Title
Renormalization group analysis of generalized Coulomb systems in two dimensions

Journal
Physics Letters B, 107(6)

ISSN
0370-2693

Authors
Kares, Robert J
Bander, Myron

Publication Date
1981-12-01

DOI
10.1016/0370-2693(81)91224-7

RENORMALIZATION GROUP ANALYSIS OF GENERALIZED COULOMB SYSTEMS IN TWO DIMENSIONS

Robert J. KARES and Myron BANDER
Department of Physics, University of California, Irvine, CA 92715, USA

Received 28 July 1981

We present a renormalization group analysis of a two dimensional Coulomb gas with internal degrees of freedom. The specific model considered has a global O(3) internal symmetry. The behavior of this system in the low temperature region is found to differ appreciably from that of the XY model.

The two dimensional Coulomb gas has been very useful in describing the vortices of the XY model [1]. However, many other two dimensional systems have classical vortex solutions, including nonabelian ones [2]. These nonabelian vortices possess additional internal degrees of freedom which correspond to the orientation of the vortex in the space of some compact group. In order to describe configurations of such nonabelian vortices it is necessary to generalize the description of the neutral Coulomb system to include gases carrying such additional internal degrees of freedom. The particular generalization which we will consider in this paper consists of assigning to each particle, i, of the gas, a generalized "charge" in the form of an n component unit vector, $\hat{t}_i$. We take the hamiltonian to be,

$$H = -\pi \sum_{i \neq j} \hat{t}_i \cdot \hat{t}_j \ln(\Lambda |x_i - x_j|) , \quad \text{with} \quad \sum_i \hat{t}_i = 0 , \quad \sum_{\alpha=1}^{n} (\hat{t}_i^{\alpha})^2 = 1 . \tag{1}$$

The case of $n = 1$ is the XY model. Here the system has a global Z_2 symmetry which consists of flipping the sign of all the charges. For $n > 1$ the system has a global $O(n)$ symmetry. In order to simplify the technical details we shall specialize to $n = 3$. This case arises naturally as a simple model for the nonabelian vortex gas in 2D $SU_2/Z_2 \times SU_2/Z_2$ spin systems [2]. In this paper we present a renormalization group (R.G.) analysis of the $n = 3$ case. Contrary to the expectation of other authors [3] we find that the O(3) gas does not exhibit Kosterlitz–Thouless type behavior [1]. Instead, for $n = 3$ the spin wave phase is destroyed at all temperatures by the appearance of vortex bound pairs of small but nonvanishing net "charge".

We begin our analysis with the grand partition function of the O(3) gas,

$$Z = \sum_{N=0}^{\infty} \frac{(\xi_0 \Lambda^2)^N}{N!} \prod_{i=1}^{N} \int d^2x_i \, d\Omega_i \exp\{-S\} , \tag{2}$$

where

$$S = -\pi\beta \sum_{i \neq j} \hat{t}_i \cdot \hat{t}_j \ln(\Lambda |x_i - x_j|) + \pi\beta \left(\sum_i \hat{t}_i \right)^2 \ln(R/a) ,$$

$d\Omega_i$ is an integration over the possible orientations of $\hat{t}_i$. ξ_0 is the density of vortices in units of $\Lambda^2 = a^{-2}$, with a the lattice spacing. $\beta = 1/T$ and R is the system size. The $\ln(R/a)$ term is included in S to enforce the constraint

$\Sigma_i \hat{t}_i = 0$.

Introducing a triplet of auxiliary fields, ψ^α, we convert (2) to a field theory in the usual fashion [4]. The result is,

$$Z = \int [\mathrm{d}\psi^\alpha] \exp\left\{\int\left[-\tfrac{1}{2}(\partial\boldsymbol{\psi}(x))^2 + \xi_0\Lambda^2 \int \exp[\mathrm{i}b\hat{t}\cdot\boldsymbol{\psi}(x)]\,\mathrm{d}\Omega\right]\mathrm{d}^2x\right\}, \tag{3}$$

where $b = 2\pi\sqrt{\beta}$. Notice that the local interaction term has the form,

$$\int \exp[\mathrm{i}b\hat{t}\cdot\boldsymbol{\psi}(x)]\,\mathrm{d}\Omega = j_0(b|\boldsymbol{\psi}(x)|)\,. \tag{4}$$

It will be convenient in the following, however, to leave the interaction in the exponential form. In fact, let us write,

$$\int \exp[\mathrm{i}b\hat{t}\cdot\boldsymbol{\psi}(x)]\,\mathrm{d}\Omega = \int \delta(|t|-1)\exp[\mathrm{i}bt\cdot\boldsymbol{\psi}(x)]\,\mathrm{d}^3t\,. \tag{5}$$

Eq. (3) provides a convenient starting point for developing a R.G. We follow a procedure outlined by Kogut [5] for the XY model. Begin by defining a cutoff version of (3),

$$Z_\Lambda = \int_{0<k<\Lambda} [\mathrm{d}\psi^\alpha(k)] \exp\left\{\int\left[-\tfrac{1}{2}(\partial\boldsymbol{\psi}_\Lambda(x))^2 + \xi_0\Lambda^2\int\delta(|t|-1)\exp[\mathrm{i}bt\cdot\boldsymbol{\psi}_\Lambda(x)]\,\mathrm{d}^3t\right]\mathrm{d}^2x\right\}. \tag{6}$$

Now we can write,

$$\psi^\alpha_\Lambda(x) = \psi^\alpha_{\Lambda'}(x) + h^\alpha(x)\,, \quad \Lambda' < \Lambda\,,$$

where $\psi^\alpha_{\Lambda'}(x)$ has only Fourier components with $0 < k < \Lambda'$ and $h^\alpha(x)$ has only components with $\Lambda' < k < \Lambda$.

We may now reorganize (6) and do the integrations over components with $\Lambda' < k < \Lambda$ to $O(\xi_0^2)$ in perturbation theory. If we then let $\Lambda' = \Lambda - \delta\Lambda$, $\delta\Lambda/\Lambda \ll 1$ and keep only terms of $O(\delta\Lambda)$ the result is,

$$Z_\Lambda = \int_{0<k<\Lambda'} [\mathrm{d}\psi^\alpha(k)] \exp\left\{\int\left\{-\tfrac{1}{2}(\partial\psi^\alpha_{\Lambda'}(x))(\partial\psi^\beta_{\Lambda'}(x))\left(\delta^{\alpha\beta} + \xi_0^2\frac{b^4a_2}{2}\frac{\delta\Lambda}{\Lambda}A^{\alpha\beta}\right)\right.\right.$$
$$\left.\left. + \Lambda'^2\int\left[\xi_0\delta(|t|-1)\left[1+\left(2-\frac{b^2|t|^2}{4\pi}\right)\frac{\delta\Lambda}{\Lambda}\right] - \xi_0^2b^2a_1\frac{\delta\Lambda}{\Lambda}B(|t|)\right]\exp[\mathrm{i}bt\cdot\boldsymbol{\psi}_{\Lambda'}(x)]\,\mathrm{d}^3t\right\}\mathrm{d}^2x\right\}, \tag{7}$$

where

$$A^{\alpha\beta} = \int\delta(|t_1|-1)\,\delta(|t_2|-1)\exp[\mathrm{i}b(t_1+t_2)\cdot\boldsymbol{\psi}_{\Lambda'}(x)](t_1\cdot t_2)\,t_1^\alpha t_2^\beta\,\mathrm{d}^3t_1\,\mathrm{d}^3t_2$$
$$= (4\pi)^2\left[\frac{j_1^2(\eta)}{\eta^2}\delta^{\alpha\beta} - b^2\psi^\alpha_{\Lambda'}(x)\,\psi^\beta_{\Lambda'}(x)\left(\frac{j_1^2(\eta)-\eta^2(j_1'(\eta))^2}{\eta^4}\right)\right], \quad \eta = b|\boldsymbol{\psi}_{\Lambda'}(x)|\,, \tag{8}$$

$$B(|t|) = \int\delta(|t/2+u|-1)\,\delta(|t/2-u|-1)\,(t/2+u)\cdot(t/2-u)\,\mathrm{d}^3u$$
$$= \begin{cases}(2\pi/|t|)(|t|^2/2-1) & |t|\leqslant 2\\ 0 & |t|>2\end{cases} \tag{9}$$

and a_1, a_2 are dimensionless constants which depend on the momentum slicing procedure [5]. In addition to the Coleman renormalization [6] notice that we have generated three new O(3) symmetric interaction terms including derivative couplings of the form $(\partial\boldsymbol{\psi})^2F_1(b|\boldsymbol{\psi}|)$ and $(\boldsymbol{\psi}\cdot\partial\boldsymbol{\psi})^2F_2(b|\boldsymbol{\psi}|)$. The behavior of these additional interactions under the R.G. will have to be determined.

In order to accomplish this we must include the new interactions in our R.G. procedure from the outset. We take for our new generic action,

$$\int\left\{-\tfrac{1}{2}(\partial\psi_\Lambda)^2\left[1+\xi_0^2\int\gamma_\Lambda(|t|)\exp(ibt\cdot\psi_\Lambda)\,\mathrm{d}^3t\right]-\tfrac{1}{2}(\psi_\Lambda\cdot\partial\psi_\Lambda)^2\,\xi_0^2\int\sigma_\Lambda(|t|)\exp(ibt\cdot\psi_\Lambda)\,\mathrm{d}^3t\right.$$
$$\left.+\Lambda^2\int[\xi_0 f_\Lambda(|t|)+\xi_0^2 g_\Lambda(|t|)]\exp(ibt\cdot\psi_\Lambda)\,\mathrm{d}^3t\right\}\mathrm{d}^2x\,, \tag{10}$$

where in order to properly keep track of the powers of ξ_0 we have written separately the $O(\xi_0)$ and $O(\xi_0^2)$ parts of the last term. Beginning with (10) we can again perform the R.G. transformation which leads to (7) keeping terms correct to $O(\xi_0^2)$. The result is a consistent set of coupled R.G. equations for the Fourier transforms $\gamma_\Lambda(|t|)$, $\sigma_\Lambda(|t|)$, $f_\Lambda(|t|)$ and $g_\Lambda(|t|)$ correct to $O(\xi_0^2)$ which we can then solve subject to the initial conditions.

$$f_{\Lambda_0}(|t|)=\delta(|t|-1)\,,\qquad \gamma_{\Lambda_0}(|t|)=\sigma_{\Lambda_0}(|t|)=g_{\Lambda_0}(|t|)=0\,. \tag{11,12}$$

The equation for $f_\Lambda(|t|)$ is simplest. It has the form,

$$\Lambda\frac{\partial f_\Lambda(|t|)}{\partial\Lambda}=\left(\frac{b^2|t|^2}{4\pi}-2\right)f_\Lambda(|t|)\,. \tag{13}$$

Then the solution of (13) subject to (11) is,

$$f_\Lambda(|t|)=(\Lambda/\Lambda_0)^{(b^2/4\pi-2)}\,\delta(|t|-1)\,. \tag{14}$$

This is the standard Coleman renormalization [6]. If we write,

$$\int f_\Lambda(|t_1|)\,f_\Lambda(|t_2|)\exp[ib(t_1+t_2)\cdot\psi]\,(t_1\cdot t_2)\,t_1^\alpha t_2^\beta\,\mathrm{d}^3t_1\,\mathrm{d}^3t_2$$
$$=\delta^{\alpha\beta}\int f_\Lambda^{(1)}(|t|)\exp(ibt\cdot\psi)\,\mathrm{d}^3t+\psi^\alpha\psi^\beta\int f_\Lambda^{(2)}(|t|)\exp(ibt\cdot\psi)\,\mathrm{d}^3t\,, \tag{15}$$

the equations for γ_Λ and σ_Λ take the form,

$$\Lambda\frac{\partial\sigma_\Lambda(|t|)}{\partial\Lambda}=\frac{b^2|t|^2}{4\pi}\,\sigma_\Lambda(|t|)+\frac{b^2}{\pi}\int_{|t|}^{\infty}s\,\sigma_\Lambda(s)\,\mathrm{d}s-\frac{b^4a_2}{2}\,f_\Lambda^{(2)}(|t|)\,, \tag{16}$$

$$\Lambda\frac{\partial\gamma_\Lambda(|t|)}{\partial\Lambda}=\frac{b^2|t|^2}{4\pi}\,\gamma_\Lambda(|t|)-\frac{1}{2\pi}\,\sigma_\Lambda(|t|)-\frac{b^4a_2}{2}\,f_\Lambda^{(1)}(|t|)\,. \tag{17}$$

Using (14) we have obtained an approximate solution of the equations subject to the initial conditions (12). We find that for $b^2>8\pi$ the behavior of γ_Λ and σ_Λ is such that

$$\int\gamma_\Lambda(|t|)\exp(ibt\cdot\psi)\,\mathrm{d}^3t\,,\quad\text{and}\quad\int\sigma_\Lambda(|t|)\exp(ibt\cdot\psi)\,\mathrm{d}^3t$$

are driven to 0 as $\Lambda\to 0$. For $b^2>8\pi$ the derivative interactions iterate to zero. Naively, one might have expected this result on dimensional grounds. However, the behavior of γ_Λ and σ_Λ near $t=0$ requires more careful study. Of particular interest is whether or not γ_Λ contains a $\delta(t)$ singularity at $t=0$. To see the significance of this point recall that the temperature renormalization in the XY model arises from the appearance in $\gamma_\Lambda(t)$ of a singular term containing $\delta(t)$. When integrated over t this $\delta(t)$ picks out the $t=0$ part of the γ_Λ term in (10) thus providing a natural redefinition of the scale of the ψ field which can then be absorbed into a redefinition of b. *In the O(3) model no such singularity appears.* The temperature renormalization is absent in the O(3) case.

The equation for $g_\Lambda(|t|)$ takes the form,

$$\Lambda \frac{\partial g_\Lambda(|t|)}{\partial \Lambda} = \left(\frac{b^2|t|^2}{4\pi} - 2\right) g_\Lambda(|t|) + b^2 a_1 \int f_\Lambda(|t/2+u|)$$
$$\times f_\Lambda(|t/2-u|)(|t/2+u)\cdot(t/2-u)\, \mathrm{d}^3u + \frac{3}{4\pi}\gamma_\Lambda(|t|) - \frac{1}{4\pi b^2}\nabla_t^2 \sigma_\Lambda(|t|)\,. \tag{18}$$

To discuss the behavior of its solution for $b^2 > 8\pi$ it is sufficient to drop the γ_Λ and σ_Λ terms. It is easy to verify that including these terms does not alter the qualitative nature of the conclusions reached. Using (14) we may write the simplified version of (18) in the form,

$$\Lambda\, \partial g_\Lambda(|t|)/\partial\Lambda = (b^2|t|^2/4\pi - 2)\, g_\Lambda(|t|) + b^2 a_1 B(|t|)(\Lambda/\Lambda_0)^{2(b^2/4\pi - 2)}\,. \tag{19}$$

This is easily solved to yield,

$$g_\Lambda(|t|) = \frac{b^2 a_1 B(|t|)\{(\Lambda/\Lambda_0)^{(b^2|t|^2/4\pi-2)} - (\Lambda/\Lambda_0)^{2(b^2/4\pi-2)}\}}{[(b^2|t|^2/4\pi - 2) - 2(b^2/4\pi - 2)]}\,, \tag{20}$$

where $B(|t|)$ is given in (9).

For $b^2 > 8\pi$ the second term in (20) is driven to zero as $\Lambda \to 0$. In the first term t's with $|t|^2 > 8\pi/b^2$ are also driven to zero. However, $g_\Lambda(|t|)$'s with $|t|^2 < 8\pi/b^2$ grow like

$$(\Lambda_0/\Lambda)^{(2-b^2|t|^2/4\pi)} \quad \text{as } \Lambda \to 0\,.$$

This result has a simple physical interpretation. For $b^2 > 8\pi$ the charges of the $n = 1$ or XY Coulomb gas form bound pairs. Because of the discrete nature of the charges in this case, each pair must have zero net charge. In the O(3) case, however, the charges are three dimensional unit vectors with continuous orientations. For $b^2 > 8\pi$ they again form bound pairs of almost oppositely oriented $\hat{t}$'s, however, unless the unit vectors are exactly antiparallel each pair will have a nonvanishing net vector charge. Hence, an O(3) pair can still have long range effects. The appearance of such pairs in the system manifests itself by the growth of $g_\Lambda(|t|)$'s for $|t|^2 < 8\pi/b^2$.

A heuristic way of seeing this result is provided by a modification of the famous Kosterlitz–Thouless argument for the appearance of vortices in the XY model [1]. In the O(3) model the Coulomb energy of an effective charge t is

$$E = \pi|t|^2 \ln(R/a)\,.$$

The IR divergent part of its configurational entropy is given by,

$$S = k \ln(R/a)^2\,.$$

So the free energy of this charge is

$$F = (\pi|t|^2 - 2kT)\ln(R/a)\,.$$

For fixed T, the formation of pairs with effective charge

$$|t|^2 < 2kT/\pi = 2/\pi\beta = 8\pi/b^2$$

is favored.

As the coefficient of $\exp(ibt\cdot\psi)$ grows for $|t|^2 < 8\pi/b^2$, we cannot follow its R.G. evolution. We would have to go to higher orders in perturbation theory. It is, however, likely that the activities of the bound pairs will approach some finite limit for $|t|^2 < 8\pi/b^2$. Thus for $b^2 > 8\pi$ the low temperature spin wave phase of the O(3) model is destroyed by the appearance of "vortex" pairs having nontrivial long range effects. This picture of the low T region raises the possibility that at sufficiently low temperatures the formation of "multi vortex" clusters might occur.

We expect similar behavior for vortices "pointing" in other compact group or coset spaces and if this picture is valid then we see that systems supporting vortices with continuous internal degrees of freedom will develop a mass gap down to $T = 0$.

This research was supported in part by the National Science Foundation.

References

[1] J.M. Kosterlitz and D.J. Thouless, J. Phys. C6 (1973) 1181;
J.M. Kosterlitz, J. Phys. C7 (1974) 1046.
[2] R.J. Kares and M. Bander, Phys. Lett. 96B (1980) 320.
[3] S.A. Bulgadaev, Phys. Lett. 87B (1979) 47.
[4] A.M. Polyakov, Nucl. Phys. B120 (1977) 429;
F.A. Schaposnik, Phys. Rev. D18 (1978) 1183.
[5] J. Kogut, Rev. Mod. Phys. 51 (1979) 659;
A.M. Polyakov, Phys. Lett. 59B (1975) 79.
[6] S. Coleman, Phys. Rev. D11 (1975) 2088;
D.J. Amit, Y. Goldschmidt and G. Grinstein, J. Phys. A13 (1980) 585.