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A SYMBOLIC REPRESENTATION FOR ANOSOV–KATOK SYSTEMS

By

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Abstract. This paper is the first of a series of papers culminating in the result that measure preserving diffeomorphisms of the disc or 2-torus are unclassifiable. It addresses another classical problem: which abstract measure preserving systems are realizable as smooth diffeomorphisms of a compact manifold? The main result gives symbolic representations of Anosov–Katok diffeomorphisms.

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1 Introduction

In 1932 J. von Neumann in [20] laid the foundations for ergodic theory. In it he expressed the likelihood that any abstract measure preserving transformation (abbreviated to MPT in this paper) is isomorphic to a continuous MPT and perhaps even to a differentiable one. Recall that two MPT's, T and S, are isomorphic if there is an invertible measure preserving mapping between the measure spaces which commutes with the actions of T and S. His brief remarks [1] eventually gave rise to one of the outstanding problems in smooth dynamics, namely:

Does every ergodic MPT with finite entropy have a smooth model?

By a smooth model is meant an isomorphic copy of the MPT which is given by smooth diffeomorphism of a compact manifold preserving a measure equivalent to the volume element. The finite entropy restriction is required by a result of A. G. Kushnirenko that showed that the entropy of any such diffeomorphism must be finite. An even more basic problem which von Neumann formulated in the same paper, was that of classifying all measure preserving transformations up to isomorphism. This problem was solved long ago for several classes of transformations that have special properties. P. Halmos and J. von Neumann showed that ergodic MPT's with pure point spectrum are classified by the unitary equivalence of the associated unitary operators defined on the L^2 by the MPT, while A. N. Kolmogorov and D. Ornstein showed that Bernoulli shifts are classified by their entropy.

One way to show that not all finite entropy ergodic MPT's have a smooth model would be to show that their classification is easier than the general classification

[1] In [20] on page 590, "Vermutlich kann sogar zu jeder allgemeinen Strömung eine isomorphe stetige Strömung gefunden werden [footnote 13], vielleicht sogar eine stetig-differentiierbare, oder gar eine mechanische. Footnote 13: Der Verfasser hofft, hierfür demnächst einen Beweis anzugeben."

problem. Set theory provides a framework for a rigorous comparison of the complexity of different equivalence relations, and thus could potentially be a tool for settling this question.

Indeed, starting in the late 1990's a different type of result began to appear that used descriptive set theoretic techniques. These anti-classification results demonstrate in a rigorous way that positive classifications, such as those described above, are not possible.

The first is due to Beleznay and Foreman [4] who showed that the class of measure distal transformations used in early ergodic theoretic proofs of Szemeredi's theorem is not a Borel set. Later, Hjorth [16] introduced the notion of turbulence and showed that there is no Borel way of attaching algebraic invariants to ergodic transformations that completely determine isomorphism. Foreman and Weiss [10] improved this result by showing that the conjugacy action of the measure preserving transformations is turbulent—hence no generic class can have a complete set of algebraic invariants.

An "anti-classification" theorem requires a precise definition of what a classification is. Informally a classification is a method of determining isomorphism between transformations perhaps by computing (in a liberal sense) other invariants for which equivalence is easy to determine. The key words here are **method** and **computing**. For negative theorems, the more liberal a notion one takes for these words, the stronger the theorem. One natural notion is the Borel/non-Borel distinction. Saying a set X or function f is Borel is a loose way of saying that membership in X or the computation of f can be done using a countable (possibly transfinite) protocol whose basic input is membership in open sets. Saying that X or f is not Borel is saying that determining membership in X or computing f cannot be done with any countable amount of resources.

In the context of classification problems, saying that an equivalence relation E on a space X is not Borel is saying that there is no countable amount of information and no countable transfinite protocol for determining, for arbitrary $x, y \in X$, whether xEy. Any such method must inherently use uncountable resources.[2]

In considering the isomorphism relation as a collection $\mathcal{I}$ of pairs (S, T) of measure preserving transformations, Hjorth showed that $\mathcal{I}$ is not a Borel set. However, the pairs of transformations he used to demonstrate this were inherently non-ergodic, leaving open the essential problem:

Is isomorphism of ergodic measure preserving transformations Borel?

[2] Many well-known classification theorems have as immediate corollaries that the resulting equivalence relation is Borel. An example of this is the Spectral Theorem, which has a consequence that the relation of Unitary Conjugacy for normal operators is a Borel equivalence relation.

This question was answered in the negative for ergodic transformations of standard measure spaces by Foreman, Rudolph and Weiss in [11]. This answer can be interpreted as saying that determining isomorphism between abstract ergodic transformations is inaccessible to countable methods that use countable amounts of information.

This series of papers culminates in a result that—even restricted to the Lebesgue measure preserving diffeomorphisms of the 2-torus—the isomorphism relation is not Borel.

Theorem 1. *If M is either the torus $\mathbb{T}^2$, the disk D or the annulus, then the measure-isomorphism relation among pairs (S, T) of ergodic measure preserving C^∞-diffeomorphisms of M is not a Borel set with respect to the C^∞-topology.*

What is in this paper? The transformations built by Foreman, Rudolph and Weiss ([11]) to prove the earlier result were based on odometers (in the sense that the Kronecker factor was an odometer). It is a well-known open problem whether it is possible to have a smooth transformation on a compact manifold that has a non-trivial odometer factor. Thus proving the anti-classification theorem in the smooth context required constructing a different collection of hard-to-classify transformations and then showing that this collection could be realized smoothly.

The new collection of transformations, the Circular Systems, are defined as symbolic systems constructed using the Circular Operator, a formal operation on words. This paper defines this class and then realizes them smoothly using the method of conjugacy originating in a famous paper of Anosov and Katok.

In fact, something much stronger is shown: Theorems 58 and 60 show that the Circular Systems exactly coincide with the isomorphism classes of the Anosov–Katok construction. We loosely summarize 58 and 60 as follows:

Theorem 2 (Main Result of this paper)**.** *Let T be an ergodic transformation on a standard measure space. Then the following are equivalent:*

(1) *T is isomorphic to a uniform Anosov–Katok diffeomorphism.*[3]
(2) *T is isomorphic to a (uniform) circular system.*

This theorem shows that a broad class of transformations can be realized as Anosov–Katok diffeomorphisms. In fact we conjecture the following:

Conjecture. Suppose that T is a zero-entropy ergodic transformation that has a Liouvillian irrational rotation of the circle as a factor. Then T is isomorphic to a uniform circular system.

[3]Built using the untwisted method of conjugacy with some minor technical assumptions.

If the conjecture is true, then all zero entropy ergodic transformations with a Liouvillean rotation factor can be realized as smooth transformations, and moreover, every zero entropy ergodic transformation is a factor of an ergodic smooth transformation.

Applications of Theorem 2. Theorem 2 is primarily useful in that it reduces questions about diffeomorphisms to combinatorial questions about symbolic shifts. Theorem 2 implies that any systematic way of building uniform circular systems with given ergodic properties automatically implies that there are ergodic measure preserving diffeomorphisms of the torus with those same properties.

In a sequel to this paper ([13]) to this paper, the collection of circular shifts is endowed with a category structure and it is shown that this category is quite large. In particular, it contains measure theoretically distal transformations of arbitrarily countable height.

Another sequel ([14]) reduces a complete analytic set (the ill-founded trees) to the isomorphism relation for circular systems. By Theorem 2 this automatically gives a reduction to the isomorphism problem for diffeomorphisms.

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2 Preliminaries

In this section we establish some of the conventions we follow in this paper. There are many sources of background information on this including any standard text such as [22] or [21].

2.1 Measure spaces. We will call separable non-atomic probability spaces **standard** measure spaces and denote them $(X, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Boolean algebra of measurable subsets of X and μ is a countably additive, non-atomic measure defined on $\mathcal{B}$. We will often identify two members of $\mathcal{B}$ that differ by a set of μ-measure 0 and seldom distinguish between $\mathcal{B}$ and the σ-algebra of classes of measurable sets modulo measure zero unless we are making a pointwise definition and need to claim it is well defined on equivalence classes.

Remark 3. von Neumann proved that every standard measure space is isomorphic to $([0, 1], \mathcal{B}, \lambda)$, where λ is Lebesgue measure and $\mathcal{B}$ is the algebra of Lebesgue measurable sets.

If $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ are measure spaces, an isomorphism between X and Y is a bijection $\phi : X \to Y$ such that ϕ is measure preserving and both ϕ and ϕ^{-1} are measurable. We will ignore sets of measure zero when discussing isomorphisms; i.e., we allow the domain and range of ϕ to be subsets of X and Y (resp.) of measure one.

A measure preserving system is a 4-tuple $(X, \mathcal{B}, \mu, T)$ where $T : X \to X$ is a measure isomorphism. A **factor map** between two measure preserving systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ is a measurable, measure preserving function $\phi : X \to Y$ such that $S \circ \phi = \phi \circ T$. A factor map is an isomorphism between systems iff ϕ is a measure isomorphism. As above we only require the domain and range of ϕ to have measure one, rather than that ϕ be one-to-one and onto.

2.2 Partitions of measurable spaces. We will be concerned with ordered countable measurable partitions of measure spaces. An ordered countable measurable partition is a sequence $\mathcal{P} = \langle P_n : n \in \mathbb{N} \rangle$ such that:

(1) each $P_n \in \mathcal{B}$,
(2) if $n \neq m$ then $P_n \cap P_m = \emptyset$,
(3) $\bigcup_n P_n$ has measure one.

We explicitly allow some of the P_n's to be measure zero. The P_n's will be called the **atoms** of the partition.

If $\mathcal{P} = \langle P_n : n \in \mathbb{N} \rangle$ and $\mathcal{Q} = \langle Q_n : n \in \mathbb{N} \rangle$ are two ordered partitions, then the partition distance is defined as follows:

$$D_\mu(\mathcal{P}, \mathcal{Q}) = \sum \mu(P_i \Delta Q_i).$$

We will frequently refer to ordered countable measurable partitions simply as **partitions**. A partition is finite iff for all large enough n, $\mu(P_n) = 0$. If we let $\mathbb{P}_n$ be the space of partitions with $\leq n$-atoms (i.e. for $m \geq n$, $\mu(P_m) = 0$), then $(\mathbb{P}_n, D_\mu)$ is a connected space.

If $\mathcal{P}$ and $\mathcal{Q}$ are two partitions, then we say that $\mathcal{Q}$ **ϵ-refines** $\mathcal{P}$ iff the atoms of $\mathcal{Q}$ can be grouped into sets $\langle S_n : n \in \mathbb{N} \rangle$ such that

$$\sum_n \mu\Big(P_n \Delta \Big(\bigcup_{i \in S_n} Q_i\Big)\Big) < \epsilon.$$

If $\mathcal{P}$ and $\mathcal{Q}$ are partitions, then $\mathcal{Q}$ **refines** $\mathcal{P}$ iff the atoms of $\mathcal{Q}$ can be grouped into sets $\langle S_n : n \in \mathbb{N}\rangle$ such that

$$\sum_n \mu\Big(P_n \Delta\Big(\bigcup_{i\in S_n} Q_i\Big)\Big) = 0.$$

In this case we will write that $\mathcal{Q} \ll \mathcal{P}$. A **decreasing sequence of partitions** is a sequence $\langle \mathcal{P}_n : n \in \mathbb{N}\rangle$ such that for all $m < n, \mathcal{P}_n \ll \mathcal{P}_m$. If $A \in \mathcal{B}$ is a measurable set and $\mathcal{P}$ is a partition, then we let $\mathcal{P} \restriction A$ be the partition of A defined as $\langle P_n \cap A : n \in \mathbb{N}\rangle$.

Definition 4. Let $(X, \mathcal{B}, \mu)$ be a measure space. We will say that a sequence of partitions $\langle \mathcal{P}_n : n \in \mathbb{N}\rangle$ **generates** (or generates $\mathcal{B}$) iff the smallest σ-algebra containing $\bigcup_n \mathcal{P}_n$ is $\mathcal{B}$ (modulo measure zero sets). If T is a measure preserving transformation, we will write $T\mathcal{P}$ for the partition $\langle Ta : a \in \mathcal{P}\rangle$. In the context of a measure preserving $T : X \to X$ we will say that a partition $\mathcal{P}$ is a **generator** for T iff $\langle T^i\mathcal{P} : i \in \mathbb{Z}\rangle$ generates $\mathcal{B}$.

We will be manipulating partitions of $[0, 1)$ and $[0, 1) \times [0, 1)$ in various ways so we develop some notation for doing so. We let $\mathcal{I}_q$ be the partition of $[0, 1)$ with atoms $\langle [i/q, (i + 1)/q) : 0 \le i < q\rangle$, and refer to $[i/q, (i + 1)/q)$ as I_i^q.[4] If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of spaces X and Y respectively, we let $\mathcal{P} \otimes \mathcal{Q}$ be the partition of $X \times Y$ given by $\{P_i \times Q_j : i, j \in \mathbb{N}\}$. To make this definition complete we need to fix in advance an arbitrary ordering of $\mathbb{N} \times \mathbb{N}$ that is used to order $\mathcal{P} \otimes \mathcal{Q}$. Finally, we use the notation $I \otimes \mathcal{Q}$ for the partition $\mathcal{P} \otimes \mathcal{Q}$ where $\mathcal{P}$ has one element I.

If $T : X \to X$ and $\mathcal{P} = \langle a_i : i \in I\rangle$ is a partition of X, then the $(T, \mathcal{P}, n)$-**name** of s is a_i if and only if $T^n(x) \in a_i$. If T is invertible, then the $(T, \mathcal{P})$-name is $s \in \mathcal{P}^{\mathbb{Z}}$ if and only if for all $n \in \mathbb{Z}, T^n(x) \in s(n)$. We suppress $\mathcal{P}$ and/or T if either is obvious from the context.

3 Presentations of measure preserving systems

Measure preserving systems occur naturally in many guises with diverse topologies. As far as is known, the Borel/non-Borel distinction for dynamical properties is the same in each of these presentations and many of the presentations have the same generic classes. (See the forthcoming paper [12].)

In this section we briefly review the properties of the presentations relevant to this paper. These are: abstract invertible preserving systems, smooth transformations preserving volume elements and symbolic systems.

[4] If $i > q$ then I_i^q refers to $I_{i'}^q$, where $i' < q$ and $i' \equiv i \mod q$.

3.1 Abstract measure preserving systems. As noted in Section 2.1 every standard measure space is isomorphic to the unit interval with Lebesgue measure. Hence every invertible measure preserving transformation of a standard measure space is isomorphic to an invertible Lebesgue measure preserving transformation on the unit interval.

In accordance with the conventions of [22] we denote the collection of measure preserving transformations of [0, 1] by MPT.[5] We note that two measure preserving transformations are identified if they are equal on sets of full measure.

We can associate to each invertible measure preserving transformation $T \in \mathrm{MPT}$ a unitary operator $U_T : L^2([0,1]) \to L^2([0,1])$ by defining $U(f) = f \circ T$. In this way MPT can be identified with a closed subgroup of the unitary operators on $L^2([0,1])$ with respect to the weak operator topology on the space of unitary transformations. This makes MPT into a Polish space. We will call this the **weak topology** on MPT (see [15]).

A concrete description of the topology can be given as follows: Let $S \in \mathrm{MPT}$, $\mathcal{P}$ be a finite measurable partition and $\epsilon > 0$. Define

$$N(S, \mathcal{P}, \epsilon) = \left\{ T \in \mathrm{MPT} : \sum_{A \in \mathcal{P}, i = \pm 1} \lambda(T^i A \Delta S^i A) < \epsilon \right\}.$$

If $\langle \mathcal{P}_n : n \in \mathbb{N} \rangle$ is a generating sequence of partitions for $\mathcal{B}$, then

$$\{N(S, \mathcal{P}_n, \epsilon) : S \in \mathrm{MPT}, n \in \mathbb{N}, \epsilon > 0\}$$

generates the weak operator topology on MPT.

We will denote the ergodic transformations belonging to MPT by $\mathcal{E}$. Halmos ([15]) showed that $\mathcal{E}$ is a dense $\mathcal{G}_\delta$ set in MPT. In particular, the weak topology makes $\mathcal{E}$ into a Polish subspace of MPT.

The following is easy to check.

Lemma 5. *Let $\langle T_n : n \in \mathbb{N} \rangle$ be a sequence of measure preserving transformations and $\langle \mathcal{P}_n : n \in \mathbb{N} \rangle$ be a generating sequence of partitions. Then the following are equivalent*

(1) *The sequence $\langle T_n : n \in \mathbb{N} \rangle$ converges to an invertible measure preserving system in the weak topology.*
(2) *For all measurable sets A, for all $\epsilon > 0$ there is an N for all $n, m > N$ and $i = \pm 1$ one has $\mu(T_n^i A \Delta T_m^i A) < \epsilon$.*
(3) *For all $\epsilon > 0, p \in \mathbb{N}$ there is an N for all $m, n > N$ such that*

$$\sum_{A \in \mathcal{P}_p, i = \pm 1} \mu(T_n^i A \Delta T_m^i A) < \epsilon.$$

[5] Recently several authors have adopted the notation $Aut(\mu)$ for the same space.

In case the sequence $\langle T_n : n \in \mathbb{N} \rangle$ converges, then we can identify the limit as the unique T such that for all measurable sets A,

$$\mu(T_n A \,\Delta\, TA) \to 0.$$

There is another topology on the collection of measure preserving transformations of X to Y for measure spaces X and Y. If $S, T : X \to Y$ are measure preserving transformations, the **uniform distance** between S and T is defined to be

$$d_U(S, T) = \mu\{x : Sx \neq Tx\}.$$

This topology refines the weak topology and is a complete, but not separable topology.

3.2 Diffeomorphisms.

Let M be a C^k-smooth compact finite-dimensional manifold and μ be a standard measure on M determined by a smooth volume element. For each k there is a Polish topology on the k-times differentiable homeomorphisms of M, the C^k-topology. The C^∞-topology is the coarsest topology refining the C^k-topology for each $k \in \mathbb{N}$. It is also a Polish topology and a sequence of C^∞-diffeomorphisms converges in the C^∞-topology if and only if it converges in the C^k-topology for each $k \in \mathbb{N}$. The C^∞ topology is also a Polish topology and we will sometimes use a Polish metric d^∞ on the diffeomorphisms inducing this topology.

The collection of μ-preserving diffeomorphisms forms a closed nowhere dense set in the C^k-topology on the C^k-diffeomorphisms, and as such inherits a Polish topology. [6] We will denote this space by $\mathrm{Diff}^k(M, \mu)$.

The measure preserving diffeomorphisms of a compact manifold can also be endowed with the weak topology, which is coarser than the C^k-topology. To see that the weak topology is coarser than the C^k-topologies, note that if M is compact and has dimension n, then M has a countable generating sequence of finite partitions into "half-open" sets whose boundaries are finite unions of submanifolds of dimension less than n. Let $\mathcal{P}$ be such a partition. Then the boundaries of the elements of $\mathcal{P}$ all have measure zero and if S and T are close in the C^k-topology, then S and T take the boundaries to very similar places. In particular, $S\mathcal{P}$ and $T\mathcal{P}$ don't differ very much.

[6] One can also consider the space of measure preserving homeomorphisms with the $\| \|_\infty$ topology, which behaves in some ways similarly.

One can also consider the space of abstract μ-preserving transformations on M with the weak topology. In [5] it is shown that the collection of a.e.-equivalence classes of smooth transformations form a Π^0_3-set ($\mathcal{G}_{\delta\sigma\delta}$) in MPT(M), and hence the collection has the Property of Baire. In particular, by invariance it is either meager or comeager.

3.3 Symbolic systems. Let Σ be a countable or finite alphabet endowed with the discrete topology. Then $\Sigma^{\mathbb{Z}}$ can be given the product topology, which makes it into a separable, totally disconnected space that is compact if Σ is finite.

Notation. If $u = \langle \sigma_0, \ldots, \sigma_{n-1} \rangle \in \Sigma^{<\infty}$ is a finite sequence of elements of Σ, then we denote the cylinder set based at k in $\Sigma^{\mathbb{Z}}$ by writing $\langle u \rangle_k$. If $k = 0$ we abbreviate this and write $\langle u \rangle$. Explicitly, $\langle u \rangle_k = \{f \in \Sigma^{\mathbb{Z}} : f \restriction [k, k+n) = u\}$. The collection of cylinder sets form a base for the product topology on $\Sigma^{\mathbb{Z}}$, thus we frequently refer to them as "basic open sets."

The shift map

$$sh : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}},$$

defined by setting $sh(f)(n) = f(n+1)$ is a homeomorphism. If μ is a shift-invariant Borel measure, then the resulting measure preserving system $(\Sigma^{\mathbb{Z}}, \mathcal{B}, \mu, sh)$ is called a **symbolic system**. The closed support of μ is a shift-invariant closed subset of $\Sigma^{\mathbb{Z}}$, called a **symbolic shift** or **sub-shift**.

We can construct symbolic shifts from arbitrary measure preserving systems as follows: If $(X, \mathcal{B}, \mu, T)$ is a measure preserving system and $\mathcal{P} = \{A_i : i \in I\}$ is a measurable partition (where I is countable or finite), let $\Sigma = \{a_i : i \in I\}$, then we can define a map

$$\phi : X \to \Sigma^{\mathbb{Z}}$$

by setting $\phi(x)(n) = a_i$ iff $T^n x \in A_i$.

The map ϕ induces a shift-invariant Borel measure $\nu = \phi^* \mu$ on $\Sigma^{\mathbb{Z}}$ by setting $\nu(B) = \mu(\phi^{-1}(B))$. The resulting invariant measure makes $(\Sigma^{\mathbb{Z}}, \mathcal{C}, \nu, sh)$ into a factor of $(X, \mathcal{B}, \mu, T)$ with factor map ϕ. Since X is standard, if $\mathcal{P}$ generates then ϕ is an isomorphism.

Remark 6. We will use the fact that we can systematically change symbols in some positions of letters in $x \in \Sigma^{\mathbb{Z}}$ to get a new element $x' \in \Sigma^{\mathbb{Z}}$ as long as the change is equivariant with the shift and the map $x \mapsto x'$ is one to one. Because the change is one to one we can copy over the measure ν to a measure ν' so that the resulting measure on $(\Sigma)^{\mathbb{Z}}$ will define an isomorphic system.

Notation. For a word $w \in \Sigma^{<\mathbb{N}}$ we will write $|w|$ for the length of w.

We want to be able to unambiguously parse elements words and elements of symbolic shifts. For this we will use construction sequences consisting of uniquely readable words.

Definition 7. Let Σ be an alphabet and $\mathcal{W}$ be a collection of finite words in Σ. Then $\mathcal{W}$ is **uniquely readable** iff whenever $u, v, w \in \mathcal{W}$ and $uv = pws$ then either p or s is the empty word.

Symbolic shifts are often described intrinsically by giving a collection of words that constitute a clopen base for the support of an invariant measure. Fix an alphabet Σ, and a sequence of uniquely readable collections of words $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ with the properties that:

(1) for each n all of the words in $\mathcal{W}_n$ have the same length q_n,
(2) each $w' \in \mathcal{W}_{n+1}$ contains each $w \in \mathcal{W}_n$ as a subword,
(3) there is a summable sequence $\langle \epsilon_n : n \in \mathbb{N} \rangle$ of positive numbers such that for each n, every word $w \in \mathcal{W}_{n+1}$ can be uniquely parsed into segments

$$u_0 w_0 u_1 w_1 \cdots w_l u_{l+1}$$

such that each $w_i \in \mathcal{W}_n$, $u_i \in \Sigma^{<\mathbb{N}}$ and for this parsing

$$\frac{\sum_i |u_i|}{q_{n+1}} < \epsilon_{n+1}. \tag{1}$$

Definition 8. A sequence $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ satisfying items (1)–(3) will be called a **construction sequence**.

Define $\mathbb{K}$ to be the collection of $x \in \Sigma^{\mathbb{Z}}$ such that every finite contiguous subword of x occurs inside some $w \in \mathcal{W}_n$. Then $\mathbb{K}$ is a closed shift-invariant subset of $\Sigma^{\mathbb{Z}}$ that is compact if Σ is finite.

Definition 9. Let $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ be a construction sequence. Then $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ is **uniform** if there are $\langle d_n : n \in \mathbb{N} \rangle$, where $d_n : \mathcal{W}_n \to (0, 1)$, and a sequence $\langle \epsilon_n : n \in \mathbb{N} \rangle$ going to zero such that for each n, all words $w \in \mathcal{W}_n$ and $w' \in \mathcal{W}_{n+1}$ if $f(w, w')$ is the number of i such that $w = w_i$,

$$\sum_{w \in \mathcal{W}_n} \left| \frac{f(w, w')}{q_{n+1}/q_n} - d_n(w) \right| < \epsilon_{n+1}. \tag{2}$$

The words u_i are often called **spacers**. The d_n are target values for the densities of n-words in $n+1$ words. The uniformity is that each n-word occurs nearly the same number of times in every $n+1$-word. If $\mathbb{K}$ is built from a uniform construction sequence we will call $\mathbb{K}$ a **uniform symbolic system**.

If $f(w, w')$ is a constant (depending on n) for all $w \in \mathcal{W}_n$, $w' \in \mathcal{W}_{n+1}$ we can take $d_n(w) = \frac{f(w,w')}{q_{n+1}/q_n}$ and satisfy Definition 9. In this case we call the construction sequence and $\mathbb{K}$ **strongly uniform**.

If $\mathbb{K} \subset \Sigma^{\mathbb{Z}}$ is a symbolic system, then an element $x \in \mathbb{K}$ is a function $x : \mathbb{Z} \to \Sigma$. If I is a finite or infinite interval in $\mathbb{Z}$, then we write $x \upharpoonright I$ for the function x restricted to this interval. In our constructions we will restrict our measures to a natural set:

Definition 10. Suppose that $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$ is a construction sequence for a symbolic system $\mathbb{K}$ with each $\mathcal{W}_n$ uniquely readable. Let S be the collection $x \in \mathbb{K}$ such that there are sequences of natural numbers $\langle a_m : m \in \mathbb{N}\rangle$, $\langle b_m : m \in \mathbb{N}\rangle$ going to infinity such that for all large enough m, $x \upharpoonright [-a_m, b_m) \in \mathcal{W}_m$.

Note that S is a dense shift-invariant $\mathcal{G}_\delta$ set.

Lemma 11. *Fix a construction sequence $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$ for a symbolic system $\mathbb{K}$ in a finite alphabet Σ. Then:*

(1) *$\mathbb{K}$ is the smallest shift-invariant closed subset of $\Sigma^{\mathbb{Z}}$ such that for all n, and $w \in \mathcal{W}_n$, $\mathbb{K}$ has non-empty intersection with the basic open interval $\langle w\rangle \subset \Sigma^{\mathbb{Z}}$.*

(2) *Suppose that $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$ is a uniform construction sequence. Then there is a unique non-atomic shift-invariant measure ν on $\mathbb{K}$ concentrating on S and this ν is ergodic.*

Proof. Item (1) is clear from the definitions. To see item (2), fix a measure ν concentrating on S. It suffices to show that the ν-measures of sets of the form $\langle u\rangle_0$ for $u \in \mathcal{W}_k$ are uniquely determined by $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$. Fix a $u \in \mathcal{W}_k$ for some k. By the Ergodic Theorem it suffices to show that for all $\epsilon > 0$ and all large enough n, if $w', w'' \in \mathcal{W}_{n+1}$, then the proportion of occurrences of u among the k-words in w' is within ϵ of the proportion of the k-words occurring in w''.

For each $w \in \mathcal{W}_n$, let $\lambda_w(u)$ be the proportion of occurrences of u among the k-words occurring in w. Then the proportion of occurrences of u among the k-words in w' is approximated up to the proportion of w' taken up by spacers (which is summably small) by

$$\sum_{w \in \mathcal{W}_n} \lambda_w(u) \frac{f_n(w, w')}{q_{n+1}/q_n}$$

and a similar approximation holds for w''. Computing:

$$\begin{aligned}
&\Big|\sum_{w\in\mathcal{W}_n}\lambda_w(u)\frac{f_n(w,w')}{q_{n+1}/q_n}-\sum_{w\in\mathcal{W}_n}\lambda_w(u)\frac{f_n(w,w'')}{q_{n+1}/q_n}\Big|\\
&\quad\le\sum_{w\in\mathcal{W}_n}\lambda_w(u)\Big|\frac{f_n(w,w')}{q_{n+1}/q_n}-\frac{f_n(w,w'')}{q_{n+1}/q_n}\Big|\\
&\quad\le\sum_{w\in\mathcal{W}_n}\Big|\frac{f_n(w,w')}{q_{n+1}/q_n}-\frac{f_n(w,w'')}{q_{n+1}/q_n}\Big|\\
&\quad\le\sum_{w\in\mathcal{W}_n}\Big(\Big|\frac{f_n(w,w')}{q_{n+1}/q_n}-d_n(w)\Big|+\Big|d_n(w)-\frac{f_n(w,w'')}{q_{n+1}/q_n}\Big|\Big)\\
&\quad\le 2\epsilon_{n+1}.
\end{aligned}$$

Taking n large enough we have shown that $\nu(\langle u\rangle_0)$ is uniquely determined. Since there is a unique measure on S, that measure must be ergodic. □

Remark 12. We make two remarks about Lemma 11.

(1) If X is a Polish space, $T : X \to X$ is a Borel automorphism and D is a T-invariant Borel set with a unique T-invariant measure on D, then that measure must be ergodic.

(2) If $(\mathbb{K}, sh)$ is an arbitrary symbolic shift then its inverse is $(\mathbb{K}, sh^{-1})$, where $sh^{-1}(f)(n) = f(n-1)$. If x is in $\mathbb{K}$, we define the reverse of x by setting $\mathrm{rev}(x)(k) = x(-k)$. We can view $(\mathbb{K}, sh^{-1})$ as the symbolic system $(\mathrm{rev}(\mathbb{K}), sh)$, where $\mathrm{rev}(\mathbb{K})$ consists of all of the reverses of elements of $\mathbb{K}$.

(3) Assuming the hypothesis of Lemma 11, the proof also shows that there is a unique non-atomic shift-invariant measure on $\mathrm{rev}(S)$ and that for this measure, which we denote ν^{-1}, we have $\nu(\langle w\rangle) = \nu^{-1}(\langle \mathrm{rev}(w)\rangle)$.

4 Circular symbolic systems

We now define a class of symbolic shifts that we call circular systems. The main result of this paper is that the circular systems are symbolic representations of the smooth diffeomorphisms defined by Anosov–Katok method of conjugacies. The construction sequences of circular systems have quite specific combinatorial properties that will be important to our understanding of the Anosov–Katok systems in the sequels to this paper.

These symbolic systems are built from construction sequences $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$, where $\mathcal{W}_{n+1}$ is the result of applying an abstract operation $\mathcal{C}$ to sequences of words from $\mathcal{W}_n$. We call these systems circular because they are closely tied to the behavior of rotations by a convergent sequence of rationals $\alpha_n = p_n/q_n$. The rational rotation by p/q permutes the $1/q$ intervals of the circle cyclically along a sequence determined by some numbers $j_i =_{def} p^{-1}i \pmod q$.[7] To have a symbolic representation of an Anosov–Katok diffeomorphism, one must be able to describe how the intervals of length $1/q_{n+1}$ are permuted by addition of α_{n+1} mod(1) in terms of the intervals of length $1/q_n$. The abstract symbolic operation $\mathcal{C}$ does this. We explain this in detail in Section 7. Theorem 60, which says that all strongly uniform systems built with $\mathcal{C}$ and a suitable coefficient sequence can be realized as measure preserving diffeomorphisms, is the part of this paper that we use in later papers to construct measure preserving diffeomorphisms with complicated combinatorial structures. Theorem 58 is the companion to Theorem 60. It says that all (unwisted, strongly uniform) Anosov–Katok diffeomorphisms can be represented by circular symbolic systems.

Let k, l, p, q be positive natural numbers with p and q relatively prime. Set $j_i \equiv_q (p)^{-1}i$ with $j_i < q$. It is easy to verify that

$$q - j_i = j_{q-i}. \tag{3}$$

The operation $\mathcal{C}$ is defined on sequences $w_0, \ldots, w_{k-1}$ of words in an alphabet $\Sigma \cup \{b, e\}$ (where we assume that neither b nor e belongs to Σ) by setting:

$$\mathcal{C}(w_0, w_1, w_2, \ldots, w_{k-1}) = \prod_{i=0}^{q-1}\prod_{j=0}^{k-1}(b^{q-j_i} w_j^{l-1} e^{j_i}). \tag{4}$$

Remark 13.

- Suppose that each w_i has length q. Then the length of $\mathcal{C}(w_0, w_1, \ldots, w_{k-1})$ is klq^2.
- Whenever e occurs in $\mathcal{C}(w_0, \ldots, w_{k-1})$ there is an occurrence of b to the left of it.
- Suppose that $n < m$ and b occurs at position n and e occurs at position m, and neither occurrence is in a w_i. Then there must be some w_i occurring between n and m.

The following unique readability lemma is used to show that many construction sequences for circular systems are strongly uniform. We will also use it when we show that our symbolic representations of diffeomorphisms come from generating partitions.

[7] We assume that p and q are relatively prime and the exponent -1 denotes the multiplicative inverse of p mod q.

Lemma 14. *Suppose that Σ is a finite or countable alphabet and that $u_0, \dots, u_{k-1}$, $v_0, \dots, v_{k-1}$ and $w_0, \dots, w_{k-1}$ are words in the alphabet $\Sigma \cup \{b, e\}$ of some fixed length $q < l/2$. Let*

$$\begin{aligned} u &= \mathcal{C}(u_0, u_1, \dots, u_{k-1}),\\ v &= \mathcal{C}(v_0, v_1, \dots, v_{k-1}),\\ w &= \mathcal{C}(w_0, w_1, \dots, w_{k-1}). \end{aligned}$$

Suppose that uv is written as pws where p and s are words in $\Sigma \cup \{b, e\}$. Then either p is the empty word and $u = w$, $v = s$, or s is the empty word and $u = p$, $v = w$.

Proof. We note that the map $i \mapsto j_i$ is one-to-one. Hence each location in the word of length klq^2 is uniquely determined by the lengths of nearby sequences of b's and e's. □

In fact something stronger is true: if $\sigma \in \Sigma$ occurs at place m in w, then m is uniquely determined by $w_0, w_1, \dots, w_{k-1}$ and the $q^l/2 + 1$ letters on either side of σ.

We now describe how to use the $\mathcal{C}$ operation to build a collection of symbolic shifts. Our systems will be defined using a sequence of natural number parameters k_n and l_n that are fundamental to the version of the Anosov–Katok construction as presented in [18].

The numbers $\langle l_n \rangle$ will be assumed to go to infinity quite rapidly.[8] From the k_n and l_n we define other sequences of numbers: $\langle p_n, q_n, \alpha_n : n \in \mathbb{N} \rangle$ (with more defined later). We begin by letting $p_0 = 0$ and $q_0 = 1$ and inductively set

$$p_{n+1} = p_n q_n k_n l_n + 1, \quad q_{n+1} = k_n l_n {q_n}^2. \tag{5}$$

Thus $p_1 = 1$ and $q_1 = k_0 l_0$. Letting $\alpha_n = p_n/q_n$ we see that

$$\frac{p_{n+1}}{q_{n+1}} = \alpha_n + \frac{1}{k_n l_n q_n^2}.$$

We note that p_n and q_n are relatively prime for $n \geq 1$ and hence it makes sense to define an integer j_i with $0 \leq j_i < q_n$ by setting[9]

$$j_i = (p_n)^{-1} i \mod q_n. \tag{6}$$

Let Σ be a non-empty finite or countable alphabet. We will construct the systems we study by building collections of words $\mathcal{W}_n$ in $\Sigma \cup \{b, e\}$ by induction.

[8] In particular, in what follows we will assume that $\sum \frac{1}{l_n}$ is finite.

[9] For $q_0 = 1$, $\mathbb{Z}/q_0\mathbb{Z}$ has one element, [0], so ${p_0}^{-1} = p_0 = 0$. Also, formally j_i should have a notation indicating that it depends on n, i.e. j_i^n. We neglect this to reduce notational complexity.

We set

$$\mathcal{W}_0 = \Sigma.$$

Having built $\mathcal{W}_n$ we choose a set of **prewords** $P_{n+1} \subseteq (\mathcal{W}_n)^{k_n}$ and form $\mathcal{W}_{n+1}$ by taking all words of the form $\mathcal{C}(w_0, w_1, \ldots, w_{k_n-1})$ with $(w_0, \ldots, w_{k_n-1}) \in P_{n+1}$.[10] It follows from Lemma 14 that each $\mathcal{W}_n$ is uniquely readable.

Strong unique readability assumption. Let $n \in \mathbb{N}$, and view $\mathcal{W}_n$ as a collection Λ_n of letters. We will say that P_{n+1} satisfies strong unique readability if and only if when viewing each element of P_{n+1} as a word with letters in Λ_n, the resulting collection of Λ_n–words is uniquely readable.

Definition 15. A construction sequence $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$ will be called **circular** if it is built using the operation $\mathcal{C}$, a circular coefficient sequence and each P_{n+1} satisfies the strong unique readability assumption.

We now show that strongly uniform circular systems are uniform.

Lemma 16. *Suppose $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$ is a circular construction sequence such that:*

(1) $\sum 1/l_n$ *is finite and*

(2) *for each n there is a number f_n such that each word $w \in \mathcal{W}_n$ occurs exactly f_n times in each word in P_{n+1}.*

Then $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$ is strongly uniform.

Proof.

For each $w \in \mathcal{W}_n$, $w' \in \mathcal{W}_{n+1}$, if we set $f(w, w') = f_n q_n (l_n - 1)$ and $d_n = (\frac{f_n}{k_n})(1 - (\frac{1}{l_n}))$ then:

$$\frac{f(w, w')}{q_{n+1}/q_n} = f_n q_n (l_n - 1)\left(\frac{q_n}{k_n l_n q_n^2}\right) = \left(\frac{f_n}{k_n}\right)\left(1 - \left(\frac{1}{l_n}\right)\right) = d_n. \qquad \square$$

Definition 17. A symbolic shift $\mathbb{K}$ constructed from a circular construction sequence will be called a **circular system**. If $\mathbb{K}$ is constructed from a (strongly) uniform circular construction sequence, then we will say that $\mathbb{K}$ is a **(strongly) uniform circular system**.

Lemma 11 gives a characterization of the support of a uniform circular system and shows that there is a unique shift-invariant measure on the set S.

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[10] Passing from $\mathcal{W}_n$ to $\mathcal{W}_{n+1}$ we use $\mathcal{C}$ with parameters $k = k_n$, $l = l_n$, $p = p_n$ and $q = q_n$. The j_i is $(p_n)^{-1} i$ modulo q_n. Strictly speaking, we should probably write $\mathcal{C}_n$ for the operation $\mathcal{C}$ at stage n that uses these parameters and write j_i as j_i^n. By Remark 13, the length of each of the words in $\mathcal{W}_{n+1}$ is q_{n+1}.

Definition 18. Suppose that $w = \mathcal{C}(w_0, w_1, \dots, w_{k-1})$. Then w consists of blocks of w_i repeated $l - 1$ times, together with some b's and e's that are not in the w_i's. The **interior** of w is the portion of w in the w_i's. The remainder of w consists of blocks of the form b^{q-j_i} and e^{j_i}. We call this portion the **boundary** of w.

In a block of the form w_j^{l-1} the first and last occurrences of w_j will be called the **boundary portion of the block** w_j^{l-1}. The other occurrences will be the **interior** occurrences.

We note that the boundary consists of sections of w made up of b's and e's. However, not all b's and e's occurring in w are in the boundary, as they may be part of a power $w_i^{l_i-1}$.

The boundary of w constitutes a small portion of the word:

Lemma 19. *The proportion of the word w written in* (4) *that belongs to its boundary is* $1/l$. *Moreover, the proportion of the word that is within q letters of its boundary is* $3/l$.

We now characterize the set $S \subset \mathbb{K}$ for circular systems and show a strong unique ergodicity result.

Lemma 20. *Let $\mathbb{K}$ be a circular system. Then:*

(1) *Let ν be a shift-invariant measure on $\mathbb{K}$. Then ν concentrates on S iff ν concentrates on the collection of $s \in \mathbb{K}$ such that $\{i : s(i) \notin \{b, e\}\}$ is unbounded in both $\mathbb{Z}^-$ and $\mathbb{Z}^+$.*

(2) *Suppose that $\mathbb{K}$ is a circular system. If ν is a non-atomic shift-invariant measure on $\mathbb{K}$, then $\nu(S) = 1$. If $\mathbb{K}$ is uniform, there is a unique non-atomic shift-invariant measure on $\mathbb{K}$.*

Proof. To start the proof we note that if e occurs at m_0 in $s \in S$, then there is an $m_1 < m_0$ and a $\sigma \in \Sigma$ such that σ occurs at m_1 in s. Similarly, if b occurs at m_0, there is an $m_1 > m_0$ and a $\sigma \in \Sigma$ such that σ occurs at m_1. To see this fix an occurrence m_0 of e in s. (The argument for b is symmetric.) Let n be the smallest natural number such that there is a $w \in \mathcal{W}_{n+1}$ occurring in s on the interval $[a, b]$ with $a \le m_0 \le b$. Then $w = \mathcal{C}(w_0, \dots, w_{k_n-1})$, for some $w_i \in \mathcal{W}_n$. Since m_0 does not occur in the first occurrence of w_0 in w, this first occurrence is at an interval $[a_0, b_0]$ with $b_0 < m_0$. Since some $\sigma \in \Sigma$ must occur in w_0 we have the m_0 as required.

From this we see that if $s \in S$ and $\{i : s(i) \notin b, e\}$ is bounded below in $\mathbb{Z}$, then s must have a left tail consisting of the letter b, and a similar statement holds for e's and right tails.

Let ν be a shift-invariant measure concentrating on S and $\langle \nu_i : i \in I\rangle$ be ν's ergodic decomposition. Let T be the set of $s \in S$ such that for some $k \in \mathbb{Z}$, $s \restriction (-\infty, k)$ is constantly b (i.e. those elements of s that are constantly b on a tail going left). Then T is a shift-invariant set. We claim $\nu(T) = 0$. If $\nu(T) \neq 0$, then for some i, $\nu_i(T) \neq 0$. Thus without loss of generality we can assume that ν is ergodic. The ergodic theorem applied to the basic open set $\langle b\rangle$ centered at 0 and the averages $\frac{1}{N}\sum_{-N+1}^{0} sh^n(\langle b\rangle)$ shows that $\nu(\langle b\rangle) = 1$. The shift invariance and countable completeness of ν implies that ν gives the constant b sequence measure one, contradicting the assumption that ν concentrates on S. The proof that ν gives the collection of $s \in S$ with a positive tail constantly e measure zero is very similar, using the ergodic averages in the positive direction.

For the reverse implication we show that the collection of $s \in \mathbb{K}$ such that $\{i : s(i) \notin \{b, e\}\}$ is unbounded in both $\mathbb{Z}^-$ and $\mathbb{Z}^+$ is a subset of S. Let s have this property. Suppose that we are given $a, b \in \mathbb{N}$. We must find an $a' > a$ and a $b' > b$ such that $s \restriction [-a', b') \in \mathcal{W}_n$. Choose an $i > a, i' > b$ such that $s(-i) \notin \{b, e\}$ and $s(i') \notin \{b, e\}$. Let n be so large that $q_n > i + i' + 1$ and consider $s \restriction [-10q_{n+1}, 10q_{n+1}]$. This must be a subword of some word w^* in $\mathcal{W}_{m+1}$ with $m \geq n + 1$. Suppose that $w^* = \mathcal{C}(w_0, \ldots, w_{k_m-1})$. Since the connected segments of the boundary of w^* are of length $q_{m+1} > i + i' + 1$, and neither $-i$ nor i' are in a position in s corresponding to the boundary of w^*, the positions of w^* corresponding to $-i$ and i^* must be in a segment of w^* of the form $w_k^{l_m-1}$. If $-i - 1$ and $i' + 1$ are positions in w^* in different copies of w_k, then they must be separated by a segment of b's and e's of length q_m, a contradiction. Hence they lie in a particular copy of w_k. Letting $-a'$ be the beginning position of that copy and b' be the end position, we have finished the proof of the first claim.

To see the second item, let ν be an ergodic non-atomic measure. Then, as in the first claim, ν gives measure zero to those elements of $\mathbb{K}$ that are constant on a tail in either direction. Hence ν concentrates on those s that have arbitrarily large positive and negative i in both directions where $s(i)$ is not in $\{b, e\}$. Hence ν concentrates on S.

If ν is an arbitrary non-atomic measure, then the measures in its ergodic decomposition have to give measure zero to those elements of $\mathbb{K}$ that are constant on a tail in either direction, and hence ν concentrates on S. Thus the last assertion follows from Lemma 11. □

We now define a canonical factor of a circular system.

Definition 21. Let $\langle k_n, l_n : n \in \mathbb{N}\rangle$ be a coefficient sequence for a circular system with $\sum 1/l_n < \infty$. Let $\Sigma_0 = \{*\}$. Define a uniform circular construction sequence such that each $\mathcal{W}_n$ has a unique element as follows:

(1) $\mathcal{W}_0 = \{*\}$ and

(2) If $\mathcal{W}_n = \{w_n\}$ then $\mathcal{W}_{n+1} = \{\mathcal{C}(w_n, w_n, \ldots, w_n)\}$.

Let $\mathcal{K}$ be the resulting circular system.

Let $\mathbb{K}$ be an arbitrary circular system with coefficients $\langle k_n, l_n\rangle$, $\sum 1/l_n < \infty$. Then $\mathbb{K}$ has a canonical factor isomorphic to $\mathcal{K}$ as we see by defining the following function:

$$\pi(x)(i) = \begin{cases} x(i) & \text{if } x(i) \in \{b, e\}, \\ * & \text{otherwise} \end{cases} \tag{7}$$

The following easy lemma justifies the terminology of Definition 21:

Lemma 22. *Let π be defined by* (7). *Then:*

(1) *$\pi : \mathbb{K} \to \mathcal{K}$ is a Lipshitz map,*

(2) *$\pi(sh^{\pm 1}(x)) = sh^{\pm 1}(\pi(x))$, and thus*

(3) *π is a factor map of $\mathbb{K}$ to $\mathcal{K}$ and $\mathbb{K}^{-1}$ to $\mathcal{K}^{-1}$.*

Definition 23. We will call $\mathcal{K}$ the **circle factor** or **rotation factor** of any circular system with the same construction coefficients $\langle k_n, l_n : n \in \mathbb{N}\rangle$.

Let p_n and q_n be defined as in (5), $\alpha_n = p_n/q_n$. Then, if $\sum 1/l_n < \infty$, the sequence of α_n converges to an irrational α. Theorem 43 says that $\mathcal{K}$ is isomorphic to a rotation by α. In the smooth realization of a circular system $\mathbb{K}$, the factor $\mathcal{K}$ corresponds to a rotation of the equator.

Because the rotation by α is a discrete spectrum, $\mathcal{K}$ is a factor of the Kronecker factor of a circular system $\mathbb{K}$. In general, it is not the whole Kronecker factor; in [14] we show that if the sequences of words w_i used by $\mathcal{C}$ satisfy randomness assumptions, then $\mathcal{K}$ coincides with the Kronecker factor.

5 Periodic processes

An important tool in the Anosov–Katok method of conjugacy is the use of periodic processes to build a measure preserving transformation. We give a somewhat simplified version of the method here—it is described in more generality in [18] and [17].

The idea behind the method is the following standard proposition ([15]):

Proposition 24. *Let T be an ergodic measure preserving transformation, $n \in \mathbb{N}$ and $\epsilon > 0$. Then there is a periodic transformation S with period n such that $d_U(S, T) < \epsilon$. In particular, T is a weak limit of periodic transformations.*

Taking this as a starting point we now describe a method for determining an ergodic transformation by using a sequence of periodic transformations. Rather than view our transformations as point-maps, we will view a periodic transformation as a periodic permutation of a partition. Our view is less general than that in [17] in that we take the cycles of the permutation to all be of the same length. Adapting to the general case is routine.

Definition 25. Let $\mathcal{P}$ be a partition of the measure space X. A **periodic process** is a permutation of the atoms of $\mathcal{P}$ such that:

(1) each cycle has the same length,
(2) the atoms in each cycle have the same measure.

If all of the atoms of $\mathcal{P}$ have the same measure, we will call $\mathcal{P}$ a **uniform** periodic process.

It is convenient to view the cycles $\mathcal{T}_1, \mathcal{T}_2, \ldots \mathcal{T}_n$ of τ as "towers." This is done by arbitrarily choosing an element B of $\mathcal{T}_i$ and designating it as the base and viewing the k^{th} level of the tower to be $\tau^k(B)$.

A slightly subtle point is that the periodic process is a map that permutes the partition and is not defined pointwise. We will frequently manipulate periodic processes τ by considering measure preserving transformations F that permute the partition in the same manner as τ does. We will call such an F a **pointwise realization** of τ.

We need to have a notion of convergence to use periodic processes to determine an ergodic transformation. We use a uniform version of ϵ-refinement.

Definition 26. Let τ be a periodic process defined on $\mathcal{P}$ and σ be a periodic process defined on $\mathcal{Q}$. We will say that σ **ϵ-approximates** τ iff there are disjoint collections of $\mathcal{Q}$-atoms $\{S_A : A \in \mathcal{P}\}$ and a set $D \subset X$ of measure less than ϵ such that for some choice of bases for the towers of τ:

(1) for each $A \in \mathcal{P}$ we have $(\bigcup S_A) \setminus D \subseteq A$,
(2) if $A \in \mathcal{P}$ is not on the top level of a τ-tower and $B \in S_A$ we have $\sigma(B) \setminus D \subseteq \tau(A)$,
and
(3) for each tower $\mathcal{T}_k$ of σ the measures of the intersections of $X \setminus D$ with each level of $\mathcal{T}_k$ are the same.

This definition is saying that after removing a set of measure less than ϵ, the action of σ is subordinate to the action of τ, except on the top level of the towers of τ. We note that in the definition of ϵ-approximation, the second and third clauses imply that we can view σ as a periodic process defined on the set X that is built by a "cutting and stacking" construction from the restriction of τ to $X \setminus D$ using subsets of D as fill sets. We make this explicit when we compute symbolic representations of limits of periodic systems in Section 7.1.

Remark 27. In the version of the Anosov–Katok construction we consider, the situation is somewhat simpler than the general construction given in Definition 26 in that the levels of the towers of τ_{n+1} are either subsets of or disjoint from the levels of the towers in τ_n. In this case $\bigcup S_A \subseteq A$.

We will use periodic processes to build an ergodic transformation in the following manner.

Lemma 28. *Let $\langle \varepsilon_n : n \in \mathbb{N}\rangle$ be a summable sequence of positive numbers and $\langle \tau_n : n \in \mathbb{N}\rangle$ be a sequence of periodic processes defined on a sequence of partitions $\langle \mathcal{P}_n : n \in \mathbb{N}\rangle$. Suppose that*

(1) *τ_{n+1} ε_n-approximates τ_n*
and

(2) *the sequence $\langle \mathcal{P}_n : n \in \mathbb{N}\rangle$ σ-generates the measure algebra.*

Then there is a unique transformation $T : X \to X$ such that for all n_0

$$\lim_{n\to\infty} \mu\left(\bigcup_{A\in\mathcal{P}_{n_0}} (\tau_n A \,\Delta\, TA)\right) = 0. \tag{8}$$

Proof. For each n, let F_n be a pointwise realization of τ_n. Using the fact that the sequence of partitions generates the σ-algebra, we can apply Lemma 5 to see that the sequence $\langle F_n : n \in \mathbb{N}\rangle$ converges in the weak topology. If T is the limit, then clearly (8) holds. □

We will call a sequence satisfying the hypothesis of Lemma 28 a **convergent sequence** of periodic processes. We note that the proof of Lemma 28 shows the following.

Lemma 29. *Let $\langle \varepsilon_n : n \in \mathbb{N}\rangle$ be a summable sequence of positive numbers. Suppose that $\langle \tau_n : n \in \mathbb{N}\rangle$ is a sequence of periodic processes converging to a measure preserving transformation T. Let $\langle F_n : n \in \mathbb{N}\rangle$ be an arbitrary sequence of measure preserving, invertible transformations such that for each n, $\sum_{A\in\mathcal{P}_n} \mu(F_n A \,\Delta\, \tau_n A) < \varepsilon_n$. Then $\langle F_n : n \in \mathbb{N}\rangle$ converges weakly to T.*

Proof. Same as Lemma 28. □

We will use the next lemma to construct isomorphisms between limits of measure preserving transformations.

Lemma 30. *Fix a summable sequence of positive numbers $\langle \varepsilon_n : n \in \mathbb{N} \rangle$. Let (X, μ) and (Y, ν) be standard measure spaces and $\langle T_n : n \in \mathbb{N} \rangle$, $\langle S_n : n \in \mathbb{N} \rangle$ be measure preserving transformations of X and Y that converge in the weak topology to T and S respectively. Suppose that $\langle \mathcal{P}_n : n \in \mathbb{N} \rangle$ is a decreasing sequence of partitions and $\langle \phi_n : n \in \mathbb{N} \rangle$ is a sequence of measure preserving transformations such that*

(1) $\phi_n : X \to Y$ *is an isomorphism between* T_n *and* S_n*,*
(2) *the sequences* $\langle \mathcal{P}_n : n \in \mathbb{N} \rangle$ *and* $\langle \phi_n(\mathcal{P}_n) : n \in \mathbb{N} \rangle$ *generate the measure algebras of X and Y respectively,*
(3) $D_\nu(\phi_{n+1}(\mathcal{P}_n), \phi_n(\mathcal{P}_n)) < \varepsilon_n$.

Then the sequence $\langle \phi_n : n \in \mathbb{N} \rangle$ converges in the weak topology to an isomorphism between T and S.

Proof. Conditions 2 and 3 verify the hypothesis of Lemma 5. Thus the sequence of ϕ_n converges. The proof that the limit is an isomorphism is similar. □

6 The Anosov–Katok method of conjugacy

We now give a brief exposition of a method developed by Anosov and Katok ([1]) for realizing abstract measure preserving systems as C^∞-transformations on the unit disk, the annulus or the two-torus ($\mathbb{T}^2$).[11] The main result of [1] is that there is an ergodic measure preserving diffeomorphism of the unit disk in $\mathbb{R}^2$ that is isomorphic to an irrational rotation of the circle. An important feature of the construction is that the irrational is Liouvillean; the irrationals for which the construction works can all be approximated rapidly by rational numbers. The Anosov–Katok method has been simplified and extended by several people ([8, 6]). We discuss only two such results.

Herman ([9]) proved that if T is a C^k-diffeomorphism ($k \geq 2$) that has Diophantine rotation number on the boundary, then there are T-invariant closed curves arbitrarily close to the boundary, a property violating ergodicity. In particular, if T is an ergodic area-preserving C^k-diffeomorphism of the unit disk measure with rotation number α on the boundary, then α is Liouvillean.

A remarkable converse of this theorem is proved in [8], where it is shown that if α is an arbitrary Liouvillean irrational, then there is a C^∞-diffeomorphism of the

[11] As noted earlier, we only describe the untwisted case of the Anosov–Katok method.

torus (or unit disk or annulus) that is isomorphic to rotation by α and has rotation number α.

Given these results, it is natural to ask if behavior opposite to that of irrational rotations can be realized as diffeomorphisms. This question was answered in the original paper of Anosov and Katok ([1]) where it is shown that there are area-preserving diffeomorphisms of the disk (or torus or annulus) that are weakly mixing. This result was extended in [7] to get weakly mixing diffeomorphisms with arbitrary Liouvillean rotation number.

These results are proved by careful examination of the norms of a convergent sequence of diffeomorphisms. We give up this control in our construction which is closer to the original untwisted Anosov–Katok method.

All of the theorems in this section are variations of the known results appearing in [18]. What is new in this paper is a symbolic representation of the untwisted Anosov–Katok systems as uniform circular systems. In applications of the results of this paper we are concerned with building circular systems with intricate combinatorial properties. To get diffeomorphisms with these properties we appeal to the results in the next few sections to see that the circular systems can be realized as measure preserving diffeomorphisms.

6.1 Abstract Anosov–Katok-method. The Anosov–Katok construction inductively defines a sequence τ_n of periodic processes that converge to the desired transformation T. From a high-level point of view, what is useful to us in later applications is that this construction allows us to insert an arbitrary finite amount of information into each τ_n and still make the construction converge to a diffeomorphism provided that a simple equivariance condition is satisfied.

We elucidate the method on the torus, for convenience. The techniques are easily modified to give symbolic representations of Anosov–Katok diffeomorphisms of the disk or annulus.

We will present the method in two stages. In the first we follow [18] very closely, repeating the discussion and using the notation there as far as possible, consistent with our later purposes. That construction has three sequences of numbers $\langle k_n : n \in \mathbb{N}\rangle$, $\langle l_n : n \in \mathbb{N}\rangle$ and $\langle s_n : n \in \mathbb{N}\rangle$ as parameters. The assertion will be that if $\langle l_n : n \in \mathbb{N}\rangle$ goes to infinity fast enough (the size of l_n depends on $\langle k_m : m \leq n\rangle$ and $\langle s_m : m \leq n+1\rangle$), $s_{n+1} \leq s_n^{k_n}$ and s_n goes to infinity, then the method creates a sequence of periodic approximations that converges to an ergodic transformation T.

Given coefficient sequences $\langle k_n\rangle$ and $\langle l_n\rangle$ we can build sequences $\langle p_n\rangle$ and $\langle q_n\rangle$

as in (5), starting with $p_0 = 0$ and $q_0 = 1$. The rationals

$$\alpha_n = \frac{p_n}{q_n} \tag{9}$$

will approximate a Liouvillean number $\alpha = \lim \alpha_n$. It is clear that this approximation is very fast once we note that

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_n}{q_n} + \frac{1}{q_{n+1}} = \alpha_n + \frac{1}{q_{n+1}}.$$

Thus if the l_n grow fast enough, α is Liouvillean.

The only condition we put on the sequence of s_n's is that s_n divides s_{n+1} and that they go to infinity.

What we call the **abstract method** defines a transformation on an abstract measure space $(X, \mathcal{B}, \mu)$ by periodic processes. Since approximation by periodic processes produces a weakly convergent sequence, we will get no information about the continuity properties of the limit.

There is an auxiliary space $\mathcal{A} = [0, 1) \times [0, 1]$ which provides the combinatorial basis for the approximations. For the abstract method we will neglect a set of measure zero and view $\mathcal{A}$ as $[0, 1) \times [0, 1)$. This is appropriate because we are not concerned about continuity properties. In the smooth case we must worry about the boundary; there the action on the two boundary segments $[0, 1) \times \{0\}$ and $[0, 1) \times \{1\}$ of the limit transformation will be identical so, a fortiori, we are working on the torus.

We let S^1 act on $\mathcal{A}$ by "rotation" on the first coordinate. We will denote this additively, viewing the rotation action on S^1 as "addition mod 1" in the x-direction of the unit square. Specifically we identify $[0, 1)$ with S^1 by the map $x \mapsto e^{2\pi ix}$. Then rotation by α corresponds to addition in the exponent. Given $\alpha \in \mathbb{R}$, we denote the "rotation" of the unit interval determined by α as $\mathcal{R}_\alpha$ and the horizontal rotation of the rectangle $\mathcal{A}$ by α as $\overline{\mathcal{R}}_\alpha$.

For positive $q, s \in \mathbb{N}$ we let $\xi_s^q = \mathcal{I}_q \otimes \mathcal{I}_s$, i.e. the partition of $\mathcal{A}$ that has atoms of the form

$$[i/q, (i+1)/q) \times [j/s, (j+1)/s)$$

with $0 \le i < q$ and $0 \le j < s$. For given sequences $\langle k_n \rangle$, $\langle l_n \rangle$ and $\langle s_n \rangle$, we simplify notation by setting $\xi_n = \mathcal{I}_{q_n} \otimes \mathcal{I}_{s_n}$. If $Z : \mathcal{A} \to X$ is an invertible measure preserving transformation, then Z defines partitions of X by setting $\zeta_s^q = Z\xi_s^q$ and $\zeta_n = Z\xi_n$. We will refer to the rectangle $[i/q_n, (i+1)/q_n) \times [j/s_n, (j+1)/s_n)$ as $R^n_{i,j}$ and call $R^n_{i,j}$ the $(i, j)^{\text{th}}$ element of ξ_n.

If $\alpha = p/q$ with p, q relatively prime, then the atoms of ξ_s^q are permuted by the action $\overline{\mathcal{R}}_\alpha$ and this permutation has s cycles, each of length q. Conjugating by Z gives a periodic process τ defined on X with partition ζ_s^q that has s towers of length q. When building periodic processes we often want to view τ as the permutation of the atoms of ζ_s^q and not as a pointwise realization of a measure preserving map.

In this paper our periodic processes will be strongly uniform and have pointwise realizations of the following form:

$$T_n = Z_n \circ \overline{\mathcal{R}}_{\alpha_n} \circ Z_n^{-1}$$

where the sequence of α_n is defined by (5) and (9) and Z_n is a measure isomorphism between $\mathcal{A}$ and X. For notational simplicity we let $Z_0 = Z$, where $Z : \mathcal{A} \to X$ is a fixed isomorphism from $(\mathcal{A}, \mathcal{B}, \lambda)$ to $(X, \mathcal{B}, \mu)$. For $n \geq 1$, Z_n will be of the form

$$Z_n = Z \circ h_1 \circ \cdots \circ h_n$$

where each h_i is a measure preserving transformation of $\mathcal{A}$ that induces a permutation of the atoms of $\mathcal{I}_{k_{i-1}q_{i-1}} \otimes \mathcal{I}_{s_i}$. Thus $Z_n : \mathcal{A} \to X$ is an invertible measure preserving transformation. Because ξ_n refines $\mathcal{I}_{k_{n-1}q_{n-1}} \otimes \mathcal{I}_{s_n}$, we can view h_n as permuting the atoms of ξ_n.

Definition 31. Since $\overline{\mathcal{R}}_{\alpha_n}$ gives a periodic process with partition ξ_n, the map T_n induces a periodic process with partition ζ_n, which we take to be τ_n. When we want to view τ_n as a collection of towers, we take the bases of τ_n to be the sets $Z_n R^n_{0,s}$, for $s < s_n$.

To start the inductive construction we let $s_0 \geq 2$ and take τ_0 to be the periodic process based on the partition ξ_0 induced by the action on ξ_0 given by $\overline{\mathcal{R}}_{\alpha_0}$ (which is the identity map). Thus τ_0 has s_0 towers of height one.

What remains is to describe how to pass from Z_n to Z_{n+1}. The trick (due to Anosov and Katok) is to note that if h_{n+1} commutes with $\overline{\mathcal{R}}_{\alpha_n}$, then

$$T_n = Z_n \overline{\mathcal{R}}_{\alpha_n} Z_n^{-1} = Z_n h_{n+1} h_{n+1}^{-1} \overline{\mathcal{R}}_{\alpha_n} Z_n^{-1} = Z_n h_{n+1} \overline{\mathcal{R}}_{\alpha_n} h_{n+1}^{-1} Z_n^{-1} = Z_{n+1} \overline{\mathcal{R}}_{\alpha_n} Z_{n+1}^{-1}$$

Consequently, if α_{n+1} is chosen sufficiently close to α_n, the map T_{n+1} will permute ξ_n very similarly to the way that T_n does. It follows that the periodic process τ_{n+1} will be very close to the periodic process τ_n. We note that determining those ϵ for which τ_{n+1} ϵ-approximates τ_n is independent of the choice of pointwise realizations T_n and T_{n+1} as it is a property of the partitions.

Summarizing, if h_{n+1} is chosen so that it commutes with $\overline{\mathcal{R}}_{\alpha_n}$ and the sequence $\langle \alpha_n : n \in \mathbb{N} \rangle$ converges fast enough, then the sequence of periodic processes we construct converges in the weak topology to an invertible measure preserving system. Moreover, we can guarantee that the α_n converge arbitrarily fast by choosing the l_n-sequence to grow fast.

Remark 32. The foregoing construction of an invertible measure preserving transformation is determined up to isomorphism by the following data:

(1) The sequences $\langle k_n, l_n, s_n : n \in \mathbb{N}\rangle$ and
(2) the maps $\langle h_n : n \in \mathbb{N}\rangle$.

Constructing h_{n+1}. We will build h_{n+1} of a special form. We choose numbers s_{n+1} and k_n with s_{n+1} and k_n being multiples of s_n and $s_n^{k_n} \geq s_{n+1}$. We think of k_n and s_{n+1} as very large. (In later applications, k_n will be chosen after s_{n+1} and will be large enough to satisfy some requirements determined by the law of large numbers.) The map h_{n+1} is taken to be a measure preserving transformation permuting ξ_{n+1} that also induces a permutation of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$, i.e. it takes atoms of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$ to atoms of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$.

Our transformations will be **untwisted** in the language of [18]. This means that h_{n+1} induces a permutation of the atoms of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$ that lie inside $[0, 1/q_n) \times [0, 1)$. Since $\overline{\mathcal{R}}_{\alpha_n}$ cyclically permutes the towers of ξ_n starting with bases of the form R^n_{0,s_n} and the h_{n+1} commute with $\overline{\mathcal{R}}_{\alpha_n}$, the h_{n+1}'s are determined by what they do on $[0, 1/q_n) \times [0, 1)$.

We start the process of defining h_{n+1} with an arbitrary permutation p of $(\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}) \upharpoonright [0, 1/q_n) \times [0, 1)$ and let h^0_{n+1} be any pointwise realization of p that gives a permutation of $\xi_{n+1} \upharpoonright [0, 1/q_n) \times [0, 1)$. We then extend h^0_{n+1} to a pointwise map on $\mathcal{A}$ that commutes with $\overline{\mathcal{R}}_{\alpha_n}$ by "copying over" equivariantly

Having defined the h_{n+1} with the numbers k_n and s_{n+1} so that it commutes with $\overline{\mathcal{R}}_{\alpha_n}$, we choose l_n large enough that α_{n+1} is very close to α_n.[12] Since h_{n+1} permutes the elements of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$ it permutes the elements of ξ_{n+1}, as well.

Lemma 33. *For each n, the partition ζ_{n+1} refines ζ_n and the collection of partitions $\{\zeta_n : n \in \mathbb{N}\}$ generates the measure algebra of X.*

Proof. Since $\langle \xi_n : n \in \mathbb{N}\rangle$ is a decreasing sequence of partitions that generates the measure algebra of $\mathcal{A}$, $\langle Z\xi_n : n \in \mathbb{N}\rangle$ is a decreasing sequence that generates the measure algebra of X. Each h_m is a permutation of ξ_n for $m \leq n$. Consequently, $Z\xi_n = Z_n\xi_n = \zeta_n$. Hence the ζ_n are decreasing and generate. □

Ergodicity. We introduce some requirements to guarantee the ergodicity of our systems. The requirements we state here are stronger than necessary,[13] but

[12] The limit of the sequence only depends on the periodic processes, and hence the permutation of ξ_n is determined by h_n, rather than h_n as a pointwise map. It will follow that there are functions $l^*_n(x_0, \ldots, x_{n-1}, y_0, \ldots, y_n, z_0, \ldots, z_{n+1})$ such that if $l_n \geq l^*_n(l_0, \ldots, l_{n-1}, k_0 \ldots, k_n, s_0, \ldots, s_{n+1})$ for all n, then the construction converges.

[13] Cleverer versions of Z_n (e.g. [7]) can be constructed that guarantee ergodicity even if every s_n is equal to 2. It is not difficult to check that these have symbolic representations very similar to the ones we are describing here. We use Requirements 1–3 to guarantee ergodicity for circular systems with fast growing l_n-sequences.

easy to verify. We postpone the proof that these requirements imply ergodicity until Theorem 58 in Section 7.5.

Here are our requirements:

Requirement 1. The sequence s_n tends to ∞.

Requirement 2. (Strong Uniformity) For each $R^n_{0,j} \in \xi_n$ and each $s < s_{n+1}$ we have that the cardinality of

$$\{t < k_n : h_{n+1}([t/k_nq_n, (t+1)/k_nq_n) \times [s/s_{n+1}, (s+1)/s_{n+1})) \subseteq R^n_{0,j}\}$$

is k_n/s_n.

Given $s < s_{n+1}$ we can associate a k_n-tuple $(j_0, \ldots, j_{k_n-1})_s$ so that

$$h_{n+1}([t/k_nq_n, (t+1)/k_nq_n) \times [s/s_{n+1}, (s+1)/s_{n+1})) \subseteq R^n_{0,j_t}. \tag{10}$$

Requirement 3. We assume that the map $s \mapsto (j_0, \ldots j_{k_n-1})_s$ is one-to-one.

Discussion. There are k_n atoms a of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$ that lie in the strip $[0, 1/q_n) \times [s/s_n, (s+1)/s_n)$. For each such a, $h_{n+1}(a) \subset R^n_{0,j}$ for some j. If we assign this j to a, then we get a sequence of j's of length k_n. Requirement 2 says that each j occurs k_n/s_n times. It follows that the proportion of the atoms of ξ_{n+1} contained in any strip $[0, 1/q_n) \times [s/s_{n+1}, (s+1)/s_{n+1})$ whose h_n-image is a subset of a given $R^n_{0,j}$ is k_n/s_n. Requirement 3 says that we get different sequences of j's for different s.

We have the following lemma which describes the mechanism for inserting arbitrary finite information into each stage of the Anosov–Katok construction.

Lemma 34. *Let $w_0, \ldots, w_{s_{n+1}-1} \subseteq \{0, 1, \ldots s_n - 1\}^{k_n}$ be words such that each i with $0 \leq i < s_n$ occurs k_n/s_n times in each w_j. Then there is an invertible measure preserving h_{n+1} commuting with $\overline{\mathcal{R}}_{\alpha_n}$ and inducing a permutation of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$ such that if j_t is the t^{th} letter of w_s, then*

$$h_{n+1}([t/k_nq_n, (t+1)/k_nq_n) \times [s/s_{n+1}, (s+1)/s_{n+1}))$$

is a subset of R^n_{0,j_t}.

Proof. Construct h^0_{n+1} as follows: The atoms of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$ partition each atom of ξ_n into $k_n(\frac{s_{n+1}}{s_n})$ pieces. Each index i occurs in each w_s exactly k_n/s_n times and there are s_{n+1} many words w_s. Hence it is possible to construct a bijection between $(\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}) \upharpoonright [0, 1/q_n) \times [i/s_n, (i+1)/s_n)$ and the occurrences of i in all of the words w_s. Ranging over i one gets a bijection b between the numbers occurring in all of the words and $(\mathcal{I}_{k_n q_n} \otimes \mathcal{I}_{s_{n+1}}) \upharpoonright [0, 1/q_n) \times [0, 1)$ such that if j is the t^{th} letter of w_s, then b associates $[t/k_n q_n, (t+1)/k_n q_n) \times [s/s_{n+1}, (s+1)/s_{n+1})$ with an element of $\mathcal{I}_{k_n q_n} \otimes \mathcal{I}_{s_{n+1}} \upharpoonright R^n_{0,j}$.

We can interpret each w_s as assigning numbers to the atoms of $\mathcal{I}_{k_n q_n} \otimes \mathcal{I}_{s_{n+1}}$ that are subsets of the strip $[0, 1/q_n) \times [s/s_{n+1}, (s+1)/s_{n+1})$. Hence b can be interpreted as a permutation of $(\mathcal{I}_{k_n q_n} \otimes \mathcal{I}_{s_{n+1}}) \upharpoonright [0, 1/q_n) \times [0, 1)$. We take h^0_{n+1} to be a pointwise realization of b and extend h^0_{n+1} equivariantly. □

6.2 Approximating partition permutations by diffeomorphisms. In this section we prove that any permutation of a matrix of rectangles can be well-approximated by a C^∞-measure preserving transformation. We note that similar results were attained independently in [3], [2] and earlier in [19]. With the goal of a self-contained exposition, we present a proof of the theorem here.

Theorem 35. *Let n horizontal lines and m vertical lines divide* $[0, 1] \times [0, 1]$ *into an array of mn equal size rectangles. Let σ be a permutation of the rectangles and $\epsilon > 0$. Then there is a C^∞, invertible, measure preserving transformation ϕ of* $[0, 1] \times [0, 1]$ *that is the identity on a neighborhood of the boundary of* $[0, 1] \times [0, 1]$ *such that for a set L of Lebesgue measure at least $1 - \epsilon$, for all rectangles R*

$$\text{If } x \in L \cap R, \text{ then } \phi(x) \in \sigma(R). \tag{11}$$

We say that σ is **ϵ-approximated** by ϕ, if σ and ϕ satisfy the conclusion of Theorem 35. The collection of permutations σ that can be ϵ-approximated for all $\epsilon > 0$ is closed under composition.

We first prove a lemma about vertical and horizontal swaps.

Lemma 36. *Consider* $[0, 2] \times [0, 1]$. *Then for any $\delta > 0$, there is a C^∞-measure preserving transformation ϕ_0 that is the identity on a neighborhood of the boundary of* $[0, 2] \times [0, 1]$ *and for all but ϵ measure sends* $[0, 1] \times [0, 1]$ *to* $[1, 2] \times [0, 1]$ *and vice versa.*

Proof. Let $D \subseteq \mathbb{R}^2$ be the disk centered at $(0, 0)$ that has area $2 - \epsilon/2$ and radius $R = R(\epsilon)$. Let $\gamma > 0$ be such that the disc of radius $R - \gamma$ has area $2 - \epsilon$. It is a standard result that for any positive γ, we can find a C^∞ function $f : [0, R] \to [0, \pi]$ such that f is identically equal to π on $[0, R - \gamma]$ and is identically equal to zero in an arbitrarily small neighborhood of R in $[0, R]$.

Let $F : D \to D$ be defined in polar coordinates by setting $F(r, \theta) = (r, \theta + f(r))$. Then F is C^∞, measure preserving, rotates the disk of radius $R - \gamma$ by π and is the identity on a neighborhood of the boundary of the disk of radius R. (See Figure 1)

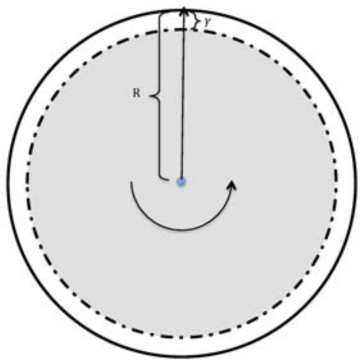


Figure 1. The transformation F.

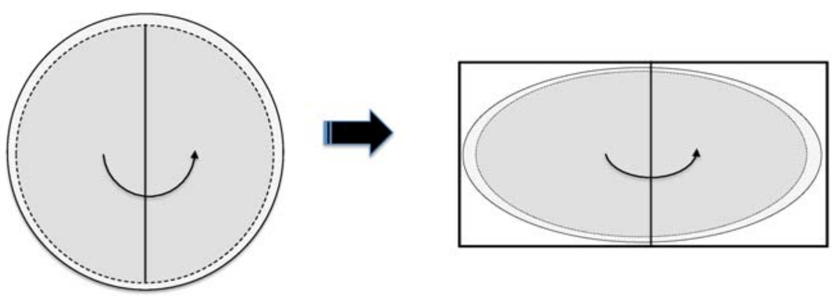


Figure 2. The transformation G.

Consider now $[0, 2] \times [0, 1]$. We can remove a set of measure $\epsilon/2$ near the boundary of $[0, 2] \times [0, 1]$, so that we are left with a flattened disk

$$D^* \subseteq [0, 2] \times [0, 1]$$

that has C^∞-boundary. By [19] there is an invertible measure preserving C^∞ function $G : D \to D^*$ that takes the left half disk to $D^* \cap ([0, 1] \times [0, 1])$ and the right half disk to $D^* \cap ([1, 2] \times [0, 1])$. (See Figure 2.)

Then $\phi_0 = GFG^{-1}$ is the desired function. □

Clearly Lemma 36 can be rescaled arbitrarily. The result analogous to Lemma 36 holds for vertically exchanging the two rectangles $[0, 1] \times [0, 1]$ and $[0, 1] \times [1, 2]$.

Proof of Theorem 35. Number the mn rectangles $\{0, \ldots, mn-1\}$ in a zig-zag fashion by giving the first row the numbers $0, 1, \ldots, m-1$ from left to right, enumerating the second row from right to left by the numbers $m, m+1, \ldots, 2m-1$, the third row from left to right with the numbers $2m, 2m+1, \ldots, 3m-1$ and so on.

0	1	2		m-2	m-1
2m-1	2m-2	2m-3	...	m+1	m
2m	2m+1	2m+2		3m-2	3m-1
	...			...	
(n-1)m	(n-1)m+1	(n-1)m+2		nm-2	nm-1

Figure 3. Labeling the partition of $[0, 1] \times [0, 1]$ into rectangles.

This allows us to view σ as a permutation of $\{0, 1, 2, \ldots, mn-1\}$. We will call permutations corresponding to exchanging vertically or horizontally adjacent rectangles **swaps**. With this numbering the swaps include all transpositions of the form $(k, k+1)$ for $0 \leq k < nm-1$. The smooth measure preserving approximations to swaps given by Lemma 36 will be called **δ-approximate** swaps.

Lemma 36 implies that for all k with $0 \leq k < mn-1$ and all $\delta > 0$ there is a C^∞ δ-approximate swap ϕ_k of the rectangles labelled k and $k+1$. This can be extended to a measure preserving diffeomorphism of $[0, 1] \times [0, 1]$ by taking ϕ_k to be the identity outside the rectangles.

Since every permutation of mn can be written as a composition of less than or equal to $(nm)^2$ transpositions of the form $(k, k+1)$, given any σ we can build ϕ by taking δ small enough and composing δ-approximate swaps corresponding to the transpositions composed to create σ. □

6.3 Smooth Anosov–Katok-method. We now show how the Anosov–Katok method of Section 6.1 can be used to construct smooth transformations isomorphic to the abstract transformations constructed in Section 6.1. In this case the measure space X will also be $\mathcal{A}$, the initial map Z will be the identity, and the spatial maps h_n will be replaced by measure preserving diffeomorphisms h_n^s that closely approximate them. Thus for $n \geq 1$, we replace the Z_n by functions

$$H_n = h_1^s \circ \cdots \circ h_n^s,$$

where the h_i^s are C^∞-measure preserving transformations of $\mathcal{A}$. We let

$$S_n = H_n \overline{\mathcal{R}}_{\alpha_n} H_n^{-1}.$$

Because we want the limit of the S_n's to be a diffeomorphism, we must consider the pointwise properties of the S_n; in particular, we are no longer working with periodic processes, but concrete realizations of measure preserving transformations. For the limit of the sequence $\langle S_n : n \in \mathbb{N}\rangle$ to be a smooth transformation it suffices to arrange that $\|S_{n+1} - S_n\|_{C^n} < \varepsilon_n$ for some summable sequence ε_n. Alternately, we fix a metric d^∞ giving the C^∞ topology; we can require that $d^\infty(S_{n+1}, S_n) < \varepsilon_n$. These, in turn, can be arranged by taking α_{n+1} sufficiently close to α_n which is done by choosing the number l_n large enough.

Remark 37. The transformations H_n will all be equal to the identity on a neighborhood of the boundary of $\mathcal{A}$. This guarantees that each S_n is equal to $\overline{\mathcal{R}}_{\alpha_{n+1}}$ on a neighborhood of the boundary of $\mathcal{A}$. In particular, we can view each S_n as a C^∞-measure preserving transformation of the torus. Thus we can view the limit transformation S as a C^∞-measure preserving transformation of the torus that is equal to rotation by α along the line determined by identifying the top and bottom boundaries of $\mathcal{A}$.

The main theorem of this section states that we can realize any transformation built by the version of the abstract Anosov–Katok method-of-conjugacy[14] as a measure preserving diffeomorphism. This theorem is implicit in the results in [18].

Theorem 38. *Suppose that $T : \mathcal{A} \to \mathcal{A}$ is a measure preserving transformation that is built by the abstract Anosov–Katok method using a parameter sequence $\langle k_n, l_n : n \in \mathbb{N}\rangle$ such that the sequence of l_n grows fast enough. Then there is a C^∞-measure preserving transformation $S : \mathcal{A} \to \mathcal{A}$ that is measure theoretically isomorphic to T.*

Proof. Fix a summable sequence $\langle \varepsilon_n : n \in \mathbb{N}\rangle$. Let $\langle T_n : n \in \mathbb{N}\rangle$ be a sequence of transformations built by the abstract Anosov–Katok method using $\langle h_n : n \in \mathbb{N}\rangle$ (thus $T_n = Z_n \overline{\mathcal{R}}_{\alpha_n} Z_n^{-1}$) and let T be the limit of the T_n in the weak topology. Using Theorem 35 we will build smooth approximations $\langle h_n^s : n \in \mathbb{N}\rangle$ and define a sequence of smooth transformations by setting H_0 to be the identity map and, for $n \geq 1$, setting $H_n = h_1^s \circ \cdots \circ h_n^s$. We show that if the h_n^s approximate h_n closely enough and $\langle \alpha_n : n \in \mathbb{N}\rangle$ converges fast enough, then the sequence $S_n = H_n \overline{\mathcal{R}}_{\alpha_n} H_n^{-1}$ converges to a C^∞-measure preserving transformation S and that S is isomorphic to T as a measure preserving transformation of $(\mathcal{A}, \mathcal{B}, \lambda)$.

To see that rapid convergence of $\langle \alpha_n : n \in \mathbb{N}\rangle$ implies that the S_n converge in the C^∞-topology, we note that since h_n^s commutes with $\overline{\mathcal{R}}_{\alpha_n}$ for each n it follows

[14] The method of proof of Theorem 38 can be easily adapted to other treatments.

that

$$S_n = H_n \overline{\mathcal{R}}_{\alpha_n} H_n^{-1} = H_n h^s_{n+1} \overline{\mathcal{R}}_{\alpha_n} (h^s_{n+1})^{-1} H_n^{-1}.$$

Hence, by the continuity of d^∞ with respect to composition, if we take α_{n+1} close enough to α_n, we can arrange that $d^\infty(S_{n+1}, S_n) < \varepsilon_n$.

Let $n \geq 0$. To define h^s_{n+1} we need to both approximate h_{n+1} and make h^s_{n+1} commute with $\overline{\mathcal{R}}_{\alpha_n}$. We use Theorem 35 to choose a smooth measure preserving, invertible $h'_{n+1} : [0, 1/q_n] \times [0, 1] \to [0, 1/q_n] \times [0, 1]$ such that:

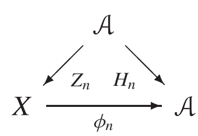
(1) If σ is the permutation of the atoms of $\mathcal{I}_{k_n q_n} \otimes \mathcal{I}_{s_n}$ inside $[0, 1/q_n) \times [0, 1)$ determined by h_{n+1}, then for all but a set of measure ϵ_{n+1}/q_n, we satisfy (11); i.e. for the atoms R of $\mathcal{I}_{k_n q_n} \otimes \mathcal{I}_{s_n}$ inside $[0, 1/q_n) \times [0, 1)$, we know that the vast majority of $x \in R$ have $h'_{n+1}(x) \in \sigma(R)$;
and

(2) the function h'_{n+1} is the identity map on a neighborhood of the boundary of $[0, 1/q_n] \times [0, 1]$.

Since h'_{n+1} is the identity on a neighborhood of the boundary of $[0, 1/q_n] \times [0, 1]$, we can copy it onto each $[i/q_n, (i+1)/q_n] \times [0, 1]$ and thereby extend it to a C^∞-measure preserving h^s_{n+1} that commutes with $\overline{\mathcal{R}}_{\alpha_n}$. If σ is the $\overline{\mathcal{R}}_{\alpha_n}$-equivariant permutation of ξ_{n+1} determined by h_{n+1}, then there is a set L_{n+1} of measure at least $1 - \epsilon_{n+1}$ such that for all atoms R of ξ_{n+1} and $x \in L_{n+1}$:

$$\text{If } x \in L_{n+1} \cap R, \text{ then } h^s_{n+1}(x) \in \sigma(R). \tag{12}$$

We now use Lemma 30 to show that T is isomorphic to S. We let $\phi_n = H_n \circ Z_n^{-1}$ and $\mathcal{P}_n = \zeta_n$.



We must verify conditions (1)–(3) of Lemma 30. It is clear that ϕ_n is an isomorphism between T_n and S_n since Z_n and H_n are isomorphisms between $\overline{\mathcal{R}}_{\alpha_{n+1}}$ and T_n and S_n respectively. Thus condition (1) is clear. To see condition (2) we must prove the following claim:

Claim. The partition $\langle \zeta_n : n \in \mathbb{N} \rangle$ generates the measure algebra of X and $\langle \phi_n(\zeta_n) : n \in \mathbb{N} \rangle$ generates the measure algebra of $\mathcal{A}$.

Proof of Claim. That the ζ_n's generate is the content of Lemma 33. We must check that the $\phi_n(\zeta_n)$ generate. Since $\phi_n = Z_n^{-1} H_n$, this is equivalent to the statement that the $H_n \xi_n$ generate.

Since the sequence ε_n is summable, the Borel–Cantelli lemma tells us that there is an increasing sequence of sets $\langle G_n : n \in \mathbb{N}\rangle$ with $G_n \subseteq \mathcal{A}$ such that the measures of $\lambda(G_n)$ approach 1 and for all $m \geq n$ h_m and h^s_m permute that partition $\xi_m \upharpoonright G_n$ the same way.

Explicitly: for all $\epsilon > 0$ we can find an n so large that

$$G_n =_{def} L_n \cap \bigcap_{m>n} (h^s_{n+1} \circ h^s_{n+2} \circ \cdots \circ h^s_m)^{-1} L_m \tag{13}$$

has measure at least $1 - \epsilon$.[15] By definition, for all $x \in G_n$ and all $m > n$, $h_{n+1} \circ \cdots \circ h_m(x) \in L_m$.

Fix a measurable set $D \subseteq \mathcal{A}$ and a $\delta > 0$. We must find a large enough n that the atoms of $H_n\xi_n$ can be used to approximate D within a set of measure δ. We first choose an n_0 so large that $\lambda(G_{n_0}) > 1 - \delta/100$. Let $D' = H^{-1}_{n_0}(D)$.

Since the ξ_n's generate, we can find an $n \geq n_0$ and a collection C' of atoms of ξ_n such that $(\bigcup C')\Delta D'$ has measure less than $\delta/100$. Hence we can find a collection of atoms of $H_{n_0}\xi_n$ whose union approximates D within $\delta/100$.

We note that for $n_0 < m \leq n$, h_m permutes the atoms of ξ_n. From the definition of G_{n_0}, for $n_0 < m \leq n$, $a \in \xi_n$ and $x \in a \cap G_{n_0}$, we know that $h^s_{n_0+1} \circ h^s_{n_0+2} \circ \cdots \circ h^s_m(x)$ belongs to the same atom of ξ as $h_{n_0+1} \circ \cdots \circ h_m(x)$ does. It follows that

$$\lambda\left(\bigcup_{a\in\xi_n} (h^s_{n_0+1} \circ h^s_{n_0+2} \circ \cdots \circ h^s_m(a) \;\Delta\; h_{n_0+1} \circ \cdots \circ h_m(a))\right) < \delta/50.$$

Since $h_{n_0+1}, \ldots, h_n$ permute the atoms of ξ_n, we can define a bijection $\sigma : \xi_n \longrightarrow \xi_n$ such that

$$\lambda\left(\bigcup_{a\in\xi_n} (h^s_{n_0+1} \circ h^s_{n_0+2} \circ \cdots \circ h^s_n(a) \;\Delta\; \sigma(a))\right) < \delta/50$$

Thus

$$\lambda\left(\bigcup_{a\in\xi_n} [H_n(a) \;\Delta\; H_{n_0}(\sigma(a))]\right) < \delta/50.$$

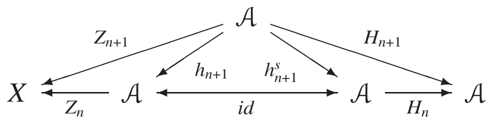
or equivalently

$$\lambda\left(\bigcup_{a\in\xi_n} [H_n(\sigma^{-1}(a)) \;\Delta\; H_{n_0}(a)]\right) < \delta/50.$$

Since D can be approximated within $\delta/100$ by a union of atoms of $H_{n_0}\xi_n$, it can be approximated within $\delta/25$ by a union of atoms of $H_n(\xi_n)$. We have verified the claim. □

[15] The L_n's are defined in (12).

To use Lemma 30, we are left with showing that $\phi_{n+1}(\zeta_n)$ ε_n-approximates $\phi_n(\zeta_n)$. Chasing the following diagram, where the leftmost and rightmost triangles commute:



we see that

$$\phi_n(\zeta_n) = H_n Z_n^{-1}(\zeta_n) = H_n h_{n+1} Z_{n+1}^{-1}(\zeta_n)$$

while

$$\phi_{n+1}(\zeta_n) = H_{n+1} Z_{n+1}^{-1}(\zeta_n) = H_n h^s_{n+1} Z_{n+1}^{-1}(\zeta_n).$$

Letting $\mathcal{Q}_n = Z_{n+1}^{-1}(\zeta_n)$, we need to see that $H_n h^s_{n+1}(\mathcal{Q}_n)$ ε_n-approximates $H_n h_{n+1}(\mathcal{Q}_n)$. This follows easily, since H_n is measure preserving and h^s_{n+1} was chosen so that $h^s_{n+1}(\mathcal{Q}_n)$ closely approximated $h_{n+1}(\mathcal{Q}_n)$. □

Remark 39. Let d^∞ be a Polish metric inducing the C^∞-topology. Then by taking α_{n+1} sufficiently close to α_n, we can arrange $d^\infty(S_{n+1}, S_n) < \varepsilon_n/4$.

What does *fast enough* mean? We do not produce explicit lower bounds on the speed of growth of the l_n's; instead we give inductive lower bounds. Here is what we mean by fast enough in the statement of Theorem 38.

Fix a summable sequence $\langle \varepsilon_n : n \in \mathbb{N} \rangle$; without loss of generality

$$\varepsilon_n/4 > \sum_{m>n} \varepsilon_m.$$

Fix a metric d^∞ inducing the C^∞-topology on the diffeomorphisms of the 2-torus. For each choice $\langle k_i : i \leq n \rangle$, $\langle l_i : i < n \rangle$, and $\langle s_i : i \leq n+1 \rangle$ there are only finitely many permutations of the relevant partitions and thus only finitely many choices for the periodic processes determined by $\langle h_i : i \leq n \rangle$. Hence there is a single number $l_n^* = l_n^*(\langle k_i : i \leq n \rangle, \langle l_i : i < n \rangle, \langle s_i : i \leq n+1 \rangle)$ such that for all $l_n \geq l_n^*$ we can choose a smooth approximation h_n^s such that

$$d^\infty(S_n, S_{n+1}) < \varepsilon_n/4. \tag{14}$$

Without loss of generality we can assume that for all n, $l_n^* > 2^n$.

Indeed we can say more. In our construction we have the inequality that $s_{n+1} \leq s_n^{k_n}$. Hence there is a sequence of bounds $b_n = b_n(|\Sigma|, \langle k_i : i < n \rangle)$ such that $s_n \leq b_n$. By ranging over all $s_n \leq b_n$ in the previous paragraph we get a sequence $l_n^* = l_n^*(\langle k_i : i < n+1 \rangle, \langle l_i : i < n \rangle)$ such that any sequence $l_n \geq l_n^*$ grows fast enough for the hypothesis of Theorem 38.

From now on, by fast enough we mean a sequence of l_n with $l_n \geq l_n^*$.

7 A symbolic representation of Anosov–Katok-systems

In this section we show two theorems. The first (Theorem 60), which is used in the sequel, is that if $\mathbb{K}$ is a strongly uniform circular system with fast growing coefficients, then $\mathbb{K}$ is isomorphic to a measure preserving Anosov–Katok diffeomorphism of $\mathcal{A}$. As usual $\mathcal{A}$ is our proxy for the unit disk, annulus or $\mathbb{T}^2$.

The second theorem (Theorem 58), which we hope is of independent interest, is that if T is built by the (untwisted) Anosov–Katok method with coefficients $\langle k_n, l_n, s_n : n \in \mathbb{N}\rangle$ and $\langle l_n : n \in \mathbb{N}\rangle$ grows fast enough, then T has a representation as a circular symbolic system with the same coefficients.

By Theorem 38, to represent the Anosov–Katok diffeomorphisms it suffices to represent the transformations built by the abstract method. We will use the notation (such as k_n, l_n, Z_n, X) from that Section 6.1. We take $X = \mathcal{A}$ and $Z_0 = Z$ to be the identity map.

7.1 Symbolic representations of periodic processes. We begin with a very general discussion of how periodic processes can be viewed as symbolic systems. Our symbolic description is a variant of standard cutting and stacking constructions where spacers are added at both the bottom (the "beginning," denoted by b's) and at the top (the "end," denoted by e's). Our representation is given explicitly for the case where all of the cycles in the periodic process have the same length. It is straightforward to adapt our analysis to the general case.

A sequence of periodic processes converging sufficiently rapidly to a transformation T gives rise to a symbolic presentation of a factor of T that has a special form. Here is how this works. Let $\langle \varepsilon_n : n \in \mathbb{N}\rangle$ be a summable sequence. Fix a sequence $\langle \tau_n : n \in \mathbb{N}\rangle$ of periodic processes converging to a transformation T with partitions $\langle \mathcal{P}_n : n \in \mathbb{N}\rangle$. Suppose that the length of the towers corresponding to τ_n is q_n and that τ_{n+1} ε_n-approximates τ_n. Since τ_{n+1} ϵ_n-approximates τ_n, we can find a set D_n (as in Definition 26) such that on the complement of D_n, the action of τ_{n+1} is subordinate to τ_n. Without loss of generality, $q_0 = 1$ and by skipping a finite number of steps at the beginning of the approximation we can assume that $\mu(\bigcup D_n) < 1/2$.

Setting $G_n = X \setminus \bigcup_{m \geq n} D_n$ we get an increasing sequence of sets $\langle G_n : n \in \mathbb{N}\rangle$ such that:

(1) the measure of G_n goes to one,
(2) on G_n, the sequence $\langle \mathcal{P}_m : m \geq n\rangle$ is a decreasing sequence of partitions, and
(3) for each tower $\mathcal{T}_k$ of τ_n, the measures of the intersections of G_n with the levels of $\mathcal{T}_k$ are all the same.

We describe a sequence of collections of sets $\mathcal{Q}_0$, $\langle B_n : n \in \mathbb{N}\rangle$ and $\langle E_n : n \in \mathbb{N}\rangle$ such that $\mathcal{Q}_0 \cup \{B_n, E_n\}$ is a partition of G_n.[16]

Let $\langle \mathcal{T}_i : i < s_0\rangle$ be the collection of towers for τ_0. Because we assume that $q_0 = 1$, each of the towers of τ_0 has height 1. Let $\mathcal{Q}_0$ be the partition of G_0 into s_0 pieces consisting of the sets in these towers of τ_0 intersected with G_0. Let $B_0 = E_0 = \emptyset$.

We inductively define the B_n and E_n so that $B_{n+1} \supseteq B_n$ and $E_{n+1} \supseteq E_n$. Suppose that B_n, E_n are defined, and we want to define B_{n+1}, E_{n+1}. By assumption, each tower of τ_{n+1} restricted to G_{n+1} consists of

(1) contiguous sequences of length q_n consisting of portions of consecutive levels of the towers of τ_n, interspersed with
(2) new levels of the towers of τ_{n+1} intersected with G_{n+1}.

The interspersed levels are first divided into maximal contiguous portions. We now arbitrarily divide the levels of each of these portions into two contiguous subcollections of levels.[17] One of these subcollections comes before the other in the natural ordering of the tower by the cyclic permutation τ_{n+1}. We view the subcollection that comes first as **ending** a block of levels coming from τ_n and the second as **beginning** the next block of levels. The set B_{n+1} consists of the union of B_n with the unions of the points in these second contiguous subcollections of levels and E_{n+1} as E_n together with the first contiguous subcollection of levels (see Figure 4).

The partition $\mathcal{Q} =_{def} \mathcal{Q}_0 \cup \{\bigcup B_n, \bigcup E_n\}$ will generate the transformations we eventually construct, but may not do so in general. We now describe a symbolic representation of the factor of T determined by $\mathcal{Q}$.

Suppose that the number of towers in the partition corresponding to τ_0 is s_0. Let Σ be an alphabet of cardinality s_0. We view Σ as indexing the partition $\mathcal{Q}_0$; e.g. if $\mathcal{Q}_0 = \{A_i : i \in I\}$ then $\Sigma = \{a_i : i \in I\}$. The alphabet of our symbolic shift will be $\{b, e\} \cup \Sigma$, where b and e are symbols not in Σ. Proceeding as in Section 3, we define a factor map $\pi : X \to (\Sigma \cup \{b, e\})^{\mathbb{Z}}$ by letting:

(1) $\phi(x)(m) = a_i$ iff $T^m(x) \in A_i$,
(2) $\phi(x)(m) = b$ iff $T^m(x) \in \bigcup_n B_n$, and
(3) $\phi(x)(m) = e$ iff $T^m(x) \in \bigcup_n E_n$.

The symbolic representation is the resulting system $((\Sigma \cup \{b, e\})^{\mathbb{Z}}, \mathcal{C}, \phi^*\mu, sh)$.

We now examine the construction of the approximations and the partition $\mathcal{Q}_0 \cup \{\bigcup B_n, \bigcup E_n\}$ to get a clear description of the support of the measure $\phi^*\mu$.

[16] The use of B's and E's correspond to the mnemonic **beginning** and **end**.

[17] If a tower $\mathcal{T}$ of τ_{n+1} both begins and ends with levels not in a tower of τ_n, then we include the new top portion with the new bottom portion and view it as one contiguous block when we do this division. This is consistent with our view of the τ_n as periodic.

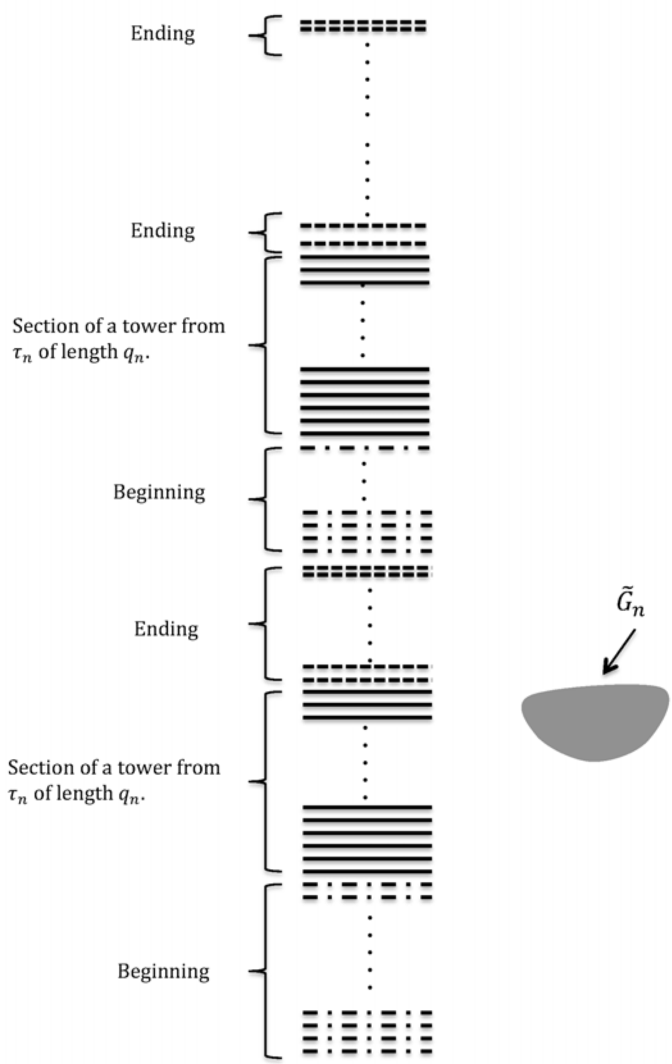


Figure 4. A typical tower in τ_{n+1} built from sections of towers in τ_n filled in with beginning and ending levels.

We inductively define a sequence of collections of words $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ with the following properties:

(1) There is a surjection ϕ_n from $\{\mathcal{T} \cap G_n : \mathcal{T}$ is a tower for τ_n and $G_n \cap \mathcal{T}$ is non-empty$\}$ to $\mathcal{W}_n$.

(2) The length of each word in $\mathcal{W}_n$ is q_n.

Let $\mathcal{W}_0 = \Sigma$. Suppose that we have defined $\mathcal{W}_n$ and ϕ_n. We want to define $\mathcal{W}_{n+1}$ and ϕ_{n+1}. The levels of each $\mathcal{S} \cap G_n$, where $\mathcal{S}$ is a tower of τ_{n+1}, come in blocks of three kinds:

(1) a contiguous block of length q_n coming from a tower of τ_n,
(2) a contiguous block of levels in $B_{n+1} \setminus B_n$, and
(3) a contiguous block of levels in $E_{n+1} \setminus E_n$.

The word in $\mathcal{W}_{n+1}$ we associate with $\mathcal{S} \cap G_n$ via ϕ_{n+1} is the word of length q_{n+1} whose j^{th} letter is v if

(1) the j^{th} level is in the k^{th} place of a block coming from a tower $\mathcal{T}$ of τ_n and the k^{th} letter of $\phi_n(\mathcal{T})$ is v, or
(2) $v = b$ and the j^{th} level is in $B_{n+1} \setminus B_n$, or
(3) $v = e$ and the j^{th} level is in $E_{n+1} \setminus E_n$.

Working as in Section 3.3, we let $\mathbb{K}$ be the collection of $x \in \Sigma^{\mathbb{Z}}$ such that every finite contiguous subword of x occurs inside some $w \in \mathcal{W}_n$. Then $\mathbb{K}$ is a closed shift-invariant set that constitutes the support of $\phi^*\mu$.

We will use this technique to represent the smooth transformations we construct. It will follow from the "Requirements 1–3" that in our representation we can choose the partition $\mathcal{Q}$ so that it generates the transformation T and the system $\mathbb{K}$ will satisfy Lemma 11. In particular, the unique non-atomic measure will be $\phi^*\mu$ and we will have a symbolic representation of our transformation T.

7.2 The dynamical and geometric orderings. Fix a rational $\alpha = p/q$ in reduced form. We endow the partition $\mathcal{I}_q$ of $[0, 1)$ with two orderings. The first is straightforward: ordering these intervals from left to right according to their left endpoint gives us the **geometric** ordering; in other words, $I_i^q < I_j^q$ iff $i < j$.

The rotation by α gives us a different ordering which we call the **dynamical** ordering, $<_d$, which we now define explicitly:

We set an interval $I_i^q <_d I_j^q$ iff there are $n < m < q$ such that $np \equiv i \mod q$ and $mp \equiv j \mod q$.

This can be rephrased conceptually as follows. The first interval in the ordering is I_0^q. Repeatedly applying $\mathcal{R}_\alpha$, we get a sequence of intervals

$$I_0, \mathcal{R}_\alpha I_0, \mathcal{R}_\alpha^2 I_0, \ldots, \mathcal{R}_\alpha^{q-1} I_0.$$

The list

$$\langle I_0, \mathcal{R}_\alpha I_0, \mathcal{R}_\alpha^2 I_0, \ldots, \mathcal{R}_\alpha^{q-1} I_0 \rangle$$

gives the dynamical ordering of $\mathcal{I}_q$.

Remark 40. If we let $j_i \equiv (p)^{-1} i \mod q$ (with $0 \le j_i < q$), then the i^{th} interval in the geometric ordering, I_i^q, is the j_i^{th} interval in the dynamical ordering.

7.3 Transects.

Wikipedia Entry: *A transect is a path along which one records and counts occurrences of the phenomena of study.*[18]

We now see how the periodic processes τ_n and τ_{n+1} compare.

7.3.1 Without the partitions.

Let $\alpha = p/q$ and $\beta = p'/q'$, where we assume $\beta = \alpha + \frac{1}{klq^2}$.[19] We compare how the dynamical ordering determined by β interacts with the dynamical ordering determined by α. In the discussion that follows we use j_i to denote $(p)^{-1}i \mod q$, i.e. we refer to the dynamical ordering of $\mathcal{I}_q$ with respect to α.

If $J = [t'/q', (t'+1)/q')$ is a subinterval of $I = [t/q, (t+1)/q)$, then $\mathcal{R}_\beta J$ is a subinterval of $\mathcal{R}_\alpha I$ unless J is the geometrically last interval in the subdivision of I into intervals of length q'. In the latter case $\mathcal{R}_\beta J$ is the first subinterval of the geometric successor of $\mathcal{R}_\alpha I$. (See Figure 5.)

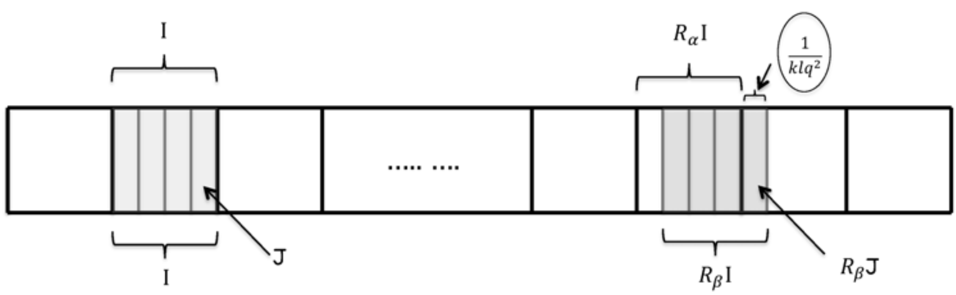


Figure 5. A diagram of $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ acting on the first coordinate of $[0, 1] \times [0, 1]$, showing where $\mathcal{R}_\alpha$, $\mathcal{R}_\beta$ send $I \times [0, 1]$ and $J \times [0, 1]$.

Restating this, if $\mathcal{R}_\alpha I$ is the j_i^{th} interval in the dynamical ordering, and J is the geometrically last subinterval of I, then $\mathcal{R}_\beta J$ is the geometrically first subinterval of the j_{i+1}^{st} interval in the dynamical ordering of $\mathcal{I}_q$.

If $J = [0, 1/q')$ is the geometrically first subinterval of $I = I_0^q$, then for $0 \le n < klq$, $\mathcal{R}_\beta^n J \subseteq \mathcal{R}_\alpha^n I_0^q$ and $\mathcal{R}_\beta^{klq} J$ is the geometrically first subinterval of I_1^q. Since $\mathcal{R}_\alpha^{mq} I = I$ for all m, we have $\mathcal{R}_\beta^{mq} J$ is a subinterval of I for $0 \le m < kl$.

In general, the n's between $mklq$ and $(m+1)klq$ can be split into three pieces:

– the **beginning** interval $[mklq, mklq + q - j_m)$ that has length $q - j_m$,

[18] Retrieved August 15, 2011.
[19] Thus $q' = klq^2$.

- the **middle** interval $[mklq+q-j_m, mklq+q-j_m+(kl-1)q)$ that has length $(kl-1)q$, and
- the **end** interval $[mklq+q-j_m+(kl-1)q, (m+1)klq)$ that has length j_m.

Assume inductively that $\mathcal{R}_\beta^{mklq}J$ is the geometrically first subinterval of I_m^q. Then for $mklq \le n < mklq+q-j_m$, $\mathcal{R}_\beta^n J$ transits the intervals in places $j_m, j_m+1, \dots, q-1$ in the dynamical ordering of $\mathcal{I}_q$. The next application of $\mathcal{R}_\beta$ puts the orbit in the first interval of $\mathcal{I}_q$ in the dynamical ordering, which coincides with the first interval in the geometric ordering.

For the middle range (those n for which $mkl+q-j_m \le n < mkl+q-j_m+(kl-1)q$), Starting as a subset of I_0^q, $\mathcal{R}_\beta^n J$ transits the intervals $\mathcal{I}_q$ following the dynamical ordering coming from α.

In the end range (those n for which $mkln+q-j_m+(kl-1)q \le n < (m+1)klq$) the $\mathcal{R}_\beta^n J$ transits the intervals in places 0 up to j_m-1 in the dynamical ordering on the I_i^q's, ending up in the geometrically last subinterval of I_m^q. Finally, we have that $\mathcal{R}_\beta^{(m+1)klq} J$ is the geometrically first subinterval of I_{m+1}^q.

We illustrate this in Figure 6 where we have three vertical columns, each representing all of the intervals of $\mathcal{I}_q$ in their $\mathcal{R}_\alpha$-dynamical ordering. The transformation $\mathcal{R}_\alpha$ moves the larger rectangles up the columns, and $\mathcal{R}_\beta$ moves the flatter rectangles. The left-hand column shows the beginning $\mathcal{I}_{q'}$ intervals (starting with the first $1/q'$-interval for simplicity), the middle column shows one pass (of many) through the middle section, and the last column shows the end portion. Note that the end portion ends in the rectangle just below where the beginning portion started. The black rectangle indicates the beginning of the next sequence after jumping over some of the $\mathcal{I}_q$ rectangles in the geometric ordering.

7.3.2 With the partitions of $[0, 1)$**.** We continue our examination of the path of $J = [0, 1/q')$ through the unit interval by considering it in light of the partition $\mathcal{I}_{kq}$. In the discussion in this section we denote I_i^q simply as I_i.

If $J' = J_j^{q'}$ is a subinterval of I_i an application of $\mathcal{R}_\beta^{lq}$ moves J' to the right, $\frac{lq}{klq^2} = \frac{1}{kq}$. In particular, it moves it from being a subinterval of an element of $\mathcal{I}_{kq}$ to the subinterval of the element of $\mathcal{I}_{kq}$ adjacent on the right.

If J' is the leftmost subinterval of I_i, then an application of $\mathcal{R}_\beta^{mlq}$ moves J' over m elements of $\mathcal{I}_{kq}$. For $0 \le m < k$, $\mathcal{R}_\beta^{mlq}$ keeps J' a subinterval of I_i, but $\mathcal{R}_\beta^{klq} J'$ is the leftmost subinterval of I_{i+1}.

We divide the atoms of $\mathcal{I}_{kq}$ into k ordered sets $w_0, \dots, w_{k-1}$ where

$$w_j = \langle I_{j+tk}^{kq} : 0 \le t < q \rangle. \tag{15}$$

Thus each w_j is the orbit under $\mathcal{R}_\alpha$ of $[j/kq, (j+1)/kq)$. We can view w_j as a

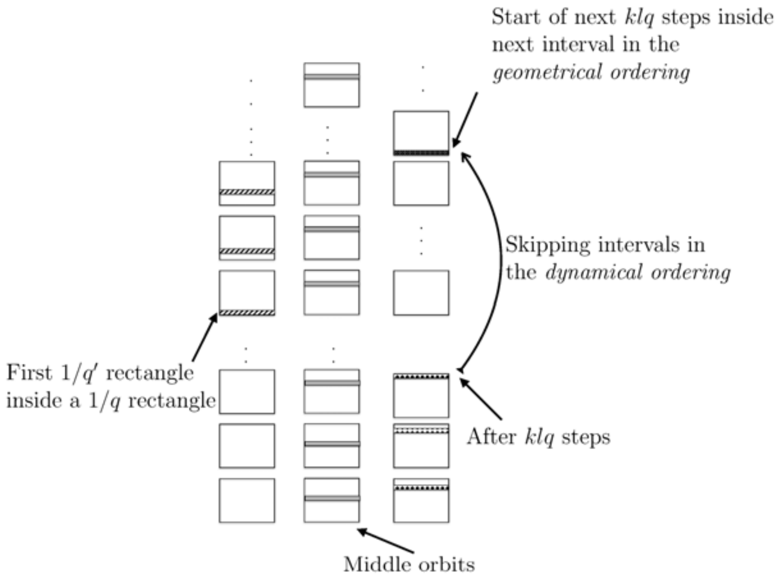


Figure 6. **With vertical orientation**: Each column is the whole $1/q$ partition in dynamical ordering. The first shows the beginning trajectory of the $1/q'$ partition, the second shows one pass of many through the middle section, and the last column is the ending portion of the $1/q'$ trajectory. After the last step the end portion will jump some $1/q$ pieces in the dynamical ordering to begin a new trajectory.

word of length q in the alphabet $\mathcal{I}_{kq}$.

We now follow our original interval J through the w_j under the iterates of $\mathcal{R}_\beta$. The $\mathcal{R}_\beta$-orbit of J has length q'. The first klq iterates are straightforward. Any number n less than klq can be written in the form $mlq + sq + t$, where $0 \leq m < k$, $0 \leq s < l$ and $0 \leq t < q$. By the remarks in the previous paragraph, $\mathcal{R}_\beta^n J$ is a subinterval of the t^{th} element w_m. We could then write the $\mathcal{I}_{kq}$-name of any point in J in the first klq iterates as

$$w_0^l w_1^l w_2^l \cdots w_{k-1}^l. \tag{16}$$

Applying $\mathcal{R}_\beta^{klq}$ to J makes it the geometrically first subinterval of I_1. The above pattern would repeat itself with respect to the partition $\mathcal{I}_{kq}$, were it not for the fact that the w_j's start in I_0, and hence $\mathcal{R}_\beta^{klq} J$ is a subinterval of the j_1^{st} element of w_0.

Thus we must use $q-j_1$ applications of $\mathcal{R}_\beta$ to bring $\mathcal{R}_\beta^{klq}J$ back to I_0. Then $(l-1)q$ more applications carry $\mathcal{R}_\beta^{klq+(q-j_1)}J$ through $l-1$ copies of w_0, and j_1 additional applications bring it back to I_1 as a subinterval of an interval in the middle of w_1.

In summary, using $q-j_1$ iterates we are back in I_0, but in the first interval in w_1. Using $(l-1)q$ more iterates carries us through $l-1$ copies of w_1, and a j_1 more puts us into the j_1^{st} element of w_2 and so on. Hence we can write the $\mathcal{I}_{kq}$ name of any point in J in iterates $klq \le n < 2klq$ in the form

$$b_0^{q-j_1}w_0^{(l-1)}e_0^{j_1}b_1^{q-j_1}w_1^{l-1}e_1^{j_1}b^{q-j_1}w_2^{l-1}e_2^{j_1}\cdots w_{k-1}^{l-1}e_{k-1}^{j_1} \tag{17}$$

where $b_j^{q-j_1}$ and $e_j^{j_1}$ are the last $q-j_1$ elements of w_j and the first j_1-elements of w_j.

The interval $\mathcal{R}_\beta^{2klq}J$ is the leftmost subinterval of I_2 of length $1/q'$. This is a subinterval of the j_2^{nd} element of the word w_0. Here the pattern repeats itself, but with j_2 playing the role of j_1 and so on.

Inductively one shows that the $\mathcal{I}_{kq}$-names of any element of J have the following form between interates $mklq$ and $(m+1)klq$ with $1 \le m < q$:

$$b_0^{q-j_m}w_0^{l-1}e_0^{j_m}b_1^{q-j_m}w_1^{l-1}e_1^{j_m}b^{q-j_m}w_2^{l-1}e_2^{j_m}\cdots w_{k-1}^{l-1}e_{k-1}^{j_m}. \tag{18}$$

Since $j_0 = 0$, we can make (16) a special case of (18) by replacing the first copy of each w_i by $b_i^{q-j_0} = b_i^q$. Using Remark 6, we see that we get an isomorphic symbolic system.

Summarizing, any point in J can have its $\mathcal{I}_{kq}$-name written in the form

$$w = \prod_{i=0}^{q-1}\prod_{j=0}^{k-1}(b_j^{q-j_i}w_j^{l-1}e_j^{j_i}) \tag{19}$$

where the b's and e's have the definition given above.

The letters occurring in particular locations of w are in a specific one-to-one correspondence with intervals in the partition $\mathcal{I}_{q'}$. So, for example, the second b in the third string of the form b^{q-j_3} uniquely labels an interval in $\mathcal{I}_{q'}$ and so on. We will make much use of the correspondence between the letters of w and the corresponding intervals.

Another observation is that if we omit the subscripts on the b's and e's, we can still decode this correspondence using the arithmetical properties of their exponents. Omitting the subscripts we see that the word w in (19) takes the form $\mathcal{C}(w_0, \ldots, w_{k-1})$.

Remark 41. By Lemma 19, for all but the $3/l$ portion of $x \in [0,1)$, the $[-q,q]$-name of x with respect to $\mathcal{I}_{kq}$ under $\mathcal{R}_\beta$ is the same as the $[-q,q]$-name of x with respect to $\mathcal{I}_{kq}$ under $\mathcal{R}_\alpha$.

7.3.3 What this means for the τ_n. The partition ξ_n slices $\mathcal{A}$ into s_n equal height horizontal strips and into q_n vertical strips over the atoms of $\mathcal{I}_{q_n}$. We examine the analysis in the previous section taking $\alpha = \alpha_n$ and $\beta = \alpha_{n+1}$ (so $p = p_n, q = q_n, p' = p_{n+1}$ and $q' = q_{n+1}$).

The action of $\overline{\mathcal{R}}_\alpha$ on $\mathcal{A}$ exactly mimics the action of $\mathcal{R}_\alpha$ on the x-axis. Hence on each horizontal strip of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$ of the form $\mathcal{I}_{k_nq_n} \times [s/s_{n+1}, (s+1)/s_{n+1})$ we get the same analysis of the comparisons of $\overline{\mathcal{R}}_{\alpha_{n+1}}$ and $\overline{\mathcal{R}}_{\alpha_n}$, as we did in comparing $\mathcal{R}_{\alpha_n}$ and $\mathcal{R}_{\alpha_{n+1}}$.

The labeling of $\mathcal{I}_{q_{n+1}}$ into words in equation (19) can be copied over to $\mathcal{I}_{q_{n+1}} \times [s/s_{n+1}, (s+1)/s_{n+1})$ and reflects the names of the partition $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$ with respect to the action of $\overline{\mathcal{R}}_{\alpha_{n+1}}$. Doing this labels some of the atoms of $\mathcal{I}_{q_{n+1}} \otimes \mathcal{I}_{s_{n+1}}$ with some b's and e's, corresponding to the boundaries of the words of the type that occur in (19). Explicitly, if a is an atom of $\mathcal{I}_{q_{n+1}}$ labeled with a b (respectively e) in (19), then all of the atoms in $\{a\} \times \mathcal{I}_{s_{n+1}}$ are labeled with a b (respectively e).

Definition 42. Let $B_0 = E_0 = \emptyset$ and let B_{n+1} (respectively E_{n+1}) be the union of B_n with the set of $x \in \mathcal{A}$ that occurs in an atom of $\mathcal{I}_{q_{n+1}} \otimes \mathcal{I}_{s_{n+1}}$ labeled with a b (respectively an e). Let $B'_{n+1} = B_{n+1} \setminus B_n$ and $E'_{n+1} = E_{n+1} \setminus E_n$.

For $n > 0$, the measure of $B'_{n+1} \cup E'_{n+1}$ is $1/l_n$. Moreover, the collection of $x \in \mathcal{A}$, whose $[-q, q]$-name with respect to the partition $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$ under $\overline{\mathcal{R}}_{\alpha_n}$ is the same as its $[-q, q]$-name with under the action of $\overline{\mathcal{R}}_{\alpha_{n+1}}$, has measure at least $1 - 3/l_n$.

We now define

$$\Gamma_n = \{x \in X : \text{for all } m > n, x \text{ does not occur in } Z_m(B'_m \cup E'_m)\}. \tag{20}$$

Then $\Gamma_{n+1} \supseteq \Gamma_n$. Since $\sum 1/l_n < \infty$, the Borel–Cantelli Lemma implies that for almost every $x \in \mathcal{A}$ there is an m for all $n > m, x \in \Gamma_n$.

We note that the sets B_n, E_n and Γ_n correspond to the B_n, E_n and G_n in Section 7.1. The B'_n, E'_n are those x that are first labeled b or e at stage n.

7.4 The factor $\mathcal{K}$ is a rotation of the circle. As a warm-up for the symbolic representation of Anosov–Katok diffeomorphisms of surfaces, we are in a position to give a one-dimensional representation of the circular system $\mathcal{K}$ given in Definition 21. Let $\langle k_n, l_n : n \in \mathbb{N}\rangle$ be a sequence of numbers such that $k_n \geq 2$ and $\sum 1/l_n < \infty$. Let p_n and q_n be defined as in (5), $\alpha_n = p_n/q_n$. Then the sequence of α_n converges to an irrational α.

Theorem 43. *Let ν be the unique non-atomic shift-invariant measure on $\mathcal{K}$. Then*

$$(\mathcal{K}, \mathcal{B}, \nu, sh) \cong (S^1, \mathcal{D}, \lambda, \mathcal{R}_\alpha),$$

where $\mathcal{R}_\alpha$ is the rotation of the circle S^1 and λ is Lebesgue measure.

In the forthcoming [13] we give a completely different (short) algebraic proof of this result. We give a geometric proof here because it gives more information that is used in [14].

Proof. Recall that the construction sequence for $\mathcal{K}$ consists of $\mathcal{W}_n = \{w_n\}$, where $w_0 = *$ and $w_{n+1} = \mathcal{C}(w_n, \ldots, w_n)$.

Let $X = [0, 1)$ and take $\mu = \lambda$. Define a sequence of uniform periodic processes that converge to $\mathcal{R}_\alpha$, by taking the n^{th} periodic process σ_n to be the cyclic permutation of $\mathcal{I}_{q_n}$ given by the dynamical ordering. The base of σ_n is $J_n = I_0^{q_n}$ and the levels are

$$\langle I_0, \mathcal{R}_{\alpha_n} I_0, \mathcal{R}^2_{\alpha_n} I_0, \ldots, \mathcal{R}^{q-1}_{\alpha_n} I_0\rangle.$$

The periodic approximation σ_n can be realized pointwise by the transformation $\mathcal{R}_{\alpha_n}$. Since the $\mathcal{R}_{\alpha_n}$'s converge to $\mathcal{R}_\alpha$ and $\sum q_n/q_{n+1} < \infty$, Lemma 28 shows that the σ_n's converge (as periodic processes) to $\mathcal{R}_\alpha$.

We now follow the method described in Section 7.1. Let $\langle G_n : n \in \mathbb{N}\rangle$ be as defined there. Let $\mathcal{Q}_0$ be the trivial partition of G_0. Labeling elements of G_0 with letter "$*$", we inductively define $\langle B_n, E_n : n \in \mathbb{N}\rangle$ and show that if $x \in G_n$ is in the bottom level of σ_n, then the $\{*, b, e\}$-name of x is w_n.

We start with σ_0, the trivial action on the tower that has one level, the partition $\mathcal{I}_1$. The next periodic process σ_1 is a single cycle of length q_1. We take the first level of σ_1 to consist of a subset of B_1 followed by $l_0 - 1$ levels which we view as levels of σ_0 concatenated without spacers, followed by a single level contained in B_1, followed by $l_0 - 1$ levels without spacers, followed by a single level contained in B_1 and so on. There are $q_1 = k_0 l_0$ many levels total, of these k_0 are equally spaced subsets of B_1. The set $E_1 = \emptyset$.

If we label the levels that are not subsets of B_1 with $*$'s and the levels of B_1 with b's, we get a string that starts with a b, is followed by $l_0 - 1$ $*$'s, followed by a b and so on, k_1 many times. Keeping in mind t hat $p_0 = 0$ and $q_0 = 1$ (so every $j_i = 0$) this has the form

$$\prod_{i<q_0} \prod_{j<k_1} b^{q_0 - j_i} *^{l_0 - 1} e^{j_i},$$

as desired.

We now describe the induction step. Passing from σ_n to σ_{n+1} we assume that the words corresponding to elements of the bottom level of σ_n are given by w_n. Following the analysis of Section 7.3.2, we can partition the levels of σ_{n+1} into contiguous segments that have names $w_n^{l_n-1}$ interspersed with b's and e's yielding the word $\mathcal{C}(w_n, w_n, \dots, w_n)$. We let B_n be the levels newly labelled with b's and E_n the levels newly labeled with e's. By Lemma 19, the measure of $B_n \cup E_n$ is $1/l$.

Let $S \subseteq \mathcal{K}$ be as in Definition 10. By Lemma 20, $\nu(S) = 1$ and for all $s \in S$, there is an N for all $n \geq N$ with $a_n, b_n \geq 0$ such that $s \restriction [-a_n, b_n) = w_n$.[20] For such an s and n, let $r_n(s) = a_n$. We interpret $r_n(s)$ as the position of s's "0" in w_n.

Supposing that $r_n(s)$ exists, we define

$$\rho_n(s) = \frac{i}{q_n} \tag{21}$$

iff $I_i^{q_n}$ is the $r_n(s)^{\text{th}}$ interval in the dynamical ordering of $\mathcal{I}_{q_n}$. (This is equivalent to $i = j_{r_n(s)}$.)[21] Equivalently, since the r_n^{th} interval in the geometric ordering is $I^{q_n}_{p_n r_n(s)}$,

$$i \equiv p_n r_n(s) \mod q_n.$$

Thus $\rho_n(s)$ is the left endpoint of the $r_n(s)^{\text{th}}$ interval in the periodic process σ_n.

Because the $r_{n+1}(s)^{\text{th}}$ letter in w_{n+1} is in the $r_n(s)^{\text{th}}$ position in a copy of w_n, we see that the $r_{n+1}(s)^{\text{th}}$ interval in the dynamical ordering of $\mathcal{I}_{q_{n+1}}$ is a subinterval of the $r_n(s)^{\text{th}}$ interval in the dynamical ordering of $\mathcal{I}_{q_n}$. It follows that

$$\rho_{n+1}(s) \geq \rho_n(s)$$

and that

$$|\rho_{n+1}(s) - \rho_n(s)| < 1/q_n.$$

Since $\sum_n 1/q_n < \infty$, the sequence $\langle \rho_n(s) : n \in \mathbb{N} \rangle$ is Cauchy. We define

$$\phi_0(s) = \lim_n \rho_n(s).$$

It is easy to check that $\phi_0(sh(s)) = \mathcal{R}_\alpha(\phi_0(s))$, and hence by the unique ergodicity of the measure ν on S

$$(\mathcal{K}, \mathcal{C}, \nu, sh) \cong ([0,1), \mathcal{B}, \lambda, \mathcal{R}_\alpha).$$

This finishes the proof. □

[20] The s's for which a_n and b_n do not exist are those s for which $s(0)$ is in the boundary portion of w_m for some $m \geq n$.

[21] Thus r_n and ρ_n both have the same subset of S as their domain and contain the same information. They map to different places $r_n : S \to \mathbb{N}$, whereas $\rho_n : S \to [0,1)$ and is the left endpoint of the r_n^{th} interval in the dynamical ordering.

The following is immediate from the proof of Theorem 43

Proposition 44. *For $x \in [0, 1)$, let $D_n(x) = j$ if x belongs to the j^{th} interval in the dynamical ordering of $\mathcal{I}_{q_n}$ (or equivalently, $D_n(x) = j$ if $x \in I^{q_n}_{jp_n}$). Then for all $s \in S$ and all large enough n,*

$$r_n(s) = D_n(\phi_0(s)).$$

7.5 A symbolic representation of the abstract Anosov–Katok systems. We now give a symbolic representation of the transformations built by the version of the Anosov–Katok technique as described in Section 6. Our symbolic representation will consist of the names of points in $\mathcal{A}$ with respect to a generating partition $\mathcal{Q}$ that we build in Section 7.1, with the addition of a systematic method of assigning b's and e's. To find the names we compute them with respect to the periodic processes τ_n and show that for every k and almost every point x, the $[-k, k]$-name of x with respect to $\mathcal{Q}$ and τ_n stabilizes for large n.

If $\mathcal{Q}$ is a partition that is refined by the levels of the towers of a periodic process τ, then the $\mathcal{Q}$-names of any pointwise realization of τ are constant on the levels of the tower. Hence we can view these names as naming the levels themselves in the periodic orbits of the action of τ on various towers. We call the resulting collection of names the $(\tau, \mathcal{Q})$-names.

We begin our exercise by fixing an arbitrary partition $\mathcal{Q}^*$ of X that is refined by the partition ζ_n and comparing the $\mathcal{Q}^*$-names of points under τ_n and τ_{n+1}.

Remark 45. For each n we take the base of the s^{th}-tower in the periodic process τ_n to be $Z_n(R^n_{0,s})$. It will follow that the word giving the $\mathcal{Q}^*$-names for the s^{th}-tower is the same as the word consisting of the $(\overline{\mathcal{R}}_{\alpha_n}, Z_n^{-1}(Q^*))$-names of the tower based at $R^n_{0,s}$ (which is the first atom of ξ_n lying on the s^{th} horizontal strip of ξ_n).

To compute the $(\tau_{n+1}, \mathcal{Q}^*)$-names, we copy $\mathcal{Q}^*$ to $\mathcal{A}$ via Z_n^{-1} to get a partition $\mathcal{P} =_{def} Z_n^{-1}\mathcal{Q}^*$. Because

$$\tau_{n+1} = Z_n(h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h_{n+1}^{-1})Z_n^{-1},$$

this reduces the problem of finding $(\tau_{n+1}, \mathcal{Q}^*)$-names to that of computing the $(h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h_{n+1}^{-1}, \mathcal{P})$-names of the towers whose levels constitute the partition $Z_n^{-1}\zeta_{n+1}$. Since $Z_n^{-1}\zeta_n = \xi_n$ and h_{n+1} permutes the atoms of ξ_{n+1}, we see that $Z_n^{-1}\zeta_{n+1} = \xi_{n+1}$.

How each rectangle moves. For notational simplicity, let $k = k_n, q = q_n$, $I_i = I_i^{q_n}$ and $J_j = I_j^{q_{n+1}}$.[22] Fix a rectangle R in ξ_{n+1}. We have two cases. The first case is that $h_{n+1}^{-1}R = R_{i,j}^{n+1}$, where J_i is not the geometrically last $1/q_{n+1}$ subinterval of an interval in $\mathcal{I}_{kq}$. (See Figure 7.)

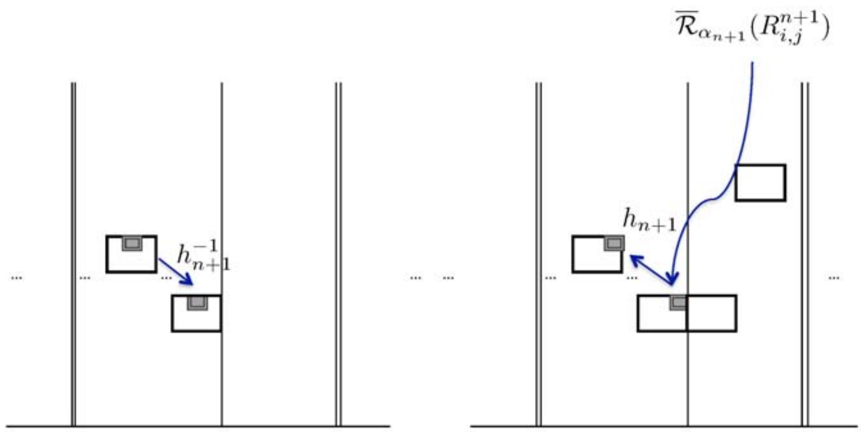


Figure 7. Case 1.

In this case know that $\overline{\mathcal{R}}_{\alpha_{n+1}}$ and $\overline{\mathcal{R}}_{\alpha_n}$ send $R_{i,j}^{n+1}$ to a rectangle whose base is a subinterval of the same element of $\mathcal{I}_{kq}$. Since h_{n+1} commutes with $\overline{\mathcal{R}}_{\alpha_n}$ and permutes the atoms of the partition $\mathcal{I}_{k_n q_n} \otimes \mathcal{I}_{s_{n+1}}$, we see that $h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h_{n+1}^{-1}R$ is a subrectangle of the same member of $\mathcal{I}_{kq} \otimes \mathcal{I}_{s_{n+1}}$ as $\overline{\mathcal{R}}_{\alpha_n}R$ is. In particular, the $\mathcal{P}$-name of $h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h_{n+1}^{-1}R$ is the same as the $\mathcal{P}$-name of $\overline{\mathcal{R}}_{\alpha_n}R$.

The second case is when J_i is the geometrically last $1/q_{n+1}$ subinterval of an interval in $\mathcal{I}_{kq}$; then $\overline{\mathcal{R}}_{\alpha_{n+1}}$ sends $R_{i,j}^{n+1}$ to the geometrically first subrectangle of a new element R' of $\mathcal{I}_{kq} \otimes \mathcal{I}_{s_{n+1}}$. Thus $h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h_{n+1}^{-1}R$ is a subset of $h_{n+1}(R')$. (See Figure 8.)

How the tower moves. Since the bases of the towers for τ_{n+1} are the sets $Z_{n+1}R_{0,s}^{n+1}$, the base for the s^{th} tower for $h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h_{n+1}^{-1}$ is of the form

$$Z_n^{-1}Z_{n+1}R_{0,s}^{n+1} = h_{n+1}R_{0,s}^{n+1}.$$

Computing:

$$(h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h_{n+1}^{-1})^t(h_{n+1}R_{0,s}^{n+1}) = h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}^t R_{0,s}^{n+1}.$$

Thus if $F_0, \ldots, F_{q_{n+1}-1}$ are the levels of the s^{th} tower for $h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h_{n+1}^{-1}$, we see that $h_{n+1}^{-1}F_t = R_{i_t,s}^{n+1}$, where $i_0 = 0$ and the sequence of intervals $\langle J_{i_t} : t < q_{n+1}\rangle$ is the orbit of J_0 under $\mathcal{R}_{\alpha_{n+1}}$.

[22] We note that the behavior of J_i's with respect to the partition $\mathcal{I}_{k_n q_n}$ is that of the transects we discussed in Section 7.3.2.

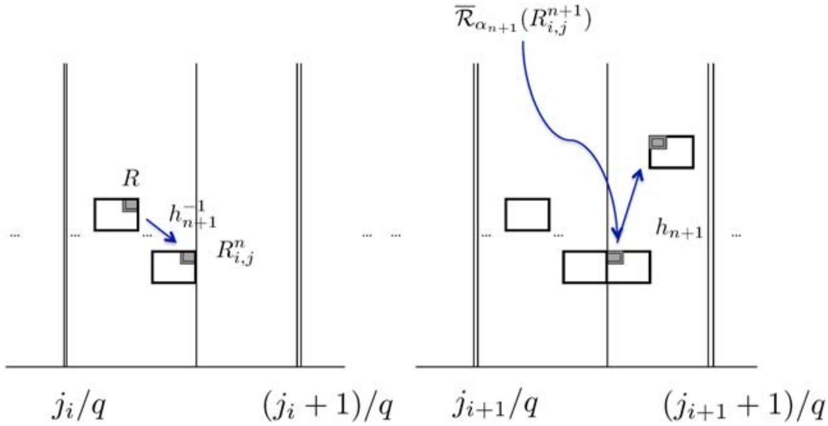


Figure 8. Case 2.

In Section 7.3.2, we labeled the intervals $\langle J_{i_t} : t < q_{n+1}\rangle$ with w's and b's and e's. From our discussion of how each rectangle moves we observe that for those t where J_{i_t} is labeled with a part of a w, the two transformations $h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h^{-1}_{n+1}$ and $\overline{\mathcal{R}}_{\alpha_n}$ move F_t to subrectangles of the same element of $\mathcal{I}_{kq}\otimes\mathcal{I}_{s_{n+1}}$ and hence the same element of $\mathcal{P}$.

Now, for $j < k$, $t < q$ and $s < s_{n+1}$, let $R_{j,t,s}$ be the rectangle

$$[(j+tk)/kq, ((j+tk)+1)/kq) \times [s/s_{n+1}, (s+1)/s_{n+1}).$$

This is the product of the t^{th} interval in the w_j of equation (15) and the interval $[s/s_{n+1}, (s+1)/s_{n+1})$. Let $u_{j,s}$ be the sequence of $\mathcal{P}$-names of the intervals

$$\langle h_{n+1}(R_{j,t,s}) : t < q\rangle. \tag{22}$$

Then $u_{j,s}$ is the sequence of names of the h_{n+1}-image of subrectangles of $[0,1)\times[s/s_{n+1}, (s+1)/s_{n+1})$ taken along the transect whose horizontal intervals form the word w_j. Because h_{n+1} commutes with $\overline{\mathcal{R}}_{\alpha_n}$, we note the following:

Remark 46. The word $u_{j,s}$ is the sequence of $\mathcal{P}$-names of the levels of the tower $\langle\overline{\mathcal{R}}^t_{\alpha_n}h_{n+1}(R^{n+1}_{i,s}) : 0 \le t < q_n\rangle$, for any $J_i \subseteq [j/kq, (j+1)/kq)$.

Following our analysis of the transects on $[0,1)$, we can now describe the $\mathcal{P}$-name of the orbit of $F_0 = h_{n+1}R^{n+1}_{0,s_0}$ under $h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h^{-1}_{n+1}$. The letter t refers to the number of applications of $h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h^{-1}_{n+1}$ and thus the level of the tower.

(1) To begin with, there is a segment of t's where the $(h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h^{-1}_{n+1}, \mathcal{P})$-name agrees with the $\overline{\mathcal{R}}_{\alpha_n}$-name $u_{0,s}$. This segment has length lq_n, as the intervals J_{i_t}

move to the right. At stage $t = lq_n$ these intervals cross a boundary for the $\mathcal{I}_{kq}$ partition. At this point the name changes to $u_{1,s}$ repeated l-times. Then $u_{2,s}$ is repeated l times and so on. This occurs k times through the $u_{t,s}$ until $t = k_n l_n q_n - 1$, where the J_{i_t} becomes the geometrically last subinterval of I_{q_n-1}. Then $J_{i_{t+1}}$ is the geometrically first subinterval of $[1/q_n, (1+1)/q_n)$.

(2) We then have a segment where the transect is labeled with $q_n - j_1$ many b's, after which $t = k_n l_n q_n + q_n - j_1$.

(3) If $t = kq_n^l + q_n - j_1$, then J_{i_t} is a subinterval of the geometrically first interval in $\mathcal{I}_{kq}$. In particular, the name of F_t is the first letter of $u_{0,s}$.

(4) At this point the $h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h_{n+1}^{-1}$-names are the same as the $\overline{\mathcal{R}}_{\alpha_n}$-names for $q(l-1)$ iterations yielding the name $u_{0,s}^{l-1}$.

(5) This is followed by a segment of length j_1 where the transect is labeled with e's.

(6) The pattern begins again with a portion of the tower where the transect is labeled with b^{q-j_1}, followed by $u_{1,s}^{l-1}$ followed by e^{j_1}. This is repeated for $u_{2,s}$, $u_{3,s}$ and so forth.

(7) At stage $t = 2k_n l_n q_n - 1$, J_{i_t} is the geometrically last subinterval of I_0. This implies that $J_{i_{t+1}}$ is the geometrically first subinterval of I_2.

(8) Here we get a block of b's of length $q - j_2$ and the pattern described in items (3)–(6) begins again with j_2 replacing j_1.

(9) The pattern described in items (3)–(8) repeat until we get to $t = klq^2 - 1$, at which point we have completed the period of $h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h_{n+1}^{-1}$.

We illustrate this with two diagrams. The levels of both Figures 9 and 10 are the rectangles of the partitions in the dynamical ordering with the $\mathcal{R}_{\alpha_n}$ and $\mathcal{R}_{\alpha_{n+1}}$ moving in the vertical direction. In Figure 9 we show how the ξ_{n+1} transects move through $\mathcal{A}$ in the global fashion at scale $1/k_n q_n$—neglecting the $1/k_n l_n q_n$ portions. The small light rectangles show the initial pass of a ξ_{n+1} atom, with the more darkly shaded rectangles showing a later pass.

In Figure 10 we magnify the first diagram to show the features at the $1/k_n l_n q_n$ scale. This is part of the darker rectangle transect from Figure 9 as it passes through a portion of width $1/k_n q_n$, going from j/kq_n to $(j+1)/kq_n$ in $1/k_n l_n q_n^2$ increments.

We have shown:

Theorem 47. *Let F_0 be the base of a tower $\mathcal{T}$ for $h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}}h_{n+1}^{-1}$. Suppose that $h_{n+1}^{-1}F_0 = R_{0,s}$. Then the $\mathcal{P}$-names of $\mathcal{T}$ agree with*

$$u =_{\text{def}} \prod_{i=0}^{q-1}\prod_{j=0}^{k-1}(b^{q-j_i}u_{j,s}^{l-1}e^{j_i})$$

on the interior of u.

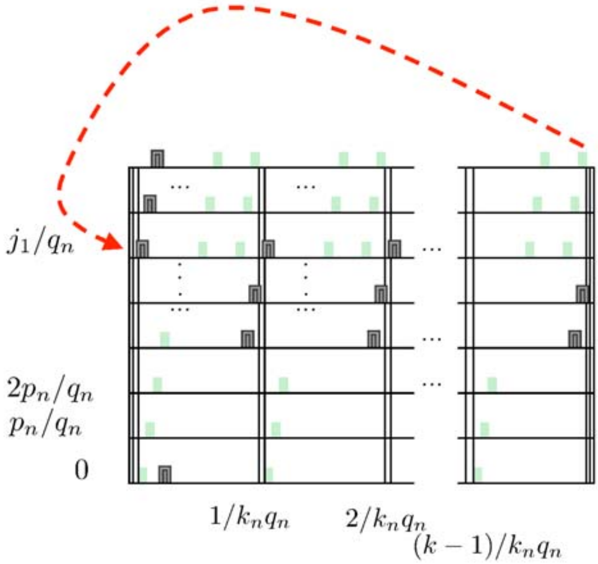


Figure 9. A coarse diagram of the transects.

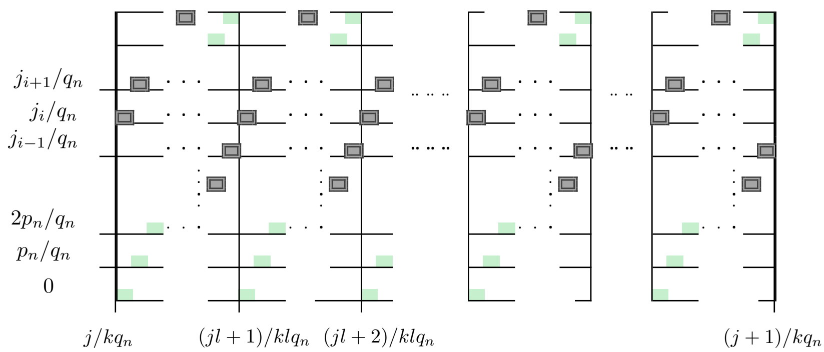


Figure 10. A finer diagram of the transects.

From this we immediately get

Corollary 48. *Suppose that* $J_i \subseteq [j/kq, (j+1)/kq - (1/q_{n+1}))$ *with* $j < kq$ *and* $R = R^n_{i,s}$. *Then the levels* $\langle \overline{\mathcal{R}}^t_{\alpha_n} h_{n+1}(R) : t < q\rangle$ *coincide with the levels* $\langle (h_{n+1}\overline{\mathcal{R}}_{\alpha_{n+1}} h^{-1}_{n+1})^t h_{n+1} R : t < q\rangle$ *in the tower for* τ_n. *In particular, their* $\mathcal{Q}^*$*-names agree.*

Proof. $J_i \subseteq [j/kq, (j+1)/kq - 1/q_{n+1})$ is equivalent to J_i being labeled with the first letter of a w_m in a transect. □

In light of Remark 19, we know:

Corollary 49. *For a set of $x \in X$ having measure at least $1 - 3/l_n$, the $(\tau_{n+1}, \mathcal{Q}^*)$ and $(\tau_n, \mathcal{Q}^*)$ names of x agree on the interval $[-q, q]$.*

We draw attention to the connection with the sets Γ_n from (20). Recall that these are the collections of points that do not get labeled with b's or e's for the first time at some stage $m > n$. With this in mind the following corollary is clear:

Corollary 50. *Suppose that $x \in \Gamma_n$ and x is on level t_n of a τ_n-tower and the level t_{n+1} of a τ_{n+1}-tower. Let w_n be the $\mathcal{Q}^*$-name of x with respect to τ_n and w_{n+1} be the $\mathcal{Q}^*$-name of x with respect to τ_{n+1}. Then $w_{n+1} \restriction [t_{n+1} - t_n, t_{n+1} + q_n - t_n) = w_n$.*

This corollary is saying that for $x \in \Gamma_n$, the $\mathcal{Q}^*$-name has stabilized on the interval of length q_n corresponding to x's position in a τ_n-tower.

The symbolic representation. Let

$$B = \{x \in X : \text{ for some } m \leq n, x \in \Gamma_n \text{ and } Z_m^{-1}x \in B_m\}$$

and

$$E = \{x \in X : \text{ for some } m \leq n, x \in \Gamma_n \text{ and } Z_m^{-1}x \in E_m\},$$

where the B_n's and E_n's are given by Definition 42 and Γ_n is defined in (20).

Let $\{A_i : i < s_0\}$ be the partition $\zeta_0 \restriction (X \setminus B \cup E)$[23] and

$$\mathcal{Q} = \{A_i : i < s_o\} \cup \{B, E\}. \tag{23}$$

We now do the following:

(1) Compute the doubly infinite names of a typical point with respect to $\mathcal{Q}$.
(2) Show that the function sending an $x \in X$ to its $\mathcal{Q}$-name has range in the set S in Definition 10.
(3) Show that " Requirements 1–3" (just before Lemma 34) imply $\mathcal{Q}$ is a generator for the transformation $T = \lim \tau_n$.

The names will be in the alphabet $\Sigma \cup \{b, e\}$, where $\Sigma = \{a_i : i < s_0\}$. Naturally, a point $x \in X$ will get a name $f \in (\Sigma \cup \{b, e\})^{\mathbb{Z}}$ with $f(n)$ being a_i if $T^n x \in A_i$ and $f(n)$ being b or e if $T^n x \in B$ or $T^n x \in E$, respectively.

To give a complete description of the $(T, \mathcal{Q})$-names of points in $\bigcup \Gamma_n$ we proceed by induction on n. If $n = 0$, then $\tau_0 = \text{id}$, and $\Gamma_0 = X \setminus (B \cup E)$. The τ_0-names of $x \in \Gamma_0$ are simply the elements of Σ.

Suppose that F_0 is the base of a tower $\mathcal{T}$ for τ_{n+1} and $Z_{n+1}R_0 = F_0$, where $R_0 = R^{n+1}_{0,s^*}$ for some $s^* < s_{n+1}$. Inductively assume that the $(\tau_n, \mathcal{Q})$-names of the towers with bases $Z_n R^n_{0,s}$ (with $s < s_n$) are $u_0, \ldots, u_{s_n - 1}$.

[23] Explicitly $\zeta_0 \restriction (X \setminus B \cup E)$ is given by the sets $Z[[0, 1) \times [s/s_0, (s + 1)/s_0)] \setminus (B \cup E)$.

Definition 51. Define a sequence of words $w_0, \dots, w_{k_n-1}$ by setting $w_j = u_s$ where

$$h_{n+1}([j/kq_n, (j+1)/kq_n) \times [s^*/s_{n+1}, (s^*+1)/s_{n+1})) \subseteq R^n_{0,s}.$$

We will say that $(w_0, \dots, w_{k_n-1})$ is the sequence of n-words associated with $\mathcal{T}$.

We define a circular system by inductively specifying the sequence $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$. Let $\mathcal{W}_0 = \{a_i : i < s_0\}$.

Suppose that we have defined $\mathcal{W}_n$. Define

$$\mathcal{W}_{n+1} = \{\mathcal{C}_{n+1}(w_0, \dots, w_{k_n-1}) : \\ (w_0, \dots, w_{k_n-1}) \text{ is associated with a tower } \mathcal{T} \text{ in } \tau_{n+1}\}.$$

We will call $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ the construction sequence associated with the Anosov–Katok construction. The following is worth noting.

Proposition 52. *Assume that the Anosov–Katok construction satisfies Requirements 1–3 in Section 6.1. Then $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ is strongly uniform.*

Theorem 53. *Suppose that T is a limit of a sequence of Anosov–Katok periodic processes with l_n growing fast enough. Then almost all $x \in X$ have $\mathcal{Q}$-names in $\mathbb{K}$, the circular system with construction sequence $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$. In particular, there is a measure ν on $\mathbb{K}$ that makes $(\mathbb{K}, \mathcal{B}, \nu, sh)$ isomorphic to the factor of X generated by $\mathcal{Q}$.*

Proof. Let M be a positive integer. Then for almost all x there is an N so large that for all $n > N$, x does not occur in the first or last M levels of any tower in τ_n.

Fix an arbitrary point $x \in \bigcup \Gamma_n$. Fix n_0 with $x \in \Gamma_{n_0}$ and consider $n \geq n_0$. Then x belongs to a level of a tower $\mathcal{T}$ of τ_n. Without loss of generality we can assume that x does not occur in the first or last M levels of $\mathcal{T}$. By Corollary 50, if x is in the t_n^{th} level of a τ_n-tower, then the T-name of x agrees with the τ_n-name of x on the interval $[-t_n, q_n - t_n)$.

Applying Theorem 47, with $\mathcal{P} = Z_n^{-1}\mathcal{Q}$, we see that a tower $\mathcal{T}$ for τ_n gets the name

$$w = \prod_{i=0}^{q_{n-1}-1} \prod_{j=0}^{k_{n-1}-1} (b^{q-j_i} w_j^{l-1} e^{j_i}),$$

where $(w_0, \dots, w_{k_{n-1}-1})$ is the sequence of words associated with $\mathcal{T}$. If x is at level t_n, then the $\mathcal{Q}$-name of x on the interval $[-t_n, q_n - t_n)$ is

$$\prod_{i=0}^{q-1} \prod_{j=0}^{k-1} (b^{q-j_i} w_j^{l-1} e^{j_i}) = \mathcal{C}_n(w_0, \dots, w_{k_{n-1}-1}).$$

Since $M < \min(t_n, q_n - t_n)$, $x \upharpoonright [-M, M]$ is a subword of some word in $\mathcal{W}_n$.

Thus, for a typical x, every finite subinterval of the $(T, \mathcal{Q})$-name for x is a subword of some $\mathcal{W}_{n+1}$. It follows that the factor of $(X, \mathcal{B}, \lambda, T)$ corresponding to the partition $\mathcal{Q}$ is a factor of the uniform circular system we defined from the sequence of $\mathcal{W}_n$'s.

Conversely, by Lemma 11, the uniform circular system $\mathbb{K}$ determined by $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ is characterized as the smallest shift invariant closed set intersecting every basic open set $\langle w \rangle$ in $(\Sigma \cup \{b, e\})^{\mathbb{Z}}$ determined by some $w \in \mathcal{W}_n$. However, each $w \in \mathcal{W}_n$ is represented on $\Gamma_n \cap \mathcal{T}$ for some $\mathcal{T}$, hence each $\langle w \rangle$ has non-empty intersection with the set of words arising from $(T, \mathcal{Q})$-names. □

We will need the following lemma that follows from the proof of Theorem 53.

Lemma 54. *Let $a_0, b_0 \in \mathbb{N}$. Then for almost all x and all large n, $x \in \Gamma_n$ and x does not occur in the first a_0 levels or last b_0 levels of a tower of τ_n. In particular, for almost all $x \in X$ there are $a > a_0$, $b > b_0$ such that the $\mathcal{Q}$-name of x restricted to the interval $[-a, b)$ belongs to $\mathcal{W}_n$.*

From Lemma 54 and Theorem 53 we conclude:

Corollary 55. *For almost all $x \in X$ the $\mathcal{Q}$-name of x is in S. In particular, if ν is the unique non-atomic shift-invariant measure on $\mathbb{K}$, then the factor Y of $(X, \mathcal{B}, \lambda, T)$ generated by $\mathcal{Q}$ is isomorphic to $(\mathbb{K}, \mathcal{B}, sh, \nu)$. In particular, there is a unique non-atomic, shift-invariant measure on Y.*

Generation. To illustrate the potential difficulty, suppose that each tower $\mathcal{T}$ of each τ_n is associated to the same sequence of $n-1$-words; then for every $x, t_1, t_2 \in [0, 1)$, the $(\tau_n, \mathcal{Q})$-names of (x, t_1) and (x, t_2) are the same. This would imply that $\mathcal{Q}$ generates a proper factor of X.

Hence if h_{n+1} does not vary enough on each horizontal strip of the form $[s/s_{n+1}, (s+1)/s_{n+1})$, the partition $\mathcal{Q}$ may not be generated. Fortunately, Requirements 1–3 (stated just before Lemma 34) are sufficient conditions to guarantee generation.

Lemma 56. *Suppose that hypothesis 1–3 (stated before Lemma 34) hold. Then $\mathcal{Q}$-generates the transformation T.*

Proof. Without loss of generality we can take $X = \mathcal{A}$ and Z to be the identity map. Since the $\langle Z_n \xi_n : n \in \mathbb{N} \rangle$ is a decreasing sequence of partitions that generate the measure algebra, and $\mu(\Gamma_n)$ increases to 1, the atoms of $\langle Z_n(\xi_n) \upharpoonright \Gamma_n : n \in \mathbb{N} \rangle$ also σ-generate the measure algebra. Thus it suffices to show that each member of a $Z_n \xi_n \upharpoonright \Gamma_n$ belongs to the smallest translation invariant σ-algebra $\mathcal{B}$ generated by $\{\bigvee_{i=-N}^{N} T^i(\mathcal{Q} \cup \{B, E\}) : N \in \mathbb{N}\}$.

Each $P \in Z_n\xi_n$ is the t^{th} level of some tower $\mathcal{T}$ for τ_n. Let $w \in (\Sigma \cup \{b, e\})^{q_n}$ be the $(\tau_n, \mathcal{Q})$-name of $\mathcal{T}$. Then the j^{th} letter of w determines an

$$S_{i_j} \in (\{B\} \cup \{E\} \cup \{A_i : i < s_0\}).$$

Since the $(\tau_n, \mathcal{Q})$-name of $\mathcal{T}$ is correct on Γ_n,

$$P \cap \Gamma_n \subseteq \bigcap_{0 \le j < q_n} T^{t-j}(S_{i_j}) \cap \Gamma_n.$$

On the other hand, since the map sending towers to names w is one-to-one, we see that

$$P \cap \Gamma_n \supseteq \bigcap_{0 \le j < q_n} T^{t-j}(S_{i_j}) \cap \Gamma_n,$$

which is what we needed to show. □

It remains to show that the hypothesis of Lemma 56 holds.

Lemma 57. *Suppose that our sequence of h_n's satisfy Requirements 1–3 in Section 6.1. Then $\mathcal{Q}$ generates the transformation T.*

Proof. We use Requirement 3 to show inductively that for all $n \ge 1$, if $\mathcal{T}$ and $\mathcal{T}'$ are two τ_n towers then the $\mathcal{Q}$-names associated with $\mathcal{T}$ and $\mathcal{T}'$ are different.

For $n = 0$ this is trivial. Suppose it is true for n; we show it for $n+1$. Let $\mathcal{T}$ and $\mathcal{T}'$ be two towers and assume that they have bases $R^{n+1}_{0,s}$ and $R^{n+1}_{0,s'}$. By Requirement 3, if $(j_0, \ldots, j_{k_n-1})$ and $(j'_0, \ldots, j'_{k_n-1})$ are the k_n tuples associated with s and s', then they are distinct. Let w_t be the $\mathcal{Q}$-name associated with the n-tower with base $R^n_{0,t}$. By induction, the w_t's are distinct. By Theorem 47, the $\mathcal{Q}$-name of $\mathcal{T}$ is $\mathcal{C}(w_{j_0}, \ldots, w_{j_{k_n-1}})$ and the $\mathcal{Q}$-name of $\mathcal{T}'$ is $\mathcal{C}(w_{j'_0}, \ldots w_{j'_{k_n-1}})$. Since $(j_0, \ldots, j_{k_n-1})$ and $(j'_0, \ldots, j'_{k_n-1})$ are different we know $\mathcal{C}(w_{j_0}, \ldots, w_{j_{k_n-1}})$ and $\mathcal{C}(w_{j'_0}, \ldots, w_{j'_{k_n-1}})$ are different.

It now follows from Lemma 56 that $\mathcal{Q}$ generates. □

Ergodicity of the Anosov–Katok systems. We can now show that abstract Anosov–Katok systems are ergodic and isomorphic to uniform circular systems.

Theorem 58. *Suppose that $(X, \mathcal{B}, \mu, T)$ is built by the Anosov–Katok method using fast growing coefficients and h_n's satisfying Requirements 1–3. Let $\mathcal{Q}$ be the partition defined in (23). Then the $\mathcal{Q}$-names describe a strongly uniform circular construction sequence $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$. Let $\mathbb{K}$ be the associated circular system and $\phi : X \to \mathbb{K}$ be the map sending each $x \in X$ to its $\mathcal{Q}$-name. Then ϕ is one-to-one on a set of μ-measure one. Moreover, there is a unique non-atomic shift-invariant measure ν concentrating on the range of ϕ, and this measure is ergodic. In particular, $(X, \mathcal{B}, \lambda, T)$ is isomorphic to $(\mathbb{K}, \mathcal{B}, \nu, sh)$ and is thus ergodic.*

Proof. Since the l_n-sequence grows fast, we know that the sequence of $\mathcal{W}_n$'s forms a uniform construction sequence and hence there is a unique shift-invariant non-atomic measure ν on the set $S \subseteq \mathbb{K}$ given in Definition 10 and ν is ergodic. By Lemma 54, the range of ϕ is a subset of the set S. Hence the factor determined by ϕ is isomorphic to $(S, \mathcal{B}, \mu, sh)$. In particular, this factor is ergodic.

Since the sequence of h_n's satisfy Requirements 1–3 the partition $\mathcal{Q}$ generates X. Hence ϕ is an isomorphism. □

Corollary 59. *If T is a diffeomorphism of the disk built using the Anosov–Katok method of conjugacy satisfying Requirements 1–3 in Section 6.1 then T is measure theoretically isomorphic to a strongly uniform circular system.*

7.6 Tying it all together. In Sections 6 and 7, we have described a class of area-preserving diffeomorphisms of the disk, annulus or torus. We have shown that, subject to some requirements (Requirements 1–3, in Section 6.1), these transformations are ergodic and have a symbolic presentation in a particular form, that of uniform circular systems.

The Anosov–Katok systems are built recursively depending on some data: some sequences of numbers k_n, l_n, s_n. Having been given these numbers, the final bit of data needed to determine the system is a sequence of permutations h_n of the partitions ξ_n. These permutations can be viewed as labeling the horizontal strips of the partition ξ_n with bases of the towers from the previous periodic process.

The numerical sequences and the labeling completely determine a construction sequence that is built recursively using an operator $\mathcal{C}$. The resulting sequence is uniform and circular and thus carries a unique non-atomic measure.

In our applications we take a different tack. We will view the results of this section as showing that uniform circular systems satisfying some minimal requirements are isomorphic to C^∞-measure preserving transformations on the disk, annulus or torus. Here is a converse to Corollary 59.

Theorem 60. *Suppose that $\langle k_n, l_n, s_n : n \in \mathbb{N}\rangle$ are sequences of natural numbers tending to infinity such that the l_n grow sufficiently fast, the s_n grow to infinity, and s_n divides both k_n and s_{n+1}.*

Let $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$ be a circular construction sequence in an alphabet $\Sigma \cup \{b, e\}$ such that:

1. *$\mathcal{W}_0 = \Sigma$, and for $n \geq 1$, $|\mathcal{W}_{n+1}| = s_{n+1}$.*
2. *(Strong Uniformity) For each $w' \in \mathcal{W}_{n+1}$, and $w \in \mathcal{W}_n$, if $w' = \mathcal{C}(w_0, \dots w_{k_n-1})$, then there are k_n/s_n many j with $w = w_j$.*

Then:

(A) *If $\mathbb{K}$ is the associated symbolic shift, then there is a unique non-atomic ergodic measure ν on $\mathbb{K}$.*

(B) *There is a C^∞-measure preserving transformation T defined on the torus (resp. disk, annulus) such that the system $(\mathcal{A}, \mathcal{B}, \lambda, T)$ is isomorphic to $(\mathbb{K}, \mathcal{B}, \nu, sh)$.*

Before we begin the proof of the theorem, we note that we do not know any a priori formulas for a growth rate for the l_n that is sufficient for the conclusion of the theorem; the growth rate is determined inductively as described in the comments at the end of Section 6.

Proof. We show that Lemma 34 allows us to inductively construct a sequence $\langle h_n : n \in \mathbb{N}\rangle$ that yields $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$ as its construction sequence. Suppose that we have defined $\langle h_{n^*} : n^* \leq n\rangle$. From the definition of circular construction sequence (Definition 15) we can find $P_{n+1} \subseteq (\mathcal{W}_n)^{k_n}$ such that $\mathcal{W}_{n+1}$ is the collection of w' such that for some sequence $(w_0, \ldots, w_{k_n-1}) \in P_{n+1}$, $w' = \mathcal{C}(w_0, \ldots, w_{k_n-1})$. Enumerate P_{n+1} as $w'_0, \ldots, w'_{s_{n+1}-1}$. Now apply Lemma 34 to get h_{n+1} from $w'_0, \ldots, w'_{s_{n+1}-1}$.

We claim that the $\langle h_n : n \in \mathbb{N}\rangle$ satisfy Requirements 1–3 Requirement 1 is just that the s_n go to infinity. Requirement 2 follows from item (2) and Requirement 3 follows since the words in P_{n+1} are distinct. □

The smooth transformation T built in Theorem 60 is determined by the collections of words $\langle P_n^T : n \in \mathbb{N}\rangle$ and our particular description of the Anosov–Katok construction. The words in P_n^T determine the maps h_n in the Anosov–Katok construction. Recall at the end of Section 6.3 we chose a summable sequence $\langle \varepsilon_n : n \in \mathbb{N}\rangle$ such that $\varepsilon_n/4 > \sum_{m>n} \varepsilon_m$ and a metric d^∞ that determined the C^∞-topology. The sequence of ε_n give estimates for the smooth approximations h_n^s to h_n and the sequence $\langle S_n : n \in \mathbb{N}\rangle$ converging to T. Equation (14) shows that $d^\infty(S_n, S_{n+1}) < \varepsilon_n/4$. From this we observe that the sequence $\langle P_n^T : n \leq M\rangle$ determines an ε_M neighborhood in which T must lie.

Conversely, different choices of P_n give quite distant h_n's and hence distant h_n^s in the C^∞-norm. We record this for use in applications.

Proposition 61. *Suppose that $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ and $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$ are construction sequences for two circular systems and M is such that $\langle \mathcal{U}_n : n \leq M\rangle = \langle \mathcal{W}_n : n \leq M\rangle$. If S and T are the smooth realizations of the circular systems using the Anosov–Katok method given in this paper, then the d^∞-distance between S and T is less than ε_M.*

Proof. Given the circular construction sequences, we have associated sequences $\langle k_n^U, l_n^U, h_n^U, s_n^U : n \in \mathbb{N}\rangle$ and $\langle k_n^W, l_n^W, h_n^W, s_n^W : n \in \mathbb{N}\rangle$ determining approximation $\langle S_n : n \in \mathbb{N}\rangle, \langle T_n : n \in \mathbb{N}\rangle$ to diffeomorphisms S, T. From the hypothesis we see $\langle k_n^U, l_n^U, h_n^U, s_n^U : n \leq M\rangle = \langle k_n^W, l_n^W, h_n^W, s_n^W : n \leq M\rangle$. Thus $S_M = T_M$. By Remark 39 and (14) we see that

$$d^\infty(S_m, S) < \varepsilon_M/2,$$
$$d^\infty(T_M, T) < \varepsilon_M/2.$$

It follows that $d^\infty(S, T) < \varepsilon_M$. □

We end with a remark that seems relevant to the classification of diffeomorphisms of the torus up to conjugacy by homeomorphisms.

Remark 62. In Theorem 3.3 of [6] it is shown that there are exactly three ergodic invariant measures on the disk D^2 with respect to the Anosov–Katok diffeomorphism constructed there. There is one concentrating on the fixed point in the center, one concentrating on the boundary (where T is a rotation), and one that gives every open set positive measure (Lebesgue measure).

Because we are working on the torus, the top and bottom lines of $\mathcal{A}$ are identified, rather than having one of them collapsed to a point. Thus we get two ergodic invariant measures. One concentrates on the "equator" of the torus—the horizontal line corresponding to the top and bottom of the annulus on which we base our construction. T restricted to this line is the rotation $\mathcal{R}_\alpha$ ($\alpha = \lim \alpha_n$). The second invariant measure is Lebesgue measure.

If ν is an invariant ergodic measure that gives every open set positive measure, then ν is Lebesgue measure. We can also prove this consequence in a different way. If T is an Anosov–Katok diffeomorphism and $\mathbb{K}$ is the circular system isomorphic to T, then there is a unique non-atomic invariant measure on $\mathbb{K}$ (Lemma 20). If

$$\phi : \mathbb{K} \to \mathbb{T}^2$$

is the isomorphism, then the range of ϕ is T invariant and has a unique non-atomic invariant measure. By considering the sets G_n defined in (13) one can establish that if ν is a T-invariant measure giving positive measure to every open set, then ν gives positive measure to the range of ϕ. It follows that if ν is ergodic, then ν is Lebesgue measure.

7.7 Two projects. Here are two projects that we believe are of interest. The first is to extend the symbolic representation given in this paper to other versions of the Anosov–Katok construction—in particular to the twisted case, or to the constructions of weakly mixing transformations in [7].

Secondly, the fact that weakly mixing transformations can be realized by the Anosov–Katok method suggests the possibility that a comeager collection of transformations (with respect to the weak topology) could be realized by a method similar to the Anosov–Katok method.

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