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Faces of Weight Polytopes, a Generalization of a Theorem of Vinberg and Koszul Algebras

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Faces of Weight Polytopes, a Generalization of a Theorem of Vinberg and Koszul Algebras

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Timothy Blake Ridenour

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To Tony Danza!.

ABSTRACT OF THE DISSERTATION

Faces of Weight Polytopes, a Generalization of a Theorem of Vinberg and Koszul Algebras

by

Timothy Blake Ridenour

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, August 2010
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Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and let V be a $\mathfrak{g}$-semisimple module. In this article, we study the category $\widehat{\mathcal{G}}$ of $\mathbb{Z}_+$-graded $\mathfrak{g} \ltimes V$-modules with finite-dimensional grade pieces. We construct and classify certain special subsets called *weak* $\mathbb{F}$*-faces* of the set of weights of V. If V is a generalized Verma module, our result allows us to recover and extend a result due to Vinberg on the classification of faces of the weight polytope.

If $\mathfrak{g}$ is semisimple and V is simple, we use the *positive* weak $\mathbb{F}$-faces of the set of weights of V to construct a large family of Koszul algebras which have finite global dimension. We are also able to construct truncated subcategories of $\widehat{\mathcal{G}}$ which are directed and highest weight.

Contents

Introduction

Motivated by the study of the category of finite-dimensional representations of the classical and quantum loop algebras associated to a simple Lie algebra, $\mathfrak{g}$, Chari and Greenstein studied the category of $\mathbb{Z}_+$-graded modules with finite-dimensional graded pieces for the polynomial current algebra $\mathfrak{g}[t]$ as well as its truncation $\mathfrak{g} \ltimes \mathfrak{g}_{ad}$ [5],[6]. For an arbitrary representation V of $\mathfrak{g}$, we can consider the semi-direct product Lie algebra $\mathfrak{g} \ltimes V$. Several well-known families of Lie algebras including cominiscule parabolic subalgebras of $\mathfrak{g}$, parabolic subalgebras of semisimple Lie algebras that arise from good gradings [8], quotients of realizations of foldings of twisted affine Lie algebras [3], and r-Takiff Lie algebras $\mathfrak{g} \otimes \mathbb{C}[t]/(t^r)$ are semi-direct product Lie algebras. It is, therefore, of interest to consider the category of finite-dimensional representations of semi-direct product Lie algebras $\mathfrak{a} \ltimes V$. In particular, we generally deal with the case where $\mathfrak{a}$ is reductive and V is acted upon semisimply by the center $\mathcal{Z}(\mathfrak{a})$ of $\mathfrak{a}$.

Unfortunately, the category of finite-dimensional modules for $\mathfrak{a} \ltimes V$ is not semisimple and the simple objects in this category may have self-extensions. If we consider the category $\widehat{\mathcal{G}}$ of $\mathbb{Z}_+$-graded modules over $\mathfrak{a} \otimes V$ with finite-dimensional graded pieces, the simple objects no longer have self-extensions, so it will be convenient to work in this category. $\widehat{\mathcal{G}}$ has enough projectives; however, the full subcategory $\mathcal{G}$ of finite-dimensional objects in $\widehat{\mathcal{G}}$ does not have enough projectives. It is of interest to determine truncated subcategories of $\widehat{\mathcal{G}}$ which do have enough projectives and whose simple objects are of finite global dimension. In particular, we wish to consider the truncated subcategories of the form $\mathcal{G}[\Gamma]$ whose objects have Jordan-Hölder factors parametrized by Γ associated to *nice* subsets Γ of the the set P^+ of dominant integral weights.

As in [5] and [6], we are particularly interested in the sets $\Gamma \subset P^+$ which are associated to certain subsets Ψ of the set of weights of V. In fact, if we choose Ψ appropriately, we are able to construct Koszul algebras of any finite global dimension. The truncated subcategories in which we are interested are those which are equivalent to the category of finite-dimensional modules over these Koszul algebras.

It is natural to ask which subsets of the set of weights of the $\mathfrak{a}$-module V give rise to these Koszul algebras. In [6] and [4], some of these sets were studied and classified for the classical finite-dimensional simple Lie algebras. In Chapter 1, we consider subsets $X \subset \mathbb{R}^n$ and study subsets of X which we call weak $\mathbb{F}$-faces and positive weak $\mathbb{F}$-faces where F is a subfield of $\mathbb{R}$. We study some combinatorial properties of these sets and give a classification in the case when the convex hull of X gives a polytope in $\mathbb{R}^n$.

In Chapter 2, we extend a result due to Vinberg [19] and give a description of weak and positive weak $\mathbb{F}$-faces for the set of weights of a generalized Verma module of a (semi-)simple Lie algebra $\mathfrak{g}$. In Chapter 3, we study some more equivalent properties of positive weak $\mathbb{F}$-faces in the set of roots of the finite-dimensional simple Lie algebras and give explicit descriptions of these sets in the case when $\mathfrak{g}$ is classical. The appendix gives a list of the positive weak $\mathbb{F}$-faces of the root system of $\mathfrak{g}$ when $\mathfrak{g}$ is an exceptional finite-dimensional simple Lie algebra.

Finally, in Chapter 4, we discuss the category $\widehat{\mathcal{G}}$ and truncated subcategories of the form $\mathcal{G}[\Gamma]$ where Γ is associated to certain positive weak $\mathbb{F}$-faces Ψ of the set of weights of V. In fact, for certain choices of Ψ, we see that we can construct Koszul algebras of finite global dimension in a natural way.

Preliminaries and Notation

Notation

Throughout this paper, we let $\mathbb{R}$ (respectively $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$) denote the real numbers (respectively the rationals, the integers, and the natural numbers). For any subset $X \subset \mathbb{R}$, we let $X_+ := X \cap [0, \infty)$, $X_{>0} := X \cap (0, \infty)$. In the case $X = \mathbb{Z}$, note that $\mathbb{Z}_{>0} = \mathbb{N}$. We will let $\mathfrak{a}$ be a $\mathbb{Z}_+$-graded Lie algebra over $\mathbb{C}$, and we will assume that $\mathfrak{a}_0 = \mathfrak{g} \oplus \mathfrak{h}_0$ is a reductive Lie algebra with semisimple part $\mathfrak{g}$ and center $\mathfrak{h}_0$.

Lie Algebras

Definition. *A vector space $\mathfrak{a}$ over a field k with a binary operation $[\ ,\ ] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ is a Lie algebra over k if*

(a) $[\ ,\ ]$ *is bilinear;*

(b) $[x, y] = -[y, x]$ *for all* $x, y \in \mathfrak{a}$*;*

(c) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all* $x, y, z \in \mathfrak{a}$ *(Jacobi identity).*

Note. *For the purposes of this work, we shall assume that all Lie algebras are over* $\mathbb{C}$*.*

A Lie algebra $\mathfrak{a}$ is $\mathbb{Z}_+$-graded if $\mathfrak{a} = \bigoplus_{i \in \mathbb{Z}_+} \mathfrak{a}_i$ with $[\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_{i+j}$ for all $i, j \in \mathbb{Z}_+$.

A subspace $\mathfrak{i}$ of a Lie algebra $\mathfrak{g}$ is called a *Lie subalgebra* of $\mathfrak{g}$ if $[x, y] \in \mathfrak{i}$ for all $x, y \in \mathfrak{i}$ and an *ideal* if $[x, y] \in \mathfrak{i}$ for all $x \in \mathfrak{g}, y \in \mathfrak{i}$. If $\mathfrak{g}$ is nonzero and the only ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$, then $\mathfrak{g}$ is a *simple* Lie algebra. The *center* of a Lie algebra $\mathfrak{g}$ is the ideal $\mathcal{Z}(\mathfrak{g}) := \{x \in \mathfrak{g} \ : \ [x, y] = 0 \ \forall y \in \mathfrak{g}\}$.

Given a Lie algebra $\mathfrak{a}$, we can define the following sequence of ideals: $\mathfrak{a}^{(0)} = \mathfrak{a}$, $\mathfrak{a}^{(i+1)} = [\mathfrak{a}^{(i)}, \mathfrak{a}^{(i)}]$ for $i \geq 0$. We say that $\mathfrak{a}$ is *solvable* if $\mathfrak{a}^{(i)} = 0$ for some $i > 0$. The unique maximal solvable ideal of $\mathfrak{a}$ is called the *radical* of $\mathfrak{a}$, $Rad(\mathfrak{a})$.

Definition. *A Lie algebra $\mathfrak{a}$ is called reductive if $Rad(\mathfrak{a}) = \mathcal{Z}(\mathfrak{a})$. If $Rad(\mathfrak{a}) = 0$, then $\mathfrak{a}$ is semisimple.*

Every reductive Lie algebra can be written as a direct sum of a semisimple Lie algebra and an abelian Lie algebra: $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}_0$ where $\mathfrak{g} = \mathfrak{a}^{(1)}$ is semisimple and $\mathfrak{h}_0 = \mathcal{Z}(\mathfrak{a})$. In particular, every semisimple Lie algebra is reductive.

Suppose that $\mathfrak{g}$ is a semisimple Lie algebra. A maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra*. We obtain a decomposition $\mathfrak{g} = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \oplus \mathfrak{h}$ where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \ \forall \ h \in \mathfrak{h}\}$. The set R is called the set of *roots* of $\mathfrak{g}$. A subset $\Delta \subset R$ is called a *base* if Δ is a basis for $\mathfrak{h}^*$ and every root $\beta \in R$ can be written in the form $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ with either $k_\alpha \in \mathbb{Z}_+$ for all $\alpha \in \Delta$ or $-k_\alpha \in \mathbb{Z}_+$ for all $\alpha \in \Delta$.

Given a base Δ, define R^+ (respectively, R^-) to be the set of roots which can be written as non-negative (respectively, non-positive) integer combinations of Δ. We identify Δ with the set $I = \{1, ..., n\}$ where $n = \dim(\mathfrak{h})$. Define $\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha$. A subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h} \oplus \mathfrak{n}^+$ is called a *parabolic subalgebra* of $\mathfrak{g}$.

Definition. *Let $\mathfrak{a}$ be a Lie algebra over $\mathbb{C}$. The invariant, symmetric bilinear form $\kappa : \mathfrak{a} \times \mathfrak{a} \to \mathbb{C}$ given by $\kappa(x, y) = Tr(\mathrm{ad}(x)\,\mathrm{ad}(y))$ is called the Killing form on $\mathfrak{a}$.*

Theorem 1. *[10] A Lie algebra $\mathfrak{a}$ is semisimple if and only if the Killing form on $\mathfrak{a}$ is* *nondegenerate:* $\forall\ x \in \mathfrak{a}\ \exists\ y \in \mathfrak{a}$ *such that* $\kappa(x, y) \neq 0$.

If $\mathfrak{a}$ is a finite-dimensional semisimple Lie algebra, we identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via the Killing form: $t_\alpha \in \mathfrak{h}$ is the element which satisfies $\alpha(h) = \kappa(t_\alpha, h)$ for all $h \in \mathfrak{h}$. In particular, there is a positive-definite symmetric bilinear form $(,) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ given by $(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$.

Representations

Let V be a vector space over $\mathbb{C}$. Define $\mathfrak{gl}(V)$ to be the Lie algebra consisting of linear transformations $V \to V$ with bracket given by the commutator $[x, y] = xy - yx$.

Definition. *Let* $\mathfrak{a}$ *be a Lie algebra over* $\mathbb{C}$. *A Lie algebra homomorphism* $\rho : \mathfrak{a} \to \mathfrak{gl}(V)$ *is called a representation of* $\mathfrak{a}$.

Note. *A representation* $\rho : \mathfrak{a} \to \mathfrak{gl}(V)$ *defines a* $\mathfrak{a}$*-module action on* V*, and, similarly, a* $\mathfrak{a}$*-module action on* V *can be used to define a representation* $\mathfrak{a} \to \mathfrak{gl}(V)$. *We will use the terminology of* $\mathfrak{a}$*-modules and representations interchangeably without further mention.*

Let V be an $\mathfrak{a}$-module. We say that $v \in V$ is $\mathfrak{a}$-*invariant* if $a.v = 0$ for all $a \in \mathfrak{a}$. Denote the set of all $\mathfrak{a}$-invariants by $V^{\mathfrak{a}}$.

Lemma 0.1. *[6, Lemma 4.1] Let* M *be a finite-dimensional* $\mathfrak{a}$*-module. There exists an isomorphism*

$$M \cong \bigoplus_{\nu \in P^+} V(\nu) \otimes (V(\nu)^* \otimes M)^{\mathfrak{a}}$$

of $\mathfrak{a}$*-modules.*

Let V be a $\mathfrak{a}$-module. Define the semi-direct product Lie algebra $\mathfrak{a} \otimes V$ by giving the vector space $\mathfrak{a} \oplus V$ the bracket operation $[(x, v), (y, w)] = ([x, y], x.w - y.v)$ for all $x, y \in \mathfrak{a}$

and $v, w \in V$. The Lie algebra $\mathbf{U}(\mathfrak{a} \ltimes V)$ is known as an *undeformed infinitesimal Hecke algebra.*

Associative Algebras

Let A be a vector space over $\mathbb{C}$, and let $id_A : A \to A$ be the identity map. A is an **associative algebra** over $\mathbb{C}$ if there exists a homomorphism $m : A \otimes A \to A$ such that $m \circ (id_A \otimes m) = m \circ (m \otimes id_A)$. In other words, A is a vector space over $\mathbb{C}$ with a multiplication which satisfies $r(xy) = (rx)y = x(ry)$ for all $r \in \mathbb{C}$ and $x, y \in A$.

Example. *The tensor algebra* $T(V) = \bigoplus_{r \in \mathbb{Z}_+} T^r(V)$ *of a vector space* V *over* $\mathbb{C}$*, where* $T^r(V)$ *is the* r*-fold tensor product of* V*, is an associative algebra over* $\mathbb{C}$*. If* I *is the ideal in* $T(V)$ *generated by* $\{x \otimes y + y \otimes x \mid x, y \in V\}$*, then the symmetric algebra* $\mathrm{Sym}(V) := T(V)/I$ *is also an associative algebra over* $\mathbb{C}$*.*

Definition. *Let* A *be a finite-dimensional associative algebra over* $\mathbb{C}$*.* A *is said to be* **basic** *if every simple* A*-module is one-dimensional.*

Let N *be the Jacobson radical of* A*.* A *is* **quasi-hereditary** *if* A *has a nonzero ideal* J *such that*

(i) J *is a projective* A*-module;*

(ii) $\mathrm{Hom}_A(J, A/J) = 0$ *and* $JNJ = 0$*;*

(iii) *If* $J \neq A$*, then* A/J *has a nonzero ideal* J' *such that* (*i*) *and* (*ii*) *hold in* A/J*.*

The following theorem due to Cline, Parshall, and Scott ([7, Theorem 3.6]) gives a useful characterization of finite-dimensional quasi-hereditary algebras.

Theorem 2. *[7, Theorem 3.6] Let* A *be a finite-dimensional algebra over a field* k*. The following are equivalent:*

(a) A is a quasi-hereditary algebra;

(b) The category of A-modules is a highest weight category;

(c) The category of finitely-generated A-modules is a highest weight category.

An algebra A is $\mathbb{Z}_+$-graded if $A = \bigoplus_{i\in\mathbb{Z}_+} A[i]$ with $A[i]A[j] \subset A[i+j]$ for all $i, j \in \mathbb{Z}_+$. Given a $\mathbb{Z}_+$-graded associative algebra over $\mathbb{C}$, we define a new associative algebra $T_{A[0]}(A[1])$ where $T_{A[0]}(A[1])[0] = A[0]$ and $T_{A[0]}(A[1])[r] = T^r_{A[0]}(A[1])$ is the r-fold tensor product of $A[1]$ over $A[0]$. A canonical homomorphism $\mathbf{m} : T_{A[0]}(A[1]) \to A$ of $\mathbb{Z}^+$-graded modules can be defined by extending the assignment $\mathbf{m}(a) = a$ for all $a \in A[k]$, $k = 0, 1$.

Definition. *A $\mathbb{Z}_+$-graded associative algebra over $\mathbb{C}$ is said to be* **quadratic** *if the homomorphism $\mathbf{m}$ is surjective and $\ker \mathbf{m}$ is generated as an A-module by $R := \ker \mathbf{m} \cap T^2_{A[0]}(A[1])$.*

The Koszul complex of a quadratic algebra A is a complex of the form

$$\cdots \xrightarrow{\partial_{r+1}} K^r_A \xrightarrow{\partial_r} K^{r-1}_A \xrightarrow{\partial_{r-1}} \cdots \xrightarrow{\partial_2} K^1_A \xrightarrow{\partial_1} K^0_A \xrightarrow{\partial_0} A$$

where $K^r_A = A \otimes_{A[0]} N^r_A$ and

$$N^r_A := \begin{cases} A[r], & r = 0, 1 \\ \bigcap_{j=0}^{r-2}(T^j_{A[0]}(A[1]) \otimes_{A[0]} R \otimes_{A[0]} T^{r-j-2}_{A[0]}(A[1]), & r \geq 2 \end{cases}.$$

The differential $\partial_r : K^r_A \to K^{r-1}_A$ is the restriction of the map $A \otimes T^r_{A[0]}(A[1]) \to A \otimes T^{r-1}_{A[0]}(A[1])$ given by extending

$$x \otimes (a_1 \otimes \ldots a_r) \mapsto xa_1 \otimes (a_2 \otimes a_r).$$

Notice that if $a_1 \otimes a_r \in N^r_A$, then $a_1 \otimes a_2 \in R$, and $m(a_1 \otimes a_2) = 0$. Furthermore, $\partial_{r-1} \circ \partial_r(x \otimes a_1 \otimes ... \otimes a_r) = xa_1a_2 \otimes (a_3 \otimes ...a_r) = 0$.

Suppose that A is a $\mathbb{Z}_+$-graded associative algebra over $\mathbb{C}$. A is **Koszul** if $A[0]$ is semisimple and admits a projective resolution of $\mathbb{Z}_+$-graded A-modules

$$\cdots \to P_2 \to P_1 \to P_0 \to A[0] \to 0$$

such that P_r is generated, as a graded A-module, by $P_r[r]$ for all $r \in \mathbb{Z}_+$.

From [1, Corollary 2.3.3], every Koszul algebra is quadratic. Furthermore, by [1, Theorem 2.6.1], a quadratic algebra A over $\mathbb{C}$ is Koszul if and only if its Koszul complex is exact.

Part I

Faces of Generalized Verma Modules

Chapter 1

Alternative Characterization of Faces

Definitions

Given $v, w \in \mathbb{R}^n$, set

$$H(v,w) := \{u \in \mathbb{R}^n \mid v \cdot (u-w) = 0\}, \quad H^+(v,w) := \{u \in \mathbb{R}^n \mid v \cdot (u-w) \geq 0\};$$

i.e., the affine hyperplane orthogonal to v containing w and the corresponding (positive) half-space. Let $\mathcal{P}$ be a convex subset of $\mathbb{R}^n$. We say that $H(v,w)$ is a **supporting hyperplane** of $\mathcal{P}$ if

$$\mathcal{P} \subset H^+(v,w) \text{ and } \mathcal{P} \cap H(v,w) \neq \emptyset.$$

A **proper face** of $\mathcal{P}$ is the intersection of $\mathcal{P}$ with a supporting hyperplane. A **face** of $\mathcal{P}$ is either a proper face or $\mathcal{P}$ itself. We will say that $\mathcal{P}$ is a **polyhedron** if it is the intersection of a finite number of affine half-spaces, and a bounded polyhedron is a **polytope**.

Given a subset $X \subset \mathbb{R}^n$ and a subfield $\mathbb{F} \subset \mathbb{R}$, we define the $\mathbb{F}$**–convex hull of** X to be the set

$$\mathrm{conv}_{\mathbb{F}}(X)=\left\{\sum_{s=1}^{k} r_s x_s \mid k\in\mathbb{Z},\ r_s\in\mathbb{F}_+,\ x_s\in X,\ \sum_{s=1}^{k} r_s=1\right\}.$$

Also define the $\mathbb{F}$**-cone of X**, $\mathrm{cone}_{\mathbb{F}}(X)$, to be the set

$$\mathrm{cone}_{\mathbb{F}}(X)=\left\{\sum_{s=1}^{k} r_s x_s\ :\ k\in\mathbb{Z}_+, r_s\in\mathbb{F}_+, x_s\in X\right\}.$$

The following is standard; see [21], for instance:

Theorem 3 (Decomposition Theorem)**.** *Let $\mathcal{P}$ be a subset of $\mathbb{R}^n$. Then,*

(i) *$\mathcal{P}$ is a polytope if and only if $\mathcal{P}=\mathrm{conv}_{\mathbb{R}}(U)$ for some finite subset $U\subset\mathbb{R}^n$.*

(ii) *$\mathcal{P}$ is a polyhedron if and only if $\mathcal{P}=\mathrm{conv}_{\mathbb{R}}(U)+\mathrm{cone}_{\mathbb{R}}(V)$ for some finite sets $U,V\subset\mathbb{R}^n$.*

In particular, the convex hull of the union of a finite set with a polytope or polyhedron is also a polytope or a polyhedron, respectively.

Results on Polyhedra

In this section we prove some general results on polyhedra. Given a subset X of $\mathbb{F}^n$ and a subfield $\mathbb{F}$ of $\mathbb{R}$, we denote by $\mathrm{conv}_{\mathbb{F}}(X)$ the $\mathbb{F}$–convex span of X. We define the relative interior of $\mathrm{conv}_{\mathbb{F}}(X)$ to be the set $\mathrm{relint}(\mathrm{conv}_{\mathbb{F}}(X))$ given by

$$\mathrm{relint}(\mathrm{conv}_{\mathbb{F}}(X)):=\{x\in X\mid \forall\, y\in X,\ \exists z\in X, t\in\mathbb{F}\cap(0,1) \text{ such that } x=ty+(1-t)z\}.$$

Definition. *Fix a subset $X\subset\mathbb{R}^n$, and a subfield $\mathbb{F}\subset\mathbb{R}$.*

(i) *We say that $Y\subset X$ is a* **weak $\mathbb{F}$-face** *of X if for every $U\subset X$,*

$$\mathrm{conv}_{\mathbb{F}}(Y)\cap\mathrm{relint}_{\mathbb{F}}(\mathrm{conv}_{\mathbb{F}}(U))\neq\emptyset \implies U\subset Y.$$

(ii) *A weak* $\mathbb{F}$*-face* $Y \subset X$ *is a* **positive weak** $\mathbb{F}$**-face** *of* X *if for every* $U \subset X$*,*

$$\operatorname{conv}_{\mathbb{F}}(Y) \cap \operatorname{relint}_{\mathbb{F}}(\operatorname{conv}_{\mathbb{F}}(U \cup \{0\})) = \emptyset.$$

Our main goal in this section is to determine the relationship between the weak $\mathbb{F}$-faces of a set X and the faces of $\operatorname{conv}_{\mathbb{F}}(X)$. The following lemma will be crucial in our examination of these sets.

Lemma 1.1. *Suppose* $X \subset \mathbb{F}^n$*. If* u *is in the* $\mathbb{F}$*-relative interior of* $\operatorname{conv}_{\mathbb{F}}(X)$ *and* $x_0 \in X$*, then there exist* $m > 0$*,* $r_0, r_1, ..r_m \in \mathbb{F}_{>0}$*, and* $x_1, ..., x_m \in X$ *such that*

$$u = r_0 x_0 + \sum_{j=1}^{m} r_j x_j, \quad r_0 + \sum_{j=1}^{m} r_j = 1.$$

Proof. Since u is in the interior of $\operatorname{conv}_{\mathbb{F}}(X)$, we can find $t \in \mathbb{F}_{>0}$ such that $u = tx_0 + (1-t)x'$, where $x' \in \operatorname{conv}_{\mathbb{F}}(X)$. By definition of $\operatorname{conv}_{\mathbb{F}}(X)$, we can write $x' = \sum_{j=1}^{m} s_j x_j$ for some $x_j \in X$ and $s_j \in \mathbb{F}_{>0}$ such that $\sum_{j=1}^{m} s_j = 1$. Solving for u, we have

$$u = tx_0 + \sum_{j=1}^{m} s_j (1-t) x_j.$$

Setting $r_0 = t$ and $r_j = s_j(1-t)$ gives the result. $\square$

Notice that if $\#X > 1$, we may choose all $x_0, x_1, \ldots, x_m$ to be distinct from u: we start by choosing any $x_0 \neq u$ in $\operatorname{conv}_{\mathbb{F}}(X)$, and proceed as above. Now, if $x_j = u$ for some $j > 0$, then we simply subtract $r_j u$ from both sides, and divide by $1 - r_j$.

The following result will also be used frequently.

Lemma 1.2. *Let* $\mathcal{P} \subset \mathbb{R}^n$ *be nonempty. A nonempty subset* $F \subset \mathcal{P}$ *is a face of* $\mathcal{P}$ *if and only if* F *is the subset of* $\mathcal{P}$ *that maximizes some linear functional* $\varphi \in (\mathbb{R}^n)^*$*.*

Proof. If F is a face of $\mathcal{P}$, $F = \mathcal{P} \cap H(v, w)$ for some supporting hyperplane $H(v, w)$. Define $\varphi : \mathbb{R}^n \to \mathbb{R}$ by $\varphi(u) = -v \cdot u$.

Suppose $u \in F$. Then, $\varphi(u) = -v \cdot u = -v \cdot w = \varphi(w)$ since $u \in H(v, w)$ gives that $v \cdot (u - w) = 0$. If $u \in \mathcal{P} \setminus F$, then $u \in H_+(v, w) \setminus H(v, w)$, so $v \cdot (u - w) > 0$. This gives $v \cdot u > v \cdot w$ and, thus, $\varphi(u) < \varphi(w)$. In particular, φ is maximized in $\mathcal{P}$ precisely on F. Similarly, if $\varphi \in (\mathbb{R}^n)^*$ is maximized in $\mathcal{P}$ on F, choose v such that $\varphi(u) = -v \cdot u$ for all $u \in \mathbb{R}^n$. Let $w \in F$. Then, $F = \mathcal{P} \cap H(v, w)$. □

Proposition 1.3. *Suppose that* $X \subset \mathbb{F}^n$. *Then* $\mathrm{conv}_{\mathbb{F}}(X) = \mathrm{conv}_{\mathbb{R}}(X) \cap \mathbb{F}^n$.

Proof. The inclusion $\mathrm{conv}_{\mathbb{F}}(X) \subset \mathrm{conv}_{\mathbb{R}}(X) \cap \mathbb{F}^n$ follows immediately since $\mathbb{F}$-linear combinations of elements of $\mathbb{F}^n$ are necessarily still in $\mathbb{F}^n$.

Suppose that $u \in \mathrm{conv}_{\mathbb{R}}(X) \cap \mathbb{F}^n$. Then, $u \in \mathrm{conv}_{\mathbb{R}}(U)$ for some finite subset $U \subset X$. By Caratheodory's theorem, u is in some r-simplex, $S \subset \mathrm{conv}_{\mathbb{R}}(U)$, such that the vertices of S are a subset of U.

Let $\{s_0, s_1, \ldots, s_r\}$ be the vertices of S. Then, $u = \sum_{i=0}^{r} n_i s_i$ for some $n_i \in \mathbb{R}_+$ with $\sum_{i=0}^{r} n_i = 1$.

Let ψ be an $\mathbb{F}$-affine transformation of $\mathbb{R}^n$ such that $\psi(s_0) = \mathbf{0}$ and $\psi(s_i) = \mathbf{e}_i$, $1 \leq i \leq r$, where $\{\mathbf{e}_i : 1 \leq i \leq n\}$ are the standard basis vectors in $\mathbb{R}^n$. Choose a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$ such that $\psi(x) = T(x) + \mathbf{b}$ for all $x \in \mathbb{R}^n$. Then,

$$\begin{aligned}
\psi(u) &= T(\textstyle\sum_{i=0}^{r} n_i s_i) + \mathbf{b} \\
&= (\textstyle\sum_{i=0}^{r} n_i T(s_i)) + \mathbf{b} \\
&= \textstyle\sum_{i=0}^{r} n_i T(s_i) + \sum_{i=0}^{r} n_i \mathbf{b} \\
&= \textstyle\sum_{i=0}^{r} n_i (T(s_i) + \mathbf{b}) \\
&= \textstyle\sum_{i=0}^{r} n_i \psi(s_i) \\
&= \textstyle\sum_{i=1}^{r} n_i \mathbf{e}_i.
\end{aligned}$$

Since ψ is $\mathbb{F}$-affine, $\psi(u) \in \mathbb{F}_+^n$, and $n_i \in \mathbb{F}_+$ for $i > 0$. Furthermore, $n_0 = 1 - \sum_{i=1}^{r} n_i \in \mathbb{F}_+$, so $u \in \mathrm{conv}_{\mathbb{F}}(X)$. □

Corollary 1.4. *Suppose $X \subset \mathbb{F}^n$, and F is a face of* $\operatorname{conv}_{\mathbb{R}}(X)$*. Then $F \cap \mathbb{F}^n$ is a face of* $\operatorname{conv}_{\mathbb{F}}(X)$*.*

Proof. Let $H(v, w)$ be a supporting hyperplane for $\operatorname{conv}_{\mathbb{R}}(X)$ such that $F = \operatorname{conv}_{\mathbb{R}}(X) \cap H(v, w)$. Then,

$$F \cap \mathbb{F}^n = \operatorname{conv}_{\mathbb{R}}(X) \cap H(v, w) \cap \mathbb{F}^n = \operatorname{conv}_{\mathbb{F}}(X) \cap H(v, w).$$

□

Polyhedra and weak $\mathbb{F}$-faces

We now prove a general result relating weak $\mathbb{F}$-faces and polyhedra.

Theorem 4. *Suppose that* $\operatorname{conv}_{\mathbb{R}}(X)$ *is a polyhedron for $X \subset \mathbb{F}^n$. Then, $Y \subset X$ is a weak $\mathbb{F}$-face if and only if $Y = X \cap F$ for some face F of* $\operatorname{conv}_{\mathbb{R}}(X)$*.*

Proof. First, suppose that $Y = X \cap F$ for some face F of $\operatorname{conv}_{\mathbb{R}}(X)$. By Lemma 1.2, one can find a linear functional $\varphi \in (\mathbb{R}^n)^*$ such that $\varphi(u) \geq \varphi(v)$ for all $u \in F$ and $v \in \operatorname{conv}_{\mathbb{R}}(X)$. Let $x_0 \in U \subset X$, and suppose $u \in \operatorname{conv}_{\mathbb{F}}(Y) \cap \operatorname{relint}(\operatorname{conv}_{\mathbb{F}}(U))$.

We can write $u = \sum_{y \in Y} s_y y$ with $s_y \in \mathbb{F}_+$ and $\sum_{y \in Y} s_y = 1$, and, thus, $\varphi(u) = \varphi(F)$. By Lemma 1.1, $u = \sum_{j=0}^m r_j x_j$ for some $r_j \in \mathbb{F}_{>0}$ and $x_j \in U$. Applying φ, we have

$$\varphi(F) = \varphi(u) = \sum_{j=0}^{m} r_j \varphi(x_j) \leq \sum_{j=0}^{m} r_j \varphi(F) = \varphi(F).$$

Since r_0 is positive, $\varphi(x_0) = \varphi(F)$, so $x_0 \in F \cap X = Y$. Since $x_0 \in U$ was arbitrary, $U \subset Y$, so $Y = X \cap F$ is a weak $\mathbb{F}$-face of X.

Now, let Y be a weak $\mathbb{F}$-face of X, and let F be the smallest face of $\operatorname{conv}_{\mathbb{R}}(X)$ such that $Y \subset F$. If $\#F \cap X = 1$, then $Y = F \cap X$ and we are done. Suppose that $\#F \cap X > 1$. Since F is minimal and $\operatorname{conv}_{\mathbb{R}}(X)$ is a polyhedron, the interior of F must contain an element $y \in \operatorname{conv}_{\mathbb{F}}(Y)$. Let $x \in F \cap X$. If $x = y$, then it is clear that $x \in Y$.

Suppose that $x \neq y$. Then, by Lemma 1.1, $y = r_0 x + \sum_{i=1}^{m} r_i x_i$ for some $r_i \in \mathbb{F}_{>0}$ and $x_i \in F \cap X$. In particular, $y \in \text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(F \cap X))$. Since Y is a weak $\mathbb{F}$-face of X, this gives that $F \cap X \subset Y$. $\square$

Positive weak $\mathbb{F}$-faces

Next, we study positive weak $\mathbb{F}$-faces. We start with an equivalent characterization.

Lemma 1.5. *For all subsets $X \subset \mathbb{R}^n$ and subfields $\mathbb{F} \subset \mathbb{R}$, the positive weak $\mathbb{F}$-faces of X are the weak $\mathbb{F}$-faces $Y \subset X$ such that Y is a weak $\mathbb{F}$-face of $X \cup \{0\}$ and $0 \notin \text{conv}_{\mathbb{F}}(Y)$.*

Proof. First, suppose that Y is a positive weak $\mathbb{F}$-face of X. Let $U \subset X \cup \{0\}$ and suppose that $\text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U)) \neq \emptyset$. Then, $U \subset X$ because, otherwise, $0 \in U$ giving $\text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}((U \setminus \{0\}) \cup \{0\})) \neq \emptyset$, which contradicts Y being a positive weak $\mathbb{F}$-face of X. Thus, $U \subset Y$ and Y is a weak $\mathbb{F}$-face of $X \cup \{0\}$. Suppose that $0 \in \text{conv}_{\mathbb{F}}(Y)$, and let $U = Y$. Then, $\text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U \cup \{0\})) = \text{relint}(\text{conv}_{\mathbb{F}}(Y)) \neq \emptyset$, which contradicts Y being a positive weak $\mathbb{F}$-face of X.

Now, suppose Y is a weak $\mathbb{F}$-face for both X and $X \cup \{0\}$ such that $0 \notin \text{conv}_{\mathbb{F}}(Y)$. Let $U \subset X$. If $\text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U \cup \{0\})) \neq \emptyset$, then $U \cup \{0\} \subset Y$ since Y is a weak $\mathbb{F}$-face of $X \cup \{0\}$. However, this is impossible since $0 \notin \text{conv}_{\mathbb{F}}(Y)$. Thus, Y is a positive weak $\mathbb{F}$-face of X. $\square$

To connect these results to the results in Chapter 4, we prove the following proposition.

Theorem 5. *Let $X \subset \mathbb{R}^n$ and $\mathbb{F}$ a subfield of $\mathbb{R}$.*

(i) *A subset Y is a weak $\mathbb{F}$-face of X if and only if the following condition holds:*

Suppose that $\sum_{y \in Y} m_y y = \sum_{x \in X} r_x x$ *with* $m_y, r_x \in \mathbb{F}_+$ $\forall y \in Y, x \in X$ *then*

$$\sum_{y \in Y} m_y = \sum_{x \in X} r_x \implies x \in Y \text{ if } r_x \neq 0.$$

(ii) *A subset* Y *is a positive weak* $\mathbb{F}$*-face of* X *if and only if (i) holds and*

$$\sum_{y \in Y} m_y y = \sum_{x \in X} r_x x \implies \sum_{y \in Y} m_y \leq \sum_{x \in X} r_x.$$

Proof. (i) ($\Leftarrow$) Suppose $U \subset X$ and $u \in \mathrm{conv}_{\mathbb{F}}(Y) \cap \mathrm{relint}(\mathrm{conv}_{\mathbb{F}}(U))$. Let $x_0 \in U$. By Lemma 1.1 and the definition of $\mathrm{conv}_F(Y)$, we can write $u = \sum_{y \in Y} m_y y = r_0 x_0 + \sum_{j=1}^{m} r_j x_j$ for some $x_j \in U$ where $m_y \in \mathbb{F} \cap [0, 1]$ for all $y \in Y$, $r_j \in \mathbb{F} \cap (0, 1)$ for $j = 0, \ldots, m$, and $\sum_{y \in Y} m_y = \sum_{j=0}^{m} r_j = 1$. Then, $r_0 \neq 0$, so $x_0 \in Y$. Since x_0 was arbitrary, $U \subset Y$.

($\Rightarrow$) Suppose that $u = \sum_{y \in Y} m_y y = \sum_{x \in X} r_x x$ and $\sum_{y \in Y} m_y = \sum_{x \in X} r_x > 0$ with $m_y, r_x \in \mathbb{F}_+$ for all $y \in Y$ and $x \in X$.

Let $U = \{x \in X \mid r_x \neq 0\}$, and consider $u' = \frac{1}{\sum_{x \in X} r_x} u \in \mathrm{conv}_{\mathbb{F}}(U)$. Notice that $u' = \sum_{y \in Y} \frac{m_y}{\sum_{y \in Y} m_y} y \in \mathrm{conv}_{\mathbb{F}}(Y)$, so it suffices to show that $u' \in \mathrm{relint}(\mathrm{conv}_{\mathbb{F}}(U))$. Furthermore, since each $x \in \mathrm{conv}_{\mathbb{F}}(U)$ is a convex sum of a finite number of elements in U, it suffices to check that for every $x_0 \in U$, there exists $y_0 \in \mathrm{conv}_{\mathbb{F}}(U)$ and $r_0 \in \mathbb{F} \cap (0, 1)$ such that $u' = r_0 x_0 + (1 - r_0) y_0$. By construction, we have $u' = \sum_{x \in U} r'_x x$ with $r'_x \in \mathbb{F} \cap (0, 1)$ and $\sum_{x \in U} r'_x = 1$. Letting $r_0 = r'_{x_0}$, we have

$$u' = r_0 x_0 + \sum_{x \neq x_0} r'_x x = r_0 x_0 + (1 - r_0) \sum_{x \neq x_0} \frac{r'_x}{1 - r_0} x.$$

It is easy to check that $y = \sum_{x \neq x_0} \frac{r'_x}{1 - r_0} x \in \mathrm{conv}_{\mathbb{F}}(U)$. In particular, $\mathrm{conv}_{\mathbb{F}}(Y) \cap \mathrm{relint}(\mathrm{conv}_{\mathbb{F}}(U)) \neq \emptyset$, so $U = \{x \in X \mid r_x \neq 0\} \subset Y$.

(ii) ($\Leftarrow$) Suppose that $\operatorname{conv}_{\mathbb{F}}(Y) \cap \operatorname{relint}(\operatorname{conv}_{\mathbb{F}}(U \cup \{0\})) \neq \emptyset$ for some $U \subset X$. Let $u \in \operatorname{conv}_{\mathbb{F}}(Y) \cap \operatorname{relint}(\operatorname{conv}_{\mathbb{F}}(U \cup \{0\}))$. By Lemma 1.1, there exist $r_0, r_1, \ldots, r_n \in \mathbb{F} \cap (0,1)$ and $x_1, \ldots, x_n \in U$ such that $\sum_{j=0}^{n} r_j x_j = 1$, and

$$u = r_0 \cdot 0 + \sum_{j=1}^{n} r_j x_j = \sum_{j=1}^{n} r_j x_j.$$

Similarly, there exist $m_y \in \mathbb{F} \cap [0,1]$ such that $\sum_{y \in Y} m_y = 1$, and $u = \sum_{y \in Y} m_y y$. However, this gives $u = \sum_{y \in Y} m_y y = \sum_{j=1}^{n} r_j x_j$ with $\sum_{j=1}^{n} r_j = 1 - r_0 < 1 = \sum_{y \in Y} m_y$, which is impossible. Thus, $\operatorname{conv}_{\mathbb{F}}(Y) \cap \operatorname{relint}(\operatorname{conv}_{\mathbb{F}}(U \cup \{0\})) = \emptyset$ for all $U \subset X$.

($\Rightarrow$) Let $u = \sum_{y \in Y} m_y y = \sum_{x \in X} r_x x$ with $m_y, r_x \in \mathbb{F} \cap [0, \infty)$. Let

$$U = \{x \in X \mid r_x \neq 0\} \neq \emptyset,$$

and suppose that $\sum_{x \in X} r_x < \sum_{y \in Y} m_y$. Define $r_0 = \sum_{y \in Y} m_y - \sum_{x \in X} r_x > 0$. Then, $u = r_0 \cdot 0 + \sum_{x \in U} r_x x$.

Let $u' = \dfrac{1}{\sum_{y \in Y} m_y} u$. Since $r_0 + \sum_{x \in U} r_x = \sum_{y \in Y} m_y$, $u' \in \operatorname{conv}_{\mathbb{F}}(U \cup \{0\})$. In fact, by an argument similar to that in part (i), $u' \in \operatorname{relint}(\operatorname{conv}_{\mathbb{F}}(U \cup \{0\}))$. However, this gives $u' \in \operatorname{conv}_{\mathbb{F}}(Y) \cap \operatorname{relint}(\operatorname{conv}_{\mathbb{F}}(U \cup \{0\}))$, which contradicts Y being a positive weak $\mathbb{F}$-face of X. Therefore, $\sum_{y \in Y} m_y \leq \sum_{x \in X} r_x$.

$\square$

Finally, this equivalent formulation of the positive weak $\mathbb{F}$-faces allows us to explain the terminology.

Theorem 6. *Suppose* $\operatorname{conv}_{\mathbb{R}}(X)$ *is a polyhedron for* $X \subset \mathbb{F}^n$. *Then* $Y \subset X$ *is a positive weak* $\mathbb{F}$*-face of* X *if and only if* Y *maximizes in* X *some linear functional* $\varphi \in (\mathbb{R}^n)^*$ *which has a positive maximum on* X.

If 0 *is in the interior of* $\mathrm{conv}_{\mathbb{R}}(X)$*, then a subset* Y *is a positive weak* $\mathbb{F}$*-face of* X *if and only if* $Y \neq X$ *and* Y *is a weak* $\mathbb{F}$*-face of* X*.*

Proof. By the Decomposition Theorem (Theorem 3), the $\mathbb{R}$-convex hull of $X \cup \{0\}$ is also a polyhedron. By Theorem 4 and Lemma 1.5, if Y is a positive weak $\mathbb{F}$-face of X, then $Y = (X \cup \{0\}) \cap F$ for some face F of $\mathrm{conv}_{\mathbb{R}}(X \cup \{0\})$, so Y maximizes some $\varphi \in (\mathbb{R}^n)^*$ in $X \cup \{0\}$ and, hence, also in X. Furthermore, since $0 \notin \mathrm{conv}_{\mathbb{R}}(Y)$, $\varphi(Y) > \varphi(0) = 0$, so φ has a positive maximum on X.

Conversely, choose $\varphi \in (\mathbb{R}^n)^*$ which is maximized in X precisely on Y and $\varphi(Y) > 0$. By Theorem 4 and Lemma 1.2, Y is a weak $\mathbb{F}$-face. Suppose that $\sum_{y \in Y} m_y y = \sum_{x \in X} r_x x$ with $m_y, r_x \in \mathbb{F}_+$ for all $y \in Y$ and $x \in X$. Applying φ, we have

$$\varphi(Y) \sum_{y \in Y} m_y = \sum_{y \in Y} m_y \varphi(y) = \sum_{x \in X} r_x \varphi(x) \leq \varphi(Y) \sum_{x \in X} r_x.$$

Since $\varphi(Y) > 0$, $\sum_{y \in Y} m_y \leq \sum_{x \in X} r_x$, and Y is a positive weak $\mathbb{F}$-face of X by Theorem 5.

Finally, suppose that $0 \in \mathrm{relint}_{\mathbb{R}}(\mathrm{conv}_{\mathbb{R}}(X))$. The result is clear if $X = \{0\}$, so suppose otherwise. By the remark immediately following Lemma 1.1, we see that $0 \in \mathrm{conv}_{\mathbb{R}}(X \setminus \{0\}) \cap \mathbb{F}^n$, so by Proposition 1.3, $0 \in \mathrm{conv}_{\mathbb{F}}(X \setminus \{0\}) \subset \mathrm{conv}_{\mathbb{F}}(X)$. Then, X is not a positive weak $\mathbb{F}$-face of itself, by Lemma 1.5. Since every positive weak $\mathbb{F}$-face is a weak $\mathbb{F}$-face, it now suffices to prove that every proper weak $\mathbb{F}$-face of X is a positive weak $\mathbb{F}$-face.

Let $Y \subsetneq X$ be a (proper) weak $\mathbb{F}$-face of X. By Theorem 4, $Y = F \cap X$, for some proper face F of $\mathrm{conv}_{\mathbb{R}}(X) = \mathrm{conv}_{\mathbb{R}}(X \cup \{0\})$. Since F is a proper face of $\mathrm{conv}_{\mathbb{R}}(X)$, $0 \notin F$, so $0 \notin Y = F \cap (X \cup \{0\})$. By Lemma 1.2, $Y \subset X$ maximizes some linear functional φ on $X \cup \{0\}$, and $0 \notin Y$. Hence $\varphi(Y) > \varphi(0) = 0$, and we are done by the first part of this result. $\square$

Chapter 2

Generalized Verma Modules

In this chapter, the results from Chapter 1 are applied to set of weights of a generalized Verma module.

Fix a complex semisimple Lie algebra $\mathfrak{g}$ of rank n and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and let $R \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Set $I = \{1, \cdots, n\}$ and fix a set $\{\alpha_i : i \in I\}$ of simple roots. Denote by R^+ the corresponding set of positive roots. Let κ be the Killing form on $\mathfrak{g}$; recall that its restriction to $\mathfrak{h}$ induces a positive definite inner product $(\ , \ )$ on the real span $\mathfrak{h}^*_{\mathbb{R}}$ of R^+. Let $\{\omega_i : i \in I\}$ be the basis of $\mathfrak{h}^*$ which satisfies $2(\alpha_i, \omega_j) = \delta_{i,j}(\alpha_i, \alpha_i)$. Since the Killing form is nondegenerate, it induces an identification of $\mathfrak{h}_{\mathbb{R}}$ with $\mathfrak{h}^*_{\mathbb{R}}$. Define $h_{\alpha_i} \in \mathfrak{h}_{\mathbb{R}}$ to be the vector identified with $2\alpha_i/(\alpha_i, \alpha_i)$; these vectors form an $\mathbb{R}$-basis of $\mathfrak{h}_{\mathbb{R}}$.

The root lattice Q (respectively, weight lattice P) is the integer span of the simple roots α_i (respectively, fundamental weights ω_i), while Q^+ (respectively, P^+) is the $\mathbb{Z}_+$-span of the simple roots (respectively, fundamental weights). Given a subset J of I, let Q_J (respectively P_J) be the $\mathbb{Z}$-span of the simple roots $\{\alpha_j : j \in J\}$ (respectively, the fundamental weights $\{\omega_j : j \in J\}$), and set $R_J^+ := R^+ \cap Q_J$.

Given any $\lambda \in \mathfrak{h}^*$, say $\lambda = \sum_{i\in I} r_i\omega_i$ with all $r_i \in \mathbb{R}$, we set

$$\operatorname{supp}\lambda := \{i \in I : r_i \neq 0\}, \quad J_\lambda := \{i \in I : \lambda(h_{\alpha_i}) \in \mathbb{Z}_+\}.$$

Clearly, $\lambda \in P^+$ if and only if $J_\lambda = I$. Finally, let $\mathcal{W}$ be the Weyl group of R, namely the subgroup of $\mathrm{Aut}(\mathfrak{h}_\mathbb{R}^*)$ generated by the simple reflections $\{s_i : i \in I\}$. Note that the inner product $(\ , \ )$ on $\mathfrak{h}_\mathbb{R}^*$ is $\mathcal{W}$-invariant.

2.1 GVM Definition

Fix a Chevalley basis $\{x_\alpha^\pm, h_i = h_{\alpha_i} : \alpha \in R^+, 1 \leq i \leq n\}$ of $\mathfrak{g}$, set $\mathfrak{n}^\pm = \bigoplus_{\alpha\in R^+} \mathbb{C}x_\alpha^\pm$, and write

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

The subalgebras $\mathfrak{n}_J^\pm$ are defined in the obvious way:

$$\mathfrak{n}_J^\pm = \bigoplus_{\alpha\in R_J^+} \mathfrak{g}_{\pm\alpha}.$$

Let $\mathfrak{p}_J$ be the parabolic Lie subalgebra of $\mathfrak{g}$ associated to J. Then we have

$$\mathfrak{p}_J = \mathfrak{n}_J^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{m}_J = \mathfrak{n}_J^- \oplus \mathfrak{h} \oplus \mathfrak{n}_J^+, \quad \mathfrak{u}_J^\pm = \bigoplus_{\alpha\in R^+\setminus R_J^+} \mathbb{C}x_\alpha^\pm,$$

where $\mathfrak{m}_J$ is reductive with semisimple part $\mathfrak{g}_J$, and $\mathfrak{u}_J^\pm$ is nilpotent. The subgroup $\mathcal{W}_J$ of $\mathcal{W}$ generated by $\{s_j : j \in J\}$ is the Weyl group of $\mathfrak{g}_J$, and we set $\rho_J = \sum_{j\in J}\omega_j$. The following is standard, but we isolate it in the form of a Lemma, since it is used frequently.

Lemma 2.1. *For $w \in \mathcal{W}_J$ and $i \notin J$, we have $w\alpha_i \in R^+$, and hence $w\alpha \in R^+$ for all $\alpha \in R^+ \setminus R_J^+$.*

Given any Lie algebra $\mathfrak{a}$, we let $\mathbf{U}(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$. The Poincare–Birkhoff–Witt theorem gives us an isomorphism of vector spaces:

$$\mathbf{U}(\mathfrak{g}) \cong \mathbf{U}(\mathfrak{n}^-) \otimes \mathbf{U}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{n}^+).$$

2.2 Properties of GVMs

We now give the definition and elementary properties of the *generalized Verma modules.* Recall that a weight module V for a reductive Lie algebra $\mathfrak{a}$ with Cartan subalgebra $\mathfrak{t}$ is one which has a decomposition

$$V = \bigoplus_{\mu \in \mathfrak{t}^*} V_\mu,$$

where $V_\mu = \{v \in V : hv = \mu(h)v, \ \ \forall \ \ h \in \mathfrak{t}\}$. We set $\operatorname{wt} V = \{\mu \in \mathfrak{t}^* : V_\mu \neq 0\}$.

Given $\lambda \in \mathfrak{h}^*$ and $J \subset J_\lambda$, the generalized Verma module $M(\lambda, J)$ is the $\mathfrak{g}$-module generated by an element m_λ with defining relations:

$$\mathfrak{n}^+ m_\lambda = 0, \quad hm_\lambda = \lambda(h)m_\lambda, \quad (x_\alpha^-)^{\lambda(h_\alpha)+1} m_\lambda = 0,$$

for all $h \in \mathfrak{h}$ and $\alpha \in R_J^+$. The following is standard - see [17]:

Proposition 2.2. *Let $\lambda \in \mathfrak{h}^*$ and $J \subset J_\lambda$.*

(i) *The $\mathfrak{g}$-module $M(\lambda, J)$ is a free $\mathbf{U}(\mathfrak{u}_J^-)$-module, and $\dim \mathbf{U}(\mathfrak{m}_J)m_\lambda < \infty$. In particular, $\operatorname{wt}(\mathbf{U}(\mathfrak{m}_J)m_\lambda)$ is a finite subset of $\mathfrak{h}^*$, and*

$$\operatorname{wt} M(\lambda, J) = \operatorname{wt}(\mathbf{U}(\mathfrak{m}_J)m_\lambda) - \left\{ \sum_{\alpha \notin R_J^+} r_\alpha \alpha : r_\alpha \in \mathbb{Z}_+ \right\}.$$

(ii) *The set $\operatorname{wt} M(\lambda, J)$ is $\mathcal{W}_J$-invariant.*

In the special case when $\lambda \in P^+$, the module $M(\lambda, I)$ is the irreducible finite-dimensional module with highest weight λ.

2.3 The Weights of a GVM

Given any subset X of $\mathfrak{h}_{\mathbb{R}}^*$ we let $\mathrm{conv}_{\mathbb{R}}(X)$ be the convex hull of X; i.e.,

$$\mathrm{conv}_{\mathbb{R}}(X) = \left\{ \sum_{s=1}^{k} r_s x_s \ : \ k \in \mathbb{Z}_+, r_s \in \mathbb{R}_+, x_s \in X, \ \sum_{s=1}^{k} r_s = 1 \right\}.$$

Also define $\mathrm{cone}_{\mathbb{R}}(X)$ to be the cone of X, i.e.,

$$\mathrm{cone}_{\mathbb{R}}(X) = \left\{ \sum_{s=1}^{k} r_s x_s \ : \ k \in \mathbb{Z}_+, r_s \in \mathbb{R}_+, x_s \in X \right\}.$$

Proposition 2.3. *For $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ and $J \subset J_\lambda$, we have*

$$\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J)) = \mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, \mathbf{U}(\mathfrak{m}_J) m_\lambda) - \mathrm{cone}_{\mathbb{R}}(R^+ \setminus R_J^+),$$

and hence $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J))$ *is a $\mathcal{W}_J$-invariant subset of $\mathfrak{h}_{\mathbb{R}}^*$.*

Proof. Let $\mu \in \mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J)$. Write $\mu = \displaystyle\sum_k s_k \mu_k$ with $\mu_k \in \mathrm{wt}\, M(\lambda, J)$, $s_k \in [0, 1]$, and $\displaystyle\sum_k s_k = 1$. By Proposition 2.2, $\mu_k = \nu_k - \displaystyle\sum_{\alpha \notin R_J^+} r_{\alpha,k} \alpha$ where $\nu_k \in \mathrm{wt}\, \mathbf{U}(\mathfrak{m}_J) m_\lambda$ and $r_{\alpha,k} \in R_+$ for all k. Replacing μ_k,

$$\begin{aligned} \mu &= \textstyle\sum_k s_k (\nu_k - \sum_{\alpha \notin R_J^+} r_{\alpha,k} \alpha) \\ &= \textstyle\sum_k s_k \nu_k - \sum_k s_k \sum_{\alpha \notin R_J^+} r_{\alpha,k} \alpha \\ &= \textstyle\sum_k s_k \nu_k - \sum_{\alpha \notin R_J^+} (\sum_k s_k r_{\alpha,k}) \alpha. \end{aligned}$$

This shows that $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J)) \subset \mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, \mathbf{U}(\mathfrak{m}_J)) - \mathrm{cone}_{\mathbb{R}}(R^+ \setminus R_J^+)$.

For the reverse inclusion, let

$$\mu = \sum_k r_k \mu_k - \sum_{\alpha \in R^+ \setminus R_J^+} m_\alpha \alpha, \quad \mu_k \in \mathrm{wt}\, \mathbf{U}(\mathfrak{m}_J) m_\lambda, \quad r_k, m_\alpha \in \mathbb{R}_+, \quad \sum_k r_k = 1.$$

If $m_\alpha = 0$ for all α, then we are done since $\mu_k \in \mathrm{wt}(M(\lambda, J))$ for all k. Hence, we may assume that $m_\alpha > 0$ for some $\alpha \in R^+ \setminus R_J^+$. Furthermore, we may assume without loss of generality that $r_1 \neq 0$. Thus, we can write

$$\mu = \sum_{k \neq 1} r_k \mu_k + r_1 (\mu_1 - \sum_\alpha \frac{m_\alpha}{r_1} \alpha).$$

Choose $t \in \mathbb{Z}_+$ such that $r = \sum_\alpha \frac{m_\alpha}{r_1} \leq t$. Since $\mu_1 - t\alpha \in \mathrm{wt}\, M(\lambda, J)$, the claim follows by noting that

$$\mu = \sum_{k \neq 1} r_i \mu_i + \left(r_1 - \frac{r_1 r}{t} \right) \mu_1 + \sum_\alpha \frac{m_\alpha}{t} (\mu_1 - t\alpha) \in \mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J)).$$

The fact that $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J))$ is $\mathcal{W}_J$-invariant is immediate from Proposition 2.2. ■

Using the Decomposition Theorem, we have the following corollary to Proposition 2.3:

Corollary 2.4. $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, \mathbf{U}(\mathfrak{m}_J) m_\lambda)$ *is a convex polytope, and* $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J))$ *is a convex polyhedron in* $\mathfrak{h}^*_{\mathbb{R}}$.

2.4 Faces and GVMs

One of the main results of this chapter is the following:

Theorem 2.1. *Let* $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$, $J \subset J_\lambda$, *and let* $F \subset \mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J))$. *Then* F *is a face of* $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J))$ *if and only if there exists a subset* I_0 *of* I *and* $w \in \mathcal{W}_J$, *such that*

$$wF = \mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J) \cap (\lambda - Q^+_{I_0})).$$

Remark. *In the case when* $\lambda \in P^+$ *and* $J = J_\lambda = I$, *Theorem 2.1 is proved in [19]. The proof supplied here is quite different.*

Proof. Let F be a face of $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J))$. By Lemma 1.2, F maximizes some linear functional $\varphi \in \mathfrak{h}^*_{\mathbb{R}}$ in $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J))$. Let $\nu \in \mathfrak{h}^*_{\mathbb{R}}$ be such that $\varphi(\mu) = (\nu, \mu)$ for all $\mu \in \mathfrak{h}^*_{\mathbb{R}}$. Choose $w \in \mathcal{W}_J$ such that $(w(\nu), \alpha_j) \geq 0$ for all $j \in J$. Consider the linear functional given by the inner product $(w(\nu), -)$. If $\mu \in \mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J)$ maximizes $(w(\nu), -)$ in $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J))$, then $w^{-1}\mu$ maximizes φ, so $w^{-1}(\mu) \in F$. In particular, this shows that wF maximizes the inner product $(w(\nu), -)$ and, hence, wF is a face of $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\, M(\lambda, J))$.

Suppose that $(w(\nu), \alpha_i) < 0$ for some $i \in I \setminus J$. Since $i \notin J$, $\lambda - r\alpha_i \in \operatorname{wt} M(\lambda, J)$ for all $r \in \mathbb{Z}_+$. However, $(w(\nu), \lambda - r\alpha_i) = (w(\nu), \lambda) - r(w(\nu), \alpha_i)$ can be arbitrarily large, which contradicts $(w(\nu), -)$ having a finite maximum in $\operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J))$. Thus, $w(\nu)$ must be in the fundamental Weyl chamber.

Write $w(\nu) = \sum_{i \in I} a_i \omega_i$ with $a_i \geq 0$ for all $i \in I$. Let $I_0 = \{i \in I \mid a_i = 0\}$. Recall that $\operatorname{wt} M(\lambda, J) \subset \lambda - Q^+$, so every $\mu \in \operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J))$ can be written in the form $\mu = \lambda - \sum_{i \in I} m_i \alpha_i$ with $m_i \in R^+$. Then,

$$\begin{aligned} (w(\nu), \mu) &= (w(\nu), \lambda - \textstyle\sum_{i \in I} m_i \alpha_i) \\ &= (w(\nu), \lambda) - \textstyle\sum_{i \in I} (w(\nu), \alpha_i) \\ &= (w(\nu), \lambda) - \textstyle\sum_{i \in I} m_i a_i \\ &\leq (w(\nu), \lambda). \end{aligned}$$

This shows that the maximal value of $(w(\nu), -)$ in $\operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J))$ is $(w(\nu), \lambda)$. Furthermore, if $\mu \in \operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J)) \cap (\lambda - Q^+_{I_0})$, then $m_i = 0$ for all $i \notin I_0$, so $(w(\nu), \mu) = (w(\nu), \lambda)$ and $\mu \in wF$. Similarly, if $\mu \notin \operatorname{conv}_{\mathbb{R}}(M(\lambda, J)) \cap (\lambda - Q^+_{I_0})$, then $m_i \neq 0$ for some $i \notin I_0$, so $(w(\nu), \mu) < (w(\nu), \lambda)$ and $\mu \notin wF$. Thus, $wF = \operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J) \cap (\lambda - Q^+_{I_0}))$.

For the converse, let $\rho_{I \setminus I_0} = \sum_{i \in I \setminus I_0} \omega_i$, and consider the linear functional φ given by

$$\varphi(\mu) = (\rho_{I \setminus I_0}, \mu).$$

The subset of $\operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J))$ that maximizes φ is precisely $\operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J) \cap (\lambda - Q^+_{I_0}))$. Hence, $\operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J) \cap (\lambda - Q^+_{I_0}))$ is a face of $\operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J))$.

Suppose that $F \subset \operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J))$ and $wF = \operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J) \cap (\lambda - Q^+_{I_0}))$ for some $w \in \mathcal{W}_J$. Notice that F maximizes the linear functional $\varphi \circ w$ where $\varphi(\mu) = (\rho_{I \setminus I_0}, \mu)$ as above; therefore, F is a face of $\operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J))$ by Lemma 1.2. ■

As a consequence, we obtain information about the set of weights of $M(\lambda, J)$ that lie in a face.

Corollary 2.5. *Let $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$, $J \subset J_\lambda$, and suppose that F is a face of* $\operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J))$*. There exist $w \in \mathcal{W}_J$ and $I_0 \subset I$ such that*

$$w(F \cap \operatorname{wt} M(\lambda, J)) = \operatorname{wt} M(\lambda, J) \cap (\lambda - Q_{I_0}^+).$$

Proof. By Theorem 2.1, there exist $w \in \mathcal{W}_J$ and $I_0 \subset I$, such that

$$wF = \operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J) \cap (\lambda - Q_{I_0}^+)).$$

By Proposition 2.2, $\operatorname{wt} M(\lambda, J)$ is $\mathcal{W}_J$-invariant, so we have

$$wF \cap \operatorname{wt} M(\lambda, J) = w(F \cap \operatorname{wt} M(\lambda, J)),$$

and the corollary follows if we prove that

$$\operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J) \cap (\lambda - Q_{I_0}^+)) \cap \operatorname{wt} M(\lambda, J) = (\lambda - Q_{I_0}^+) \cap \operatorname{wt} M(\lambda, J).$$

The fact that the right-hand side is contained in the left-hand side follows from $\operatorname{wt} M(\lambda, J) \cap (\lambda - Q_{I_0}^+) \subset \operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J) \cap (\lambda - Q_{I_0}^+))$.

For the reverse inclusion, given $\mu \in \operatorname{conv}_{\mathbb{R}}(\operatorname{wt} M(\lambda, J) \cap (\lambda - Q_{I_0}^+)) \cap \operatorname{wt} M(\lambda, J)$, we write

$$\mu = \lambda - \sum_{i \in I} n_i \alpha_i = \sum_{s=1}^{k} a_s(\lambda - \sum_{i \in I_0} m_{si} \alpha_i),$$

where $n_i \in \mathbb{Z}_+$ for $i \in I$, $m_{si} \in \mathbb{Z}_+$ for $1 \leq s \leq k$ and $i \in I_0$, and $a_s \in \mathbb{R}_+$, with $\sum_{s=1}^k a_s = 1$. Simplifying the expression above gives

$$\sum_{i \in I} n_i \alpha_i = \sum_{i \in I_0} (\sum_{s=1}^{k} a_s m_{si}) \alpha_i.$$

Using the linear independence of α_i, $i \in I$, we see that $n_i = 0 \ \forall i \notin I_0$, and, hence, $\mu \in \lambda - Q_{I_0}^+$. The reverse inclusion is proved, and we are done. $\square$

2.5 An Alternative Characterization

Another corollary of the above theorem is

Corollary 2.6. *F is a face of* $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}(M(\lambda, J)))$ *if and only if*

$$F = \mathrm{conv}_{\mathbb{R}}(w(\mathrm{wt}\,\mathbf{U}(\mathfrak{m}_{I_0 \cap J})m_\lambda)) - \mathrm{cone}_{\mathbb{R}}\, w(R^+_{I_0} \setminus R^+_{I_0 \cap J})$$

for some $w \in \mathcal{W}_J$ and $I_0 \subset I$.

Proof. We first prove the following statement (which generalizes Proposition 2.3):

For all $I_0, J \subset I$,

$$\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\,M(\lambda, J) \cap (\lambda - Q^+_{I_0})) = \mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\,\mathbf{U}(\mathfrak{m}_{I_0 \cap J})m_\lambda) - \mathrm{cone}_{\mathbb{R}}(R^+_{I_0} \setminus R^+_{I_0 \cap J}). \quad (2.1)$$

To prove this equation, note that $\mathrm{wt}(M(\lambda, J)) \cap (\lambda - Q^+_{I_0})$ is the same as the set of weights of the $\mathfrak{g}_{I_0}$-submodule $\mathbf{U}(\mathfrak{g}_{I_0})m_\lambda$. Restricting our attention to $\mathfrak{g}_{I_0}$, we see that, as in Proposition 2.2,

$$\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\,\mathbf{U}(\mathfrak{g}_{I_0})m_\lambda) = \mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\,\mathbf{U}(\mathfrak{m}_{I_0 \cap J})m_\lambda) - \left\{ \sum_{\alpha \in R^+_{I_0} \setminus R^+_{I_0 \cap J}} r_\alpha \alpha \ : \ r_\alpha \in \mathbb{R}_+ \right\}.$$

This proves (2.1). Now, by Theorem 2.1, if F is a face, there exist $w \in \mathcal{W}_J$ and $I_0 \subset I$ such that $w^{-1}(F) = \mathrm{conv}_{\mathbb{R}}(\mathrm{wt}(M(\lambda, J)) \cap (\lambda - Q^+_{I_0}))$. The result then follows from (2.1) and the linearity of w. ■

Using Theorem 2.1, we see that the vertices of $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\,M(\lambda, J))$ are the elements of $\mathcal{W}_J(\lambda)$. It is well-known that a polytope is the convex hull of its vertices, so $\mathrm{conv}_{\mathbb{R}}(\mathrm{wt}\,\mathbf{U}(\mathfrak{m}_J)m_\lambda) = \mathrm{conv}_{\mathbb{R}}(\mathcal{W}_J(\lambda))$. Using this, we see that the above corollary is a special case of the following more general result - which also generalizes a result of Vinberg in [19].

Proposition 2.7. *Let $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$, and let $J \subset I$. Suppose $\mathbf{P}(\lambda, J)$ is the polyhedron in $\mathfrak{h}_{\mathbb{R}}^*$ given by*

$$\mathbf{P}(\lambda, J) = \mathrm{conv}_{\mathbb{R}}(\mathcal{W}_J(\lambda)) - \mathrm{cone}_{\mathbb{R}}(R^+ \setminus R_J^+).$$

Then F is a face of $\mathbf{P}(\lambda, J)$ if and only if

$$w(F) = \mathrm{conv}_{\mathbb{R}}(\mathcal{W}_{J\cap I_0}(\lambda')) - \mathrm{cone}_{\mathbb{R}}(R_{I_0}^+ \setminus R_{J\cap I_0}^+),$$

for some $w \in \mathcal{W}_J$ and $I_0 \subset I$, where $\lambda' \in \mathcal{W}_J(\lambda)$ satisfies $\lambda'(h_j) \geq 0 \ \forall \ j \in J$.

Proof. The proof goes through as in the proof of Theorem 2.1 once we note that if $\mu \in \mathbf{P}(\lambda, J)$, then $\lambda' - \mu \in \mathbb{R}_+\Delta$. For this, it suffices to show that $\lambda' - w(\lambda) \in \mathbb{R}_+\Delta$ for all $w \in \mathcal{W}_J$.

Consider the partial order on $\mathfrak{h}_{\mathbb{R}}^*$ given by $\mu \preccurlyeq \nu$ if and only if $\nu - \mu \in \mathbb{R}_+\Delta$. Recall that the intersection of the fundamental Weyl chamber and the Weyl orbit of any nonzero element in $\mathfrak{h}_{\mathbb{R}}^*$ contains exactly one element. In particular, if $w \in \mathcal{W}_J$ and $w(\lambda) \neq \lambda'$, then $w(\lambda)(h_j) < 0$ for some $j \in J$, whence $w(\lambda) \prec s_j(w(\lambda))$.

Since the set $\mathcal{W}_J(\lambda)$ is finite, it must contain a maximal element with respect to the partial order. We have shown that $w(\lambda) \neq \lambda'$ is not maximal, so λ' must be the unique maximal element in $\mathcal{W}_J(\lambda)$ in this partial order. In other words, $\lambda' - w(\lambda) \in \mathbb{R}_+\Delta$ for all $w \in \mathcal{W}_J$. ■

Chapter 3

Irreducible Ad-Nilpotent Ideals

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, and let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$ be its Levi decomposition, where $\mathfrak{m}$ is the reductive part and $\mathfrak{u}$ is the unipotent radical of $\mathfrak{p}$. By restricting the adjoint action of $\mathfrak{g}$ to $\mathfrak{m}$, an $\mathfrak{m}$-module structure may be induced on $\mathfrak{u}$. The center $\mathfrak{z}$ of $\mathfrak{m}$ acts semisimply on $\mathfrak{u}$. An unpublished result of Kostant (a proof can be found in [11] and [20]) shows that the distinct $\mathfrak{z}$-eigenspaces of $\mathfrak{u}$ are irreducible $\mathfrak{m}$-modules. To describe these irreducible $\mathfrak{m}$-modules, choose a positive root system R^+ and a proper subset $J \subset I$ of simple roots such that

$$\mathfrak{p} = \mathfrak{h} \oplus_{\alpha \in R^+} \mathfrak{g}_\alpha \oplus_{\alpha \in R_J^+} \mathfrak{g}_{-\alpha},$$

$$\mathfrak{m} = \mathfrak{h} \oplus_{\alpha \in R_J^+} \mathfrak{g}_{\pm\alpha},$$

$$\mathfrak{u} = \oplus_{\alpha \in R^+ \setminus R_J^+} \mathfrak{g}_\alpha.$$

Define an equivalence relation $\sim$ on R^+ by $\alpha \sim \beta$ if and only if $\alpha - \beta \in Q_J$ where Q_J is the $\mathbb{Z}$-span of Δ_J. An irreducible $\mathfrak{m}$-submodule of $\mathfrak{u}$ is the direct sum of root spaces which lie in a fixed equivalence class. The following is a standard simple exercise, but a proof is provided for completeness.

Lemma 3.1. *There exists a unique ad-nilpotent ideal $\mathfrak{i}_0$ of $\mathfrak{p}$ which is irreducible as an*

$\mathfrak{m}$*-module. In fact,* $\mathfrak{i}_0 = \bigoplus_{\alpha\sim\theta} \mathfrak{g}_\alpha$ *and* $\mathfrak{i}_0$ *is abelian.*

Proof. Assume we have fixed R^+ and J as in the discussion preceding the Lemma. Let $\alpha, \gamma \in R^+$ with $\alpha+\gamma \in R^+$. If $\alpha \sim \theta$, then $\gamma \in R_J^+$ since $\theta - (\alpha+\gamma) \in Q^+$. Hence, $\alpha + \gamma \sim \theta$. It also follows that if $\alpha \sim \theta$ and $\gamma \sim \theta$, then $\alpha+\gamma \notin R^+$ since $\theta - (\alpha+\gamma) = (\theta-\alpha) + (\theta-\gamma) - \theta \notin Q^+$. Furthermore, if $\alpha \sim \theta$ and $\beta \in R_J^+$, then $\theta - (\alpha - \beta) \in Q_J$ so $\alpha - \beta \sim \theta$, which shows that $\mathfrak{i}_0$ is an abelian ideal of $\mathfrak{p}$. The fact that $\mathfrak{i}_0$ is an irreducible $\mathfrak{m}$-module is a special case of the result of Kostant mentioned before the Lemma.

If $\mathfrak{i}$ is any other ad-nilpotent ideal of $\mathfrak{p}$, then $x_\theta \in \mathfrak{i}$. Hence, $\mathfrak{i}$ contains the irreducible $\mathfrak{m}$-module generated by x_θ; i.e., $\mathfrak{i}_0 \subset \mathfrak{i}$.

□

We call $\mathfrak{i}_0$ the irreducible ad-nilpotent ideal of $\mathfrak{p}$. In other words, given a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{h}$, a certain set of roots, namely the roots that determine the irreducible ad-nilpotent ideal, can be associated canonically to $\mathfrak{p}$. It is not difficult to see that different parabolic subalgebras could give rise to the same set of roots. For example, let $\mathfrak{g}$ be of type A_5. Suppose that R^+ is the standard set of positive roots, and let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be the parabolic subalgebras given by $J = \{1,5\}$ and $J = \{1,3,5\}$ respectively. For both $\mathfrak{p}_1$ and $\mathfrak{p}_2$, the set of roots that determine the irreducible ad-nilpotent ideal is $\{\theta, \theta-\alpha_1, \theta-\alpha_5, \theta-\alpha_1-\alpha_5\}$.

In this chapter, we give a necessary and sufficient condition for a subset $S \subset R$ to determine the unique irreducible ad-nilpotent ideal $\mathfrak{i}_S$ in some parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. As a corollary, we prove that given such a subset, there exists a parabolic subalgebra $\mathfrak{p}_S$ whose unique irreducible ad-nilpotent ideal is given by S and, if $\mathfrak{p}$ is any other parabolic subalgebra whose unique irreducible ad-nilpotent ideal is given by S, then $\mathfrak{p} \subset \mathfrak{p}_S$. As a further consequence of our result, we write down explicitly all subsets of R of the classical simple finite-dimensional Lie algebras which satisfy this condition. For the exceptional simple finite-dimensional Lie algebras, the sets which satisfy this condition are written explicitly up to Weyl-conjugation.

The following theorem follows easily from our discussion of weak $\mathbb{F}$-faces:

Theorem 7. *Let $S \subset R$. The following are equivalent:*

(i) *S is a positive weak $\mathbb{F}$-face in R.*

(ii) *$wS = (\theta - Q_J^+) \cap R$ for some $w \in \mathcal{W}$ and $J \subsetneq I$.*

(iii) *S maximizes some linear functional inside R; i.e.,*

$$S = \{\alpha \in R \mid (\mu, \alpha) \geq (\mu, \beta) \ \forall \ \beta \in R\}$$

for some $\mu \in P$ such that $(\mu, \alpha) > 0$ for some $\alpha \in R$.

(iv) *$S = \{\alpha \in R \mid (\rho_S, \alpha) \geq (\rho_S, \beta) \ \forall \ \beta \in R\}$, where $2\rho_S = \sum_{\alpha \in S} \alpha$ and $\max \rho_S > 0$.*

(v) *For all $\alpha, \beta \in S$, $\alpha + \beta \notin R \cup \{0\}$, and if $\gamma \in R \setminus S$, then $(\gamma + R) \cap (S + S) = \emptyset$.*

Before proving the theorem, we establish the following corollary. Given a non-zero $\lambda \in P$, let $\mathfrak{p}_\lambda$ be the corresponding parabolic subalgebra of $\mathfrak{g}$; i.e.,

$$\mathfrak{p}_\lambda = \mathfrak{m}_\lambda \oplus \mathfrak{u}_\lambda, \quad \mathfrak{m}_\lambda = \mathfrak{h} \bigoplus_{\{\alpha \mid (\lambda, \alpha) = 0\}} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{u}_\lambda = \bigoplus_{\{\alpha \mid (\lambda, \alpha) > 0\}} \mathfrak{g}_\alpha.$$

Corollary 3.2. *Let $S \subset R$ satisfy the equivalent conditions from Theorem 7. Then,*

(i) *The unique irreducible ideal $\mathfrak{p}_{\rho_S}$ is given by S.*

(ii) *If $\mathfrak{p}$ is another parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ whose unique irreducible* ad-*nilpotent ideal is given by S, then $\mathfrak{p} \subset \mathfrak{p}_{\rho_S}$.*

Proof. Let $S = \{\alpha \in R \mid (\rho_S, \alpha) = \max \rho_S\}$. To prove (i), it suffices by Lemma 3.1 to show that

$$\mathfrak{i}_{\rho_S} := \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$$

is an ad-nilpotent ideal in $\mathfrak{p}_{\rho_S}$ which is an irreducible $\mathfrak{m}_{\rho_S}$-module.

Suppose that $(\rho_S, \alpha) = \max \rho_S$ and $(\rho_S, \beta) > 0$. Then, $\alpha + \beta \notin R$ since $(\rho_S, \alpha + \beta) > (\rho_S, \alpha)$, which shows that $[\mathfrak{u}_{\rho_S}, \mathfrak{i}_{\rho_S}] = 0$. On the other hand, if $(\rho_S, \beta) = 0$ and $\alpha \pm \beta \in R$, then $(\rho_S, \alpha \pm \beta) = (\rho_S, \alpha) = \max \rho_S$, and, hence, $[\mathfrak{m}_{\rho_S}, \mathfrak{i}_{\rho_S}] \subset \mathfrak{i}_{\rho_S}$. In particular, $\mathfrak{i}_{\rho_S}$ is an ad-nilpotent ideal in $\mathfrak{p}_{\rho_S}$.

Choose a positive root system R^+ so that ρ_S is dominant with respect to R^+, in which case, $S \subset R^+$ and $\theta \in S$. To show that $\mathfrak{i}_{\rho_S}$ is irreducible, suppose that $\alpha \in S$ and let $\theta - \alpha = \sum_{j=1}^{k} \beta_j$ with $\beta_j \in R^+$ for all j. Since $\theta \in S$ and $\alpha \in S$, we have $(\rho_S, \theta - \alpha) = 0$, so $(\rho_S, \beta_j) = 0$ for all $1 \leq j \leq k$. In other words, $\alpha \sim \theta$, and, hence, $\mathfrak{i}_{\rho_S}$ is the unique irreducible ad-nilpotent ideal in $\mathfrak{p}_{\rho_S}$.

To prove (ii), suppose that $\mathfrak{p}$ is a parabolic subalgebra such that $\mathfrak{i}_{\rho_S}$ is the unique irreducible ad-nilpotent ideal in $\mathfrak{p}$. It suffices to show that $(\rho_S, \alpha) \geq 0$ for all $\alpha \in R$ such that $\mathfrak{g}_\alpha \subset \mathfrak{p}$ and that $(\rho_S, \alpha) = 0$ if $\mathfrak{g}_{\pm\alpha} \subset \mathfrak{p}$. If $\alpha \in R$ and $(\beta, \alpha) \geq 0$ for all $\beta \in S$, then $(\rho_S, \alpha) \geq 0$. Suppose that $\mathfrak{g}_\alpha \subset \mathfrak{p}$ and $(\beta, \alpha) < 0$ for some $\beta \in S$. Since $\beta + \alpha \in R$, $\mathfrak{g}_{\beta+\alpha} \subset \mathfrak{i}_{\rho_S}$ since $\mathfrak{i}_{\rho_S}$ is an ad-nilpotent ideal of $\mathfrak{p}$. In particular, this gives that $\beta + \alpha \in S$, so

$$(\rho_S, \beta + \alpha) = (\rho_S, \beta) = \max \rho_S$$

which implies that $(\rho_S, \alpha) = 0$. Finally, suppose that $\mathfrak{g}_{\pm\alpha} \subset \mathfrak{p}$. Then, either $(\beta, \alpha) = 0$ for all $\beta \in S$ or there exists a $\beta \in S$ such that either $(\beta, \alpha) < 0$ or $(\beta, -\alpha) < 0$. In either case, we have shown that $(\rho_S, \alpha) = 0$.

$\square$

The remainder of this chapter is devoted to proving Theorem 7. The equivalence of the first three conditions follows from Corollary 2.5 and Theorem 6. It is trivial that (*iv*) implies (*iii*), and, using Theorem 5, it follows that (*i*) implies (*v*). It now only remains to show that (*v*) implies (*iv*). To prove this implication, we shall need the following result. Note that $(\alpha, \beta) \geq 0$ for all $\alpha, \beta \in S$ since $\alpha + \beta \notin R$.

Lemma 3.3. *Let $\alpha, \beta \in R$. Assume that β is a long root and $(\alpha, \beta) = 0$. There exists $\gamma, \gamma' \in R$ with $\gamma \notin \{\alpha, \beta\}$ such that $\alpha + \beta = \gamma + \gamma'$.*

Proof. Let $\{\alpha_1 : i \in I\}$ be the set of simple roots, and assume without loss of generality that α is simple, say $\alpha = \alpha_{i_0}$. Set $I_0 = \{i_0\}$. For $k \geq 1$, define subsets I_k recursively by

$$I_k = \{i \in I : (\alpha_i, \alpha_j) < 0 \text{ for some } j \in I_{k-1}\}.$$

Since $\mathfrak{g}$ is simple, $I = \bigcup_{k \geq 0} I_k$. Choose m minimal such that there exists $i_m \in I_m$ with $(\beta, \alpha_{i_m}) \neq 0$. Since $i_m \notin I_{m-1}$ we pick $i_{m-1} \in I_{m-1}$ with $(\alpha_{i_m}, \alpha_{i_{m-1}}) < 0$. Again, $i_{m-1} \notin I_{m-2}$ since otherwise we would have $i_m \in I_{m-1}$. In other words, we can choose elements $i_k \in I_k \backslash I_{k-1}$, with $1 \leq k < m$ such that $(\beta, \alpha_{i_k}) = 0$ and$(\alpha_{i_k}, \alpha_{i_{k+1}}) < 0$. Hence $\alpha_{i_p} + \cdots + \alpha_{i_k} \in R$ for all $0 \leq p \leq k \leq m$. If $(\beta, \alpha_{i_m}) < 0$, then $(\beta, \alpha_{i_p} + \cdots + \alpha_{i_m}) < 0$ for all $0 \leq p \leq m$ and so $\gamma = \alpha_{i_0} + \alpha_{i_1} + \cdots + \alpha_{i_m} + \beta \in R$. Since $\gamma - \alpha_{i_0}, \gamma - \beta \in R$, they are in particular non–zero and the Lemma follows from

$$\alpha_{i_0} + \beta = (\alpha_{i_0} + \alpha_{i_1} + \cdots + \alpha_{i_m} + \beta) - (\alpha_{i_1} + \cdots + \alpha_{i_m}).$$

If $(\beta, \alpha_{i_m}) > 0$, then $\gamma = \alpha_{i_0} + \alpha_{i_1} + \cdots + \alpha_{i_m} \in R$ and also $\gamma' = \beta - (\alpha_{i_1} + \cdots + \alpha_{i_m}) \in R$. It remains to prove that $\gamma \notin \{\alpha, \beta\}$. It is clear that $\gamma \neq \alpha_{i_0}$ since $\gamma - \alpha_{i_0} \in R$. Suppose that $\gamma = \beta$. Then we have

$$(\beta, \beta) = (\beta, \alpha_{i_0} + \alpha_{i_1} + \cdots + \alpha_{i_m}) = (\beta, \alpha_{i_m}).$$

Since β is a long root this implies that we must have $\beta = \alpha_{i_m} = \gamma$. But this is impossible since $\alpha_{i_0} + \alpha_{i_1} + \cdots + \alpha_{i_{m-1}} \in R$ and the proof of the Lemma is complete.

$\square$

We first prove that (iv) implies (i) when S is a subset of the set of long roots in R. For $\alpha \in S$, set

$$\alpha^{\perp} = \{\gamma \in S : (\alpha, \gamma) = 0\}.$$

We claim that for all $\alpha, \beta \in S$, we have,

$$\#\alpha^{\perp} = \#\beta^{\perp}. \tag{3.1}$$

Assuming the claim, the proof that $S = S(\rho_S)$ is completed as follows. Since S consists of long roots, we assume that $(\delta, \delta) = 2$ if $\delta \in S$ and also that $(\delta, \beta) \leq 1$ for all $\beta \in R \backslash \{\delta\}$. If $\gamma \in S$ and $\gamma \neq \delta$, then we have $0 \leq (\gamma, \delta) \leq 1$ and so

$$(\rho_S, \gamma) = \sum_{\delta \in S} (\delta, \gamma) = \#S + 1 - \#\gamma^{\perp}, \quad \gamma \in S.$$

Suppose now that $\gamma' \notin S$. If $(\gamma, \delta) \leq 0$ for all $\delta \in S$, then $(\rho_S, \gamma) \leq 0$ and hence $\gamma' \notin S(\rho_S)$. Suppose that $(\gamma', \delta) > 0$ for some $\delta \in S$. If $\delta_1 \in \delta^{\perp}$ is such that $(\delta_1, \gamma') > 0$ then we would have that $\delta_1 - \gamma' + \delta \in R$. Since

$$\delta_1 + \delta = (\delta_1 - \gamma' + \delta) + \gamma$$

it would follow that $\gamma' \in S$ which is a contradiction. Hence, $(\gamma', \delta^{\perp}) = 0$. But now, we have

$$(\rho_S, \gamma') = \sum_{\delta_1 \in \delta^{\perp}} (\delta_1, \gamma') + \sum_{\delta_1 \in S \backslash \delta^{\perp})} (\delta_1, \gamma') \leq \#S - \#\delta^{\perp} < \#S + 1 - \#\delta^{\perp} = \rho_S(\delta).$$

Hence, $\gamma' \notin S(\rho_S)$ proving that $S = S(\rho_S)$.

It suffices to prove (3.1) when $(\alpha, \beta) > 0$, for, if $(\alpha, \beta) = 0$, then by Lemma 3.3, we can choose $\gamma, \gamma' \in R$ such that $\alpha + \beta = \gamma + \gamma'$. By the conditions on S, this means that

$\gamma, \gamma' \in S$. Using the fact that S consists of long roots, we see also that

$$(\alpha, \gamma) + (\beta, \gamma) = (\gamma, \gamma) + (\gamma, \gamma') \geq 2,$$

which in turn implies that $(\alpha, \gamma) > 0$ and $(\beta, \gamma) > 0$, which gives

$$\#\alpha^{\perp} = \#\gamma^{\perp} = \#\beta^{\perp}.$$

Assume now that $(\alpha, \beta) > 0$. Let s_α, s_β be the reflections in W corresponding to the roots α and β. It suffices to prove that

$$s_\alpha s_\beta(\alpha^{\perp}) \subset \beta^{\perp}. \tag{3.2}$$

If $\gamma \in \alpha^{\perp}$, it is easy to calculate that

$$s_\alpha s_\beta(\gamma) = \begin{cases} \gamma, & (\gamma, \beta) = 0, \\ \gamma - \beta + \alpha, & (\gamma, \beta) = 1. \end{cases}.$$

In the first case, $\gamma \in S(\beta)$. In the second case we have $\alpha + \gamma = (\gamma - \beta + \alpha) + \beta$ which implies that $\gamma - \beta + \alpha \in S$. Since S consists of long roots, we get $(\gamma - \beta + \alpha, \beta) = 0$ and (3.2) follows.

It remains to consider the case where S contains a short root. It suffices to consider the case when S contains the highest short root since for each $w \in \mathcal{W}$, wS satisfies (v) from Theorem 7 if and only if S does. Suppose first that $\mathfrak{g}$ is of type G_2, and let α_1, α_2 be the simple roots with α_2 being short. Assume that $\alpha_1 + 2\alpha_2 \in S$. Since $2(\alpha_1 + 2\alpha_2) = (2\alpha_1 + 3\alpha_2) + \alpha_2$ and $2\alpha_1 + 3\alpha_2 \in R$, it follows that $\alpha_2 \in S$, but this is impossible since $\alpha_1 + 3\alpha_2 \in R$. Hence, the result is vacuously true for G_2. Suppose now that $\mathfrak{g}$ is of type F_4 and suppose that S contains the highest short root. Then, a similar argument proves

$$S = R \cap (\theta - Q^+_{\{1,2,3\}}), \quad \rho_S = 7\omega_4.$$

In this section, we explicitly write down all of the sets that satisfy the equivalent conditions of Theorem 7 for the classical simple finite-dimensional Lie algebras. For an explicit description of these sets (up to Weyl conjugation) in the case of the exceptional Lie algebras, see Appendix. For the purposes of the following proposition, it is convenient to use a slightly different notation for the set of roots. Let $\{\epsilon_i \mid 1 \le i \le n+1\}$ be an orthonormal basis for $\mathbb{R}^{n+1}$. For $\mathfrak{g}$ of type A_n, we set $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \le i \le n$. For $\mathfrak{g}$ of types B_n, C_n, and D_n, set $\alpha_i = \epsilon_i - \epsilon_{i+1}$ if $1 \le i \le n-1$. Finally, we set $\alpha_n = \epsilon_n$, $\alpha_n = 2\epsilon_n$ and $\alpha_n = \epsilon_{n-1} + \epsilon_n$ for $\mathfrak{g}$ of types B_n, C_n and D_n, respectively.

Proposition 3.4. *Let $S \subset R$ be such that if $\alpha, \beta \in S$, then $\alpha + \beta \notin R \cup \{0\}$ and if $\gamma \in R \setminus S$, then $(\gamma + R) \cap (S + S) = \emptyset$.*

(i) *If $\mathfrak{g}$ is of type A_n, then there exist disjoint subsets $\mathbf{i}$, $\mathbf{j}$ of $\{1, \ldots, n+1\}$ such that*

$$S = \{\epsilon_i - \epsilon_j : i \in \mathbf{i}, j \in \mathbf{j}\} \qquad \rho_S = |\mathbf{j}| \sum_{p=1}^{|\mathbf{i}|} \epsilon_{i_p} - |\mathbf{i}| \sum_{p=1}^{|\mathbf{j}|} \epsilon_{j_p}.$$

(ii) *If $\mathfrak{g}$ is of type C_n, there exist disjoint subsets $\mathbf{i}$ and $\mathbf{j}$ of $\{1, \ldots, n\}$ such that*

$$S = \{\epsilon_{i_1} + \epsilon_{i_2} : i_1, i_2 \in \mathbf{i}\} \cup \{-(\epsilon_{j_1} + \epsilon_{j_2}) : j_1, j_2 \in \mathbf{j}\} \cup \{\epsilon_i - \epsilon_j : i \in \mathbf{i}, j \in \mathbf{j}\},$$
$$\rho_S = (|\mathbf{i}| + |\mathbf{i}| + 1) \left(\sum_{i \in \mathbf{i}} \epsilon_i - \sum_{j \in \mathbf{j}} \epsilon_j \right).$$

(iii) *If $\mathfrak{g}$ is of type B_n, then either*

(a) there exist disjoint subset $\mathbf{i}$ and $\mathbf{j}$ of $\{1, \ldots, n\}$ such that

$$S = \{\epsilon_i - \epsilon_j : i \in \mathbf{i}, j \in \mathbf{j}\} \cup \{\epsilon_k + \epsilon_\ell : k, \ell \in \mathbf{i}, k \neq \ell\} \cup \{-(\epsilon_k + \epsilon_\ell) : k, \ell \in \mathbf{j}, k \neq \ell\},$$
$$\rho_S = (|\mathbf{i}| + |\mathbf{i}| - 1) \left(\sum_{i \in \mathbf{i}} \epsilon_i - \sum_{j \in \mathbf{j}} \epsilon_j \right),$$

(b) or there exists $i \in I$ such that

$$\pm S = \{\epsilon_i\} \cup \{\epsilon_i \pm \epsilon_j : i \neq j, 1 \le j \le n\}, \rho_S = \pm(2n-1)\epsilon_i.$$

(iv) *If $\mathfrak{g}$ is of type D_n, then either*

(a) there exist disjoint subset $\mathbf{i}$ *and* $\mathbf{j}$ *of* $\{1, \ldots, n\}$ *such that*

$$S = \{\epsilon_i - \epsilon_j : i \in \mathbf{i}, j \in \mathbf{j}\} \cup \{\epsilon_k + \epsilon_\ell : k, \ell \in \mathbf{i}, k \neq \ell\} \cup \{-(\epsilon_k + \epsilon_\ell) : k, \ell \in \mathbf{j}, k \neq \ell\},$$
$$\rho_S = (|\mathbf{i}| + |\mathbf{i}| - 1) \left(\sum_{i \in \mathbf{i}} \epsilon_i - \sum_{j \in \mathbf{j}} \epsilon_j \right),$$

(b) or there exists $i \in I$ such that

$$\pm S = \{\epsilon_i \pm \epsilon_j : i \neq j, 1 \leq j \leq n\}, \rho_S = \pm(2n-2)\epsilon_i.$$

Proof. The fact that these are the sets S that satisfy the condition follows from a case by case consideration of the set of roots. Similarly, in each case, it is a simple task to check that $S = \{\alpha \in R \mid (\rho_S, \alpha) = \max \rho_S\}$. For instance, if $\mathfrak{g}$ is of type A_n, for each $\alpha \in S$, $(\rho_S, \alpha) = |\mathbf{i}| + |\mathbf{j}|$. If $\alpha = \epsilon_k - \epsilon_\ell \notin S$, then either $k \notin \mathbf{i}$ or $\ell \notin \mathbf{j}$ in which case, $(\rho_S, \alpha) < |\mathbf{i}| + |\mathbf{j}|$.

□

Part II

Koszul algebras

Chapter 4

Faces and Koszul Algebras

4.1 The main results

For this chapter, fix a complex semisimple Lie algebra $\mathfrak{g}$. Also fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, a base of simple roots $\Delta := \{\alpha_i : i \in I\}$ inside the root system $R = R^+ \coprod -R^+ \subset \mathfrak{h}^*$ and Weyl group W (with longest element $w_\circ$). Let $P \supset Q$ denote the weight and root lattices respectively, and P^+ the set of dominant integral weights. Finally, fix a finite-dimensional $\mathfrak{g}$-module V that is not a direct sum of trivial $\mathfrak{g}$-modules (i.e., $\mathrm{wt}(V) \neq \{0\}$.) We now imitate some of the constructions in [6].

Definition. (i) *Define* $\mathbf{V} := \bigoplus_{\mu \in P^+} V(\mu)$, $\mathbf{V}^{\circledast} := \bigoplus_{\mu \in P^+} V(\mu)^*$, *where* $V(\mu)$ *is the finite-dimensional simple* $\mathfrak{g}$*-module of highest weight* μ*.*

(ii) *Given an associative* $\mathbb{C}$*-algebra* A*, define* $\mathbf{A} := A \otimes \mathbf{V}^{\circledast} \otimes \mathbf{V}$*. We will freely identify* $\mathbf{A}$ *with* $\mathbf{V}^{\circledast} \otimes A \otimes \mathbf{V}$ *without further mention.*

The next result is taken from [6]. We need it to present our final main result for $A = \mathrm{Sym}(V)$.

Lemma 4.1.

(i) $\mathbf{A}$ *is an associative algebra under the multiplication given by linearly extending* $(a \otimes f \otimes v)(b \otimes g \otimes w) := g(v)ab \otimes f \otimes w$.

(ii) *If* $A = \bigoplus_{k \geq 0} A[k]$ *is a grading on* A, *then* $\mathbf{A} = \bigoplus_{k \geq 0} A[k] \otimes \mathbf{V}^{\circledast} \otimes \mathbf{V}$ *is a grading on* $\mathbf{A}$.

(iii) *If* A *is a* $\mathbb{Z}_+$*-graded* $\mathfrak{g}$*-module algebra such that* $\mathfrak{g}(A[k]) \subset A[k]$ $\forall k \geq 0$, *then* $\mathbf{A}$ *has the same property.*

(iv) *The natural embedding* $\mathbf{V}^{\circledast} \otimes \mathbf{V} \to \operatorname{End} \mathbf{V}$ *of* $\mathfrak{g}$*-modules given by extending* $f \otimes v \mapsto (w \mapsto f(w)v)$ *is an anti-homomorphism of algebras. Given* $\mu \in P^+$, *it restricts to an isomorphism of algebras* $V(\mu)^* \otimes V(\mu) \to (\operatorname{End} V(\mu))^{op}$.

The natural embedding $\mathbf{V}^{\circledast} \otimes \mathbf{V} \to \operatorname{End} \mathbf{V}^{\circledast}$ *given by* $f \otimes v \mapsto (g \mapsto g(v)f)$ *is a homomorphism of algebras. Given* $\mu \in P^+$, *it restricts to an isomorphism of algebras* $V(\mu)^* \otimes V(\mu) \to \operatorname{End} V(\mu)^*$*).*

(v) *Under this isomorphism, the preimage of the identity element in* $(\operatorname{End} V(\mu))^{op}$ *or* $\operatorname{End} V(\mu)^*$ *is the canonical* $\mathfrak{g}$*-invariant element* $1_\mu \in V(\mu)^* \otimes V(\mu)$.

(vi) *For* $\mu, \nu \in P^+$, *we have*

$$1_\mu 1_\nu = \delta_{\mu,\nu}, \qquad 1_\mu(\mathbf{V}^{\circledast} \otimes \mathbf{V}) = V(\mu)^* \otimes \mathbf{V}, \qquad (\mathbf{V}^{\circledast} \otimes \mathbf{V})1_\mu = \mathbf{V}^{\circledast} \otimes V(\mu).$$

(vii) *Let* 1_μ *denote* $1_A \otimes 1_\mu \in \mathbf{A}$. *Then,* $1_\mu \mathbf{A} 1_\nu = A \otimes V(\mu)^* \otimes V(\nu)$.

Proof. (i) This follows from a straightforward calculation:

$$\begin{aligned} ((a_1 \otimes f_1 \otimes v_1)(a_2 \otimes f_2 \otimes v_2))(a_3 \otimes f_3 \otimes v_3) &= (f_2(v_1)a_1a_2 \otimes f_1 \otimes v_2)(a_3 \otimes f_3 \otimes v_3) \\ &= f_3(v_2)f_2(v_1)a_1a_2a_3 \otimes f_1 \otimes v_3. \end{aligned}$$

Similarly,

$$\begin{aligned}(a_1 \otimes f_1 \otimes v_1)((a_2 \otimes f_2 \otimes v_2)(a_3 \otimes f_3 \otimes v_3)) &= (a_1 \otimes f_1 \otimes v_1)(f_3(v_2)a_2a_3 \otimes f_2 \otimes v_3) \\ &= f_3(v_2)f_2(v_1)a_1a_2a_3 \otimes f_1 \otimes v_3.\end{aligned}$$

(ii) This follows immediately from the definition of the multiplication.

(iii) $\mathfrak{g}(\mathbf{A}[k]) \subset (\mathfrak{g}.A[k]) \otimes \mathbf{V}^{\circledast} \otimes \mathbf{V}) + A[k] \otimes \mathfrak{g}.(\mathbf{V}^{\circledast}) \otimes \mathbf{V} + A[k] \otimes \mathbf{V}^{\circledast} \otimes \mathfrak{g}.\mathbf{V} \subset A[k] \otimes \mathbf{V}^{\circledast} \otimes \mathbf{V} = \mathbf{A}[k]$.

(iv) Let $\varphi_1 : \mathbf{V}^{\circledast} \otimes \mathbf{V} \to \operatorname{End} \mathbf{V}$ be the natural embedding. It suffices to show that $\varphi_1((f_1 \otimes v_1)(f_2 \otimes v_2)) = \varphi_1(f_2 \otimes v_2) \circ \varphi_1(f_1 \otimes v_1)$ for $f_1, f_2 \in \mathbf{V}^{\circledast}$ and $v_1, v_2 \in \mathbf{V}$. Let $w \in \mathbf{V}$. Then,

$$\begin{aligned}\varphi_1((f_1 \otimes v_1)(f_2 \otimes v_2))(w) &= \varphi_1(f_2(v_1)f_1 \otimes v_2)(w) \\ &= f_2(v_1)f_1(w)v_2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi_1(f_2 \otimes v_2) \circ \varphi_1(f_1 \otimes v_1)(w) &= \varphi_1(f_2 \otimes v_2)(f_1(w)v_1) \\ &= f_2(f_1(w)v_1)v_2 \\ &= f_1(w)f_2(v_1)v_2.\end{aligned}$$

Thus, the natural embedding $\mathbf{V}^{\circledast} \otimes \mathbf{V} \to \operatorname{End} \mathbf{V}$ is an anti-homomorphism of algebras.

Similarly, let $\varphi_2 : \mathbf{V}^{\circledast} \otimes \mathbf{V} \to \operatorname{End} \mathbf{V}^{\circledast}$ be the natural embedding. It suffices to show that $\varphi_2((f_1 \otimes v_1)(f_2 \otimes v_2)) = \varphi_2(f_1 \otimes v_1) \circ \varphi_2(f_2 \otimes v_2)$ for $f_1, f_2 \in \mathbf{V}^{\circledast}$ and $v_1, v_2 \in \mathbf{V}$. Let $g \in \mathbf{V}^{\circledast}$. Then,

$$\begin{aligned}\varphi_2((f_1 \otimes v_1)(f_2 \otimes v_2))(g) &= \varphi_2(f_2(v_1)f_1 \otimes v_2)(g) \\ &= g(v_2)f_2(v_1)f_1\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi_2(f_1\otimes v_1)\circ\varphi_2(f_2\otimes v_2)(g) &= \varphi_2(f_1\otimes v_1)(g(v_2)f_2) \\ &= g(v_2)f_2(v_1)f_1\end{aligned}.$$

Thus, the natural embedding $\mathbf{V}^{\circledast}\otimes\mathbf{V}\to\operatorname{End}\mathbf{V}^{\circledast}$ is a homomorphism of algebras.

(v) Let $id_{V(\mu)}\in(\operatorname{End}V(\mu))^{op}$ be the identity morphism. Note that $(g.id_{V(\mu)})(v) = g.(id_{V(\mu)}(v)) - id_{V(\mu)}(g.v) = g.v - g.v = 0$ for all $g\in\mathfrak{g}$. In particular, $id_{V(\mu)}$ is $\mathfrak{g}$-invariant. The preimage of $id_{V(\mu)}$ in $V(\mu)^*\otimes V(\mu)$ under the isomorphism must also be $\mathfrak{g}$-invariant.

(vi) Observe that $(f\otimes v)(g\otimes w)=0$ if $v\in V(\mu)$, $g\in V(\nu)^*$ with $\mu\neq\nu$. This shows that $1_\mu 1_\nu=0$ if $\mu\neq\nu$. Under the isomorphism, $1_\mu 1_\mu$ maps to the composition $id_{V(\mu)}\circ id_{V(\mu)} = id_{V(\mu)}$. Since 1_μ is the preimage of the identity element in $(\operatorname{End}V(\mu))^{op}$ by definition, this shows that $1_\mu 1_\mu = 1_\mu$.

Write $1_\mu=\sum_i \xi_i\otimes u_i$ with $\xi_i\in V(\mu)^*$ and $u_i\in V(\mu)$. It follows that $1_\mu(f\otimes v)=0$ if $f\notin V(\mu)^*$, and, similarly, $(f\otimes v)1_\mu=0$ if $v\notin V(\mu)$.

Note that, for any $f\in V(\mu)^*$, $\sum_i f(u_i)\xi_i = id_{V(\mu)^*}(f) = f$ by the isomorphism $V(\mu)^*\otimes V(\mu)\to\operatorname{End}V(\mu)^*$. Similarly, for any $v\in V(\mu)$, $\sum_i\xi_i(v)u_i = id_{V(\mu)}(v)=v$ by the isomorphism $V(\mu)^*\otimes V(\mu)\to(\operatorname{End}V(\mu))^{op}$. In particular, $1_\mu(f\otimes v) = \sum_i f(u_i)\xi_i\otimes v = f\otimes v$ for all $f\in V(\mu)^*$, $v\in\mathbf{V}$, and $(g\otimes w)1_\mu = g\otimes\sum_i\xi_i(w)u_i = g\otimes w$ for all $g\in\mathbf{V}^{\circledast}$, $w\in V(\mu)$.

Finally, note that $\mathbf{V}^{\circledast}\otimes\mathbf{V}=\sum_{\lambda\in P^+}V(\lambda)^*\otimes\mathbf{V}=\mathbf{V}^{\circledast}\otimes\sum_{\nu\in P^+}V(\mu)$. The result now follows.

(vii) This follows from (vi):

$$\begin{aligned} 1_\mu \mathbf{A} 1_\nu &= ((1_A \otimes 1_\mu)(A \otimes \mathbf{V}^{\circledast} \otimes \mathbf{V}))(1_A \otimes 1_\nu) \\ &= (A \otimes V(\mu)^* \otimes \mathbf{V})(1_A \otimes 1_\nu) \\ &= A \otimes V(\mu)^* \otimes V(\nu). \end{aligned}$$

□

Definition. *1. Suppose $\Psi \subset \mathrm{wt}(V)$ is nonempty.*

(a) Define $\leq_\Psi$ via $\mu \leq_\Psi \nu$ in $\mathfrak{h}^* \Leftrightarrow \nu - \mu \in \mathbb{Z}_+ \Psi$, and define

$$d_\Psi(\mu, \nu) := \min\{\sum_{\beta \in \Psi} m_\beta : \nu - \mu = \sum_{\beta \in \Psi} m_\beta \beta,\ m_\beta \in \mathbb{Z}_+\ \forall \beta\}.$$

(b) Given $\mu, \nu \in P^+$, define $[\mu, \nu]_\Psi := (\leq_\Psi \nu) \cap (\mu \leq_\Psi)$, where

$$\leq_\Psi \nu := \{\eta \in P^+ : \eta \leq_\Psi \nu\}, \qquad \mu \leq_\Psi := \{\eta \in P^+ : \mu \leq_\Psi \eta\}.$$

(c) Define $N_\Psi := \sum_{\mu \in \Psi} \dim(V)_\mu$.

2. If A is $\mathbb{Z}_+$-graded and $\nu \leq_\Psi \mu$ in P^+, define

$$\mathbf{A}_\Psi(\mu, \nu) := 1_\mu \mathbf{A}[d_\Psi(\nu, \mu)] 1_\nu,$$

and for $F \subset P^+$, define $\mathbf{A}_\Psi(F) := \bigoplus_{\nu \leq_\Psi \mu \in F} \mathbf{A}_\Psi(\mu, \nu)$.

3. If, in addition, A is a $\mathfrak{g}$-module algebra such that $\mathfrak{g}(A[k]) \subset A[k]$ $\forall k \geq 0$, then define

$$\mathbf{A}_\Psi^{\mathfrak{g}}(\mu, \nu) := (\mathbf{A}_\Psi(\mu, \nu))^{\mathfrak{g}}, \quad \mathbf{A}_\Psi^{\mathfrak{g}}(F) := (\mathbf{A}_\Psi(F))^{\mathfrak{g}}, \quad \mathbf{A}_\Psi^{\mathfrak{g}} := \mathbf{A}_\Psi^{\mathfrak{g}}(P^+).$$

For instance, if $V = \mathfrak{g}_{\mathrm{ad}}$ and $\Psi \subset R^+$, then $N_\Psi = |\Psi|$, as in [6].

We are now ready to state our main theorem, which this chapter is devoted to proving. The proofs involve various categories of graded $\mathfrak{g} \ltimes V$-modules, with finite-dimensional graded pieces.

Theorem 4.1. *Suppose $\Psi = F \cap \mathrm{wt}(V(\lambda))$ where F is some proper face of the weight polytope corresponding to $V(\lambda)$.*

Denote by $\mathbf{S}$ the algebra $\mathbf{A}$ with $A = \mathrm{Sym}(V)$. Given $\mu, \nu \in P^+$, the algebras $\mathbf{S}^{\mathfrak{g}}_{\Psi}(\leq_\Psi \nu), \mathbf{S}^{\mathfrak{g}}_{\Psi}(\mu \leq_\Psi), \mathbf{S}^{\mathfrak{g}}_{\Psi}([\mu,\nu]_\Psi)$, and $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ are Koszul with global dimension at most N_Ψ. Moreover, there exist $\mu \leq_\Psi \nu \in P^+$ such that these bounds on the global dimension are attained.

We now write down the other main results of this chapter, which will be used to prove Theorem 4.1 as well. (We will prove the next three results in greater generality below.)

Theorem 4.2. *Let $\widehat{\mathcal{G}}$ be the full subcategory of all $\mathfrak{g}$-semisimple $\mathbb{Z}_+$-graded $\mathfrak{g} \ltimes V$-modules with finite-dimensional pieces, and let $\mathcal{G}$ be the full subcategory of finite-dimensional objects in $\widehat{\mathcal{G}}$. The set of simple objects in either category is in bijection with $\Lambda := P^+ \times \mathbb{Z}_+$. Moreover, for all $M \in \widehat{\mathcal{G}}$, the module $\mathbb{P}(M) := \mathbf{U}(\mathfrak{g} \ltimes V) \otimes_{\mathbf{U}(\mathfrak{g})} M$ is projective in $\widehat{\mathcal{G}}$.*

These projective modules are used to construct explicit projective covers and resolutions in $\widehat{\mathcal{G}}$.

Theorem 4.3. *Given $(\mu, r) \in \Lambda$ and $0 \leq j \leq \dim V$, define $P_j(\mu, r) := \mathbb{P}(\wedge^j(V) \otimes V(\mu, r))$, where $V(\mu, r)$ is simple in $\widehat{\mathcal{G}}$ and corresponds to (μ, r). Then*

$$0 \to P_{\dim V}(\mu, r) \to \cdots \to P_0(\mu, r) \to V(\mu, r) \to 0$$

is a projective resolution of $V(\mu, r)$ in $\widehat{\mathcal{G}}$. Thus, for all $j \in \mathbb{Z}_+$ and $(\mu, r), (\nu, s) \in \Lambda$,

$$\mathrm{Ext}^j_{\widehat{\mathcal{G}}}(V(\mu, r), V(\nu, s)) \cong \begin{cases} \mathrm{Hom}_{\mathfrak{g}}(\wedge^j(V) \otimes V(\mu), V(\nu)) & \text{if } j = s - r; \\ 0, & \text{otherwise.} \end{cases}$$

For our last main result, we need some more definitions.

Definition.

1. *Given a set of simple objects* $\Gamma \subset \Lambda$*, define* $\mathcal{G}[\Gamma]$ *to be the full subcategory of* $\mathcal{G}$ *whose objects have all Jordan-Hölder factors only in* Γ.

2. *Define a partial order* $\preccurlyeq$ *on* Λ*, which is generated by* $(\mu, r) \preccurlyeq (\mu + \beta, r + 1)$*, for all* $\beta \in \mathrm{wt}(V)$.

3. *Suppose* $\mathcal{C}$ *is an abelian finite length category, whose objects and morphism sets are* $\mathbb{C}$*-vector spaces, and whose morphism sets are all finite-dimensional. We say that* $\mathcal{C}$ *is directed if*

   (a) *The simple objects in* $\mathcal{C}$ *are parametrized by a poset* $(\Pi, \leq)$*, such that all sets* $\{\xi \in \Pi : \xi < \tau\}$ *are finite (for all* $\tau \in \Pi$*).*

   (b) *For all simple objects* $\xi, \tau \in \mathcal{C}$ *(and abusing notation, in* Π *as well),* $\mathrm{Ext}^1_{\mathcal{C}}(\xi, \tau) \neq 0 \implies \xi < \tau$.

4. *(From [7].) Let* $\mathcal{C}$ *be an abelian finite length category with finitely many simples whose objects and morphisms are* $\mathbb{C}$*-vector spaces. Let* Π *be the set of isomorphism classes of simple objects. By abuse of notation, we also use* Π *to denote the simple objects. We say that a partial order* $\leq$ *on* Π *makes* $\mathcal{C}$ *a highest weight category if*

   (a) *For each* $\tau \in \Pi$*, there exists an object* $\Delta(\tau) \twoheadrightarrow \tau$ *in* $\mathcal{C}$*, such that the Jordan-Hölder factors* ξ *of the kernel satisfy* $\xi < \tau$.

   (b) *For each* $\tau \in \Pi$*, there exists a projective cover* $P(\tau) \twoheadrightarrow \Delta(\tau) \twoheadrightarrow \tau$*, such that the kernel of the first map has a finite filtration with factors* $\Delta(\xi)$*, each of which satisfies:* $\xi > \tau$.

Theorem 4.4. *For all* $\Gamma \subset \Lambda$*,* $\mathcal{G}[\Gamma]$ *is directed. If* $\Gamma \subset (\Lambda, \preccurlyeq)$ *is finite and interval-closed, then* $\mathcal{G}[\Gamma]$ *is a highest weight category with* $\Delta(\tau) = \tau \ \forall \tau \in \Gamma$*.* $\mathcal{G}[\Gamma]$ *is anti-equivalent* *to the category of modules over a quasi-hereditary, finite-dimensional, basic, $\mathbb{Z}_+$-graded $\mathbb{C}$-algebra $\mathfrak{B}(\Gamma)$.*

4.2 Preliminaries

Notation. Unless otherwise specified, all tensor products are over $\mathbb{C}$. Suppose $\mathfrak{a}$ is a $\mathbb{Z}_+$-graded Lie algebra, $\mathfrak{a} = \bigoplus_{n \geq 0} \mathfrak{a}_n$, where $\mathfrak{a}_0$ is a non-abelian reductive Lie algebra, and each $\mathfrak{a}_n$ is a finite-dimensional completely reducible $\mathfrak{a}_0$-module. Define $\mathfrak{a}_+ := \bigoplus_{n>0} \mathfrak{a}_n$. Let $\mathfrak{a}_0 = \mathfrak{g} \oplus \mathfrak{h}_0$, where $\mathfrak{g} := [\mathfrak{a}_0, \mathfrak{a}_0] = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is semisimple (with this triangular decomposition), and $\mathfrak{h}_0 = \mathfrak{Z}(\mathfrak{a}_0)$. Let $Q^+ \subset \mathfrak{h}^*$ denote $\mathbb{Z}_+ R^+ = \mathbb{Z}_+ \Delta$. Fix a Chevalley basis $x_\alpha^\pm$ for $\alpha \in R^+$ and set $h_\alpha := [x_\alpha, x_{-\alpha}]$. Let $\{\alpha_i : i \in I\}$ denote the set of simple roots, and define $h_i := h_{\alpha_i}$ $\forall i \in I$.

Given a weight $\lambda \in P^+ \times \mathfrak{h}_0^*$, one defines the $\mathfrak{a}_0$-module, $V(\lambda)$, as being generated by a vector v_λ modulo the relations

$$\mathfrak{n}^+ v_\lambda = 0; \qquad h.v_\lambda = \lambda(h) v_\lambda \ \forall\ h \in \mathfrak{h} \oplus \mathfrak{h}_0; \qquad (x_{\alpha_i}^-)^{\lambda(h_i)+1} v_\lambda = 0 \ \forall i \in I.$$

Finite dimensional examples of this setup include r-Takiff Lie algebras $\mathfrak{g}[t]/(t^r)$ where $r \in \mathbb{Z}_+$ and $\mathfrak{g}$ is semisimple, semidirect product Lie algebras $\mathfrak{g} \ltimes V$ where $\mathfrak{g}$ is reductive and V is a completely reducible $\mathfrak{g}$-module, and parabolic Lie algebras associated to good gradings of semisimple Lie algebras [8].

Classification of graded or finite-dimensional simple modules

We now develop our theory in this setup (in parallel to [5, 6]), with the eventual intention of applying it to *undeformed infinitesimal Hecke algebras* $\mathbf{U}(\mathfrak{a}_0 \ltimes \mathfrak{a}_1)$ [12, 14]. Our first result describes all simple finite-dimensional $\mathfrak{a}$-modules (or graded $\mathfrak{a}$-modules with finite-dimensional pieces) for certain $\mathfrak{a}$ - in particular, when $\mathfrak{a}$ is finite-dimensional.

Proposition 4.2. *Suppose M is a simple $\mathfrak{a}$-module.*

(i) *Suppose $\mathfrak{a}$ and M are finite-dimensional. Let $\mathfrak{a}_+^{\mathfrak{g}}$ be the sum of all trivial $\mathfrak{g}$-submodules of $\mathfrak{a}_+$, and $\mathfrak{a}'$ be its complement as a $\mathfrak{g}$-submodule of $\mathfrak{a}_+$. Then $M \cong V(\lambda)$ for some $\lambda|_{\mathfrak{h}}$ (as $\mathfrak{g}$-modules), with $\mathfrak{a}'.M = 0$ and where $\mathfrak{h}_0 \oplus \mathfrak{a}_+^{\mathfrak{g}}$ acts by a character on all of M.*

(ii) *If M is $\mathbb{Z}_+$-graded with finite-dimensional pieces, then M is of the form $V(\lambda)$ and concentrated in a unique degree $r \in \mathbb{Z}_+$ (where $\lambda \in P^+ \times \mathfrak{h}_0^*$). For distinct (λ, r), these simple modules are pairwise nonisomorphic and*

$$\mathrm{Hom}_{\widehat{\mathcal{G}}}(V(\lambda, r), V(\mu, s)) = 0, \quad \text{if } (\lambda, r) \neq (\mu, s)$$
$$\mathrm{Hom}_{\widehat{\mathcal{G}}}(V(\lambda, r), V(\lambda, r) \cong \mathbb{C}.$$

Thus, $\mathcal{G}$ may be perceived as a non-semisimple deformation of the category of finite-dimensional $\mathfrak{g}$-modules. Moreover, a special case of the last part motivates the study of graded representations:

Corollary 4.3. *Suppose $\mathfrak{a}$ is finite-dimensional and $\mathfrak{a}_+^{\mathfrak{g}} = 0$. Then every simple finite-dimensional M is of the form $V(\lambda)$ for some λ (so $\mathfrak{a}_+ M = 0$).*

(Thus, this is similar to the situation when M is graded.) For example, this result could be applied to r-Takiff Lie algebras $\mathfrak{g}[t]/(t^r)$ for any $r > 0$, where $\mathfrak{g}$ is semisimple.

Proof of Proposition 4.2. Define $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{h}_0 \oplus \mathfrak{n}^+ \oplus \mathfrak{a}_+$. Notice that $\mathfrak{b}_1 = [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{n}^+ \oplus \mathfrak{a}_+$. Since $\mathfrak{a}$ is finite dimensional, $\mathfrak{b}_1$ is solvable; hence, $\mathfrak{b}$ is solvable as well. By Lie's theorem, $\mathfrak{b}$ has a simultaneous eigenvector $m_\lambda \in M$ with $\mathfrak{h}$-weight λ.

We now claim that $\mathfrak{n}^+ \oplus \mathfrak{a}'$ kills m_λ. To see this, note that $\mathfrak{n}^+ = [\mathfrak{h}, \mathfrak{n}^+] \subset [\mathfrak{b}, \mathfrak{b}]$, which kills m_λ. Similarly, $\mathfrak{a}'$ has a basis of $\mathfrak{h}$-weight vectors a_ν, and if $\nu \neq 0$, then $a_\nu m_\lambda = 0$, since it has weight $\nu + \lambda \neq \lambda$. It remains to show that $\mathfrak{a}'_0.m_\lambda = 0$. By the structure of finite-dimensional representations of semisimple Lie algebras, every $a_0' \in \mathfrak{a}_0'$ is in $[\mathfrak{n}^+_\mu, \mathfrak{a}'_{-\mu}]$ for some $0 \neq \mu \in \mathfrak{h}^*$ (since there is no copy of the trivial $\mathfrak{g}$-representation in $\mathfrak{a}'$). This proves the claim.

Note that $\mathfrak{h}_0 \oplus \mathfrak{a}_+^{\mathfrak{g}}$ acts by a character (η, say) on m_λ and commutes with all of $\mathfrak{g}$. Hence, $M = \mathbf{U}(\mathfrak{n}^-)m_\lambda$ is of the form $V(\lambda)$, and $\mathfrak{h}_0 \oplus \mathfrak{a}_+^{\mathfrak{g}}$ acts by η on all of $M = V(\lambda)$.

The second part is now standard. Suppose M is graded with finite-dimensional pieces. Then, it has a lowest (nonzero) graded component, say $M[r]$. One checks that $\bigoplus_{s>r} M[s]$ is then a proper $\mathfrak{a}$-submodule of M. By the simplicity of M, $\bigoplus_{s>r} M[s]$ is trivial, so $M = M[r]$ as desired. Moreover, $M[r]$ is a finite-dimensional $\mathfrak{g}$-module, so we are done by the previous part (and $\mathfrak{a}_+$ acts by 0). $\square$

Truncations by grading; tensor structure

We now mimic the constructions in [5] in this more general setup. First, we make the following observation: since $\mathfrak{a}_0$ is reductive, all finite-dimensional $\mathfrak{a}_0$-modules need not be completely reducible; however, if we restrict to modules that are semisimple for $\mathfrak{h}_0 = \mathfrak{z}(\mathfrak{a}_0)$, then this does hold. Thus, we will work henceforth with $\mathfrak{h}_0$-semisimple modules only. (If need be, the reader may assume that we work with all modules, assuming that $\mathfrak{a}_0$ is *semisimple*.) We start by defining three subcategories of $\mathfrak{h}_0$-semisimple $\mathfrak{a}$-modules:

- Let $\mathcal{F}(\mathfrak{a}_0)$ denote the (full sub)category of finite-dimensional $\mathfrak{h}_0$-semisimple $\mathfrak{a}_0$-modules. This is a semisimple tensor category.

- Let $\widehat{\mathcal{G}}$ be the category of all $\mathbb{Z}_+$-graded $\mathfrak{a}$-modules V with finite-dimensional $\mathfrak{h}_0$-semisimple components $V[r]$ (so $V[r] \in \mathcal{F}(\mathfrak{a}_0)\ \forall r$).

- Let $\mathcal{G}$ (defined previously only in a special case: $\mathfrak{a}_+ = \mathfrak{a}_1 = V(\lambda)$) be the full subcategory of finite dimensional objects in $\widehat{\mathcal{G}}$.

In both $\mathcal{G}$ and $\widehat{\mathcal{G}}$, the morphisms are graded maps of $\mathfrak{a}$-modules.

A covariant functor $\mathrm{ev} : \mathcal{F}(\mathfrak{a}_0) \to \mathcal{G}(\subset \widehat{\mathcal{G}})$ can be defined by $\mathrm{ev}(V)[k] := 0$ if $k > 0$, and $\mathrm{ev}(V)[0] := V$ with $\mathfrak{a}_+ \mathrm{ev}(V) := 0$. Now define the *grading shifts* $\tau_s : \widehat{\mathcal{G}} \to \widehat{\mathcal{G}}$ by $(\tau_s V)[k] := V[k-s]\ \forall V \in \widehat{\mathcal{G}}, k, s \in \mathbb{Z}_+$. Finally, define

$$V(\lambda, r) := \tau_r(\mathrm{ev}(V(\lambda)))\ \forall \lambda \in P^+ \times \mathfrak{h}_0^*,\ r \in \mathbb{Z}_+.$$

One now defines $\mathcal{G}_{\leq s}$ as the full subcategory of $\widehat{\mathcal{G}}$ whose objects V satisfy $V[r] = 0\ \forall r > s$. Then $\mathcal{G}_{\leq s} \subset \mathcal{G}_{\leq r} \subset \mathcal{G}\ \forall s \leq r \in \mathbb{Z}_+$. Moreover, if we define $V_{>s} := \bigoplus_{r>s} V[r]$ for $V \in \widehat{\mathcal{G}}$, then $V_{\leq s} = V/V_{>s} \in \mathcal{G}_{\leq s}$. Furthermore, if $f \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(V, W)$, then $V_{>s}$ is contained in the kernel of the canonical morphism $\bar{f} : V \to W_{\leq s}$; hence, there is a natural morphism $f_{\leq s} \in \mathrm{Hom}_{\mathcal{G}_{\leq s}}(V_{\leq s}, W_{\leq s})$.

Lemma 4.4. *The functor $F : \widehat{\mathcal{G}} \to \mathcal{G}_{\leq r}$ defined by the assignment $V \mapsto V_{\leq r}$ and $f \mapsto f_{\leq r}$ is full, exact, and essentially surjective.*

Proof. Let $V \in \mathcal{G}_{\leq r}$. Considering V as an object in $\widehat{\mathcal{G}}$, we have $V_{>r} = 0$; hence, $V \cong V_{\leq r}$ as objects in $\mathcal{G}_{\leq r}$.

Let $0 \to V \xrightarrow{f} X \xrightarrow{g} W \to 0$ be a short exact sequence in $\widehat{\mathcal{G}}$. Since g is a surjective morphism in $\widehat{\mathcal{G}}$, $g(X_{>r}) = W_{>r}$. Thus, $0 \to V_{>r} \xrightarrow{f_{>r}} X_{>r} \xrightarrow{g_{>r}} W_{>r} \to 0$ is a short exact sequence in $\widehat{\mathcal{G}}$ also. Applying the Snake Lemma, we get that $0 \to V_{\leq r} \xrightarrow{f_{\leq r}} X_{\leq r} \xrightarrow{g_{\leq r}} W_{\leq r} \to 0$ is exact in $\widehat{\mathcal{G}}$. Since $\mathcal{G}_{\leq r}$ is a full subcategory of $\widehat{\mathcal{G}}$, the last sequence is also an exact sequence in $\mathcal{G}_{\leq r}$.

Finally, for objects V and W in $\widehat{\mathcal{G}}$, suppose that $\varphi \in \mathrm{Hom}_{\mathcal{G}_{\leq r}}(V_{\leq r}, W_{\leq r})$. Let $\pi_{V,r} : V \to V_{\leq r}$ and $\pi_{W,r} : W \to W_{\leq r}$ be the projection maps. Since each graded component of V is finite-dimensional, we can construct a morphism $f : V \to W$ in $\widehat{\mathcal{G}}$ such that $\varphi \circ \pi_{V,r} = \pi_{W,r} \circ f$. Then, $\varphi = f_{\leq r}$ by definition.

Thus, F is a full, exact, and essentially surjective functor.

$\square$

Next, one defines a tensor product on $\widehat{\mathcal{G}}$ via

$$(V \otimes W)[k] := \bigoplus_{i \in \mathbb{Z}_+} V[i] \otimes W[k-i],$$

with the convention that $V[i] := 0 \ \forall i < 0$. Then, similar to [5], we have

Lemma 4.5. *Suppose $\mathfrak{a}$ is $\mathbb{Z}_+$-graded, as above.*

(i) $\mathbf{U}(\mathfrak{a}_+) \in \widehat{\mathcal{G}}$.

(ii) *If $V \in \mathcal{G}_{\leq r}$ and $W \in \mathcal{G}_{\leq s}$, then $V \otimes W \in \mathcal{G}_{\leq r+s}$, whence $\mathcal{G} \subset \widehat{\mathcal{G}}$ are tensor categories.*

(iii) $\tau_s V = V \otimes V(0, s)$ *and* $\tau_{r+s}(V \otimes W) = \tau_r(V) \otimes \tau_s(W)$ *for* $r, s \in \mathbb{Z}_+,\ V, W \in \widehat{\mathcal{G}}$.

(iv) *For all $s \geq r$, $V \in \widehat{\mathcal{G}}$, and $M = M[r] \in \mathcal{G}$, $(V \otimes M)_{\leq s} = V_{\leq s-r} \otimes M$.*

Proof. (i) Let $x_i \in \mathfrak{a}_{j_i}$ for $1 \leq i \leq k$. We define a grading on $\mathbf{U}(\mathfrak{a}_+)$ by stipulating that the grade of the element $x_1 x_2 ... x_k \in \mathbf{U}(\mathfrak{a}_+)$ is $\sum_{i=1}^{k} j_i$. Furthermore, we say that $\mathbf{U}(\mathfrak{a}_+)[0] = \mathbb{C}$. Suppose that $x_1 \in \mathfrak{a}_{j_1}$ and $x_2 \in \mathfrak{a}_{j_2}$. By construction, $x_1 x_2 - x_2 x_1 \in \mathbf{U}(\mathfrak{a}_+)[j_1 + j_2]$. Since $\mathfrak{a}$ is a graded lie algebra, $[x_1, x_2] \in \mathfrak{a}_{j_1+j_2}$; hence, $[x_1, x_2] \in \mathbf{U}(\mathfrak{a}_+)[j_1 + j_2]$. In particular, the grading is well-defined.

For each $n > 0$, notice that $\mathbf{U}(\mathfrak{a}_+)[n]$ is generated by elements of the form $x_1 x_2 ... x_k$ with $x_i \in \mathfrak{a}_{j_i}$ such that $\sum_{i=1}^{k} j_i = n$. Since each $j_i > 0$ and each $\mathfrak{a}_{j_i}$ is a finite-dimensional $\mathfrak{a}_0$-module, we must have that $\mathbf{U}(\mathfrak{a}_+)[n]$ is a finite dimensional $\mathfrak{a}_0$-module. Thus, $\mathbf{U}(\mathfrak{a}_+) \in \widehat{\mathcal{G}}$.

(ii) It suffices to show that $(V \otimes W)[k] = 0$ for all $k > r + s$. Since $V \in \mathcal{G}_{\leq r}$ and $W \in \mathcal{G}_{\leq s}$, we can write $V = \bigoplus_{i=1}^{r} V[i]$ and $W = \bigoplus_{j=1}^{s} W[j]$. Then, if $k > r + s$,

$(V \otimes W)[k] = \bigoplus_{i\in\mathbb{Z}_+} V[i] \otimes W[k-i] = 0$ since for each $i \in \mathbb{Z}_+$ either $i > r$ or $k - i > s$.

(iii) Let $V \in \widehat{\mathcal{G}}$. Then, for $i \in \mathbb{Z}_+$, $\tau_s V[i] = V[i-s]$. On the other hand, $V \otimes V(0,s)[i] = \bigoplus_{k\in\mathbb{Z}_+} V[k] \otimes V(0,s)[i-k]$. Note that $V(0,s)[i-k] = 0$ except when $i-k = s$. Thus, $V \otimes V(0,s)[i] = V[i-s] \otimes V(0,s) \cong V[i-s]$. Since the graded components of the modules are equal (isomorphic), $\tau_s V = V \otimes V(0,s)$.

Suppose $V, W \in \widehat{\mathcal{G}}$. Let $i \in \mathbb{Z}_+$. By definition,

$$\tau_{r+s}(V \otimes W)[i] = (V \otimes W)[i-r-s] = \bigoplus_{k\in\mathbb{Z}_+} V[k] \otimes W[i-r-s-k].$$

Similarly,

$$(\tau_r V \otimes \tau_s W)[i] = \bigoplus_{j\in\mathbb{Z}_+} \tau_r V[j] \otimes \tau_s W[i-j] = \bigoplus_{j\in\mathbb{Z}_+} V[j-r] \otimes W[i-j-s].$$

In the second case, recall that $V[j-r] = 0$ if $j - r < 0$. Letting $k = j - r$, we have

$$(\tau_r V \otimes \tau_s W)[i] = \bigoplus_{k\in\mathbb{Z}_+} V[k] \otimes W[i-r-s-k] = \tau_{r+s}(V \otimes W)[i].$$

(iv) First, consider the natural map $V \otimes M \to V_{\le s-r} \otimes M$. Note that $(V \otimes M)[k] = \bigoplus_{s\in\mathbb{Z}_+} V[k-s] \otimes M[s] = V[k-r] \otimes M[r] = V[k-r] \otimes M$. The assertion now follows.

□

4.3 Projectives, resolutions, and truncated subcategories

We begin with some more notation. By Proposition 4.2, the set of *simple objects* in both $\mathcal{G}$ and $\widehat{\mathcal{G}}$ is parametrized by $\Lambda := P^+ \times \mathfrak{h}_0^* \times \mathbb{Z}_+$. We define a partial order $\preccurlyeq$ on Λ as follows:

First, say that (μ, s) *covers* (λ, r) if $s > r$ and $\mu - \lambda \in \mathrm{wt}(\mathfrak{a}_{s-r})$. Define $\preccurlyeq$ to be the transitive closure of this relation. In particular, $(\lambda, r) \preccurlyeq (\mu, r) \Rightarrow \lambda = \mu$.

Given $V \in \mathcal{G}$, define $[V : V(\lambda, r)]$ to be the multiplicity of $V(\lambda, r)$ in a composition series for V; given $W \in \widehat{\mathcal{G}}$, define $[W : V(\lambda, r)] := [W_{\leq r} : V(\lambda, r)]$ and $\Lambda(W) := \{(\lambda, r) \in \Lambda : [W : V(\lambda, r)] > 0\}$. Notice that $[W : V(\lambda, r)] = \dim \mathrm{Hom}_{\mathfrak{a}_0}(V(\lambda), W[r])$.

Our first result involves the construction of projective objects in $\widehat{\mathcal{G}}$. For each $(\lambda, r) \in \Lambda$, define $P(\lambda, r) := \mathbf{U}(\mathfrak{a}) \otimes_{\mathbf{U}(\mathfrak{a}_0)} V(\lambda, r)$, whence, as vector spaces, $P(\lambda, r) \cong \mathbf{U}(\mathfrak{a}_+) \otimes V(\lambda, r)$. Thus, $P(\lambda, r) \in \widehat{\mathcal{G}}$, and

$$P(\lambda, r)[k] = \mathbf{U}(\mathfrak{a}_+)[k - r] \otimes V(\lambda, r) \ \forall k.$$

Note that $P(\lambda, r)[r] = 1 \otimes V(\lambda, r)$ is generated as an $\mathfrak{a}_0$-module by $p_{\lambda,r} := 1 \otimes v_{\lambda,r}$, the unique maximal vector (up to scalar multiplication).

Proposition 4.6. *Let* $(\lambda, r) \in \Lambda$.

(i) *$P(\lambda, r)$ is generated as an $\mathfrak{a}$-module by $p_{\lambda,r}$, modulo the relations*

$$\mathfrak{n}^+ p_{\lambda,r} = (h - \lambda(h)) p_{\lambda,r} = 0; \qquad (x_{\alpha_i}^-)^{\lambda(h_i)+1} p_{\lambda,r} = 0 \ \forall i \in I, \ h \in \mathfrak{h} \oplus \mathfrak{h}_0.$$

(ii) *If $V \in \widehat{\mathcal{G}}$, then $\mathbb{P}(V) := \mathbf{U}(\mathfrak{a}) \otimes_{\mathbf{U}(\mathfrak{a}_0)} V = \mathbf{U}(\mathfrak{a}_+) \otimes V$ is projective in $\widehat{\mathcal{G}}$ and surjects onto V.*

(iii) *$P(\lambda, r) = \mathbb{P}(V(\lambda, r)) \cong P(0, 0) \otimes V(\lambda, r)$ in $\widehat{\mathcal{G}}$.*

(iv) *Given $r \leq s$ in $\mathbb{Z}_+$, the modules $P(\lambda, r)_{\leq s}$ are projective in $\mathcal{G}_{\leq s}$, and $P(\lambda, r)_{\leq s} \cong P(0, 0)_{\leq s-r} \otimes V(\lambda, r)$.*

(v) *As $\mathfrak{a}_0$-modules, we have $P(0, 0)[k] \cong \mathbf{U}(\mathfrak{a}_+)[k] \cong S^{(k)}(\mathfrak{a})$, where*

$$S^{(k)}(\mathfrak{a}) := \bigoplus_{\substack{(r_1, \ldots, r_k) \in \mathbb{Z}_+^k : \\ \sum_{j=1}^k j r_j = k}} \mathrm{Sym}^{r_1}(\mathfrak{a}_1) \otimes \cdots \otimes \mathrm{Sym}^{r_k}(\mathfrak{a}_k).$$

(vi) *Let* $(\mu, s) \in \Lambda$ *and* $K(\lambda, r)$ *be the kernel of the canonical projection* $P(\lambda, r) \twoheadrightarrow V(\lambda, r)$*. Then* $[K(\lambda, r) : V(\mu, s)] \neq 0 \Rightarrow (\lambda, r) \prec (\mu, s)$*.*

(vii) *If* $V \in \widehat{\mathcal{G}}$*, then* $\dim \operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), V) = [V : V(\lambda, r)]$ *and*

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), V) \cong \operatorname{Hom}_{\mathfrak{a}_0}(V(\lambda), V[r]).$$

Proof. (i) $\mathbf{U}(\mathfrak{a}).p_{\lambda,r} = \mathbf{U}(\mathfrak{a}).(1 \otimes v_{\lambda,r}) = \mathbf{U}(\mathfrak{a}) \otimes_{\mathbf{U}(\mathfrak{a}_0)} V(\lambda, r)$ where the last equality comes from the fact that $V(\lambda, r)$ is generated by $v_{\lambda,r}$ as an $\mathfrak{a}_0$-module. The relations follow immediately since $v_{\lambda,r}$ satisfies the same relations.

(ii) Similar to the discussion preceding the proposition, $\mathbb{P}(V) \cong \mathbf{U}(\mathfrak{a}_+) \otimes V$. Then, for $V \in \widehat{\mathcal{G}}$, $\mathbb{P}(V) \in \widehat{\mathcal{G}}$ since $\mathbf{U}(\mathfrak{a}_+) \in \widehat{\mathcal{G}}$. To show $\mathbb{P}(V)$ is projective in $\widehat{\mathcal{G}}$, suppose $\pi : M \twoheadrightarrow N$ in $\widehat{\mathcal{G}}$ and $\eta : \mathbb{P}(V) \to N$. Restrict η to get a map $\eta : V \to N$ as $\mathfrak{a}_0$-modules. Now consider π as a map of $\mathfrak{a}_0$-modules. By complete reducibility of $\mathfrak{a}_0$-modules, π has a splitting $\varphi : N \to M$, so we have an $\mathfrak{a}_0$-module map $\varphi \circ \eta : V \to M$. This induces the desired map of $\mathfrak{a}$-modules (where we forget the $\mathfrak{a}_+$-action on $V \subset \mathbb{P}(V)$ now; $\mathfrak{a}_+$ acts on $\mathbf{U}(\mathfrak{a})$ by left multiplication):

$$\widetilde{\varphi} : \mathbb{P}(V) = \mathbf{U}(\mathfrak{a})V = \mathbf{U}(\mathfrak{a}) \otimes_{\mathbf{U}(\mathfrak{a}_0)} V \to \mathbf{U}(\mathfrak{a})M = M.$$

Thus, $\mathbb{P}(V)$ is projective for all $V \in \widehat{\mathcal{G}}$. Moreover, $\mathrm{id}_V : V \to V$ is an $\mathfrak{a}_0$-module map, so it extends as above to a surjection $\mathbb{P}(V) \twoheadrightarrow V$ in $\widehat{\mathcal{G}}$. In fact, we have $0 \to \mathfrak{a}_+\mathbf{U}(\mathfrak{a}_+) \otimes V \to \mathbb{P}(V) \to V \to 0$ in $\widehat{\mathcal{G}}$.

(iii) As vector spaces, we have $P(\lambda, r) \cong \mathbf{U}(\mathfrak{a}_+) \otimes V(\lambda, r)$. Similarly, $P(0, 0) \otimes V(\lambda, r) \cong \mathbf{U}(\mathfrak{a}_+) \otimes V(0, 0) \otimes V(\lambda, r)$. Since $V(0, 0) \cong \mathbb{C}$, this shows that $P(\lambda, r) \cong P(0, 0) \otimes V(\lambda, r)$ as vector spaces.

Consider $p_{0,0} \otimes v_{\lambda,r}$. Note that $\mathbf{U}(\mathfrak{a}).(p_{0,0} \otimes v_{\lambda,r}) = P(0, 0) \otimes V(\lambda, r)$. We show that $p_{0,0} \otimes v_{\lambda,r}$ satisfies the defining relations for $P(\lambda, r)$.

Let $x \in \mathfrak{n}^+$. Then,

$$x.(p_{0,0} \otimes v_{\lambda,r}) = x.p_{0,0} \otimes v_{\lambda,r} + p_{0,0} \otimes x.v_{\lambda,r} = 0.$$

Let $h \in \mathfrak{h}$. We have

$$h.(p_{0,0} \otimes v_{\lambda,r}) = h.p_{0,0} \otimes v_{\lambda,r} + p_{0,0} \otimes h.v_{\lambda,r} = 0 + p_{0,0} \otimes \lambda(h)v_{\lambda,r} = \lambda(h)p_{0,0} \otimes v_{\lambda,r}.$$

Finally, note that $x_{\alpha_i}^{-}.p_{0,0} = 0$ and consider $(x_{\alpha_i}^{-})^{\lambda(h_i)+1}.(p_{0,0} \otimes v_{\lambda,r})$:

$$(x_{\alpha_i}^{-})^{\lambda(h_i)+1}.(p_{0,0} \otimes v_{\lambda,r}) = (x_{\alpha_i}^{-})^{\lambda(h_i)+1}.p_{0,0} \otimes v_{\lambda,r} + p_{0,0} \otimes (x_{\alpha_i}^{-})^{(\lambda(h_i)+1}.v_{\lambda,r} = 0.$$

This gives a surjective morphism $P(\lambda, r) \to P(0,0) \otimes V(\lambda, r)$ in $\widehat{\mathcal{G}}$, and the result now follows.

(iv) Recall from Lemma 4.4 that the assignment $V \to V_{\leq s}$ is an exact functor. Since $P(\lambda, r)$ is projective in $\widehat{\mathcal{G}}$ and $P(\lambda, r)_{\leq s} \neq 0$ for $r \leq s$, $P(\lambda, r)_{\leq s}$ is projective in $\mathcal{G}_{\leq s}$.

Now, since $P(\lambda, r) \cong P(0,0) \otimes V(\lambda, r)$, $P(\lambda, r)_{\leq s} \cong (P(0,0) \otimes V(\lambda, r))_{\leq s}$. Applying Lemma 4.5, $(P(0,0) \otimes V(\lambda, r))_{\leq s} = P(0,0)_{\leq s-r} \otimes V(\lambda, r)$, which proves the claim.

(v) From (4.3), $P(0,0)[k] = \mathbf{U}(\mathfrak{a}_+)[k] \otimes V(0,0)$, so $P(0,0)[k] \cong \mathbf{U}(\mathfrak{a}_+)[k]$ since $V(0,0) \cong \mathbb{C}$. It remains to show the second isomorphism. The case $k = 0$ is obvious since $\mathbf{U}(\mathfrak{a}_+)[0] = \mathbb{C}$. Let $k > 0$. For $s \geq 0$, define $\mathbf{U}(\mathfrak{a}_+)_{\leq s}$ to be the subset of $\mathbf{U}(\mathfrak{a}_+)$ spanned by the set

$$\{a_1 a_2 ... a_j \mid a_i \in \mathfrak{a}_+ \ 1 \leq i \leq j \leq s\}.$$

Set $\mathbf{U}(\mathfrak{a}_+)[k]_{\leq s} := \mathbf{U}(\mathfrak{a}_+)[k] \cap \mathbf{U}(\mathfrak{a}_+)_{\leq s}$.

Notice that $\mathbf{U}(\mathfrak{a}_+)[k]_{\leq s} \subset \mathbf{U}(\mathfrak{a}_+)[k]_{\leq t}$ for $s < t$. Furthermore, by construction, $\mathbf{U}(\mathfrak{a}_+)[k]_{\leq s}\mathbf{U}(\mathfrak{a}_+)[k]_{\leq t} \subset \mathbf{U}(\mathfrak{a}_+)[k]_{\leq s+t}$, so the subspaces $\mathbf{U}(\mathfrak{a}_+)[k]_{\leq s}$ form an increasing $\mathfrak{a}_0$-equivariant filtration of $\mathbf{U}(\mathfrak{a}_+)[k]$. Since $\mathbf{U}(\mathfrak{a}_+)[k]_{\leq r}$ is finite-dimensional, we have the following isomorphism of $\mathfrak{a}_0$-modules:

$$\begin{aligned}(\mathbf{U}(\mathfrak{a}_+))[k]_{\leq r} &\cong_{\mathfrak{a}_0} (\mathbf{U}(\mathfrak{a}_+))[k]_{\leq r-1} \oplus (\mathbf{U}(\mathfrak{a}_+))[k]_{\leq r}/(\mathbf{U}(\mathfrak{a}_+))[k]_{\leq r-1} \\ &\cong_{\mathfrak{a}_0} (\mathbf{U}(\mathfrak{a}_0))[k]_{\leq r-1} \oplus S^r(\mathfrak{a}_+)[k]\end{aligned}$$

where the second isomorphism is a consequence of the PBW theorem. By a downward induction on r, it follows that

$$\mathbf{U}(\mathfrak{a}_+)[k] \cong_{\mathfrak{a}_0} \bigoplus_{r=1}^{k} S^r(\mathfrak{a}_+)[k] = \mathrm{Sym}(\mathfrak{a}_+)[k].$$

Let $\mathbf{n} = (n_s \geq n_{s-1} \geq ... \geq n_1 > 0)$ be a partition of k. Define $V(\mathbf{n})$ to be the subspace of $\mathrm{Sym}(\mathfrak{a}_+)[k]$ spanned by elements of the form $x_s x_{s-1}...x_1$ where $x_j \in \mathfrak{a}_j$. Let $x_s x_{s-1}...x_1 \in V(\mathbf{n})$ and $x_0 \in \mathfrak{a}_0$. Then,

$$x_0.(x_s x_{s-1}...x_1) = \sum_{i=1}^{s} x_s x_{s-1}...x_{i+1}[x_0, x_i]x_{i-1}...x_1 \in V(\mathbf{n}).$$

For each $\mathbf{n}$, $V(\mathbf{n})$ is therefore an $\mathfrak{a}_0$-submodule of $\mathrm{Sym}(\mathfrak{a}_+)[k]$.

We now have that $\mathrm{Sym}(\mathfrak{a}_+)[k] = \oplus V(\mathbf{n})$ where the sum is taken over all partitions of k. For each partition $\mathbf{n}$ of k, we have a corresponding k-tuple $(r_1, r_2, ..., r_k) \in \mathbb{Z}^k_{\geq 0}$ defined by setting r_i equal to the number of times i occurs in the partition, and, thus, $\sum_{j=1}^{k} j r_j = k$. Note that for each partition $\mathbf{n}$, $V(\mathbf{n})$ is precisely the natural embedding of $\mathrm{Sym}^{r_1}(\mathfrak{a}_1) \otimes \mathrm{Sym}^{r_2}(\mathfrak{a}_2) \otimes \ldots \mathrm{Sym}^{r_k}(\mathfrak{a}_k)$ into $\mathrm{Sym}(\mathfrak{a}_+)[k]$, and the result follows.

(vi) $[K(\lambda, r) : V(\mu, s)] = \dim \mathrm{Hom}_{\mathfrak{a}_0}(V(\mu), K(\lambda, r)[s])$, so we must have $r < s$ since $K(\lambda, r)[k] = 0$ for all $k \leq r$. Suppose that $[K(\lambda, r) : V(\mu, s)] \neq 0$. By the previous observation, it follows that $K(\lambda, r)[s]$ contains a highest weight vector of highest weight μ. Since $K(\lambda, r)[s] = P(\lambda, r)[s]$ for all $s > r$, $K(\lambda, r)[s] = \mathbf{U}(\mathfrak{a}_+)[s - r] \otimes$

$V(\lambda, r)$. By Steinberg's formula [10], $\mu - \lambda$ is a weight of $\mathbf{U}(\mathfrak{a}_+)[s-r]$. Finally, note that the weights of $\mathbf{U}(\mathfrak{a}_+)[s-r]$ are of the form $\sum_{i=1}^k \mu_i$ with $\mu_i \in \mathrm{wt}(\mathfrak{a}_{j_i})$ and $\sum_{i=1}^k j_i = s - r$. Thus, $(\lambda, r) \prec (\mu, s)$.

(vii) We first show that $\mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), V) \cong \mathrm{Hom}_{\mathfrak{a}_0}(V(\lambda), V)$. Since $P(\lambda, r)[r] \cong V(\lambda, r)$, if $f \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), V)$, the restriction $f : P(\lambda, r)[r] \to V[r]$ defines a morphism in $\mathrm{Hom}_{\mathfrak{a}_0}(V(\lambda), V[r])$. $P(\lambda, r)$ is generated as an $\mathfrak{a}$-module by $p_{\lambda,r}$, so every morphism in $\mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), V)$ is completely determined by its action on $p_{\lambda,r}$. If $g \in \mathrm{Hom}_{\mathfrak{a}_0}(V(\lambda), V[r])$, define $f \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), V)$ via $f(p_{\lambda,r}) = g(v_\lambda)$ where $v_\lambda \in V(\lambda)_\lambda$ is nonzero.

From the comments preceding the proposition, we know that $[V : V(\lambda, r)] = \dim \mathrm{Hom}_{\mathfrak{a}_0}(V(\lambda), V[r])$. Since $\mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), V) \cong \mathrm{Hom}_{\mathfrak{a}_0}(V(\lambda), V)$, we are done.

□

We can now show that $\widehat{\mathcal{G}}$ has enough projectives.

Proposition 4.7. *If $V = V[r] \in \widehat{\mathcal{G}}$, then V is semisimple in $\widehat{\mathcal{G}}$, and $\mathbb{P}(V)$ is its projective cover in $\widehat{\mathcal{G}}$. In particular, $P(\lambda, r)$ is the projective cover of its unique simple quotient $V(\lambda, r)$ in $\widehat{\mathcal{G}}$*

Proof. Since $V \in \widehat{\mathcal{G}}$, $V[r]$ is a finite-dimensional $\mathfrak{a}_0$-module, and we can write $V[r] = \bigoplus_{i=1}^n V_i$ with each V_i a simple $\mathfrak{a}_0$-module. For each i, $\tau_r(ev(V_i))$ is a simple module in $\widehat{\mathcal{G}}$. The fact that V is semisimple in $\widehat{\mathcal{G}}$ now follows by noting that $V = \bigoplus_{i=1}^n \tau_r(ev(V_i))$. To see that $\mathbb{P}(V)$ is the projective cover of V in $\widehat{\mathcal{G}}$, suppose that P is any other projective object in $\widehat{\mathcal{G}}$ which surjects onto V. The natural non-zero map $P \to \mathbb{P}(V)$ must be surjective in degree r. Moreover, $\mathbb{P}(V)$ is generated in degree r, so $\mathbb{P}(V)$ is a quotient of P, and the result follows. □

Since $\mathfrak{a} = \mathfrak{g} \ltimes V$ is just a special case of $\mathfrak{a}$ considered in this section, we have enough information to prove Theorem 4.2 in more generality:

Proof of Theorem 4.2. By Proposition 4.2(ii), the set of isomorphism classes simple objects in $\widehat{\mathcal{G}}$ (and $\mathcal{G}$) is in bijective correspondence with Λ. The fact that $\mathbb{P}(M)$ is projective in $\widehat{\mathcal{G}}$ for all $M \in \widehat{\mathcal{G}}$ was shown in Proposition 4.6(ii). $\square$

We have now shown that $\widehat{\mathcal{G}}$ has enough projectives. We note, however, that $\mathcal{G}$ does not have enough projectives. In fact, the category $\mathcal{G}$ does not contain any projective objects:

Proof. Suppose that P is projective in $\mathcal{G}$. Because P is a finite-dimensional $\mathfrak{a}_0$-module, P must have a simple quotient $V(\mu, s)$, and $P[k] = 0$ for all $k > n$ for some $n \in \mathbb{Z}$.
Consider the module $M := P(\mu, s)_{\leq n+1} \in \mathcal{G}$ where $P(\mu, s)$ is the projective cover of $V(\mu, s)$ in $\widehat{\mathcal{G}}$. M surjects onto $V(\mu, s)$ in $\mathcal{G}$. Since P is projective in $\mathcal{G}$ by assumption, we now have a non-zero map $f : P \to M$ in $\mathcal{G}$.
Notice that M is cyclic since it is a quotient of the cyclic module $P(\mu, s)$, and, thus, the map $f : P \to M$ is surjective. However, this is impossible since $0 = f(P[n+1]) \subset M[n+1] \neq 0$. $\square$

Next, we produce a projective resolution of each simple object inside $\widehat{\mathcal{G}}$. Given $(\mu, r) \in \Lambda$ and $j \in \mathbb{Z}_+$, define

$$P_j(\mu, r) := \mathbb{P}((\wedge^j \mathfrak{a}_+) \otimes V(\mu, r)) = \mathbf{U}(\mathfrak{a}_+) \otimes (\wedge^j \mathfrak{a}_+) \otimes V(\mu, r) \in \widehat{\mathcal{G}}.$$

In particular, $P_0(\mu, r) = P(\mu, r), P_j(\mu, r) = P_j(0, 0) \otimes V(\mu, r)$, and $P_j(\mu, r)[s]$ equals 0 if $s < r + j$.

Now define $d_j : P_j(\mu, r) \to P_{j-1}(\mu, r)$ for $j > 0$ by linearly extending the following map:

$d_j(u \otimes (\wedge_{i=1}^j x_i) \otimes v) := \theta_1 + \theta_2$, where $u \in \mathbf{U}(\mathfrak{a}_+), v \in V(\mu, r), x_i \in \mathfrak{a}_+ \ \forall i$, and

$$\begin{aligned} \theta_1 &:= \sum_{s=1}^{j} (-1)^{s-1} u x_s \otimes (\wedge_{i \neq s} x_i) \otimes v, \\ \theta_2 &:= \sum_{s<t} (-1)^{s+t} u \otimes ([x_s, x_t] \wedge \wedge_{i \neq s,t} x_i) \otimes v. \end{aligned}$$

Also define $d_0 : P_0(\mu, r) \to V(\mu, r)$ via $d_0(u \otimes v) := uv$. Thus, $d_j = D \otimes \mathrm{id}_{V(\mu,r)}$, where D is the Koszul differential on the Chevalley-Eilenberg complex (tensored with $V(\mu, r)$). In particular, D is exact.

Proposition 4.8. (i) *For all $j \in \mathbb{Z}_+$ and $(\mu, r) \in \Lambda$, $P_j(\mu, r)$ is projective in $\widehat{\mathcal{G}}$, and, if $[P_j(\mu, r) : V(\lambda, s)] > 0$, then $(\mu, r) \preccurlyeq (\lambda, s)$.*

(ii) *The following is a projective resolution of $V(\mu, r)$ in $\widehat{\mathcal{G}}$:*

$$\dots \xrightarrow{d_{n+1}} P_n(\mu, r) \xrightarrow{d_n} P_{n-1}(\mu, r) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} P_0(\mu, r) \xrightarrow{d_0} V(\mu, r) \to 0.$$

Proof.

(i) By Proposition 4.6, $P_j(\mu, r)$ is projective for each j. Suppose $[P_j(\lambda, r) : V(\mu, s)] > 0$. Then, $\mathrm{Hom}_{\mathfrak{a}_0}(V(\mu), P_j(\lambda, r)[s]) \neq 0$, whence $r \leq s$. Therefore, $P_j(\lambda, r)[s] = (\mathbf{U}(\mathfrak{a}_+ \otimes \wedge^j \mathfrak{a}_+)[s - r] \otimes V(\lambda, r)$ has a highest weight vector of highest weight μ. It now follows from an argument analogous to that from Proposition 4.6(vi) that $(\lambda, r) \preccurlyeq (\mu, s)$.

(ii) We first check that every d_j is an $\mathfrak{a}$-module map. It is clear that d_0 is an $\mathfrak{a}$-module map, so we need to check for $j > 0$.

Let $a \in \mathfrak{a}$, $u \in \mathbf{U}(\mathfrak{a}_+)$, $v \in V(\lambda, r)$ and $x_i \in \mathfrak{a}_+$ for $1 \leq i \leq j$. Consider $a.d_j(u \otimes \wedge_{i=1}^j x_j \otimes v)$:

$$
\begin{aligned}
a.d_j(u \otimes \wedge_{i=1}^{j} x_j \otimes v) &= a.\textstyle\sum_{s=1}^{j}(-1)^{s-1} u x_s \otimes (\wedge_{i\neq s} x_i) \otimes v + \\
& \quad a.\textstyle\sum_{s<t}(-1)^{s+t} u \otimes ([x_s, x_t] \wedge \wedge_{i\neq s,t} x_i \\
&= (\textstyle\sum_{s=1}^{j}(-1)^{s-1} a u x_s \otimes (\wedge_{i\neq s} x_i) \otimes v + \\
& \quad \textstyle\sum_{s<t}(-1)^{s+t} a u \otimes ([x_s, x_t] \wedge \wedge_{i\neq s,t} x_i) \otimes v)
\end{aligned}
$$

This is, by definition, $d_j(au \otimes \wedge_{i=1}^{j} x_j \otimes v) = d_j(a.(u \otimes \wedge_{i=1}^{j} x_j \otimes v))$, so d_j is an $\mathfrak{a}$-module map.

Hence, d_j is a morphism in $\widehat{\mathcal{G}}$ for all $j \geq 0$. From the discussion preceding this proposition, the sequence is now exact in $\widehat{\mathcal{G}}$.

□

Corollary 4.9. *For all $j \in \mathbb{Z}_+$, $d_j(u \otimes a \otimes v) = (u \otimes 1)d_j(1 \otimes a \otimes v)$ for all $u \in \mathbf{U}(\mathfrak{a}_+)$, $a \in \wedge^j \mathfrak{a}_+$, $v \in V(\mu, r)$. In particular, $(\operatorname{im} d_j)[s] = 0$ $\forall s < j + r$, and*

$$(\operatorname{im} d_{s-r})[s] \cong_{\mathfrak{a}_0} P_{s-r}(\mu, r)[s] \cong_{\mathfrak{a}_0} \bigwedge^{s-r} \mathfrak{a}_1 \otimes V(\mu).$$

Proof. [(As in [6].)] For the first statement, note that the $\mathfrak{a}$-action is just given by left multiplication in the first term. If $u \in \mathbf{U}(\mathfrak{a}_+)$, then

$$u.(1 \otimes a \otimes v) = (u \otimes 1)(1 \otimes a \otimes v) = u \otimes a \otimes v$$

for all $a \in \wedge^j \mathfrak{a}_+$ and $v \in V(\mu, r)$. Since d_j is an $\mathfrak{a}$-module map, we have

$$d_j(u \otimes a \otimes v) = d_j(u.(1 \otimes a \otimes v) = u.d_j(1 \otimes a \otimes v) = (u \otimes 1)d_j(1 \otimes a \otimes v).$$

Next, for each $j \in \mathbb{Z}_+$, $P_j(\mu, r)[s] = 0$ for all $s < r + j$. Since $(\operatorname{im} d_j)[s] = d_j(P_j(\mu, r)[s])$, we must have $(\operatorname{im} d_j)[s] = 0$ if $s < r + j$.

Finally, suppose $s = r + j$ (i.e., $j = s - r$). By exactness, $(\ker d_j)[s] = (\operatorname{im} d_{j+1})[s]$, and $(\operatorname{im} d_{j+1})[s] = 0$ since $s < r + (j + 1)$. Hence, $d_{s-r}(P_{s-r}(\mu, r)[s]) \cong_{\mathfrak{a}_0} P_{s-r}(\mu, r)[s]$. □

Finally, we address the issue of truncating. For each $\Gamma \subset \Lambda$, define $\widehat{\mathcal{G}}[\Gamma]$ to be the full subcategory of $\widehat{\mathcal{G}}$ whose objects are the $V \in \widehat{\mathcal{G}}$ which satisfy $\Lambda(V) \subset \Gamma$. We also define $\mathcal{G}[\Gamma]$ to be the full subcategory of $\widehat{\mathcal{G}}[\Gamma]$ whose objects are finite dimensional. We now show that for suitable subsets $\Gamma \subset \Lambda$, the truncated subcategory $\widehat{\mathcal{G}}[\Gamma]$ still has enough projectives. A few prerequisites are required for this.

Proposition 4.10. *Given $V = \bigoplus_{k\in\mathbb{Z}_+} V[k] \in \widehat{\mathcal{G}}$, suppose that V' is an $\mathfrak{a}_0$-submodule of V, $V(\lambda, r)$ is an $\mathfrak{a}_0$-submodule of $V[r]$, and $V(\mu, s)$ is an $\mathfrak{a}_0$-submodule of $V[s]$ for some $\lambda, \mu \in P^+ \times \mathfrak{h}_0^*$ and $r < s$ in $\mathbb{Z}_+$.*

(i) *If $\mathfrak{a}_{s-r}.v_\lambda \subset V'$ where $v_\lambda \in V(\lambda, r)_\lambda$, then $\mathfrak{a}_{s-r}.V(\lambda, r) \subset V'$.*

(ii) *If the composite ξ of the action and projection $\mathfrak{a}_{s-r} \otimes V(\lambda, r) \to V[s] \twoheadrightarrow V(\mu, s)$ is nonzero, then (μ, s) covers (λ, r).*

(iii) *If $\mathrm{Ext}^1_{\mathcal{G}}(V(\lambda, r), V(\mu, s)) \neq 0$, then (μ, s) covers (λ, r).*

In the proof, we need the Borel subalgebra $\mathfrak{b}^+ := \mathfrak{h} \oplus \mathfrak{h}_0 \oplus \mathfrak{n}^+$ inside $\mathfrak{a}_0$, where $\mathfrak{n}^+$ is the positive nilpotent subalgebra inside $\mathfrak{g} \subset \mathfrak{a}_0$.

Proof.

(i) Consider $v \in V(\lambda, r)$, $v = b_1 \dots b_k v_\lambda$ with all $b_i \in \mathfrak{a}_0$. We prove the claim by induction on k. For $k = 0$, we have $\mathfrak{a}_{s-r}.v_\lambda \subset V'$ by assumption. Suppose we know the result for fewer than k terms from $\mathfrak{a}_0$ and $v = b_1 \dots b_k v_\lambda$. Let $a \in \mathfrak{a}_{s-r}$. Then,

$$a.v = a.b_1 \dots b_k v_\lambda = \sum_i b_1 \dots b_{i-1}[a, b_i]b_{i+1} \dots b_k v_\lambda + (b_1 \dots b_k)av_\lambda.$$

For all i, $[a, b_i] \in [\mathfrak{a}_{s-r}, \mathfrak{a}_0] \subset \mathfrak{a}_{s-r}$, so each summand is in V' by induction.

(ii) We first claim that the composite map $\xi : \mathfrak{a}_{s-r} \otimes V(\lambda, r) \to V(\mu, s)$ is surjective.

To see this, first note that the action map $\mathfrak{a}_{s-r} \otimes V(\lambda, r) \to V[s]$ is an $\mathfrak{a}_0$-module map:

Let $a_0 \in \mathfrak{a}_0$, $a_{s-r} \in \mathfrak{a}_{s-r}$ and $v \in V(\lambda, r)$. Then,

$$a_0.(a_{s-r} \otimes v) = ([a_0, a_{s-r}] \otimes v + a_{s-r} \otimes a_0.v \mapsto [a_0, a_{s-r}].v + a_{s-r}.a_0.v = a_0.(a_{s-r}.v)$$

The projection $V[s] \twoheadrightarrow V(\mu, s)$ is also an $\mathfrak{a}_0$-module map, so ξ is an $\mathfrak{a}_0$-module map. Since ξ has nonzero image inside $V(\mu, s)$, ξ must be onto.

Furthermore, since ξ is nonzero, $\mathfrak{a}_{s-r}.V(\lambda, r) \neq 0$; hence, the maximal vector $v_\lambda \in V(\lambda, r)$ is not killed by all of $\mathfrak{a}_{s-r}$. Since $\mathfrak{a}_{s-r} \otimes \mathbb{C}v_\lambda$ is a $\mathbf{U}(\mathfrak{b}^+)$-submodule of $\mathfrak{a}_{s-r} \otimes V(\lambda, r)$, $\xi(\mathfrak{a}_{s-r} \otimes \mathbb{C}v_\lambda)$ is a nonzero $\mathbf{U}(\mathfrak{b}^+)$-submodule of $V(\mu, s)$. Since $\xi(\mathfrak{a}_{s-r} \otimes \mathbb{C}v_\lambda)$ is finite dimensional, it must contain a highest weight vector in $V(\mu, s)$. In particular, $v_\mu \in \xi(\mathfrak{a}_{s-r} \otimes V(\lambda, r))$ where $v_\mu \in V(\mu, s)_\mu$.

Inside $V[s]$, $\mathfrak{a}_{s-r}.v_\lambda$ contains an element of weight μ, which gives $\mu - \lambda \in \mathrm{wt}(\mathfrak{a}_{s-r})$. Thus, $(\lambda, r) \prec (\mu, s)$.

(iii) Suppose $0 \to V(\mu, s) \to V \to V(\lambda, r) \to 0$ is an extension in $\mathcal{G}$.

If $r = s$, then the extension is an extension of $\mathfrak{a}_0$-modules, and, thus, it splits by $\mathfrak{a}_0$-semisimplicity.

If $r > s$, $V[s]$ is an $\mathfrak{a}$-submodule of V. $V[r]$ is an $\mathfrak{a}$-module complement to $V[s] \cong V(\mu, s)$, and the extension splits.

Suppose that $r < s$ and $\mathfrak{a}_{s-r}.V[r] = 0$. Then, $V[r]$ is an $\mathfrak{a}$-submodule of V, and the extension splits.

Thus, if the extension splits, we must have $r < s$ and $\mathfrak{a}_{s-r}.V[r] \neq 0$. By the previous part, (μ, s) covers (λ, r).

$\square$

Definition. *Suppose $V \in \widehat{\mathcal{G}}$ and $\Gamma \subset \Lambda$.*

1. *Define V_Γ^+ to be the set $V_\Gamma^+ := \{v \in V[r]_\lambda : (\lambda, r) \in \Gamma, \mathfrak{n}^+ v = 0\}$.*

2. *Define $V_\Gamma := \mathbf{U}(\mathfrak{a}_0)V_\Gamma^+$, and $V^\Gamma := V/V_{\Lambda\setminus\Gamma}$.*

3. *Let $V \xrightarrow{f} U$ be a morphism in $\widehat{\mathcal{G}}$. Define the $\mathfrak{a}_0$-module map $f_\Gamma : V_\Gamma \to U_\Gamma$ by $f_\Gamma = f|_{V_\Gamma}$. Since $f(V_{\Lambda\setminus\Gamma}) \subset U_{\Lambda\setminus\Gamma}$, we also have a natural map $f^\Gamma : V^\Gamma \to U_\Gamma$.*

Our next two results are useful for producing enough projectives in truncated subcategories $\mathcal{G}[\Gamma]$.

Proposition 4.11. *Suppose $V \in \widehat{\mathcal{G}}$ and $\Gamma \subset \Lambda$.*

(i) *V_Γ and V^Γ are $\mathbb{Z}_+$-graded $\mathfrak{a}_0$-modules with all Jordan-Hölder factors in Γ. If Γ is finite, then V_Γ and V^Γ are finite-dimensional.*

(ii) *If V_Γ is not an $\mathfrak{a}$-submodule of V, then there exist $(\nu, s) \in \Lambda(V) \setminus \Gamma$ and $(\lambda, r) \in \Gamma \cap \Lambda(V)$ such that (ν, s) covers (λ, r).*

Proof. (i) V_Γ is an $\mathfrak{a}_0$-module by definition. Define $V_\Gamma[k]$ to be the $\mathfrak{a}_0$-module generated by the $v \in V_\Gamma^+ \cap V[k]$. This defines a grading of V_Γ as an $\mathfrak{a}_0$-module. Suppose that $V(\mu, s)$ is a Jordan-Hölder factor of V_Γ, so $\mathrm{Hom}_{\mathfrak{a}_0}(V(\mu), V_\Gamma[s]) \neq 0$. In particular, there is a highest weight vector v in $V_\Gamma[s]$ of highest weight μ, whence $(\mu, s) \in \Gamma$ by definition.

The analogous statements about V^Γ follow similarly.

Suppose now that Γ is finite. Each graded component of V is finite-dimensional, so V_Γ^+ is finite since it consists of the highest weight elements from a finite number of graded components of V. Finally, consider V^Γ. If $V^\Gamma[s] \neq 0$, then $V[s] \neq V_{\Lambda\setminus\Gamma}[s]$, or, in other words, $V_\Gamma[s] \neq 0$. Since Γ is finite, this can only occur for a finite number of s. Thus, V^Γ is finite-dimensional as well.

(ii) Let $v \in V_\Gamma^+$. If $\mathfrak{a}_k.v \subset V_\Gamma$ for all k, then by Proposition 4.10, $\mathfrak{a}.\mathbf{U}(\mathfrak{a}_0)v \subset V_\Gamma$. Since V_Γ is not an $\mathfrak{a}$-submodule of V by assumption, there exists $k \in \mathbb{Z}_+$ and $v \in V_\Gamma^+$ such that $a_k.v \notin V_\Gamma$ for some $a_k \in \mathfrak{a}_k$.

Without loss of generality, we may choose v to be in $V[r]_\lambda$ for some $(\lambda, r) \in \Gamma$, whence $a_k.v \in V[r+k] \setminus V_\Gamma$. Choose an $\mathfrak{a}_0$-module complement U of V_Γ in $V \in \widehat{\mathcal{G}}$. Then, there exists $(\nu, r+k)$ such that the composite $\mathfrak{a}_0$-module map

$$\xi : \mathfrak{a}_k \otimes V(\lambda, r) \to V \twoheadrightarrow U \twoheadrightarrow U[r+k] \twoheadrightarrow V(\nu, r+k)$$

is nonzero. By Proposition 4.10, $(\nu, r+k)$ covers (λ, r).

$\square$

Now suppose that Γ is interval-closed as well.

Proposition 4.12. (i) *If for each $(\lambda, r) \in \Lambda(V)\setminus\Gamma$, there exists $(\mu, s) \in \Gamma$ with $(\lambda, r) \prec (\mu, s)$, then $V_\Gamma \in \widehat{\mathcal{G}}[\Gamma]$ and $V^{\Lambda\setminus\Gamma} \in \widehat{\mathcal{G}}[\Lambda \setminus \Gamma]$.*

(ii) *Suppose that, for each $(\lambda, r) \in \Lambda(V) \setminus \Gamma$, there exists $(\mu, s) \in \Gamma$ with $(\lambda, r) \succ (\mu, s)$. Then, $V_{\Lambda\setminus\Gamma} \in \widehat{\mathcal{G}}[\Lambda \setminus \Gamma]$ and $V^\Gamma \in \widehat{\mathcal{G}}[\Gamma]$.*

(iii) *Suppose $0 \to V \xrightarrow{f} U \xrightarrow{g} W \to 0$ is a short exact sequence in $\widehat{\mathcal{G}}$. Then, U satisfies the condition from (i)(respectively, (ii)) if and only if V and W both satisfy the condition from (i) (respectively, (ii)). Moreover, if U satisfies the condition from (i)(respectively, (ii)), then $0 \to V_\Gamma \xrightarrow{f_\Gamma} U_\Gamma \xrightarrow{g_\Gamma} W_\Gamma \to 0$ (respectively, $0 \to V^\Gamma \xrightarrow{f^\Gamma} U^\Gamma \xrightarrow{g^\Gamma} W^\Gamma \to 0$) is a short exact sequence in $\widehat{\mathcal{G}}[\Gamma]$.*

Proof.

(i) By Proposition 4.11(i), it only remains to show that V_Γ is an $\mathfrak{a}$-module in this case. Suppose V_Γ is not an $\mathfrak{a}$-submodule of V. By Proposition 4.11(ii), we can find $(\nu, k) \in \Lambda \setminus \Gamma$ and $(\lambda, r) \in \Gamma$ such that $(\lambda, r) \prec (\nu, k)$. By assumption, there exists $(\mu, s) \in \Gamma$ such that $(\nu, k) \prec (\mu, s)$; however, this contradicts the interval-closure of Γ. Thus, V_Γ is an $\mathfrak{a}$-module.

(ii) This follows by emulating the proof of the previous part: Suppose that $V_{\Lambda\setminus\Gamma}$ is not an $\mathfrak{a}$-module. By Proposition 4.11(ii), we can find $(\nu, k) \in \Gamma$ and $(\lambda, r) \in \Lambda \setminus \Gamma$ such that $(\lambda, r) \prec (\nu, k)$. By assumption, there exists $(\mu, s) \in \Lambda \setminus \Gamma$ such that $(\mu, s) \prec (\nu, k)$. However, this contradicts the fact that Γ is interval-closed, so $V_{\Lambda\setminus\Gamma}$ is an $\mathfrak{a}$-module.

(iii) As $\mathfrak{a}_0$-modules, we have that $U[k] = V[k] \oplus W[k]$ for all $k \in \mathbb{Z}_+$. It follows that $\mathrm{Hom}_{\mathfrak{a}_0}(V(\lambda), U[k]) \neq 0$ if and only if at least one of $\mathrm{Hom}_{\mathfrak{a}_0}(V(\lambda), V[k])$, $\mathrm{Hom}_{\mathfrak{a}_0}(V(\lambda), W[k])$ is non-zero, so $\Lambda(U) = \Lambda(V) \cup \Lambda(W)$. The first part of the statement now follows.

As for the second part, if U satisfies (i), then $V_\Gamma \subset U_\Gamma$ in $\widehat{\mathcal{G}}[\Gamma]$. As graded $\mathfrak{a}_0$-modules, we already have $U_\Gamma = V_\Gamma \oplus W_\Gamma$, so we must have a short exact sequence in $\widehat{\mathcal{G}}[\Gamma]$. The case when (ii) holds is similar.

$\square$

We now use these preliminary results to obtain enough projectives in certain truncated subcategories.

Proposition 4.13. *Suppose Γ is finite and interval-closed. Assume $(\lambda, r), (\mu, s) \in \Gamma$.*

(i) *$P(\lambda,r)^\Gamma$ is the projective cover of $V(\lambda,r)$ in $\mathcal{G}[\Gamma]$. In particular, $\mathcal{G}[\Gamma]$ has enough projectives.*

(ii) $[P(\lambda,r):V(\mu,s)] = [P(\lambda,r)^\Gamma : V(\mu,s)] = \dim \operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\mu,s)^\Gamma, P(\lambda,r)^\Gamma)$.

(iii) *The map $f \mapsto f^\Gamma$ defines an isomorphism*

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda,r),P(\mu,s)) \to \operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\lambda,r)^\Gamma, P(\mu,s)^\Gamma).$$

(iv) *For all $(\mu,s) \in \Gamma$ and $j \in \mathbb{Z}_+$, $P_j(\mu,s)^\Gamma \in \mathcal{G}[\Gamma]$. Moreover, the induced exact sequence*

$$\dots \xrightarrow{d_{n+1}^\Gamma} P_n(\mu,s)^\Gamma \xrightarrow{d_n^\Gamma} P_{n-1}(\mu,s)^\Gamma \xrightarrow{d_{n-1}^\Gamma} \dots \xrightarrow{d_1^\Gamma} P_0(\mu,s)^\Gamma \xrightarrow{d_0^\Gamma} V(\mu,s) \to 0$$

is a finite projective resolution of $V(\mu,s)$ in $\mathcal{G}[\Gamma]$.

Proof.

(i) By Proposition 4.6(vi), $(\lambda,r) \preccurlyeq (\nu,k)$ for all $(\nu,k) \in \Lambda(P(\lambda,r))$. In particular, for each $(\mu,s) \in \Lambda(P(\lambda,r))\backslash\Gamma$, $(\lambda,r) \prec (\mu,s)$. By Proposition 4.12(ii), $P(\lambda,r)^\Gamma \in \widehat{\mathcal{G}}[\Gamma]$. Furthermore, since Γ is finite, $P(\lambda,r)^\Gamma$ is finite-dimensional by Proposition 4.11(i).

Let K be the kernel of the surjection $P(\lambda,r) \twoheadrightarrow P(\lambda,r)^\Gamma$. Given $(\mu,s) \in \Gamma$, apply $\operatorname{Hom}_{\widehat{\mathcal{G}}}(-,V(\mu,s))$ to the short exact sequence

$$0 \to K \to P(\lambda,r) \to P(\lambda,r)^\Gamma \to 0$$

in $\widehat{\mathcal{G}}$ to obtain the long exact sequence

$$\begin{aligned}\dots \to \operatorname{Hom}_{\widehat{\mathcal{G}}}(K,V(\mu,s)) &\to \operatorname{Ext}^1_{\widehat{\mathcal{G}}}(P(\lambda,r)^\Gamma, V(\mu,s)) \\ &\to \operatorname{Ext}^1_{\widehat{\mathcal{G}}}(P(\lambda,r),V(\mu,s)) \to \dots,\end{aligned}$$

Since $P(\lambda, r)$ is projective in $\widehat{\mathcal{G}}$, $\operatorname{Ext}^1_{\widehat{\mathcal{G}}}(P(\lambda, r), V(\mu, s)) = 0$. By Proposition 4.12, $K = P(\lambda, r)_{\Lambda\setminus\Gamma}$ as objects in $\widehat{\mathcal{G}}$; thus, $\operatorname{Hom}_{\widehat{\mathcal{G}}}(K, V(\mu, s)) = 0$ for all $(\mu, s) \in \Gamma$. Combining these gives that $\operatorname{Ext}^1_{\widehat{\mathcal{G}}}(P(\lambda, r)^\Gamma, V(\mu, s)) = 0$ for all $(\mu, s) \in \Gamma$. Furthermore, since $P(\lambda, r)^\Gamma \in \mathcal{G}[\Gamma]$, we have

$$\operatorname{Ext}^1_{\mathcal{G}[\Gamma]}(P(\lambda, r)^\Gamma, V(\mu, s)) = \operatorname{Ext}^1_{\widehat{\mathcal{G}}}(P(\lambda, r)^\Gamma, V(\mu, s)) = 0,$$

for all $(\mu, s) \in \Gamma$, so $P(\lambda, r)^\Gamma$ is projective in $\mathcal{G}[\Gamma]$. Finally, $P(\lambda, r)^\Gamma$ is a cyclic $\mathfrak{a}$-module since it is a quotient of the cyclic $\mathfrak{a}$-module $P(\lambda, r)$. If $P' \in \mathcal{G}[\Gamma]$ is projective in $\mathcal{G}[\Gamma]$ and P' surjects onto $V(\lambda, r)$, then there is a natural non-zero $\mathcal{G}[\Gamma]$ morphism $P' \to P(\lambda, r)^\Gamma$. The restriction of the map to $P'[r] \to P(\lambda, r)^\Gamma \cong V(\lambda, r)$ must be surjective, so the morphism $P' \to P(\lambda, r)^\Gamma$ is a surjection since $P(\lambda, r)^\Gamma$ is generated in degree r. Thus, $P(\lambda, r)^\Gamma$ is the projective cover of $V(\lambda, r)$ in $\mathcal{G}[\Gamma]$.

(ii) The first equality follows from Proposition 4.11. The second inequality follows by noting that $P(\mu, s)^\Gamma$ is generated by $P(\mu, s)^\Gamma[s] \cong V(\mu, s)$. Proceeding as in the proof of the last part of Proposition 4.6, we see that $\operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\mu, s)^\Gamma, P(\lambda, r)^\Gamma) = \operatorname{Hom}_{\mathfrak{a}_0}(V(\mu), P(\lambda, r)^\Gamma[s])$, which gives the result.

(iii) Choose a nonzero $f \in \operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), P(\mu, s))$. Then, $f(p_{\lambda,r}) \in P(\mu, s)^+_\Gamma$, which gives $f^\Gamma \neq 0$. Thus, the map $\operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), P(\mu, s)) \to \operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\lambda, r)^\Gamma, P(\mu, s)^\Gamma)$ given by $f \mapsto f^\Gamma$ is injective.

By the previous part, we know that $[P(\mu, s) : V(\lambda, r)] = \dim \operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\lambda, r)^\Gamma, P(\mu, s)^\Gamma)$. By Proposition 4.6(vii), we also know that $[P(\mu, s) : V(\lambda, r)] = \dim \operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), P(\mu, s))$. In particular, $\operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), P(\mu, s))$ and $\operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\lambda, r)^\Gamma, P(\mu, s)^\Gamma)$ have the same dimension, whence the map is an isomorphism.

(iv) By Proposition 4.8(i), if $(\mu, s) \in \Lambda(P_j(\lambda, r)) \setminus \Gamma$, then $(\lambda, r) \preccurlyeq (\mu, s)$.

It only remains to show that the resolution terminates after finitely many steps. For each $V \in \widehat{\mathcal{G}}$, $V^{\Gamma}[k] \neq 0$ only if $(\lambda, k) \in \Gamma$ for some λ. Note that $P_j(\lambda, r)[k] = 0$ for all $k < r + j$. If $s \in \mathbb{Z}_+$ is the largest such that $(\lambda, s) \in \Gamma$ for some λ, then $P_j(\lambda, r)^{\Gamma} = 0$ for all $j > s - r$.

□

4.4 Undeformed infinitesimal Hecke algebras

For this section, we assume that $\mathfrak{a}_+ = \mathfrak{a}_1$ is an $\mathfrak{a}_0$-semisimple finite-dimensional module (i.e., $\mathfrak{a}_1 \in \mathcal{F}(\mathfrak{a}_0)$). There are many examples of such algebras, including cominuscule parabolics and quotients of realizations of type-ADE affine Lie algebras associated to the folding of the Dynkin diagram given by an order 2 diagram automorphism.

Clearly, $\mathfrak{a}_1$ is an abelian ideal in $\mathfrak{a}$, of graded degree 1. Thus, $P_j(\lambda, r) = \mathbb{P}(\wedge^j \mathfrak{a}_1 \otimes V(\lambda, r))$ is generated as an $\mathfrak{a}$-module by its degree $r + j$ component. This motivates our search for Koszulity in the picture.

We begin by restating and expanding on some of the results from the previous section in this simple case. For all $(\mu, r) \in \Lambda$, the kernel $K(\mu, r)$ of the canonical surjection $P(\mu, r) \twoheadrightarrow V(\mu, r)$ is generated as an $\mathfrak{a}$-module by $K(\mu, r)[r + 1] = \mathfrak{a}_1 \otimes V(\mu, r)$.

In this case, $\mathfrak{a}_1$ is an abelian subalgebra of $\mathfrak{a}$. As $\mathfrak{a}_0$-modules, we know $P(\mu, r)[r + k] \cong \mathbf{U}(\mathfrak{a}_1)[k] \otimes V(\mu, r)$; hence, $P(\mu, r)[k] \cong \mathrm{Sym}^k(\mathfrak{a}_1) \otimes V(\mu, r)$ as $\mathfrak{a}_0$-modules.

Furthermore, by Proposition 4.6(vii), we have

$$\mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\nu, s), P(\mu, r)) \cong \mathrm{Hom}_{\mathfrak{a}_0}(V(\nu), \mathrm{Sym}^{s-r}(\mathfrak{a}_1) \otimes V(\mu)).$$

If $N = \dim \mathfrak{a}_1$, then $\wedge^j \mathfrak{a}_1 = 0$ for all $j > N$. Therefore, $0 \to P_N(\mu, r) \to \cdots \to P_0(\mu, r) \to V(\mu, r) \to 0$ is a finite projective resolution of $V(\mu, r)$ in $\widehat{\mathcal{G}}$.

We now compute all Ext groups between simple objects in $\widehat{\mathcal{G}}$ (as in [5, 6]):

Proposition 4.14. *For all $j \in \mathbb{Z}_+$ and $(\mu, r), (\nu, s) \in \Lambda$,*

$$\operatorname{Ext}^j_{\widehat{\mathcal{G}}}(V(\mu,r), V(\nu,s)) \cong \begin{cases} \operatorname{Hom}_{\mathfrak{a}_0}(\wedge^j \mathfrak{a}_1 \otimes V(\mu), V(\nu)) & \text{if } j = s-r; \\ \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Truncating the above projective resolution to obtain the short exact sequence $0 \to \operatorname{im} d_j \to P_{j-1}(\mu, r) \xrightarrow{d_{j-1}} \operatorname{im} d_{j-1} \to 0$ in $\widehat{\mathcal{G}}$ yields

$$\operatorname{Ext}^j_{\widehat{\mathcal{G}}}(V(\mu,r), V(\nu,s)) \cong \operatorname{Ext}^1_{\widehat{\mathcal{G}}}(\operatorname{im} d_{j-1}, V(\nu,s)).$$

Moreover, applying $\operatorname{Hom}_{\widehat{\mathcal{G}}}(-, V(\nu,s))$ gives the exact sequence

$$0 \to \operatorname{Hom}_{\widehat{\mathcal{G}}}(\operatorname{im} d_{j-1}, V(\nu,s)) \to \operatorname{Hom}_{\widehat{\mathcal{G}}}(P_{j-1}(\mu,r), V(\nu,s)) \to$$
$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(\operatorname{im} d_j, V(\nu,s)) \to \operatorname{Ext}^1_{\widehat{\mathcal{G}}}(\operatorname{im} d_{j-1}, V(\nu,s)) \to 0.$$

We now claim that $\operatorname{Hom}_{\widehat{\mathcal{G}}}(\operatorname{im} d_j, V(\nu,s)) = 0$ if $j \neq s - r$. To see this, first note that, by Corollary 4.9, $\operatorname{im} d_j$ has no graded component of degree less than $j + r$, whence $\operatorname{Hom}_{\widehat{\mathcal{G}}}(\operatorname{im} d_j, V(\nu,s)) = 0$ if $j < s - r$.

Moreover, by Corollary 4.9 again, for $v \in (\operatorname{im} d_j)[s]$,

$$v = \sum_p (u_p \otimes 1) d_j(1 \otimes w_p),$$

with $w_p \in \wedge^j \mathfrak{a}_1 \otimes V(\mu, r)$ and $u_p \in \mathbf{U}(\mathfrak{a}_1)[s - r - j]$ for all p. If $j + r < s$ and $f \in \operatorname{Hom}_{\widehat{\mathcal{G}}}(\operatorname{im} d_j, V(\nu,s))$, then $f(d_j(1 \otimes w_p)) = 0$ $\forall p$ since $d_j(1 \otimes w_p) \in (\operatorname{im} d_j)[r + j]$, whence $f(v) = 0$. This proves the claim, and, thus,

$$\operatorname{Ext}^j_{\widehat{\mathcal{G}}}(V(\mu,r), V(\nu,s)) \cong \operatorname{Ext}^1_{\widehat{\mathcal{G}}}(\operatorname{im} d_{j-1}, V(\nu,s)) = 0 \ \forall j \neq s - r.$$

Now suppose $j = s - r$. By Proposition 4.6, $P_{s-r-1}(\mu, r)$ is the projective cover of $\wedge^{s-r-1} \mathfrak{a}_1 \otimes V(\mu, r)$ in $\widehat{\mathcal{G}}$ since $\wedge^{s-r-1} \mathfrak{a}_1 \otimes V(\mu, r)$ is concentrated in degree $s - 1$. By the note preceding the lemma, we know that $P_{s-r-1}(\mu, r)$ is generated in degree $s - 1$;

thus, every morphism $P_{s-r-1}(\mu, r)$ is defined by its action on $P_{s-r-1}(\mu, r)[s-1]$. In particular, $\mathrm{Hom}_{\widehat{\mathcal{G}}}(P_{s-r-1}(\mu, r), V(\nu, s)) = 0$; hence,

$$
\begin{aligned}
& \mathrm{Ext}^{s-r}_{\widehat{\mathcal{G}}}(V(\mu, r), V(\nu, s)) \cong \mathrm{Ext}^{1}_{\widehat{\mathcal{G}}}(\mathrm{im}\, d_{s-r-1}, V(\nu, s)) \\
\cong\ & \mathrm{Hom}_{\widehat{\mathcal{G}}}(\mathrm{im}\, d_{s-r}, V(\nu, s)) \\
\cong\ & \mathrm{Hom}_{\mathfrak{a}_0}((\mathrm{im}\, d_{s-r})[s], V(\nu)) \\
\cong\ & \mathrm{Hom}_{\mathfrak{a}_0}((\wedge^{s-r}\mathfrak{a}_1) \otimes V(\mu), V(\nu)),
\end{aligned}
$$

where the final isomorphism follows from Corollary 4.9. □

Notice that this is a general case of Theorem 4.3; thus,

Proof of Theorem 4.3. This follows from Proposition 4.14. □

We next prove a result on the Ext-groups inside truncated subcategories $\mathcal{G}[\Gamma]$ where Γ is a finite, interval closed subset of Λ.

Proposition 4.15. *Suppose $\Gamma \subset \Lambda$ is finite and interval-closed. For all $(\mu, r), (\nu, s) \in \Gamma$ and $j \geq 0$,*

$$\mathrm{Ext}^{j}_{\mathcal{G}[\Gamma]}(V(\mu, r), V(\nu, s)) \cong \mathrm{Ext}^{j}_{\widehat{\mathcal{G}}}(V(\mu, r), V(\nu, s)).$$

Proof. Let $(\mu, s) \in \Gamma$. By Proposition 4.8, we have a projective resolution $P_\bullet(\mu, s)$ of $V(\mu, s)$ in $\widehat{\mathcal{G}}$, while Proposition 4.13 shows that there is a projective resolution $P_\bullet(\mu, s)^\Gamma$ for $V(\lambda, r)$ in $\mathcal{G}[\Gamma]$. Moreover, for all $(\lambda, r) \in \Gamma$, $\mathrm{Hom}_{\widehat{\mathcal{G}}}(P_j(\mu, s), V(\lambda, r)) \cong \mathrm{Hom}_{\mathcal{G}[\Gamma]}(P_j(\mu, s)^\Gamma, V(\lambda, r))$ by Proposition 4.13. Now using Proposition 4.12, the natural map between the following complexes

$$\mathrm{Hom}_{\widehat{\mathcal{G}}}(P_\bullet(\mu, s), V(\lambda, r)) \to \mathrm{Hom}_{\mathcal{G}[\Gamma]}(P_\bullet(\mu, s)^\Gamma, V(\lambda, r))$$

is an isomorphism. □

We end this section by proving Theorem 4.4 in greater generality.

Proof of Theorem 4.4. For each $\Gamma \subset \Lambda$, the objects of $\mathcal{G}[\Gamma]$ are the finite-dimensional $\mathbb{Z}_+$-grade $\mathfrak{a}$-modules with Jordan-Hölder factors in Γ. By finite-dimensionality, all objects in $\mathcal{G}[\Gamma]$ must also be finite length, whence $\mathcal{G}[\Gamma]$ is a finite length category. Restricting the partial order $\preccurlyeq$ on Λ to Γ, we know that the simple objects in $\mathcal{G}[\Gamma]$ are parametrized by the poset $(\Gamma, \preccurlyeq)$. For each $(\lambda, r) \in \Gamma$, $\{(\mu, s) \in \Gamma \ : \ (\mu, s) \prec (\lambda, r)\}$ is finite since each $\mathfrak{a}_i$ is finite-dimensional. Proposition 4.10 shows that $(\Gamma, \prec)$ is directed, so $\mathcal{G}[\Gamma]$ is directed for all $\Gamma \subset \Lambda$.

Next, if Γ is finite and interval-closed under $\preccurlyeq$, then Proposition 4.13 shows that each simple $V(\lambda, r) = \Delta(V(\lambda, r)) \in \mathcal{G}[\Gamma]$ has a projective cover $P(\lambda, r)^\Gamma$. Moreover, $[P(\lambda, r)^\Gamma : V(\mu, s)] = [P(\lambda, r), V(\mu, s)]$, so, if $(\mu, s) \in \Gamma$ and $[P(\lambda, r)^\Gamma : V(\mu, s)] \neq 0$, then $(\lambda, r) \prec (\mu, s)$ by Proposition 4.6. Thus, $\mathcal{G}[\Gamma]$ is a highest weight category.

Finally, suppose Γ is finite and interval closed, whence $\mathcal{G}[\Gamma]$ is a highest weight category. Define

$$P(\Gamma) := \bigoplus_{(\lambda,r)\in\Gamma} P(\lambda, r), \quad \mathfrak{B}(\Gamma) := \operatorname{End}_{\widehat{\mathcal{G}}} P(\Gamma), \quad \mathfrak{B}^\Gamma(\Gamma) := \operatorname{End}_{\mathcal{G}[\Gamma]}(P(\Gamma)^\Gamma).$$

We can rewrite these to obtain

$$\mathfrak{B}(\Gamma) = \bigoplus_{(\mu,r),(\nu,s)\in\Gamma} \operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\mu, r), P(\nu, s))$$

and, similarly,

$$\mathfrak{B}^\Gamma(\Gamma) = \bigoplus_{(\mu,r),(\nu,s)\in\Gamma} \operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\mu, r)^\Gamma, P(\nu, s)^\Gamma).$$

Using Proposition 4.6 and Proposition 4.11, it follows that

$$\begin{aligned} \mathfrak{B}(\Gamma) &= \bigoplus_{(\mu,r)\preccurlyeq_\Psi(\nu,s)\in\Gamma} \operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\mu, r), P(\nu, s)) \\ &\cong \bigoplus_{(\mu,r)\preccurlyeq_\Psi(\nu,s)\in\Gamma} \operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\mu, r)^\Gamma, P(\nu, s)^\Gamma) \\ &= \mathfrak{B}^\Gamma(\Gamma). \end{aligned}$$

Notice that $\mathfrak{B}(\Gamma)$ is graded via

$$\mathfrak{B}(\Gamma)[k] = \bigoplus_{(\lambda,r),(\mu,r-k)\in\Gamma} \mathrm{Hom}_{\mathcal{G}}(P(\lambda,r), P(\mu,r-k)).$$

It is standard that

$$\mathrm{Hom}_{\mathcal{G}[\Gamma]}(P(\Gamma)^{\Gamma}, -) : \mathcal{G}[\Gamma] \to \mathrm{Mod}-\mathfrak{B}(\Gamma)$$

is an equivalence of categories. It follows that $\mathfrak{B}(\Gamma)$ is finite-dimensional from the fact $\mathcal{G}[\Gamma] \subset \mathcal{G}$. Since $\mathcal{G}[\Gamma]$ is a highest weight category, $\mathrm{Mod}-\mathfrak{B}(\Gamma)$ is also a highest weight category, so by [7],$\mathfrak{B}(\Gamma)$ is quasi-hereditary. Simple modules over $\mathfrak{B}(\Gamma)$ correspond to simple modules $V(\lambda, r)$ in $\mathcal{G}[\Gamma]$. Thus, the simple modules in $\mathfrak{B}(\Gamma)$ are given by

$$\mathrm{Hom}_{\mathcal{G}}(P(\Gamma)^{\Gamma}, V(\lambda,r)) = \mathrm{Hom}_{\mathcal{G}}(P(\lambda,r)^{\Gamma}, V(\lambda,r)) \cong \mathbb{C},$$

whence $\mathfrak{B}(\Gamma)$ is basic.

Finally, by Proposition 4.14, the global dimension of $\mathfrak{B}(\Gamma)$ is at most $\max\limits_{(\lambda,r),(\mu,s)\in\Gamma} |r-s|$.

$\square$

4.5 Faces of polytopes and Koszul algebras

We now work toward proving Theorem 4.1.

Henceforth, $\mathfrak{a}_0 = \mathfrak{g}$ is a semisimple Lie algebra, and $\mathfrak{a}_+ = \mathfrak{a}_1 \in \mathcal{F}(\mathfrak{a}_0)$, with $[\mathfrak{h}, \mathfrak{a}_1] \neq 0$. In other words, $\mathfrak{a}_1$ is not a sum of trivial $\mathfrak{g}$-modules, and, equivalently, $\mathcal{S} := \mathrm{wt}(\mathfrak{a}_1) \neq \{0\}$. Fix a nonempty positive weak $\mathbb{Q}$-face $\Psi \subset \mathcal{S}$. Note that $\mathcal{S}$ is $\mathcal{W}$-stable, so the sum of all of the elements in $\mathcal{S}$ is $\mathcal{W}$-stable. Thus, the origin is an interior point of the convex hull of $\mathcal{S}$, so by Theorem 5, Ψ satisfies the following condition in $\mathcal{S}$:

$$\text{If } \sum_{\alpha\in\Psi} a_\alpha \alpha = \sum_{\beta\in\mathcal{S}} b_\beta \beta, \text{ for } a_\alpha, b_\beta \in \mathbb{Z}_+, \tag{4.1}$$

$$\text{then } \sum_{\alpha} a_\alpha \leq \sum_{\beta} b_\beta, \text{ with equality if and only if } \beta \in \Psi \text{ whenever } b_\beta > 0.$$

Given a weight $\xi \in P$ of $\mathfrak{g}$, define $\mathcal{S}(\xi) := \{\mu \in \mathcal{S} : (\mu, \xi) = \max_{\mathcal{S}} \xi\}$.

For convenience, we recall the following from Definition 4.1:

If $\Psi \subset \mathrm{wt}(\mathfrak{a}_1)$ and $\mu, \nu \in \mathfrak{h}^*$, we say $\mu \leq_\Psi \nu$ if and only if $\nu - \mu \in \mathbb{Z}_+\Psi$. Furthermore, if $\mu \leq_\Psi \nu$, then

$$d_\Psi(\mu, \nu) := \min\{\sum_{\beta \in \Psi} m_\beta : \nu - \mu = \sum_{\beta \in \Psi} m_\beta \beta,\ m_\beta \in \mathbb{Z}_+\ \forall \beta\}.$$

In the following proposition, we define a partial order $\preccurlyeq_\Psi$ on Λ which is a refinement of the partial order $\preccurlyeq$. Notice that $(\mu, r) \preccurlyeq (\lambda, s)$ if and only if $\lambda - \mu = \sum_{\nu \in \mathrm{wt}(\mathfrak{a}_1)} m_\nu \nu$ with $m_\nu \in \mathbb{Z}_+$ for all $\nu \in \mathrm{wt}(\mathfrak{a}_1)$ and $\sum_{\nu \in \mathrm{wt}(\mathfrak{a}_1)} m_\nu = s - r$.

Proposition 4.16. *Suppose $\Psi \subset \mathcal{S} = \mathrm{wt}(\mathfrak{a}_1)$ is a positive weak $\mathbb{Q}$-face of $\mathrm{wt}(\mathfrak{a}_1)$.*

(i) *$\leq_\Psi$ is a partial order on $\mathfrak{h}^*$. For all $\eta \leq_\Psi \mu \leq_\Psi \nu$,*

$$d_\Psi(\eta, \mu) + d_\Psi(\mu, \nu) = d_\Psi(\eta, \nu). \tag{4.2}$$

(ii) *Ψ induces a partial order $\preccurlyeq_\Psi$ on $\Lambda = P^+ \times \mathbb{Z}_+$ given by*

$$(\eta, r) \preccurlyeq_\Psi (\mu, r') \text{ if and only if } \eta \leq_\Psi \mu \text{ and } d_\Psi(\eta, \mu) = r' - r.$$

If $\Gamma \subset \Lambda$ is interval-closed under $\preccurlyeq_\Psi$, then, if $(\mu, r), (\lambda, r'') \in \Gamma$ and $(\mu, r) \preccurlyeq_\Psi (\lambda, r'')$,

$$(\mu, r) \preccurlyeq (\nu, r') \preccurlyeq (\lambda, r'') \Rightarrow (\nu, r') \in \Gamma.$$

(iii) *Suppose $\xi \in P^+$ with $\max_{\mathcal{S}} \xi > 0$. If we set $\Psi := \mathcal{S}(\xi)$, then $\{\mu \in P^+ : \mu \leq_\Psi \eta\}$ is finite for every $\eta \in P^+$. Similarly, if $\Psi := \mathcal{S}(-\xi)$ or $\mathcal{S}(w_\circ(\xi))$ where $w_\circ$ is the longest element of W, then $\{\mu \in P^+ : \eta \leq_\Psi \mu\}$ is finite for every $\eta \in P^+$.*

Proof. (i) For all $\Psi \subset \mathcal{S}$, it is clear that $\leq_\Psi$ is reflexive and transitive:

Suppose that $\mu, \nu, \lambda \in \mathfrak{h}^*$. Since $\mu - \mu = 0 \in \mathbb{Z}_+\Psi$, $\mu \leq_\Psi \mu$. If $\mu \leq_\Psi \nu \leq_\Psi \lambda$, $\lambda - \nu \in \mathbb{Z}_+\Psi$ and $\nu - \mu \in \mathbb{Z}_+\Psi$, and transitivity follows by noting that $\lambda - \mu = (\lambda - \nu) + (\nu - \mu) \in \mathbb{Z}_+\Psi$.

Now, suppose Ψ is a positive weak $\mathbb{Q}$-face of $\mathrm{wt}(\mathfrak{a}_1)$ and suppose that $\mu \leq_\Psi \nu$ and $\nu \leq_\Psi \mu$. Let $\nu - \mu = \sum_{\beta\in\Psi} r_\beta \beta$ and $\mu - \nu = \sum_{\beta\in\Psi} m_\beta \beta$ with $r_\beta, m_\beta \in \mathbb{Z}_+$ for all β. Combining these gives

$$0 = \sum_{\beta\in\Psi} (r_\beta + m_\beta)\beta.$$

Since Ψ satisfies (4.1), we have $0 = r_\beta + m_\beta$, so $r_\beta = m_\beta = 0$. In particular, $\nu - \mu = 0$ which gives antisymmetry.

Next, given that $\mu \leq_\Psi \nu$, suppose that

$$\nu - \mu = \sum_{\beta\in\Psi} r_\beta \beta = \sum_{\beta\in\Psi} m_\beta \beta.$$

Applying (4.1) to each sum gives $\sum_{\beta\in\Psi} r_\beta \leq \sum_{\beta\in\Psi} m_\beta \leq \sum_{\beta\in\Psi} r_\beta$, which shows that $d_\Psi(\mu, \nu)$ is taken over a singleton set. This uniqueness and the fact that $\lambda - \mu = (\lambda - \nu) + (\nu - \mu)$ shows that $d_\Psi(\mu, \nu) + d_\Psi(\nu, \lambda) = d_\Psi(\mu, \lambda)$.

(ii) Reflexivity and asymmetry both follow from the fact that $d_\Psi(\mu, \mu) = 0$. Transitivity follows from equation (4.2): if $(\mu, r) \preccurlyeq_\Psi (\nu, r') \preccurlyeq_\Psi (\lambda, r'')$, then $\mu \leq_\Psi \lambda$ and $d_\Psi(\mu, \lambda) = d_\Psi(\mu, \nu) + d_\Psi(\nu, \lambda) = (r' - r) + (r'' - r') = r'' - r$.

Now, suppose $(\mu, r) \preccurlyeq (\nu, s) \preccurlyeq (\lambda, t)$ and $(\mu, r) \preccurlyeq_\Psi (\lambda, t) \in \Gamma$. Then, by the remark preceding the proposition, we can write $\lambda - \nu = \displaystyle\sum_{i=1}^{t-s} \beta_i$ and $\nu - \mu = \displaystyle\sum_{j=1}^{s-r} \eta_j$, where $\beta_i, \eta_j \in \mathrm{wt}(\mathfrak{a}_1)$ for all i, j.

Since $\lambda - \mu = (\lambda - \nu) + (\nu - \mu)$ and $(t - s) + (s - r) = t - r = d_\Psi(\mu, \lambda)$, $\beta_i, \eta_j \in \Psi$ for all i, j. This gives $(\mu, r) \preccurlyeq_\Psi (\nu, s) \preccurlyeq_\Psi (\lambda, t)$, so $(\nu, s) \in \Gamma$.

(iii) If $\mu, \xi \in P^+$, then writing μ as a nonnegative rational linear combination of simple roots (by [10, Exercise 13.8]), $(\mu, \xi) \geq 0$. If $\mu \leq_\Psi \eta$, then $(\xi, \eta - \mu) = k \max_{\mathcal{S}} \xi$ for some $k \geq 0$. We now have $0 \leq (\xi, \mu) = (\xi, \eta) - k \max_{\mathcal{S}} \xi \leq (\xi, \eta)$. In particular, $\{\mu \in P^+ \mid \mu \leq_\Psi \eta\}$ is the intersection of the discrete set P^+ with the compact set $\{\lambda \in \mathfrak{h}^* \mid 0 \leq (\xi, \lambda) \leq (\xi, \eta)\}$. Thus, $\{\mu \in P^+ \mid \mu \leq_\Psi \eta\}$ is finite.

The result for $\Psi = \mathcal{S}(-\xi)$ now follows in a similar fashion by noting that $\eta \leq_\Psi \mu$ implies that $(\eta, -\xi) \leq (\mu, -\xi) \leq 0$.

Finally, consider $\Psi = \mathcal{S}(w_\circ(\xi))$. Recall that $(\eta, w_\circ(\xi)) = (w_\circ^{-1}(\eta), \xi)$, so $\max_{\mathcal{S}} w_\circ(\xi) \geq 0$. Furthermore, since $-w_\circ(\xi) \in P^+$, for every $\mu \in P^+$, we must have $(\xi, \mu) \leq 0$. For all $\mu \in P^+$ such that $\eta \leq_\Psi \mu$, we now have $(\xi, \eta) \leq (\xi, \mu) \leq 0$. Thus, $\{\mu \in P^+ \mid \eta \leq_\Psi \mu\}$ is again the intersection of P^+ with a compact set; hence, it must be finite.

$\square$

We mention that the following is also a partial order on Λ: $(\eta, r) \preccurlyeq'_\Psi (\mu, r')$ if $\mu \leq_\Psi \eta$ and $d_\Psi(\mu, \eta) = r' - r$. If $\mathfrak{a}_1 = \mathfrak{g}_{ad}$, this partial order, $\preccurlyeq'_\Psi$, is the one considered in [6]; however, we will work with $\preccurlyeq_\Psi$ exclusively.

In what follows, we need some standard results on semisimple Lie algebras and their finite-dimensional modules. We quote the following lemma from [6].

Lemma 4.17. *Suppose $\mathfrak{g}$ is a complex semisimple Lie algebra, $V \in \mathcal{F}(\mathfrak{g})$, and $\lambda, \mu \in P^+$. Let $\{x_{\alpha_i}^\pm, h_i : i \in I\}$ be Chevalley generators of $\mathfrak{g}$. Define $V^+ := \{v \in V : \mathfrak{n}^+ v = 0\}$.*

(i) $\dim \operatorname{Hom}_{\mathfrak{g}}(V(\lambda), V) = \dim(V^+ \cap V_\lambda)$.

(ii) *As vector spaces,*

$$\operatorname{Hom}_{\mathfrak{g}}(V \otimes V(\mu), V(\lambda)) \cong \{v \in V_{\lambda-\mu} : (x_{\alpha_i}^+)^{\mu(h_i)+1} v = 0 = (x_{\alpha_i}^-)^{\lambda(h_i)+1} v\}.$$

We now proceed as in [6]. Our first result involves $\mathfrak{g}$-modules. Recall the following from Definition 4.1:

Define $N_\Psi := \sum_{\mu\in\Psi} \dim(\mathfrak{a}_1)_\mu$. Given $\mu, \nu \in P^+$, define $[\mu,\nu]_\Psi := (\leq_\Psi \nu) \cap (\mu \leq_\Psi)$, where

$$\leq_\Psi \nu := \{\eta \in P^+ : \eta \leq_\Psi \nu\}, \qquad \mu \leq_\Psi := \{\eta \in P^+ : \mu \leq_\Psi \eta\}.$$

Lemma 4.18. *Suppose $\Psi \subset \mathrm{wt}(\mathfrak{a}_1)$ is a weak $\mathbb{Q}$-face of $\mathrm{wt}(\mathfrak{a}_1)$. Define $\lambda_\Psi \in P$ by* $\lambda_\Psi := \sum_{\mu\in\Psi} (\dim(\mathfrak{a}_1)_\mu)\mu$.

(i) *If $\nu, \nu+\lambda_\Psi \in P^+$, then $\dim \mathrm{Hom}_{\mathfrak{g}}(\bigwedge^{N_\Psi} \mathfrak{a}_1 \otimes V(\nu), V(\nu+\lambda_\Psi)) \leq 1$.*

(ii) *Given $\eta \in P^+$, there exists $\nu \in P^+$ such that $\nu, \nu+\lambda_\Psi \in P^+$ are both regular, $\eta \leq \nu$, and the above bound is attained.*

Proof. Applying Lemma 4.17, $\mathrm{Hom}_{\mathfrak{g}}(\bigwedge^{N_\Psi} \mathfrak{a}_1 \otimes V(\nu), V(\nu+\lambda_\Psi))$ has the same dimension as $\{v \in (\bigwedge^{N_\Psi} \mathfrak{a}_1)_{\lambda_\Psi} \mid (x_{\alpha_i}^+)^{\nu(h_i)+1}.v = (x_{\alpha_i}^-)^{(\nu+\lambda_\Psi)(h_i)+1}.v = 0\}$. It follows that

$$\dim \mathrm{Hom}_{\mathfrak{g}}\left(\bigwedge^{N_\Psi} \mathfrak{a}_1 \otimes V(\nu), V(\nu+\lambda_\Psi)\right) \leq \dim \left(\bigwedge^{N_\Psi} \mathfrak{a}_1\right)_{\lambda_\Psi}.$$

Suppose that $\wedge_{i=1}^{N_\Psi} x_i \in (\bigwedge^{N_\Psi} \mathfrak{a}_1)_{\lambda_\Psi}$ with $x_i \in (\mathfrak{a}_1)_{\beta_i}$. Then, $\sum_{i=1}^{N_\Psi} \beta_i = \lambda_\Psi$. By (4.1), $\beta_i \in \Psi$ for all i. Therefore, $(\bigwedge^{N_\Psi} \mathfrak{a}_1)_{\lambda_\Psi} = \bigwedge^{\mu\in\Psi}(\wedge^{\dim(\mathfrak{a}_1)_\mu}(\mathfrak{a}_1)_\mu)$ is one-dimensional, and the first statement follows.

For the second statement, define $c_i := \eta(h_i) \in \mathbb{Z}_+$ and $d_i := \lambda_\Psi(h_i) \in \mathbb{Z}$. Let $\mathbf{v}$ be a non-zero vector in $(\bigwedge^{N_\Psi} \mathfrak{a}_1)_{\lambda_\Psi}$. Since $\bigwedge^{N_\Psi} \mathfrak{a}_1$ is a finite-dimensional $\mathfrak{a}_0$-module, there exists $k > 0$ such that $c_i + d_i + k \in \mathbb{N}$ for all i and

$$(x_{\alpha_i}^+)^{c_i+k+1}.\mathbf{v} = (x_{\alpha_i}^-)^{c_i+d_i+k+1}.\mathbf{v} = 0.$$

Let $\rho := \sum_{i\in I} \omega_i$, and define $\nu = \eta + 2k\rho \geq \eta$. Then, $\nu(h_i) = \eta(h_i) + 2k\rho(h_i) = c_i + 2k \geq 2k > 0$ and $(\nu+\lambda_\Psi)(h_i) = c_i + d_i + 2k > 0$ for all i. Furthermore, from Lemma 4.17, $\mathrm{Hom}_{\mathfrak{g}}(\bigwedge^{N_\Psi} \mathfrak{a}_1 \otimes V(\mu), V(\mu+\lambda_\Psi)) \cong \mathbb{C}\mathbf{v}$. $\square$

We now adapt the earlier analysis to this setting. Fix some $(\lambda, r) \in \Lambda$, and define

$$\Lambda(\preccurlyeq_\Psi (\lambda, r)) := \{(\nu, s) \in \Lambda : (\nu, s) \preccurlyeq_\Psi (\lambda, r)\}.$$

Let $\Gamma \subset \Lambda(\preccurlyeq_\Psi (\lambda, r))$ be interval-closed under $\preccurlyeq_\Psi$. We now show several results concerning the category $\mathcal{G}[\Gamma]$, which provide a key step towards proving Theorem 4.1.

Lemma 4.19. *Suppose that $\Gamma \subset \Lambda(\preccurlyeq_\Psi (\lambda, r))$ is finite and interval-closed under $\preccurlyeq_\Psi$.*

(i) *If $\operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\nu, t)^\Gamma, P(\eta, s)^\Gamma) \neq 0$, then $(\eta, s) \preccurlyeq_\Psi (\nu, t)$.*

(ii) *If $\operatorname{Ext}^j_{\mathcal{G}[\Gamma]}(V(\eta, s), V(\nu, t)) \neq 0$, then $(\eta, s) \preccurlyeq_\Psi (\nu, t)$ and $j = d_\Psi(\eta, \nu)$.*

(iii) $\operatorname{gldim} \mathcal{G}[\Gamma] \leq N_\Psi$. *For some $(\lambda, r) \in \Lambda$ and some $\Gamma \subset \Lambda(\preccurlyeq_\Psi (\lambda, r)$, the upper bound is attained.*

Proof. First, note that Γ is also interval-closed under $\preccurlyeq$, whence all the earlier results apply here:

Suppose $(\nu_1, s_1), (\nu_2, s_2) \in \Gamma$ and $(\nu_1, s_1) \preccurlyeq (\lambda, s) \preccurlyeq (\nu_2, s_2)$. Then, $(\nu_1, s_1) \preccurlyeq (\nu_2, s_2) \preccurlyeq (\mu, r)$, so by Proposition 4.16, $(\nu_1, s_1) \preccurlyeq_\Psi (\nu_2, s_2)$ and, furthermore, $(\lambda, s) \in \Gamma$.

(i) Suppose $\operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\nu, t)^\Gamma, P(\eta, s)^\Gamma) \neq 0$. Then, it follows that $\operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\nu, t), P(\eta, s)) \neq 0$, and, by Proposition 4.14, $\operatorname{Hom}_{\mathfrak{g}}(V(\nu), \operatorname{Sym}^{t-s}(\mathfrak{a}_1) \otimes V(\eta)) \neq 0$. Using Lemma 4.17, ν is a highest weight of $\operatorname{Sym}^{t-s}(\mathfrak{a}_1) \otimes V(\eta)$. Hence, $\nu - \eta$ is a weight of $\operatorname{Sym}^{t-s}(\mathfrak{a}_1)$, and we can write $\nu - \eta = \displaystyle\sum_{i=1}^{t-s} \xi_i$ where $\xi_i \in \operatorname{wt}(\mathfrak{a}_1)$ for all i. On the other hand, $(\eta, s), (\nu, t) \preccurlyeq_\Psi (\lambda, r)$, which gives

$$\lambda - \eta = \sum_{\mu \in \Psi} m_\mu \mu, \ \sum_{\mu \in \Psi} m_\mu = r - s; \qquad \lambda - \nu = \sum_{\mu \in \Psi} n_\mu \mu, \ \sum_{\mu \in \Psi} n_\mu = r - t.$$

Then, $\lambda - \eta = (\lambda - \nu) + (\nu - \eta)$, which gives

$$\lambda - \eta = \sum_{\mu \in \Psi} m_\mu \mu = \sum_{\mu \in \Psi} n_\mu \mu + \sum_{i=1}^{t-s} \xi_i.$$

Furthermore, $\sum_{\mu\in\Psi} m_\mu = r-s = (r-t)+(t-s) = \sum_{\mu\in\Psi} n_\mu\mu + \sum_{i=1}^{t-s} 1$. By (4.1),$\xi_i \in \Psi \; \forall i$ whence $(\eta, s) \preccurlyeq_\Psi (\nu, t)$.

(ii) Suppose $\mathrm{Ext}^j_{\mathcal{G}[\Gamma]}(V(\eta, s), V(\nu, t)) \neq 0$. By Proposition 4.14 and Proposition 4.15, $j = t - s$ and $\mathrm{Hom}_{\mathfrak{a}_0}(\wedge^j \mathfrak{a}_1 \otimes V(\eta), V(\nu)) \neq 0$. Using Lemma 4.17, this means that $\nu - \eta$ is a weight of $\wedge^j \mathfrak{a}_1$. The result now follows from an argument identical to that for the previous part.

(iii) Since $\mathcal{G}[\Gamma]$ is finite length, it suffices to show that if $(\eta, s), (\nu, t) \in \Gamma$, then

$$\mathrm{Ext}^j(V(\eta, s), V(\nu, t)) = 0, \qquad \forall\, j > N_\Psi.$$

Now if $\mathrm{Ext}^j_{\mathcal{G}[\Gamma]}(V(\eta, s), V(\nu, t)) \neq 0$, then, by the previous part, $\wedge^j \mathfrak{a}_1 \neq 0$. In particular, this means that $j \leq N_\Psi$, which proves that $\mathrm{gldim}\,\mathcal{G}[\Gamma] \leq N_\Psi$.

Next, use Lemma 4.18 to produce $\nu \in P^+$ such that

$$\dim \mathrm{Hom}_{\mathfrak{a}_0}\left(\bigwedge^{N_\Psi} \mathfrak{a}_1 \otimes V(\nu), V(\nu + \lambda_\Psi) \right) = 1.$$

Choose $r \in \mathbb{Z}_+$, and suppose $(\nu, r), (\nu + \lambda_\Psi) \in \Gamma$. By Proposition 4.14 and Proposition 4.15, $\mathrm{Ext}^{N_\Psi}_{\mathcal{G}[\Gamma]}(V(\mu, r), V(\mu + \lambda_\Psi, r + N_\Psi)) \neq 0$, so $\mathrm{gldim}\,\mathcal{G}[\Gamma] = N_\Psi$.

$\square$

The following result is shown just like its analogue in [6].

Lemma 4.20. *Let A be an associative $\mathbb{C}$-algebra; define $\mathbf{A} := A \otimes \mathbf{V} \otimes \mathbf{V}^{\circledast}$. Suppose $\Psi \subset \mathrm{wt}(\mathfrak{a}_1)$ satisfies* (4.1).

(i) *If A is a $\mathbb{Z}_+$-graded algebra, then $\mathbf{A}_\Psi(F)$ is a graded subalgebra of $\mathbf{A}_\Psi(G)$, for all $F \subset G \subset P^+$.*

(ii) *Suppose A is also a $\mathfrak{g}$-module algebra with $\mathfrak{g}(A[k]) \subset A[k]\ \forall k$. Then $\mathbf{A}^{\mathfrak{g}}$ is also a $\mathbb{Z}_+$-graded algebra: $\mathbf{A}^{\mathfrak{g}}[k] := (\mathbf{A}[k])^{\mathfrak{g}}$. Moreover, if $F \subset G \subset P^+$, then $\mathbf{A}_\Psi^{\mathfrak{g}}(F)$ is a graded subalgebra of $\mathbf{A}_\Psi^{\mathfrak{g}}(G)$.*

Proof. (i) Recall that $\mathbf{A}_\Psi(F) = \bigoplus_{\nu \leq_\Psi \mu \in F} \mathbf{A}_\Psi(\mu,\nu)$. Define $\mathbf{A}_\Psi(F)[k] = \bigoplus_{\substack{\mu,\nu \in F : \nu \leq_\Psi \mu \\ d_\Psi(\nu,\mu) = k}} \mathbf{A}_\Psi(\mu,\nu)$.

Then, if $\mu,\nu,\lambda,\eta \in F$ and $\nu \leq_\Psi \mu, \eta \leq_\Psi \lambda$,

$$\begin{aligned} \mathbf{A}_\Psi(\mu,\nu)\mathbf{A}_\Psi(\lambda,\eta) &= 1_\mu \mathbf{A}[d_\Psi(\nu,\mu)]1_\nu 1_\lambda \mathbf{A}[d_\Psi(\eta,\lambda)]1_\eta \\ &\subset \delta_{\nu,\lambda} 1_\mu \mathbf{A}[d_\Psi(\nu,\mu)]\mathbf{A}[d_\Psi(\eta,\lambda)]1_\eta \end{aligned}.$$

If $\nu \neq \lambda$, then $\mathbf{A}_\Psi(\mu,\nu)\mathbf{A}_\Psi(\lambda,\eta) = 0$. If $\lambda = \nu$, then $\eta \leq_\Psi \mu$ and

$$\begin{aligned} \mathbf{A}_\Psi(\mu,\nu)\mathbf{A}_\Psi(\nu,\eta) &\subset 1_\mu \mathbf{A}[d_\Psi(\nu,\mu)]\mathbf{A}[d_\Psi(\eta,\nu)]1_\eta \\ &\subset 1_\mu \mathbf{A}[d_\Psi(\nu,\mu) + d_\Psi(\eta,\nu)]1_\eta \\ &= 1_\mu \mathbf{A}[d_\Psi(\eta,\mu)]1_\eta \end{aligned},$$

where the last step follows by Proposition 4.16. Therefore, $\mathbf{A}_\Psi(F)$ is $\mathbb{Z}_+$-graded for all $F \subset P^+$.

(ii) It suffices to show that $\mathbf{A}_\Psi^{\mathfrak{g}}(F)$ is graded for all $F \subset P^+$. For $k \in \mathbb{Z}_+$, let $\mathbf{A}_\Psi^{\mathfrak{g}}(F)[k] = (\mathbf{A}_\Psi(F)[k])^{\mathfrak{g}}$. We only need to show that $\mathbf{A}_\Psi^{\mathfrak{g}}(F)[j]\mathbf{A}_\Psi^{\mathfrak{g}}(F)[k] \subset \mathbf{A}_\Psi^{\mathfrak{g}}(F)[j+k]$ for all $j,k \in \mathbb{Z}_+$. However, this follows immediately from the grading on $\mathbf{A}_\Psi(F)$ and the fact that the product of two $\mathfrak{g}$-invariant element of $\mathbf{A}$ is also $\mathfrak{g}$-invariant.

□

We are now ready to approach Theorem 4.1. We need to relate the category $\mathcal{G}[\Gamma]$ where Γ is finite and interval-closed under $\preccurlyeq_\Psi$ with an algebra of the form $\mathbf{S}_\Psi^{\mathfrak{g}}(F)$ for some F. Define the projection map $\pi_1 : \Lambda = P^+ \times \mathbb{Z}_+ \to P^+$.

Lemma 4.21. *Fix $(\mu, r) \in \Lambda$, and let $\Gamma \subset \Lambda((\mu, r) \preccurlyeq_\Psi)$ be a finite, interval-closed subset. Then, $\mathfrak{B}(\Gamma)^{op} \cong \mathbf{S}^{\mathfrak{g}}_\Psi(\pi_1(\Gamma))$, and* $\operatorname{gldim} \mathfrak{B}(\Gamma) \leq N_\Psi$.

Proof. By Proposition 4.14 and Lemma 4.19, if $\operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\nu, s)^\Gamma, P(\mu, r)^\Gamma) \neq 0$, then $(\mu, r) \preccurlyeq_\Psi (\nu, s)$ and $\operatorname{Hom}_{\mathfrak{g}}(V(\nu), \operatorname{Sym}^{s-r}(\mathfrak{a}_1) \otimes V(\mu)) \neq 0$. Moreover, as vector spaces, we have

$$1_\nu \mathbf{S}^{\mathfrak{g}}_\Psi[d_\Psi(\mu,\nu)]1_\mu = (V(\nu)^* \otimes \operatorname{Sym}^{d_\Psi(\mu,\nu)}(\mathfrak{a}_1) \otimes V(\mu))^{\mathfrak{g}}$$

$$\begin{aligned} &\cong \operatorname{Hom}_{\mathfrak{g}}(V(\nu), \operatorname{Sym}^{d_\Psi(\mu,\nu)}(\mathfrak{a}_1) \otimes V(\mu)) \\ &\cong \operatorname{Hom}_{\mathcal{G}}(P(\nu, r + d_\Psi(\mu,\nu)), P(\mu, r)). \end{aligned}$$

However, the product of the terms $1_\nu \mathbf{S}^{\mathfrak{g}}_\Psi[d_\Psi(\mu,\nu)]1_\mu$ is from left to right, whereas composition of terms in $\operatorname{Hom}_{\mathcal{G}}(P(\nu, r + d_\Psi(\mu,\nu)), P(\mu, r))$ is from right to left. This shows that $\mathfrak{B}(\Gamma)^{op} \cong \mathbf{S}^{\mathfrak{g}}_\Psi(\pi_1(\Gamma))$. The bound on the global dimension follows from Theorem 4.3 and Lemma 4.19. □

Theorem 8. *If $\Gamma \subset \Lambda((\mu, r) \preccurlyeq_\Psi)$ is finite and interval closed under $\preccurlyeq_\Psi$, then $\mathfrak{B}(\Gamma)^{op} \cong \mathbf{S}^{\mathfrak{g}}_\Psi(\pi_1(\Gamma))$ is Koszul as a graded $\mathbb{C}$-algebra.*

Remark. *Similar statements to Theorem 8 and Lemma 4.21 exist for finite, interval-closed subsets Γ of $\Lambda((\mu, r) \preccurlyeq_\Psi)$, and the proofs are similar.*

Proof. Let $F = \pi_1(\Gamma)$, and let $A = \mathbf{S}^{\mathfrak{g}}_\Psi(F)$. For each $(\mu, s) \in \Gamma$, let S_μ be the simple left $\mathbf{S}^{\mathfrak{g}}_\Psi(F)$-module corresponding to the idempotent 1_μ, and let $P_\mu := \mathbf{S}^{\mathfrak{g}}_\Psi(F)1_\mu$ be its projective cover. Then, $[P_\mu : S_\nu] = \dim(1_\nu \mathbf{S}^{\mathfrak{g}}_\Psi(F)1_\mu)$. The subalgebra $1_\nu \mathbf{S}^{\mathfrak{g}}_\Psi(F)[j]1_\mu = (V(\nu)^* \otimes \operatorname{Sym}^j(\mathfrak{a}_1) \otimes V(\mu))^{\mathfrak{g}}$ is isomorphic to $\operatorname{Hom}_{\mathfrak{g}}(V(\nu), \sum^j(\mathfrak{a}_1) \otimes V(\mu))$, so

$$\dim(1_\nu \mathbf{S}^{\mathfrak{g}}_\Psi(F)[j]1_\mu) \neq 0 \implies \mu \leq_\Psi \nu,\ j = d_\Psi(\nu, \mu).$$

By Theorem 4.3, the category, $\mathcal{C}$, of finite-dimensional $\mathbf{S}_\Psi^{\mathfrak{g}}(F)$-modules is equivalent to the category $\mathcal{G}[\Gamma]$. By Lemma 4.19,

$$\mathrm{Ext}^j_{\mathcal{C}}(S_\nu, S_\mu) \neq 0 \implies \nu \leq_\Psi \mu,\ j = d_\Psi(\nu, \mu).$$

This shows that the all products of the matrices $H(A,t)$ and $H(E(A),t)$ are lower triangular.

For $\nu \leq_\Psi \mu \in F$,

$$\begin{aligned}
&(H(E(A), -t)H(A,t))_{\mu,\nu} \\
&= \sum_{\xi \in F} H(E(A), -t)_{\mu,\xi} H(A,t)_{\xi,\nu} \\
&= \sum_{\nu \leq_\Psi \xi \leq_\Psi \mu} (-1)^{d_\Psi(\xi,\mu)} t^{d_\Psi(\xi,\mu)+d_\Psi(\nu,\xi)} [P_\nu : S_\xi] \dim \mathrm{Ext}^{d_\Psi(\xi,\mu)}_{\mathcal{C}}(S_\xi, S_\mu) \\
&= t^{d_\Psi(\nu,\mu)} \sum_{j \geq 0} \sum_{\nu \leq_\Psi \xi \leq_\Psi \mu} (-1)^j [P_\nu : S_\xi] \dim \mathrm{Ext}^j_{\mathcal{C}}(S_\xi, S_\nu) \\
&= t^{d_\Psi(\nu,\mu)} \sum_{j \geq 0} (-1)^j \dim \mathrm{Ext}^j_{\mathcal{C}}(P_\nu, S_\mu) \\
&= \delta_{\nu,\mu} t^{d_\Psi(\nu,\mu)} = \delta_{\nu,\mu}
\end{aligned} .$$

Thus, the matrix $H(E(A), -t)$ is the inverse of the matrix $H(A,t)$, so $A = \mathbf{S}_\Psi^{\mathfrak{g}}(F)$ is Koszul by [1, Theorem 2.11.1].

□

4.6 Proof of the main result

For this section, $\mathfrak{g} = \mathfrak{a}_0$ is semisimple, $\mathfrak{a}_1$ is a finite-dimensional $\mathfrak{g}$-module with $\mathcal{S} := \mathrm{wt}(\mathfrak{a}_1) \neq \{0\}$, and $\Psi := \mathcal{S}(\xi)$ for $\xi \in -P^+$.

By Proposition 4.16, for all $(\eta, r) \in \Lambda$, $\pi_1(\Lambda((\eta,r) \preccurlyeq_\Psi)) = \{\mu \in P^+ : \eta \leq_\Psi \mu\}$ is finite. We use this observation without mention throughout the rest of this section.

Proposition 4.22. *The algebras $\mathbf{S}_\Psi^{\mathfrak{g}}(\mu \leq_\Psi), \mathbf{S}_\Psi^{\mathfrak{g}}([\mu,\nu]_\Psi), \mathbf{S}_\Psi^{\mathfrak{g}}(\leq_\Psi \nu), \mathbf{S}_\Psi^{\mathfrak{g}}$ all have left* *global dimension at most N_Ψ, and there exist $\mu \leq_\Psi \nu$ in Λ for which this bound is attained.*

Proof. By Theorem 8 and Lemma 4.21, the result holds for the first two algebras.

Suppose $\mu \in P^+$, let S_μ be the simple left $\mathbf{S}_\Psi^{\mathfrak{g}}$-module corresponding to the idempotent $1_\mu \in \mathbf{S}_\Psi^{\mathfrak{g}}$. Its projective cover in the category, $\mathbf{S}_\Psi^{\mathfrak{g}} - mod_f$, of finite-dimensional left $\mathbf{S}_\Psi^{\mathfrak{g}}$-modules is $P_\mu := \mathbf{S}_\Psi^{\mathfrak{g}} 1_\mu$. As in the proof of Lemma 4.21, $\mu \leq_\Psi \nu$ if $1_\nu \mathbf{S}_\Psi^{\mathfrak{g}} 1_\mu \neq 0$, so we have

$$\mathbf{S}_\Psi^{\mathfrak{g}} 1_\mu = \bigoplus_{\mu \leq_\Psi \nu} 1_\nu \mathbf{S}_\Psi^{\mathfrak{g}} 1_\mu = \mathbf{S}_\Psi^{\mathfrak{g}}(\mu \leq_\Psi) 1_\mu.$$

As in Proposition 4.6, we see that $[P_\mu : S_\nu] > 0 \implies \mu \leq_\Psi \nu$. This yields a projective resolution

$$\cdots \to P_\mu^2 \to P_\mu^1 \to P_\mu \to S_\mu \to 0$$

of S_μ in $\mathbf{S}_\Psi^{\mathfrak{g}} - mod_f$ such that $[P_\mu^j : S_\nu] > 0 \implies \mu \leq_\Psi \nu$. Therefore, we can consider this as a projective resolution of S_μ in the category of finite-dimenional $\mathbf{S}_\Psi^{\mathfrak{g}}(\mu \leq_\Psi)$ modules, which we denote by $\mathbf{S}_\Psi^{\mathfrak{g}}(\mu \leq_\Psi) - mod_f$.

Thus, for all $\mu, \nu \in P^+$, we have

$$\dim \operatorname{Ext}^j_{\mathbf{S}_\Psi^{\mathfrak{g}} - mod_f}(S_\nu, S_\mu) = \begin{cases} \dim \operatorname{Ext}^j_{\mathbf{S}_\Psi^{\mathfrak{g}}(\mu \leq_\Psi) - mod_f}(S_\mu, S_\nu)), & \mu \leq_\Psi \nu \\ 0, & \text{otherwise} \end{cases}$$

By Lemma 4.21, we know that $\dim \operatorname{Ext}^j_{\mathbf{S}_\Psi^{\mathfrak{g}}(\mu \leq_\Psi)}(S_\mu, S_\nu) = 0$ for all $j > N_\Psi$. Therefore, $\mathbf{S}_\Psi^{\mathfrak{g}}$ has global dimension less than or equal to N_Ψ.

The result for $\mathbf{S}_\Psi^{\mathfrak{g}}(\leq_\Psi \nu)$ is proved similarly, replacing $\mu \leq_\Psi$ by $[\mu, \nu]_\Psi$.

□

We will conclude this section with a proof of Theorem 4.1. To that end, we will need the following proposition.

Proposition 4.23. *For all $\mu \leq_\Psi \nu \in P^+$, the algebras $\mathbf{S}_\Psi^\mathfrak{g}$, $\mathbf{S}_\Psi^\mathfrak{g}(\mu \leq_\Psi)$, $\mathbf{S}_\Psi^\mathfrak{g}([\mu,\nu]_\Psi)$, and $\mathbf{S}_\Psi^\mathfrak{g}(\leq_\Psi \nu)$ are all quadratic.*

Proof. By [1, Corollary 2.3.3] and Theorem 8, $\mathbf{S}_\Psi^\mathfrak{g}(\mu \leq_\Psi)$ and $\mathbf{S}_\Psi^\mathfrak{g}([\mu,\nu]_\Psi)$ are quadratic. It remains to show that $\mathbf{S}_\Psi^\mathfrak{g}$ and $\mathbf{S}_\Psi^\mathfrak{g}(\leq_\Psi \nu)$ are quadratic. Consider the canonical map $\Pi_S : T(\mathfrak{a}_1) \to \mathrm{Sym}(\mathfrak{a}_1)$, and let $\mathbf{\Pi_S} : \mathbf{T} \to \mathbf{S}$ be the map $\mathbf{\Pi_S} = 1 \otimes \Pi_S \otimes 1$. We naturally identify the $\mathfrak{a}_0$-submodule of $T^2(\mathfrak{a}_1)$ generated by the set $\{x \otimes y - y \otimes x \mid x, y \in \mathfrak{a}_1\}$ with $\bigwedge^2(\mathfrak{a}_1)$, so $\ker \Pi_S$ is the ideal of $T(\mathfrak{a}_1)$ generated by the set $\bigwedge^2(\mathfrak{a}_1)$. For each $F \subset P^+$ that is interval closed with respect to $\leq_\Psi$,

$$\begin{aligned}
\mathbf{\Pi_S}(T_\Psi(F)) &= \mathbf{\Pi_S}\big(\bigoplus_{\nu \leq_\Psi \eta \in F} (V(\eta)^* \otimes T(\mathfrak{a}_1)[d_\Psi(\nu,\eta)] \otimes V(\nu)\big) \\
&= \bigoplus \nu \leq_\Psi \eta \in F V(\eta)^* \otimes \Pi_S(T(\mathfrak{a}_1)[d_\Psi(\nu, eta]) \otimes V(\nu) \\
&= \bigoplus_{\nu \leq_\Psi \eta \in F} V(\eta)^* \otimes S(\mathfrak{a}_1)[d_\Psi(\nu,\eta)] \otimes V(\nu) \\
&= \mathbf{S}_\Psi(F).
\end{aligned}$$

Furthermore, since $\mathbf{\Pi_S}$ is an $\mathfrak{a}_0$-module map, we can consider the restriction $\mathbf{\Pi}_\mathbf{S}^\mathfrak{g}|_F := \mathbf{\Pi}_\mathbf{S}^\mathfrak{g}|_{\mathbf{T}_\Psi^\mathfrak{g}(F)} : \mathbf{T}_\Psi^\mathfrak{g}(F) \to \mathbf{S}_\Psi^\mathfrak{g}(F)$. For $k = 0, 1$, we know that $\mathbf{T}_\Psi^\mathfrak{g}(F)[k] = \mathbf{S}_\Psi^\mathfrak{g}(F)[k]$, so $\ker \mathbf{\Pi}_\mathbf{S}^\mathfrak{g}|_F[k] = 0$. Suppose $k \geq 2$. Then,

$$\ker \mathbf{\Pi}_\mathbf{S}^\mathfrak{g}|_F \cap \mathbf{T}_\Psi^\mathfrak{g}(F)[k] = \bigoplus_{\substack{\mu \leq_\Psi \nu \\ d_\Psi(\mu,\nu)=k}} (V(\mu)^* \otimes \left(\sum_{j=0}^{k-2} T^j(\mathfrak{a}_1 \otimes \bigwedge^2(\mathfrak{a}_1) \otimes T^{k-j-2}(\mathfrak{a}_1) \right) \otimes V(\nu))^\mathfrak{g}.$$

By Lemma 0.1, for each $\mu \leq_\Psi \nu \in F$ with $d_\Psi(\mu,\nu) = k$ and $0 \leq j \leq k-2$, we have

$$\begin{gathered}
(V(\nu)^* \otimes T^j(\mathfrak{a}_1) \otimes \bigwedge^2(\mathfrak{a}_1) \otimes T^{k-j-2}(\mathfrak{a}_1) \otimes V(\mu))^\mathfrak{g} = \\
\sum_{\lambda,\xi \in P^+} (V(\nu)^* \otimes T^j(\mathfrak{a}_1) \otimes V(\lambda))^\mathfrak{g} (V(\lambda)^* \otimes \bigwedge^2(\mathfrak{a}_1) \otimes V(\xi))^\mathfrak{g} (V(\xi)^* \otimes T^{k-j-2}(\mathfrak{a}_1) \otimes V(\mu))^\mathfrak{g}.
\end{gathered}$$

Applying Lemma 4.17, we see that

$$\begin{aligned}
(V(\nu)^* \otimes T^j(\mathfrak{a}_1) \otimes V(\lambda))^\mathfrak{g} \neq 0 &\implies \lambda \leq \nu \\
(V(\lambda)^* \otimes \bigwedge^2(\mathfrak{a}_1) \otimes V(\xi))^\mathfrak{g} \neq 0 &\implies \xi \leq \lambda \\
(V(\xi)^* \otimes T^{k-j-2}(\mathfrak{a}_1) \otimes V(\mu))^\mathfrak{g} \neq 0 &\implies \mu \leq \xi.
\end{aligned}$$

In fact, we have that $(\mu,0) \preccurlyeq (\xi,k-j-2) \preccurlyeq (\lambda,k-j) \preccurlyeq (\nu,k)$. Applying Proposition 4.16, $\mu \leq_\Psi \xi \leq_\Psi \lambda \leq_\Psi \nu$, and $d_\Psi(\xi,\lambda) = (k-j)-(k-j-2) = 2$. Thus, $(V(\nu)^* \otimes T^j(\mathfrak{a}_1) \otimes \bigwedge^2(\mathfrak{a}_1) \otimes T^{k-j-2}(\mathfrak{a}_1) \otimes V(\mu))^{\mathfrak{g}}$ is in the ideal of $\mathbf{T}_\Psi^{\mathfrak{g}}(F)$ generated by $\ker \mathbf{\Pi}_{\mathbf{S}}^{\mathfrak{g}}|_F \cap \mathbf{T}_\Psi^{\mathfrak{g}}(F)[2]$.

In particular, we have shown that $\ker \mathbf{\Pi}_{\mathbf{S}}^{\mathfrak{g}}|_F$ is generated by $\ker \mathbf{\Pi}_{\mathbf{S}}^{\mathfrak{g}}|_F \cap \mathbf{T}_\Psi^{\mathfrak{g}}(F)[2]$. To finish the proof, we only need to show that

$$\mathbf{T}_\Psi^{\mathfrak{g}}(F)[2] \cong \mathbf{T}_\Psi^{\mathfrak{g}}(F)[1] \otimes_{\mathbf{T}_\Psi^{\mathfrak{g}}(F)[0]} \mathbf{T}_\Psi^{\mathfrak{g}}(F)[1] = \mathbf{S}_\Psi^{\mathfrak{g}}(F)[1] \otimes_{\mathbf{S}_\Psi^{\mathfrak{g}}(F)[0]} \mathbf{S}_\Psi^{\mathfrak{g}}(F)[1].$$

To verify this, first recall that

$$\mathbf{T}_\Psi^{\mathfrak{g}}(F)[1] = \bigoplus_{\substack{\mu \leq_\Psi \nu \in F; \\ d_\Psi(\mu,\nu)=1}} 1_\nu (\mathbf{V}^{\circledast} \otimes \mathfrak{a}_1 \otimes \mathbf{V})^{\mathfrak{g}} 1_\mu.$$

From this, we obtain

$$\begin{aligned}
&\mathbf{T}_\Psi^{\mathfrak{g}}(F)[1] \otimes_{\mathbf{T}_\Psi^{\mathfrak{g}}(F)[0]} \mathbf{T}_\Psi^{\mathfrak{g}}(F)[1] \\
&= \bigoplus_{\substack{\mu \leq_\Psi \nu, \xi \leq_\Psi \lambda \in F; \\ d_\Psi(\mu,\nu)=d_\Psi(\xi,\lambda)=1}} (1_\lambda (\mathbf{V}^{\circledast} \otimes \mathfrak{a}_1 \otimes \mathbf{V})^{\mathfrak{g}} 1_\xi \otimes_{\mathbf{T}_\Psi^{\mathfrak{g}}(F)[0]} 1_\nu (\mathbf{V}^{\circledast} \otimes \mathfrak{a}_1 \otimes \mathbf{V})^{\mathfrak{g}} 1_\mu) \\
&= \bigoplus_{\substack{\mu \leq_\Psi \nu, \xi \leq_\Psi \lambda \in F; \\ d_\Psi(\mu,\nu)=d_\Psi(\xi,\lambda)=1}} (1_\lambda (\mathbf{V}^{\circledast} \otimes \mathfrak{a}_1 \otimes \mathbf{V})^{\mathfrak{g}} 1_\xi \otimes_{\mathbf{T}_\Psi^{\mathfrak{g}}(F)[0]} 1_\nu 1_\nu (\mathbf{V}^{\circledast} \otimes \mathfrak{a}_1 \otimes \mathbf{V})^{\mathfrak{g}} 1_\mu) \\
&= \bigoplus_{\substack{\mu \leq_\Psi \nu, \xi \leq_\Psi \lambda \in F; \\ d_\Psi(\mu,\nu)=d_\Psi(\xi,\lambda)=1}} (1_\lambda (\mathbf{V}^{\circledast} \otimes \mathfrak{a}_1 \otimes \mathbf{V})^{\mathfrak{g}} 1_\xi 1_\nu \otimes_{\mathbf{T}_\Psi^{\mathfrak{g}}(F)[0]} 1_\nu (\mathbf{V}^{\circledast} \otimes \mathfrak{a}_1 \otimes \mathbf{V})^{\mathfrak{g}} 1_\mu) \\
&= \bigoplus_{\substack{\mu \leq_\Psi \nu \leq_\Psi \lambda \in F; \\ d_\Psi(\mu,\nu)=d_\Psi(\nu,\lambda)=1}} (1_\lambda (\mathbf{V}^{\circledast} \otimes \mathfrak{a}_1 \otimes \mathbf{V})^{\mathfrak{g}} 1_\nu \otimes_{\mathbf{T}_\Psi^{\mathfrak{g}}(F)[0]} 1_\nu (\mathbf{V}^{\circledast} \otimes \mathfrak{a}_1 \otimes \mathbf{V})^{\mathfrak{g}} 1_\mu) \quad , \\
&\cong \bigoplus_{\substack{\mu \leq_\Psi \nu \leq_\Psi \lambda \in F; \\ d_\Psi(\mu,\nu)=d_\Psi(\nu,\lambda)=1}} (1_\lambda (\mathbf{V}^{\circledast} \otimes \mathfrak{a}_1 \otimes \mathbf{V})^{\mathfrak{g}} 1_\nu \otimes 1_\nu (\mathbf{V}^{\circledast} \otimes \mathfrak{a}_1 \otimes \mathbf{V})^{\mathfrak{g}} 1_\mu) \\
&= \bigoplus_{\substack{\mu \leq_\Psi \nu \leq_\Psi \lambda \in F; \\ d_\Psi(\mu,\nu)=d_\Psi(\nu,\lambda)=1}} ((V(\lambda)^* \otimes \mathfrak{a}_1 \otimes V(\nu))^{\mathfrak{g}} \otimes (V(\nu)^* \otimes \mathfrak{a}_1 \otimes V(\mu))^{\mathfrak{g}}) \\
&\cong \bigoplus_{\substack{\mu \leq_\Psi \lambda \in F; \\ d_\Psi(\mu,\lambda)=2}} ((V(\lambda)^* \otimes \mathfrak{a}_1 \otimes \mathfrak{a}_1 \otimes V(\mu))^{\mathfrak{g}})
\end{aligned}$$

where the last isomorphism follows from Lemma 0.1 and Proposition 4.16. Finally, notice that, by definition,

$$\bigoplus_{\substack{\mu\leq_\Psi\lambda\in F;\\ d_\Psi(\mu,\lambda)=2}}((V(\lambda)^*\otimes\mathfrak{a}_1\otimes\mathfrak{a}_1\otimes V(\mu))^{\mathfrak{g}}=\bigoplus_{\substack{\mu\leq_\Psi\lambda\in F;\\ d_\Psi(\mu,\lambda)=2}}((V(\lambda)^*\otimes T^2(\mathfrak{a}_1)\otimes V(\mu))^{\mathfrak{g}}=\mathbf{T}_\Psi^{\mathfrak{g}}(F)[2].$$

Therefore, $\ker\mathbf{\Pi}_{\mathbf{S}}^{\mathfrak{g}}|_F$ is generated by

$$\ker\mathbf{\Pi}_{\mathbf{S}}^{\mathfrak{g}}|_F\cap(\mathbf{T}_\Psi^{\mathfrak{g}}(F)[1]\otimes_{\mathbf{T}_\Psi^{\mathfrak{g}}(F)[0]}\mathbf{T}_\Psi^{\mathfrak{g}}(F)[1]),$$

whence $\mathbf{S}_\Psi^{\mathfrak{g}}(F)$ is quadratic.

□

We can now give the long awaited proof of Theorem 4.1. It remains to show that the algebras in question are Koszul. Theorem 8 gives the result for $\mathbf{S}_\Psi^{\mathfrak{g}}(\mu\leq_\Psi)$ and $\mathbf{S}_\Psi^{\mathfrak{g}}([\mu,\nu]_\Psi)$. By Proposition 4.23, $\mathbf{S}_\Psi^{\mathfrak{g}}$ and $\mathbf{S}_\Psi^{\mathfrak{g}}(\leq_\Psi\nu)$ are quadratic, so, by [1, Theorem 2.6.1], we only need to show that the Koszul complex is exact for $\mathbf{S}_\Psi^{\mathfrak{g}}$ and $\mathbf{S}_\Psi^{\mathfrak{g}}(\leq_\Psi\nu)$.

Proof of Theorem 4.1. Let $A=\mathbf{S}_\Psi^{\mathfrak{g}}$ and let $F=P^+$ or $F=\leq_\Psi\nu$. Then, for any $\mu\in P^+$, $A(F\cap(\mu\leq_\Psi))$ is Koszul, whence it has an exact Koszul complex. We now wish to consider the relationship between the Koszul complex of $A(F)$ and $A((\mu\leq_\Psi)\cap F)$. Let $N^r(F)=N^r_{A(F)}$, and let $K^r(F)=A(F)\otimes_{A(F)[0]}N^r$ be the r^{th} term in the Koszul complex for $A(F)$. Notice that $N^r(F)=\bigoplus_{\mu\in P^+}N^r(F)1_\mu$. For each $\mu\in\mathcal{P}^+$, we see that

$$N^r(F)1_\mu=\bigoplus_{\mu\leq_\Psi\nu\in P^+}1_\nu N^r(F)1_\mu\cong N^r(F\cap(\mu\leq_\Psi))1_\mu.$$

Furthermore, $K^r(F)=\bigoplus_{\mu\in P^+}A(F)\otimes_{A(F)[0]}N^r(F)1_\mu:=\bigoplus_{\mu\in P^+}K^r_\mu(F)$, so it suffices to show that $\ker\partial_r|_{K^r_\mu(F)}\subset\partial_{r+1}(K^{r+1}_\mu(F))$ for each $\mu\in P^+$ and $r\in\mathbb{Z}^+$. Recall that

$1_\nu 1_\eta = \delta_{\nu,\eta}$ for all $\nu, \eta \in P^+$. Applying 4.6, we now have

$$\begin{aligned}
K_\mu^r(F) &= \bigoplus_{\mu \leq_\Psi \nu} A(F) \otimes_{A(F)[0]} 1_\nu N^r(F) 1_\mu \\
&= \bigoplus_{\mu \leq_\Psi \nu, \xi} A(F) \otimes_{A(F)[0]} 1_\nu 1_\xi N^r(F) 1_\mu \\
&= \bigoplus_{\mu \leq_\Psi \nu, \xi} A(F) 1_\nu \otimes_{A(F)[0]} 1_\xi N^r(F) 1_\mu \\
&= \left(\bigoplus_{\mu \leq_\Psi \nu} A(F) 1_\nu \right) \otimes_{A(F)[0]} \left(\bigoplus_{\mu \leq_\Psi \xi} 1_\xi N^r(F) 1_\mu \right) \\
&\cong A(F \cap (\mu \leq_\Psi)) \otimes_{A(F \cap (\mu \leq_\Psi))[0]} N^r(F \cap (\mu \leq_\Psi)).
\end{aligned}$$

Furthermore, if f_r is the isomorphism for each r, we have that $f_r \circ \partial_{r+1}|_{K_\mu^r(F)} = \partial_{r+1} \circ f_{r+1}$ for all $r \geq 0$. Thus, we obtain $\ker \partial|_{K_\mu^r(F)} \subset \partial_{r+1}(K_\mu^{r+1}(F))$. Since this is true for all $\mu \in P^+$, the Koszul complex for $\mathbf{S}_\Psi^{\mathfrak{g}}(F)$ is exact.

$\square$

Conclusions

We have introduced and studied weak $\mathbb{F}$-faces and positive weak $\mathbb{F}$-faces of subsets $X \subset \mathbb{R}^n$. In the case when the closed convex hull of X is a polytope, we showed that weak $\mathbb{F}$-faces of a set X are precisely those sets which are the intersection of X with a face of its convex hull. We have also shown that the intersection of X with a proper face of its convex hull is a positive weak $\mathbb{F}$-face of X when the origin is in the interior of the convex hull of X. This has allowed us to study these subsets for the set of weights of generalized Verma modules (GVM). In particular, we know that the subsets of the weights of a GVM which maximize an $\mathbb{F}$-linear functional are precisely the weak $\mathbb{F}$-faces. If the maximum of the linear functional on the set of weights is positive, then the set which maximizes the linear functional is a positive weak $\mathbb{F}$-face.

We have also considered the category $\widehat{\mathcal{G}}$ of $\mathbb{Z}_+$-graded representations (with finite-dimensional graded pieces) of the semi-direct product Lie algebra $\mathfrak{g} \ltimes V$. When V was a finite-dimensional simple $\mathfrak{g}$-module, we used certain weak $\mathbb{Q}$-faces, Ψ, of the set of weights of V to construct associative algebras that are subalgebras of an algebra $\mathbf{S}_\Psi^{\mathfrak{g}}$ and are Koszul of finite global dimension.

We expect that these results can be extended to integrable modules over affine Kac-Moody algebras. We also expect these results to be useful in the study of the representation theory of undeformed infinitesimal Hecke algebras.

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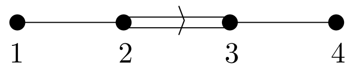
Appendix

For each of the exceptional finite-dimensional simple Lie algebras, we explicitly write down the sets which satisfy Theorem 7 (up to Weyl conjugacy). For each set S, we also write down $\rho = \sum_{\beta \in S} \beta$. It is an easy calculation to check that $(\rho, -)$ is maximized in $\operatorname{wt} V(\theta)$ precisely on S.

F_4

Use the following labelling for the simple roots:



Using this ordering, $\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \omega_1$ is the highest root. The positive weak $\mathbb{F}$-faces of $\operatorname{wt} V(\theta) = R \cup \{0\}$ are the Weyl conjugates of the following subsets:

1. $\{\theta\}$, $\rho = \omega_1$;

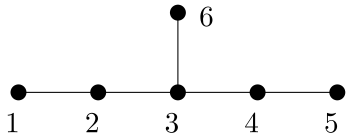
2. $\{\theta, \theta - \alpha_1\}$, $\rho = \omega_2$;

3. $\{\theta, \theta - \alpha_1, \theta - \alpha_1 - \alpha_2\}$, $\rho = 2\omega_3$;

4. $\{\theta, \theta - \alpha_1, \theta - \alpha_1 - \alpha_2, \theta - \alpha_1 - \alpha_2 - \alpha_3, \theta - \alpha_1 - \alpha_2 - 2\alpha_3, \theta - \alpha_1 - 2\alpha_2 - 2\alpha_3, \theta - 2\alpha_1 - 2\alpha_2 - 2\alpha_3\}$, $\rho = 7\omega_4$.

E_6

Use the following labelling for the simple roots:



With this ordering, $\theta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 = \omega_6$ is the highest root. The positive weak $\mathbb{F}$-faces of $\operatorname{wt} V(\theta) = R \cup \{0\}$ are the Weyl conjugates of the following subsets:

1. $\{\theta\}$, $\rho = \omega_6$;

2. $\{\theta, \theta - \alpha_6\}$, $\rho = \omega_3$;

3. $\{\theta, \theta - \alpha_6, \theta - \alpha_3 - \alpha_6\}$, $\rho = \omega_2 + \omega_4$;

4. $\{\theta, \theta - \alpha_6, \theta - \alpha_3 - \alpha_6, \theta - \alpha_2 - \alpha_3 - \alpha_6\}$, $\rho = \omega_1 + 2\omega_4$;

5. $\{\theta, \theta - \alpha_6, \theta - \alpha_3 - \alpha_6, \theta - \alpha_3 - \alpha_4 - \alpha_6\}$, $\rho = 2\omega_2 + \omega_5$;

6. $\{\theta, \theta - \alpha_6, \theta - \alpha_3 - \alpha_6, \theta - \alpha_2 - \alpha_3 - \alpha_6, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_6\}$, $\rho = 3\omega_4$;

7. $\{\theta, \theta - \alpha_6, \theta - \alpha_3 - \alpha_6, \theta - \alpha_3 - \alpha_4 - \alpha_6, \theta - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6\}$, $\rho = 3\omega_2$;

8. $\{\theta, \theta - \alpha_6, \theta - \alpha_3 - \alpha_6, \theta - \alpha_2 - \alpha_3 - \alpha_6, \theta - \alpha_3 - \alpha_4 - \alpha_6, \theta - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_6, \theta - \alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_6, \theta - \alpha_2 - 2\alpha_3 - \alpha_4 - 2\alpha_6\}$, $\rho = 4\omega_1 + 4\omega_5$;

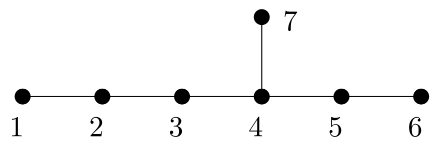
9. $\{\theta, \theta - \alpha_6, \theta - \alpha_3 - \alpha_6, \theta - \alpha_2 - \alpha_3 - \alpha_6, \theta - \alpha_3 - \alpha_4 - \alpha_6, \theta - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_6, \theta - \alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_6, \theta - \alpha_2 - 2\alpha_3 - \alpha_4 - 2\alpha_6, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_6, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_6, \theta - \alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_6, \theta - \alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4 - 2\alpha_6, \theta - \alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_6, \theta - \alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - 2\alpha_6, \theta - \alpha_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4 - 2\alpha_6, \theta - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4 - 2\alpha_6\}$, $\rho = 12\omega_5$.

10. $\{\theta, \theta-\alpha_6, \theta-\alpha_3-\alpha_6, \theta-\alpha_2-\alpha_3-\alpha_6, \theta-\alpha_3-\alpha_4-\alpha_6, \theta-\alpha_2-\alpha_3-\alpha_4-\alpha_6, \theta-\alpha_2-2\alpha_3-\alpha_4-\alpha_6, \theta-\alpha_2-2\alpha_3-\alpha_4-2\alpha_6, \theta-\alpha_2-\alpha_3-\alpha_5-\alpha_6, \theta-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6, \theta-\alpha_2-2\alpha_3-\alpha_4-\alpha_5-\alpha_6, \theta-\alpha_2-2\alpha_3-\alpha_4-\alpha_5-2\alpha_6, \theta-2\alpha_2-2\alpha_3-\alpha_4-\alpha_5-\alpha_6, \theta-2\alpha_2-2\alpha_3-\alpha_4-\alpha_5-2\alpha_6, \theta-2\alpha_2-3\alpha_3-\alpha_4-\alpha_1-2\alpha_6, \theta-2\alpha_2-3\alpha_3-2\alpha_4-\alpha_5-2\alpha_6\}$,

$\rho = 12\omega_1$

E_7

Use the following labelling for the simple roots:



With this ordering, $\theta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 = \omega_6$ is the highest root. The positive weak $\mathbb{F}$-faces of $\operatorname{wt} V(\theta) = R \cup \{0\}$ are the Weyl conjugates of the following subsets:

1. $\{\theta\}$, $\rho = \omega_6$.

2. $\{\theta, \theta - \alpha_6\}$, $\rho = \omega_5$.

3. $\{\theta, \theta - \alpha_6, \theta - \alpha_5 - \alpha_6\}$, $\rho = \omega_4$.

4. $\{\theta, \theta - \alpha_6, \theta - \alpha_5 - \alpha_6, \theta - \alpha_4 - \alpha_5 - \alpha_6\}$, $\rho = \omega_3 + \omega_7$.

5. $\{\theta, \theta - \alpha_6, \theta - \alpha_5 - \alpha_6, \theta - \alpha_4 - \alpha_5 - \alpha_6, \theta - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6\}$, $\rho = \omega_2 + 2\omega_7$.

6. $\{\theta, \theta - \alpha_6, \theta - \alpha_5 - \alpha_6, \theta - \alpha_4 - \alpha_5 - \alpha_6, \theta - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7\}$, $\rho = 2\omega_3$.

7. $\{\theta, \theta-\alpha_6, \theta-\alpha_5-\alpha_6, \theta-\alpha_4-\alpha_5-\alpha_6, \theta-\alpha_3-\alpha_4-\alpha_5-\alpha_6, \theta-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6\}$, $\rho = \omega_1 + 3\omega_7$.

8. $\{\theta, \theta - \alpha_6, \theta - \alpha_5 - \alpha_6, \theta - \alpha_4 - \alpha_5 - \alpha_6, \theta - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7, \theta - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6, \theta - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7, \theta - \alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6 - \alpha_7, \theta - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6 - \alpha_7, \theta - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7\}$, $\rho = 5\omega_2$.

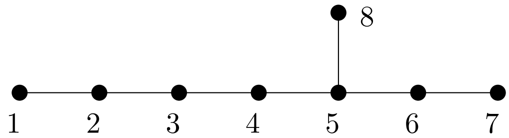
9. $\{\theta, \theta - \alpha_6, \theta - \alpha_5 - \alpha_6, \theta - \alpha_4 - \alpha_5 - \alpha_6, \theta - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6, \theta - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6\}$, $\rho = 4\omega_7$.

10. $\{\theta, \theta-\alpha_6, \theta-\alpha_5-\alpha_6, \theta-\alpha_4-\alpha_5-\alpha_6, \theta-\alpha_4-\alpha_5-\alpha_6-\alpha_7, \theta-\alpha_3-\alpha_4-\alpha_5-\alpha_6, \theta-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7, \theta-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6, \theta-\alpha_3-2\alpha_4-\alpha_5-\alpha_6-\alpha_7, \theta-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-\alpha_7, \theta-\alpha_3-2\alpha_4-2\alpha_5-2\alpha_6-\alpha_7, \theta-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7, \theta-\alpha_2-\alpha_3-4\alpha_4-\alpha_5-\alpha_6-\alpha_7, \theta-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-\alpha_7, \theta-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-2\alpha_6-\alpha_7, \theta-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6-\alpha_7, \theta-\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-\alpha_7, \theta-\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-2\alpha_6-\alpha_7, \theta-\alpha_2-2\alpha_3-3\alpha_4-2\alpha_5-\alpha_6-\alpha_7, \theta-\alpha_2-2\alpha_3-3\alpha_4-2\alpha_5-2\alpha_6-\alpha_7, \theta-\alpha_2-2\alpha_3-3\alpha_4-3\alpha_5-2\alpha_6-\alpha_7, \theta-\alpha_2-2\alpha_3-3\alpha_4-2\alpha_5-\alpha_6-2\alpha_7, \theta-\alpha_2-2\alpha_3-3\alpha_4-2\alpha_5-2\alpha_6-2\alpha_7, \theta-\alpha_2-2\alpha_3-3\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7, \theta-\alpha_2-2\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7, \theta-\alpha_2-3\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7, \theta-2\alpha_2-3\alpha_3-4\alpha_4-3\alpha_5-2\alpha_6-2\alpha_7\}$,

$\rho = 18\omega_1$.

E_8

Use the following labelling for the simple roots:



With this ordering, $\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = \omega_1$ is the highest root. The positive weak $\mathbb{F}$-faces of $\operatorname{wt} V(\theta) = R \cup \{0\}$ are the Weyl conjugates of the following subsets:

1. $\{\theta\}$, $\rho = \omega_1$

2. $\{\theta, \theta - \alpha_1\}$, $\rho = \omega_2$

3. $\{\theta, \theta - \alpha_1, \theta - \alpha_1 - \alpha_2\}$, $\rho = \omega_3$

4. $\{\theta, \theta - \alpha_1, \theta - \alpha_1 - \alpha_2, \theta - \alpha_1 - \alpha_2 - \alpha_3\}$, $\rho = \omega_4$

5. $\{\theta, \theta - \alpha_1, \theta - \alpha_1 - \alpha_2, \theta - \alpha_1 - \alpha_2 - \alpha_3, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4\}$, $\rho = \omega_5$

6. $\{\theta, \theta-\alpha_1, \theta-\alpha_1-\alpha_2, \theta-\alpha_1-\alpha_2-\alpha_3, \theta-\alpha_1-\alpha_2-\alpha_3-\alpha_4, \theta-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5\}$, $\rho = \omega_6 + \omega_8$

7. $\{\theta, \theta - \alpha_1, \theta - \alpha_1 - \alpha_2, \theta - \alpha_1 - \alpha_2 - \alpha_3, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6\}$, $\rho = \omega_7 + 2\omega_8$

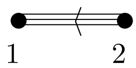
8. $\{\theta, \theta - \alpha_1, \theta - \alpha_1 - \alpha_2, \theta - \alpha_1 - \alpha_2 - \alpha_3, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_8\}$, $\rho = 2\omega_6$

9. $\{\theta, \theta - \alpha_1, \theta - \alpha_1 - \alpha_2, \theta - \alpha_1 - \alpha_2 - \alpha_3, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6, \theta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7\}$, $\rho = 3\omega_8$

10. $\{\theta, \theta-\alpha_1, \theta-\alpha_1-\alpha_2, \theta-\alpha_1-\alpha_2-\alpha_3, \theta-\alpha_1-\alpha_2-\alpha_3-\alpha_4, \theta-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5, \theta-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6, \theta-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_8, \theta-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_8, \theta-\alpha_1-\alpha_2-\alpha_3-\alpha_4-2\alpha_5-\alpha_6-\alpha_8, \theta-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-\alpha_8, \theta-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-\alpha_8, \theta-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-\alpha_8, \theta-2\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-\alpha_6-\alpha_8\}$, $\rho = 7\omega_7$.

G_2

Use the following ordering for the simple roots:



With this ordering, $\theta = 3\alpha_1 + 2\alpha_2 = \omega_2$ is the highest root. The positive weak $\mathbb{F}$-faces of $\mathrm{wt}\, V(\theta) = R \cup \{0\}$ are the following sets:

1. $\pm\{3\alpha_2 + 2\alpha_2\}$, $\rho = \pm\omega_2$,

2. $\pm\{\alpha_2\}$, $\rho = \pm(-3\omega_1 + \omega_2)$,

3. $\pm\{3\alpha_1 + \alpha_2\}$, $\pm(3\omega_1 - \omega_2)$,

4. $\pm\{\alpha_2, 3\alpha_1 + 2\alpha_2\}$, $\rho = \pm(-2\omega_1 + 2\omega_2)$,

5. $\pm\{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$, $\rho = \pm(2\omega_1)$,

6. $\pm\{-\alpha_2, 3\alpha_1 + \alpha_2\}$, $\rho = \pm(4\omega_1 - 2\omega_2)$.