{Im}, p_0}(X) - (\mathrm{Re}\, p_0)_\varepsilon(X)\right) + \mathcal{O}(\delta^2\varepsilon) \\ = \mathrm{Re}\, p_0(X) - \delta(\mathrm{Re}\, p_0)_\varepsilon(X) &+ \frac{\delta\varepsilon}{T}\int_0^T \frac{\mathrm{Re}\, p_0(\exp{(tH_{\mathrm{Im}\, p_0})}(X))}{|\exp{(tH_{\mathrm{Im}\, p_0})}(X) - X_j|^2}\, dt + \mathcal{O}(\delta^2\varepsilon),\end{aligned} \tag{3.2.41}$$

provided that $T > 0$ is chosen sufficiently small. To understand the right hand side of

(3.2.41), we infer from the proof of Proposition 3.2.1 that

$$\frac{1}{T}\int_0^T \frac{\operatorname{Re} p_0(\exp(tH_{\operatorname{Im} p_0})(X))}{|\exp(tH_{\operatorname{Im} p_0})(X)-X_j|^2}\,dt \\ = \frac{1}{T}\int_0^T \frac{\operatorname{Re} q_j(\exp(tH_{\operatorname{Im} q_j})(X-X_j))}{|\exp(tH_{\operatorname{Im} p_0})(X)-X_j|^2}\,dt + \mathcal{O}(|X-X_j|). \quad (3.2.42)$$

Furthermore, using (3.2.6), (3.2.7), and (3.2.9) we see that

$$\frac{1}{|\exp(tH_{\operatorname{Im} p_0})(X)-X_j|^2} - \frac{1}{\left|\exp(tH_{\operatorname{Im} q_j})(X-X_j)\right|^2} = \mathcal{O}\left(\frac{t\,|X-X_j|^3}{|X-X_j|^4}\right)$$

and we get therefore

$$\frac{1}{T}\int_0^T \frac{\operatorname{Re} p_0(\exp(tH_{\operatorname{Im} p_0})(X))}{|\exp(tH_{\operatorname{Im} p_0})(X)-X_j|^2}\,dt = f_j(X-X_j)+\mathcal{O}(|X-X_j|), \quad (3.2.43)$$

where

$$f_j(Y) = \frac{1}{T}\int_0^T \frac{\operatorname{Re} q_j(\exp(tH_{\operatorname{Im} q_j})(Y))}{\left|\exp(tH_{\operatorname{Im} q_j})(Y)\right|^2}\,dt. \quad (3.2.44)$$

The assumption (3.1.14) implies that the non-negative homogeneous of degree 0 function f_j satisfies

$$f_j(Y) \geq \frac{1}{C}, \quad Y \neq 0, \quad (3.2.45)$$

for some constantc$C>0$. We get, combining (3.2.41), (3.2.43), and (3.2.45) that

$$\operatorname{Re}(\widetilde{p}_0(X+i\delta H_{G_\varepsilon}(X))) = \operatorname{Re} p_0(X) - \delta(\operatorname{Re} p_0)_\varepsilon(X) \\ + \delta\varepsilon f_j(X-X_j) + \mathcal{O}(\delta^2\varepsilon + \delta\varepsilon\,|X-X_j|) \\ \geq \frac{\delta\varepsilon}{C} - \mathcal{O}(\delta^2\varepsilon) - \mathcal{O}(\delta\varepsilon\,|X-X_j|). \quad (3.2.46)$$

Here we have also used that $\operatorname{Re} p_0 - \delta(\operatorname{Re} p_0)_\varepsilon \geq 0$, for $0<\delta\leq 1$. We conclude that there exist some positive constants $0 < \delta_0 \leq 1$, $C > 1$, $\widetilde{C} > 1$ such that for all $0 < \delta \leq \delta_0$ and all $0 < \varepsilon \leq \varepsilon_0$, we have

$$\mathrm{Re}\,(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))) \geq \frac{\delta\varepsilon}{\widetilde{C}}, \quad 8\varepsilon \leq |X - X_j|^2 \leq 1/C. \tag{3.2.47}$$

Combining (3.2.40) and (3.2.47), we see that in order to cover a sufficiently small but fixed neighborhood of the critical set $\mathcal{C}$, it suffices to consider the intermediate region where $\varepsilon/2 \leq |X - X_j|^2 \leq 8\varepsilon$, for some j. Here we can write, similarly to (3.2.41),

$$\begin{aligned}\mathrm{Re}\,(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))) &= \mathrm{Re}\,p_0(X) + \delta\,(\langle(\mathrm{Re}\,p_0)_\varepsilon\rangle_{T,\mathrm{Im},p_0}(X) - (\mathrm{Re}\,p_0)_\varepsilon(X)) + \mathcal{O}(\delta^2\varepsilon) \\ &= \mathrm{Re}\,p_0(X) - \delta(\mathrm{Re}\,p_0)_\varepsilon(X) \\ &+ \frac{\delta}{T}\int_0^T g\left(\frac{|\exp(tH_{\mathrm{Im}\,p_0})(X) - X_j|^2}{\varepsilon}\right)\mathrm{Re}\,p_0(\exp(tH_{\mathrm{Im}\,p_0})(X))\,dt + \mathcal{O}(\delta^2\varepsilon).\end{aligned} \tag{3.2.48}$$

In this region we have

$$g\left(\frac{|\exp(tH_{\mathrm{Im}\,p_0})(X) - X_j|^2}{\varepsilon}\right) \asymp 1,$$

uniformly with respect to the parameters $0 < \varepsilon \leq \varepsilon_0$ and $t \in [0, T]$, provided that the positive constant T is chosen sufficiently small. It is clear therefore that we get the lower bound

$$\mathrm{Re}\,(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))) \geq \frac{\delta\epsilon}{\mathcal{O}(1)}, \tag{3.2.49}$$

also in the region where $\varepsilon/2 \leq |X - X_j|^2 \leq 8\varepsilon$.

To summarize the discussion so far, we have shown that there exist positive constants $C > 1$, $\widetilde{C} > 1$, $0 < \varepsilon_0 \leq 1$, $0 < \delta_0 \leq 1$, such that the weight function $G_\varepsilon \in C^\infty(\mathbf{R}^{2n}; \mathbf{R})$

introduced in (3.2.26), satisfies for all $0 < \varepsilon \leq \varepsilon_0$, $0 < \delta \leq \delta_0$,

$$\operatorname{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))\right) \geq \frac{\delta\, \operatorname{dist}(X, \mathcal{C})^2}{\widetilde{C}}, \quad \operatorname{dist}(X, \mathcal{C}) \leq \varepsilon^{1/2}, \tag{3.2.50}$$

$$\operatorname{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))\right) \geq \frac{\delta\, \varepsilon}{\widetilde{C}}, \quad \varepsilon^{1/2} \leq \operatorname{dist}(X, \mathcal{C}) \leq \frac{1}{C}. \tag{3.2.51}$$

It remains finally to consider the region where $\operatorname{dist}(X, \mathcal{C}) \geq \frac{1}{C}$, and here, in view of (3.1.4), we may restrict the attention to the closed ball $|X| \leq C$. Observing that

$$\begin{aligned}(\operatorname{Re} p_0)_\varepsilon(X) &= \chi_j(X) \frac{\varepsilon}{|X - X_j|^2} \operatorname{Re} p_0(X) + \varepsilon\,(1 - \chi_j(X))\operatorname{Re} p_0(X) \\ &\geq \frac{1}{\mathcal{O}(1)}\, \varepsilon \operatorname{Re} p_0(X), \quad X \in \Omega_j,\ \operatorname{dist}(X, \mathcal{C}) \geq \frac{1}{C}\end{aligned} \tag{3.2.52}$$

and recalling that $(\operatorname{Re} p_0)_\varepsilon(X) = \varepsilon \operatorname{Re} p_0(X)$ for $X \notin \bigcup_{j=1}^{N} \Omega_j$, we obtain in view of (3.1.17), by repeating the arguments above,

$$\operatorname{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))\right) \geq \frac{\delta\, \varepsilon}{\widetilde{C}}, \quad \operatorname{dist}(X, \mathcal{C}) \geq \frac{1}{C}. \tag{3.2.53}$$

Let $\chi \in C_0^\infty(\mathbf{R}^{2n}; [0, 1])$ be such that $\chi = 1$ on a large compact set, and let us set

$$\widetilde{G}_\varepsilon = \chi G_\varepsilon \in C_0^\infty(\mathbf{R}^{2n}; \mathbf{R}). \tag{3.2.54}$$

We may then observe that the estimates (3.2.50), (3.2.51), (3.2.53) are still valid with G_ε replaced by $\widetilde{G}_\varepsilon$, since the support of $\nabla \chi$ is confined to the elliptic region, in view of (3.1.4). The discussion in this section can therefore be summarized in the following result, which is a natural extension of [22, Proposition 3], see also [20].

Theorem 3.2.2. Let $P(x, \xi; h) \in S(1)$ be such that the assumptions (3.1.2), (3.1.3), (3.1.4), (3.1.11), (3.1.14), (3.1.17) hold, and let $\widetilde{p}_0 \in C^\infty(\mathbf{C}^{2n})$ be an almost holomorphic extension of the semiclassical leading symbol p_0 of P, supported in a tubular neighborhood of $\mathbf{R}^{2n}$, bounded together with all derivatives. There exist constants $\widetilde{C} > 1$, $0 < \delta_0 \leq 1$, $0 < \varepsilon_0 \leq 1$, and a function $\widetilde{G}_\varepsilon \in C_0^\infty(\mathbf{R}^{2n}; \mathbf{R})$ depending on the parameter $0 < \varepsilon \leq \varepsilon_0$, satisfying

$$\partial^\alpha G_\varepsilon = \mathcal{O}(\varepsilon^{1-\frac{|\alpha|}{2}}), \quad |\alpha| \leq 2,$$

uniformly, such that we have for all $0 < \varepsilon \leq \varepsilon_0$, $0 < \delta \leq \delta_0$,

$$\operatorname{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))\right) \geq \frac{\delta \operatorname{dist}(X, \mathcal{C})^2}{\widetilde{C}}, \quad \operatorname{dist}(X, \mathcal{C}) \leq \varepsilon^{1/2},$$

$$\operatorname{Re}\left(\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))\right) \geq \frac{\delta\, \varepsilon}{\widetilde{C}}, \quad \operatorname{dist}(X, \mathcal{C}) \geq \varepsilon^{1/2}.$$

Bibliography

[1] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. **14** (1961), 187–214.

[2] R. Berman, B. Berndtsson, and J. Sjöstrand, *A direct approach to Bergman kernel asymptotics for positive line bundles*, Ark. för Matematik, **46** (2008), 197–217.

[3] B. Berndtsson, *An introduction to things* $\overline{\partial}$, Analytic and algebraic geometry, 7–76, IAS/Park City Math. Ser., 17, Amer. Math. Soc., Providence, RI, 2010.

[4] L. Boutet de Monvel, *Hypoelliptic operators with double characteristics and related pseudo-differential operators*, Comm. Pure Appl. Math. **27** (1974), 585–639.

[5] L. Boutet de Monvel and J. Sjöstrand. *Sur la singularité des noyaux de Bergman et de Szegö*, Journées Équations aux Dérivées Partielles de Rennes, Astérisque **34-35** (1976), 123–164.

[6] D. Catlin, *The Bergman kernel and a theorem of Tian*, Analysis and geometry in several complex variables (Katata, 1997), 1–23, Trends Math. Birkhäuser Boston, MA, 1999.

[7] L. Charles, *Berezin-Toeplitz operators, a semi-classical approach*, Comm. Math. Phys. **239** (2003), 1–28.

[8] L. Coburn, *Fock space, the Heisenberg group, heat flow and Toeplitz operators*, Chapter 1 (pp. 1–15) in *Handbook of analytic operator theory*, ed. Kehe Zhu, CRC Press, 2019.

[9] L. Coburn, M. Hitrik, and J. Sjöstrand, *Positivity, complex FIOs, and Toeplitz operators*, Pure Appl. Anal. **1** (2019), 327–357.

[10] X. Dai, K. Liu, X. Ma, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom. **72** (2006), 1—41.

[11] A. Deleporte, *Toeplitz operators with analytic symbols*, J. Geom. Analysis **31** (2021), 3915—3967.

[12] A. Deleporte, M. Hitrik, and J. Sjöstrand, *A direct approach to the analytic Bergman projection*, Ann. Fac. Sci. Toulouse Math., to appear.

[13] J.-M. Delort, *F.B.I. transformation. Second microlocalization and semilinear caustics*, Lecture Notes in Mathematics 1522, Springer-Verlag, Berlin, 1992.

[14] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, Cambridge University Press, Cambridge, 1999.

[15] M. Engliš, *The asymptotics of a Laplace integral on a Kähler manifold*, J. Reine Angew. Math. **528** (2000), 1–39.

[16] M. Engliš, *Weighted Bergman kernels and quantization*, Comm. Math. Phys. **227** (2002), 211—241.

[17] A. Grigis and J. Sjöstrand, *Front d'onde analytique et sommes de carrés de champs de vecteurs*, Duke Math. J. **52** (1985), 35–51.

[18] A. Grigis and J. Sjöstrand, *Microlocal analysis for differential operators*, Cambridge University Press, 1994.

[19] F. Hérau, J. Sjöstrand, and C. Stolk, *Semiclassical analysis for the Kramers-Fokker-Planck equation*, Comm. PDE, **30** (2005), 689–760.

[20] F. Hérau, M. Hitrik, and J. Sjöstrand, *Tunnel effect for Kramers-Fokker-Planck type operators*, Ann. Henri Poincaré **9** (2008), 209–274.

[21] M. Hitrik and K. Pravda-Starov *Spectra and semigroup smoothing for non-elliptic quadratic operators*, Math. Ann., **344** (2009), 801–846.

[22] M. Hitrik and K. Pravda-Starov, *Semiclassical hypoelliptic estimates for non-selfadjoint operators with double characteristics*, Comm. P.D.E., **35** (2010), 988–1028.

[23] M. Hitrik and K. Pravda-Starov, *Eigenvalues and subelliptic estimates for non-selfadjoint semiclassical operators with double characteristics*, Annales Inst. Fourier, **63** (2013), 985–1032.

[24] M. Hitrik and J. Sjöstrand, *Two minicourses on analytic microlocal analysis*, "Algebraic and Analytic Microlocal Analysis", Springer Proceedings in Mathematics and Statistics **269** (2018), 483–540.

[25] M. Hitrik and J. Sjöstrand, *Analytic second microlocalization: a semiglobal approach*, work in progress.

[26] M. Hitrik and M. Stone, *Asymptotics for Bergman projections with smooth weights: a direct approach*, `arxiv:2105.14402`.

[27] C.-Y. Hsiao and G. Marinescu, *Asymptotics of spectral function of lower energy forms and Bergman kernel of semi-positive and big line bundles*, Comm. Anal. Geom. **22** (2014), 1—108.

[28] L. Hörmander, *L^2 estimates and existence theorems for the $\overline{\partial}$ operator*, Acta Math. **113** (1965), 89—152.

[29] L. Hörmander, *Fourier Integral Operators*, Lectures at the Nordic Summer School of Mathematics, 1969, with comments by J. Sjöstrand, 2018, `https://portal.research.lu.se/portal/en/publications/fourier-integral-operators(7730e675-f8df-4464-9cfe-2faeb4619441).html`.

[30] L. Hörmander, *The analysis of linear partial differential operators* (vol. I–IV), Springer Verlag, 1985.

[31] L. Hörmander, *Notions of convexity*, Birkhäuser, 1994.

[32] C. Kenig, J. Sjöstrand, and G. Uhlmann, *The Calderón problem with partial data*, Ann. of Math., **165** (2007), 567–591.

[33] X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Prog. Math., vol. 254, Birkhäuser Verlag, Basel, 2007.

[34] J. N. Mather, *On Nirenberg's proof of Malgrange's preparation theorem*, Proc. Liverpool Singularities-Symp. I, Lecture Notes in Math. **192** (1971), 116–120, Springer, Berlin, 1971.

[35] A. Melin and J. Sjöstrand, *Fourier integral operators with complex-valued phase functions*, Fourier integral operators and partial differential equations, Lecture Notes in Math. **459** (1975), 120—223, Springer, Berlin, 1975.

[36] A. Melin and J. Sjöstrand, *Determinants of pseudodifferential operators and complex deformations of phase space*, Methods Appl. Anal. **9** (2002), 177—237.

[37] O. Rouby, J. Sjöstrand, and S. Vũ Ngoc, *Analytic Bergman operators in the semiclassical limit*, Duke Math. J. **16** (2020), 3033—3097.

[38] J. Sjöstrand, *Parametrices for pseudodifferential operators with multiple characteristics*, Ark. för Matematik **12** (1974), 85-130.

[39] J. Sjöstrand, *Singularités analytiques microlocales*, Asterisque 1982.

[40] J. Sjöstrand, *Function spaces associated to global* I-*Lagrangian manifolds*, Structure of solutions of differential equations, Katata/Kyoto, 1995, World Scientific, 1996.

[41] F. Trèves, *Introduction to Pseudodifferential and Fourier Integral Operators*, (vol. I–II), Plenum Publ. Co. 1982.

[42] S. Zelditch, *Szegö kernels and a theorem of Tian*, Int. Math. Res. Not. **6** (1998), 317—331.