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Formalized Harmony

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by

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ABSTRACT

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The progress in electronic and computer sciences has transformed music, introducing new techniques and tools that have completely impacted the way we compose, perform, and distribute music. As creators and consumers, our experience with music has been shaped by these new technologies, resulting in a rapid evolution of the art form.

However, Western music theory has remained largely unchanged, with technology settling into the current system and formalizing its rules in each protocol and platform developed. Since Western music, and specifically its harmonic rules, were developed in a technological context that has since changed, it stands to reason that the theory should evolve as well. Several artists have expanded their artistic practice by exploring new systems, such as serialism, stochastic music, and microtonality. Composers such as Xenakis, Partch, Johnston, Tenney and Barlow have laid the foundation for a new computational system in music.

Understanding the perceived root phenomenon is essential to explain the most fundamental structures in twelve-tone equal temperament or 5-limit just intonation, as well as proposing new chords for new tunings. Through a music perception study conducted on

41 participants and 10 different tunings empirically establishes harmonic dualism, negative harmony and Barlow's harmonicity as the key factors for the perception of the harmonic root and the origin of the major and minor triads, and proposes a model as a basis for a tuning-agnostic – trans-spatial – music theory.

As a result, a set of rules and algorithms are introduced to expand music's harmonic system through computation. This thesis proposes a tuning-agnostic music theory, organizing and codifying the principles and rules of harmony into a systematic and recognizable form.

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I. Introduction

A. Problem statement

Computation and electronics have enabled the development of tools and instruments capable of producing harmonies beyond the traditional twelve-tone system. However, the process of abstracting, formalizing, and codifying the rules of harmony in a tuning-agnostic manner involves many unknown steps.

There are several ways in which the axioms of Western music theory can be translated into higher-level structures. However, the theories that support these structures lack coherence outside of the twelve-tone system. Therefore, further research is needed to develop a comprehensive understanding of how to apply harmonic principles beyond traditional tuning systems.

B. Relevance of the research

To compose using unexplored tunings and develop a music theory that is independent of specific tunings, it is crucial to define a model capable of predicting the perceived harmonic root and identifying the perceived fundamental inversion for basic chords in these spaces. Moreover, since current theories on the fundamental bass and the origin of the minor triad diverge when applied to extended just intonation, exploring new spaces can provide a better understanding of the space in which current music practice occurs.

As Thomson (1993) pointed out, empirical studies have yet to confirm the phenomenal reality that our percepts confidently describe. This thesis aims to explain these phenomena and proposes a model for generating the fundamental chords in any tuning. This model represents the first step towards developing a trans-spatial music theory, which proposes a common theoretical framework for different tunings and temperaments.

II. Literature review

The tuning of musical instruments has evolved since antiquity (Rasch, 2002) offering different solutions to the existence of commas — the difference resulting from tuning one note in two different ways (Grove, 1900). Tuning gradually evolved to temperament in the fifteenth century, and after exploring different kinds of unequal temperaments, the twelve-tone equal temperament became our present tuning system (Barbour, 1951).

Simpler just intervals are perceived as more consonant (Helmholtz, 1863) because of the interference of partials generating roughness (Plomp and Levelt, 1965). That's why many musicians develop their practice using exclusively just intonation. On the other hand, tones whose difference is smaller than a minimum detectable percentage depending on pitch and sensation level are perceived as the same tone (Shower and Biddulph, 1931), and "there is a very strong propensity for the ear to try to fit what it hears into one or a small number of harmonic series" (Erlich, 1997). So some of the properties of simple just intervals may be transferred to their tempered approximations.

Just intonation systems are usually defined by specifying a prime limit, indicating the greatest prime factor (Erlich, 2004). The standard temperament western music uses is an approximation of a 5-limit just tuning, so those systems with limits greater than 5 are known as extended just intonation (Fonville, 1991).

Some theoretical contributions are independent of the prime limit and helpful for the analysis and design of just intonation systems. Johnston defined n-dimensional lattices as a way to represent tones, intervals and scales in a topological space, and one-dimensional projections of those spaces as a way to represent these same structures in the span of a single octave (Johnston, 1977). Partch had previously divided these lattices defining otonality and utonality (over-number and under-number tonality) (Partch, 1947), establishing the main formalization of harmonic dualism in extended just intonation.

Harmonic dualism is “a school of theoretical thought which holds that the minor triad has a natural origin different from that of the major triad, but of equal validity” (Snyder, 1980). It pretends to explain the origin of “the minor triad in a reverse sense from the explanation of the major triad” (Jorgenson, 1963). While the major triad is an upward (positive direction of the lattice) construction, the minor triad would be symmetrically built in the downward direction. Zarlino and Riemann have been historically the main advocators of harmonic dualism (Zarlino, 1558; Riemann, 1896). Rameau initially proposed his monist[1] theory of the divided fifth as an alternative to Zarlino’s dualism (Rameau, 1722), he later moved to a dualist approach (Rameau, 1737) and finally concluded that the generating tone in minor triads is not the perceived root (Rameau, 1750). In Levy’s words: “generator and fundamental are divorced” (Levy, 1985).

Despite being a universally perceived phenomenon and even considered by some authors essential for the existence of music (Day, 1885), the perceptual origin of the harmonic root is still discussed. Dualists have tried to explain why the generating tone is not the one perceived as the root in many ways. From a psychoacoustical perspective, Tartini anticipated the theory of virtual pitch with his theory of the terzi suoni risultanti (Tartini, 1754), von Oettingen defended harmonic dualism based on the overtone series (von Oettingen, 1866), Indy — and similarly Godley (Godley, 1952) — demonstrated that the minor triad is cardinally inverse to the major (Indy, 1912), and Hauptmann even used

[1] Monism is the opposite of dualism. Levy would say “polarity theorist” instead of dualist and “turbidity theorist” instead of monist. [Levy(1985)]

Hegelian metaphysics to defend dualism (Hauptmann, 1853). More recently, Terhardt's virtual pitch (Terhardt, 1984) succeeded Hindemith's difference tone theory (Hindemith, 1937) and is the generally accepted one. This establishes a contradiction in music theory because while dualism is generally accepted, the most accepted theory for the fundamental root is virtual pitch, which is based on the overtone series, therefore it is a monist theory.

Aware of this contradiction, Ernst Levy proposed the concept of polarity to reconcile harmonic dualism and the fundamental root (Levy, 1985), and derived a correlation between tones and chords in the "minor side" and the "major side", describing it in terms of "telluric gravity". This is a dualist idea but with a critical variation: the frontier between the "major" and "minor" side is not the root tone, but a middle point between the root and the fifth. Levy's theory of harmony is especially relevant because it offers a solution to the dualist problem with the perceived root without resorting to the harmonic series. Steve Coleman developed Levy's ideas under the name of "symmetrical movement" considering "tonal centers in terms of spatial geometry" (Coleman, n.d.). In recent days, Jacob Collier has been the biggest proponent of this theory, using the term "negative harmony" and simplifying it as two different ways to approach the key center: "the fourths-side of the circle of fifths, and the fifths side of the circle of fifths".

Once new tunings and their respective theory are proposed, new instruments, interfaces and algorithms are needed to compose and perform. Kirck defined a two-dimensional pitch-class space (Kirck 1987) to map Ben Johnston's notation (Johnston 1978) to just intervals. Sabat proposed Micromælodeon as a microtuning algorithm using Tenney's Harmonic Distance and a lookup table with 3997 intervals (Sabat 2008).

Stange at al. did a great analysis on previous dynamically adaptive tuning systems which used logical operations and proposed instead a mathematical system of linear equations (Stange et al. 2017). Trueman et al. studied how a playful interface — bitKlavier — could be used to implement composed and adaptive tunings (Trueman et al. 2020).

III. Methods

A. Harmonic spaces

Johnston's n-dimensional lattices can be further developed to connect negative harmony with extended just intonation. I am proposing a tuning-agnostic theory of harmonic spaces based on Tenney's pitch-class spaces as a topological framework, Levy's and Coleman's negative harmony as an extention of harmonic dualism, and Barlow's number theory (Barlow, 1999; Barlow, 2012).

There are two kinds of harmonic spaces: pitch spaces, and pitch-class spaces.

1. Pitch spaces

The relationship between just tones in a pitch set can be represented by defining n-dimensional non-Euclidean (with taxicab distance[2]) discrete harmonic spaces assigning each lattice to a prime number (Tenney, 1983; Erlich, 2004). Pitch spaces are defined by the prime numbers attached to their dimensions in the form [p_1, p_2, ..., p_n]. The value of each discrete coordinate (x_1, x_2, ⋯, x_n) in these spaces is:

$$I = \prod_{i=1}^{n} p_i^{x_i} \tag{1}$$

For example, [2, 3] is a pitch space with two dimensions: one for the number 2, and another one for the number 3. Every discrete coordinate in this space corresponds to a tone:

[2] In taxicab geometry, the usual distance function of Euclidean geometry ($d = a^2 + b^2$) is replaced by the rectilinear distance: $d = a + b$.

$(-1,1) \in [2,3] = 2^{-1} \cdot 3^{1} = {}^{3}/_{2}$

$(2,-1) \in [2,3] = 2^{2} \cdot 3^{-1} = {}^{4}/_{3}$

The same model can be used for pitch spaces with more dimensions, or composed of different prime numbers:

$(2,1,-1) \in [2,3,5] = 2^{2} \cdot 3^{1} \cdot 5^{-1} = {}^{6}/_{5}$

$(-2,1) \in [7,11] = 7^{-2} \cdot 11^{1} = {}^{11}/_{49}$

2. Pitch-class spaces

Pitch-class spaces are lower-dimensional projections of pitch spaces, which only include tones contained in the interval of equivalence (equave). Pitch-class spaces are defined by the prime numbers attached to their dimensions and by the interval of equivalence in the form $[p_1, p_2, ..., p_n]$→equave. The value of each discrete coordinate $(x_1, x_2, \cdots, x_n)$ in these spaces is:

$$I = \frac{\prod_{i=1}^{n} p_i^{x_i}}{E^{\left\lfloor \log\left(\prod_{i=1}^{n} p_i^{x_i}\right)/\log E \right\rfloor}}$$

(2)

For example, [3,5]→2 is a two-dimensional pitch-class space, a projection of the [2, 3, 5] pitch space. Every discrete coordinate in this harmonic space corresponds to a tone or interval between the unison and the equave:

$(1,0) \in [3,5] \to 2 = 3^{1} \cdot 5^{0} / 2^{\lfloor \log(3^{1} \cdot 5^{0} \cdot)/\log(2) \rfloor} = {}^{3}/_{2}$

$(-1,-1) \in [3,5] \to 2 = 3^{-1} \cdot 5^{-1} / 2^{\lfloor \log(3^{-1} \cdot 5^{-1} \cdot)/\log(2) \rfloor} = {}^{16}/_{15}$

The same model can be used for pitch-class spaces with more dimensions, composed of different prime numbers, or using different equaves:

$$(1,1,-1) \in [3,5,7] \rightarrow 2 = 3^1 \cdot 5^1 \cdot 7^{-1} / 2^{\lfloor \log(3^1 \cdot 5^1 \cdot 7^{-1} \cdot)/\log(2) \rfloor} = {}^{15}/_{14}$$

$$(1,-1) \in [5,7] \rightarrow 3 = 5^1 \cdot 7^{-1} / 3^{\lfloor \log(5^1 \cdot 7^{-1} \cdot)/\log(3) \rfloor} = {}^{15}/_{7}$$

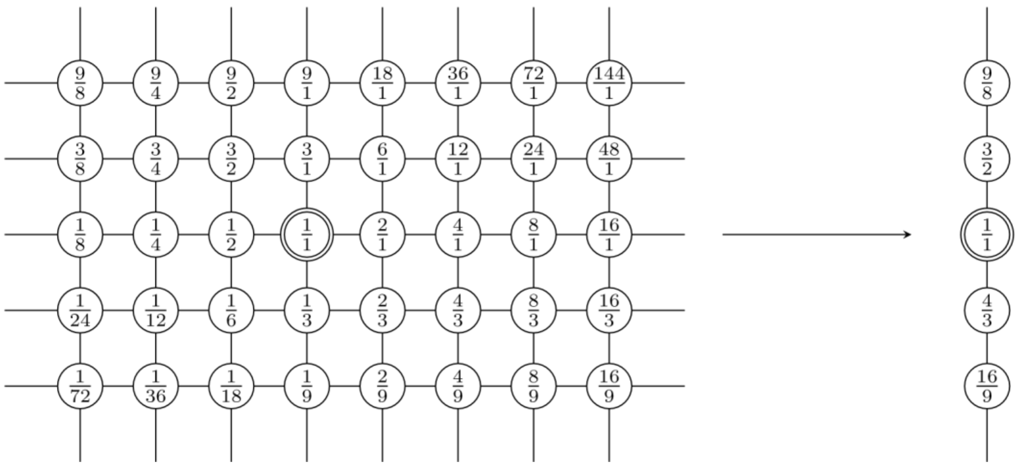

Figure 1. Section of the [2, 3] pitch space (left) and its projection in the [3]→2 pitch-class space. The double-circled tones represent the root.

Note that when harmonic spaces are defined, the prime numbers are always expressed from lowest to highest. Therefore, in a pitch space [p_1, p_2, ..., p_n] or a pitch-class space [p_1, p_2, ..., p_n]→equave:

$$p_i < p_{i+1} \tag{3}$$

B. Intervals and chords

Intervals can be otonal or utonal (Partch, 1947), depending on their direction on the lattice. Intervals in the positive direction are otonal, and the ones in the negative direction are utonal. For example, the space [3, 5]→2 has two fundamental intervals of each kind: ${}^{3}/_{2}$ and ${}^{5}/_{4}$ are otonal, and ${}^{4}/_{3}$ and ${}^{8}/_{5}$ are utonal.

The combination of fundamental intervals of the same kind (otonal or utonal) generates fundamental chords. For example, in [3, 5]→2, $3/2$ and $5/4$ generate the major triad (otonal chord), and $4/3$ and $8/5$ generate the minor triad (utonal chord). Here's where the divergence between the generating tone and perceived root comes into play. While in a C minor chord the perceived root is C, actually the generating tone is G, so the generating intervals are $4/3$ and $8/5$ instead of $6/5$ and $3/2$.

Chords can be notated in the form A:B:C where B/A and C/A are the intervals composing the chord. For example, the major triad in [3, 5]→2 is 4:5:6 and the minor 10:12:15. But in fact, in the case of minor chords, it would be more insightful and simple to express them in a downward fashion: C:B:A where C/A and C/B are the intervals composing the chord. This way, the minor triad would be 6:5:4. This notation can be extended to chords with more tones, for example: A:B:C:D being B/A, C/A and D/A the intervals.

In the early 70s, Heinz Bohlen used the twelfth (tritave) and a new triad (3:5:7) to design a 13-tone 7-limit tuning (Bohlen, 1972; Bohlen, 1978). This same tuning appears in its tempered version in van Prooijen's theory of equal-tempered scales (Prooijen, 1978) and was independently discovered by Pierce (Mathews and Roberts, 1984), who together with Mathews, proposed a major (3:5:7) and a minor (15:21:35 = 7:5:3) chord[3] (Mathews and Pierce, 1987). This tuning is known as the Bohlen-Pierce Scale.

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[3] Influenced by Elaine Walker [Walker(2001)], there's the wrong belief that the minor triad Mathews and Pierce proposed was 5:7:9 because they used this triad in combination with the major (3:5:7) to produce their scales: "The scales were described in 1978 (Bohlen) and rediscovered by Pierce after he hears the results of listening tests with two nontraditional triads in which the frequency ratios of the notes in the triads are 3:5:7 and

Mathews and Pierce – like Indy and Godley – realized that the minor triad is a downward version of the major triad and applied this same logic when proposing the basic chords for the Bohlen-Pierce Scale (Mathews and Pierce, 1987). It was clear to them that the equivalent to the major triad in [5,7]→3 should be 3:5:7 (they didn't justify this choice over the other inversions) and then proposed their inverse 7:5:3 (15:21:35) as the equivalent to the minor chord. Whether or not this was the right approach hasn't been proven yet.

C. Perception study

41 participants have been asked which chord in a set of chords played by a piano they consider "the most harmonic, consonant (less dissonant), stable, fundamental, basic" one. All the chords in the set are inversions of the same kind of chord (otonal or utonal), and they all share the same root. For example, when asked for the otonal chord in the [3,5]→2 pitch-class space, the options were:

- 3:4:5 - C1 F1 A1 C2
- 4:5:6 - C1 E1 G1 C2
- 5:6:8 - C1 D#1 G#1 C2

5:7:9". But actually the minor chord they propose is the negative complement of the major: "A given key has 'major' and 'minor' chords [..] a major chord is defined as a 6-step interval below a 4-step interval. It is the equal-tempered approximation to a 3:5:7 chord. A minor chord is a 4-step interval below a 6-step interval." [Mathews and Pierce(1987)] A 4-step interval in the Bohlen-Pierce Scale has a 7⁄5 ratio and a 10-step (4-step + 6-step) has a 7⁄3 ratio, so the resulting chord is 7:5:3 (or 15:21:35). In a later publication this distinction becomes more evident: "The BP scale has three special triads, the first being a 'major' chord [...]. This triad approximates the 3:5:7 chord. The 'minor' triad [...] has a lower interval of four steps and an upper interval of six steps. The third triad, which approximates the 5:7:9 chord..."[Mathews et al.(1988)].

The study asked about the otonal and utonal chords of 10 different pitch-class spaces, so had 20 questions in total. The chosen spaces are all the two-dimensional spaces up to 11-limit with an octave or a tritave: [3,5]→2, [3,7]→2, [3,11]→2, [5,7]→2, [5,11]→2, [5,7]→3, [5,11]→3 and [7,11]→3; and the higher-dimensional extensions of [3,5]→2: [3,5,7]→2 and [3,5,7,11]→2.

The participants interacted with an application and could play each chord as many times as they wished and in the order they wanted. The buttons were randomly sorted for each participant, so the chords weren't initially in any specific order. In addition, they had the option not to select any in case they couldn't choose one.

The musical ability of the participants has been measured using the Ollen Musical Sophistication Index (OMSI) (Ollen, 2006). Results don't show any significant correlation between the musical ability and the randomness/predictability of the participants' responses.

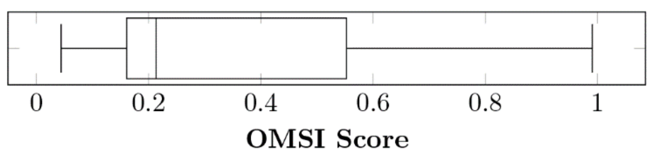


Figure 2. OMSI Score distribution shows most of the participants have a low musical ability.

IV. Results

Results show that in the 83.33% of the cases, the chords chosen as the most fundamental inversions of the otonal or utonal chords have the same structure independently of the pitch-class space they exist in. Being $(0_1, ..., 0_n)$ the harmonic root in an n-dimensional pitch-class space, the structure of the otonal chord is:

$$(0, 0, 0, \ldots, 0_n)_0, (1, 0, 0, \ldots, 0_n)_1, (0, 1, 0, \ldots, 0_n)_2, (0, 0, 1, \ldots, 0_n)_3, \ldots, (0, 0, 0, \ldots, 1_n)_n$$

And the structure of the utonal chord is:

$$(0, 0, 0, \ldots, 0_n)_0, (1, 0, 0, \ldots, 0_n)_1, (1, -1, 0, \ldots, 0_n)_2, (1, 0, -1, \ldots, 0_n)_3, \ldots, (1, 0, 0, \ldots, -1_n)_n$$

For example, in two-dimensional pitch-class spaces, the otonal chord is ((0,0), (1,0), (0,1)) while the utonal one is ((0,0), (1,0), (1,-1)).

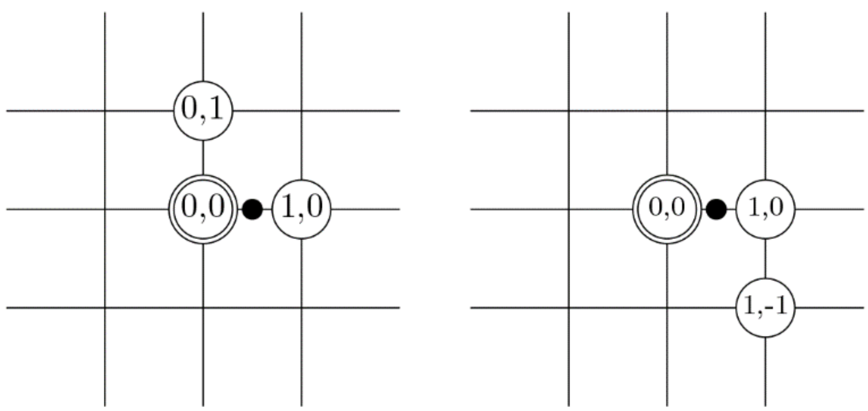


Figure 3. Otonal chord (left) and utonal chord (right) in a two-dimensional pitch-class space. The black dot represents the center of symmetry. Since $p_i < p_{i+1}$, the x axis corresponds to the lowest prime number.

The structure of the perceived fundamental inversions of otonal and utonal chords in a pitch-class space is symmetric, being (0.5, 0, ..., 0) the center of symmetry. In other words, perceptually, pitch-class spaces have a center of symmetry between the tonic and the closest tone considering Tenney's harmonic distance (Tenney, 1983).

This model extends the theory of negative harmony to new spaces, formalizing Coleman's idea of "tonal centers in terms of spatial geometry".

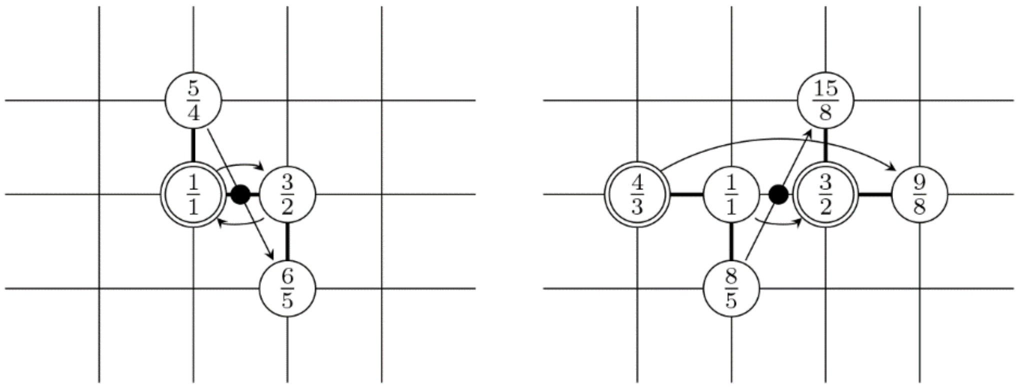


Figure 4. At the left, the symmetric relation between a C major (1⁄1,5⁄4,3⁄2) and a C minor (1⁄1,6⁄5,3⁄2) and at the right, the symmetric relation between an F minor (4⁄3,1⁄1,8⁄5) and a G major (3⁄2,15⁄8,9⁄8) in the [3,5]→2 pitch-class space. Thicker lines represent the intervals within the chord. Arrows represent how tones are reflected over the point of symmetry.

V. Evaluation and assessment

A. Discussion

To evaluate to what extent the center of symmetry is relevant, and the proposed model is accurate, we can compare it with other theories, metrics or models (Appendix A contains the value for each chord in each model):

- **Tenney's harmonic distance:** Tenney studied the history of consonance and dissonance (Tenney, 1988) and proposed harmonic distance (Tenney, 1983) to measure the distance between two points in the harmonic lattice considering that they were non-Euclidean spaces.

  Tenney's harmonic distance formula is normalized so the octave's distance is 1. To extend this property to spaces with other equaves (E), the formula can be adapted:

  $$\delta\left(\frac{a}{b}\right) = log(a \cdot b)/\log 2 \ \rightarrow \ \delta\left(\frac{a}{b}\right) = log(a \cdot b)/\log E \ \mid \ E > 1 \tag{4}$$

  The total harmonic distance of a chord is the summation of all the distances of its intervals. The fundamental chord would be the one with the smallest harmonic distance.

- **Barlow's harmonicity** (Barlow 1987, Barlow 1999, Barlow 2012): Barlow defined indigestibility (ξ) as a way to measure the simplicity of numbers, and with that, proposed a harmonicity (H) formula for just intervals.

$$\xi(N) = 2\sum_{i=1}^{\infty}\frac{n_i(p_i - 1)^2}{p_i} \qquad H(\frac{a}{b}) = \frac{sgn(\xi(a) - \xi(b))}{\xi(a) + \xi(b)}$$

(4)

The harmonicity of chords is the summation of the module of the mutual inharmonicities between every tone.

- **Sethares' dissonance** (Sethares, 1998)**:** Sethares proposed a parametrization of the Plomp and Levelt dissonance curves:

$$d(f_1, f_2, l_1, l_2) \; = \; l_{12} \cdot (e^{-3.51p} - e^{-5.75p}) \qquad l_{12} = max(l_1, l_2)$$

(5)

$$p = \frac{0.24}{0.0207 \cdot f_{min} + 18.96}(f_{max} - f_{min})$$

(6)

The dissonance (*d*) of a chord is the summation of the dissonance between each of the partials, including the root tone. In this study, a spectrum with 7 harmonic partials over the fundamental tone has been considered.

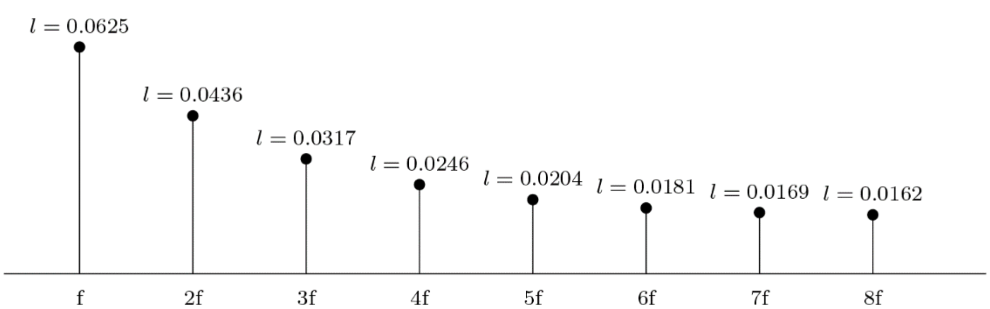


Figure 5. Considered partials for the calculation of dissonance, being l the loudness of each partial.

- **Erlich's harmonic entropy** (Erlich, 1997): This measure is based on the probability that a certain interval will be perceived as a specific rational ratio. If one interval can be perceived as many different rational ratios, the entropy will be higher, while if the probability it is perceived as a specific ratio is very high, then the entropy will be low. There are different ways to measure harmonic entropy. In this case, the evaluated interval is compared to all the intervals in J:

$$J = \left\{ \frac{a}{b} \middle| 1 \leq \frac{a}{b} \leq 3 \middle| a < 28 \middle| b < 28 \middle| gdc(a,b) = 1 \right\}$$

(7)

Depending on the distance to the evaluated interval, each of the rational intervals has a probability to be perceived. A normal distribution is assumed:

With the distribution, the unnormalized probability can be calculated:

$$Q_j(i) = \frac{S\big(cents(j) - cents(i)\big)}{\sqrt{j_n \cdot j_d}} \qquad j = \frac{j_n}{j_d}$$

(8)

And then normalized:

$$P_j(i) = Q_j(i) \Big/ \sum_{j \in J} Q_j(i)$$

(9)

Finally, the harmonic entropy is:

$$HE(i) = -\sum_{j \in J} P_j(i) \cdot \log(P_j(i))$$

(10)

- **Terhardt's virtual pitch** (Terhardt, 1984): Considering the fundamental inversion the one that first appears in the harmonic series. For example, 3:4:5 appears before 4:5:6, and 10:12:15 appears before 12:15:20.
- **Dual virtual pitch:** In the case of otonal chords, considering the fundamental inversion the one that first appears in the harmonic series, and for utonal chords, the one that first appears in the subharmonic series. For example, 3:4:5 appears before 4:5:6 (like in the virtual pitch theory), but 5:4:3 (12:15:20) appears before 6:5:4 (10:12:15).
- **Carmen Parker's drone:** Considering the basic inversion A:B:C or C:B:A (otonal or utonal) when $A = equave^n$ ($n \in \mathbb{N}$). For example, in [5,11]→3, the fundamental otonal inversions would be 9:11:15 and the utonal inversion would be 15:11:9 (33:45:55) because $9 = 3^2$.
- **Mathew's et al. dissonance** (Tenney, 1983): In their study in chords for the Bohlen-Pierce Scale ([5,7]→3), Mathews et al. proposed a dissonance (D) formula for triads:

$$D(i) = F(|f_1 - f_2|) + F(|f_2 - f_3|) + F(|f_3 - f_1|)$$

(11)

$$F(x) = \begin{cases} x, & x \leq q \\ 0, & x \geq s \\ q\left(1 - \dfrac{x-q}{s-q}\right), & x \geq q \end{cases} \qquad s \geq q$$

(12)

They proposed different values for *s* and *q* depending if the participants were trained or untrained, but in any case, as they noted: "only a few 1-step intervals make a nonzero contribution to the sum". So, for most of the chords, the total dissonance is 0. For this reason, this model hasn't been used in the comparison.

For a preliminary data analysis, the questions in which most participants responded they could not identify a fundamental chord have not been considered. It's the case of the utonal chords in [3,5,7]→2 and [3,5,7,11]→2, for which 31% and 50% of participants could not identify a fundamental inversion (See Appendix B).

The remaining questions have been divided by limit (5-limit, 7-limit or 11-limit), tonality (otonal or utonal) and equave (octave or tritave). None of the considered models performs better than the proposed model in any of these categories. All the right predictions that the next most accurate models (harmonic distance, harmonicity, entropy and virtual pitch) do, are also predicted by the proposed model.

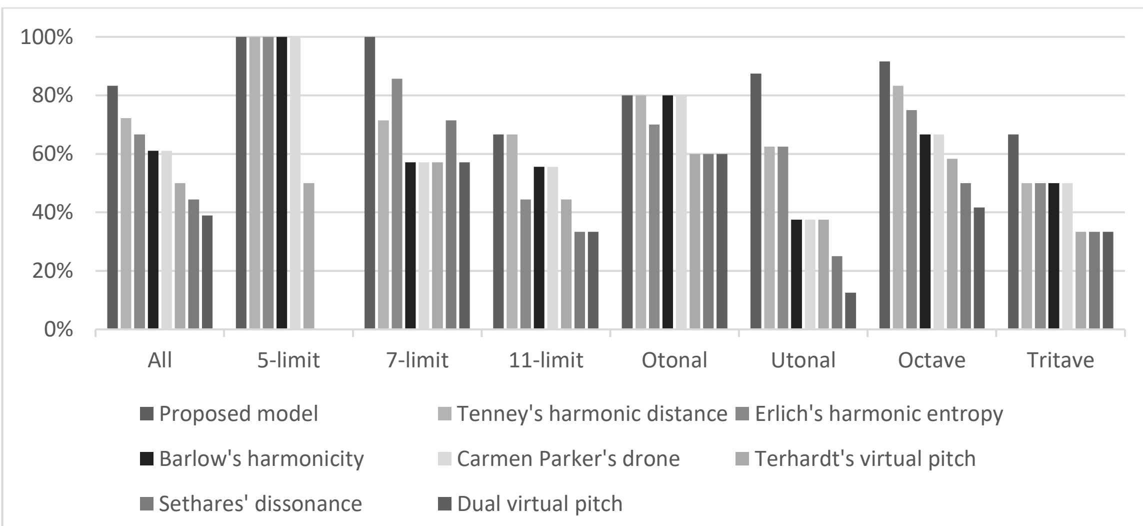


Figure 6. Accuracy of the models in different categories.

The difference becomes especially evident for utonal chords, which constitute the 50% of all chords. While most of the models' accuracy falls dramatically, in some cases to levels below the randomness threshold, the proposed model maintains the same high accuracy.

B. Modelization

Tenney's harmonic distance and Barlow's harmonicity can be adapted to consider the center of symmetry and become useful models to predict the perceived fundamental chord in a pitch-class space (See Appendix C for the derivations of these formulas). Being X a coordinate in a pitch-class space,

$$X = \left(x_{1,} \ldots, x_i\right) \in [p_{1,} \ldots, p_i] \rightarrow E$$

(13)

the fundamental chords would be those whose overall harmonic distance to the center of symmetry is lower:

$$\delta_{cs}(X) = \log\left(p_1^{|2x_1-1|} \cdot \prod_{i=2}^{n} p_i^{|2x_i|} \cdot E^{\left|\left\lfloor \log\left(p_1^{2x_1-1} \cdot \prod_{i=2}^{n} p_i^{2x_i}\right)/\log(E)\right\rceil\right|}\right) \Big/ \log(E)$$

(14)

Or which harmonicity (considering the center of symmetry the origin) is higher:

$$|H_{cs}(X)| = \xi\left(p_1^{|2x_1-1|} \cdot \prod_{i=2}^{n} \left\{p_i^{|2x_i|}\right\} \cdot E^{\left|\left\lfloor \log\left(p_1^{2x_1-1} \cdot \prod_{i=2}^{n} p_i^{2x_i}\right)/\log E\right\rceil\right|}\right)^{-1}$$

(15)

$$\xi(X) = 2\sum_{i=1}^{n} \frac{x_i(p_i - 1)^2}{p_i}$$

(16)

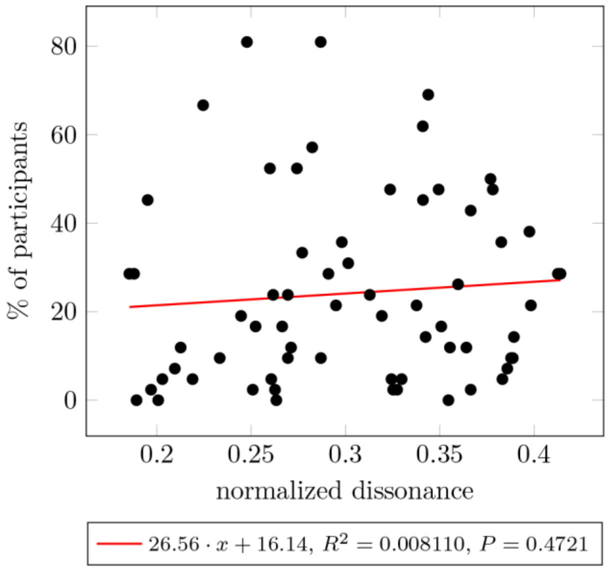
Applying these models to the 10 spaces included in the study, their prediction is always coincident, having a success rate of 83.33%, while Tenney's harmonic distance could only predict 72.22% of the chosen fundamental chords (+11.11%), and Barlow's harmonicity just 61.11% (+22.22%)[4].

With these new formulas, a secondary data analysis can be done by quantitatively comparing the correlation between the metrics and participants' choices. Only models offering a quantitative evaluation of each chord can be used, so just Sethares' dissonance, Erlich's harmonic entropy, Tenney's harmonic distance, Barlow's harmonicity and the new two formulas have been studied. The exclusion of the other theories and models can be justified by their bad results in the preliminary study. The value assigned to each of the chords by these formulas has been normalized with the other chords of the same tonality in each space, so for example, the summation of the normalized dissonance of the utonal or otonal chords is 1 in each space.

Using linear regression, the correlation between the metrics and participants' choices has been measured, being the absolute value of the line's slope indicative of the level of correlation. R^2 indicates how good the line regression approximates the data; the higher, the better. P value indicates how significant the slope is, the lower P, the better, being $P < 0.0001$ the optimal value.

While dissonance shows no significant correlation with the data (Figure 7), harmonic entropy does (Figure 8), but it is small compared to other metrics.

[4] The fact that the proposed adaptations are useful to predict the perceived fundamental inversion does not mean that they are better than the original formulas. The original formulas are valuable for the purpose they were originally designed for, and here, they're just being manipulated to show the existence of a perceptual center of symmetry.



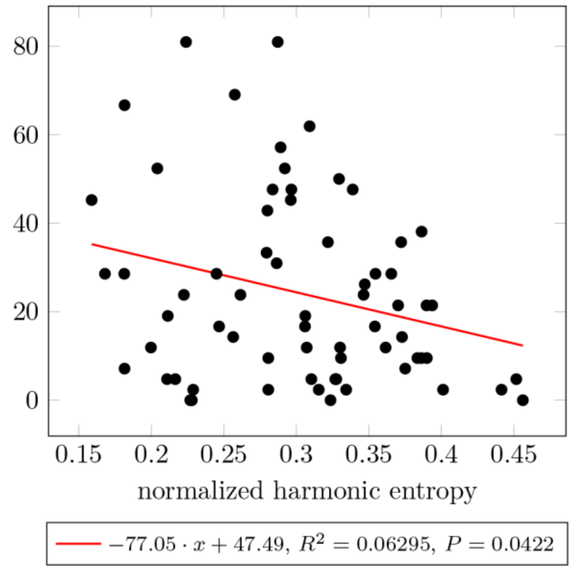


Figure 7. Correlation between the dissonance of a chord and the percentage of participants choosing it. The low slope, low R^2 and high P value mean dissonance is not significant.

Figure 8. Correlation between the harmonic entropy of a chord and the percentage of participants choosing it. Despite P value is high, a small significant correlation can be considered.

It's also the case for Tenney's harmonic distance (Figure 9). Linear regression shows a small correlation that's actually not significantly deviated from horizontal. However, with the modified version of the harmonic distance formula considering the center of symmetry, the correlation is almost doubled and becomes significant (Figure 10).

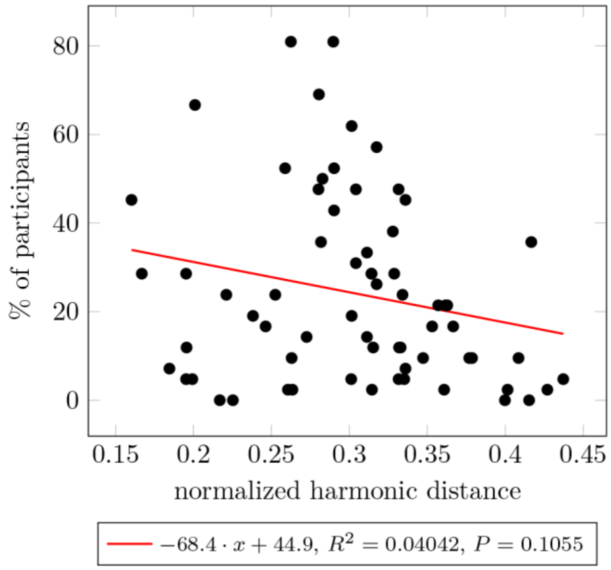


Figure 9. Correlation between the overall harmonic distance of a chord and the percentage of participants choosing it. The low slope, low R^2 and high P value indicate harmonic distance is not significant.

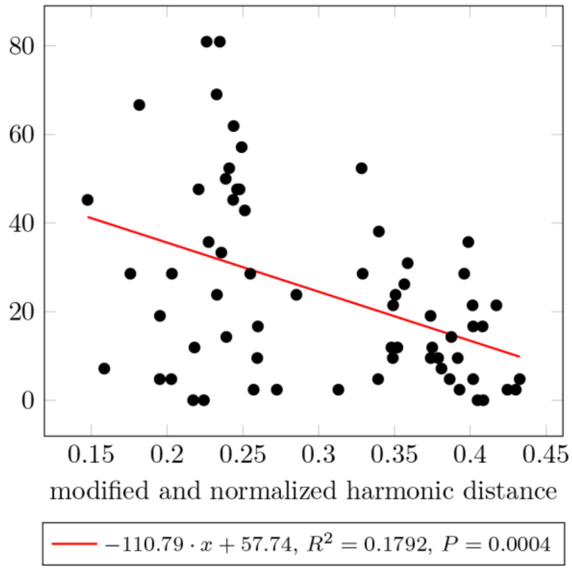
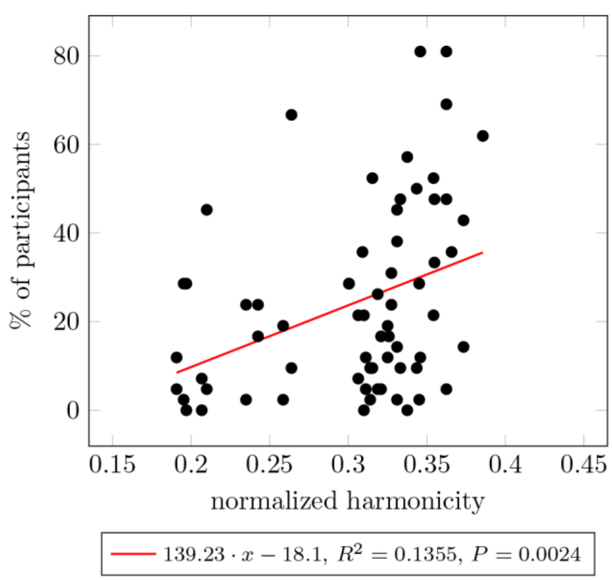


Figure 10. Correlation with the overall modified harmonic distance of a chord. Low P value and notable slope indicate a significant correlation with the harmonic distance to the center of symmetry.

With Barlow's harmonicity (Figure 11), the correlation is bigger than with any of the previous models, and it is even greater with the modified harmonicity formula relative to the perceptual center of symmetry (Figure 12).



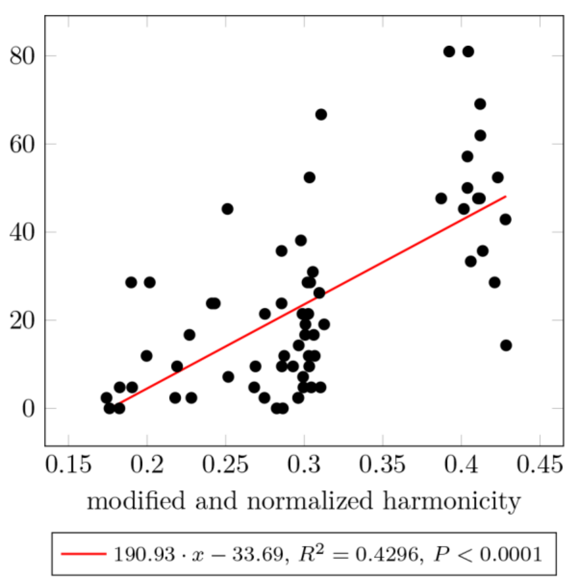


Figure 11. Correlation between the harmonicity of a chord and the percentage of participants choosing it. Adequate R^2 and P value allow to stablish a strong correlation with Barlow's indigestibility and harmonicity.

Figure 12. Correlation with the modified harmonicity of a chord. Optimal P value, high R^2 and great slope means there's a very strong correlation with the harmonicity relative to the center of symmetry.

VI. Future work

A. *Temperament, adaptive tunings, scales and interfaces*

Pitch-class spaces define a topological space to work with harmonic structures and a set of rules that can be used for algorithmic composition, but not all musicians do or will have access to the technical tools and knowledge to design and implement computer programs for music composition. For this new theoretical framework to permeate into contemporary music composition, more technical and conceptual instruments must be developed.

There are four key elements that need to be developed also in a tuning-agnostic (trans-spatial) way to transform contemporary music practice: temperament, adaptive tunings, scales and interfaces.

Temperament bridges the infinity of pitch-class spaces with a finite set, adaptive tunings allow us to keep the purity of perfect intervals and the practicality of temperament, and a theory of scales will allow for the development of intuitive and expressive trans-spatial music interfaces.

1. Temperament

Tones in a pitch-class space tend to have an exponential distribution between the unison and the equave. That's why equal temperaments are useful to approximate just intonation pitch-class spaces. Every pitch-class space can be approximated by an equal-temperament (Balzano 1980).

Equal temperaments are expressed defining an equave and the number of tones in the form E_t^e, where e is the equave and t the number of tones (van Prooijen 1978). For example, the temperament used in western music with 12 equal divisions of the octave would be expressed as E_2^{12}, and the Bohlen-Pierce Scale, which divides the tritave in 13 equal steps would be expressed as E_3^{13}.

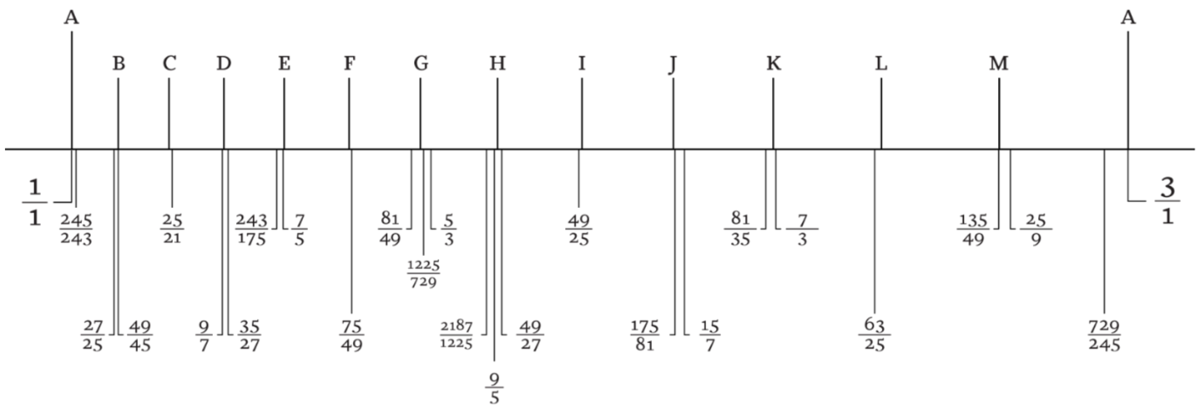


Figure 13. Pitch continuum between $^1/_1$ and $^3/_1$ with tones from the [5, 7]→3 pitch class space and the notes of the best approximation to this space: E_3^{13}, the Bohlen-Pierce Scale.

Cents are irrational intervals that are equivalent to exactly the 1/1.200th part of an octave. They’re used to measure the size of an interval. The notion of cents is based on the twelve-tone equal temperament and the octave but can be adapted to be relative to any equave and temperament. Being $^a/_b$ the interval, *e* the equave and *t* the number of tones in the temperament:

$$Relative\ cents = 100 \cdot t \frac{\log(a/b)}{\log e} \tag{17}$$

The distance in relative cents between a pitch-class space and an equal temperament can be calculated algorithmically. This way, the best temperaments for a just intonation pitch-class space can be found. Figure 3 shows how different temperaments approximate different pitch-class spaces. Darker cells correspond to approximations with a lower distance, so better approximations. Different temperaments can be good to approximate the same pitch-class space, for example, E_2^{12}, E_2^{31} and E_2^{34} are all good for [3, 5]→2.

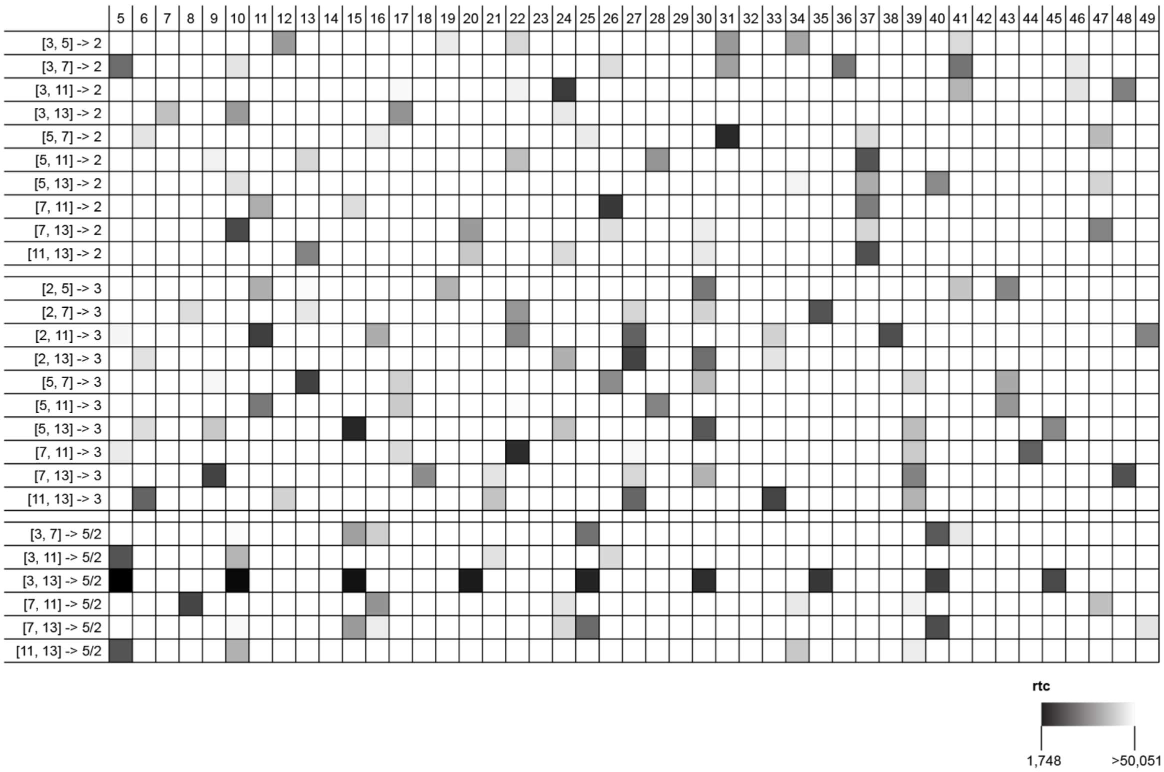


Figure 14. Table with the distance in relative cents between pitch-class spaces (rows) and equal temperaments (columns).

Once the best approximation for a pitch-class space has been found, each just tone can be assigned to a note in the temperament algorithmically. In Figure 15, tones in [5, 7]→3 (left) are mapped to notes in E_3^{13} (right).

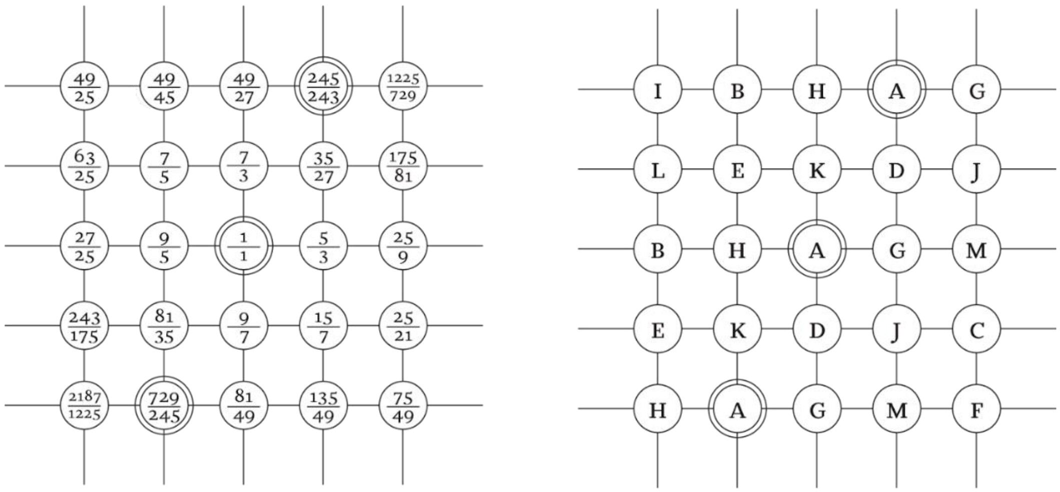


Figure 15. Mapping between the tones in the [5, 7]→3 pitch-class space and the notes in E_3^{13}.

When working parallelly with just and tempered intervals, it can be useful to define a difference between tones and notes. Each of the coordinates in a harmonic space is a tone. Their values can be for example $^1/_1$, $^3/_2$, or $^7/_5$. Temperaments are not composed by tones but by notes. Their names can be numbers or characters expressing their position in the temperament.

2. Adaptive tuning

Since pitch-class spaces are infinite, there are infinite just tones equivalent to the same note in a temperament. Depending on the harmonic context, the same note in a temperament may be performed using one or another tone. For example, in the space [5, 7]→3 approximated with E_3^{13}, two melodies composed by the same notes are using actually different tones (therefore frequencies) depending on the order of the notes:

$$[A, K, D, A] = [(0, 0), (0, 1), (1, 1), (1, 2)] = [\frac{1}{1}, \frac{7}{3}, \frac{35}{27}, \frac{245}{243}]$$

$$[A, D, K, A] = [(0, 0), (0, -1), (-1, -1), (-1, -2)] = [\frac{1}{1}, \frac{9}{7}, \frac{81}{35}, \frac{729}{735}]$$

In this adaptive tuning system, the chosen tone is the one closer to the previous one in the pitch-class space. In the first melody, K is (0, 1) because the previous note A is (0, 0). But in the second melody, K is (-1, -1) because the previous note D is (0, -1) (see Figure 15 as a reference).

There are different possible algorithms to create adaptive tunings using pitch-class spaces, but all of them are based on the distance from a tonal center. The previous example is using the simplest method possible: setting the position of the last tone played as the tonal center, but more accurate and complex algorithms may be more convenient when playing chords or multiple melodies at the same time.

The compositions resulting from this method exhibit a high level of consonance. Figure 16 displays two spectral visualizations of the same just intonation chord progression. The first image shows the progression with a static just intonation tuning, while the second image displays the same chord progression with an adaptive tuning. The circles indicate the points where the most significant differences can be observed. The adaptive version is evidently more consonant.

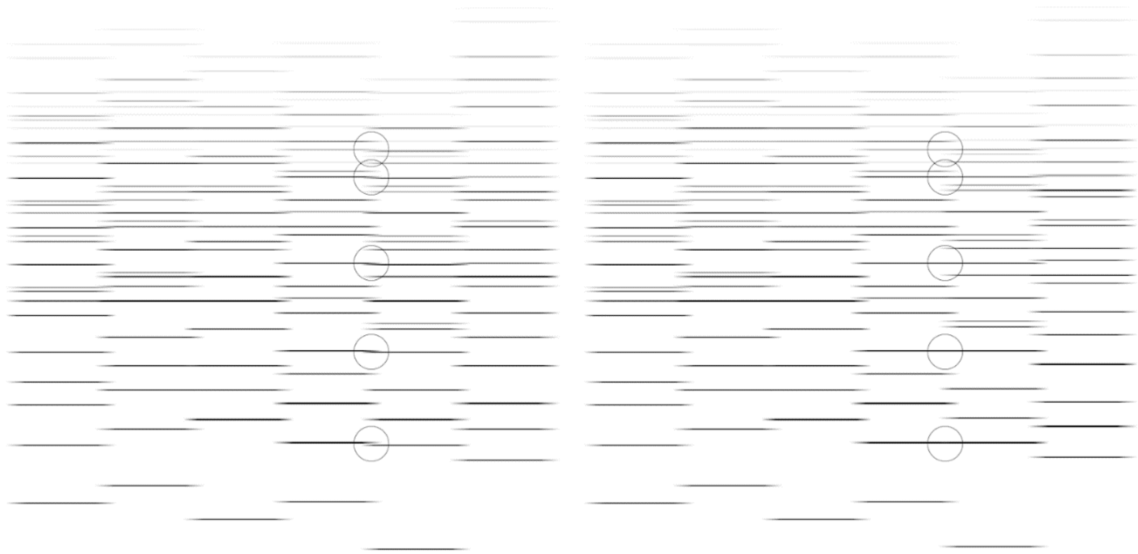


Figure 16. Spectrogram of a chord progression with static (left) and adaptive (right) tuning.

3. Scales and interfaces

Temperaments and adaptive tunings make the development of interfaces or instruments much simpler. For example, to compose or perform using a pitch-class space with infinite tones per equave like [3, 5, 7]→2, we just would need an interface with as many keys per equave as 12, 31 or 53. But we know from our experience in Western music that this is still a high number of notes, and that it's desirable to find a subset which despite stablishing a constraint it's creatively empowering. In the case of the twelve-tone system, the seven notes of the major scale (C, D, E, F, G, A, B) have been used to develop more expressive and intuitive interfaces, like the piano.

There are many properties that the major scale has that could explain why it was considered the most important subset of E_2^{12}, and why it has become one of the most important building blocks of Western music. But when we try to find these properties in structures in other tunings, it will happen what already happened when we tried to find the equivalent to the major and minor triads: each property points to a different structure. So the different theories that try to explain the origin of the major scale, diverge when applied to new tunings.

It could seem logic to apply the same methodology applied in this thesis to find the fundamental chords in any harmonic space, but scales introduce new challenges that require to find a new approach.

A tuning-agnostic study of scales for extended just intonation could be enough to write one and more thesis, so I won't develop it in depth here, but I'll point out to the main ideas that should be considered in this research:

- **Diatonicity and harmonicity:** One of the main characteristics of the major scale, is that it is diatonic, which means that it only contains two kinds of steps: whole-steps and half-steps. This property has a strong impact in harmony and melody. Another important but less noticed property is that its structure in a pitch-class space is very compact, which increases its harmonicity. When studying scales in new harmonic spaces, an emerging problem is that the diatonic scales are not always the most compact, and most of the times they're not compact at all. This raises a friction between the importance of diatonicity and the importance of harmonicity that should be addressed.
- **Translation, symmetry, and modes:** The major scale is especially powerful because of its modes. In a pitch-class space, modes are translations of the structure. That means that by translating the major scale in its pitch-class space, we can obtain all the other modes: Lydian, Phrygian… Coincidentally or not, the

minor scale (the Aeolian mode) is also the symmetrical structure of the major scale, considering the perceptual center of symmetry described in this thesis. But not all structures can obtain its symmetrical (negative) version by translation, and in some spaces the only structures with this property don't have any of the properties mentioned in the previous point.

The proposed approach for this challenge is using an artificial selection process to replicate the cumulative selection process that curated the major scale through history. The hope is that new powerful scales will emerge in the span of a study, instead of the span of centuries.

Scales in temperaments with n tones, can be understood as points in n-dimensional spaces (not harmonic spaces, a different kind of space). For example, in a temperament with 4 tones, scales would exist in a 4-dimensional space, and could be expressed by their coordinates in this space.

[1, 0, 1, 0] would be a scale containing the first and the third notes of the temperament. With this scale, an interface in the style of a piano could be designed, where the first and the third notes of the equave are white, while the second and the fourth ones are black. In a user study, participants could be asked to play with this unknown interface using both the white and black keys, and their performance could be computationally studied. The result would be a distribution showing which notes have been used the most, for example: [34, 1, 33, 32]. This result would show that the participant hasn't been using the scale that generated the interface, but another one: [1, 0, 1, 1]. Therefore, their performance could be understood as a vector in this space: [0, 0, 0, 1].

The interfaces presented to the participants would be stochastically defined by the previous participants response. The result of accumulating data would be a field showing the direction where the most expressive scales and interfaces are in these spaces.

This strategy can be useful to find a correlation between the previously mentioned properties, and dimensions or sections in these spaces, in a way a tuning-agnostic (trans-spatial) theory for scales can be proposed.

B. Higher-dimensional prime-spaces

The second direction proposed for future research is more general and pertains to the potential relevance of higher-dimensional prime spaces beyond the domain of music. This thesis thoroughly investigates pitch-class spaces, which include all rational numbers within an interval of equivalence, and briefly introduces pitch spaces, which comprise all rational numbers.

While these spaces only contain rational numbers because they are discrete, if we allow them to be continuous, they will also contain irrational numbers, i.e., all real numbers. Such a space could be referred to as a prime space, which would organize all real numbers in a manner that could be more meaningful depending on the context.

In prime spaces, all prime numbers create a hyper-sphere of radius 1, and the "primeness" of any real number can be calculated based on its distance from the center. This "primeness" is similar to the concept of Indigestibility proposed by Clarence Barlow. Another interesting aspect of prime spaces is that, due to their higher dimensions, additional operations and metrics can be applied to numbers. For instance, it is possible to calculate the angle between numbers in prime spaces.

These operations and metrics may be potentially useful in fields where simplicity is essential, like physics, biology, mathematics, cryptography, or computer science.

VII. Conclusion

The simplicity of numbers (digestibility) and the center of symmetry are the main factors in our perception of fundamental chords. The correlation with Sethares' dissonance, therefore Plomp and Levelt's roughness, has been proven to be null.

For centuries, theorists have attempted to explain most harmonic phenomena by basing their theories on the overtone series, but they have consciously and recklessly ignored the fact that the harmonics 7, 11, and 13 are not typically used in the creation of chords or scales. Meanwhile, larger numbers like 32, 45, and even 64 are used despite not being audible partials. Had they not ignored this fact, they would have realized that prime numbers play a key role in our perception of harmony. The successive progression of integers in the harmonic series is just a lower dimensional projection, a shadow like those in the cave, of n-dimensional spaces formed by primes.

Therefore, the common assertion that "small intervals are perceived as harmonic" should be substituted by "simple intervals are perceived as harmonic", understanding simplicity in the terms Barlow defined (in)digestibility.

Also the pitch-continuum is a lower-dimensional projection of the spaces where the harmonic structures are generated. Since some properties – like symmetry – are preserved in the projection, cardinality has been considered relevant in harmony, but actually, the cardinal analysis of tones in music theory should be avoided and substituted by the analysis in pitch-class spaces.

The current study supports negative harmony as an extension of harmonic dualism, and proves it is a tuning-agnostic (trans-spatial) perceptual phenomenon. It empirically demonstrates that there's a perceptual pattern in the different pitch-class spaces, even in those like [3,5]→2 and [7,11]→3 that don't have any interval in common; and proposes a model for this pattern. This proves that a trans-spatial music theory and practice is possible.

VIII. References

1. G. J. Balzano, *The Group-Theoretic Description of 12-Fold and Microtonal Pitch Systems*, Computer Music Journal , Winter, 4(4), 66–84 (1980).
2. M. Barbour, *Tuning and Temperament: A Historical Survey*, Michigan State College Press (1951).
3. C. Barlow and H. Lohner, *Two Essays on Theory,* Computer Music Journal, 11(1) (1987).
4. C. Barlow, *On the Quantification of Harmony and Metre* from *The Ratio Book* (C. Barlow), pp. 2-8 (1999).
5. C. Barlow, *On Musiquantics*, pp. 38-39 (2012).
6. H. Bohlen, *Versuch über den Aufbau eines tonalen Systems auf der Basis einer 13-stufigen Skala* (1972).
7. H. Bohlen, *13 Tonstufen in der Duodezime*, Acustica, 39(2), 75–86 (1978).
8. S. Coleman, *Symmetrical Movement Concept* (n.d.).
9. A. Day, *Treatise on Harmony* (1885).
10. P. Erlich, *On Harmonic Entropy*, (1997).
11. P. Erlich, *A Middle Path Between Just Intonation and the Equal Temperaments Part 1* (2004).
12. J. Fonville, *Ben Johnston's Extended Just Intonation: A Guide for Interpreters*, Perspectives of New Music 29(2), 106–137 (1991).
13. E. Godley, *The Minor Triad,* Music & Letters, 33(4), 285–295 (1952).
14. G. Grove, *A Dictionary of Music and Musicians,* Oxford University Press (1900).

15. M. Hauptmann, *Die Natur der Harmonik und der Metrik*, Zur Theorie der Musik (1853).

16. H. L. F. Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music* (1863).

17. P. Hindemith, *The Craft of Musical Composition*, Associated Music Publishers, Inc. (1937).

18. V. Indy, *Cours de Composition Musicale* (1912).

19. B. Johnston, *Rational Structure in Music*, Proceedings of the American Society of University Composers 11/12(10), 102–118 (1977).

20. B. Johnston, *Two Sonnets of Shakespeare*, Baltimore: Smith Publications (1978).

21. D. Jorgenson *A Résumé of Harmonic Dualism*, Music & Letters, 44(1), 31-42 (1963).

22. G. T. Kirck, *Computer Realization of Extended Just Intonation Compositions*, Computer Music Journal, 11(1), 69–75 (1987).

23. E. Levy, *A Theory of Harmony* (1985).

24. M. V. Mathews and J. R. Pierce, *The Acquisition of Musical Percepts with a New Scale*, Report No. STAN-M-40, Center for Computer Research in Music and Acoustics (1987).

25. M. V. Mathews, J. R. Pierce, A. Reeves and L. A. Roberts, *Theoretical and experimental explorations of the Bohlen-Pierce scale*, Journal of the Acoustical Society of America, 84(4), 1214–1222 (1988).

26. M. V. Mathews and L. A. Roberts, *Intonation sensitivity for traditional and nontraditional chords*, Journal of the Acoustical Society of America 75(3), 952–959 (1984).

27. J. E. Ollen, *A Criterion-Related Validity Test of Selected Indicators of Musical Sophistication Using Expert Ratings*, Doctoral dissertation, Ohio State University (2006).

28. H. Partch, *Genesis of a Music*, Da Capo Press (1947).

29. R. Plomp and W. J. M. Levelt, *Tonal Consonance and Critical Bandwidth*. The Journal of the Acoustical Society of America 38(4), 548-560 (1965).

30. K. van Prooijen, *A Theory of Equal-Tempered Scales*. Interface 7, 45–56 (1978).

31. J. Rameau, *Traité de l'Harmonie Réduite à ses Principes Naturelles* (1722).

32. J. Rameau, *Generation Harmonique ou Traité de Musique Theorique et Pratique* (1737).

33. J. Rameau, *Démonstration du Principe de l'Harmonie* (1750).

34. R. Rasch, *Tuning and temperament*, from Western Music Theory (T. Christensen), pp. 193–222, Cambridge University Press (2002).

35. H. Riemann, *Harmony Simplified* (1896).

36. M. Sabat, *An algorithm for real-time harmonic microtuning* (2008).

37. W. A. Sethares, Tuning, Timbre, Spectrum, Scale. Springer (1998).

38. E. Shower and R. Biddulph, *Differential Pitch Sensitivity of the Ear,* The Journal of the Acoustical Society of America, 3(2), 275–287 (1931).

39. J. L. Snyder, *Harmonic Dualism and the Origin of the Minor Triad*, Indiana theory review 4(1), 45-78 (1980).

40. K. Stange, C. Wick and H. Hinrichsen, *Playing Music in Just Intonation - A Dynamically Adapting Tuning Scheme*. Computer Music Journal, 42(3) (2017).

41. G. Tartini, *Trattato di Musica* (1754).

42. J. Tenney, *John Cage and the Theory of Harmony* (1983).

43. J. Tenney, *A history of consonance and dissonance*. Excelsior (1988).

44. D. Trueman, A. Bhatia, M. Mulshine and T. Trevisan, *Tuning Playfully: Composed and Adaptive Tunings in bitKlavier*, Computer Music Journal, 43(2–3) (2020).

45. E. Terhardt, *The Concept of Musical Consonance: A Link between Music and Psychoacoustics*, An Interdisciplinary Journal 1(3), 276–295 (1984).

46. W. Thomson, *The Harmonic Root: A Fragile Marriage of Concept and Percept*. Music Perception: An Interdisciplinary Journal 10(4), 385–415 (1993).

47. A. von Oettingen, *Harmoniesystem in dualer Entwickelung* (1866).

48. E. Walker, *The Bohlen-Pierce Scale: Continuing Research* (2001).

49. G. Zarlino, *Le Istituzioni Harmoniche* (1558).

Appendix A: Quantitative chord evaluation

Harm. = Barlow's harmonicity

Disson. = Sethares' dissonance

Distan. = Tenney's harmonic distance

Entro. = Erlich's harmonic entropy

Dista.* = Modified harmonic distance considering the center of symmetry

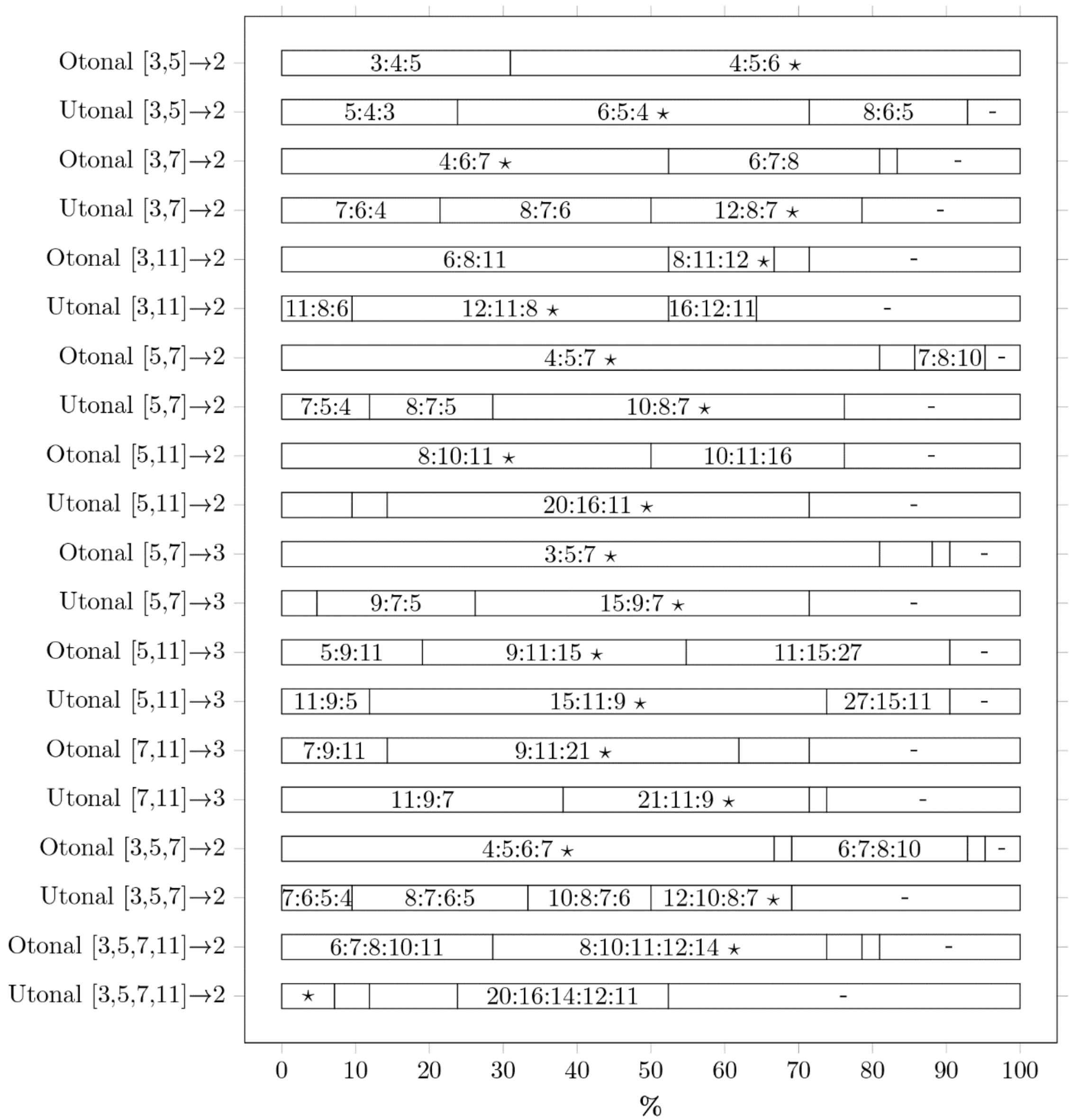
Harm.* = Modified harmonicity considering the center of symmetry

Space	Chord	Harm.	Disson.	Distan.	Entro.	Dista.*	Harm.*
[3, 5]→2	3:4:5	0,4436	0,0852	7,4919	2,1304	23,7386	0,5689
	4:5:6	0,4911	0,0972	6,9069	1,9158	15,3987	0,7670
	5:6:8	0,4200	0,1002	10,2288	3,3928	27,0426	0,5259
	5:4:3	0,4436	0,0885	8,2288	2,5764	23,2125	0,5299
	6:5:4	0,4911	0,0988	7,4919	2,1095	16,3987	0,7642
	8:6:5	0,4200	0,0955	8,9069	2,7532	26,5687	0,5616
[3, 7]→2	4:6:7	0,4258	0,0811	7,3923	1,7767	17,3696	0,7300
	6:7:8	0,3612	0,1286	8,9773	3,0854	23,7094	0,5212
	7:8:12	0,4149	0,1021	12,1997	3,8422	30,9843	0,4739
	7:6:4	0,4258	0,0916	10,1997	3,3923	25,1542	0,4666
	8:7:6	0,3612	0,1286	9,3923	3,1802	28,5395	0,5155
	12:8:7	0,4149	0,0904	8,9773	2,1319	18,3696	0,7147
[3,11]→2	6:8:11	0,3073	0,0985	9,6294	2,7085	27,0136	0,4777
	8:11:12	0,3637	0,1230	9,0444	2,3786	19,6738	0,6747
	11:12:16	0,3031	0,1376	14,5038	4,1894	35,5926	0,4223
	11:8:6	0,3073	0,0967	12,5038	3,6173	30,7625	0,4235
	12:11:8	0,3637	0,1314	9,6294	2,5974	20,6738	0,6742
	16:12:11	0,3031	0,1306	11,0444	3,0618	30,8437	0,4770
[5,7]→2	4:5:7	0,2604	0,0967	9,1293	2,9935	20,5805	0,3817
	5:7:8	0,2416	0,1093	10,4512	3,4080	33,8682	0,2962
	7:8:10	0,2509	0,1310	11,9366	4,0261	33,1952	0,2950
	7:5:4	0,2604	0,0913	9,9366	3,2024	30,8391	0,3015
	8:7:5	0,2416	0,1181	11,1293	3,6926	35,2244	0,3009
	10:8:7	0,2509	0,1273	10,4512	3,5326	21,5805	0,3806
[5,11]→2	8:10:11	0,2053	0,1451	10,7814	3,5954	22,8846	0,3456
	10:11:16	0,1906	0,1385	12,1033	3,7881	34,1724	0,2649
	11:16:20	0,2017	0,1014	15,2408	3,5305	38,8035	0,2451
	11:10:8	0,2053	0,1497	13,2408	4,1844	33,4474	0,2442
	16:11:10	0,1906	0,1273	12,7814	3,5743	38,5285	0,2653
	20:16:11	0,2017	0,1090	12,1033	3,1553	23,8846	0,3452
[5,7]→3	3:5:7	0,2474	0,0678	8,2992	1,9843	12,9374	0,3343
	5:7:9	0,2092	0,1055	10,6211	3,3226	21,7973	0,2474
	7:9:15	0,2260	0,1002	12,6915	3,5540	22,4799	0,2449
	7:5:3	0,2474	0,0712	9,5216	2,7493	19,4098	0,2476
	9:7:5	0,2092	0,1088	11,4691	3,4879	23,8674	0,2471
	15:9:7	0,2260	0,0931	10,6211	2,6237	13,9374	0,3320
[5,11]→3	5:9:11	0,1684	0,0939	11,2732	2,9401	22,6201	0,2218

	9:11:15	0,1895	0,1125	10,5362	3,0903	13,7602	0,3049
	11:15:27	0,1601	0,0877	15,5806	3,5757	24,1256	0,2106
	11:9:5	0,1684	0,1054	12,4107	3,4729	21,0555	0,2118
	15:11:9	0,1895	0,1073	11,2732	3,0824	14,7602	0,3039
	27:15:11	0,1601	0,0839	13,7062	3,0508	24,6902	0,2216
[7,11]→3	7:9:11	0,1417	0,1235	12,2441	3,8401	25,7640	0,1686
	9:11:21	0,1519	0,1027	11,0217	3,0544	14,6790	0,2335
	11:21:27	0,1343	0,0911	16,0661	3,4071	26,0444	0,1665
	11:9:7	0,1417	0,1258	12,8961	3,9798	22,5869	0,1706
	21:11:9	0,1519	0,0877	12,2441	2,8778	15,6790	0,2326
	27:21:11	0,1343	0,1030	14,1916	3,4439	28,2215	0,1696
[3,5,7]→2	4:5:6:7	0,7041	0,1889	11,7142	3,3430	26,5984	0,9051
	5:6:7:8	0,6269	0,2209	15,3581	5,1680	39,8861	0,6643
	6:7:8:10	0,6471	0,2201	12,8842	4,0955	34,1082	0,7079
	7:8:10:12	0,6898	0,2110	18,3290	5,8085	45,8278	0,6345
	7:6:5:4	0,7041	0,1963	15,3290	5,1674	37,9978	0,6312
	8:7:6:5	0,6269	0,2266	14,7142	4,8130	41,7683	0,6949
	10:8:7:6	0,6471	0,2122	14,3581	4,5427	38,0561	0,6538
	12:10:8:7	0,6898	0,2058	13,8842	3,8919	28,5984	0,9010
[3,5,7,11]→2	6:7:8:10:11	0,8143	0,3757	18,9286	5,6836	47,7819	0,8106
	7:8:10:11:12	0,8550	0,4011	24,5957	7,7081	60,9464	0,7328
	8:10:11:12:14	0,8681	0,3901	18,1737	5,3721	40,1022	1,0095
	10:11:12:14:16	0,7888	0,4376	22,1395	7,3233	53,0338	0,7654
	11:12:14:16:20	0,8075	0,3936	29,5520	7,7406	69,8588	0,7001
	11:10:8:7:6	0,8143	0,3782	25,5520	7,6722	59,0288	0,7033
	12:11:10:8:7	0,8550	0,4187	20,9286	6,1398	43,1022	1,0049
	14:12:11:10:8	0,8681	0,4056	22,5957	7,1296	55,1163	0,7294
	16:14:12:11:10	0,7888	0,4251	22,1737	6,7545	59,2722	0,7975
	20:16:14:12:11	0,8075	0,3707	22,1395	6,1317	55,2037	0,7582

Appendix B: Distribution of participants' choices

* = chord chosen by the proposed model as the most fundamental.



Appendix C: Derivation of the modified formulas

C.1 Harmonic distance to the perceptual center of symmetry

Tenney's harmonic distance formula is normalized so the octave's distance is 1. To extend this property to spaces with other equaves (E), the formula has to be adapted:

$$\delta\left(\frac{a}{b}\right) = log(a \cdot b)/\log 2 \;\rightarrow\; \delta\left(\frac{a}{b}\right) = log(a \cdot b)/\log E \;\mid\; E > 1$$

(4)

The product of the numerator and denominator of the interval can be expressed depending on the coordinates in a pitch-class space:

$$1 < I < E \quad \mid \quad I = \frac{a}{b} = \frac{\prod_{i=1}^{n} p_i^{x_i}}{E^{\left\lfloor \log\left(\prod_{i=1}^{n} p_i^{x_i}\right)/\log E \right\rfloor}} \quad \Rightarrow \quad a \cdot b$$

$$= \prod_{i=1}^{n} \left\{p_i^{|x_i|}\right\} \cdot E^{\left|\left\lfloor \log\left(\prod_{i=1}^{n} p_i^{x_i}\right)/\log E \right\rfloor\right|}$$

(19)

Being this the resulting harmonic distance formula for pitch-class spaces:

$$\delta(X) = \log\left(\prod_{i=1}^{n} \left\{p_i^{|x_i|}\right\} \cdot E^{\left|\left\lfloor \log\left(\prod_{i=1}^{n} p_i^{x_i}\right)/\log E \right\rfloor\right|}\right)\Big/ \log E$$

(20)

$$X = \left(x_{1,} \dots, x_i\right) \in [p_{1,} \dots, p_i] \rightarrow E$$

(15)

Finally, the coordinates can be mapped to make (0.5, 0, ..., 0_i) the origin:

$$\delta_{cs}(X) = \log\left(p_1^{|2x_1-1|} \cdot \prod_{i=2}^{n} p_i^{|2x_i|} \cdot E^{\left|\left\lfloor \log\left(p_1^{2x_1-1} \cdot \prod_{i=2}^{n} p_i^{2x_i}\right)/\log E \right\rceil\right|}\right) \Big/ \log E$$

(18)

C.2 Harmonicity relative to the perceptual center of symmetry

The same process can be applied to Barlow's harmonicity:

$$\xi(N) = 2\sum_{i=1}^{\infty} \frac{n_i(p_i - 1)^2}{p_i} \qquad H(\frac{a}{b}) = \frac{sgn(\xi(a) - \xi(b))}{\xi(a) + \xi(b)}$$

(4)

Since the harmonicity of structures such as chords or scales is the summation of the module of the mutual inharmonicites between every tone, the indigestibility formula can be simplified by removing the numerator, which makes no difference after applying the module:

$$\left|H\left(\frac{a}{b}\right)\right| = \frac{1}{\xi(a)+\xi(b)}$$

(21)

Applying the property $\xi(a \cdot b) = \xi(a) + \xi(b)$:

$$\left|H\left(\frac{a}{b}\right)\right| = \frac{1}{\xi(a \cdot b)}$$

(22)

Again, this expression can be transformed to receive coordinates in a pitch-class space as an input:

$$|H(X)| = \xi\left(\prod_{i=1}^{n}\left\{p_i^{|x_i|}\right\}\cdot E^{\left|\left\lfloor \log\left(\prod_{i=1}^{n} p_i^{x_i}\right)/\log E\right\rfloor\right|}\right)^{-1}$$

(23)

$$\xi(X) = 2\sum_{i=1}^{n}\frac{x_i(p_i-1)^2}{p_i}$$

(17)

$$X = \left(x_{1,} \ldots, x_i\right) \in [p_{1,} \ldots, p_i] \to E$$

(15)

And finally, the coordinates can be mapped to make (0.5,0,...,0_i) the origin:

$$|H_{cs}(X)| = \xi\left(p_1^{|2x_1-1|}\cdot\prod_{i=2}^{n}\left\{p_i^{|2x_i|}\right\}\cdot E^{\left|\left\lfloor \log\left(p_1^{2x_1-1}\cdot\prod_{i=2}^{n} p_i^{2x_i}\right)/\log E\right\rfloor\right|}\right)^{-1}$$

(16)