Title
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Publication Date
1962-08-29

UCRL-10445

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UCRL-10445

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Contract No. W-7405-eng-48

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ABSTRACT

Some criteria are established to test whether invariance under a continuous transformation will lead to a conservation law. The scale transformation is examined as a special case, and shown not only to lead to no general conservation law, but also in fact to be of a trivial nature. This is due to the rather artifical way in which scale invariance is usually introduced. A theory is then constructed by introducing an internal coordinate of dimension (length) in order to allow only the dimensionaless ratio of lengths to enter, and by exploiting the gauge-like structure of the scale transformation. In this theory the scale transformation does lead to a new conserved current (as well as to an "almost conserved" one), and the internal coordinate is shown to play the same role for the scale transformation as the internal coordinate spin plays for the case of rotations.

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I. INTRODUCTION

It seems to be inherently reasonable to require that the laws of physics be independent of the size of the units used by the physicist to measure them. In fact (at least in a flat geometry), this would appear to be a rather trivial statement, with a change in scale implying nothing more than the usual conversion of, say, centimeters to inches. The only theoretical problem seems to be whether to multiply or divide by 2.54.

However, there are other types of restrictions that we place on physical theories, which seem superficially quite similar in nature to scale invariance, and yet which have rather profound consequences. For example, the requirement that the location of the origin of the coordinate system be irrelevant leads to conservation of momentum, and the requirement that the orientation of the axes be irrelevant leads to conservation of angular momentum.

The question arises then why the scale invariance of a theory appears to be trivial and devoid of physical significance--is it due to a property of the scale transformation itself, or merely to the manner in which we choose to incorporate it into physics? To answer this question we shall examine in Section II the connection between invariance principles and conservation laws, and establish some criteria that an invariance principle should satisfy in order to yield a conservation law. Unfortunately scale invariance as usually formulated fails rather dismally to meet these criteria. Because of this, scale invariance has been relegated to the role of a formal operation which has been studied rather extensively in the literature,[1] primarily in connection with applications to the virial theorem and a "scale parameter" perturbation theory, but not with respect to any possible physical consequences.

Next, in Section III, we develop some simple formal apparatus for describing scale transformations and develop a few consequences of scale invariance.

If a theory were to contain a meaningful form of scale invariance, it would have to be formulated in terms of ratios of lengths (see Sec. II), just as a translation-invariant theory contains only the differences between coordinates. In Section IV, the fact that a change in scale resembles a gauge transformation is used to build a model theory in which an internal coordinate with the dimension "length" is introduced for the purpose of having only dimensionless ratios enter the theory.

Finally, in Section V, the scale invariance of this theory is examined and shown to be nontrivial. Not only does scale invariance then lead to a new conservation law, but also the internal coordinate is shown to enter the theory in a manner exactly analogous to the way in which the internal coordinate spin enters for rotational invariance.

In this light, the usual manner of introducing scale invariance appears to be as artifical as would be the attempt to discuss the rotational invariance of a vector field by considering only the orbital angular momentum, where the effects of the spin would have to be eliminated by introducing an external parameter and forcing it to vary in just such manner as to cancel the spin (and in the scale transformation, the "conversion of units" plays just such a role).

II. INVARIANCE PRINCIPLES AND CONSERVATION LAWS

The fact that a theory exhibits invariance with respect to a particular infinitesimal transformation does not necessarily imply that there exists a conservation law. Even when there does exist a conservation law, it does not have to be of the form of a local conservation law--that is, a law stating that the divergence of some tensor vanishes everywhere, yielding a continuity equation. It may merely assert that some particular integral over all space vanishes, without any local consequences.

If there is no conservation law at all, the invariance is rather trivial, and we shall call it "external" invariance.[2] It occurs when the theory depends upon an external parameter (such as an origin of coordinates), which is chosen to vary under the infinitesimal transformation in such a way as to cancel the effect due to the variation in the coordinates. For example, consider a system described by an action integral $A = \int \mathcal{L}(x - \xi)dx$, where ψ is an external origin introduced into the theory. Then under the translation $x \to x + \delta\sigma$, $\xi \to \xi + \delta\sigma$, the action remains invariant, but this invariance places no restriction on $\mathcal{L}$ and leads to no conservation law.

Another simple example of such a theory would be a rotationally invariant description of the temperature above the surface of the earth. Assume $T = T_0 e^{-\alpha z}$, where z represents vertical distance from the (flat) earth. This can be written in the invariant form $T = T_0 e^{-\alpha \underset{\sim}{k} \cdot \underset{\sim}{r}}$, where $\underset{\sim}{k}$ is a unit vector pointing vertically upward, and $\underset{\sim}{r}$ is the vector from some point 0 on the surface to the point in space. Then, under a rotation about 0, where $\underset{\sim}{r} \to R\underset{\sim}{r}$, $\underset{\sim}{k} \to R\underset{\sim}{k}$, the scalar product is invariant. Such a theory is rotationally invariant but implies nothing further, and in fact even contains a preferred axis.

For a theory to be free of such trivial external invariances, the external parameters that enter into it (such as masses, charges, and the velocity of light) should remain unaffected by the infinitesimal transformation in question. If such is the case, then we can proceed to derive the usual conservation laws. We assume that the theory is described by an action

$$A = \int dx\, \mathcal{L}(u_i, \partial u_i/\partial x^\mu) \,, \tag{1}$$

where the u_i represent the various fields occurring in the theory. Let the coordinates x^μ be subject to the transformation

$$x^\mu \to x^{\mu'} = x^{\mu'}\,(x^\nu, \omega_k) \,, \tag{2}$$

where the ω_k, $(k = 1 \cdots s)$ represent s parameters. It suffices to consider the infinitesimal transformations

$$x^{\mu'} = x^\mu + \delta x^\mu, \quad \delta x^\mu = \sum_k a_k{}^\mu\, \delta\omega_k \,. \tag{3}$$

Under this transformation law the fields transform as

$$u_i(x) \to u'_i(x') = u_i(x) + \delta u_i, \quad \delta u_i = \sum_k \alpha_{ik}\, \delta\omega_k \,. \tag{4}$$

The variation $\bar{\delta} u_i(x)$ in the form of the function at the point x is therefore

$$\bar{\delta} u_i \equiv u'_i(x) - u_i(x) = \sum_k \left[\alpha_{ik} - (\partial u_i/\partial x^\mu)\, a_k{}^\mu\right] \delta\omega_k \,. \tag{5}$$

(We use the summation convention for Greek indices which label the coordinates.)

The usual procedure for deriving the variation in the action[3] due to the infinitesimal transformations (3) and (4) yields

$$\delta A = - \sum_k \left(\int dx\, \partial_\mu \theta_k{}^\mu \right) \delta\omega_k \, , \tag{6}$$

where

$$\theta_k{}^\mu = \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_j)} \left[(\partial_\nu u_j) a_k{}^\nu - \alpha_{jk} \right] - \mathcal{L} a_k{}^u \, , \tag{7}$$

(we shall often use the notation $\partial_\mu \equiv \partial/\partial x^\mu$), and where the limits of integration in Eq. (6) can be any arbitrary region of space-time. Then the condition $\delta A = 0$ leads to the conservation laws

$$\partial_\mu \theta_k{}^\mu = 0, \quad (k = 1, \cdots, s) \, . \tag{8}$$

However, it may happen that for the transformation in question we do not demand that the variation in Eq. (6) vanish over any arbitrary volume, but only over some region of special interest, such as a volume sufficiently large that the fields vanish on its surfaces. Then we cannot generally conclude anything as strong as Eq. (8) but must be content with the weaker conservation laws

$$\int dx\, \partial_\mu \theta_k{}^\mu = 0, \quad (k = 1, \cdots , s) \, . \tag{9}$$

Of course Eq. (8) might follow even in the latter case, and we shall examine this possibility. Let us assume that the Lagrangian $\mathcal{L}(x)$ satisfies the simple law

$$\mathcal{L}'(x') = \mathcal{L}(x) \, . \tag{10}$$

Then the variation δA over an arbitrary region of space-time can be written

$$\delta A = \int dx' \mathcal{L}'(x') - \int dx \ \mathcal{L}(x)$$

$$= \int dx \ [J(x,\omega)-1] \ \mathcal{L}(x)$$

$$= \sum_k \ dx \ (\partial J/\partial\omega_k)_{\omega_k=0} \ \mathcal{L}(x) \ \delta\omega_k \ , \qquad (11)$$

where $J(x,\omega)$ is the Jacobian $\partial[x^{\nu'}(x,\omega)]/\partial(x^{\mu})$. Numerically we have

$$\left(\frac{\partial J}{\partial\omega_k}\right)_{\omega_k=0} = \left\|\frac{\partial a_k{}^{\mu}}{\partial x^{\nu}}\right\| , \ (k = 1, \cdots , s) . \qquad (12)$$

The first line of Eq. (11) is true, because in the derivation of δA the boundaries of the region also take part in the variation. Equation (6) yields δA over an arbitrary region regardless of whether δA vanishes or not, so that we have

$$-\partial_{\mu}\Theta_k{}^{\mu} = \left(\frac{\partial J}{\partial\omega_k}\right)_{\omega_k=0} \mathcal{L}(x) . \qquad (13)$$

Then the local conservation law follows provided that either

(a) $\left(\partial J/\partial\omega_k\right)_{\omega_k=0} = 0$

or

(b) $\mathcal{L}(x) = 0$,

which may happen by virtue of the field equation--it need not be identically zero in the variational sense. In the special cases of translational invariance, where $x^{\mu} \rightarrow x^{\mu} + \lambda^{\mu}$, and rotational invariance, where $x^{\mu} \rightarrow a_{\nu}{}^{\mu}(\alpha,\beta,\gamma;v)x^{\nu}$, we have $J = 1$, independently of λ^{μ} or of the angles α, β, γ and velocity v. Thus, the local conservation laws follow from condition (a).

Another possible case leading to the local conservation laws, Eq. (8), even if neither condition (a) nor (b) holds, occurs when the simple Eq. (10) no longer holds, but rather transforms as

$$\mathcal{L}'(x') = (1/J)\mathcal{L}(x). \tag{14}$$

Equation (14) implies the local laws immediately, from the first line of Eq. (11).

An interesting situation arises in the case of a quantized field theory, because of fluctuations caused by the uncertainty principle. For example, momentum conservation is expressed as the constancy of some integral over all of 3-space, and indeed the momentum density integrated over a finite volume does not commute with the observables of the theory, and must be consistent with the condition $\delta p \cdot \delta x \sim \hbar$. However, the theory is locally invariant in our sense [i.e., Eq. (8) holds for the stress tensor], because translation invariance follows for the action integral defined over an arbitrary space-time region, so that our remarks apply, and we have $\partial_{\mu} T_{\nu}^{\mu} = 0$. A further subtlety occurring in quantized theories is the existence of a class of invariants, such as the charge, for which finite volume integrals do commute with the observables, but this distinction does not affect anything discussed in this paper and will not be considered further.[4]

In this discussion the parameters $\delta\omega_k$ have been assumed not to be functions of the coordinates x^{μ}. For those cases in which this is not true, such as the gauge invariance of the electromagnetic (e-m) field, the quantities $\delta\omega_k$ in Eq. (6) appear under the integral as arbitrary functions, and local invariance [i.e., Eq. (8)] follows, with Θ_k^{μ} still formally given by Eq. (7). However, in such cases there are also terms of the form $\delta(\partial_\mu \omega_k)$. The coefficients of these terms vanish, though, so long as the fields are correctly coupled in a gauge-invariant manner,[5] and therefore need not be explicitly considered. The theory of Section IV will be of this type.

If we apply the foregoing considerations to the scale transformation, we see that the very fact that the equations of physics have dimensional units attached implies that the scale transformation is trivial. For example, an external parameter like mass determines a Compton wavelength, and under a change of scale this wavelength changes accordingly, making the theory externally invariant.

The obvious remedy would be to have only the ratio of lengths enter the theory, as these would be unaffected by a scale transformation. In fact, if two lengths x_1 and x_2 enter a theory, and the theory is to be invariant under the transformation $x_1 \to x_1(1 + \delta\sigma)$, $x_2 \to x_2(1 + \delta\sigma)$, then we have

$$\delta\mathcal{L}(x_1, x_2) = \frac{\partial\mathcal{L}}{\partial x_1} x_1 \delta\sigma + \frac{\partial\mathcal{L}}{\partial x_2} x_2 \delta\sigma$$

$$= \left(\frac{\partial\mathcal{L}}{\partial \ln x_1} + \frac{\partial\mathcal{L}}{\partial \ln x_2}\right) \delta\sigma ,$$

and the general solution of the equation $\delta\mathcal{L} = 0$ is

$$\mathcal{L} = \mathcal{L}(\ln x_1 - \ln x_2) = \mathcal{L}(x_1/x_2) .$$

For this reason, in the theory of Sec. IV, we introduce an internal degree of freedom κ, whose dimension is $(\text{length})^{-1}$, and demand that $\mathcal{L}$ be of the form $\mathcal{L}(x\kappa)$.

It should be mentioned that there is a class of physical theories that are scale-invariant by virtue of the fact that no scale-dependent external parameters enter--for example, the field theories describing free massless particles. Although these theories are free from any external invariance, they possess the slightly complicating feature that for the transformation $x^\mu \to x^\mu(1 + \delta\sigma)$ we have the Jacobian $J = (1 + n\delta\sigma)$, where n is the dimension of the space, and therefore $\partial J/\partial\sigma = n$. However, in the usual 4-dimensional theories the Lagrangian density has the dimensions $(\text{length})^{-4}$ and does not transform by Eq. (10), but rather by Eq. (14). Thus these theories can lead to local conservation laws, as will be discussed in the next section. The theory of Section IV will have the advantage of being capable of including massive fields, as well as being intrinsically dimensionless, which the usual massless theories are not.

III. SOME PROPERTIES OF THE SCALE TRANSFORMATION

Consider a field $\psi(x)$ with the dimensions $(\text{length})^n$. The variable x is the dimensionless number X/a, where X represents an actual physical length, independent of the coordinate system, and a is the size of the unit used for measurement. Now, if the size of the unit a is changed to a/λ, then the same length X is represented in the new units by $X/(a/\lambda) = \lambda x = x'$, and the field $\psi'(x')$ measured by an observer using the new units is related to the old field $\psi(x)$ by the equation

$$\psi(x) = \frac{1}{\lambda^n}\ \psi'(x') \ . \tag{15}$$

Equation (15) is the equivalent of the transformation law for tensors:

$$T'_{\kappa\cdots\omega}(x') = a_\kappa{}^k \cdots a_\omega{}^z\ T_{k\cdots z}(x) \ .$$

In the relativistic case, "length" as used above pertains to the 4-vector x^μ: $(ct, \underset{\sim}{x})$.

A simple illustration of Eq. (15) would be the case in which $\psi(x)$ describes the height of a water wave with the dimension (length) measured in inches. Let the height of a wave at $x = 2$ be 3 in. Then $\psi(2) = 3$. If now an observer were to measure the same wave in 1/2-in. units, he would write $\psi'(4) = 6$, or $\psi(2) = 1/2\ \psi'(4)$.

For a nonrelativistic quantum-mechanical wave function,[6] Eq. (15) reads

$$\psi(\underset{\sim}{x}) = \lambda^{3/2}\ \psi'(\underset{\sim}{x}') \ . \tag{16}$$

Similarily, for a relativistic Fermi or Bose field, Eq. (15) takes the form

$$\begin{aligned} \psi_F(x) &= \lambda^{3/2}\ \psi'_F(x') , \\ \phi_B(x) &= \lambda\ \phi'_B(x') \ . \end{aligned} \tag{17}$$

Under a scale transformation, the argument $\underset{\sim}{x}$ of the nonrelativistic wave function $\psi(\underset{\sim}{x})$ transforms as

$$\underset{\sim}{x} \to \underset{\sim}{x}' = \underset{\sim}{x}(1 + \delta\sigma) \ . \tag{18}$$

(The infinitesimal expression $(1 + \delta\sigma)$ becomes the finite expression $e^\sigma \equiv \lambda$.) The wave function of transformed coordinates becomes

$$\begin{aligned} \psi(\underset{\sim}{x} + \underset{\sim}{x}\ \delta\sigma) &= \psi(\underset{\sim}{x}) + \delta\sigma\ \underset{\sim}{x}\cdot\underset{\sim}{\nabla}\ \psi(\underset{\sim}{x}) \\ &= \psi(\underset{\sim}{x}) + i\ \delta\sigma\ \underset{\sim}{r}\cdot\underset{\sim}{p}\ \psi(\underset{\sim}{x}) \ . \end{aligned} \tag{19}$$

The operator generating this change is $\mathbf{r}\cdot\mathbf{p}$. However, this operator is not Hermitian, so we introduce the operator

$$\Lambda = \frac{1}{2}\sum_i \left[r_i , p_i\right]_+ = \underset{\sim}{r}\cdot\underset{\sim}{p} + \frac{3}{2i} \; . \tag{20}$$

For a finite transformation (in one variable) we have

$$e^{\sigma x\, \partial/\partial x}\, \psi(x) = \psi(xe^{\sigma}) , \tag{21}$$

which may be proved by expanding both sides in a Taylor series in σ. Thus, we have

$$U_\sigma\, \psi(\underset{\sim}{x}) \equiv e^{i\sigma\Lambda}\, \psi(\underset{\sim}{x}) = e^{3\sigma/2}\, \psi(\underset{\sim}{x}e^{\sigma}) , \tag{22}$$

so that the variation in the wave function at a point can be written

$$\psi(\underset{\sim}{x}) = U_\sigma\, \psi'(\underset{\sim}{x}) , \tag{23}$$

or

$$\bar{\delta}\psi(\underset{\sim}{x}) = \psi'(\underset{\sim}{x}) - \psi(\underset{\sim}{x}) = -i\, \delta\sigma\, \Lambda\, \psi(\underset{\sim}{x}) , \tag{24}$$

which agrees with the result of a direct application of Eq. (5). The operator Λ obeys the commutation rule $[\Lambda, \ell n\, x_i] = 1/i$, which says that a scale transformation can be interpreted as a displacement in $\ell n\, x_i$, just as the momentum operator produces a displacement in x_i itself.

It should be noted that Eq. (16) does not depend upon our use of a Cartesian coordinate system, and that the factor $\lambda^{3/2}$ in Eq. (16) merely represents a factor $\sqrt{J}$, where J is the Jacobian $\partial(\underset{\sim}{x}')/\partial(\underset{\sim}{x})$. In differential form Eq. (16) reads

$$\delta\psi = \psi'(\underset{\sim}{x}') - \psi(\underset{\sim}{x}) = -3/2\, \psi(\underset{\sim}{x})\, \delta\sigma . \tag{25}$$

In this form we can see that the scale transformation is essentially of the form of a gauge transformation and has no physical significance, in accordance with the remarks in Section II.

For a set of relativistic fields $u_k(x)$ of dimension $(\text{length})^{n_k}$ transforming under a scale transformation according to Eq. (15), the variation in the action can be calculated from Eq. (16). By using $\delta x^\mu = x^\mu\, \delta\sigma$ and $\delta u_k = n_k\, u_k\, \delta\sigma$, we obtain

$$\delta A = -\int d^4x\, \partial_\mu \left\{ \sum_k \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu u_k)} \left(x^\nu \partial_\nu u_k - n_k u_k \right) \right] - \mathcal{L}\delta_\nu{}^\mu\, x^\nu \right\}$$

$$= -\int d^4x\, \partial_\mu \left[x^\nu\, T_\nu{}^\mu - \sum_k n_k\, u_k\, \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_k)} \right]$$

$$= -\int d^4x\, \partial_\mu\, \Theta^\mu , \tag{26}$$

where $T_\nu{}^\mu$ is the stress-energy tensor. For a field that transformed as $u(x) = u'(x')$, and for which $\mathcal{L}(x) = 0$, we would obtain the conservation law

$$\partial_\mu (x^\nu\, T_\nu{}^\mu) = T_\mu{}^\mu = 0; \int d^3x\, x^\nu\, P_\nu = \text{const.}, \tag{27}$$

where P_ν is the momentum density, $T_\nu{}^0$.

For a massless relativistic field the Lagrangian satisfies the transformation law

$$\mathcal{L}(x) = \lambda^4 \mathcal{L}'(x') = \frac{1}{J} \mathcal{L}'(x') ,$$
$$dx' \mathcal{L}'(x') = dx \mathcal{L}(x) , \tag{28}$$

so that Eq. (26) should lead to the local conservation law

$$\partial_\mu \Theta^\mu = 0 . \tag{29}$$

In fact,

$$\partial_\mu \Theta^\mu = T_\mu{}^\mu - \sum_k n_k \left[u_k \frac{\partial \mathcal{L}}{\partial u_k} + (\partial_\mu u_k) \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_k)} \right]$$

$$= -4\mathcal{L} - \sum_k \left[n_k u_k \frac{\partial}{\partial u_k} + (n_k - 1)(\partial_\mu u_k) \frac{\partial}{\partial(\partial_\mu u_k)} \right] . \tag{30}$$

To derive Eq. (30) we have used the field equations and energy conservation,

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu u_k)} \right] = \frac{\partial \mathcal{L}}{\partial u_k} ; \quad \partial_\mu T_\nu{}^\mu = 0 . \tag{31}$$

For a Lagrangian which is a function only of the fields and their derivatives, such that every term has the same dimensions, $(\text{length})^{-4}$, so that Eq. (28) is satisfied, the right-hand side of Eq. (30) vanishes because it is merely the Euler equation for a homogeneous function (all terms having the same dimensionality), and local invariance holds.

A massless scalar field has the Lagrangian $\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)$, and the quantity Θ^μ is

$$\Theta^\mu = x^\nu T_\nu^{\ \mu} + \frac{1}{2} g^{\mu\nu} \partial_\nu \phi^2 \ . \tag{32}$$

For a massless charged scalar field, we have

$$\Theta^\mu = x^\nu T_\nu^{\ \mu} + g^{\mu\nu} \partial_\nu (\phi^* \phi) \ ; \tag{33}$$

and for a massless Dirac field,

$$\Theta^\mu = x^\nu T_\nu^{\ \mu} \ . \tag{34}$$

As an illustration of some of the remarks in Section II, we can introduce a "truncated" scale transformation for the three above-mentioned fields--the transformation $x^\mu \to x^\mu e^\sigma$, while $\phi'(x') = \phi(x)$. Under this transformation, neither $\mathcal{L}(x)$ nor $dx\,\mathcal{L}(x)$ is invariant. Nevertheless, we have $\delta A = 0$ if the action integral is taken over all space and the fields are assumed to vanish in remote regions. Then one has the integral invariant $\int d^4x\, T_\mu^{\ \mu}$ for all three fields. However, for this transformation the condition $\mathcal{L}(x) = 0$ still leads to local invariance, and that is why the Dirac fields satisfies $\partial_\mu (x^\nu T_\nu^{\ \mu}) = T_\mu^{\ \mu} = 0$, whereas for the scalar fields an extra term must be added for a local conservation law to hold.

Finally let us calculate the scale transformation properties of a massive Dirac particle coupled to the e-m field, A_μ. The field equations are

$$i\,\gamma^{\mu}\,\partial_{\mu}\psi(x) - \kappa\,\psi(x) = \alpha\,a^{\mu}(x)\,\gamma_{\mu}\,\psi(x)\,,$$

$$\Box a^{\mu}(x) = -\,\overline{\psi}(x)\,\gamma^{\mu}\,\psi(x)\,, \tag{35}$$

where $a^{\mu} = A^{\mu}/e$ (e being the electronic charge), α is the fine-structure constant, and κ is the reciprocal Compton wavelength of the fermion. Under the transformation

$$x^{\mu} \to x^{\mu}\lambda \equiv x^{\mu\prime}\,,$$
$$\psi(x) \to \lambda^{3/2}\,\psi'(x')\,,$$
$$a^{\mu}(x) \to \lambda\,a^{\mu\prime}(x')\,,$$

these equations become

$$i\,\gamma^{\mu}\,\partial'_{\mu}\,\psi'(x') - (\kappa/\lambda)\,\psi'(x') = \alpha\,a^{\mu\prime}\,\gamma_{\mu}\,\psi'(x')$$

and

$$\Box'\,a^{\mu\prime}(x') = -\,\overline{\psi}'(x')\,\gamma^{\mu}\,\psi'(x')\,. \tag{36}$$

Thus we have the transformation laws

$$\lambda^{3/2}\,\psi'(x',\kappa) = \psi(x,\lambda\kappa)$$

$$\lambda\,a^{\mu\prime}(x',\kappa) = a^{\mu}(x,\lambda\kappa)\,, \tag{37}$$

which are more complicated than that of Eq. (15) because of the change in κ, which shows explicitly the external nature of the scale transformation for massive fields. In the next section we attempt to formulate a model theory that will overcome this defect.

IV. A SCALE-INVARIANT THEORY OF PARTICLES WITH MASS

We attempt to construct a scale-invariant theory by introducing an internal coordinate whose dimension is length. This allows us to formulate the theory in terms of dimensionless quantities by exploiting the qauge-like structure of the scale transformation.

To illustrate the procedure, we use the case of a massive Dirac free field. Equation (35) (without the e-m coupling) can be written in the form

$$i\,\gamma^{\mu}(\partial_{\mu}/\kappa)\,\kappa^{-3/2}\,\psi(x) - \kappa^{-3/2}\,\psi(x) = 0\ . \tag{38}$$

If we introduce the dimensionless quantities

$$x_1 \equiv x\kappa\ , \tag{39a}$$

and

$$\psi_1(x_1,\kappa) \equiv \kappa^{-3/2}\,\psi(x,\kappa)\ , \tag{39b}$$

Eq. (38) becomes

$$i\,\gamma^{\mu}\,\partial_{1\mu}\,\psi_1(x_1) - \psi_1(x_1) = 0\ . \tag{40}$$

Equation (40) is dimensionless, but it is dependent upon a particular Compton wavelength, namely $1/\kappa$. To be able to provide for a particle of any mass, we write

$$i\,\gamma^{\mu}\,\partial_{1\mu}\psi_1 - \mu\psi_1 = 0\ , \tag{41}$$

where μ is a dimensionless parameter actually representing the ration of the wavelength of the particle to the arbitrary original wavelength, $1/\kappa$. Thus when we vary κ in the theory, we actually vary the size of the unit of length, while the mass of the particle, μ, in dimensionless units, remains fixed.

Because it is dimensionless, the field of Eq. (39) no longer transforms under the scale transformation $x^\mu \to x^\mu e^\sigma$ according to Eq. (37), but rather as

$$\psi'_1(x', \kappa) = \psi_1(x, \kappa e^\sigma)$$

or

$$\psi'_1(x_1, \kappa e^{-\sigma}) = \psi_1(x_1, \kappa) . \tag{42}$$

Thus we can change the scale by making a variation in κ, leaving x_1 fixed, as well as by making a variation in x, for fixed κ. Alternatively, we note that the field is invariant under the simultaneous transformation $x^\mu \to x^\mu e^\sigma$, $\kappa \to \kappa e^{-\sigma}$.

In the rest of this section we use only the field $\psi_1(x_1)$ and subsequently we will drop the subscript 1. Equation (42) in infinitesimal form reads

$$\psi'(x, \kappa) = \psi(x, \kappa) + \delta\sigma\, \kappa \frac{\partial}{\partial \kappa} \psi(x, \kappa) ,$$

$$\psi'(x, \tau) = \psi(x, \tau) + \delta\sigma \frac{\partial \psi}{\partial \tau} (x, \tau) , \tag{43}$$

$$\bar{\delta}\psi = \delta\sigma \frac{\partial \psi}{\partial \tau} (x, \tau) ,$$

where we have introduced the notation $\tau = \ln \kappa$ for convenience, and the symbol $\bar{\delta}$ now refers to the change in the form of ψ at the point (x, τ) rather than merely at the point (x). The final step we take is to make an independent scale transformation at every point in space, and to cancel the effect of this transformation we introduce a vector field $b_\mu(x, \tau)$ by a slight extension of the method of Yang and Mills.[7,8]

We note that under the transformation

$$\kappa \to \kappa' = \kappa\, e^{-g\sigma(x)} ,$$
$$\tau \to \tau' = \tau - g\sigma(x) , \qquad (44)$$

where g is a dimensionless coupling constant, the function $\psi(x)$ is transformed as

$$\psi(x, \tau) \to \psi'(x, \tau') = \psi(x, \tau) . \qquad (45)$$

We introduce a new field $b_\mu(x, \tau)$, which transforms as

$$b_\mu(x, \tau) \to b'_\mu(x, \tau') = b(x, \tau) - \partial_\mu \sigma(x) , \qquad (46)$$

so that $b_\mu(x, \tau)$ undergoes a gauge transformation during the change of scale. The infinitesimal form of this equation is

$$\bar{\delta}\, b_\mu \equiv b'_\mu(x, \tau) - b_\mu(x, \tau) = g\, \delta\sigma \frac{\partial b_\mu}{\partial \tau}(x, \tau) - \partial_\mu \delta\sigma . \qquad (47)$$

Then to couple the field b_μ to the field ψ, we must make the replacement

$$\partial_\mu \psi \to (\partial_\mu - g b_\mu \frac{\partial}{\partial \tau})\psi . \qquad (48)$$

Thus we have

$$[\partial_\mu - gb'_\mu(\tau') \frac{\partial}{\partial\tau'}]\psi'(\tau') = \left\{\partial_\mu - g[b_\mu(\tau) - \partial_\mu\sigma] \frac{\partial}{\partial\tau'}\right\} \psi'(\tau')$$

$$= [\partial_\mu - gb_\mu \frac{\partial}{\partial\tau}]\psi(\tau) . \qquad (49)$$

Equation (49) follows from the fact that

$$\partial_\mu \psi'(x,\tau') = \bar{\partial}_\mu \psi'(x,\tau') - g(\partial_\mu\sigma) \frac{\partial\psi'}{\partial\tau'}(x,\tau') ,$$

$$\bar{\partial}_\mu \equiv (\partial_\mu)_{\tau' \text{ fixed}} , \qquad (50)$$

and the relation

$$\frac{\partial\psi'(\tau')}{\partial\tau'} = \frac{\partial\psi(\tau)}{\partial\tau} .$$

These operations are analogous to those used in introducing the e-m field,[9] where the equivalent relations are

$$\psi \to \psi e^{-ie\sigma(x)} ,$$

$$A_\mu \to A_\mu - \partial_\mu\sigma , \qquad (51)$$

and

$$\partial_\mu \to \partial_\mu - ieA_\mu .$$

In the e-m case the field A_μ obeys the subsidiary condition

$$\partial_\mu A^\mu = 0 , \qquad (52)$$

where $\Box\sigma = 0$, and the relevant gauge-invariant quantity is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{53}$$

For our b_μ field, the condition $\partial_\mu b^\mu = 0$ would give a condition on σ which would make σ a function of τ. However, we may choose the gauge-invariant condition

$$\partial_\mu b^\mu - g b_\mu \frac{\partial b^\mu}{\partial \tau} \equiv D_\mu b^\mu = 0 , \tag{54}$$

and now σ may be any function of $\overline{x}$ satisfying the conditions

$$\Box \sigma = 0 . \tag{55}$$

The gauge-invariant field tensor is

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu b_\nu - \partial_\nu b_\mu + g\left(b_\nu \frac{\partial b_\mu}{\partial \tau} - b_\mu \frac{\partial b_\nu}{\partial \tau}\right) \\ &= D_\mu b_\nu - D_\nu b_\mu . \end{aligned} \tag{56}$$

The quantities τ, ψ, $D_\mu b^\mu$, and $F_{\mu\nu}$ all transform as

$$f(x,\tau) \rightarrow f'(x,\tau') = f(x,\tau) . \tag{57}$$

The quantity b_μ acquires an extra gradient under the transformation; however, when properly coupled, via Eq. (48), it makes the Lagrangian transform according to Eq. (57). The action is given by

$$A = \int d^4x \int d\tau \, \mathcal{L}(x,\tau) , \tag{58}$$

and is invariant under the transformation. It should be mentioned that the range of the variable τ is from $-\infty$ to $+\infty$, which follows from its definition as $\ell n\ \kappa$ and from the fact that κ is intrinsically positive, both in its original role as a wavelength and in its new role as the size of the length unit. We place on all fields that depend upon τ the condition that they vanish sufficiently fast at $\tau = \pm\infty$. Note that an important difference between the b_μ field and the e-m field is that there is no i in the operator D_μ, so that it has the same form when operating on both ψ and $\overline{\psi}$.

The field equations are of the form

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu u)}\right] + \frac{\partial}{\partial\tau}\left[\frac{\partial \mathcal{L}}{\partial(\partial u/\partial\tau)}\right] - \frac{\partial \mathcal{L}}{\partial u} = 0, \quad (u = \psi, \overline{\psi}, b_\nu) . \tag{59}$$

For the free-field Lagrangian for b_μ we choose

$$\mathcal{L}_0 = -(1/4)F_{\mu\nu}\,F^{\mu\nu} , \tag{60}$$

which yields the field equations

$$D_\mu F^{\mu\nu} - 2g\,\frac{\partial b_\mu}{\partial\tau}\,F^{\mu\nu} = 0 . \tag{61}$$

(For gauge invariance the term b_μ must occur only in the operator D_μ, but $\partial b_\mu/\partial\tau$ may appear.) In the presence of a Dirac field the Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_0 - \frac{1}{2i}\,[\overline{\psi}\,\gamma^\mu\,D_\mu\psi + i\mu\overline{\psi}\psi] + \frac{1}{2i}\,[(D_\mu\overline{\psi})\gamma^\mu\psi - i\mu\overline{\psi}\psi] . \tag{62}$$

This Lagrangian leads to the equations

$$\gamma^\mu D_\mu \psi - \frac{g}{2} \frac{\partial b_\mu}{\partial \tau} \gamma^\mu \psi + i\mu\psi = 0 ,$$

$$D_\mu \bar{\psi}\gamma^\mu - \frac{g}{2} \frac{\partial b_\mu}{\partial \tau} \bar{\psi}\gamma^\mu - i\mu\bar{\psi} = 0 , \tag{63}$$

$$D_\mu F^{\mu\nu} - 2g \frac{\partial b_\mu}{\partial \tau} F^{\mu\nu} = gG^\nu ,$$

where

$$G^\nu = \frac{1}{2i} [\frac{\partial \bar{\psi}}{\partial \tau} \gamma^\nu \psi - \bar{\psi} \gamma^\nu \frac{\partial \psi}{\partial \tau}] . \tag{64}$$

The last equation of the Eqs. (63) leads to current conservation,

$$\partial_\nu \mathcal{J}^\nu = 0; \mathcal{J}^\nu = G^\nu + \frac{\partial}{\partial \tau} [b_\mu F^{\mu\nu}] + \frac{\partial b_\mu}{\partial \tau} F^{\mu\nu} . \tag{65}$$

The ordinary charge current,

$$j^\mu = \bar{\psi} \gamma^\mu \psi , \tag{66}$$

satisfies the equation

$$\partial_\mu j^\mu - g \frac{\partial}{\partial \tau} [b_\mu j^\mu] = 0 , \tag{67}$$

which leads to current conservation when integrated over theinternal variable τ:

$$\partial_\mu \int j^\mu d\tau = 0 . \tag{68}$$

Thus there are two conserved currents in the theory, $\int j^\mu d\tau$ and $\mathcal{J}^\mu$.

The latter is a function of (x^μ, τ), whereas the former is a function of the x^μ alone. The other matter "current," G^μ, satisfies the equation

$$\partial_\mu G^\mu - g \frac{\partial}{\partial\tau} [b_\mu G^\mu] - g \frac{\partial b_\mu}{\partial\tau} G^\mu = 0 , \qquad (69)$$

which is equivalent to Eq. (65), so that $\int G^\mu d\tau$ is conserved only to lowest order in g and thus plays the role of an "almost conserved" current in the theory. The quantity G^μ vanishes for a free Dirac field (i.e., one uncoupled to b_μ).

Energy conservation takes the form

$$\partial_\mu T_\nu^{\ \mu} + \frac{\partial T_\nu}{\partial\tau} = 0 , \qquad (70)$$

where

$$T_\nu = \sum_k \frac{\partial \mathcal{L}}{\partial(\partial u_k/\partial\tau)} \partial_\nu u_k , \qquad (71)$$

and $T_\nu^{\ \mu}$ is the usual stress tensor. Thus, the divergence of the energy-momentum tensor vanishes when integrated over the internal coordinate--that is,

$$\partial_\mu \int d\tau \, T_\nu^{\ \mu} = 0 . \qquad (72)$$

For use in the next section, we note that under the transformation $x^\mu \to x^\mu e^\sigma$, κ fixed, the variation in the action, Eq. (58), is given by

$$\begin{aligned} \delta A &= \int d^4x \int d\tau [\partial_\mu (x^\nu T_\nu^{\ \mu}) + \frac{\partial}{\partial\tau} (x^\nu T_\nu)] \delta\sigma \\ &= \int d^4x \int d\tau \, T_\mu^{\ \mu} \, \delta\sigma . \end{aligned} \qquad (73)$$

V. SCALING PROPERTIES OF THE THEORY

In order to consider the scaling properties of the theory it will be convenient for us to reintroduce the dimensional variables x and κ, remembering that the theory of Section IV was formulated for the field $\psi_1(x_1, \kappa)$, defined by Eq. (39). This theory was formulated to be invariant under the transformation

$$x^\mu \to x^\mu e^{g\sigma} = x^{\mu\prime} ,$$

$$\kappa \to \kappa e^{-g\sigma} = \kappa' , \tag{74}$$

where x and $1/\kappa$ both have the dimension of length, rather than invariant merely under the transformation $x^\mu \to x^\mu e^\lambda$, with κ fixed.

Equation (42) can be used to determine the transformation properties of $\psi_1(x_1, \kappa)$ under the transformation (74), as

$$\psi'_1(xe^{g\sigma} \kappa e^{-g\sigma}, \kappa e^{-g\sigma}) = \psi_1(x\kappa, \kappa) \tag{75}$$

or

$$\psi'_1(x_1, \kappa') = \psi_1(x_1, \kappa) . \tag{76}$$

Equation (75) says that only κ changes under the transformation (74) but not the dimensionless variable x_1. This of course is equivalent to Eq. (45), the starting point of the theory (remember that in the last section, "x" stands for "x_1").

We shall now compute what it is that is conserved under the transformation (74). To do so, we must be aware that the transformation of the fields, Eq. (47), is not actually a gauge transformation, but rather a combination of a gauge and a coordinate transformation, and it will be important to take into account the variation of the boundary of the region of integration. Since the boundary does vary, and also because the field b_μ multiplies a derivative $\frac{\partial\psi}{\partial\tau}$ of the field ψ, the formalism of Utiyama[5] must be slightly extended, but does give the correct result.

We note that the derivatives of b_μ appear only in the combination $F_{\mu\nu}$, which transforms as Eq. (57), so that for $\tau \to \tau - g\delta\sigma$ we may write

$$\bar{\delta}\, F_{\mu\nu} = g\,\delta\sigma \frac{\partial F_{\mu\nu}}{\partial\tau} . \tag{77}$$

The only other appearance of the field b_μ is in the operator D_μ, so the explicit variation $\bar{\delta} b_\mu$ refers only to those terms of the Lagrangian which couple b_μ to the fields, ψ, $\bar{\psi}$. Finally we use the symbol u to collectively represent ψ and $\bar{\psi}$, and any term in δ of the form f(u) is to be interpreted as $f(\psi) + f(\bar{\psi})$.

We now list all relevant variations that are induced by the transformation (74):

$$\begin{aligned}
&\delta x_1 = 0 ,\\
&\delta\tau = -g\,\delta\sigma ,\\
&\bar{\delta} u = \frac{\partial u}{\partial\tau}\, g\,\delta\sigma ,\\
&\bar{\delta}\partial_\mu u = (\partial_\mu \frac{\partial u}{\partial\tau})\, g\,\delta\sigma - \frac{\partial u}{\partial\tau}\, g\,\delta(\partial_\mu\sigma) ,\\
&\bar{\delta}\, \frac{\partial u}{\partial\tau} = \frac{\partial^2 u}{\partial\tau^2}\, g\,\delta\sigma ,\\
&\bar{\delta}\, b_\mu = \frac{\partial b_u}{\partial\tau}\, g\,\delta\sigma - \delta(\partial_\mu\sigma) ,\\
&\bar{\delta}\, F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial\tau}\, g\,\delta\sigma .
\end{aligned} \tag{78}$$

In these equations we have used the fact that the operators δ and ∂_μ commute. All the fields listed undergo coordinate transformations (i.e., $\tau \to \tau'$), and their variations are given by Eq. (5). However, the fields b_μ and $\partial_\mu u$ also undergo gauge transformations. The equation for $\bar{\delta}\, b_\mu$ is just Eq. (47). The gauge part of the transformation of $\partial_\mu u$ is given by Eq. (50),

$$\delta(\partial_\mu u) \equiv \partial_\mu u'(\tau') - \partial_\mu u(\tau) = -g \frac{\partial u}{\partial \tau} \delta(\partial_\mu \sigma) ,$$

and the variation $\bar{\delta}(\partial_\mu u)$ follows from Eq. (5).

If now the Lagrangian is considered to be a function of all the parameters on the left-hand sides of Eqs. (78), we may write

$$\delta \mathcal{L} = g\, \delta\sigma \left[-\frac{\partial \mathcal{L}}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial u} \frac{\partial u}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)} \partial_\mu \left(\frac{\partial u}{\partial \tau}\right) + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial \tau}\right)} \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial \mathcal{L}}{\partial b_\mu} \frac{\partial b_\mu}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} \frac{\partial F^{\mu\nu}}{\partial \tau} \right]$$

$$- \delta(\partial_\mu \sigma) \left[g \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)} \frac{\partial u}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial b_\mu} \right] + \mathcal{L} \frac{\partial(\delta\tau)}{\partial \tau} . \qquad (79)$$

In Eq. (79), the variations $\delta\sigma$ and $\delta(\partial_\mu \sigma)$ are assumed to be taken independently. Then, as was promised in Section II, the coefficient of $\delta(\partial_\mu \sigma)$ vanishes automatically if the field b_μ is coupled to $\psi, \bar{\psi}$ via Eq. (48). The last term is the contribution from the variation of the boundary,[3] but vanishes because $\delta\sigma$ is a function of the x^μ alone, which shows the importance of taking the subsidiary condition in the form of Eq. (54), as was explained previously. If we use the equation of motion, Eq. (59), to replace the term $\partial\mathcal{L}/\partial u$, we arrive at

$$\delta \mathcal{L} = \delta\sigma \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial_\mu u} \frac{\partial u}{\partial \tau} \right) + \frac{\partial}{\partial \tau} \left(\frac{\partial \mathcal{L}}{\partial(\partial u/\partial \tau)} \right) + \frac{\partial \mathcal{L}}{\partial b_\mu} \frac{\partial b_\mu}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \frac{\partial F_{\mu\nu}}{\partial \tau} - \frac{\partial \mathcal{L}}{\partial \tau} \right] . \tag{80}$$

Inserting the explicit form of the Lagrangian, Eq. (62), into Eq. (80), we find

$$\delta \mathcal{L} = \delta\sigma \left[\partial_\mu G^\mu - g \frac{\partial}{\partial \tau} (b_\mu G^\mu) - g\, G^\mu \frac{\partial b_\mu}{\partial \tau} - \frac{1}{2} F_{\mu\nu} \frac{\partial F^{\mu\nu}}{\partial \tau} - \frac{\partial \mathcal{L}}{\partial \tau} \right] . \tag{81}$$

It happens that the terms in $\psi, \bar{\psi}$ in the Lagrangian vanish because of the equations of motion, so that the last two terms of Eq. (81) cancel. Then the condition $\delta \mathcal{L} = 0$ gives exactly Eq. (69), which is equivalent to Eq. (65), $\partial_\mu \mathcal{J}^\mu = 0$. Thus we see that conservation of the current $\mathcal{J}^\mu$ arises by virtue of the scale invariance of the theory.

Finally, we should like to point out that the introduction of an internal coordinate κ is exactly analogous to the introduction of the internal coordinate spin in the case of angular momentum. In that case, one no longer requires the transformation law

$$\bar{\delta}\phi = \delta\underset{\sim}{\omega} \cdot \underset{\sim}{L}\phi , \tag{82}$$

where the operator $\underset{\sim}{L}$ transforms the spatial coordinates, but requires rather that

$$\bar{\delta}\phi_{(k)} = \delta\underset{\sim}{\omega} \cdot \underset{\sim}{L}\phi_{(k)} + \delta\underset{\sim}{\omega} \cdot \underset{\sim}{S}\phi_{(k)} , \tag{83}$$

where the operator $\underset{\sim}{S}$ is independent of x but rearranges the internal coordinates (k). One then derives the conservation law

$$\partial_\mu M^{\mu\nu\lambda} = \partial_\mu \left(T^{\mu\nu} x^\lambda - T^{\mu\lambda} x^\nu \right) + \partial_\mu S^{\mu\nu\lambda} = 0 , \tag{84}$$

where the term in parentheses comes from the operator $\underset{\sim}{L}$, and angular-momentum conservation comes about only by virtue of the combined operation $\underset{\sim}{J} = \underset{\sim}{L} + \underset{\sim}{S}$.

In the case of the scale transformation, we have the transformation law (74) that states

$$\bar{\delta}\psi_1 = \delta\sigma\, x^\mu \partial_\mu \psi_1 - \delta\sigma \frac{\partial}{\partial\tau} \psi_1 , \tag{85}$$

analogously to Eq. (81). Then Eq. (85) corresponds to a more general scale transformation because of the internal coordinate. The first term corresponds to the "orbital" part of the transformation, and the second term corresponds to a rearrangement of the internal coordinates. Now the coordinate x occurs only in the dimensionless form x_1, and Eq. (85) can be broken into the form

$$\bar{\delta}\psi_1 = \kappa\, \delta x^\mu (\partial/\partial x_1{}^\mu)\psi_1 + x_1{}^\mu (\partial/\partial x_1{}^\mu)\psi_1\, \delta\tau - \delta\sigma \frac{\partial}{\partial\tau} \psi_1 . \tag{86}$$

The variation under a scale transformation in x_1 is given by Eq. (73), so that we have

$$\partial_\mu \Theta^\mu = T_\mu{}^\mu(x_1)\, \delta\sigma + \left[\partial_\mu \mathcal{J}^\mu - T_\mu{}^\mu(x_1) \right] \delta\sigma . \tag{87}$$

In Eq. (87) the first term comes from the variation in x, and the second term from the variation in κ (or τ). Thus, in exactly the same way as the total J determines angular-momentum conservation, the "total scale transformation" (i.e., variation of all dimensional coordinates x^μ and κ) determines conservation of $\mathcal{J}^\mu$.

Although there is no evidence that nature has any use for the specific theory developed above, our main point is that the trivial appearance of the scale transformation in physics is illusory, and that it can be made into a powerful and, ultimately, perhaps, even useful tool.

We would like to thank Dr. Albert Finn for reading portions of the manuscript, and Professor Geoffrey F. Chew and Dr. David L. Judd for the hospitality extended by the Lawrence Radiation Laboratory.

FOOTNOTES AND REFERENCES

*Work performed under the auspices of the U. S. Atomic Energy Commission during the tenure of a National Science Foundation Post-Doctoral Fellowship.

1. A good summary of the usual treatments of the scale transformation can be found in P. Löwdin, J. Mol. Spectr. 3, 46 (1959), which also contains a good bibliography on the subject.

2. The remarks on "external" invariance contained in this section are presumably known to everyone; however, they are particularly relevant to the scale transformation and so are stated explicitly.

3. See, for example, N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields, (interscience Publishers, Inc., New York, 1959), Sec. 2.5.

4. A discussion of such points is contained in the book, W. Thirring, Principles of Quantum Electrodynamics (Academic Press, Inc., New York, 1958).

5. See R. Utiyama, Phys. Rev. 101, 1597 (1956).

6. For particles in finite enclosures, the transformation of the boundaries is very important. However, we are considering only the case in which $\psi(x)$, $\nabla\psi(x) \rightarrow 0$ at the boundaries.

7. The method was introduced by C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1956), for the case of isotopic spin, and extended to general Lie groups by R. Utiyama, reference 5. The individual Lie groups are discussed by S. L. Glashow and M. Gell-Mann, Ann. Phys. 15, 437 (1961). The present case is for a continuous parameter, τ, upon which the new field b_μ may depend.

8. The method may be applied for any continuous parameter and does not depend on the scale transformation.

9. However, there is a rather subtle conceptual difference, which is discussed in the next section.