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On differentiating maps induced by functional calculus and applications to free stochastic calculus

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On differentiating maps induced by functional calculus and applications to free stochastic calculus

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

Mathematics

by

Evangelos A. Nikitopoulos

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Professor Ioan Bejenaru
Professor Adrian Ioana
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2024

The Dissertation of Evangelos A. Nikitopoulos is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2024

DEDICATION

To my wife, the love of my life, Eva Loeser;

my parents, Alison and Dimitris Nikitopoulos;

my brother, Anthony Nikitopoulos; and

the memory of my friend, Emily Zhu

EPIGRAPH

If you find a new function,
you should differentiate it.

Mantra of Bruce K. Driver

You can't always get what you want.
You can't always get what you want.
You can't always get what you want.
But if you try sometimes,
well, you might find
you get what you need.

The Rolling Stones

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PREFACE

I was led to the problem of differentiating maps induced by functional calculus by an Itô-type formula in free stochastic calculus—namely, [BS98, Prop. 4.3.4]—the reinterpretation and extension of which ended up being the subject of [Nik22] and thus Chapter 7. Before I started graduate school, my advisors and their coauthor, Brian Hall, had tried to use this formula for some of their calculations in (the first version of) [DHK22]. Ultimately, they found the formula too computationally inflexible and resorted to a power series–based argument using the polynomial version of the formula, [BS98, Prop. 4.3.2]. Unsatisfied with this, they asked me to look into it. Eventually—with the help of research notes from one of my advisors, Bruce Driver, and some discussions with Adrian Ioana—I was led to the vast and rich literature on multiple operator integrals (MOIs), surveyed helpfully in [ST19]. After learning about MOIs and their applications, e.g., to differentiating maps induced by functional calculus, I was able to make the key connection: The terms in the formula [BS98, Prop. 4.3.4] are MOIs and have much more computational flexibility than it seems at first glance.

The rest is history. Well, actually, the rest is this dissertation. It is based primarily on the papers [Nik22, Nik23a, Nik23b, Nik23c] and upgrades to the results therein. The main upgrades are (1) a generalization of many of the main results in [Nik23c] to symmetrically normed ideals of unital C^*-algebras via the introduction and study of "Varopoulos C^k functions" in Chapter 3 and (2) a more streamlined proof of [Nik23b, Cor. 4.2.11 & Thm. 4.2.12] (i.e., Corollary 5.6.10 and Theorem 5.6.11) via Theorem 5.2.7. To motivate and, in some sense, complete the story, I also included a chapter on differentiating maps induced by the holomorphic functional calculus.

A word on formatting. There are two formatting quirks of which the reader should take note. The first is the numbering scheme. It is common for (labeled) display relations to be numbered independently from definitions, lemmas, propositions, theorems, etc. In this dissertation, however, the numbering scheme includes relations. For example, the first three numbered items in §3.3 are a definition, a relation, and a proposition; they are labeled Definition 3.3.1, (3.3.2), and Proposition 3.3.3. Second, the standing assumptions for each chapter are declared right at the beginning in "Standing assumptions" environments. When in doubt about, e.g., "what H is," please check the beginning of the chapter.

ACKNOWLEDGMENTS

I have many people to thank for supporting me throughout graduate school. First and foremost is my loving and supportive wife, Eva Loeser. Thank you for being my partner in crime. Thank you for being my travel and adventure buddy. Thank you for pushing me to be a better version of myself. Thank you, Eva, for everything. You're brilliant, intuitive, kind, patient, persistent, and endlessly enthusiastic—you're my ray of sunshine. I love you. I am so lucky to be married to you.

Second are my advisors, Bruce Driver and Todd Kemp, whose mentorship has impacted me tremendously. Thank you both for all you've taught me, for always being so generous with your time, for all your indispensable guidance, and for facilitating my transition from student to independent researcher with so much care. You texted, called, or met with me whenever I asked. You expertly balanced giving me advice and encouraging me to make my own decisions. You have always done right by me, and I will always be grateful.

Bruce, you've generously provided me with copious writings—expository notes, research notes, written feedback, etc.—cajoling me in the right direction. You've taught me to reflect after proving new results. You have completely transformed my concept of what it means to understand (after repeatedly toppling my "understanding" of various things with simple, seemingly innocuous questions). Perhaps most importantly, you've inducted me into the cults of the Daniell integral and the microfiber cloth eraser.

Todd, you have been the source of ludicrously many amazing opportunities. Since essentially "Day 1," you've been setting me up for success in ways both large—arranging me transformative, multi-month research visits to Toulouse and Saarbrücken—and small—asking me just a few weeks into my first quarter if I wanted to sub for your graduate course on smooth manifolds. You've also always been there to refine a proposal or application draft with me, which undoubtedly helped me get my NSF Graduate Research Fellowship and postdoc offers. Finally, you have taught me the invaluable skill of managing a research *program*, as opposed to a single research project.

Bruce and Todd, thank you both once again. I look forward to many more years of learning from and working with you.

Third, I am grateful to Michael Hartz, Adrian Ioana, David Jekel, Junekey Jeon, Ed McDonald, Jacob Sterbenz, and the anonymous referee of [Nik22] for a variety of particularly helpful comments and conversations. Michael helped me understand the multivariate holomorphic functional calculus and brought [Cur88] to my attention, thereby aiding the development in Chapter 2. When I was in the early stages of understanding the free Itô formula in [BS98], Adrian suggested I look up Anna Skripka for information on multiple operator integrals (MOIs); this led me to find [ST19], which became an essential resource for my research. David brought to my attention his parallel work on his space $C^k_{\mathrm{nc}}(\mathbb{R})$ of "noncommutative C^k functions" in his dissertation, [Jek20]. I've spent untold hours discussing vector-valued integrals with Junekey. In addition to sharpening my understanding, these discussions helped improve §A.1 significantly. Junekey also brought Lemma 6.2.9 to my attention. The first preprint of [Nik23c] worked only with the C^*-algebra $\mathcal{A} = B(H)$ but included a full treatment of the "separation of variables" approach to defining MOIs in $B(H)$ with H not necessarily separable. Ed informed me the latter was of independent interest and encouraged me to publish it on its own. This led me to write the paper [Nik23b] and allowed me to make room for the general C^*-algebra versions of the results in [Nik23c]. Ed also brought [DDSZ20] to my attention, which was helpful for the research in [Nik23a]. Jacob suggested the function κ from §3.7 as a compactly supported Hölder continuous function with non-integrable Fourier transform. Finally, the anonymous referee's careful review of [Nik22] led to improvements to both [Nik22] and [Nik23c].

Fourth, I have my family to thank. Mom and Μπαμπά, thank you και ευχαριστώ for always being there for me, for fostering my independence and curiosity, και που μου έμαθες ελληνικά. (Συγνώμη, αυτό το καταραμένο πρόγραμμα δεν γράφει τόσο καλά στα ελληνικά.) You're the best parents I could ever ask for. Anthony, thanks for keeping me honest and being an inspiration. You're an exceptional brother, and I'm so proud of the person you've grown up to be. Amy and Doug, Mom- and Dad-in-Law, thank you both for welcoming me with open arms into your family and for being there for us during graduate school. I love you all so much.

Last but not least are my friends. They say you are who you hang out with. I hope that's true because I've spent the last several years around some wonderful people. Here are just a few specific shoutouts: to Itai Maimon, for countless hours at the gym, ethics debates, and heart-to-hearts; to TY Mathers, for (horror) movies, Popeyes, and initiating my stint in climbing; to Isaac Phillips (aka Farm Isaac), for going above and beyond as my best man; to Brandon Zborowski, for being an extraordinary officiant at my and Eva's wedding; to Izak Oltman (aka Running Izak), for your intensity, wacky adventures, and engaging conversations; to Varun Khurana, for your boundless spirit and *hilarious* impression of me; to Laura and Matt Crum (and Mara, Lita, Bowie, and Binx Crum) for cocktail and movie nights, camping trips, and cuddle puddles; to Cathy Wahlenmayer (aka Queen of The Nest) and Nate Conlon, for being The Great Gatherers of the old guard and new blood, respectively; to Ryan Mike and Davide Provasoli, for being excellent hosts and all-around phenomenal people; to Somak Maitra (aka SoMacDaddyO), for being one of the most helpful people I have ever known; to Soumya Ganguly, for touching my and Eva's hearts with your celebration of our wedding; to Sawyer Robertson, for that El Cajon Mountain hike and way too many absurdly long phone calls; to Andrés Rodríguez Rey, for so, so many kind words and for buying me a coffee one day (maybe); to Chineze Christopher, for teaching me to make doro wat; to Dave DePew, for Grinder Gym, Thirsty Thursdays, and initiating my stint in armwrestling; to Alberto Dayan, for hospitably taking me and Eva all over Saarbrücken; to Jeff, for 47 pears; and to The Campfire Crew, for the pilgrimage of a lifetime.

One more friend to whom I am indebted is Emily Zhu, who tragically passed away in the summer of 2023. Together with a few other math grad students, Eva included, I worked with Emily for many hours a week for more than two years on the Mathematics Graduate Student Council (MGSC) and the Association for Women and Mathematics (AWM). Throughout that time and thereafter, Emily became a close friend I cherished enormously. To honor her memory, I'll share three things that come to mind when I think of her.

First of all, I think of birds. It is no secret to anyone who knew her well that she loved birds and, more generally, that she revered and celebrated all that is cute or beautiful. This oh-so-pure reverence, which she shared generously with those around her, has given me a precious gift: the habit of "stopping to smell the flowers" figuratively, "stopping to watch the birds" literally. Ever since a few months after meeting her, she has popped into my head every single time I see a bird, thereby forcing me to pause and appreciate it.

Second, I think of Excel graphs. The reason traces back to the first proposal I worked on with Emily for MGSC, shortly after the start of the pandemic. Before then, MGSC reports and proposals contained charts and graphs auto-generated by Google Forms. They looked bad. Since I thought this could make us seem less professional, I proposed that we make our own charts and graphs. So I spent hours clumsily making Excel generate them automatically instead of Google Forms. They still looked bad. Emily would never have put it so indelicately, so she took a look at them and said something like, “They look good, but I think I might be able to make them even better.” By the next meeting, she had designed a custom style for the charts, complete with a meticulously chosen color palette and font. They looked beautiful. This little anecdote highlights two things I admired about Emily: her exceptional artistic sense and her prodigious competency in everything she did. These facets of Emily will always inspire me to do my best.

Third, I think of the online messaging platform Discord, to which Emily introduced me—again, shortly after the start of the pandemic—when she proposed to MGSC that we should make a Discord for the math department as a way to keep our community connected when we were forced to be physically isolated from each other. Most of those involved in MGSC at the time responded lukewarmly to the suggestion. Some of us, like me, were troglodytes when it came to technology and were suspicious of the change. But Emily felt strongly about it, so she calmly, gently, level-headedly persuaded us to give it a go. And the thing is, when Emily felt strongly about something, she was usually right. Right she was! She gave the math department the gift of being able to stay connected when we needed it most. Her push for and creation of the math department Discord exemplified not only her deep care for her community but also her distinct way of helping people be more open-minded.

I have learned a lot from Emily, and I am a better person as a result. For that, I am eternally grateful.

This concludes the portion of my acknowledgments coming from my own heart and mind. Below is the more legalistic portion, so to speak. While this portion is valuable, I cannot take credit for the language.

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Chapter 3, in part, is a reprint of the material as it appears in "Noncommutative C^k functions and Fréchet derivatives of operator functions" (2023). Nikitopoulos, Evangelos A. *Expositiones Mathematicae*, 41, 115–163. Also, Chapter 3, in part, is being prepared for submission for publication of the material. Nikitopoulos, Evangelos A.

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Chapter 6, in part, is a reprint of the material as it appears in "Higher derivatives of operator functions in ideals of von Neumann algebras" (2023). Nikitopoulos, Evangelos A. *Journal of Mathematical Analysis and Applications*, 519, 126705.

Chapter 7, in part, is a reprint of the material as it appears in "Itô's formula for noncommutative C^2 functions of free Itô processes" (2022). Nikitopoulos, Evangelos A. *Documenta Mathematica*, 27, 1447–1507.

Appendix B, in part, is a reprint of the material as it appears in "Higher derivatives of operator functions in ideals of von Neumann algebras" (2023). Nikitopoulos, Evangelos A. *Journal of Mathematical Analysis and Applications*, 519, 126705.

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PUBLICATIONS

E. A. Nikitopoulos, *Multiple operator integrals in non-separable von Neumann algebras*, Journal of Operator Theory **89** (2023), 361–427.

E. A. Nikitopoulos, *Noncommutative C^k functions and Fréchet derivatives of operator functions*, Expositiones Mathematicae **41** (2023), 115–163.

E. A. Nikitopoulos, *Higher derivatives of operator functions in ideals of von Neumann algebras*, Journal of Mathematical Analysis and Applications **519** (2023), 126705.

E. A. Nikitopoulos, *Itô's formula for noncommutative C^2 functions of free Itô processes*, Documenta Mathematica **27** (2022), 1447–1507, Erratum: Documenta Mathematica **28** (2023), 1275–1277.

ABSTRACT OF THE DISSERTATION

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A combination of the method of perturbation formulas and polynomial approximation is employed to compute the k^{th} derivatives of (1) maps on symmetrically normed ideals of a unital Banach algebra induced by a holomorphic function, (2) maps on the self-adjoint elements of symmetrically normed ideals of a unital C^*-algebra induced by functions of a real variable that are "slightly better than C^k," and (3) maps on the self-adjoint elements of integral symmetrically normed ideals of a von Neumann algebra induced by functions of a real variable that are "slightly better than C^k and Lipschitz." Along the way, the "separation of variables" approach to defining multiple operator integrals on non-separable Hilbert spaces is developed. As an application to free probability, a free Itô formula of P. Biane and R. Speicher is extended, reinterpreted, and made more computationally flexible.

Introduction

If A is a (possibly unbounded) linear operator on a Banach space or element of a Banach algebra, a **functional calculus** for A is a "well-behaved" map Π_A from a collection $\mathcal{F}_A$ of scalar functions to the space of operators or Banach algebra to which A belongs. The point of Π_A is to provide a sensible definition of $f(A)$ for $f \in \mathcal{F}_A$, so "well-behaved" usually entails a combination of algebraic properties, e.g., linearity and multiplicativity, and analytic properties, e.g., continuity. The functional calculi of interest in this dissertation are as follows:

- the holomorphic functional calculus for an element of a unital Banach algebra (§2.1),
- the continuous functional calculus for a self-adjoint—more generally, normal—element of a unital C^*-algebra (§3.2), and
- the Borel functional calculus for an unbounded self-adjoint—more generally, normal—operator on a complex Hilbert space affiliated with a von Neumann algebra (§4.2).

For all these functional calculi Π_A, the dependence of $f(A) = \Pi_A(f)$ on f is an elementary, even definitional, matter. However, the dependence of $f(A)$ on A is sometimes difficult to analyze. For example, relating the smoothness properties of $A \mapsto f(A)$ to those of f can be a delicate matter.

As a warm-up, let us consider the holomorphic case, where no significant difficulties arise. Let $\mathcal{B}$ be a unital Banach algebra, let $U \subseteq \mathbb{C}$ be an open set, and let $\mathcal{B}_U$ be the set of $a \in \mathcal{B}$ such that the spectrum $\sigma(a)$ of a is contained in U. If $a \in \mathcal{B}_U$, then the **holomorphic functional calculus** for a is the unique continuous, unital algebra homomorphism $H_a^U \colon \operatorname{Hol}(U) \to \mathcal{B}$ sending the inclusion $\iota_U \colon U \hookrightarrow \mathbb{C}$ to a (§2.1). The standard construction of H_a^U is via a Cauchy-type integral formula:

$$f(a) \coloneqq H_a^U(f) = \frac{1}{2\pi i}\int_\Gamma f(z)\,(z-a)^{-1}\,\mathrm{d}z \in \mathcal{B}, \qquad f \in \operatorname{Hol}(U),$$

where Γ is any cycle surrounding $\sigma(a)$ in U. Now, for $f \in \mathrm{Hol}(U)$, write $f_{\mathcal{B}} \colon \mathcal{B}_U \to \mathcal{B}$ for the map $a \mapsto f(a)$ induced by f via this holomorphic functional calculus. We claim $f_{\mathcal{B}} \colon \mathcal{B}_U \to \mathcal{B}$ is holomorphic. Indeed, we can differentiate the definition of $f(a)$. If $a \in \mathcal{B}_U$ and $b \in \mathcal{B}$, then

$$\begin{aligned}\partial_b f_{\mathcal{B}}(a) &= \frac{1}{2\pi i}\partial_b \int_\Gamma f(z)\,(z-a)^{-1}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\Gamma f(z)\,\partial_b(z-a)^{-1}\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\Gamma f(z)\,(z-a)^{-1}b\,(z-a)^{-1}\,\mathrm{d}z,\end{aligned}$$

where ∂_b denotes differentiation in direction b. (The technical details are unimportant for the present discussion.) Differentiating under the integral in this way repeatedly yields the following.

Theorem 1 (Holomorphic case)**.** *If $f \in \mathrm{Hol}(U)$, then $f_{\mathcal{B}} \in \mathrm{Hol}(\mathcal{B}_U;\mathcal{B})$, and*

$$\partial_{b_k}\cdots\partial_{b_1} f_{\mathcal{B}}(a) = \frac{1}{2\pi i}\sum_{\pi\in S_k}\int_\Gamma f(z)\,(z-a)^{-1}b_{\pi(1)}\cdots(z-a)^{-1}b_{\pi(k)}\,(z-a)^{-1}\,\mathrm{d}z, \qquad a \in \mathcal{B}_U,\ b_j \in \mathcal{B},$$

where S_k is the symmetric group on k letters.

The k^{th} derivative formula above is worth pondering. To this end, we introduce more notation. Write $\#_k \colon \mathcal{B}^{\hat{\otimes}_\pi(k+1)} \to B_k\big(\mathcal{B}^k;\mathcal{B}\big)$ for the bounded linear map, written $u\#_k b \coloneqq \#_k(u)[b]$, determined by

$$(a_1\otimes\cdots\otimes a_{k+1})\#_k[b_1,\ldots,b_k] = a_1b_1\cdots a_kb_ka_{k+1}, \qquad a_i, b_j \in \mathcal{B}.$$

Here, $\hat{\otimes}_\pi$ is the projective tensor product of Banach spaces, and $B_k\big(\mathcal{B}^k;\mathcal{B}\big)$ is the space of bounded k-linear maps $\mathcal{B}^k \to \mathcal{B}$. Now, fix $a \in \mathcal{B}_U$, and write $\tilde{a}_i \coloneqq 1^{\otimes(i-1)}\otimes a\otimes 1^{\otimes(k+1-i)} \in \mathcal{B}^{\hat{\otimes}_\pi(k+1)}$ for all $i \in \{1,\ldots,k+1\}$. In this notation, Theorem 1 says

$$\begin{aligned}\partial_{b_k}\cdots\partial_{b_1} f_{\mathcal{B}}(a) &= \sum_{\pi\in S_k}\left(\frac{1}{2\pi i}\int_\Gamma f(z)\,(z-a)^{-1}\otimes\cdots\otimes(z-a)^{-1}\,\mathrm{d}z\right)\#_k\big[b_{\pi(1)},\ldots,b_{\pi(k)}\big] \\ &= \sum_{\pi\in S_k}\left(\frac{1}{2\pi i}\int_\Gamma f(z)\,(z-\tilde{a}_1)^{-1}\cdots(z-\tilde{a}_{k+1})^{-1}\,\mathrm{d}z\right)\#_k\big[b_{\pi(1)},\ldots,b_{\pi(k)}\big]\end{aligned}$$

for all $b_1,\ldots,b_k \in \mathcal{B}$. Next, we explain how to write $(2\pi i)^{-1}\int_\Gamma f(z)\,(z-\tilde{a}_1)^{-1}\cdots(z-\tilde{a}_{k+1})^{-1}\,\mathrm{d}z$ in terms of a multivariate version of the holomorphic functional calculus (§2.4).

For $f \in \mathrm{Hol}(U)$ and $k \in \mathbb{N}_0$, define

$$f^{[k]}(\boldsymbol{\lambda}) \coloneqq \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\lambda_1)\cdots(z-\lambda_{k+1})}\,\mathrm{d}z, \qquad \boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_{k+1}) \in U^{k+1}.$$

(Strictly speaking, the right-hand side is not defined for all $\boldsymbol{\lambda} \in U^{k+1}$. We sweep this under the rug for now; please see §1.3 for a proper treatment.) Then $f^{[k]}$ is a holomorphic function of $k+1$ variables—i.e., $f^{[k]} \in \mathrm{Hol}\left(U^{k+1}\right)$—that is characterized by the following properties:

- $f^{[0]} = f$ (by Cauchy's integral formula); and
- if $k \in \mathbb{N}$ and $\lambda_1,\dots,\lambda_{k+1} \in U$ are distinct, then

$$f^{[k]}(\lambda_1,\dots,\lambda_{k+1}) = \frac{f^{[k-1]}(\lambda_1,\dots,\lambda_k) - f^{[k-1]}(\lambda_1,\dots,\lambda_{k-1},\lambda_{k+1})}{\lambda_k - \lambda_{k+1}}. \tag{2}$$

We shall take this recursion to be the *definition* of $f^{[k]}(\lambda_1,\dots,\lambda_{k+1})$ whenever f is an arbitrary function defined on a subset of $\mathbb{C}$. More precisely, if $S \subseteq \mathbb{C}$ and $f\colon S \to \mathbb{C}$ is any function, then $f^{[0]} \coloneqq f$, and $f^{[k]}$ is defined recursively by Equation (2). The function $f^{[k]}$ is called the $\boldsymbol{k}^{\mathbf{th}}$ **divided difference** of f.

Theorem 3. *Suppose $f \in \mathrm{Hol}(U)$, $\mathbf{a} = (a_1,\dots,a_{k+1}) \in \mathcal{B}_U^{k+1}$, and $\tilde{a}_i \coloneqq 1^{\otimes(i-1)} \otimes a_i \otimes 1^{\otimes(k+1-i)}$ for all $i \in \{1,\dots,k+1\}$. If we define*

$$f_\otimes^{[k]}(\mathbf{a}) \coloneqq f^{[k]}(\tilde{a}_1,\dots,\tilde{a}_{k+1}) \in \mathcal{B}^{\hat{\otimes}_\pi(k+1)}$$

via the multivariate holomorphic functional calculus (§2.4), which makes sense because $\mathcal{B}^{\hat{\otimes}_\pi(k+1)}$ is a unital Banach algebra and $[\tilde{a}_i,\tilde{a}_j] = 0$ for all $i,j \in \{1,\dots,k+1\}$, then

$$f_\otimes^{[k]}(\mathbf{a}) = \frac{1}{2\pi i}\int_\Gamma f(z)\,(z-\tilde{a}_1)^{-1}\cdots(z-\tilde{a}_{k+1})^{-1}\,\mathrm{d}z.$$

In particular, by Theorem 1,

$$\partial_{b_k}\cdots\partial_{b_1} f_{\mathcal{B}}(a) = \sum_{\pi \in S_k} f_\otimes^{[k]}(\underbrace{a,\dots,a}_{k+1\text{ times}})\#_k\left[b_{\pi(1)},\dots,b_{\pi(k)}\right], \qquad a \in \mathcal{B}_U,\ b_j \in \mathcal{B}.$$

Generalizations of Theorems 1 and 3 are proven in Chapter 2.

Next, we move to the "real C^k case." Let $\mathcal{A}$ be a unital C^*-algebra, and write $\mathcal{A}_{\text{sa}}$ for the real Banach space of self-adjoint elements of $\mathcal{A}$, i.e., $\mathcal{A}_{\text{sa}} = \{a \in \mathcal{A} : a^* = a\}$. If $a \in \mathcal{A}_{\text{sa}}$, then the **continuous functional calculus** for a is the unique (isometric) unital $*$-homomorphism $\Phi_a \colon C(\sigma(a)) \to \mathcal{A}$ such that $\Phi_a\big(\iota_{\sigma(a)}\big) = a$ (§3.2). As in the holomorphic case, we write $f(a) \coloneqq \Phi_a(f) \in \mathcal{A}$ for all $f \in C(\sigma(a))$.

Now, if $f \colon \mathbb{R} \to \mathbb{C}$ is a continuous function, then we write $f_{\mathcal{A}} \colon \mathcal{A}_{\text{sa}} \to \mathcal{A}$ for the map $a \mapsto f(a) = \big(f|_{\sigma(a)}\big)(a)$ induced by f via the continuous functional calculus. It is elementary to prove that $f_{\mathcal{A}} \colon \mathcal{A}_{\text{sa}} \to \mathcal{A}$ is continuous. Indeed, if $f(\lambda) = \sum_{i=0}^n c_i\, \lambda^i \in \mathbb{C}[\lambda]$ is a polynomial, then $f_{\mathcal{A}}(a) = \sum_{i=0}^n c_i\, a^i$, so the conclusion is obvious. If $f \in C(\mathbb{R})$ is arbitrary, then Weierstrass's approximation theorem provides a sequence $(q_n(\lambda))_{n\in\mathbb{N}}$ in $\mathbb{C}[\lambda]$ converging to f uniformly on compact subsets of $\mathbb{R}$. Since the functional calculus $\Phi_a \colon C(\sigma(a)) \to \mathcal{A}$ is an isometry, $(q_n)_{\mathcal{A}} \to f_{\mathcal{A}}$ uniformly on bounded subsets of $\mathcal{A}_{\text{sa}}$ as $n \to \infty$. Thus, $f_{\mathcal{A}}$ is continuous.

It therefore is natural to wonder whether $f \in C^k(\mathbb{R})$ implies $f_{\mathcal{A}} \in C^k(\mathcal{A}_{\text{sa}}; \mathcal{A})$ whenever $k \in \mathbb{N}$. It turns out this is not generally true. Take $\mathcal{A} = B(H)$, where H is an infinite-dimensional complex Hilbert space, in which case $f_{\mathcal{A}} = f_{B(H)}$ is called the **operator function** induced by f. By [AP16, Thm. 1.2.9], if $f \in C(\mathbb{R})$ and $f_{B(H)} \in C^1(B(H)_{\text{sa}}; B(H))$, then f is **locally operator Lipschitz**, i.e., $f_{B(H)}|_{\{a\in B(H)_{\text{sa}}:\|a\|\leq r\}}$ is Lipschitz with respect to the operator norm $\|\cdot\|$ whenever $r > 0$. Yu. B. Farforovskaya showed in [Far72, Far76] that there exist functions $f \in C^1(\mathbb{R})$ that are *not* locally operator Lipschitz. In particular, there exist functions $f \in C^1(\mathbb{R})$ such that $f_{B(H)} \notin C^1(B(H)_{\text{sa}}; B(H))$. Please see [Pel85] and [AP16, §1.2 & §1.5] for more information.

To elucidate the difficulties with differentiating operator functions and to motivate some of this dissertation's constructions and results, we examine the finite-dimensional case, i.e., we take $H = \mathbb{C}^n$, in which case $B(H) = \mathrm{M}_n(\mathbb{C}) = \{n \times n \text{ complex matrices}\}$ and $f_{B(H)} = f_{\mathrm{M}_n(\mathbb{C})}$ is called the **matrix function** induced by f. If $a \in \mathrm{M}_n(\mathbb{C})$ and $\lambda \in \sigma(a) = \{\text{eigenvalues of } a\}$, then we define $P_\lambda^a \in \mathrm{M}_n(\mathbb{C})$ to be the orthogonal projection onto the λ-eigenspace of a. The spectral theorem from linear algebra has a nice restatement in terms of the **spectral resolution** $\big\{P_\lambda^a : \lambda \in \sigma(a)\big\}$ of a: A matrix $a \in \mathrm{M}_n(\mathbb{C})$ is normal ($a^*a = aa^*$) if and only if $P_\lambda^a P_\mu^a = \delta_{\lambda\mu}\, P_\lambda^a$ for all $\lambda, \mu \in \sigma(a)$ and $\sum_{\lambda\in\sigma(a)} P_\lambda^a = I_n$, in which case $a = \sum_{\lambda\in\sigma(a)} \lambda\, P_\lambda^a$. Consequently, we can write down a nice expression for the continuous functional calculus in the algebra $\mathcal{A} = \mathrm{M}_n(\mathbb{C})$. Indeed, if $a \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}$—more generally, if a is normal—then

$$f(a) = \sum_{\lambda \in \sigma(a)} f(\lambda)\, P_\lambda^a \tag{4}$$

for any (continuous) function $f \colon \sigma(a) \to \mathbb{C}$.

Theorem 5 (Matrix function derivatives)**.** *If $f \in C^k(\mathbb{R})$, then $f_{\mathrm{M}_n(\mathbb{C})} \in C^k(\mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}; \mathrm{M}_n(\mathbb{C}))$, and*

$$\partial_{b_k} \cdots \partial_{b_1} f_{\mathrm{M}_n(\mathbb{C})}(a) = \sum_{\pi \in S_k} \sum_{\boldsymbol{\lambda} \in \sigma(a)^{k+1}} f^{[k]}(\boldsymbol{\lambda})\, P_{\lambda_1}^a b_{\pi(1)} \cdots P_{\lambda_k}^a b_{\pi(k)} P_{\lambda_{k+1}}^a, \qquad a, b_i \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}. \tag{6}$$

Above and throughout, we write $\boldsymbol{\lambda} \coloneqq (\lambda_1, \ldots, \lambda_{k+1})$.

This result is due essentially to Yu. L. Daletskii and S. G. Krein [DK56], though it was proven in approximately the above form by F. Hiai as [Hia10, Thm. 2.3.1]. We discuss two proofs in Chapter 3.

Now, let us ponder Equation (6) to hint at the technical difficulties in the infinite-dimensional case. First, in view of Equation (4), it appears as though $\partial_{b_k} \cdots \partial_{b_1} f_{\mathrm{M}_n(\mathbb{C})}(a)$ is a symmetrization of the $\#_k$-action on $(b_1, \ldots, b_k)$ of the tensor

$$f^{[k]}\big(a \otimes I_n^{\otimes k}, \ldots, I_n^{\otimes k} \otimes a\big) \in \mathrm{M}_n(\mathbb{C})^{\otimes(k+1)},$$

defined using multivariate continuous functional calculus [DL90, App., §5]. (Given our discussion of Theorems 1 and 3, perhaps this does not come as a surprise.) This can be made rigorous in the finite-dimensional case but not in general. To understand why, let $\mathcal{A}$ be our arbitrary unital C^*-algebra. Recall that $\#_k$ is defined on $\mathcal{A}^{\hat{\otimes}_\pi(k+1)}$, which is a Banach algebra but not necessarily a C^*-algebra. Since continuous functional calculus is defined only in C^*-algebras, it is not generally possible to make sense of $f^{[k]}\big(a \otimes 1^{\otimes k}, \ldots, 1^{\otimes k} \otimes a\big)$ in $\mathcal{A}^{\hat{\otimes}_\pi(k+1)}$ for an arbitrary $f \in C^k(\mathbb{R})$ and element $a \in \mathcal{A}_{\mathrm{sa}}$.[1] In Chapter 3, we overcome this difficulty by requiring that $f^{[k]} \colon \mathbb{R}^{k+1} \to \mathbb{C}$ is "slightly better than continuous."

[1]The most natural setting for $f^{[k]}(a \otimes 1^{\otimes k}, \ldots, 1^{\otimes k} \otimes a)$ is the minimal C^*-tensor product $\mathcal{A}^{\otimes_{\min}(k+1)}$, but $\#_k$ is not defined on this algebra. In fact, $\#_k$ is not even defined on the maximal C^*-tensor product $\mathcal{A}^{\otimes_{\max}(k+1)}$.

Definition 7 (Varopoulos C^k functions)**.** A function $f \in C^k(\mathbb{R})$ is called **Varopoulos $\boldsymbol{C^k}$** if $f^{[k]}\big|_{[-r,r]^{k+1}} \in C([-r,r])^{\hat{\otimes}_\pi(k+1)}$ for all $r > 0$.[2] In this case, write $f \in VC^k(\mathbb{R})$.

We thoroughly study the space $VC^k(\mathbb{R})$ in Chapter 3. In particular, we show that if f is "slightly better than C^k," e.g., if f belongs to the Besov space $\dot{B}_1^{k,\infty}(\mathbb{R})$ (Definition 3.6.1) or the Hölder space $C_{\text{loc}}^{k,\varepsilon}(\mathbb{R})$ (Definition 3.6.13), then f is Varopoulos C^k. We also show that polynomials are dense in $VC^k(\mathbb{R})$ in an appropriate sense, which implies that $VC^k(\mathbb{R})$ may be identified with the space $C_{\text{nc}}^k(\mathbb{R})$ introduced and briefly studied by D. A. Jekel in [Jek20, Ch. 18]; please see Remark 3.4.13 for more information.

Now, let $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in \mathcal{A}_{\text{sa}}^{k+1}$. If $r := \max\{\|a_i\| : i \in \{1, \ldots, k+1\}\}$, then $C(\sigma(a_1))\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\sigma(a_{k+1})) \hookrightarrow C([-r,r])^{\hat{\otimes}_\pi(k+1)}$, so that

$$f_{\otimes}^{[k]}(\mathbf{a}) := \big(\Phi_{a_1}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \Phi_{a_{k+1}}\big)\big(f^{[k]}\big|_{\sigma(a_1)\times\cdots\times\sigma(a_{k+1})}\big) \in \mathcal{A}^{\hat{\otimes}_\pi(k+1)} \tag{8}$$

makes sense whenever $f \in VC^k(\mathbb{R})$.

Theorem 9 (Real C^k case)**.** *If $f \in VC^k(\mathbb{R})$, then $f_{\mathcal{A}} \in C^k(\mathcal{A}_{\text{sa}}; \mathcal{A})$, and*

$$\partial_{b_k} \cdots \partial_{b_1} f_{\mathcal{A}}(a) = \sum_{\pi \in S_k} f_{\otimes}^{[k]}(\underbrace{a, \ldots, a}_{k+1 \text{ times}}) \#_k \big[b_{\pi(1)}, \ldots, b_{\pi(k)}\big], \qquad a, b_i \in \mathcal{A}_{\text{sa}},$$

where $f_{\otimes}^{[k]}(a, \ldots, a)$ is defined as in Equation (8)*.*

A generalization of Theorem 9 is proven in Chapter 3. Combining this generalization with the paragraph before the statement of Theorem 9 yields extensions and improvements of results in [Pel06, ACDS09].

Next, we ponder Equation (6) in a different way to hint at a different set of technical difficulties. Let H be a complex Hilbert space. In the spectral theorem for self-adjoint—or normal—operators on H, the matrices $\{P_\lambda^a : \lambda \in \sigma(a)\}$ are replaced by a projection-valued measure $P^a \colon \mathcal{B}_{\sigma(a)} \to B(H)$ on the Borel subsets of $\sigma(a)$, and the sum in Equation (4) becomes an integral, i.e., $f(a) = \int_{\sigma(a)} f(\lambda)\, P^a(\mathrm{d}\lambda)$ whenever $f \colon \sigma(a) \to \mathbb{C}$ is continuous (or even measurable,

[2] Here, we take for granted that if $\Omega_1, \ldots, \Omega_m$ are compact Hausdorff spaces, then $C(\Omega_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\Omega_m)$ can be identified as a subalgebra of $C(\Omega_1 \times \cdots \times \Omega_m)$ called the **Varopoulos algebra** (§3.3).

§4.2). Taking this view and naïvely turning sums into integrals, Equation (6) becomes

$$\partial_{b_k}\cdots\partial_{b_1} f_{B(H)}(a) = \sum_{\pi\in S_k} \underbrace{\int_{\sigma(a)}\cdots\int_{\sigma(a)}}_{k+1 \text{ times}} f^{[k]}(\boldsymbol{\lambda})\, P^a(\mathrm{d}\lambda_1)\, b_{\pi(1)}\cdots P^a(\mathrm{d}\lambda_k)\, b_{\pi(k)}\, P^a(\mathrm{d}\lambda_{k+1}) \quad (10)$$

for all $a, b_1, \ldots, b_k \in B(H)_{\mathrm{sa}}$ (perhaps even with a unbounded). However, standard theory only allows for the integration of *scalar-valued* functions against projection-valued measures; while the innermost integral $\int_{\sigma(a)} f^{[k]}(\lambda_1,\ldots,\lambda_{k+1})\, P^a(\mathrm{d}\lambda_1)$ makes sense using standard theory, it already is unclear how to integrate the map $\lambda_2 \mapsto \int_{\sigma(a)} f^{[k]}(\lambda_1,\lambda_2,\ldots,\lambda_{k+1})\, P^a(\mathrm{d}\lambda_1)\, b_{\pi(1)}$ against P^a. It therefore is unclear how even to interpret—let alone prove—Equation (10) in the infinite-dimensional case. In their seminal paper [DK56], Daletskii and Krein dealt with this by using a Riemann–Stieltjes-type construction to define $\int_s^t \Phi(r)\, P^a(\mathrm{d}r) \in B(H)$ for certain *operator-valued* functions $\Phi\colon [s,t] \to B(H)$, where $\sigma(a) \subseteq [s,t]$. This approach, which requires rather stringent regularity assumptions on Φ, allowed them to make sense of the right-hand side of Equation (10) as an iterated operator-valued integral, i.e., a "multiple operator integral," when $f \in C^{2k}(\mathbb{R})$. Furthermore, they proved Equation (10) (with $b_1 = \cdots = b_k$) for $f \in C^{2k}(\mathbb{R})$.

As we have seen already, the assumption $f \in C^{2k}(\mathbb{R})$ is far too strong. Historically (and when differentiating at unbounded operators, discussed below), the key to relaxing it is finding a different way to interpret the multiple operator integral (MOI) on the right-hand side of Equation (10). For our purposes, the right way to do so is to use the "separation of variables approach" developed originally for separable H in [Pel06, ACDS09]; this approach is extended to general, i.e., not necessarily separable, H in Chapter 5. For much more information about MOIs and their applications, please see A. Skripka and A. Tomskova's book [ST19].

Next, we briefly address the generalizations of Theorems 1, 3, and 9 mentioned above as well as the case of unbounded operators. Recall that $\mathcal{B}$ is a unital Banach algebra. Let $\mathcal{I} \subseteq \mathcal{B}$ be an ideal of $\mathcal{B}$, i.e., a linear subspace such that $arb \in \mathcal{I}$ whenever $r \in \mathcal{I}$ and $a, b \in \mathcal{B}$. If $\|\cdot\|_{\mathcal{I}}$ is a norm on $\mathcal{I}$ such that $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a complex Banach space, the inclusion $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \hookrightarrow (\mathcal{B}, \|\cdot\|)$ is bounded, and $\|arb\|_{\mathcal{I}} \leq \|a\|\|r\|_{\mathcal{I}}\|b\|$ whenever $r \in \mathcal{I}$ and $a, b \in \mathcal{B}$, then $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a **symmetrically normed ideal** of $\mathcal{B}$, written $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{B}$. As can be seen by considering the case when f is a polynomial, it is reasonable to expect that if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{B}$, $a \in \mathcal{B}_U$, and $f \in \mathrm{Hol}(U)$, then

$f(a+b)-f(a) \in \mathcal{I}$ whenever $b \in \mathcal{I}_{U,a} \coloneqq \{c \in \mathcal{I} : a+c \in \mathcal{B}_U\}$, and the map

$$\mathcal{I}_{U,a} \ni b \mapsto f_{a,\mathcal{I}}(b) \coloneqq f(a+b) - f(a) \in \mathcal{I}$$

is holomorphic with respect to $\|\cdot\|_{\mathcal{I}}$. This is, indeed, the case and is established in Chapter 2. Similarly, it is proven in Chapter 3 that if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a symmetrically normed ideal of the unital C^*-algebra $\mathcal{A}$, $a \in \mathcal{A}_{\text{sa}}$, and $f \in VC^k(\mathbb{R})$, then the map

$$\mathcal{I}_{\text{sa}} \coloneqq \mathcal{I} \cap \mathcal{A}_{\text{sa}} \ni b \mapsto f_{a,\mathcal{I}}(b) \coloneqq f(a+b) - f(a) \in \mathcal{I}$$

is well defined and C^k with respect to $\|\cdot\|_{\mathcal{I}}$. Furthermore, appropriate modifications of the formulas in Theorems 1, 3, and 9 hold for the "perturbed" maps $b \mapsto f(a+b) - f(a)$.

One can also try to differentiate the map $b \mapsto f(a+b) - f(a)$ when a is an unbounded operator. To be more specific, let H be a complex Hilbert space, let $\mathcal{M} \subseteq B(H)$ be a von Neumann algebra, let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\text{s}} \mathcal{M}$, and let a be an unbounded operator on H affiliated with $\mathcal{M}$ (Definition 4.2.16). If $f : \mathbb{R} \to \mathbb{C}$ is Lipschitz and $b \in \mathcal{M}$, then $f(a+b) - f(a)$ is densely defined; if f is slightly better than Lipschitz, then $f(a+b) - f(a)$ extends to a bounded linear operator on H belonging to $\mathcal{M}$. In this case, we may consider the question of when the **perturbed operator function** $\mathcal{I}_{\text{sa}} \ni b \mapsto f_{a,\mathcal{I}}(b) \coloneqq f(a+b) - f(a) \in \mathcal{I}$ is well defined and C^k with respect to $\|\cdot\|_{\mathcal{I}}$. This is the focus of Chapter 6. Therein, we use the MOI results from Chapter 5 to prove formulas like Equation (10) for perturbed operator functions. Our results generalize and improve the best-known such results from [Pel06, ACDS09].

Finally, in Chapter 7, we apply MOIs and derivative formulas like Equation (10) to free stochastic calculus. Specifically, we extend, reinterpret, and make more computationally flexible a free Itô-type formula of P. Biane and R. Speicher [BS98].

Dissertation summary

With the discussion above in mind, here are the problems considered in this dissertation.

(P.1) Let $\mathcal{B}$ be a unital Banach algebra, and let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be a symmetrically normed ideal of $\mathcal{B}$. If $a \in \mathcal{B}$, $U \subseteq \mathbb{C}$ is an open set containing $\sigma(a)$, and $f : U \to \mathbb{C}$ is a holomorphic function, when can one say that the map $\mathcal{I} \ni b \mapsto f(a+b) - f(a) \in \mathcal{I}$ is well defined and holomorphic (with respect to $\|\cdot\|_{\mathcal{I}}$) in a neighborhood of $0 \in \mathcal{I}$? In this case, how does one compute its derivatives?

(P.2) Let $\mathcal{A}$ be a unital C^*-algebra, and let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be a symmetrically normed ideal of $\mathcal{A}$. If $a \in \mathcal{A}_{\text{sa}}$, i.e., $a^* = a$, and $f\colon \mathbb{R} \to \mathbb{C}$ is k-times continuously differentiable (C^k), when can one say that the map $\mathcal{I}_{\text{sa}} := \mathcal{A}_{\text{sa}} \cap \mathcal{I} \ni b \mapsto f(a+b) - f(a) \in \mathcal{I}$ is well defined and C^k (with respect to $\|\cdot\|_{\mathcal{I}}$)? In this case, how does one compute its derivatives?

(P.3) Let $\mathcal{M}$ be a von Neumann algebra, and let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be a symmetrically normed ideal of $\mathcal{M}$. If a is a self-adjoint operator affiliated with $\mathcal{M}$ and $f\colon \mathbb{R} \to \mathbb{C}$ is C^k, when can one say that the map $\mathcal{I}_{\text{sa}} \ni b \mapsto f(a+b) - f(a) \in \mathcal{I}$ is well defined and C^k? In this case, how does one compute its derivatives?

This dissertation's main contributions to these problems are as follows.

(C.1) The map in question in (P.1) is always well defined and holomorphic in a neighborhood of $0 \in \mathcal{I}$, and its k^{th} derivative may be computed in terms of the k^{th} divided difference $f^{[k]}$ of f via theories of multivariate holomorphic functional calculus. This is the subject of Chapter 2. The methods therein serve as motivation for those used in (C.2) and (C.3).

(C.2) In Chapter 3, we prove that if $f\colon \mathbb{R} \to \mathbb{C}$ is Varopoulos C^k (Definition 7), then the map in question in (P.2) is well defined and C^k, and its k^{th} derivative may be computed in terms of $f^{[k]}$ via a projective tensor product–valued kind of multivariate continuous functional calculus. Our results vastly generalize (the bounded cases) of results from [Pel06, ACDS09]. This dissertation also seems to be the first place bounded operators (elements of C^*-algebras) are treated by themselves, i.e., not as special cases of unbounded operators. As a result, we are able to simplify previous developments substantially.

(C.3) In Chapter 6, we prove that if $\mathcal{I}$ is an integral symmetrically normed ideal (a new notion, Definition 6.2.2), $f \in \dot{B}_1^{k,\infty}(\mathbb{R})$, and f' is bounded, then the map in question in (P.3) is well defined and C^k, and its k^{th} derivative may be computed in terms of $f^{[k]}$ via multiple operator integrals (Chapter 5). We also use vector-valued integral techniques and the theory of symmetric operator spaces (§6.3) to provide large classes of interesting examples of integral symmetrically normed ideals. Our results vastly generalize those of [Pel06] and, in symmetric operator space–induced examples of interest, generalize and dramatically weaken the regularity hypotheses in the results of [ACDS09], thereby making substantial progress on [ST19, Prob. 5.3.22]; please see §6.1 for details.

Hidden in the discussion of (C.2) and (C.3) is the fact that the results in [Pel06, ACDS09] are only for (von Neumann algebras in) $B(H)$, where H is a separable complex Hilbert space. Our results, however, never require separability assumptions due to the following additional contribution of this dissertation.

(C.4) In Chapter 5, we develop the "separation of variables" approach to defining multiple operator integrals (MOIs) on Hilbert spaces that are not necessarily separable. Previously, only separable Hilbert spaces had been treated. The general case requires a great deal of technical care with vector- and operator-valued integrals.

Finally, we apply our results to free probability—specifically, free stochastic calculus.

(C.5) In Chapter 7, we explore a connection between free stochastic calculus and the theory of MOIs by proving an Itô formula for Varopoulos C^2 functions—more generally, noncommutative C^2 functions (§3.8)—of self-adjoint free Itô processes. Specifically, we reinterpret the free Itô formula [BS98, Prop. 4.3.4] of Biane–Speicher by identifying the terms therein as MOIs. This enables us to enlarge the class of functions for which one can formulate and prove a free Itô formula as well as to improve the computational flexibility of the theory greatly.

Chapter 1

Background I

In this chapter, we lay out background material that is essential to the entire dissertation. (Additional background relevant only to certain chapters is covered later.) Specifically, we cover elementary aspects of infinite-dimensional calculus (§1.1 and §1.2), divided differences (§1.3), Banach algebras (§1.4), and projective tensor products (§1.5). The exposition assumes the reader is comfortable with topological vector spaces; please see [Rud91, Pt. I] for the relevant material.

Standing assumptions. Fix a choice of base field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Unless otherwise specified, all vector spaces are $\mathbb{F}$-vector spaces, and all linear maps are $\mathbb{F}$-linear. In §1.1, $(\Omega, \mathscr{F}, \mu)$ is a measure space, and V is a Hausdorff, locally convex topological vector space (HLCTVS) with topological dual V^*. In §1.2, $k \in \mathbb{N}$; $V_1, \ldots, V_k, V, W$ are normed vector spaces; and $U \subseteq V$ is an open set. In §1.4, $\mathbb{F} = \mathbb{C}$ always. In §1.5, $k \in \mathbb{N}$, and $V_1, W_1, \ldots, V_k, W_k, V, W$ are Banach spaces.

1.1 Vector-valued integrals

Here, we begin a discussion of a "weak" notion of V-valued integration that we shall continue in §5.3. In most of this dissertation, we cite external sources for the proofs of well-known results. However, we err on the side of proving rather than citing results on vector-valued integrals, as they play a central role in several delicate arguments.

Notation 1.1.1 (σ-algebras)**.** If S is a set, then $\Omega^S := \{\text{functions } S \to \Omega\}$ and $2^S := \{\text{subsets of } S\}$. If $\mathscr{S} \subseteq \Omega^S$, then $\sigma(\mathscr{S}) \subseteq 2^S$ is the smallest σ-algebra on S with respect to which all members of $\mathscr{S}$ are measurable. If X is a topological space, then $\mathcal{B}_X^a := \sigma(C(X;\mathbb{R}))$ is its Baire σ-algebra, and $\mathcal{B}_X$ is its Borel σ-algebra. Unless otherwise specified, a topological space carries its Borel σ-algebra.

Let $(\Xi, \mathscr{G})$ be another measurable space. Note that a function $f\colon \Xi \to S$ is $(\mathscr{G}, \sigma(\mathscr{S}))$-measurable if and only if $s \circ f\colon \Xi \to \Omega$ is $(\mathscr{G}, \mathscr{F})$-measurable whenever $s \in \mathscr{S}$. Also, $\mathcal{B}_X^a \subseteq \mathcal{B}_X$.

Definition 1.1.2 (Weak measurability and integrability)**.** A map $F\colon \Omega \to V$ is **weakly measurable** if it is $(\mathscr{F}, \sigma(V^*))$-measurable. A weakly measurable map $F\colon \Omega \to V$ is **weakly** or **Gel'fand–Pettis (μ-)integrable** if $\int_\Omega |\ell \circ F| \, \mathrm{d}\mu < \infty$ whenever $\ell \in V^*$ and there exists a (necessarily unique) vector $\int_\Omega F \, \mathrm{d}\mu = \int_\Omega F(\omega) \, \mu(\mathrm{d}\omega) \in V$ such that

$$\ell\left(\int_\Omega F \, \mathrm{d}\mu\right) = \int_\Omega (\ell \circ F) \, \mathrm{d}\mu, \qquad \ell \in V^*. \tag{1.1.3}$$

In this case, $\int_\Omega F \, \mathrm{d}\mu \in V$ is the **weak** or **Gel'fand–Pettis (μ-)integral** of F.

The uniqueness of $\int_\Omega F \, \mathrm{d}\mu$ is a consequence of the fact that V^* separates points, i.e., $v = 0$ if and only if $\ell(v) = 0$ whenever $\ell \in V^*$; please see [Rud91, Thm. 3.4]. Also, $\sigma(V^*) \subseteq \mathcal{B}_V^a$, so Baire measurable maps $\Omega \to V$ are weakly measurable. Finally, by the comment after Notation 1.1.1, $F\colon \Omega \to V$ is weakly measurable if and only if $\ell \circ F\colon \Omega \to \mathbb{F}$ is measurable whenever $\ell \in V^*$.

Example 1.1.4 (Finite-dimensional case)**.** When $V = \mathbb{F}^n$, $F\colon \Omega \to V$ is weakly measurable (respectively, integrable) if and only if its components are measurable (respectively, integrable, in which case the components of $\int_\Omega F \, \mathrm{d}\mu$ are the integrals of the respective components of F).

Definition 1.1.5 (Simple and σ-simple functions)**.** Let Ξ be a set. A function $f\colon \Omega \to \Xi$ is **simple** (respectively, **σ-simple**) if $f(\Omega)$ is finite (respectively, countable) and

$$f^{-1}(\xi) = \{\omega \in \Omega : f(\omega) = \xi\} \in \mathscr{F}, \qquad \xi \in \Xi.$$

A simple map $F\colon \Omega \to V$ is **(μ-)integrable** if $\mu\big(F^{-1}(V \setminus \{0\})\big) < \infty$.

Note that if $f\colon \Omega \to \Xi$ is σ-simple, then f is $(\mathscr{F}, 2^\Xi)$-measurable.

Example 1.1.6 (Simple maps)**.** If $F\colon \Omega \to V$ is σ-simple, then F is $(\mathscr{F}, 2^V)$-measurable and thus weakly measurable. Also, if $F\colon \Omega \to V$ is simple and μ-integrable, then F is weakly μ-integrable. Indeed, define

$$w \coloneqq \sum_{v \in F(\Omega)} v \, \mu\big(F^{-1}(v)\big) = \sum_{v \in V} v \, \mu\big(F^{-1}(v)\big) \in V,$$

where $0 \cdot \infty := 0 \in V$. (Note that we have broken the standard notational convention for scalar multiplication; we shall do so regularly without further comment.) If $\ell \in V^*$, then

$$\begin{aligned}\ell(w) &= \sum_{v \in F(\Omega)} \ell(v)\, \mu\big(F^{-1}(v)\big) = \sum_{c \in \ell(F(\Omega))} c \sum_{v \in F(\Omega) : \ell(v) = c} \mu\big(F^{-1}(v)\big) \\ &= \sum_{c \in \ell(F(\Omega))} c\, \mu\big((\ell \circ F)^{-1}(c)\big) = \int_\Omega (\ell \circ F)\, \mathrm{d}\mu.\end{aligned}$$

(Above, we used that μ is finitely additive and the collection $\big\{F^{-1}(v) : v \in F(\Omega),\ \ell(v) = c\big\}$ is a partition of $(\ell \circ F)^{-1}(c)$.) In other words, $w = \int_\Omega F\, \mathrm{d}\mu$.

Proposition 1.1.7 (Basic properties)**.** *Let W be another HLCTVS.*

(i) *Linear combinations of weakly measurable (respectively, integrable) maps are weakly measurable (respectively, integrable, and the weak integral is a linear operation).*

(ii) *If $F \colon \Omega \to V$ is weakly measurable (respectively, integrable) and $T \colon V \to W$ is a continuous linear map, then $TF = T \circ F \colon \Omega \to W$ is weakly measurable (respectively, integrable, in which case $T \int_\Omega F\, \mathrm{d}\mu = \int_\Omega TF\, \mathrm{d}\mu$).*

Proof. As we encourage the reader to verify, these properties follow easily from the definitions and the fact that Equation (1.1.3) characterizes $\int_\Omega F\, \mathrm{d}\mu$. □

Next, we study one situation in which weak integrals exist and behave exceptionally well.

Definition 1.1.8 (Strong measurability and integrability)**.** A map $F \colon \Omega \to V$ is **strongly** or **Bochner** measurable if there exists a sequence $(F_n)_{n \in \mathbb{N}}$ of simple maps $\Omega \to V$ converging pointwise to F. If, in addition, we can arrange that F_n is μ-integrable for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \int_\Omega \alpha(F_n - F)\, \mathrm{d}\mu = 0 \tag{1.1.9}$$

whenever α is a continuous seminorm on V, then F is **strongly** or **Bochner (*μ*-)integrable**.

Since simple maps $\Omega \to V$ are Baire measurable and the pointwise limit of a sequence of Baire measurable maps is Baire measurable, strongly measurable maps are Baire measurable. In particular, if $F \colon \Omega \to V$ is strongly measurable and $\alpha \colon V \to \mathbb{R}_+ := [0, \infty)$ is a continuous seminorm, then $\alpha(F)\colon \Omega \to \mathbb{R}_+$ is measurable. Consequently, the integral in Equation (1.1.9) makes sense. If, in addition, F is strongly integrable, then $\int_\Omega \alpha(F)\,\mathrm{d}\mu < \infty$. Indeed, if $(F_n)_{n\in\mathbb{N}}$ is as in the second part of Definition 1.1.8, then $\int_\Omega \alpha(F)\,\mathrm{d}\mu \le \int_\Omega \alpha(F-F_n)\,\mathrm{d}\mu + \int_\Omega \alpha(F_n)\,\mathrm{d}\mu < \infty$ for sufficiently large n. Finally, suppose that $\mathscr{S} \subseteq \mathbb{R}_+^V$ is a collection of seminorms generating the topology of V. If α is an arbitrary continuous seminorm on V, then there exist a $C \ge 0$ and $\alpha_1,\ldots,\alpha_m \in \mathscr{S}$ such that $\alpha \le C\sum_{i=1}^m \alpha_i$. It follows that if Equation (1.1.9) holds whenever $\alpha \in \mathscr{S}$, then Equation (1.1.9) holds whenever α is an arbitrary continuous seminorm on V.

Proposition 1.1.10 (Bochner integral)**.** *Suppose V is sequentially complete, $F\colon \Omega \to V$ is strongly integrable, and $(F_n)_{n\in\mathbb{N}}$ is as in the second part of Definition 1.1.8.*

(i) $\left(\int_\Omega F_n\,\mathrm{d}\mu\right)_{n\in\mathbb{N}}$ *converges in V, and its limit is called the* ***Bochner (μ-)integral*** *of F.*

(ii) *F is weakly integrable, and $\int_\Omega F\,\mathrm{d}\mu = \lim_{n\to\infty}\int_\Omega F_n\,\mathrm{d}\mu$.*

(iii) (Triangle inequality) *If $\alpha\colon V \to \mathbb{R}_+$ is a continuous seminorm, then*

$$\alpha\left(\int_\Omega F\,\mathrm{d}\mu\right) \le \int_\Omega \alpha(F)\,\mathrm{d}\mu.$$

Proof. First, if $G\colon \Omega \to V$ is an integrable simple map and $\alpha\colon V \to \mathbb{R}_+$ is any seminorm, then

$$\alpha\left(\int_\Omega G\,\mathrm{d}\mu\right) = \alpha\left(\sum_{v\in G(\Omega)} v\,\mu\big(G^{-1}(v)\big)\right) \le \sum_{v\in G(\Omega)} \alpha(v)\,\mu\big(G^{-1}(v)\big) = \int_\Omega \alpha(G)\,\mathrm{d}\mu.$$

The last identity above holds by a calculation like the one in Example 1.1.6. We conclude that the triangle inequality holds in this case. With this in mind, we take each item in turn.

(i) If $\alpha\colon V \to \mathbb{R}_+$ is a continuous seminorm and $n,m \in \mathbb{N}$, then

$$\begin{aligned}\alpha\left(\int_\Omega F_n\,\mathrm{d}\mu - \int_\Omega F_m\,\mathrm{d}\mu\right) &= \alpha\left(\int_\Omega (F_n - F_m)\,\mathrm{d}\mu\right) \le \int_\Omega \alpha(F_n - F_m)\,\mathrm{d}\mu \\ &\le \int_\Omega \alpha(F_n - F)\,\mathrm{d}\mu + \int_\Omega \alpha(F - F_m)\,\mathrm{d}\mu \to 0\end{aligned}$$

as $n,m \to \infty$ by the previous paragraph and the assumptions on $(F_n)_{n\in\mathbb{N}}$. We conclude that $\left(\int_\Omega F_n\,\mathrm{d}\mu\right)_{n\in\mathbb{N}}$ is Cauchy in V. Since V is sequentially complete, $\left(\int_\Omega F_n\,\mathrm{d}\mu\right)_{n\in\mathbb{N}}$ converges in V.

(ii) We already observed after Definition 1.1.8 that F is weakly measurable. Now, let $\ell \in V^*$, and write $v := \lim_{n\to\infty} \int_\Omega F_n \, \mathrm{d}\mu$ for the Bocher integral of F. Since $|\ell|$ is a continuous seminorm, $\int_\Omega |\ell \circ F| \, \mathrm{d}\mu < \infty$, and

$$\begin{aligned} \left| \ell(v) - \int_\Omega (\ell \circ F) \, \mathrm{d}\mu \right| &= \lim_{n\to\infty} \left| \ell\left(\int_\Omega F_n \, \mathrm{d}\mu \right) - \int_\Omega (\ell \circ F) \, \mathrm{d}\mu \right| \\ &= \lim_{n\to\infty} \left| \int_\Omega \ell \circ (F_n - F) \, \mathrm{d}\mu \right| \leq \lim_{n\to\infty} \int_\Omega |\ell \circ (F_n - F)| \, \mathrm{d}\mu = 0. \end{aligned}$$

Thus, F is weakly integrable, and $v = \int_\Omega F \, \mathrm{d}\mu$.

(iii) Take the limit as $n \to \infty$ in the inequality $\alpha\big(\int_\Omega F_n \, \mathrm{d}\mu \big) \leq \int_\Omega \alpha(F_n) \, \mathrm{d}\mu$ from the first paragraph of the proof. □

Corollary 1.1.11 (Dominated convergence theorem)**.** *Suppose V is sequentially complete and $(F_n)_{n\in\mathbb{N}}$ is a sequence of strongly integrable maps $\Omega \to V$. If $(F_n)_{n\in\mathbb{N}}$ converges pointwise to a map $F \colon \Omega \to V$ and*

$$\int_\Omega \sup_{n\in\mathbb{N}} \alpha(F_n) \, \mathrm{d}\mu < \infty$$

whenever α is a continuous seminorm on V, then F is weakly integrable, and

$$\int_\Omega F \, \mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega F_n \, \mathrm{d}\mu. \tag{1.1.12}$$

Sketch of proof. Since F is the pointwise limit of a sequence of weakly measurable maps, it is weakly measurable. By Fatou's lemma, $\int_\Omega \alpha(F) \, \mathrm{d}\mu < \infty$ whenever α is a continuous seminorm on V. By the triangle inequality and the scalar-valued dominated convergence theorem, $\big(\int_\Omega F_n \, \mathrm{d}\mu \big)_{n\in\mathbb{N}}$ is Cauchy and therefore convergent in V. By repeating the proof of Proposition 1.1.10(ii), we see that F is weakly integrable and Equation (1.1.12) holds. □

Remark 1.1.13. Actually, by Theorem 1.1.17 below, the map F from Corollary 1.1.11 is strongly integrable if V is a Fréchet space (a metrizable, complete, locally convex topological vector space). Also, the triangle inequality and dominated convergence theorems presented above are special cases of more general results for weak integrals that we shall not need until Chapter 5; please see Propositions 5.3.3 and 5.3.4.

Proposition 1.1.10 enables us to construct the Bochner integral (against a finite measure) of a continuous map from a compact interval to a sequentially complete HLCTVS.

Notation 1.1.14 (Partitions)**.** Fix $a, b \in \mathbb{R}$ such that $a < b$ and a map $F\colon [a,b] \to V$.

(i) $\mathcal{P}_{[a,b]}$ is the set of partitions of the interval $[a,b]$, i.e., the set of finite subsets of $[a,b]$ containing a and b. If $\Pi \in \mathcal{P}_{[a,b]}$ and $t \in \Pi$, then $t_- \in \Pi$ is the member of Π to the left of t; precisely, $a_- \coloneqq a$ and $t_- \coloneqq \max\{s \in \Pi : s < t\}$ for $t \in \Pi \setminus \{a\}$. Also, $\Delta t \coloneqq t - t_-$, $\Delta_t F \coloneqq F(t) - F(t_-)$, and $|\Pi| \coloneqq \max\{\Delta s : s \in \Pi\}$ is the mesh of Π.

(ii) $\mathcal{P}^*_{[a,b]}$ is the set of augmented partitions of $[a,b]$, i.e., the set of pairs (Π,ξ), where $\Pi \in \mathcal{P}_{[a,b]}$ and $\xi\colon \Pi \to [a,b]$ satisfies $t_* \coloneqq \xi(t) \in [t_-, t]$ whenever $t \in \Pi$. If $(\Pi,\xi) \in \mathcal{P}^*_{[a,b]}$, then

$$F^{(\Pi,\xi)} \coloneqq F(a)\, 1_{\{a\}} + \sum_{t \in \Pi} F(t_*)\, 1_{(t_-,t]}\colon [a,b] \to V.$$

The sets $\mathcal{P}_{[a,b]}$ and $\mathcal{P}^*_{[a,b]}$ are frequently directed by refinement. We direct them instead using the mesh $|\cdot|$, i.e., $\Pi \le \Pi'$ and $(\Pi,\xi) \le (\Pi',\xi')$ whenever $|\Pi| \ge |\Pi'|$. Below, we record what it means for nets indexed by $\mathcal{P}_{[a,b]}$ and $\mathcal{P}^*_{[a,b]}$ to converge; we leave the proof to the reader.

Lemma 1.1.15 (Limits as $|\Pi| \to 0$)**.** *Let X be a topological space, and fix a net $x\colon \mathcal{P}_{[a,b]} \to X$. For $y \in X$, $\lim_{\Pi \in \mathcal{P}_{[a,b]}} x(\Pi) = y$ holds if and only if for all open neighborhoods U of y, there exists a $\delta > 0$ such that $|\Pi| < \delta$ implies $x(\Pi) \in U$; this happens if and only if for every sequence $(\Pi_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_{[a,b]}$ such that $\lim_{n\to\infty} |\Pi_n| = 0$, we have $\lim_{n\to\infty} x(\Pi_n) = y$. In this case, we write $\lim_{|\Pi|\to 0} x(\Pi) = y$ or say that $x(\Pi) \to y$ as $|\Pi| \to 0$. One can characterize and notate the convergence of nets $x^*\colon \mathcal{P}^*_{[a,b]} \to X$ similarly.*

Example 1.1.16 (Continuous maps)**.** Take $(\Omega,\mathscr{F}) \coloneqq \big([a,b], \mathcal{B}_{[a,b]}\big)$, and assume $\mu([a,b]) < \infty$. We claim that if $F\colon [a,b] \to V$ is continuous, then F is strongly integrable. Indeed, if $(\Pi,\xi) \in \mathcal{P}^*_{[a,b]}$ and α is a continuous seminorm on V, then

$$\alpha\big(F^{(\Pi,\xi)} - F\big) = \sum_{t\in\Pi} \alpha(F(t_*) - F)\, 1_{(t_-,t]} \le \sup\{\alpha(F(s) - F(t)) : |s-t| \le |\Pi|\} \xrightarrow{|\Pi|\to 0} 0$$

because F is continuous and $[a,b]$ is compact. Since $F^{(\Pi,\xi)}\colon [a,b] \to V$ is a simple map, F is strongly measurable. By the same estimate,

$$\int_{[a,b]} \alpha\big(F^{(\Pi,\xi)} - F\big)\,\mathrm{d}\mu \le \mu([a,b]) \sup\{\alpha(F(s)-F(t)) : |s-t| \le |\Pi|\} \xrightarrow{|\Pi|\to 0} 0.$$

Thus, F is strongly integrable, as claimed. By Proposition 1.1.10 (and Lemma 1.1.15), more is true whenever V is sequentially complete: F is weakly integrable, and

$$\int_{[a,b]} F\,\mathrm{d}\mu = \lim_{|\Pi|\to 0} \int_{[a,b]} F^{(\Pi,\xi)}\,\mathrm{d}\mu = F(a)\,\mu(\{a\}) + \lim_{|\Pi|\to 0} \sum_{t\in\Pi} F(t_*)\,\mu((t_-,t]).$$

If μ is the Lebesgue measure, then we write

$$\int_a^b F(t)\,\mathrm{d}t \coloneqq \int_{[a,b]} F\,\mathrm{d}\mu = \lim_{|\Pi|\to 0} \sum_{t\in\Pi} F(t_*)\,\Delta t.$$

The right-hand side above is, of course, the (V-valued) Riemann integral of F.

In many situations, e.g., in Chapter 2, integrals as in Example 1.1.16 are enough. In others, e.g., in Chapter 3, a more general criterion for the existence of Bochner integrals is required. We end this section by quoting such a criterion—in fact, an alternative characterization of strong measurability and integrability. (In yet other situations, e.g., in Chapter 5, the Bochner integral is insufficient, and other results on the existence of weak integrals are required.)

Theorem 1.1.17 (Pettis's measurability theorem)**.** *Suppose V is metrizable.*

(i) *If $F\colon \Omega \to V$ is strongly measurable, then F is Borel measurable, and $F(\Omega)$ is a separable subset of V. If F is weakly measurable and $F(\Omega)$ is separable, then F is strongly measurable.*

(ii) *A strongly measurable map $F\colon \Omega \to V$ is strongly integrable if and only if $\int_\Omega \alpha(F)\,\mathrm{d}\mu < \infty$ whenever α is a continuous seminorm on V.*

This result is well known and present in many books when V is a Banach space; please see, e.g., [Coh13, App. E]. When V is only a metrizable, locally convex topological vector space, the proof is similar, but the details are more involved. For the sake of completeness, and because the author is unaware of a reference that treats this level of generality, we provide a proof in Appendix A (specifically, §A.1).

1.2 Fréchet derivatives

Here, we review some definitions and facts related to Fréchet differentiability and derivatives. Specifically, we define (higher) Fréchet differentiability, list some basic properties of Fréchet derivatives, study a topology on a certain space of C^k maps, and conduct a brief discussion of holomorphicity and the convergence of nets of holomorphic maps. We also compute the derivatives of a homogeneous polynomial as an important example.

Notation 1.2.1 (Bounded multilinear maps)**.** If $T\colon V_1 \times \cdots \times V_k \to W$ is a k-linear map, then

$$\|T\|_{B_k(V_1\times\cdots\times V_k;W)} \coloneqq \sup\{\|T[v_1,\ldots,v_k]\|_W : v_i \in V_i,\ \|v_i\|_{V_i} \leq 1,\ i \in \{1,\ldots,k\}\} \in [0,\infty]$$

is the operator norm of T, and $B_k(V_1 \times \cdots \times V_k; W)$ is the space of bounded k-linear maps $V_1 \times \cdots \times V_k \to W$, i.e., the space of k-linear maps $V_1 \times \cdots \times V_k \to W$ with finite operator norm. Also, $(B(V_1;W), \|\cdot\|_{V_1\to W}) \coloneqq \big(B_1(V_1;W), \|\cdot\|_{B_1(V_1;W)}\big)$, $B(W) \coloneqq B(W;W)$, and $B_0\big(V^0;W\big) \coloneqq W$. Finally, if $\mathbb{F}$ is needed in the notation, then $B^{\mathbb{F}}$ will be used in place of B.

Note that $B_k(V_1 \times \cdots \times V_k; W) \cong B(V_k; B_{k-1}(V_1 \times \cdots \times V_{k-1}; W))$ via the isometry

$$T \mapsto (v_k \mapsto ((v_1,\ldots,v_{k-1}) \mapsto T[v_1,\ldots,v_k])). \tag{1.2.2}$$

In particular, by induction, if W is a Banach space, then so is $B_k(V_1 \times \cdots \times V_k; W)$.

Definition 1.2.3 (Fréchet differentiability)**.** Let $F\colon U \to W$ be a map, and fix $v \in U$.

(i) The map F is (**once**) **Fréchet differentiable at** $\boldsymbol{v}$ if there exists a (necessarily unique) $DF(v) = D^1F(v) \in B(V;W) = B^{\mathbb{F}}(V;W)$ such that

$$\lim_{\substack{h\to 0\\ h\in V}} \frac{\|F(v+h) - F(v) - DF(v)h\|_W}{\|h\|_V} = 0. \tag{1.2.4}$$

If F is Fréchet differentiable at points in U, then F is (**once**) **Fréchet differentiable in** $\boldsymbol{U}$, and the map $U \ni w \mapsto DF(w) \in B(V;W)$ is its (**first**) **Fréchet derivative**.

(ii) Higher Fréchet differentiability is defined recursively: For $k \geq 2$, F is $\boldsymbol{k}$**-times Fréchet differentiable at** $\boldsymbol{v}$ if it is $(k-1)$-times Fréchet differentiable in a neighborhood—say

U for simplicity—of v and $D^{k-1}F\colon U \to B_{k-1}\big(V^{k-1};W\big)$ is Fréchet differentiable at v. In this case, $D^kF(v)$ is the element of $B_k\big(V^k;W\big)$ mapping to $D\big(D^{k-1}F\big)(v)$ under the isomorphism $B_k\big(V^k;W\big) \cong B\big(V;B_{k-1}\big(V^{k-1};W\big)\big)$ from Relation (1.2.2). If F is k-times Fréchet differentiable at all points in U, then F is ***k*-times Fréchet differentiable in *U***, and the map $U \ni w \mapsto D^kF(w) \in B_k\big(V^k;W\big)$ is its ***k*th Fréchet derivative**.

(iii) If F is k-times Fréchet differentiable in U and $D^kF\colon U \to B_k\big(V^k;W\big)$ is continuous, then F is ***k*-times continuously (Fréchet) differentiable**—or ***C^k*** for short—**in *U***.

(iv) The map F is **holomorphic in *U*** if $\mathbb{F} = \mathbb{C}$ and F is C^1 in U. Explicitly, V and W are complex normed vector spaces, and $DF(v)\colon V \to W$ is $\mathbb{C}$-linear in this case.

We shall omit "in U" from the terminology whenever confusion is unlikely. Also, we set the following notation for spaces of maps.

(v) If X and Y are topological spaces, then $C(X;Y) = C^0(X;Y)$ is the space of continuous maps $X \to Y$, $BC(X;W) = BC^0(X;W)$ is the space of bounded continuous maps $X \to W$, $C(X) = C^0(X) \coloneqq C(X;\mathbb{C})$, and $BC(X) = BC^0(X) \coloneqq BC(X;\mathbb{C})$.

(vi) $C^k(U;W)$ is the space of C^k maps $U \to W$, $C^\infty(U;W) \coloneqq \bigcap_{n\in\mathbb{N}} C^n(U;W)$ is the space of smooth maps $U \to W$, and $C^n(U) \coloneqq C^n(U;\mathbb{C})$ whenever $n \in \mathbb{N} \cup \{\infty\}$. (In the previous sentence, $\mathbb{C}$ is viewed as an $\mathbb{F}$-vector space.) If $\mathbb{F} = \mathbb{C}$, then $\mathrm{Hol}(U;W)$ is the space of holomorphic maps $U \to W$, and $\mathrm{Hol}(U) \coloneqq \mathrm{Hol}(U;\mathbb{C})$.

As an important example that will be of use to us, we compute the Fréchet derivatives of a homogeneous polynomial, i.e., a multilinear map evaluated diagonally.

Notation 1.2.5. Let $m \in \mathbb{N}_0$.

(i) If S is a set and $s \in S$, then $s_{(m)} \coloneqq (s,\dots,s) \in S^m$. To be clear, $s_{(0)}$ is the empty list.

(ii) Is $\alpha = (\alpha_1,\dots,\alpha_m) \in \mathbb{N}_0^m$ is a multi-index, then $|\alpha| \coloneqq \alpha_1 + \cdots + \alpha_m$ is its order.

(iii) S_m is the symmetric group on m letters, i.e., the group of permutations of $\{1,\dots,m\}$.

(iv) If $T \in B_m(V^m;W)$, then

$$S(T)[v_1,\dots,v_m] \coloneqq \frac{1}{m!}\sum_{\pi\in S_m} T\big[v_{\pi(1)},\dots,v_{\pi(m)}\big], \qquad v_1,\dots,v_m \in V.$$

Proposition 1.2.6 (Derivatives of a homogeneous polynomial)**.** *Fix $n \in \mathbb{N}_0$ and $T \in B_n(V^n; W)$. If $P\colon V \to W$ is defined by $v \mapsto T\big[v_{(n)}\big] = S(T)\big[v_{(n)}\big]$, then $P \in C^\infty(V;W)$, and*

$$D^k P(v)[h_1,\ldots,h_k] = 1_{k\le n}\frac{n!}{(n-k)!}S(T)\big[h_1,\ldots,h_k,v_{(n-k)}\big] \tag{1.2.7}$$
$$= \sum_{\pi\in S_k}\sum_{|\alpha|=n-k} T\big[v_{(\alpha_1)},h_{\pi(1)},\ldots,v_{(\alpha_k)},h_{\pi(k)},v_{(\alpha_{k+1})}\big], \qquad v,h_i \in V.$$

The final sum above is over multi-indices $\alpha \in \mathbb{N}_0^{k+1}$ with order $n-k$, and empty sums are zero.

Proof. We prove Equation (1.2.7) and leave the remaining elementary combinatorics to the reader. To this end, we may and do assume that T is symmetric, i.e., $T = S(T)$. We proceed by induction on k. For the base case ($k=1$), observe that if $v,h \in V$, then

$$P(v+h)-P(v) = \sum_{i=0}^{n-1}\Big(T\big[(v+h)_{(i+1)},v_{(n-i-1)}\big]-T\big[(v+h)_{(i)},v_{(n-i)}\big]\Big)$$
$$= \sum_{i=0}^{n-1} T\big[(v+h)_{(i)},h,v_{(n-i-1)}\big] = \sum_{i=0}^{n-1} T\big[h,(v+h)_{(i)},v_{(n-i-1)}\big]$$

because $P(v) = T\big[v_{(n)}\big]$, the first sum telescopes, and T is n-linear (and symmetric). It follows from the bounded n-linearity of T that

$$\frac{1}{\|h\|_V}\big\|P(v+h)-P(v)-n\,T\big[h,v_{(n-1)}\big]\big\|_W$$
$$\le \sum_{i=0}^{n-1}\big\|T\big[\cdot,(v+h)_{(i)},v_{(n-i-1)}\big]-T\big[\cdot,v_{(n-1)}\big]\big\|_{V\to W} \xrightarrow{h\to 0} 0.$$

Thus, P is once continuously differentiable, and the desired derivative formula holds. This completes the proof of the base case. Now, assume we have proven the result with $k \ge 1$. If $k \ge n$, then the induction hypothesis says $D^kP(v)$ is constant in $v \in V$. Therefore, $D^{k+1}P \equiv 0$, which completes the induction step in this case. Next, assume $k < n$, and define

$$T_0[\mathbf{h},\mathbf{v}] \coloneqq \frac{n!}{(n-k)!}T[\mathbf{h},\mathbf{v}] \in W, \qquad (\mathbf{h},\mathbf{v}) \in V^k \times V^{n-k} = V^n.$$

Then the map $V^{n-k} \ni \mathbf{v} \mapsto T_1[\mathbf{v}] \coloneqq T_0[\cdot,\mathbf{v}] \in B_k\big(V^k;W\big)$ is bounded, $(n-k)$-linear, and symmetric, and the induction hypothesis says precisely that $D^kP(v) = T_1\big[v_{(n-k)}\big]$ for all $v \in V$. Consequently, the base case applies to D^kP and says

$$D\big(D^kP\big)(v)h = (n-k)\,T_1\big[h, v_{(n-k-1)}\big] = (n-k)\,T_0\big[\cdot, h, v_{(n-k-1)}\big], \qquad v, h \in V.$$

A moment's reflection on the definition of T_0 yields the desired formula for $D^{k+1}P$. □

We now focus on the necessary general theory. To begin, we list some basic properties.

Proposition 1.2.8 (Basic properties)**.** *Let $F\colon U \to W$ be a map, and let $v \in U$.*

(i) *If F is Fréchet differentiable at v, then F is continuous at v.*

(ii) *Linear combinations of maps that are k-times Fréchet differentiable at v are k-times Fréchet differentiable at v, and $F \mapsto D^kF(v)$ is a linear operation.*

(iii) (Directional derivatives) *If F is k-times Fréchet differentiable at v, then*

$$D^kF(v)[h_1, \ldots, h_k] = \partial_{h_k} \cdots \partial_{h_1} F(v) \coloneqq \frac{\mathrm{d}}{\mathrm{d}s_k}\Big|_{s_k=0} \cdots \frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0} F(v + s_1h_1 + \cdots + s_kh_k)$$

for all $h_1, \ldots, h_k \in V$.

(iv) *If F is k-times Fréchet differentiable at v, then $D^kF(v)$ is* ***symmetric****: If $\pi \in S_k$ and $h_1, \ldots, h_k \in V$, then $D^kF(v)\big[h_{\pi(1)}, \ldots, h_{\pi(k)}\big] = D^kF(v)[h_1, \ldots, h_k]$.*

Proof. The first two items follow easily from the definitions. The third item follows by induction and testing Equation (1.2.4) on $h = th_1$ with $t \to 0$. The final item is [HJ14, Thm. 1.76]. □

Next, we introduce a topology on a space of C^k maps with a certain boundedness property.

Definition 1.2.9 ($C^k_{\mathrm{bb}}(U;W)$ and its topology)**.** The space of continuous maps $U \to W$ that are bounded on closed balls contained in U is denoted by $C_{\mathrm{bb}}(U;W) = C^0_{\mathrm{bb}}(U;W)$. If $k \in \mathbb{N} \cup \{\infty\}$, then $C^k_{\mathrm{bb}}(U;W)$ is the space of C^k maps $F\colon U \to W$ such that $D^iF \in C_{\mathrm{bb}}\big(U; B_i\big(V^i;W\big)\big)$ whenever $0 \le i < k+1$; here and throughout, $D^0F \coloneqq F$. If $k \in \mathbb{N}_0 \cup \{\infty\}$, then $C^k_{\mathrm{bb}}(U;W)$ is endowed with the $\boldsymbol{C^k_{\mathrm{bb}}}$ **topology**: the locally convex topology generated by the seminorms

$$C^k_{\mathrm{bb}}(U;W) \ni F \mapsto \sup_{w \in \bar{B}_r(v)} \big\|D^iF(w)\big\|_{B_i(V^i;W)} \in \mathbb{R}_+, \qquad 0 \le i < k+1,\ \bar{B}_r(v) \subseteq U.$$

Example 1.2.10 (Finite-dimensional case)**.** Recall that closed balls in V are compact if and only if V is finite-dimensional. In this case, if $k \in \mathbb{N}_0 \cup \{\infty\}$, then $C^k_{\mathrm{bb}}(U;W) = C^k(U;W)$, and the C^k_{bb} topology is the topology of locally uniform convergence of all derivatives of order strictly less than $k+1$, i.e., the $\boldsymbol{C^k}$ **topology**.

Example 1.2.11 (Homogeneous polynomials)**.** By Proposition 1.2.6, if $n \in \mathbb{N}_0$, $T \in B_n(V^n;W)$, and $P \colon V \to W$ is defined by $v \mapsto T\big[v_{(n)}\big]$, then $P \in C^\infty_{\mathrm{bb}}(V;W)$.

To prove the basic topological properties of $C^k_{\mathrm{bb}}(U;W)$, we record a criterion for the Fréchet differentiability of the limit of a sequence of Fréchet differentiable maps.

Theorem 1.2.12 (Differentiability of a limit [HJ14, Thm. 1.85])**.** *Assume W is complete and U is convex and bounded. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of k-times Fréchet differentiable maps $U \to W$. If $\big(D^kF_n\big)_{n\in\mathbb{N}}$ converges uniformly and there exist $x_0,\ldots,x_{k-1} \in U$ such that $\big(D^iF_n(x_i)\big)_{n\in\mathbb{N}}$ converges for all $i \in \{0,\ldots,k-1\}$, then there exists a k-times Fréchet differentiable map $F \colon U \to W$ such that $\big(D^iF_n\big)_{n\in\mathbb{N}}$ converges uniformly to D^iF whenever $i \in \{0,\ldots,k\}$.*

Proposition 1.2.13 (Properties of $C^k_{\mathrm{bb}}(U;W)$)**.** *Let $k \in \mathbb{N}_0 \cup \{\infty\}$.*

(i) *$C^k_{\mathrm{bb}}(U;W)$ is an HLCTVS. If W is complete, then so is $C^k_{\mathrm{bb}}(U;W)$.*

(ii) *If there is a countable family $\mathscr{S}$ of closed balls contained in U such that every closed ball contained in U is covered by finitely many members of $\mathscr{S}$, then $C^k_{\mathrm{bb}}(U;W)$ is metrizable.*

Proof. We take both items in turn.

(i) The C^k_{bb} topology is defined by a collection of seminorms that separates points, so $C^k_{\mathrm{bb}}(U;W)$ is locally convex and Hausdorff. Now, assume W is complete. We show that $C^k_{\mathrm{bb}}(U;W)$ is complete when $k<\infty$. The $k=\infty$ case follows from the finite-k case.

Let $\mathcal{C}$ be a closed ball contained in U, and consider the map

$$C^k_{\mathrm{bb}}(U;W) \ni F \mapsto T_{\mathcal{C}}F \coloneqq \big(F|_{\mathcal{C}} = D^0F|_{\mathcal{C}},\ldots,D^kF|_{\mathcal{C}}\big) \in X \coloneqq \prod_{i=0}^k BC\big(\mathcal{C};B_i\big(V^i;W\big)\big).$$

Also, for $v \in \mathcal{C}$, we define $\mathrm{ev}_v \colon X \to \prod_{i=0}^k B_i\big(V^i;W\big)$ by $(G_0,\ldots,G_k) \mapsto (G_0(v),\ldots,G_k(v))$, i.e., by evaluation at v. By definition of $C^k_{\mathrm{bb}}(U;W)$ and X, $T_{\mathcal{C}}$ and ev_v are continuous linear maps.

Next, let $(F_i)_{i\in I}$ be a Cauchy net in $C^k_{\mathrm{bb}}(U;W)$, and retain the ball $\mathcal{C}\subseteq U$ from the previous paragraph. Since $T_{\mathcal{C}}$ is continuous, $(T_{\mathcal{C}}F_i)_{i\in I}$ is a Cauchy net in X. Since X is complete, $(T_{\mathcal{C}}F_i)_{i\in I}$ converges to an element $\big(F^0_{\mathcal{C}},\dots,F^k_{\mathcal{C}}\big)\in X$. Since the evaluation maps are continuous, $F^j_{\mathcal{C}}$ is the pointwise limit of $\big(D^jF_i|_{\mathcal{C}}\big)_{i\in I}$ for all $j\in\{0,\dots,k\}$. Consequently, if $j\in\{0,\dots,k\}$, then there exists a map $F^j\in C_{\mathrm{bb}}\big(U;B_j\big(V^j;W\big)\big)$ such that $F^j|_{\mathcal{C}}=F^j_{\mathcal{C}}$ for all closed balls $\mathcal{C}\subseteq U$.

Finally, we claim that if $\mathcal{C}=\bar{B}_r(v)$ is a closed ball contained in U, then $F^0|_{B_r(v)}=F^0_{\mathcal{C}}|_{B_r(v)}$ is k-times Fréchet differentiable, and $D^j\big(F^0|_{B_r(v)}\big)=F^j_{\mathcal{C}}|_{B_r(v)}=F^j|_{B_r(v)}$ for all $j\in\{1,\dots,k\}$. Indeed, since X is first countable and $(T_{\mathcal{C}}F_i)_{i\in I}$ converges to $\big(F^0_{\mathcal{C}},\dots,F^k_{\mathcal{C}}\big)$, there exists a sequence $(i_n)_{n\in\mathbb{N}}$ in I such that $(T_{\mathcal{C}}F_{i_n})_{n\in\mathbb{N}}$ converges to $\big(F^0_{\mathcal{C}},\dots,F^k_{\mathcal{C}}\big)$ as well. By definition of $T_{\mathcal{C}}$, an appeal to Theorem 1.2.12 completes the proof of the claim and thus also this item.

(ii) Under the stated hypothesis, the C^k_{bb} topology is generated by the countable family $\big\{F\mapsto\sup_{w\in\mathcal{C}}\big\|D^iF(w)\big\|_{B_i(V^i;W)}:0\le i<k+1,\,\mathcal{C}\in\mathscr{S}\big\}$ of seminorms that separates points. The desired result follows. □

Note that the hypothesis of Proposition 1.2.13(ii) is satisfied when $U=V$ or V is finite-dimensional. In the former case, we can take $\mathscr{S}=\big\{\bar{B}_n(0):n\in\mathbb{N}\big\}$; in the latter, we can take $\mathscr{S}$ to be a countable family of closed balls in U whose interiors cover U.

We end this section with some results on holomorphic maps.

Theorem 1.2.14 (Characterizations of holomorphicity [HJ14, Thm. 1.160])**.** *Let $\mathbb{F}=\mathbb{C}$, and assume V and W are complete. A map $F\colon U\to W$ is holomorphic if and only if it is C^∞ in U (with $\mathbb{C}$-multilinear Fréchet derivatives since $\mathbb{F}=\mathbb{C}$), if and only if it is analytic in U. Please see [HJ14, Def. 1.154] for the definition of analyticity.*

Remark 1.2.15 (Taylor series expansion)**.** By [HJ14, Thms. 1.140 & 1.146], more is true under the hypotheses of Theorem 1.2.14. Specifically, if $F\in\mathrm{Hol}(U;W)$ and $w\in U$, then there exists an $r>0$ such that $\bar{B}_r(w)\subseteq U$ and

$$F(v)=\sum_{n=0}^{\infty}\frac{1}{n!}\partial^n_{v-w}F(w)=\sum_{n=0}^{\infty}\frac{1}{n!}D^nF(w)\big[(v-w)_{(n)}\big],\qquad v\in\bar{B}_r(w),$$

where the series on the right-hand side converges absolutely uniformly on $\bar{B}_r(w)$.

The essence of the first equivalence in Theorem 1.2.14 is that we can upgrade C^1 to C^∞ in the complex case. The essence of the next result is that we can upgrade the local uniform convergence of a net of holomorphic maps to the local uniform convergence of all its derivatives.

Theorem 1.2.16 (Convergence of holomorphic maps)**.** *Let $\mathbb{F} = \mathbb{C}$, and assume V and W are complete. If $(F_i)_{i \in I}$ is a net in* $\mathrm{Hol}(U; W)$ *converging locally uniformly to $F \colon U \to W$ (i.e., for all $v \in U$, there exists a neighborhood $U_0 \subseteq U$ of v such that $(F_i|_{U_0})_{i \in I}$ converges uniformly to $F|_{U_0}$), then $F \in$* $\mathrm{Hol}(U; W)$*, and $\big(D^k F_i\big)_{i \in I}$ converges locally uniformly to $D^k F$ whenever $k \in \mathbb{N}$.*

Sketch of proof. Fix $v \in U$, let $\varepsilon > 0$ be such that $\bar{B}_\varepsilon(v) \subseteq U$ and $\big(F_i|_{\bar{B}_\varepsilon(v)}\big)_{i \in I}$ converges uniformly to $F|_{\bar{B}_\varepsilon(v)}$, and define $\delta \coloneqq \varepsilon/2$. By Cauchy's estimates [HJ14, Cor. 1.164] (and the polarization formula for multilinear maps [HJ14, Prop. 1.11]), if $i, j \in I$ and $k \in \mathbb{N}$, then

$$\left\| D^k F_i(w) - DF_j^k(w) \right\|_{B_k(V^k; W)} \leq \frac{k^k}{\delta^k} \sup_{u \in \bar{B}_\delta(w)} \|F_i(u) - F_j(u)\|_W, \qquad w \in \bar{B}_\delta(v),$$

because $F_i - F_j \in \mathrm{Hol}(U; W)$ and $\bar{B}_\delta(w) \subseteq \bar{B}_\varepsilon(v) \subseteq U$ whenever $w \in \bar{B}_\delta(v)$. Consequently,

$$\sup_{w \in \bar{B}_\delta(v)} \left\| D^k F_i(w) - DF_j^k(w) \right\|_{B_k(V^k; W)} \leq \frac{k^k}{\delta^k} \sup_{u \in \bar{B}_\varepsilon(v)} \|F_i(u) - F_j(u)\|_W.$$

It follows that $\big(D^k F_i|_{\bar{B}_\delta(v)}\big)_{i \in I}$ is uniformly Cauchy and therefore uniformly convergent. An appeal to Theorem 1.2.12 then completes the proof. (We encourage the reader to fill in the details of the last two sentences.) □

Corollary 1.2.17. *Under the assumptions of Theorem 1.2.14,* $\mathrm{Hol}(U; W) \cap C_{\mathrm{bb}}(U; W)$ *is a closed linear subspace of $C_{\mathrm{bb}}(U; W)$.*

Proof. This is immediate from Theorem 1.2.16 because a C_{bb}-convergent net is automatically locally uniformly convergent. □

By combining Example 1.2.10, Proposition 1.2.13, and Corollary 1.2.17, we see that if $\mathbb{F} = \mathbb{C}$, V is finite-dimensional, and W is a Banach space, then $\mathrm{Hol}(U; W)$ is a Fréchet space with the topology of locally uniform convergence.

1.3 Divided differences

In this section, we define divided differences and collect their relevant properties.

Definition 1.3.1 (Divided differences)**.** Let $S \subseteq \mathbb{C}$. For a function $f\colon S \to \mathbb{C}$, recursively define $f^{[0]} := f$ and, for $k \in \mathbb{N}$ and distinct $\lambda_1, \ldots, \lambda_{k+1} \in S$,

$$f^{[k]}(\lambda_1, \ldots, \lambda_{k+1}) := \frac{f^{[k-1]}(\lambda_1, \ldots, \lambda_k) - f^{[k-1]}(\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1})}{\lambda_k - \lambda_{k+1}}.$$

The function $f^{[k]}$ is the $\boldsymbol{k}^{\mathbf{th}}$ **divided difference** of f.

Notation 1.3.2 (Σ_m, Δ_m, and ρ_m)**.** If $m \in \mathbb{N}$, then

$$\Sigma_m := \big\{\vec{s} = (s_1, \ldots, s_m) \in \mathbb{R}_+^m : |\vec{s}\,| = s_1 + \cdots + s_m \leq 1\big\}, \quad \text{and}$$
$$\Delta_m := \big\{\mathbf{t} = (t_1, \ldots, t_{m+1}) \in \mathbb{R}_+^{m+1} : t_1 + \cdots + t_{m+1} = 1\big\}.$$

Also, ρ_m is the pushforward of the m-dimensional Lebesgue measure on Σ_m by the homeomorphism $\Sigma_m \ni \vec{s} \mapsto (\vec{s}, 1 - |\vec{s}\,|) \in \Delta_m$. Explicitly, ρ_m is the Borel measure on Δ_m characterized by

$$\int_{\Delta_m} \varphi(\mathbf{t})\, \rho_m(\mathrm{d}\mathbf{t}) = \int_{\Sigma_m} \varphi(\vec{s}, 1 - |\vec{s}\,|)\, \mathrm{d}\vec{s}, \qquad \varphi \in \ell^\infty(\Delta_m, \mathcal{B}_{\Delta_m}).$$

In particular, $\rho_m(\Delta_m) = 1/m!$, as the reader may verify.

Proposition 1.3.3 (Basic properties)**.** *Fix $S \subseteq \mathbb{C}$, functions $f, g\colon S \to \mathbb{C}$, $k \in \mathbb{N}$, and distinct $\lambda_1, \ldots, \lambda_{k+1} \in S$. Also, write $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_{k+1})$.*

(i) $f^{[k]}(\boldsymbol{\lambda}) = \sum_{i=1}^{k+1} f(\lambda_i) \prod_{j \neq i} (\lambda_i - \lambda_j)^{-1}$*. In particular, $f^{[k]}$ is symmetric.*

(ii) (Product rule) $(fg)^{[k]}(\boldsymbol{\lambda}) = \sum_{i=0}^{k} f^{[i]}(\lambda_1, \ldots, \lambda_{i+1})\, g^{[k-i]}(\lambda_{i+1}, \ldots, \lambda_{k+1})$.

(iii) *Suppose $S \subseteq \mathbb{R}$ is an open interval or $S \subseteq \mathbb{C}$ is open and convex. If $f \in C^k(S)$, then*

$$f^{[k]}(\boldsymbol{\lambda}) = \int_{\Delta_k} f^{(k)}(\mathbf{t} \cdot \boldsymbol{\lambda})\, \rho_k(\mathrm{d}\mathbf{t}) = \int_{\Sigma_k} f^{(k)}\left(\sum_{i=1}^{k} s_i \lambda_i + \left(1 - \sum_{i=1}^{k} s_i\right) \lambda_{k+1}\right) \mathrm{d}s_1 \cdots \mathrm{d}s_k.$$

In particular, if U is an open subset of $\mathbb{R}$ or $\mathbb{C}$ and $h \in C^k(U)$, then $h^{[k]}$ extends uniquely to a continuous function $U^{k+1} \to \mathbb{C}$. We use the same notation for this extension.

Proof. All three items are proven by induction on k. The arguments for the first two items are straightforward and left to the reader. For the third item, suppose $f \in C^1(S)$, and let $\lambda_1, \lambda_2 \in S$ be distinct. By the fundamental theorem of calculus,

$$f^{[1]}(\lambda_1, \lambda_2) = \frac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2} = \int_0^1 f'(s_1\lambda_1 + (1 - s_1)\lambda_2)\, \mathrm{d}s_1.$$

This establishes the base case. Now, assume the desired formula holds on $C^k(S)$ with $k \geq 1$. If $f \in C^{k+1}(S)$ and $\lambda_1, \ldots, \lambda_{k+2} \in S$ are distinct, then, writing $\vec{\mu} := (\lambda_1, \ldots, \lambda_k) \in S^k$, we have

$$\begin{aligned} f^{[k+1]}\big(\vec{\mu}, \lambda_{k+1}, \lambda_{k+2}\big) &= \int_{\Sigma_k} \frac{f^{(k)}\big(\vec{\mu}\cdot\vec{s} + (1 - |\vec{s}\,|)\lambda_{k+1}\big) - f^{(k)}\big(\vec{\mu}\cdot\vec{s} + (1 - |\vec{s}\,|)\lambda_{k+2}\big)}{\lambda_{k+1} - \lambda_{k+2}}\, \mathrm{d}\vec{s} \quad (1.3.4) \\ &= \int_{\Sigma_k} (1 - |\vec{s}\,|)\left(f^{(k)}\right)^{[1]}\big(\vec{\mu}\cdot\vec{s} + (1 - |\vec{s}\,|)\lambda_{k+1}, \vec{\mu}\cdot\vec{s} + (1 - |\vec{s}\,|)\lambda_{k+2}\big)\, \mathrm{d}\vec{s} \\ &= \int_{\Sigma_k} (1 - |\vec{s}\,|) \int_0^1 f^{(k+1)}\big(\vec{\mu}\cdot\vec{s} + (1 - |\vec{s}\,|)(t\lambda_{k+1} + (1-t)\lambda_{k+2})\big)\, \mathrm{d}t\, \mathrm{d}\vec{s} \quad (1.3.5) \\ &= \int_{\Sigma_k} \int_0^{1-|\vec{s}\,|} f^{(k+1)}\big(\vec{\mu}\cdot\vec{s} + s_{k+1}\lambda_{k+1} + (1 - |\vec{s}\,| - s_{k+1})\lambda_{k+2}\big)\, \mathrm{d}s_{k+1}\, \mathrm{d}\vec{s} \quad (1.3.6) \\ &= \int_{\Sigma_{k+1}} f^{(k+1)}\left(\sum_{i=1}^{k+1} s_i\lambda_i + \left(1 - \sum_{i=1}^{k+1} s_i\right)\lambda_{k+2}\right) \mathrm{d}s_1 \cdots \mathrm{d}s_{k+1}. \end{aligned}$$

Equation (1.3.4) holds by definition of $f^{[k+1]}$ and the inductive hypothesis, Equation (1.3.5) holds by the base case, and Equation (1.3.6) holds by the change of variable $s_{k+1} := (1 - |\vec{s}\,|)t$. $\square$

It is perhaps worth noting that the product rule in Proposition 1.3.3(ii) is somewhat strange, as the left-hand side $(fg)^{[k]}(\lambda_1, \ldots, \lambda_{k+1})$ is symmetric in $(\lambda_1, \ldots, \lambda_{k+1})$, while the terms $f^{[i]}(\lambda_1, \ldots, \lambda_{i+1})\, g^{[k-i]}(\lambda_{i+1}, \ldots, \lambda_{k+1})$ in the decomposition are not.

Now, here is a useful consequence of Proposition 1.3.3(iii).

Corollary 1.3.7. *Retain the assumptions from Proposition 1.3.3(iii). If $f \in C^k(S)$, then*

$$\left|f^{[k]}(\boldsymbol{\lambda})\right| \leq \frac{1}{k!} \sup\left\{\left|f^{(k)}(s)\right| : s \in \operatorname{conv}(\lambda_1, \ldots, \lambda_{k+1})\right\}, \qquad \boldsymbol{\lambda} \in S^{k+1},$$

where $\operatorname{conv}(\lambda_1, \ldots, \lambda_{k+1})$ *is the convex hull of* $\{\lambda_1, \ldots, \lambda_{k+1}\}$. *Furthermore,*

$$f^{(k)}(\lambda) = k!\, f^{[k]}\big(\lambda_{(k+1)}\big) = k!\, f^{[k]}(\lambda, \ldots, \lambda), \qquad \lambda \in S.$$

Proof. Both claimed relations follow from Proposition 1.3.3(iii) and the fact that $\rho_k(\Delta_k) = (k!)^{-1}$. For the inequality, we also need the observation that $\{\mathbf{t} \cdot \boldsymbol{\lambda} : \mathbf{t} \in \Delta_k\} = \operatorname{conv}(\lambda_1, \ldots, \lambda_{k+1})$. $\square$

Next, here are some important example calculations of divided differences.

Example 1.3.8 (Polynomials)**.** Let $n \in \mathbb{N}_0$, and define $p_n(\lambda) \coloneqq \lambda^n \in \mathbb{C}[\lambda] \subseteq \operatorname{Hol}(\mathbb{C})$. We claim that if $k \in \mathbb{N}$, then

$$p_n^{[k]}(\boldsymbol{\lambda}) = \sum_{|\gamma|=n-k} \boldsymbol{\lambda}^\gamma = \sum_{\gamma \in \mathbb{N}_0^{k+1} : |\gamma|=n-k} \lambda_1^{\gamma_1} \cdots \lambda_{k+1}^{\gamma_{k+1}}, \qquad \boldsymbol{\lambda} \in \mathbb{C}^{k+1}. \tag{1.3.9}$$

Since $p_n^{[k]}$ is continuous, it suffices to establish Equation (1.3.9) for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{k+1}) \in \mathbb{C}^{k+1}$ such that $\lambda_1, \ldots, \lambda_{k+1}$ are distinct. To do so, we proceed by induction on k. The base case is the well-known identity

$$p_n^{[1]}(\lambda_1, \lambda_2) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \sum_{i=0}^{n-1} \lambda_1^i \lambda_2^{n-1-i} = \sum_{\gamma_1+\gamma_2=n-1} \lambda_1^{\gamma_1} \lambda_2^{\gamma_2}, \qquad \lambda_1 \neq \lambda_2.$$

Now, suppose Equation (1.3.9) holds. If λ_{k+2} is distinct from $\lambda_1, \ldots, \lambda_{k+1}$, then

$$\begin{aligned} p_n^{[k+1]}(\lambda_1, \ldots, \lambda_{k+2}) &= \frac{p_n^{[k]}(\lambda_1, \ldots, \lambda_{k+1}) - p_n^{[k]}(\lambda_1, \ldots, \lambda_k, \lambda_{k+2})}{\lambda_{k+1} - \lambda_{k+2}} \\ &= \sum_{|\gamma|=n-k} \lambda_1^{\gamma_1} \cdots \lambda_k^{\gamma_k} \frac{\lambda_{k+1}^{\gamma_{k+1}} - \lambda_{k+2}^{\gamma_{k+1}}}{\lambda_{k+1} - \lambda_{k+2}} \\ &= \sum_{|\gamma|=n-k} \left(\sum_{\delta_1+\delta_2=\gamma_{k+1}-1} \lambda_1^{\gamma_1} \cdots \lambda_k^{\gamma_k} \lambda_{k+1}^{\delta_1} \lambda_{k+2}^{\delta_2} \right) \\ &= \sum_{\tilde{\gamma} \in \mathbb{N}_0^{k+2} : |\tilde{\gamma}|=n-k-1} \lambda_1^{\tilde{\gamma}_1} \cdots \lambda_{k+2}^{\tilde{\gamma}_{k+2}} \end{aligned}$$

by the inductive hypothesis and the base case. This completes the proof of the claim. In particular, if $p(\lambda) \in \mathbb{C}[\lambda]$, then $p^{[k]}(\lambda_1, \ldots, \lambda_{k+1}) \in \mathbb{C}[\lambda_1, \ldots, \lambda_{k+1}]$, for all $k \in \mathbb{N}$.

Example 1.3.10 (Rational functions)**.** Let $U \subseteq \mathbb{C}$ be an open set, and fix $z_0 \in \mathbb{C} \setminus U$. Define

$$r_{z_0}(\lambda) \coloneqq \frac{1}{z_0 - \lambda}, \qquad \lambda \in U.$$

Of course, $r_{z_0} \in \mathrm{Hol}(U)$. By an easy induction argument, if $k \in \mathbb{N}$, then

$$r_{z_0}^{[k]}(\boldsymbol{\lambda}) = \frac{1}{(z_0 - \lambda_1) \cdots (z_0 - \lambda_{k+1})}, \qquad \boldsymbol{\lambda} \in U^{k+1}. \tag{1.3.11}$$

By combining Equation (1.3.11) with the previous example, the product rule in Proposition 1.3.3(ii), and the fundamental theorem of algebra, we see that if $r(\lambda)$ is a rational function with poles outside of U and $k \in \mathbb{N}$, then there exist and polynomials $q_1(\lambda), \ldots, q_{k+1}(\lambda) \in \mathbb{C}[\lambda]$ with roots outside of U and a multivariate polynomial $P(\boldsymbol{\lambda}) \in \mathbb{C}[\boldsymbol{\lambda}] = \mathbb{C}[\lambda_1, \ldots, \lambda_{k+1}]$ such that

$$r^{[k]}(\boldsymbol{\lambda}) = \frac{P(\boldsymbol{\lambda})}{q_1(\lambda_1) \cdots q_{k+1}(\lambda_{k+1})}, \qquad \boldsymbol{\lambda} \in U^{k+1}.$$

This observation will come in handy in §2.4.

Notation 1.3.12. Let $(\Omega, \mathscr{F})$ be a measurable space.

(i) $\ell^0(\Omega, \mathscr{F})$ is the set of $(\mathscr{F}, \mathcal{B}_{\mathbb{C}})$-measurable functions $\Omega \to \mathbb{C}$, and $\ell^\infty(\Omega, \mathscr{F})$ is the set of $f \in \ell^0(\Omega, \mathscr{F})$ such that $\|f\|_{\ell^\infty(\Omega)} \coloneqq \sup_{\omega \in \Omega} |f(\omega)| < \infty$.

(ii) $M(\Omega, \mathscr{F})$ is the space of complex measures on $(\Omega, \mathscr{F})$. If $\mu \in M(\Omega, \mathscr{F})$, then $|\mu|$ is the variation measure of μ, and $\|\mu\| \coloneqq |\mu|(\Omega)$.

Definition 1.3.13 (Wiener space)**.** If $\mu \in M(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and $k \in \mathbb{N}_0$, then $\mu_{(k)} \coloneqq \int_{\mathbb{R}} |\xi|^k \, |\mu|(\mathrm{d}\xi)$ is the "k^{th} moment" of $|\mu|$. The $\boldsymbol{k}^{\mathbf{th}}$ **Wiener space** $W_k(\mathbb{R})$ is the set of functions $f \colon \mathbb{R} \to \mathbb{C}$ such that there exists a (necessarily unique) $\mu \in M(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ with $\mu_{(k)} < \infty$ and $f(\lambda) = \int_{\mathbb{R}} e^{i\lambda\xi} \, \mu(\mathrm{d}\xi)$ for all $\lambda \in \mathbb{R}$.

Example 1.3.14 (Wiener space)**.** Let $k \in \mathbb{N}$. If $f = \int_{\mathbb{R}} e^{i \cdot \xi} \, \mu(\mathrm{d}\xi) \in W_k(\mathbb{R})$, then $f \in C^k(\mathbb{R})$. More specifically,

$$f^{(k)}(\lambda) = \int_{\mathbb{R}} \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} e^{i\lambda\xi} \, \mu(\mathrm{d}\xi) = \int_{\mathbb{R}} (i\xi)^k e^{i\lambda\xi} \, \mu(\mathrm{d}\xi), \qquad \lambda \in \mathbb{R}. \tag{1.3.15}$$

In particular,

$$f^{[k]}(\boldsymbol{\lambda}) = \int_{\Delta_k} \int_{\mathbb{R}} (i\xi)^k e^{i(\mathbf{t} \cdot \boldsymbol{\lambda})\xi} \, \mu(\mathrm{d}\xi) \, \rho_k(\mathrm{d}\mathbf{t}), \qquad \boldsymbol{\lambda} \in \mathbb{R}^{k+1}, \tag{1.3.16}$$

by Proposition 1.3.3(iii).

We end this section with another useful expression for the divided differences of a holomorphic function. To this end, we review some complex analysis.

Definition 1.3.17 (Cycles, trace, and index)**.** Let $U \subseteq \mathbb{C}$ be an open set. A **cycle** Γ in U is a finite collection $(\gamma_i \colon [a_i, b_i] \to U)_{i=1}^n$ of piecewise C^1 closed curves in U. The **trace** of Γ is the set $\Gamma^* := \bigcup_{i=1}^n \gamma_i([a_i, b_i]) \subseteq U$. If V is a complex Fréchet space and $\varphi \colon \Gamma^* \to V$ is continuous, then

$$\int_\Gamma \varphi(z)\, \mathrm{d}z := \sum_{i=1}^n \int_{a_i}^{b_i} \varphi(\gamma_i(t))\, \dot{\gamma}_i(t)\, \mathrm{d}t \in V.$$

Finally,

$$\mathrm{Ind}_\Gamma(z) := \frac{1}{2\pi i} \int_\Gamma \frac{1}{w - z}\, \mathrm{d}w, \qquad z \in \mathbb{C} \setminus \Gamma^*,$$

is the **index** or **winding number** (**function**) of Γ.

Theorem 1.3.18 (Properties of Ind_Γ [Rud86, Thm. 10.10])**.** *If $U \subseteq \mathbb{C}$ is an open set and Γ is a cycle in U, then $\mathrm{Ind}_\Gamma(z) \in \mathbb{Z}$ for all $z \in \mathbb{C} \setminus \Gamma^*$, $\mathrm{Ind}_\Gamma \colon \mathbb{C} \setminus \Gamma^* \to \mathbb{Z}$ is constant on each connected component of $\mathbb{C} \setminus \Gamma^*$, and $\mathrm{Ind}_\Gamma(z) = 0$ whenever $|z|$ is sufficiently large.*

Theorem 1.3.19 (Cauchy's integral formula)**.** *Let $U \subseteq \mathbb{C}$ be an open set, let V be a complex Banach space, and let $F \colon U \to V$ be a holomorphic map. If Γ is a cycle in U such that $\mathrm{Ind}_\Gamma(z) = 0$ whenever $z \in \mathbb{C} \setminus U$, then*

$$\mathrm{Ind}_\Gamma(z)\, F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(w)}{w - z}\, \mathrm{d}w, \qquad z \in U \setminus \Gamma^*.$$

Proof. The scalar case ($V = \mathbb{C}$) is [Rud86, Thm. 10.35, Eq. (2)]. The general case follows from the scalar case applied to the functions $\{\ell \circ F : \ell \in V^*\}$. □

We are now prepared for the final result of this section.

Proposition 1.3.20 (Holomorphic divided differences)**.** *Let $U \subseteq \mathbb{C}$ be an open set, and let Γ be a cycle in U. If $f \in \mathrm{Hol}(U)$ and $k \in \mathbb{N}_0$, then*

$$(\mathrm{Ind}_\Gamma\, f)^{[k]}(\boldsymbol{\lambda}) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - \lambda_1) \cdots (z - \lambda_{k+1})}\, \mathrm{d}z, \qquad \boldsymbol{\lambda} \in (U \setminus \Gamma^*)^{k+1}.$$

In particular, if $V := \{z \in U \setminus \Gamma^* : \mathrm{Ind}_\Gamma(z) = 1\}$*, then*

$$f^{[k]}(\boldsymbol{\lambda}) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\lambda_1)\cdots(z-\lambda_{k+1})}\,\mathrm{d}z, \qquad \boldsymbol{\lambda} \in V^{k+1}.$$

Consequently, $f^{[k]} \in \mathrm{Hol}\left(U^{k+1}\right)$.

Proof. As with essentially all properties of the k^{th} divided difference, we proceed by induction on k. The base case ($k = 0$) is precisely Cauchy's integral formula. Now, write $g := \mathrm{Ind}_\Gamma f$, and assume the desired formula for $g^{[k]}$ holds with $k \geq 0$. If $\lambda_1, \ldots, \lambda_{k+2} \in U \setminus \Gamma^*$ are distinct, then

$$\begin{aligned} g^{[k+1]}(\lambda_1,\ldots,\lambda_{k+2}) &= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\lambda_1)\cdots(z-\lambda_k)(\lambda_{k+1}-\lambda_{k+2})}\left(\frac{1}{z-\lambda_{k+1}} - \frac{1}{z-\lambda_{k+2}}\right)\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\lambda_1)\cdots(z-\lambda_{k+2})}\,\mathrm{d}z \end{aligned}$$

by definition of $g^{[k+1]}$ and the inductive hypothesis. An appeal to the continuity of both sides in $(\lambda_1, \ldots, \lambda_{k+2})$ completes the proof. □

1.4 Banach algebras

In this section, we lay out a few basics of Banach algebras, spectral theory, and resolvents. We also say a few words about C^*-algebras and von Neumann (or W^*-)algebras.

Definition 1.4.1 (Banach algebra)**.** A **Banach algebra** is a complex Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ together with a(n associative) $\mathbb{C}$-algebra structure on $\mathcal{B}$ satisfying

$$\|ab\|_{\mathcal{B}} \leq \|a\|_{\mathcal{B}}\|b\|_{\mathcal{B}}, \qquad a, b \in \mathcal{B}.$$

We often shall say "$\mathcal{B}$ is a Banach algebra" in this case, keeping the norm and algebra structure implicit. The Banach algebra $\mathcal{B}$ is **unital** if the underlying algebra is unital and $\|1\|_{\mathcal{B}} = 1$.

Example 1.4.2 (Bounded operators)**.** If V is a complex Banach space, then $B(V)$ is a unital Banach algebra with the operator norm and the usual algebra structure in which composition is the product operation. In particular, if $n \in \mathbb{N}$, then $\mathrm{M}_n(\mathbb{C}) := \{n \times n \text{ complex matrices}\} \cong B(\mathbb{C}^n)$ is a unital Banach algebra.

We now move on to the spectral theory of an element of a unital Banach algebra.

Notation 1.4.3 (Invertible elements)**.** If $\mathcal{B}$ is a unital Banach algebra, then $\mathcal{B}_{\mathrm{inv}}$ is the group of multiplicatively invertible elements of $\mathcal{B}$, i.e., the set of $b \in \mathcal{B}$ such that there exists a (necessarily unique) $b^{-1} \in \mathcal{B}$ such that $bb^{-1} = b^{-1}b = 1$.

Definition 1.4.4 (Resolvent and spectrum)**.** Let $\mathcal{B}$ be a unital Banach algebra, and fix $a \in \mathcal{B}$. The set $\rho(a) = \rho_{\mathcal{B}}(a) := \{\lambda \in \mathbb{C} : \lambda - a = \lambda 1 - a \in \mathcal{B}_{\mathrm{inv}}\}$ is the **resolvent set** of a; its complement $\sigma(a) = \sigma_{\mathcal{B}}(a) := \mathbb{C} \setminus \rho(a)$ is the **spectrum** of a. The **spectral radius** of a is the number $r(a) = r_{\mathcal{B}}(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}$, and the **resolvent** of a is the map $\rho(a) \ni \lambda \mapsto R_\lambda(a) := (\lambda - a)^{-1} \in \mathcal{B}_{\mathrm{inv}}$.

Theorem 1.4.5 (Properties of $\mathcal{B}_{\mathrm{inv}}$, $\rho(a)$, and $\sigma(a)$)**.** *Let $\mathcal{B}$ be a unital Banach algebra.*

(i) (Geometric series) *If $a \in \mathcal{B}$ and $\sum_{n=0}^{\infty} \|a^n\|_{\mathcal{B}} < \infty$, then $1 - a \in \mathcal{B}_{\mathrm{inv}}$, and*

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

For example, this is the case if $\|a\|_{\mathcal{B}} < 1$, in which case $\left\|(1-a)^{-1}\right\|_{\mathcal{B}} \leq 1/(1 - \|a\|_{\mathcal{B}})$.

(ii) *If $U \subseteq \mathbb{C}$ is an open set, then $\mathcal{B}_U := \{a \in \mathcal{B} : \sigma(a) \subseteq U\}$ is open in $\mathcal{B}$. In particular, $\mathcal{B}_{\mathrm{inv}} = \mathcal{B}_{\mathbb{C}\setminus\{0\}}$ is open in $\mathcal{B}$.*

(iii) (Resolvent identity) *If $a, b \in \mathcal{B}$, $\lambda \in \rho(a)$, and $\mu \in \rho(b)$, then*

$$R_\lambda(a) - R_\mu(b) = R_\lambda(a)\,(\mu - \lambda + a - b)\,R_\mu(b).$$

(iv) *The resolvent set $\rho(a) \subseteq \mathbb{C}$ is open, the resolvent $\rho(a) \ni \lambda \mapsto R_\lambda(a) \in \mathcal{B}$ is holomorphic, and $\sigma(a)$ is (closed and) nonempty.*

(v) (Gel'fand's spectral radius formula) *If $a \in \mathcal{B}$, then*

$$r(a) = \inf_{n \in \mathbb{N}} \|a^n\|_{\mathcal{B}}^{\frac{1}{n}} = \lim_{n \to \infty} \|a^n\|_{\mathcal{B}}^{\frac{1}{n}}.$$

In particular, $r(a) \leq \|a\|_{\mathcal{B}}$, so $\sigma(a) \subseteq \mathbb{C}$ is compact.

Please see [Rud91, Ch. 10] for proofs of all the items in the theorem above. The first four items are not so difficult to prove. Also, while Gel'fand's spectral radius formula takes some work to prove, the inequality $r(a) \leq \|a\|_{\mathcal{B}}$ is easy to establish.

Next, we briefly discuss some special classes of more highly structured Banach algebras. For the next definition, recall that a function $f\colon S \to S$ on a set S is an **involution** if $f \circ f = \mathrm{id}_S$.

Definition 1.4.6 (Various $*$-algebras)**.** Let $\mathcal{A}$ be a $\mathbb{C}$-algebra.

(i) A **$*$-operation** on $\mathcal{A}$ is an involution $*\colon \mathcal{A} \to \mathcal{A}$, written $a^* := *(a)$ for $a \in \mathcal{A}$, such that $(\lambda a)^* = \bar{\lambda}\, a^*$, $(a+b)^* = a^* + b^*$, and $(ab)^* = b^* a^*$ for all $\lambda \in \mathbb{C}$ and $a, b \in \mathcal{A}$. In this case, $\mathcal{A}$ is a **$*$-algebra**. Now, suppose $\mathcal{B}$ is another $*$-algebra. A **$*$-homomorphism** is an algebra homomorphism $\pi\colon \mathcal{A} \to \mathcal{B}$ such that $\pi(a^*) = \pi(a)^*$ for all $a \in \mathcal{A}$.

(ii) A **Banach $*$-algebra** is a Banach algebra with an isometric $*$-operation.

(iii) A Banach $*$-algebra $\mathcal{A}$ is a ***C*-algebra*** if

$$\|a^* a\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}^2, \qquad a \in \mathcal{A}.$$

This is the ***C*-identity*** or ***C*-condition***.

(iv) If $\mathcal{A}$ is a unital C^*-algebra and there exists a complex Banach space V such that $\mathcal{A}$ is isometrically isomorphism to V^*, then $\mathcal{A}$ is a ***W*-algebra***.

Example 1.4.7 (Bounded operators on Hilbert spaces)**.** If H is a complex Hilbert space, then $B(H)$ is a unital C^*-algebra with the adjoint $*$-operation. In particular, if $n \in \mathbb{N}$, then $\mathrm{M}_n(\mathbb{C})$ is a C^*-algebra with the conjugate-transpose $*$-operation.

Example 1.4.8 (Continuous functions, commutative C^*-algebras)**.** If X is a locally compact Hausdorff space, then $C_0(X) = \{f \in C(X) : f \text{ vanishes at } \infty\}$ is a commutative C^*-algebra with the norm $\|\cdot\|_{\ell^\infty(X)}$ and pointwise operations. Furthermore, $C_0(X)$ is unital if and only if X is compact, in which case $C_0(X) = C(X)$ and $\sigma(f) = \operatorname{im} f = f(X)$ for all $f \in C(X)$. Finally, if $\mathcal{A}$ is a commutative C^*-algebra, then there exists a (unique-up-to-homeomorphism) locally compact Hausdorff space X and an isometric $*$-isomorphism $\pi\colon \mathcal{A} \to C_0(X)$; if $\mathcal{A}$ is also unital, then X is compact, in which case $C_0(X) = C(X)$. Please see [Con90, §VIII.2] for details.

Example 1.4.9 (Essentially bounded functions, commutative W^*-algebras)**.** Let $(\Omega, \mathscr{F}, \mu)$ be a measure space such that μ is not identically zero. Then $L^\infty(\mu)$ is a unital, commutative C^*-algebra under pointwise μ-almost everywhere operations, and

$$\sigma(f) = \mu\text{-}\operatorname{ess\,ran} f = \{\lambda \in \mathbb{C} : \mu(\{\omega \in \Omega : |f(\omega) - \lambda| < \varepsilon\}) > 0 \text{ for all } \varepsilon > 0\}, \qquad f \in L^\infty(\mu).$$

Now, define $M \colon L^\infty(\mu) \to L^1(\mu)^*$ by

$$(Mf)(g) := \int_\Omega fg \, \mathrm{d}\mu, \qquad f \in L^\infty(\mu), \; L^1(\mu).$$

By [Fol99, Prop. 6.13], if μ is semifinite, then M is an isometry. It is not hard to show, as we encourage the reader to do, that if μ is not semifinite, then M is not injective. Thus, M is injective if and only if μ is semifinite, in which case M is an isometry. I. E. Segal established in [Seg51, Thm. 5.1] several equivalent characterizations of the situation in which M is surjective as well. For instance, M is an isometric isomorphism if and only if μ is localizable,[1] e.g., if μ is σ-finite. From this discussion, we learn that if μ is localizable, then $L^\infty(\mu)$ is a commutative W^*-algebra. Finally, by [Sak71, Prop. 1.18.1], if $\mathcal{M}$ is a commutative W^*-algebra, then there exists a localizable measure space $(\Xi, \mathscr{G}, \nu)$ and an isometric $*$-isomorphism $\pi \colon \mathcal{M} \to L^\infty(\nu)$.

Remark 1.4.10 (Boundedness of $*$-homomorphisms between C^*-algebras)**.** In the last two examples, we used the term "isometric $*$-isomorphism." It turns out this term is redundant: [Con90, Thm. VIII.4.8] says that if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and $\pi \colon \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, then $\|\pi\|_{\mathcal{A}\to\mathcal{B}} \leq 1$, and π is injective if and only if it is an isometry.

We end this section with "concrete" characterizations of C^*-algebras and W^*-algebras.

Definition 1.4.11. A ***C*-subalgebra*** of a C^*-algebra is a (topologically) closed subalgebra that is closed under $*$. A **concrete *C*-algebra*** is a C^*-subalgebra of $B(H)$, where H is a complex Hilbert space. A **von Neumann algebra** is a unital concrete C^*-algebra $\mathcal{M} \subseteq B(H)$ that is closed in the weak operator topology on $B(H)$ (Definition 4.1.1(i)).

[1]The measure μ is **localizable** if μ is semifinite and every collection of measurable sets has a "μ-essential union," i.e., for every $\mathscr{S} \subseteq \mathscr{F}$, there exists a set $S \in \mathscr{F}$ such that $\mu(G \setminus S) = 0$ for all $G \in \mathscr{S}$ and whenever $S_0 \in \mathscr{F}$ satisfies $\mu(G \setminus S_0) = 0$ for all $G \in \mathscr{F}$, we have $\mu(S \setminus S_0) = 0$.

Of course, a C^*-subalgebra of a C^*-algebra is a C^*-algebra in its own right. In particular, a concrete C^*-algebra is a C^*-algebra. Now, here is an algebraic characterization of when a concrete C^*-algebra is a von Neumann algebra.

Definition 1.4.12 (Commutant and bicommutant)**.** Let H be a complex Hilbert space. If $S \subseteq B(H)$, then

$$S' := \{a \in B(H) : [a, s] := as - sa = 0 \text{ for all } s \in S\}$$

is the **commutant** of S, and $S'' := (S')'$ is the **bicommutant** of S.

Observe that if $S \subseteq B(H)$ is closed under the adjoint, then $S' \subseteq B(H)$ is a von Neumann algebra, and $S'' \subseteq B(H)$ is a von Neumann algebra containing S—in fact, the smallest such von Neumann algebra.

Theorem 1.4.13 (Von Neumann's bicommutant theorem [Dix81, Cor. I.3.2])**.** *A $*$-subalgebra $\mathcal{M} \subseteq B(H)$ is a von Neumann algebra if and only if $\mathcal{M}'' = \mathcal{M}$.*

Example 1.4.14 (Essentially bounded functions again)**.** Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and this time, define $M \colon L^\infty(\mu) \to B(L^2(\mu))$ by $M(f)g := fg \in L^2(\mu)$ for all $f \in L^\infty(\mu)$ and $g \in L^2(\mu)$. Then M is injective if and only if μ is semifinite, in which case M is an isometry. Now, another one of the conditions in [Seg51, Thm. 5.1] equivalent to μ's being localizable is that (μ is semifinite and) $M(L^\infty(\mu))' = M(L^\infty(\mu))$ in $B(L^2(\mu))$. Consequently, if μ is localizable, then $M(L^\infty(\mu))'' = M(L^\infty(\mu))' = M(L^\infty(\mu))$, so $M(L^\infty(\mu))$ is a von Neumann algebra by von Neumann's bicommutant theorem.

Finally, we state two fundamental results: one justifying the term "concrete C^*-algebra" and the other saying that von Neumann algebras are "concrete W^*-algebras."

Theorem 1.4.15 (Gel'fand–Naimark–Segal)**.** *If $\mathcal{A}$ is a (unital) C^*-algebra, then there exists a complex Hilbert space H and an injective (unital) $*$-homomorphism $\pi \colon \mathcal{A} \to B(H)$. By Remark 1.4.10, π is an isometry, so $\pi(\mathcal{A})$ is a (unital) concrete C^*-algebra.*

Theorem 1.4.16 (Sakai)**.** *Every von Neumann algebra is a W^*-algebra. Conversely, if $\mathcal{M}$ is a W^*-algebra, then there exists a complex Hilbert space H and an injective, unital $*$-homomorphism $\pi \colon \mathcal{M} \to B(H)$ such that $\pi(\mathcal{M}) \subseteq B(H)$ is a von Neumann algebra.*

Theorem 1.4.15 is often called the GNS theorem. For proofs of both the GNS theorem and Sakai's theorem, please see [Sak71, §1.15 & §1.16]. Also, since it will be useful to us in later chapters, we discuss the first statement in Sakai's theorem more in §4.1; specifically, the predual of a von Neumann algebra is described in Theorem 4.1.2(iv).

In much of the modern literature, "abstract" C^*-algebras are preferred to concrete C^*-algebras, while von Neumann algebras are preferred to W^*-algebras. For the most part, this dissertation will follow suit.

1.5 Projective tensor products

Finally, we briefly discuss projective tensor products of Banach spaces. For a proper treatment, please see [Rya02, Ch. 2].

Notation 1.5.1. Write $\otimes = \otimes^{\mathbb{F}}$ for the algebraic $\mathbb{F}$-tensor product. For $u \in V_1 \otimes \cdots \otimes V_k$, define

$$\pi(u) := \inf\left\{ \sum_{n=1}^{N} \prod_{i=1}^{k} \|v_{i,n}\|_{V_i} : v_{i,n} \in V_i,\ u = \sum_{n=1}^{N} v_{1,n} \otimes \cdots \otimes v_{k,n} \right\}.$$

Proposition 1.5.2. *π is a norm on $V_1 \otimes \cdots \otimes V_k$, and $\pi(v_1 \otimes \cdots \otimes v_k) = \|v_1\|_{V_1} \cdots \|v_k\|_{V_k}$ whenever $v_1 \in V_1, \ldots, v_k \in V_k$.*

In the $k = 2$ case, this is [Rya02, Prop. 2.1]. The same proof works in the general case.

Definition 1.5.3 (Projective tensor product)**.** The (**Banach space**) **projective tensor product** $\big(V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k, \|\cdot\|_{V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k}\big)$ is the completion of the normed vector space $(V_1 \otimes \cdots \otimes V_k, \pi)$.

The primary virtue of the projective tensor product is that it satisfies a topological version of the universal property of the algebraic tensor product: It boundedly linearizes bounded multilinear maps (Notation 1.2.1).

Proposition 1.5.4 (Universal property of $\hat{\otimes}_\pi$)**.** *If $T \in B_k(V_1 \times \cdots \times V_k; W)$, then there exists a unique $\tilde{T} \in B\big(V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k; W\big)$ such that $\tilde{T}(v_1 \otimes \cdots \otimes v_k) = T[v_1, \ldots, v_k]$ for all $v_1 \in V_1, \ldots, v_k \in V_k$. Furthermore, $\|\tilde{T}\|_{V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k \to W} = \|T\|_{B_k(V_1 \times \cdots \times V_k; W)}$.*

We leave the proof to the reader. This has a number of useful consequences. We list them and, once again, leave their proofs to the reader.

Corollary 1.5.5 (Projective tensor product Banach algebra)**.** *Suppose* $\mathbb{F} = \mathbb{C}$*. If* $\mathcal{B}_1, \ldots, \mathcal{B}_k$ *are Banach* (*$*$-*)*algebras, then* $\mathcal{B}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{B}_k$ *is a Banach* (*$*$-*)*algebra with the product determined by*

$$(a_1 \otimes \cdots \otimes a_k)(b_1 \otimes \cdots \otimes b_k) = (a_1 b_1) \otimes \cdots \otimes (a_k b_k), \qquad a_i, b_i \in \mathcal{B}_i,$$

(and the $*$*-operation determined by* $(a_1 \otimes \cdots \otimes a_k)^* = a_1^* \otimes \cdots \otimes a_k^*$ *for all* $a_1 \in \mathcal{B}_1, \ldots, a_k \in \mathcal{B}_k$*).*

Corollary 1.5.6 (Projective tensor product of bounded linear maps)**.** *If* $T_i \in B(V_i; W_i)$ *for all* $i \in \{1, \ldots, k\}$*, then there exists a unique* $T_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi T_k \in B\big(V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k; W_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi W_k\big)$ *such that*

$$\big(T_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi T_k\big)(v_1 \otimes \cdots \otimes v_k) = T_1 v_1 \otimes \cdots \otimes T_k v_k, \qquad v_i \in V_i.$$

Furthermore,

$$\big\|T_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi T_k\big\|_{V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k \to W_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi W_k} = \|T_1\|_{V_1 \to W_1} \cdots \|T_k\|_{V_k \to W_k}.$$

If, in addition, V_i *and* W_i *are Banach* (*$*$-*)*algebras and* T_i *is a* (*$*$-*)*homomorphism for all* $i \in \{1, \ldots, k\}$*, then* $T_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi T_k$ *is a* (*$*$-*)*homomorphism.*

Corollary 1.5.7 (Commutativity and associativity)**.** *The linear isomorphisms*

$$V_2 \otimes V_1 \otimes \cdots \otimes V_k \to V_1 \otimes \cdots \otimes V_k \to V_1 \otimes (V_2 \otimes \cdots \otimes V_k)$$

extend uniquely to isometric isomorphisms

$$V_2 \hat{\otimes}_\pi V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k \to V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k \to V_1 \hat{\otimes}_\pi \big(V_2 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k\big).$$

Corollary 1.5.8 (# operations)**.** *Suppose* $\mathbb{F} = \mathbb{C}$*, and let* $\mathcal{B} = \mathcal{B}_1, \ldots, \mathcal{B}_k$ *be Banach algebras.*

(i) *There exists a unique bounded linear map* $\#_k \colon \mathcal{B}^{\hat{\otimes}_\pi (k+1)} \to B_k\big(\mathcal{B}^k; \mathcal{B}\big)$ *such that*

$$\#_k(a_1 \otimes \cdots \otimes a_{k+1})[b_1, \ldots, b_k] = a_1 b_1 \cdots a_k b_k a_{k+1}, \qquad a_i, b_j \in \mathcal{B}.$$

Furthermore, the operator norm of $\#_k$ *is at most one.*

(ii) *For every* $i \in \{1, \ldots, k\}$, *there exists a unique bounded linear map*

$$\#_{k,i} \colon \mathcal{B}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{B}_i \hat{\otimes}_\pi \mathcal{B}_i \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{B}_k \to B\big(\mathcal{B}_i; \mathcal{B}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{B}_k\big)$$

such that

$$\#_{k,i}(a_1 \otimes \cdots \otimes a_i \otimes b_i \otimes a_{i+1} \cdots \otimes a_k)c = a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i c b_i \otimes a_{i+1} \otimes \cdots \otimes a_k$$

whenever $a_j \in \mathcal{B}_j$ *for all* $j \in \{1, \ldots, k\}$ *and* $b_i, c \in \mathcal{B}_i$. *Furthermore, the operator norm of* $\#_{k,i}$ *is at most one.*

Notation 1.5.9 (# operations)**.** Retain the setup of Corollary 1.5.8. If $u \in \mathcal{B}^{\hat{\otimes}_\pi (k+1)}$, $b \in \mathcal{B}^k$, $v \in \mathcal{B}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{B}_i \hat{\otimes}_\pi \mathcal{B}_i \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{B}_k$, and $c \in \mathcal{B}_i$, then

$$u \#_k b \coloneqq \#_k(u)[b] \quad \text{and} \quad v \#_{k,i} c \coloneqq \#_{k,i}(v)c.$$

Also, we shall write $\# \coloneqq \#_1$ and $u \# b \coloneqq u \#_1 b$ in the $k = 1$ case.

The operations in Corollary 1.5.8 play a prominent role both technically and conceptually in this dissertation, so the reader is advised to write out some examples.

We end this section with a useful concrete description of the projective tensor product.

Theorem 1.5.10 (Series description of projective tensor product)**.** *If* $u \in V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k$, *then there exist sequences* $(v_{1,n})_{n \in \mathbb{N}} \in V_1^{\mathbb{N}}, \ldots, (v_{k,n})_{n \in \mathbb{N}} \in V_k^{\mathbb{N}}$ *such that*

$$\sum_{n=1}^{\infty} \|v_{1,n}\|_{V_1} \cdots \|v_{k,n}\|_{V_k} < \infty \quad \text{and} \quad u = \sum_{n=1}^{\infty} v_{1,n} \otimes \cdots \otimes v_{k,n} \quad \text{in} \quad V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k. \tag{1.5.11}$$

(Recall that $\|v_{1,n}\|_{V_1} \cdots \|v_{k,n}\|_{V_k} = \|v_{1,n} \otimes \cdots \otimes v_{k,n}\|_{V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k}$.*) Moreover, if* $\varepsilon > 0$, *then we may choose* $\{(v_{i,n})_{n \in \mathbb{N}} : i \in \{1, \ldots, k\}\}$ *such that*

$$\sum_{n=1}^{\infty} \|v_{1,n}\|_{V_1} \cdots \|v_{k,n}\|_{V_k} \leq \|u\|_{V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k} + \varepsilon$$

as well.

Sketch of proof. We proceed by induction on k. The $k = 2$ case is [Rya02, Prop. 2.8]. Now, assume the result for $k \geq 2$ tensorands, and let $V \coloneqq V_2 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k$. We shall use Corollary 1.5.7 freely without comment. By the $k = 2$ case, if $u \in V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k = V_1 \hat{\otimes}_\pi V$ and $\varepsilon > 0$, then there exist sequences $(v_{1,n})_{n \in \mathbb{N}} \in V_1^{\mathbb{N}}$ and $(u_n)_{n \in \mathbb{N}} \in V^{\mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} \|v_{1,n}\|_{V_1} \|u_n\|_V \leq \|u\|_{V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k} + \frac{\varepsilon}{2} \text{ and } u = \sum_{n=1}^{\infty} v_{1,n} \otimes u_n.$$

Now, if $n \in \mathbb{N}$, then there exist sequences $(v_{2,m}^n)_{m \in \mathbb{N}} \in V_2^{\mathbb{N}}, \ldots, (v_{k,m}^n)_{m \in \mathbb{N}} \in V_k^{\mathbb{N}}$ such that

$$\sum_{m=1}^{\infty} \left\|v_{2,m}^n\right\|_{V_2} \cdots \left\|v_{k,m}^n\right\|_{V_k} \leq \|u_n\|_V + \frac{\varepsilon}{2^{n+1}(1 + \|v_{1,n}\|_{V_1})} \text{ and } u_n = \sum_{m=1}^{\infty} v_{2,m}^n \otimes \cdots \otimes v_{k,m}^n$$

by the induction hypothesis. But then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \|v_{1,n}\|_{V_1} \prod_{i=2}^{k} \left\|v_{i,m}^n\right\|_{V_i} \leq \|u\|_{V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k} + \varepsilon \text{ and } u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} v_{1,n} \otimes v_{2,m}^n \otimes \cdots \otimes v_{k,m}^n.$$

The result follows. □

Remark 1.5.12. Observe that may take the sequences $\{(v_{i,n})_{n \in \mathbb{N}} : i \in \{1, \ldots, k\}\}$ in Theorem 1.5.10 to be bounded as well. Indeed, by rescaling, we can ensure that $\|v_{i,n}\|_{V_i} \leq 1$ whenever $i \in \{2, \ldots, k\}$ and $\|v_{1,n}\|_{V_1} \cdots \|v_{k,n}\|_{V_k} = \|v_{1,n}\|_{V_1}$ for all $n \in \mathbb{N}$. In this case, the sequences $\{(v_{i,n})_{n \in \mathbb{N}} : i \in \{2, \ldots, k\}\}$ are clearly bounded. Since

$$\sum_{n=1}^{\infty} \|v_{1,n}\|_{V_1} = \sum_{n=1}^{\infty} \|v_{1,n}\|_{V_1} \cdots \|v_{k,n}\|_{V_k} < \infty,$$

$(v_{1,n})_{n \in \mathbb{N}}$ is bounded as well.

As an immediate consequence of Theorem 1.5.10, we get that

$$\|u\|_{V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k} = \inf \left\{ \sum_{n=1}^{\infty} \prod_{i=1}^{k} \|v_{i,n}\|_{V_i} : (v_{i,n})_{n \in \mathbb{N}} \in V_i^{\mathbb{N}} \text{ satisfy Relation (1.5.11)} \right\} \tag{1.5.13}$$

for all $u \in V_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi V_k$.

Chapter 2

Warm-up: Holomorphic functional calculus

In this chapter, we discuss the holomorphic functional calculus for an element of a unital Banach algebra and compute the higher derivatives of maps on (symmetrically normed ideals of) the algebra induced, via this functional calculus, by a holomorphic function. As we shall see in Chapters 3 and 6, the method we use here to differentiate the holomorphic functional calculus is quite common—nearly universal—for differentiating maps arising from other functional calculi. In the holomorphic case, there are no serious technical obstacles to the method's implementation, so it is a good setting for a first demonstration. Therefore, the reader should consider this chapter as motivation for the aforementioned later chapters.

Standing assumptions. Throughout, $\mathcal{B}$ is a unital Banach algebra with norm $\|\cdot\|_{\mathcal{B}} = \|\cdot\|$, $U \subseteq \mathbb{C}$ is a nonempty open set, all vector spaces are complex, and all linear maps are $\mathbb{C}$-linear. In §2.3, $k \in \mathbb{N}_0$. In §2.4, $k \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $U_1, \ldots, U_m \subseteq \mathbb{C}$ are nonempty open sets.

2.1 Definition and examples

The holomorphic functional calculus allows us to plug elements of $\mathcal{B}$ with spectrum contained in U into holomorphic functions defined on U. In other words, it enables us to define $f(a) \in \mathcal{B}$ for $f \in \operatorname{Hol}(U)$ and $a \in \mathcal{B}$ such that $\sigma(a) \subseteq U$. Here is the fundamental result.

Theorem 2.1.1 (Holomorphic functional calculus)**.** *If $a \in \mathcal{B}_U = \{b \in \mathcal{B} : \sigma(b) \subseteq U\}$, then there exists a unique continuous, unital algebra homomorphism $H_a^U \colon \operatorname{Hol}(U) \to \mathcal{B}$ such that $H_a^U(\iota_U) = a$, where $\iota_U \colon U \hookrightarrow \mathbb{C}$ is the inclusion. (Recall that* Hol(U) *is a complex Fréchet space with the topology of locally uniform convergence.)*

Proof of uniqueness. Let $\Phi\colon \operatorname{Hol}(U) \to \mathcal{B}$ be a unital algebra homomorphism mapping ι_U to a, and write

$$\mathcal{R}_U \coloneqq \{\text{rational functions with poles outside of } U\}, \tag{2.1.2}$$

viewed as a subset of $\operatorname{Hol}(U)$. We claim that Φ is uniquely determined on $\mathcal{R}_U$. To see this, note first that if $p(z) \coloneqq \sum_{i=0}^{n} c_i\, z^i$ is a polynomial, then $p|_U = \sum_{i=0}^{n} c_i\, \iota_U^i$. Since Φ is a unital homomorphism sending ι_U to a, we get $\Phi(p|_U) = \sum_{i=0}^{n} c_i\, \Phi(\iota_U)^i = \sum_{i=0}^{n} c_i\, a^i$. Thus, Φ is uniquely determined on polynomials. Next, if $f \in \operatorname{Hol}(U)$ does not vanish, then f is invertible in $\operatorname{Hol}(U)$ with inverse $1/f$. Since Φ is a unital homomorphism, $\Phi(f)$ is invertible in $\mathcal{B}$, and $\Phi(f)^{-1} = \Phi(1/f)$. In particular, if $q(z)$ is a polynomial that does not vanish in U, then $r(z) \coloneqq p(z)/q(z)$ is a rational function with poles outside of U, and $\Phi(r|_U) = \Phi(p|_U)\, \Phi(q|_U)^{-1}$. This proves the claim (and confirms that $\Phi(r|_U)$ is given by "plugging a into r").

Suppose, in addition, that Φ is continuous and $\Psi\colon \operatorname{Hol}(U) \to \mathcal{B}$ is another continuous, unital algebra homomorphism taking ι_U to a. By the previous paragraph, Φ and Ψ agree on $\mathcal{R}_U$. By Runge's theorem [Rud86, Thm. 13.9], $\mathcal{R}_U$ is dense in $\operatorname{Hol}(U)$. Thus, by the continuity of Φ and Ψ, $\Phi = \Psi$ on all of $\operatorname{Hol}(U)$. This completes the proof of the uniqueness of H_a^U. □

The construction of H_a^U requires some complex analysis, most of which we reviewed at the end of §1.3. In addition to this material, we need Cauchy's theorem and a result on the existence of cycles surrounding compact sets.

Theorem 2.1.3 (Cauchy's theorem)**.** *Let V be a complex Banach space, and let $F\colon U \to V$ be a holomorphic map. If Γ_1 and Γ_2 are cycles in U such that $\operatorname{Ind}_{\Gamma_1}(z) = \operatorname{Ind}_{\Gamma_2}(z)$ whenever $z \in \mathbb{C} \setminus U$, then $\int_{\Gamma_1} F(z)\,\mathrm{d}z = \int_{\Gamma_2} F(z)\,\mathrm{d}z$. If $\operatorname{Ind}_{\Gamma_1}(z) = 0$ whenever $z \in \mathbb{C} \setminus U$, then $\int_{\Gamma_1} F(z)\,\mathrm{d}z = 0$.*

Proof. The scalar case ($V = \mathbb{C}$) is [Rud86, Thm. 10.35, Eqs. (3) & (5)]. The general case follows from the scalar case applied to the functions $\{\ell \circ F : \ell \in V^*\}$. □

Theorem 2.1.4 (Existence of surrounding cycles [Rud86, pf. of Thm. 13.5])**.** *If $K \subseteq U$ is compact, then there exists a cycle Γ in $U \setminus K$ such that $\operatorname{Ind}_{\Gamma}(z) = 0$ whenever $z \in \mathbb{C} \setminus U$ and $\operatorname{Ind}_{\Gamma}(z) = 1$ whenever $z \in K$. In other words, Γ **surrounds** K in U.*

By Cauchy's integral formula (Theorem 1.3.19), if K and Γ are as in Theorem 2.1.4, then

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,\mathrm{d}w, \qquad f \in \mathrm{Hol}(U),\ z \in K. \tag{2.1.5}$$

Observe that if $a \in \mathcal{B}_U$ and $K = \sigma(a) \subseteq U$, then the right-hand side of Equation (2.1.5) makes sense with $z = a$; just replace the scalar $(w-z)^{-1}$ with the resolvent $(w-a)^{-1} \in \mathcal{B}$. As we shall see momentarily, doing so constitutes the definition of $H_a^U(f) \in \mathcal{B}$.

Proof of existence in Theorem 2.1.1. Fix $a \in \mathcal{B}_U$, and let Γ be as in Theorem 2.1.4 with $K = \sigma(a)$. Define

$$H_a^U(f) \coloneqq \frac{1}{2\pi i}\int_\Gamma f(w)\,(w-a)^{-1}\,\mathrm{d}w \in \mathcal{B}, \qquad f \in \mathrm{Hol}(U). \tag{2.1.6}$$

The integral in Equation (2.1.6) exists because the map $U \setminus \sigma(a) \ni w \mapsto f(w)\,(w-a)^{-1} \in \mathcal{B}$ is holomorphic—in particular, continuous. We must prove that the map $\mathrm{Hol}(U) \ni f \mapsto H_a^U(f) \in \mathcal{B}$ is a continuous, unital algebra homomorphism satisfying $H_a^U(\iota_U) = a$. The linearity of H_a^U is obvious, and the continuity of H_a^U is an easy consequence of the dominated convergence theorem. It remains to prove that $H_a^U(1) = 1$, $H_a^U(\iota_U) = a$, and $H_a^U(fg) = H_a^U(f)\,H_a^U(g)$.

To begin, we first claim that if $z_0 \in \mathbb{C} \setminus U$ and $f \in \mathrm{Hol}(U)$, then

$$H_a^U\big((z_0 - \iota_U)^{-1} f\big) = (z_0 - a)^{-1} H_a^U(f). \tag{2.1.7}$$

Indeed, by the resolvent identity, if $w \in \rho(a)$, then

$$(w-a)^{-1} = (z_0-a)^{-1} + (z_0-w)(z_0-a)^{-1}(w-a)^{-1}.$$

Since $(z_0 - \iota_U)^{-1} f \in \mathrm{Hol}(U)$ and $\mathrm{Ind}_\Gamma(z) = 0$ whenever $z \in \mathbb{C} \setminus U$, this gives

$$\int_\Gamma \frac{f(w)}{z_0-w}\,(w-a)^{-1}\,\mathrm{d}w = (z_0-a)^{-1}\underbrace{\int_\Gamma \frac{f(w)}{z_0-w}\,\mathrm{d}w}_{=0} + (z_0-a)^{-1}\int_\Gamma f(w)\,(w-a)^{-1}\,\mathrm{d}w$$

by Cauchy's theorem. Dividing the equation above by $2\pi i$ yields Equation (2.1.7).

Next, we claim that

$$H_a^U(\iota_U^n) = a^n, \qquad n \in \mathbb{N}_0. \tag{2.1.8}$$

Indeed, write $\mathbb{D}_R \coloneqq \{z \in \mathbb{C} : |z| < R\}$. By Theorems 1.3.18 and 1.4.5(v), if $R > \|a\|$ is sufficiently large, then $\mathrm{Ind}_\Gamma(z) = 0$ whenever $z \in \mathbb{C} \setminus \mathbb{D}_R$, and $\sigma(a) \subseteq \mathbb{D}_R$. Let Γ_R be the cycle consisting of the single counterclockwise circle $[0,1] \ni t \mapsto R\,e^{2\pi i t} \in \mathbb{C}$. Now, the map $\mathbb{D}_{2R}\backslash\sigma(a) \ni w \mapsto w^n\,(w-a)^{-1} \in \mathcal{B}$ is holomorphic, $\mathrm{Ind}_{\Gamma_R}(z) = 0 = \mathrm{Ind}_\Gamma(z)$ whenever $z \in \mathbb{C}\backslash\mathbb{D}_{2R}$, and $\mathrm{Ind}_{\Gamma_R}(z) = 1 = \mathrm{Ind}_\Gamma(z)$ whenever $z \in \sigma(a)$. Therefore, by Cauchy's theorem, the geometric series expansion of $(w-a)^{-1}$ (Theorem 1.4.5(i)), and the dominated convergence theorem,

$$\int_\Gamma w^n\,(w-a)^{-1}\,\mathrm{d}w = \int_{\Gamma_R} w^n\,(w-a)^{-1}\,\mathrm{d}w = \sum_{k=0}^{\infty}\left(\int_{\Gamma_R}\frac{w^n}{w^{k+1}}\,\mathrm{d}w\right)a^k. \tag{2.1.9}$$

By an elementary calculation,

$$\int_{\Gamma_R} w^{n-k-1}\,\mathrm{d}w = 2\pi i\,\delta_{kn}, \qquad k \in \mathbb{N}_0.$$

Consequently, Equation (2.1.9) reads

$$\int_\Gamma w^n\,(w-a)^{-1}\,\mathrm{d}w = 2\pi i\,a^n.$$

Dividing both sides by $2\pi i$ yields Equation (2.1.8).

Finally, we prove that H_a^U is multiplicative. Let $\mathcal{R}_U \subseteq \mathrm{Hol}(U)$ be as in Equation (2.1.2), and write $\mathscr{M} \coloneqq \big\{(f,g) \in \mathrm{Hol}(U) \times \mathrm{Hol}(U) : H_a^U(fg) = H_a^U(f)\,H_a^U(g)\big\}$. By the continuity of H_a^U and multiplication in $\mathcal{B}$, $\mathscr{M}$ is closed in $\mathrm{Hol}(U) \times \mathrm{Hol}(U)$. It is easy to see from Equations (2.1.7) and (2.1.8), the linearity of H_a^U, and the fundamental theorem of algebra (to factor the denominator of a rational function) that $\mathcal{R}_U \times \mathcal{R}_U \subseteq \mathscr{M}$. Since, once again, $\mathcal{R}_U$ is dense in $\mathrm{Hol}(U)$, we conclude that $\mathscr{M} = \mathrm{Hol}(U) \times \mathrm{Hol}(U)$. This completes the proof. □

Definition 2.1.10 (Holomorphic functional calculus)**.** If $a \in \mathcal{B}_U$, then the map H_a^U is the (**Dunford–Riesz**) **holomorphic functional calculus** for a, and we write

$$f(a) \coloneqq H_a^U(f) \in \mathcal{B}, \qquad f \in \mathrm{Hol}(U)$$

Let us now study some examples. First, observe that the uniqueness part of Theorem 2.1.1 implies that if $V \subseteq \mathbb{C}$ is an open set such that $\sigma(a) \subseteq V \subseteq U$, then

$$H_a^U(f) = H_a^V(f|_V), \qquad f \in \mathrm{Hol}(U). \tag{2.1.11}$$

In other words, $f(a) \in \mathcal{B}$ depends on the germ of f at $\sigma(a)$; this justifies the exclusion of U in the notation $f(a) = H_a^U(f)$ and provides flexibility in examples.

Example 2.1.12 (Series)**.** If $R > r(a)$, where $r(a)$ is the spectral radius of a, then $\sigma(a) \subseteq \mathbb{D}_R$. Take $U \coloneqq \mathbb{D}_R$. Any $f \in \mathrm{Hol}(U)$ expands as the series

$$f(z) = \sum_{n=0}^{\infty} c_n\, z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \qquad z \in U = \mathbb{D}_R,$$

that converges in $\mathrm{Hol}(U)$. Thus,

$$f(a) = H_a^U\left(\sum_{n=0}^{\infty} c_n\, \iota_U^n\right) = \sum_{n=0}^{\infty} c_n\, H_a(\iota_U)^n = \sum_{n=0}^{\infty} c_n\, a^n \in \mathcal{B}.$$

One of the most common examples of this form is $f(z) = e^z$, which yields the exponential of a:

$$e^a = \sum_{n=0}^{\infty} \frac{1}{n!} a^n.$$

More generally,

$$f(a) = \sum_{n=0}^{\infty} c_n\, (a - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (a - z_0)^n \in \mathcal{B}, \qquad f \in \mathrm{Hol}(\mathbb{D}_S(z_0)),$$

whenever $\sigma(a) \subseteq \mathbb{D}_S(z_0) = \{z \in \mathbb{C} : |z - z_0| < S\}$.

Next, we compute $f(A)$ for a square matrix A and a holomorphic function f defined on a neighborhood of the eigenvalues of A using the Jordan–Chevalley decomposition of A. To this end, in the following result, we compute $f(a + b) \in \mathcal{B}$ for commuting elements $a, b \in \mathcal{B}$ with b nilpotent ($b^n = 0$ for some $n \in \mathbb{N}$). We provide two proofs: an illuminating but slightly highfalutin one using our construction of the holomorphic functional calculus, i.e., Equation (2.1.6), and an elementary but opaque one using the uniqueness part of Theorem 2.1.1.

Proposition 2.1.13 (Perturbation by a commuting nilpotent)**.** *If $a \in \mathcal{B}_U$, $b \in \mathcal{B}$ is nilpotent, and $[a,b]=0$, then $\sigma(a+b)=\sigma(a)$, and*

$$f(a+b) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\, b^n, \qquad f \in \mathrm{Hol}(U).$$

Of course, since b is nilpotent, the series above is a finite sum.

First proof. Suppose b is nilpotent and $[a,b]=0$. First, we make a useful observation, the verification of which we leave to the reader: a is invertible if and only if $a+b$ is invertible, in which case

$$(a+b)^{-1} = \sum_{n=0}^{\infty} (-1)^n a^{-(n+1)} b^n. \tag{2.1.14}$$

The identity $\sigma(a+b)=\sigma(a)$ follows from this observation applied to $a-\lambda$ ($\lambda \in \mathbb{C}$) in place of a.

Next, let Γ be as in Theorem 2.1.4 with $K=\sigma(a)$. If $f \in \mathrm{Hol}(U)$, then, by Equations (2.1.6) and (2.1.14),

$$\begin{aligned} f(a+b) &= \frac{1}{2\pi i} \int_\Gamma f(w)\,(w-a-b)^{-1}\,\mathrm{d}w \\ &= \frac{1}{2\pi i} \int_\Gamma f(w) \sum_{n=0}^{\infty} (w-a)^{-(n+1)} b^n \,\mathrm{d}w \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\Gamma f(w)\,(w-a)^{-(n+1)}\,\mathrm{d}w \right) b^n. \end{aligned}$$

Therefore, if we can show that

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\Gamma f(w)\,(w-a)^{-(n+1)}\,\mathrm{d}w, \qquad f \in \mathrm{Hol}(U), \tag{2.1.15}$$

then the proof is complete. To this end, define $V \coloneqq \{z \in U \setminus \Gamma^* : \mathrm{Ind}_\Gamma(z) = 1\}$. By differentiating Cauchy's integral formula repeatedly, we see that if $f \in \mathrm{Hol}(U)$ and $n \in \mathbb{N}_0$, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w, \qquad z \in V. \tag{2.1.16}$$

Now, since the map $\Gamma^* \ni w \mapsto f(w)\,(w-\iota_V)^{-(n+1)} \in \mathrm{Hol}(V)$ is continuous and $\mathrm{Hol}(V)$ is a complex Fréchet space, the integral $\int_\Gamma f(w)\,(w-\iota_V)^{-(n+1)}\,\mathrm{d}w \in \mathrm{Hol}(V)$ exists. By applying the continuous linear functionals $\{\mathrm{Hol}(V) \ni g \mapsto g(z) \in \mathbb{C} : z \in V\}$ to this integral, we conclude that Equation (2.1.16) may be rewritten as the identity

$$f^{(n)}\big|_V = \frac{n!}{2\pi i}\int_\Gamma f(w)\,(w-\iota_V)^{-(n+1)}\,\mathrm{d}w \tag{2.1.17}$$

in $\mathrm{Hol}(V)$. Finally, since Γ surrounds $\sigma(a)$ (in U), $\sigma(a) \subseteq V$ by definition of V. By Equation (2.1.11), $f^{(n)}(a) = f^{(n)}|_V(a) = H_a^V(f^{(n)}|_V)(a)$ whenever $f \in \mathrm{Hol}(U)$, i.e., we may use V as our reference open set. Since $H_a^V\colon \mathrm{Hol}(V) \to \mathcal{B}$ is a continuous, unital algebra homomorphism that maps ι_V to a, we conclude from Equation (2.1.17) that

$$\begin{aligned} f^{(n)}(a) = H_a^V\big(f^{(n)}|_V\big) &= \frac{n!}{2\pi i} H_a^V\left(\int_\Gamma f(w)\,(w-\iota_V)^{-(n+1)}\,\mathrm{d}w\right) \\ &= \frac{n!}{2\pi i}\int_\Gamma f(w)\,H_a^V\big((w-\iota_V)^{-(n+1)}\big)\,\mathrm{d}w = \frac{n!}{2\pi i}\int_\Gamma f(w)\,(w-a)^{-(n+1)}\,\mathrm{d}w, \end{aligned}$$

as desired. □

Second proof. We take the first paragraph of the first proof, namely Equation (2.1.14), as the starting point for this proof as well. Define

$$H(f) \coloneqq \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\, b^n \in \mathcal{B}, \qquad f \in \mathrm{Hol}(U). \tag{2.1.18}$$

By Theorem 2.1.1, it suffices to prove that $H\colon \mathrm{Hol}(U) \to \mathcal{B}$ is a continuous, unital algebra homomorphism that maps ι_U to $a+b$. Since the maps $\mathrm{Hol}(U) \ni f \mapsto f^{(n)} \in \mathrm{Hol}(U)$ and $\mathrm{Hol}(U) \ni f \mapsto f(a) \in \mathcal{B}$ are continuous and linear, H is continuous and linear. Also, if $f \equiv 1$, then $f^{(n)} \equiv 0$ whenever $n \geq 1$, so Equation (2.1.18) reads $H(f) = f(a) = 1$; if $f = \iota_U$, then $f' \equiv 1$, and $f^{(n)} \equiv 0$ whenever $n \geq 2$, so Equation (2.1.18) reads $H(f) = f(a) + f'(a)\,b = a + b$.

It remains to prove that H is multiplicative. To this end, we first explain briefly that $f(a)\,b = b\,f(a)$ whenever $f \in \mathrm{Hol}(U)$. Indeed, if $\mathcal{R}_U \subseteq \mathrm{Hol}(U)$ is as in Equation (2.1.2), then it is easy to see by a direct computation that $f(a)\,b = b\,f(a)$ whenever $f \in \mathcal{R}_U$. The claim then follows from the density of $\mathcal{R}_U$ in $\mathrm{Hol}(U)$ and the continuity of H_a^U and multiplication. (Alternatively, $f(a)\,b = b\,f(a)$ can be read off immediately from Equation (2.1.6).)

Finally, if $f, g \in \mathrm{Hol}(U)$, then

$$H(fg) = \sum_{n=0}^{\infty} \frac{1}{n!}(fg)^{(n)}(a)\, b^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)\, g^{(n-k)}(a)\, b^n \tag{2.1.19}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} f^{(k)}(a)\, b^k\, g^{(n-k)}(a)\, b^{n-k} \tag{2.1.20}$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{k!(n-k)!} f^{(k)}(a)\, b^k\, g^{(n-k)}(a)\, b^{n-k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)\, b^k \sum_{n=k}^{\infty} \frac{1}{(n-k)!} g^{(n-k)}(a)\, b^{n-k} = H(f)\, H(g).$$

In Equation (2.1.19), we used the product rule and the homomorphism property; and in Equation (2.1.20), we used the previous paragraph. This completes the proof. □

Example 2.1.21 (Matrices)**.** In this example, we take $\mathcal{B} \coloneqq \mathrm{M}_n(\mathbb{C})$, $A \in \mathcal{B}$, and $U \subseteq \mathbb{C}$ to be the union of finitely many disjoint disks centered at the eigenvalues of A. There exist unique matrices $D, N \in \mathcal{B}$ such that D is diagonalizable, N is nilpotent, $DN = ND$, and $A = D + N$. This is called the **Jordan–Chevalley decomposition** of A, and it is covered in [Hum72, §4.2]. Its relation to "the" Jordan normal form of A is that there exists an invertible matrix $S \in \mathcal{B}$ such that $D_0 \coloneqq S^{-1}DS$ is the diagonal part of the Jordan normal form of A and $N_0 \coloneqq S^{-1}NS$ is the strictly upper triangular part of the Jordan normal form of A.

Let D, N, S, D_0, and N_0 be as above with $D_0 = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. It is easy to see from the uniqueness part of Theorem 2.1.1 or Equation (2.1.6) (and Cauchy's integral formula) that

$$f(D) = f(SD_0S^{-1}) = Sf(D_0)S^{-1} = S\,\mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))S^{-1}, \qquad f \in \mathrm{Hol}(U).$$

Consequently, we get from Proposition 2.1.13 that if $f \in \mathrm{Hol}(U)$, then

$$f(A) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(D)\, N^k = \sum_{k=0}^{\infty} S f^{(k)}(D_0) S^{-1} (SN_0S^{-1})^k$$

$$= S\left(\sum_{k=0}^{\infty} \mathrm{diag}\left(f^{(k)}(\lambda_1), \ldots, f^{(k)}(\lambda_n)\right) N_0^k\right) S^{-1}.$$

This gives a way to compute $f(A)$ explicitly.

We end this section by computing the spectrum of $f(a)$, as this will come in handy later.

Theorem 2.1.22 (Spectral mapping theorem). *$\sigma(f(a)) = f(\sigma(a))$ for all $a \in \mathcal{B}_U$ and $f \in \mathrm{Hol}(U)$.*

Proof. It suffices to prove that $f(a) \in \mathcal{B}_{\mathrm{inv}}$ if and only if $f(\lambda) \neq 0$ whenever $\lambda \in \sigma(a)$. Indeed, since $\sigma(a)$ is compact, if $f(\lambda) \neq 0$ for all $\lambda \in \sigma(a)$, then there exists an open set $V \subseteq U$ containing $\sigma(a)$ such that $f(\lambda) \neq 0$ for all $\lambda \in V$. Therefore, $g \coloneqq f|_V \in \mathrm{Hol}(V)$ is invertible in $\mathrm{Hol}(V)$, and by Equation (2.1.11), $f(a) = g(a) \in \mathcal{B}_{\mathrm{inv}}$ with $f(a)^{-1} = g(a)^{-1} = (1/g)(a)$. Conversely, suppose there exists a $\lambda \in \sigma(a)$ such that $f(\lambda) = 0$. If $h \coloneqq f^{[1]}(\lambda, \cdot) \in \mathrm{Hol}(U)$, then $f(\mu) = (\lambda - \mu)\, h(\mu)$ for all $\mu \in U$, i.e., $f = (\lambda - \iota_U)\, h$. Consequently, $f(a) = (\lambda - a)\, h(a)$. Since $[\lambda - a, h(a)] = 0$ and $\lambda - a$ is not invertible, it is a basic algebra fact that the product $f(a) = (\lambda - a)\, h(a)$ cannot be invertible. This completes the proof. □

2.2 Symmetrically normed ideals

In this section, we introduce the normed ideals of interest: symmetrically normed ideals. First, recall that an **ideal** of a $\mathbb{C}$-algebra $\mathcal{A}$ is a linear subspace $\mathcal{I} \subseteq \mathcal{A}$ such that $arb \in \mathcal{I}$ whenever $a, b \in \mathcal{A}$ and $r \in \mathcal{I}$.

Definition 2.2.1 (Symmetrically normed ideals). Let $\mathcal{I} \subseteq \mathcal{B}$ be an ideal, and suppose $\|\cdot\|_{\mathcal{I}}$ is a norm on $\mathcal{I}$. The pair $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a **Banach ideal** of $\mathcal{B}$ if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a Banach space and the inclusion $\iota_{\mathcal{I}} \colon (\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \hookrightarrow (\mathcal{B}, \|\cdot\|)$ is bounded; in this case, we write $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq \mathcal{B}$ and $C_{\mathcal{I}} \coloneqq \|\iota_{\mathcal{I}}\|_{\mathcal{I} \to \mathcal{B}} \in [0, \infty)$. If, in addition,

$$\|arb\|_{\mathcal{I}} \leq \|a\| \|r\|_{\mathcal{I}} \|b\|, \qquad a, b \in \mathcal{B},\ r \in \mathcal{I},$$

then $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a **symmetrically normed ideal** of $\mathcal{B}$, and we write $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{B}$ or $\mathcal{I} \trianglelefteq_{\mathrm{s}} \mathcal{B}$ when confusion is unlikely.

Remark 2.2.2. Beware that definitions of a symmetrically normed ideal vary in the literature. Sometimes, it is required that $C_{\mathcal{I}} = 1$. Sometimes, $\mathcal{B}$ is required to be a von Neumann or C^*-algebra, and $\mathcal{I}$ is required to be a $*$-ideal with $\|r^*\|_{\mathcal{I}} = \|r\|_{\mathcal{I}}$ for all $r \in \mathcal{I}$. Sometimes, even more requirements are imposed. We take the above minimal definition because it is all we need. Please see §6.2 for more information on this matter.

Example 2.2.3 (Closed ideals)**.** If $\mathcal{I} \subseteq \mathcal{B}$ is a closed ideal, then $(\mathcal{I}, \|\cdot\|) = (\mathcal{I}, \|\cdot\|_{\mathcal{B}}) \trianglelefteq_{\mathrm{s}} \mathcal{B}$. In particular, the **trivial ideals**, $\mathcal{I} = \{0\}$ and $\mathcal{I} = \mathcal{B}$, are symmetrically normed ideals.

We shall see *many* more interesting examples of symmetrically normed ideals (of von Neumann algebras) in Chapter 6. For now, we collect some basic properties for later use.

Notation 2.2.4. If $a \in \mathcal{B}$ and $S \subseteq \mathcal{B}$, then $S_{U,a} \coloneqq \{s \in S : \sigma(a+s) \subseteq U\} = \{s \in S : a+s \in \mathcal{B}_U\}$.

Proposition 2.2.5. *If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq \mathcal{B}$ and $a \in \mathcal{B}$, then $\mathcal{I}_{U,a} \subseteq \mathcal{I}$ is an open set in $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$.*

Proof. The set $\mathcal{I}_{U,a}$ is the inverse image of the open set $\mathcal{B}_U \subseteq \mathcal{B}$ under the map $\mathcal{I} \ni b \mapsto a+b \in \mathcal{B}$, which is continuous because $\iota_{\mathcal{I}} \colon (\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \hookrightarrow (\mathcal{B}, \|\cdot\|)$ is continuous. $\square$

Now, please consult Corollary 1.5.8 for the notation in the next result.

Proposition 2.2.6. *Let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{B}$, and fix $k \in \mathbb{N}$. If $u \in \mathcal{B}^{\hat{\otimes}_\pi (k+1)}$, $i \in \{1, \ldots, k\}$, and $b = (b_1, \ldots, b_k) \in \mathcal{B}^{i-1} \times \mathcal{I} \times \mathcal{B}^{k-i}$, then $u \#_k b \in \mathcal{I}$, and*

$$\|u \#_k b\|_{\mathcal{I}} \leq \|u\|_{\mathcal{B}^{\hat{\otimes}_\pi (k+1)}} \|b_i\|_{\mathcal{I}} \prod_{j \neq i} \|b_j\|.$$

In particular, if $b = (b_1, \ldots, b_k) \in \mathcal{I}^k$, then $\|u \#_k b\|_{\mathcal{I}} \leq C_{\mathcal{I}}^{k-1} \|u\|_{\mathcal{B}^{\hat{\otimes}_\pi (k+1)}} \prod_{j=1}^k \|b_j\|_{\mathcal{I}}$.

Proof. Let $b = (b_1, \ldots, b_k) \in \mathcal{B}^{i-1} \times \mathcal{I} \times \mathcal{B}^{k-i}$. By definition of a symmetrically normed ideal, if $a_1, \ldots, a_{k+1} \in \mathcal{B}$ and $u \coloneqq a_1 \otimes \cdots \otimes a_{k+1}$, then

$$\begin{aligned}\|u \#_k b\|_{\mathcal{I}} = \|a_1 b_1 \cdots a_k b_k a_{k+1}\|_{\mathcal{I}} &\leq \|a_1 b_1 \cdots a_{i-1} b_{i-1} a_i\| \, \|b_i\|_{\mathcal{I}} \|a_{i+1} b_i \cdots b_k a_{k+1}\| \\ &\leq \|a_1\| \cdots \|a_{k+1}\| \, \|b_i\|_{\mathcal{I}} \prod_{j \neq i} \|b_j\|.\end{aligned}$$

The result then follows from the universal property of the projective tensor product and the continuity of $\iota_{\mathcal{I}} \colon (\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \hookrightarrow (\mathcal{B}, \|\cdot\|)$. $\square$

2.3 Perturbation and derivative formulas

Let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{B}$, let $a \in \mathcal{B}_U$, and let $f \in \mathrm{Hol}(U)$. The main goal of this section is to use "perturbation formulas" to compute the derivatives of the map $\mathcal{I}_{U,a} \ni b \mapsto f(a+b) - f(a) \in \mathcal{I}$. We begin by introducing the central object appearing in the formulas.

Lemma 2.3.1. *Fix $m \in \mathbb{N}$, and let $\mathcal{B}_1, \ldots, \mathcal{B}_m$ be unital Banach algebras. If $i \in \{1, \ldots, m\}$, $a \in \mathcal{B}_i$, and $\tilde{a} \coloneqq 1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes(m-i)} \in \mathcal{C} \coloneqq \mathcal{B}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{B}_m$, then $\sigma_{\mathcal{B}_i}(a) = \sigma_{\mathcal{C}}(\tilde{a})$.*

Proof. It suffices to show that $a \in (\mathcal{B}_i)_{\text{inv}}$ if and only if $\tilde{a} \in \mathcal{C}_{\text{inv}}$. If $a \in (\mathcal{B}_i)_{\text{inv}}$, then $1^{\otimes(i-1)} \otimes a^{-1} \otimes 1^{\otimes(m-i)} \in \mathcal{C}$ is the inverse of $\tilde{a}$, so $\tilde{a} \in \mathcal{C}_{\text{inv}}$. Conversely, suppose $\tilde{a} \in \mathcal{C}_{\text{inv}}$. For each $j \in \{1, \ldots, m\} \setminus \{i\}$, let $\ell_j \in \mathcal{B}_j^*$ be such that $\ell_j(1) = 1$. Now, define

$$T \coloneqq \ell_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \ell_{i-1} \hat{\otimes}_\pi \mathrm{id}_{\mathcal{B}_i} \hat{\otimes}_\pi \ell_{i+1} \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \ell_m \in B(\mathcal{C}; \mathcal{B}_i).$$

Using the universal property of $\hat{\otimes}_\pi$, it is easy to see that

$$T[u\tilde{a}] = (Tu)\, a \;\text{ and }\; T[\tilde{a}u] = a\, Tu, \qquad u \in \mathcal{C}.$$

Also, writing $\mathbf{1}$ for the unit in $\mathcal{C}$, we have $T\mathbf{1} = 1$. Finally, let $u \coloneqq \tilde{a}^{-1} \in \mathcal{C}$. We claim that $a \in (\mathcal{B}_i)_{\text{inv}}$ with $b \coloneqq Tu = a^{-1}$. Indeed, $ab = T[\tilde{a}u] = T\mathbf{1} = 1$, and $ba = T[u\tilde{a}] = T\mathbf{1} = 1$. Thus, $ab = 1 = ba$, as claimed. This completes the proof. $\square$

Notation 2.3.2. Let $a_1, \ldots, a_{k+1} \in \mathcal{B}_U$, and write $\mathbf{a} \coloneqq (a_1, \ldots, a_{k+1}) \in \mathcal{B}_U^{k+1}$. Define

$$f^{[k]}(\mathbf{a}) \coloneqq \frac{1}{2\pi i} \int_\Gamma f(z)\,(z - a_1)^{-1} \cdots (z - a_{k+1})^{-1}\, \mathrm{d}z \in \mathcal{B}, \qquad f \in \mathrm{Hol}(U), \tag{2.3.3}$$

where Γ is any cycle surrounding $\sigma(a_1) \cup \cdots \cup \sigma(a_{k+1})$ in U. (Please see Theorem 2.1.4.) Now, suppose $\mathcal{B}_i$ is a unital Banach algebra and $a_i \in (\mathcal{B}_i)_U$ for all $i \in \{1, \ldots, k+1\}$, and write

$$\tilde{a}_i \coloneqq 1^{\otimes(i-1)} \otimes a_i \otimes 1^{\otimes(k+1-i)} \in (\mathcal{B}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{B}_{k+1})_U, \qquad i \in \{1, \ldots, k+1\}.$$

For $f \in \mathrm{Hol}(U)$, define $f_\otimes^{[k]}(a_1, \ldots, a_{k+1}) \coloneqq f^{[k]}(\tilde{a}_1, \ldots, \tilde{a}_{k+1})$, i.e.,

$$f_\otimes^{[k]}(a_1, \ldots, a_{k+1}) = \frac{1}{2\pi i} \int_\Gamma f(z)\,(z - a_1)^{-1} \otimes \cdots \otimes (z - a_{k+1})^{-1}\, \mathrm{d}z \in \mathcal{B}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{B}_{k+1},$$

where Γ is any cycle surrounding $\sigma_{\mathcal{B}_1}(a_1) \cup \cdots \cup \sigma_{\mathcal{B}_{k+1}}(a_{k+1})$ in U. (Note that we are using Lemma 2.3.1 implicitly in the last two sentences.)

By Cauchy's theorem, the definition of $f^{[k]}(\mathbf{a})$ in Equation (2.3.3) is independent of the choice of Γ. Also, by Proposition 1.3.20, the element $f^{[k]}(\mathbf{a}) \in \mathcal{B}$ appears to be the function $f^{[k]} \in \mathrm{Hol}\left(U^{k+1}\right)$ applied to the $(k+1)$-tuple $\mathbf{a} \in \mathcal{B}^{k+1}$ via a kind of *multivariate* holomorphic functional calculus. As we discuss in the next section, this can be made precise when $[a_i, a_j] = 0$ for all $i, j \in \{1, \ldots, k+1\}$, e.g., for the $(k+1)$-tuple $(\tilde{a}_1, \ldots, \tilde{a}_{k+1})$ in the second part of Notation 2.3.2. Please see Definition 2.4.5, Lemma 2.4.6, and Theorem 2.4.7 specifically. This point about multivariate holomorphic functional calculus is not essential for our purposes. Regardless, here are some examples lending additional credence to this view. The uninterested reader may skip to Proposition 2.3.7.

Proposition 2.3.4. *Let* $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in \mathcal{B}^{k+1}$. *If* $n \in \mathbb{N}_0$ *and* $p_n(\lambda) \coloneqq \lambda^n \in \mathbb{C}[\lambda]$, *then*

$$p_n^{[k]}(\mathbf{a}) = \sum_{|\alpha| = n-k} a_1^{\alpha_1} \cdots a_{k+1}^{\alpha_{k+1}}.$$

Recall that empty sums are zero.

Proof. This is a generalization of the proof of Equation (2.1.8), so we shall be brief. If $R > \max\{\|a_i\| : i \in \{1, \ldots, k+1\}\}$, then the cycle Γ_R consisting of the single counterclockwise circle $[0,1] \ni t \mapsto R\, e^{2\pi i t} \in \mathbb{C}$ surrounds $\sigma(a_1) \cup \cdots \cup \sigma(a_{k+1})$ in $\mathbb{C} \setminus \mathbb{D}_{2R}$. Thus,

$$p_n^{[k]}(\mathbf{a}) = \frac{1}{2\pi i} \int_{\Gamma_R} z^n \, (z - a_1)^{-1} \cdots (z - a_{k+1})^{-1} \, \mathrm{d}z.$$

Consequently, by the geometric series expansion of $(z - a_i)^{-1}$ (Theorem 1.4.5(i)) and the dominated convergence theorem,

$$\begin{aligned}
p_n^{[k]}(\mathbf{a}) &= \frac{1}{2\pi i} \int_{\Gamma_R} z^n \left(\sum_{\alpha_1 = 0}^{\infty} z^{-\alpha_1 - 1} a_1^{\alpha_1} \right) \cdots \left(\sum_{\alpha_{k+1} = 0}^{\infty} z^{-\alpha_{k+1} - 1} a_{k+1}^{\alpha_{k+1}} \right) \mathrm{d}z \\
&= \frac{1}{2\pi i} \sum_{\alpha \in \mathbb{N}_0^{k+1}} \left(\int_{\Gamma_R} z^{n - |\alpha| - k - 1} \, \mathrm{d}z \right) a_1^{\alpha_1} \cdots a_{k+1}^{\alpha_{k+1}} \\
&= \sum_{\alpha \in \mathbb{N}_0^{k+1}} \delta_{n, |\alpha| + k} \, a_1^{\alpha_1} \cdots a_{k+1}^{\alpha_{k+1}} = \sum_{|\alpha| = n-k} a_1^{\alpha_1} \cdots a_{k+1}^{\alpha_{k+1}},
\end{aligned}$$

as desired. $\square$

Proposition 2.3.5. *Suppose* $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in \mathcal{B}_U^{k+1}$ *and* $[a_i, a_j] = 0$ *for all* $i, j \in \{1, \ldots, k+1\}$*. Also, fix* $z_0 \in \mathbb{C} \setminus U$*, and write* $r_{z_0} \coloneqq (z_0 - \iota_U)^{-1} \in \mathrm{Hol}(U)$*. If* $f \in \mathrm{Hol}(U)$*, then*

$$(r_{z_0} f)^{[k]}(\mathbf{a}) = \sum_{i=0}^{k} (z_0 - a_{i+1})^{-1} \cdots (z_0 - a_{k+1})^{-1} f^{[i]}(a_1, \ldots, a_{i+1}).$$

In particular, $r_{z_0}^{[k]}(\mathbf{a}) = (z_0 - a_1)^{-1} \cdots (z_0 - a_{k+1})^{-1}$.

Proof. We proceed by induction on $k \geq 0$. The $k = 0$ case follows from Equation (2.1.6) and the multiplicativity of the holomorphic functional calculus. Now, assume the formula for $k \in \mathbb{N}_0$, and let $k \in \mathbb{N}$. If $F_i(w) \coloneqq (w - a_1)^{-1} \cdots (w - a_i)^{-1}$, then

$$\begin{aligned}
(r_{z_0} f)^{[k]}(\mathbf{a}) &= \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{z_0 - w} F_{k+1}(w)\, \mathrm{d}w = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{z_0 - w} (w - a_{k+1})^{-1} F_k(w)\, \mathrm{d}w \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{z_0 - w} \big((z_0 - a_{k+1})^{-1} + (z_0 - w)(z_0 - a_{k+1})^{-1}(w - a_{k+1})^{-1}\big) F_k(w)\, \mathrm{d}w \\
&= (z_0 - a_{k+1})^{-1} \underbrace{\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{z_0 - w} F_k(w)\, \mathrm{d}w}_{=(r_{z_0} f)^{[k-1]}(a_1, \ldots, a_k)} + (z_0 - a_{k+1})^{-1} \underbrace{\frac{1}{2\pi i} \int_\Gamma f(w)\, F_{k+1}(w)\, \mathrm{d}w}_{=f^{[k]}(\mathbf{a})} \\
&= (z_0 - a_{k+1})^{-1} \sum_{i=0}^{k-1} (z_0 - a_{i+1})^{-1} \cdots (z_0 - a_k)^{-1} f^{[i]}(a_1, \ldots, a_{i+1}) + (z_0 - a_{k+1})^{-1} f^{[k]}(\mathbf{a}) \\
&= \sum_{i=0}^{k} (z_0 - a_{i+1})^{-1} \cdots (z_0 - a_{k+1})^{-1} f^{[i]}(a_1, \ldots, a_{i+1})
\end{aligned}$$

by the resolvent identity and the induction hypothesis. This completes the proof. $\square$

Finally, here is an example related to Proposition 2.1.13; please see Remark 2.3.12.

Example 2.3.6. Let $a \in \mathcal{B}_U$, and fix a cycle Γ surrounding $\sigma(a)$ in U. Also, recall from Notation 1.2.5(i) that $a_{(k+1)} = (a, \ldots, a) \in \mathcal{B}_U^{k+1}$. If $b_1, \ldots, b_k \in \mathcal{B}$ commute with a and $f \in \mathrm{Hol}(U)$, then

$$\begin{aligned}
f_\otimes^{[k]}\big(a_{(k+1)}\big) \#_k [b_1, \ldots, b_k] &= \frac{1}{2\pi i} \int_\Gamma f(z)\, (z - a)^{-1} b_1 \cdots (z - a)^{-1} b_k (z - a)^{-1}\, \mathrm{d}z \\
&= \left(\frac{1}{2\pi i} \int_\Gamma f(z)\, (z - a)^{-(k+1)}\, \mathrm{d}z \right) b_1 \cdots b_k = \frac{1}{k!} f^{(k)}(a)\, b_1 \cdots b_k
\end{aligned}$$

by Equation (2.1.15).

We now begin to work in earnest toward our derivative formulas. Here is how the method of perturbation formulas works. Given a formalism for $f_{\otimes}^{[k]}(a_1,\ldots,a_{k+1})\#_k[b_1,\ldots,b_k]$, one needs two ingredients to differentiate the map $a \mapsto f(a)$ (or $b \mapsto f(a+b)-f(a)$). The first is the establishment of perturbation formulas: identities—taking place in $\mathcal{B}$, $\mathcal{B}^{\hat{\otimes}_\pi (k+1)}$, or whatever space is relevant for the given formalism—resembling the recursive definition of $f^{[k+1]}$ in terms of $f^{[k]}$. The second is an appropriate continuity property of $f_{\otimes}^{[k]}(a_1,\ldots,a_{k+1})\#_k[b_1,\ldots,b_k]$ in the arguments $(a_1,\ldots,a_{k+1})$. Here is the first ingredient in the holomorphic case.

Proposition 2.3.7 (Perturbation formulas)**.** *If $f \in \mathrm{Hol}(U)$, then*

$$f(a)-f(b)=f_{\otimes}^{[1]}(a,b)\#[a-b], \qquad a,b \in \mathcal{B}_U.$$

Now, suppose $\mathcal{B}_1,\ldots,\mathcal{B}_{k+1}$ are unital Banach algebras. If $f \in \mathrm{Hol}(U)$, then

$$f_{\otimes}^{[k]}(\mathbf{a})-f_{\otimes}^{[k]}(\mathbf{b})=\sum_{i=1}^{k+1} f_{\otimes}^{[k+1]}(a_1,\ldots,a_i,b_i,\ldots,b_{k+1})\#_{k+1,i}[a_i-b_i]$$

for all $\mathbf{a}=(a_1,\ldots,a_{k+1}),\mathbf{b}=(b_1,\ldots,b_{k+1}) \in (\mathcal{B}_1)_U \times \cdots \times (\mathcal{B}_{k+1})_U$

Proof. Fix $a,b \in \mathcal{B}_U$ and $\mathbf{a},\mathbf{b} \in (\mathcal{B}_1)_U \times \cdots \times (\mathcal{B}_{k+1})_U$ as in the statement. Also, let Γ be a cycle surrounding the compact set $\sigma(a)\cup\sigma(b)\cup\bigcup_{i=1}^{k+1}(\sigma(a_i)\cup\sigma(b_i))$ in U. Finally, recall

$$R_z(a)=(z-a)^{-1}, \qquad z \in \rho(a).$$

By Equation (2.1.6) and the resolvent identity,

$$\begin{aligned}
f(a)-f(b) &= \frac{1}{2\pi i}\int_\Gamma f(z)\,(R_z(a)-R_z(b))\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\Gamma f(z)\,R_z(a)(a-b)R_z(b)\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\Gamma f(z)\,(R_z(a)\otimes R_z(b))\#[a-b]\,\mathrm{d}z \\
&= \left(\frac{1}{2\pi i}\int_\Gamma f(z)\,R_z(a)\otimes R_z(b)\,\mathrm{d}z\right)\#[a-b] \\
&= f_{\otimes}^{[1]}(a,b)\#[a-b].
\end{aligned}$$

This is the first desired formula. Next, by the resolvent identity once again,

$$
\begin{aligned}
f_{\otimes}^{[k]}(\mathbf{a}) - f_{\otimes}^{[k]}(\mathbf{b}) &= \sum_{j=1}^{k+1} \big(f_{\otimes}^{[k]}(a_1,\ldots,a_j,b_{j+1}\ldots,b_{k+1}) - f_{\otimes}^{[k]}(a_1,\ldots,a_{j-1},b_j,\ldots,b_{k+1})\big) \\
&= \frac{1}{2\pi i}\sum_{j=1}^{k+1}\int_\Gamma f(z)\, R_z(a_1)\otimes\cdots\otimes R_z(a_{j-1})\otimes(\underbrace{R_z(a_j)-R_z(b_j)}_{R_z(a_j)(a_j-b_j)R_z(b_j)})\otimes R_z(b_{j+1})\otimes\cdots\otimes R_z(b_{k+1})\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\sum_{j=1}^{k+1}\int_\Gamma f(z)\,(R_z(a_1)\otimes\cdots\otimes R_z(a_j)\otimes R_z(b_j)\otimes\cdots\otimes R_z(b_{k+1}))\#_{k+1,j}[a_j-b_j]\,\mathrm{d}z \\
&= \sum_{j=1}^{k+1}\left(\frac{1}{2\pi i}\int_\Gamma f(z)\,R_z(a_1)\otimes\cdots\otimes R_z(a_j)\otimes R_z(b_j)\otimes\cdots\otimes R_z(b_{k+1})\,\mathrm{d}z\right)\#_{k+1,j}[a_j-b_j] \\
&= \sum_{j=1}^{k+1} f_{\otimes}^{[k+1]}(a_1,\ldots,a_j,b_j,\ldots,b_{k+1})\#_{k+1,j}[a_j-b_j],
\end{aligned}
$$

as desired. □

Here is the second ingredient.

Proposition 2.3.8 (Continuous perturbation property)**.** *Suppose $\mathcal{B}_1,\ldots,\mathcal{B}_{k+1}$ are unital Banach algebras. The map $(\mathcal{B}_1)_U\times\cdots\times(\mathcal{B}_{k+1})_U \ni \mathbf{a}\mapsto f^{[k]}(\mathbf{a})\in\mathcal{B}_1\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi\mathcal{B}_{k+1}$ is continuous.*

Proof. We begin with an observation. Write $(\mathcal{C},\|\cdot\|_{\mathcal{C}}) := \big(\mathcal{B}_1\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi\mathcal{B}_{k+1},\|\cdot\|_{\mathcal{B}_1\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi\mathcal{B}_{k+1}}\big)$, let $\mathbf{a},\mathbf{b}\in(\mathcal{B}_1)_U\times\cdots\times(\mathcal{B}_{k+1})_U$, and fix a cycle Γ surrounding the compact set $\bigcup_{i=1}^{k+1}\sigma(a_i)$ in U. If $\varepsilon := \sum_{i=1}^{k+1}\|a_i-b_i\|_{\mathcal{B}_i}$ is sufficiently small, then Γ surrounds $\bigcup_{i=1}^{k+1}(\sigma(a_i)\cup\sigma(b_i))$ in U, and

$$
\begin{aligned}
\big\|f_{\otimes}^{[k]}(\mathbf{b})\big\|_{\mathcal{C}} &\le \frac{1}{2\pi}\int_\Gamma\prod_{i=1}^{k+1}\|R_z(b_i)\|_{\mathcal{B}_i}\,|\mathrm{d}z| \\
&= \frac{1}{2\pi}\int_\Gamma\prod_{i=1}^{k+1}\big\|R_z(a_i)(1-(b_i-a_i)R_z(a_i))^{-1}\big\|_{\mathcal{B}_i}\,|\mathrm{d}z| \\
&\le \frac{1}{2\pi}\int_\Gamma\prod_{i=1}^{k+1}\frac{\|R_z(a_i)\|_{\mathcal{B}_i}}{1-\|R_z(a_i)\|_{\mathcal{B}_i}\|b_i-a_i\|_{\mathcal{B}_i}}\,|\mathrm{d}z| \\
&\le \frac{\ell(\Gamma)}{2\pi}\prod_{i=1}^{k+1}\frac{c_i}{1-c_i\|a_i-b_i\|_{\mathcal{B}_i}},
\end{aligned}
$$

where $\ell(\Gamma)$ is the sum of the lengths of the curves comprising Γ and $c_i := \sup_{z\in\Gamma^*}\|R_z(a_i)\|_{\mathcal{B}_i}$, by the triangle inequality, the resolvent identity, and Theorem 1.4.5(i). For this argument to work, we need $\|a_i - b_i\|_{\mathcal{B}_i} < 1/c_i$ and ε to be small enough that Γ surrounds $\bigcup_{i=1}^{k+1}(\sigma(a_i) \cup \sigma(b_i))$ in U. Next, let $(\mathbf{a}_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathcal{B}_1)_U \times \cdots \times (\mathcal{B}_{k+1})_U$ converging to $\mathbf{a}$. By Proposition 2.3.7 and the estimate we just proved, if Γ is a cycle surrounding $\bigcup_{i=1}^{k+1}\sigma(a_i)$ in U and $n \in \mathbb{N}$ is sufficiently large, then

$$\begin{aligned}\big\|f_\otimes^{[k]}(\mathbf{a}) - f_\otimes^{[k]}(\mathbf{a}_n)\big\|_{\mathcal{C}} &\leq \sum_{i=1}^{k+1}\big\|f_\otimes^{[k+1]}(a_1,\ldots,a_i,a_{n,i},\ldots,a_{n,k+1})\#_{k,i}[a_i - a_{n,i}]\big\|_{\mathcal{C}} \\ &\leq \sum_{i=1}^{k+1}\big\|f_\otimes^{[k+1]}(a_1,\ldots,a_i,a_{n,i},\ldots,a_{n,k+1})\big\|_{\mathcal{B}_1\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi\mathcal{B}_i\hat{\otimes}_\pi\mathcal{B}_i\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi\mathcal{B}_{k+1}}\|a_i - a_{n,i}\|_{\mathcal{B}_i} \\ &\leq \frac{\ell(\Gamma)}{2\pi}\sum_{i=1}^{k+1}\|a_i - a_{n,i}\|_{\mathcal{B}_i} c_1\cdots c_i\prod_{j=i}^{k+1}\frac{c_j}{1 - c_j\|a_j - a_{n,j}\|_{\mathcal{B}_j}} \xrightarrow{n\to\infty} 0,\end{aligned}$$

as desired. $\square$

We now move on to the main result of this chapter.

Lemma 2.3.9. *If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{B}$, $a, b \in \mathcal{B}_U$, $f \in \mathrm{Hol}(U)$, and $a - b \in \mathcal{I}$, then $f(a) - f(b) \in \mathcal{I}$.*

Proof. By Proposition 2.2.6, $u\#c \in \mathcal{I}$ whenever $u \in \mathcal{B}\hat{\otimes}_\pi\mathcal{B}$ and $c \in \mathcal{I}$, so the conclusion follows from the first formula in Proposition 2.3.7. $\square$

Theorem 2.3.10 (Derivatives of holomorphic functional calculus)**.** *Let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{B}$. If $a \in \mathcal{B}_U$ and $f \in \mathrm{Hol}(U)$, then the map*

$$\mathcal{I}_{U,a} \ni b \mapsto f_{a,\mathcal{I}}(b) \coloneqq f(a+b) - f(a) \in \mathcal{I}$$

is holomorphic with respect to $\|\cdot\|_{\mathcal{I}}$. (This map is well defined by Lemma 2.3.9.) Furthermore, if $b \in \mathcal{I}_{U,a}$ and $b_1, \ldots, b_k \in \mathcal{I}$, then

$$\begin{aligned}\partial_{b_k}\cdots\partial_{b_1} f_{a,\mathcal{I}}(b) &= \sum_{\pi\in S_k} f_\otimes^{[k]}\big((a+b)_{(k+1)}\big)\#_k\big[b_{\pi(1)},\ldots,b_{\pi(k)}\big] \\ &= \frac{1}{2\pi i}\sum_{\pi\in S_k}\int_\Gamma f(z)\,(z-a-b)^{-1}b_{\pi(1)}\cdots(z-a-b)^{-1}b_{\pi(k)}(z-a-b)^{-1}\,\mathrm{d}z,\end{aligned}$$

where Γ is any cycle surrounding $\sigma(a+b)$ in U.

Proof. Let $b \in \mathcal{I}_{U,a}$, and let $h \in \mathcal{I}$ be such that $b+h \in \mathcal{I}_{U,a}$. We prove the claimed derivative formula by induction on k. For the base case, note that

$$\begin{aligned}
\varepsilon(h) &\coloneqq \frac{1}{\|h\|_{\mathcal{I}}}\big\|f_{a,\mathcal{I}}(b+h) - f_{a,\mathcal{I}}(b) - f_{\otimes}^{[1]}(a+b,a+b)\#h\big\|_{\mathcal{I}} \\
&= \frac{1}{\|h\|_{\mathcal{I}}}\Big\|f(a+b+h) - f(a+b) - f_{\otimes}^{[1]}(a+b,a+b)\#h\Big\|_{\mathcal{I}} \\
&= \frac{1}{\|h\|_{\mathcal{I}}}\Big\|f_{\otimes}^{[1]}(a+b+h,a+b)\#h - f_{\otimes}^{[1]}(a+b,a+b)\#h\Big\|_{\mathcal{I}} \\
&\leq \Big\|f_{\otimes}^{[1]}(a+b+h,a+b) - f_{\otimes}^{[1]}(a+b,a+b)\Big\|_{\mathcal{B}\hat{\otimes}_{\pi}\mathcal{B}} \xrightarrow{\|h\|_{\mathcal{I}}\to 0} 0
\end{aligned}$$

by Propositions 2.3.7, 2.2.6, and 2.3.8. Now, assume the claimed derivative formula for the k^{th} derivative. If $b_1,\ldots,b_k \in \mathcal{I}$ and $b_{k+1} \coloneqq h$, then

$$\begin{aligned}
\varepsilon(b_1,\ldots,b_{k+1}) &\coloneqq \frac{1}{\|h\|_{\mathcal{I}}}\Bigg\|\partial_{b_k}\cdots\partial_{b_1}f_{a,\mathcal{I}}(b+h) - \partial_{b_k}\cdots\partial_{b_1}f_{a,\mathcal{I}}(b) \\
&\qquad - \sum_{\sigma\in S_{k+1}} f_{\otimes}^{[k+1]}\big((a+b)_{(k+2)}\big)\#_{k+1}\big[b_{\sigma(1)},\ldots,b_{\sigma(k+1)}\big]\Bigg\|_{\mathcal{I}} \\
&= \frac{1}{\|h\|_{\mathcal{I}}}\Bigg\|\sum_{\pi\in S_k}\big(f_{\otimes}^{[k]}\big((a+b+h)_{(k+1)}\big) - f_{\otimes}^{[k]}\big((a+b)_{(k+1)}\big)\big)\#_k\big[b_{\pi(1)},\ldots,b_{\pi(k)}\big] \\
&\qquad - \sum_{\sigma\in S_{k+1}} f_{\otimes}^{[k+1]}\big((a+b)_{(k+2)}\big)\#_{k+1}\big[b_{\sigma(1)},\ldots,b_{\sigma(k+1)}\big]\Bigg\|_{\mathcal{I}} \\
&= \frac{1}{\|h\|_{\mathcal{I}}}\Bigg\|\sum_{\pi\in S_k}\sum_{i=1}^{k+1} f_{\otimes}^{[k+1]}\big((a+b+h)_{(i)},(a+b)_{(k+2-i)}\big)\#_{k+1}\big[b_{\pi(1)},\ldots,b_{\pi(i-1)},h,b_{\pi(i)},\ldots,b_{\pi(k)}\big] \\
&\qquad - \sum_{\pi\in S_k}\sum_{i=1}^{k+1} f_{\otimes}^{[k+1]}\big((a+b)_{(k+2)}\big)\#_{k+1}\big[b_{\pi(1)},\ldots,b_{\pi(i-1)},h,b_{\pi(i)},\ldots,b_{\pi(k)}\big]\Bigg\|_{\mathcal{I}} \\
&\leq k!\,\|b_1\|\cdots\|b_k\|\sum_{i=1}^{k+1}\Big\|f_{\otimes}^{[k+1]}\big((a+b+h)_{(i)},(a+b)_{(k+2-i)}\big) - f_{\otimes}^{[k+1]}\big((a+b)_{(k+2)}\big)\Big\|_{\mathcal{B}^{\hat{\otimes}_{\pi}(k+2)}} \\
&\leq k!\,C_{\mathcal{I}}^{k}\|b_1\|_{\mathcal{I}}\cdots\|b_k\|_{\mathcal{I}}\sum_{i=1}^{k+1}\Big\|f_{\otimes}^{[k+1]}\big((a+b+h)_{(i)},(a+b)_{(k+2-i)}\big) - f_{\otimes}^{[k+1]}\big((a+b)_{(k+2)}\big)\Big\|_{\mathcal{B}^{\hat{\otimes}_{\pi}(k+2)}}
\end{aligned}$$

by the induction hypothesis and Propositions 2.3.7 and 2.2.6. Writing

$$F(a)[b_1,\ldots,b_{k+1}] \coloneqq \sum_{\sigma\in S_{k+1}} f_{\otimes}^{[k+1]}\big(a_{(k+2)}\big)\#_{k+1}\big[b_{\sigma(1)},\ldots,b_{\sigma(k+1)}\big], \qquad a\in\mathcal{B}_U,\ b_i\in\mathcal{I},$$

we then conclude from Proposition 2.3.8 that

$$\frac{1}{\|h\|_{\mathcal{I}}}\Big\|D^k f_{a,\mathcal{I}}(b+h)+D^k f_{a,\mathcal{I}}(b)-F(a+b)\Big\|_{B_k(\mathcal{I}^k;\mathcal{I})}$$
$$\leq k!\,C_{\mathcal{I}}^k\sum_{i=1}^{k+1}\big\|f_{\otimes}^{[k+1]}\big((a+b+h)_{(i)},(a+b)_{(k+2-i)}\big)-f_{\otimes}^{[k+1]}\big((a+b)_{(k+2)}\big)\big\|_{\mathcal{B}^{\hat{\otimes}_\pi(k+2)}}\xrightarrow{\|h\|_{\mathcal{I}}\to 0}0.$$

This completes the proof. □

Corollary 2.3.11. *If $f\in \mathrm{Hol}(U)$, then the map $f_{\mathcal{B}}\colon \mathcal{B}_U\to\mathcal{B}$ defined (via the holomorphic functional calculus) by $a\mapsto f(a)$ is holomorphic. Furthermore,*

$$\partial_{b_k}\cdots\partial_{b_1}f_{\mathcal{B}}(a)=\sum_{\pi\in S_k}f_{\otimes}^{[k]}\big(a_{(k+1)}\big)\#_k\big[b_{\pi(1)},\ldots,b_{\pi(k)}\big],\qquad a\in\mathcal{B}_U,\ b_i\in\mathcal{B}.$$

Proof. Apply Theorem 2.3.10 with $(\mathcal{I},\|\cdot\|_{\mathcal{I}})=(\mathcal{B},\|\cdot\|)$ and $a=0$. □

Remark 2.3.12 (Taylor series expansion)**.** Fix $a\in\mathcal{B}_U$ and $f\in\mathrm{Hol}(U)$. By combining Remark 1.2.15 and Theorem 2.3.10, we get that if $b\in\mathcal{I}_{U,a}$ is sufficiently near $0\in\mathcal{I}_{U,a}$, then

$$f(a+b)-f(a)=\sum_{n=0}^{\infty}\frac{1}{n!}\partial_b^n f_{a,\mathcal{I}}(0)=\sum_{n=1}^{\infty}f_{\otimes}^{[n]}\big(a_{(n+1)}\big)\#_n\big[b_{(n)}\big],$$

where the series above converges absolutely in $(\mathcal{I},\|\cdot\|_{\mathcal{I}})$. In particular, if $[a,b]=0$ as well, then Example 2.3.6 gives

$$f(a+b)-f(a)=\sum_{n=1}^{\infty}\frac{1}{n!}f^{(n)}(a)\,b^n,$$

which generalizes the formula from Proposition 2.1.13.

2.4 A word on multivariate holomorphic functional calculus

In this section, we make precise that $f^{[k]}(\mathbf{a})$, as defined in Equation (2.3.3), is the holomorphic function $f^{[k]}$ of $k+1$ variables applied to the $(k+1)$-tuple $\mathbf{a}$. This necessitates the development of a "baby" multivariate holomorphic functional calculus. A proper treatment of "adult" multivariate functional calculi is out of the scope of this dissertation; we refer the reader to [Cur88] for more information and references.

Notation 2.4.1. For each $i \in \{1, \ldots, m\}$, let S_i be a nonempty set. Also, write $S \coloneqq S_1 \times \cdots \times S_m$ and $V \coloneqq U_1 \times \cdots \times U_m \subseteq \mathbb{C}^m$.

(i) If $f_i \in \mathbb{C}^{S_i}$ for all $i \in \{1, \ldots, m\}$, then

$$(f_1 \otimes \cdots \otimes f_m)(\mathbf{s}) \coloneqq f_1(s_1) \cdots f_m(s_m), \qquad \mathbf{s} = (s_1, \ldots, s_m) \in S.$$

Of course, $f_1 \otimes \cdots \otimes f_m \in \mathbb{C}^S$.[1]

(ii) Recall from Equation (2.1.2) that $\mathcal{R}_U \subseteq \mathrm{Hol}(U)$ is the set of rational functions with poles outside of U. Define

$$\mathcal{R}_V \coloneqq \mathrm{span}\left\{r_1 \otimes \cdots \otimes r_m : r_1 \in \mathcal{R}_{U_1}, \ldots, r_m \in \mathcal{R}_{U_m}\right\} \subseteq \mathrm{Hol}(V).$$

Also, define $\mathrm{Hol}_0(V)$ to be the closure of $\mathcal{R}_V$ in $\mathrm{Hol}(V)$.

By Runge's theorem,

$$\{f_1 \otimes \cdots \otimes f_m : f_1 \in \mathrm{Hol}(U_1), \ldots, f_m \in \mathrm{Hol}(U_m)\} \subseteq \mathrm{Hol}_0(V).$$

Regardless, if $m \geq 2$, then $\mathrm{Hol}_0(V) \subsetneq \mathrm{Hol}(V)$ in general. This is part of what complicates holomorphic functional calculus in the multivariate case. To avoid this and other complications, we construct a functional calculus defined only on $\mathrm{Hol}_0(V)$.

Theorem 2.4.2 ("Baby" multivariate holomorphic functional calculus)**.** *Write $V \coloneqq U_1 \times \cdots \times U_m$. Also, suppose $\mathbf{a} = (a_1, \ldots, a_m) \in \mathcal{B}_{U_1} \times \cdots \times \mathcal{B}_{U_m}$ is such that $[a_i, a_j] = 0$ for all $i, j \in \{1, \ldots, m\}$. There exists a unique continuous, unital algebra homomorphism $H_{\mathbf{a}}^V \colon \mathrm{Hol}_0(V) \to \mathcal{B}$ that maps the coordinate function $1^{\otimes(i-1)} \otimes \iota_{U_i} \otimes 1^{\otimes(m-i)}$ to a_i for all $i \in \{1, \ldots, m\}$.*

Proof. By the argument from the proof of uniqueness in Theorem 2.1.1, if $\Phi, \Psi \colon \mathrm{Hol}_0(V) \to \mathcal{B}$ are two unital algebra homomorphism sending $\mathbf{z} \mapsto z_i$ to a_i for all $i \in \{1, \ldots, m\}$, then $\Phi = \Psi$ on $\mathcal{R}_V$. Since $\mathcal{R}_V$ is dense in $\mathrm{Hol}_0(V)$, if Φ and Ψ are also continuous, then $\Phi = \Psi$ on $\mathrm{Hol}_0(V)$.

[1] Using a result like [Rya02, Prop. 1.2] (and the comments thereafter), one can show that the linear map $\mathbb{C}^{S_1} \otimes \cdots \otimes \mathbb{C}^{S_m} \to \mathbb{C}^S$ determined by $f_1 \otimes \cdots \otimes f_m \mapsto ((s_1, \ldots, s_m) \mapsto f_1(s_1) \cdots f_m(s_m))$ is injective, so our notation is justified.

Now, for each $i \in \{1, \ldots, m\}$, let Γ_i be a cycle in surrounding $\sigma(a_i)$ in U_i, and define

$$H_{\mathbf{a}}^V(\varphi) := \frac{1}{(2\pi i)^m} \int_{\Gamma_m} \cdots \int_{\Gamma_1} \varphi(z_1, \ldots, z_m)\,(z_1 - a_1)^{-1} \cdots (z_m - a_m)^{-1}\,\mathrm{d}z_1 \cdots \mathrm{d}z_m, \qquad \varphi \in \mathrm{Hol}_0(V).$$

Clearly, $H_{\mathbf{a}}^V \colon \mathrm{Hol}_0(V) \to \mathcal{B}$ is linear. By the dominated convergence theorem, $H_{\mathbf{a}}^V$ is continuous. By Equation (2.1.6), if $f_i \in \mathrm{Hol}(U_i)$ for all $i \in \{1, \ldots, m\}$, then

$$\begin{aligned} H_{\mathbf{a}}^V(f_1 \otimes \cdots \otimes f_m) &= \frac{1}{(2\pi i)^m} \int_{\Gamma_m} \cdots \int_{\Gamma_1} f_1(z_1)\,(z_1 - a)^{-1} \cdots f_m(z_m)\,(z_m - a_m)^{-1}\,\mathrm{d}z_1 \cdots \mathrm{d}z_m \\ &= \left(\frac{1}{2\pi i} \int_{\Gamma_1} f_1(z)\,(z - a_1)^{-1}\,\mathrm{d}z \right) \cdots \left(\frac{1}{2\pi i} \int_{\Gamma_m} f_m(z)\,(z - a_m)^{-1}\,\mathrm{d}z \right) \\ &= f_1(a_1) \cdots f_m(a_m). \end{aligned} \tag{2.4.3}$$

In particular, $H_{\mathbf{a}}^V$ is unital and maps $1^{\otimes(i-1)} \otimes \iota_{U_i} \otimes 1^{\otimes(m-i)}$ to a_i for all $i \in \{1, \ldots, m\}$. Finally, we show that $H_{\mathbf{a}}^V$ is multiplicative. Indeed, observe that if $a \in \mathcal{B}_{U_1}$, $b \in \mathcal{B}_{U_2}$, and $[a, b] = 0$, then $[f(a), g(b)] = 0$ for all $f \in \mathrm{Hol}(U_1)$ and $g \in \mathrm{Hol}(U_2)$. Consequently, if $f_i, g_i \in \mathrm{Hol}(U_i)$ for all $i \in \{1, \ldots, m\}$, $\varphi := f_1 \otimes \cdots \otimes f_m$, and $\psi := g_1 \otimes \cdots \otimes g_m$, then

$$\begin{aligned} H_{\mathbf{a}}^V(\varphi\psi) &= H_{\mathbf{a}}^V(f_1 g_1 \otimes \cdots \otimes f_m g_m) = (f_1 g_1)(a_1) \cdots (f_m g_m)(a_m) \\ &= f_1(a_1)\,g_1(a_1) \cdots f_m(a_m)\,g_m(a_m) = f_1(a_1) \cdots f_m(a_m)\,g_1(a_1) \cdots g_m(a_m) \\ &= H_{\mathbf{a}}^V(f_1 \otimes \cdots \otimes f_m)\,H_{\mathbf{a}}^V(g_1 \otimes \cdots \otimes g_m) = H_{\mathbf{a}}^V(\varphi)\,H_{\mathbf{a}}^V(\psi) \end{aligned}$$

by Equation (2.4.3) and the properties of the (single-variate) holomorphic functional calculus. It follow that $H_{\mathbf{a}}^V$ is multiplicative on the subalgebra $\mathcal{R}_V \subseteq \mathrm{Hol}_0(V)$. Since $\mathcal{R}_V$ is dense in $\mathrm{Hol}_0(V)$ and $H_{\mathbf{a}}^V$ is continuous, we are done. □

Remark 2.4.4. It is actually the case that if

$$\Phi(\varphi) := \frac{1}{(2\pi i)^m} \int_{\Gamma_m} \cdots \int_{\Gamma_1} \varphi(z_1, \ldots, z_m)\,(z_1 - a_1)^{-1} \cdots (z_m - a_m)^{-1}\,\mathrm{d}z_1 \cdots \mathrm{d}z_m, \qquad \varphi \in \mathrm{Hol}(V),$$

then $\Phi \colon \mathrm{Hol}(V) \to \mathcal{B}$ is a unital, continuous algebra homomorphism sending $\mathbf{z} \mapsto z_i$ to a_i for all $i \in \{1, \ldots, m\}$, but it takes slightly more work to prove that Φ is multiplicative. More seriously, uniqueness is a delicate issue for functional calculi defined on all of $\mathrm{Hol}(V)$. Unlike the single-variate case, the proper formulation of uniqueness results, e.g., [Put83, Thm. 1], requires the introduction of more refined notions of "joint spectrum" for m-tuples of (commuting) elements, e.g., the Taylor joint spectrum [Tay70b, Tay70a] or the Harte spectrum [Har72b, Har72a]. Once again, we refer the interested reader to [Cur88] for more information and references.

Definition 2.4.5 ("Baby" multivariate holomorphic functional calculus)**.** The map $H_{\mathbf{a}}^V$ from Theorem 2.4.2 is the (**baby**) **holomorphic functional calculus** for the m-tuple $\mathbf{a} \in \mathcal{B}^m$, and

$$\varphi(\mathbf{a}) \coloneqq H_{\mathbf{a}}^V(\varphi) \in \mathcal{B}, \qquad \varphi \in \mathrm{Hol}_0(V) = \mathrm{Hol}_0(U_1 \times \cdots \times U_m).$$

We are now prepared to formulate and prove the main result of this section.

Lemma 2.4.6. *If $f \in \mathrm{Hol}(U)$, then $f^{[k]} \in \mathrm{Hol}_0\left(U^{k+1}\right)$.*

Proof. Write $\mathcal{D}_k \colon \mathrm{Hol}(U) \to \mathrm{Hol}\left(U^{k+1}\right)$ for the k^{th} divided difference map $f \mapsto f^{[k]}$. As was observed at the end of Example 1.3.10, $\mathcal{D}_k\mathcal{R}_U \subseteq \mathcal{R}_{U^{k+1}} \subseteq \mathrm{Hol}_0\left(U^{k+1}\right)$. By Proposition 1.3.20, the map $\mathcal{D}_k \colon \mathrm{Hol}(U) \to \mathrm{Hol}\left(U^{k+1}\right)$ is continuous. Since $\mathcal{R}_U$ is dense in $\mathrm{Hol}(U)$ and $\mathrm{Hol}_0\left(U^{k+1}\right)$ is closed in $\mathrm{Hol}\left(U^{k+1}\right)$, we conclude that $\mathcal{D}_k\,\mathrm{Hol}(U) \subseteq \overline{\mathcal{D}_k\mathcal{R}_U} \subseteq \mathrm{Hol}_0\left(U^{k+1}\right)$, as desired. $\square$

Theorem 2.4.7 (Justification of Notation 2.3.2)**.** *Let $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in \mathcal{B}_U^{k+1}$ be such that $[a_i, a_j] = 0$ for all $i, j \in \{1, \ldots, k+1\}$. If Γ is a cycle surrounding $\bigcup_{i=1}^{k+1} \sigma(a_i)$ in U, then*

$$f^{[k]}(\mathbf{a}) = H_{\mathbf{a}}^{U^{k+1}}\left(f^{[k]}\right) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)\,(z-a_1)^{-1}\cdots(z-a_{k+1})^{-1}\,\mathrm{d}z, \qquad f \in \mathrm{Hol}(U).$$

Proof. By the uniqueness part of Theorem 2.4.2, if $V \subseteq U$ is an open set such that $\mathbf{a} \in \mathcal{B}_V^{k+1}$, then $\varphi(\mathbf{a}) = H_{\mathbf{a}}^{V^{k+1}}\left(\varphi|_{V^{k+1}}\right)$ for all $\varphi \in \mathrm{Hol}_0\left(U^{k+1}\right)$. If $V \coloneqq \{z \in U \setminus \Gamma^* : \mathrm{Ind}_\Gamma(z) = 1\}$, then Proposition 1.3.20 implies

$$f^{[k]}\big|_{V^{k+1}} = \frac{1}{2\pi i}\int_\Gamma f(z)\left((z-\iota_V)^{-1}\right)^{\otimes(k+1)}\mathrm{d}z \in \mathrm{Hol}_0\left(V^{k+1}\right), \qquad f \in \mathrm{Hol}(U), \tag{2.4.8}$$

where the right-hand side is Bochner integral in the Fréchet space $\mathrm{Hol}_0\left(V^{k+1}\right)$. Applying the continuous homomorphism $H_{\mathbf{a}}^{V^{k+1}}$ to both sides of Equation (2.4.8) then completes the proof. $\square$

Chapter 3

Differentiating at bounded operators

Let $\mathcal{A}$ be a unital C^*-algebra. In this chapter, we discuss the continuous functional calculus for normal elements of $\mathcal{A}$ and compute the higher derivatives of maps on (the self-adjoints of) symmetrically normed ideals of $\mathcal{A}$ induced, via the continuous functional calculus, by sufficiently regular functions of a real variable. Specifically, we introduce and study a space $VC^k(\mathbb{R}) \subseteq C^k(\mathbb{R})$ of "Varopoulos C^k functions" such that the following result holds: If $\mathcal{I}$ is a symmetrically normed ideal of $\mathcal{A}$, $a \in \mathcal{A}_{\mathrm{sa}}$, and $f \in VC^k(\mathbb{R})$, then the map $\mathcal{I}_{\mathrm{sa}} \ni b \mapsto f(a+b) - f(a) \in \mathcal{I}$ is well defined and C^k, and the formula for its k^{th} derivative may be written in terms of a projective tensor product–valued kind of multivariate functional calculus. Furthermore, we prove that $VC^k(\mathbb{R})$ contains all functions for which comparable results are known. Specifically, $VC^k(\mathbb{R})$ contains the homogeneous Besov space $\dot{B}_1^{k,\infty}(\mathbb{R})$ and the Hölder space $C_{\mathrm{loc}}^{k,\varepsilon}(\mathbb{R})$. We highlight, however, that the results in this chapter are the first of their kind to be proven for an arbitrary symmetrically normed ideal of an arbitrary unital C^*-algebra. At the end of the chapter, we give an invitation to the theory of multiple operator integrals (MOIs) by introducing and studying a space $NC^k(\mathbb{R}) \subseteq C^k(\mathbb{R})$ of "noncommutative C^k functions" containing $VC^k(\mathbb{R})$ and such that if $f \in NC^k(\mathbb{R})$, then the map $\mathcal{A}_{\mathrm{sa}} \ni a \mapsto f(a) \in \mathcal{A}$ is C^k, and the formula for its k^{th} derivative can be written in terms of MOIs.

Standing assumptions. Throughout, $m, k \in \mathbb{N}$. In §3.2 and §3.5, $\mathcal{A}$ is a unital C^*-algebra, and $\|\cdot\|_{\mathcal{A}} = \|\cdot\|$. In §3.3, $\Omega_1, \ldots, \Omega_m$ are compact Hausdorff spaces, and $\Omega \coloneqq \Omega_1 \times \cdots \times \Omega_m$. In §3.8, $\Omega_1, \ldots, \Omega_m$ are Polish spaces (complete, separable metric spaces), $\Omega \coloneqq \Omega_1 \times \cdots \times \Omega_m$, $\mathcal{A}$ is a unital C^*-algebra, H is a complex Hilbert space, and $\mathcal{M} \subseteq B(H)$ is a von Neumann algebra.

3.1 Introduction

Let $a \in \mathcal{A}_{\mathrm{sa}} := \{b \in \mathcal{A} : b^* = b\}$. The **continuous functional calculus** for a is the unique (isometric) unital $*$-homomorphism $\Phi_a \colon C(\sigma(a)) \to \mathcal{A}$ sending the inclusion $\iota_{\sigma(a)} \colon \sigma(a) \hookrightarrow \mathbb{C}$ to a. We discuss its construction in the next section. If $\sigma(a) \subseteq S \subseteq \mathbb{C}$, then we write

$$f(a) := \Phi_a\big(f|_{\sigma(a)}\big) \in \mathcal{A}, \qquad f \in C(S).$$

Recall from the dissertation introduction that if $f \in C(\mathbb{R})$, then the map

$$\mathcal{A}_{\mathrm{sa}} \ni a \mapsto f_{\mathcal{A}}(a) := f(a) = \Phi_a\big(f|_{\sigma(a)}\big) \in \mathcal{A}$$

is continuous; however, it is not generally true that $f \in C^k(\mathbb{R})$ implies $f_{\mathcal{A}} \in C^k(\mathcal{A}_{\mathrm{sa}}; \mathcal{A})$. In particular, it is not generally true that if $a \in \mathcal{A}_{\mathrm{sa}}$, $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{A}$ (Definition 2.2.1), and $f \in C^k(\mathbb{R})$, then the map $\mathcal{I}_{\mathrm{sa}} := \mathcal{I} \cap \mathcal{A}_{\mathrm{sa}} \ni b \mapsto f(a+b) - f(a) \in \mathcal{I}$ is well defined and C^k with respect to $\|\cdot\|_{\mathcal{I}}$. We deal with this by asking f to be slightly more regular than C^k. To shed some light on our approach, we examine the matrix case, i.e., we take $\mathcal{A} = \mathrm{M}_n(\mathbb{C})$.

Notation 3.1.1. If $a \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}$ and $\lambda \in \sigma(a) = \{\text{eigenvalues of } a\}$, then $P_\lambda^a \in \mathrm{M}_n(\mathbb{C})$ is the orthogonal projection onto the λ-eigenspace of a. If $\mathbf{a} = (a_1, \ldots, a_m) \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}^m$ and $\varphi \colon \sigma(a_1) \times \cdots \times \sigma(a_m) \to \mathbb{C}$ is any function, then

$$\varphi_{\otimes}(\mathbf{a}) := \sum_{\boldsymbol{\lambda} \in \sigma(a_1) \times \cdots \times \sigma(a_m)} \varphi(\boldsymbol{\lambda})\, P_{\lambda_1}^{a_1} \otimes \cdots \otimes P_{\lambda_m}^{a_m} \in \mathrm{M}_n(\mathbb{C})^{\otimes m}.$$

Above, $\otimes$ is the tensor product over $\mathbb{C}$, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$.

Theorem 3.1.2 (Derivatives of perturbed matrix functions)**.** *Suppose $\mathcal{A} \subseteq \mathrm{M}_n(\mathbb{C})$ is a unital $*$-subalgebra, and let $\mathcal{I} \trianglelefteq \mathcal{A}$. If $a \in \mathcal{A}$ and $f \in C^k(\mathbb{R})$, then the **perturbed matrix function** $f_{a,\mathcal{I}} \colon \mathcal{I}_{\mathrm{sa}} \to \mathcal{I}$ defined by $b \mapsto f(a+b) - f(a)$ is well defined and C^k. Furthermore,*

$$\partial_{b_k} \cdots \partial_{b_1} f_{a,\mathcal{I}}(b) = \sum_{\pi \in S_k} f_{\otimes}^{[k]}\big((a+b)_{(k+1)}\big) \#_k \big[b_{\pi(1)}, \ldots, b_{\pi(k)}\big], \qquad b, b_i \in \mathcal{I}_{\mathrm{sa}}, \tag{3.1.3}$$

where $(a+b)_{(k+1)} = (a+b, \ldots, a+b) \in \mathcal{A}^{k+1}$ (Notation 1.2.5(i)) and $\#_k$ is as in Notation 1.5.9.

With $\mathcal{I} = \mathcal{A} = \mathrm{M}_n(\mathbb{C})$, this result is due essentially to Yu. L. Daletskii and S. G. Krein [DK56], though it was proven in approximately the above form by F. Hiai as [Hia10, Thm. 2.3.1]. One way to prove Theorem 3.1.2 is to use the method of perturbation formulas; please see the proof of [ST19, Thm. 5.3.2] for this kind of argument. This is currently the standard approach to proving such results since it can be adapted to differentiating operator functions at unbounded operators; please see, e.g., [dPS04, Pel06, ACDS09, AP16, Pel16, CLMSS19, LMS20, LMM21] as well as Chapter 6. The classical approach (of Daletskii–Krein) is by polynomial approximation: Establish Equation (6) first when f is a polynomial, and then deduce the general case from the density of polynomials in $C^k(\mathbb{R})$. The details of both methods provide important inspiration for this chapter. Since we already saw an example of the method of perturbation formulas in Chapter 2, we go through the polynomial approximation argument in §3.10.

Looking at Notation 3.1.1 and Equation (3.1.3), we can see what must be done in the general case. In view of the fact that

$$f(a) = \sum_{\lambda \in \sigma(a)} f(\lambda)\, P_\lambda^a, \qquad a \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}},\ f \in C(\sigma(a)) = \mathbb{C}^{\sigma(a)},$$

it seems as though $\varphi_\otimes(\mathbf{a}) \in \mathrm{M}_n(\mathbb{C})^{\otimes m}$, as defined in Notation 3.1.1, is the m-variate (continuous) function $\varphi \colon \sigma(a_1) \times \cdots \times \sigma(a_m) \to \mathbb{C}$ applied to the m-tuple

$$\big(I_n^{\otimes(i-1)} \otimes a_i \otimes I_n^{\otimes(m-i)}\big)_{i=1}^m \in \big(\mathrm{M}_n(\mathbb{C})^{\otimes m}\big)^m$$

of commuting elements. To make sense of this when $\mathrm{M}_n(\mathbb{C})$ is replaced by our arbitrary unital C^*-algebra $\mathcal{A}$ and $\mathrm{M}_n(\mathbb{C})^{\otimes m}$ replaced by $\mathcal{A}^{\hat{\otimes}_\pi m}$ (so that we may apply the # operations), we ask that the function $\varphi \colon \sigma(a_1) \times \cdots \sigma(a_m) \to \mathbb{C}$ be slightly better than continuous. More precisely, we ask that $\varphi \in C(\sigma(a_1)) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\sigma(a_m))$. The algebra $C(\sigma(a_1)) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\sigma(a_m))$ has a concrete description as a subalgebra of $C(\sigma(a_1) \times \cdots \times \sigma(a_m))$ called the **Varopoulos algebra**, which we study in §3.3. For functions in the Varopoulos algebra, we can define the kind of functional calculus we need. (Please review Corollary 1.5.6.)

Notation 3.1.4 (Projective tensor product functional calculus)**.** If $\mathbf{a} = (a_1, \ldots, a_m) \in \mathcal{A}_{\mathrm{sa}}^m$ and $\varphi \in C(\sigma(a_1)) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\sigma(a_m))$, then $\varphi_\otimes(\mathbf{a}) \coloneqq \big(\Phi_{a_1} \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \Phi_{a_m}\big)(\varphi) \in \mathcal{A}^{\hat{\otimes}_\pi m}$.

Now, to ensure that Equation (3.1.3) makes sense for general $\mathcal{A}$ and $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{A}$, we simply demand that $f^{[k]}\big|_{[-r,r]^{k+1}} \in C([-r,r])^{\hat{\otimes}_\pi (k+1)}$ for all $r > 0$.

Definition 3.1.5 (Varopoulos C^k functions)**.** A function $f \in C^k(\mathbb{R})$ is **Varopoulos $\boldsymbol{C^k}$**, written $f \in VC^k(\mathbb{R})$, if $f^{[k]}\big|_{[-r,r]^{k+1}} \in C([-r,r])^{\hat{\otimes}_\pi (k+1)}$ for all $r > 0$.

It turns out $C^k(\mathbb{R}) \subseteq VC^{k-1}(\mathbb{R})$ (taking $VC^0(\mathbb{R}) \coloneqq C(\mathbb{R})$), so that $VC^{k-1}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$. We conduct a thorough study of $VC^k(\mathbb{R})$, including a natural topology it carries, in §3.4, §3.6, and §3.7. This chapter's first main result comes in the form of a summary of this study, including some examples of Varopoulos C^k functions that paint the picture that a function only has to be "slightly better than C^k" to be Varopoulos C^k. To state our result, we note that $W_k(\mathbb{R})$ is the k^{th} Wiener space (Definition 1.3.13), $\dot{B}_q^{s,p}(\mathbb{R})$ is the homogeneous (s,p,q)-Besov space (Definition 3.6.1), and $C_{\mathrm{loc}}^{k,\varepsilon}(\mathbb{R})$ is the space of C^k functions such that $f^{(k)}$ is locally ε-Hölder continuous (Definition 3.6.13). In addition, if $\mathcal{S} \subseteq C^k(\mathbb{R})$, then $\mathcal{S}_{\mathrm{loc}}$ is defined to be the set of all $f \in C^k(\mathbb{R})$ such that for all $r > 0$, there exists $g \in \mathcal{S}$ such that $g|_{[-r,r]} = f|_{[-r,r]}$.

Theorem 3.1.6 (A study of $VC^k(\mathbb{R})$)**.** *If $k \in \mathbb{N}$ and $\varepsilon > 0$, then*

(i) $\dot{B}_1^{k,\infty}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$,

(ii) $C_{\mathrm{loc}}^{k,\varepsilon}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$,

(iii) $W_k(\mathbb{R})_{\mathrm{loc}} \subsetneq VC^k(\mathbb{R}) \subsetneq C^k(\mathbb{R})$, *and*

(iv) $W_k(\mathbb{R})$ *and* $\mathbb{C}[\lambda]$ *are dense subspaces of* $VC^k(\mathbb{R})$.

Proof. The first item is part of Theorem 3.6.10. The second item (ii) is Theorem 3.6.17. The first containment in the third item (but not its strictness) follows from Example 3.4.3 and Proposition 3.4.4(ii); an example demonstrating its strictness is given in Theorem 3.7.1. An example demonstrating the strictness of the second containment in the third item is given in Theorem 3.9.1. Finally, the fourth item is Theorem 3.4.12. □

Remark 3.1.7. As a consequence of Theorem 3.1.6(iv), $VC^k(\mathbb{R})$ may be identified with the space $C_{\mathrm{nc}}^k(\mathbb{R})$ introduced and briefly studied by D. A. Jekel in [Jek20, Ch. 18]. Please see Remark 3.4.13 for more information.

Our second main result concerns the higher differentiability of (perturbed) maps induced via the continuous functional calculus by Varopoulos C^k functions.

Theorem 3.1.8 (Derivatives of perturbed operator functions)**.** *Let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{A}$. If $a \in \mathcal{A}_{\mathrm{sa}}$ and $f \in VC^k(\mathbb{R})$, then the* ***perturbed operator function*** *$f_{a,\mathcal{I}} \colon \mathcal{I}_{\mathrm{sa}} \to \mathcal{I}$ defined by $b \mapsto f(a+b)-f(a)$ is well defined and belongs to $C^k_{\mathrm{bb}}(\mathcal{I}_{\mathrm{sa}};\mathcal{I})$ (Definition 1.2.9) with respect to $\|\cdot\|_{\mathcal{I}}$. Furthermore,*

$$\partial_{b_k} \cdots \partial_{b_1} f_{a,\mathcal{I}}(b) = \sum_{\pi \in S_k} f_{\otimes}^{[k]}\big((a+b)_{(k+1)}\big) \#_k \big[b_{\pi(1)}, \ldots, b_{\pi(k)}\big], \qquad b, b_i \in \mathcal{I}_{\mathrm{sa}}.$$

Remark 3.1.9. The term "(perturbed) operator function" is used because of the historical importance of the case when $\mathcal{A} = B(H)$, where H is a complex Hilbert space.

Corollary 3.1.10 (Derivatives of operator functions)**.** *If $f \in VC^k(\mathbb{R})$, then the* ***operator function*** *$f_{\mathcal{A}} \colon \mathcal{A}_{\mathrm{sa}} \to \mathcal{A}$ defined by $a \mapsto f(a)$ belongs to $C^k_{\mathrm{bb}}(\mathcal{A}_{\mathrm{sa}};\mathcal{A})$. Furthermore,*

$$\partial_{b_k} \cdots \partial_{b_1} f_{\mathcal{A}}(a) = \sum_{\pi \in S_k} f_{\otimes}^{[k]}\big(a_{(k+1)}\big) \#_k \big[b_{\pi(1)}, \ldots, b_{\pi(k)}\big], \qquad a, b_i \in \mathcal{A}_{\mathrm{sa}}.$$

Proof. Apply Theorem 3.1.8 with $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) = (\mathcal{A}, \|\cdot\|)$ and $a = 0$. $\square$

Inspired by the two proofs of Theorem 3.1.2 mentioned above, we provide two proofs of Theorem 3.1.8 in §3.5. Together, Theorems 3.1.6(i) and 3.1.8 yield a vast generalization of previously known results on the k-times differentiability of perturbed operator functions. Indeed, let H be a separable complex Hilbert space. The case when $\mathcal{I} = \mathcal{A} = B(H)$ and $f \in \dot{B}_1^{k,\infty}(\mathbb{R})$ was established in [Pel06]; and the case when $\mathcal{A} \subseteq B(H)$ is a von Neumann algebra, $\mathcal{I}$ has property (F) (§6.1), and $f \in W_{k+1}(\mathbb{R})$ is established in [ACDS09]. These results are discussed in more depth in §6.1. The only other such result in the literature that we do not recover is the case when $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) = \big(\mathcal{S}_p(H), \|\cdot\|_{\mathcal{S}_p}\big)$ is the ideal of Schatten p-class operators (Definition 4.3.1) with $1 < p < \infty$. In this case, perturbed operator functions induced by C^k functions are well defined and C^k in the Schatten p-norms; please see [LMS20].

The papers referenced in the previous paragraph make use of multiple operator integrals (MOIs), which are prominent in Chapters 5–7. In §3.8, we provide an MOI-based approach to the polynomial approximation argument for computing higher derivatives of operator functions.

3.2 Continuous functional calculus

In §2.1, we covered the basics of the holomorphic functional calculus for an element a of a unital Banach algebra. In this section, we show that this calculus can be extended to all continuous functions on $\sigma(a)$ when the Banach algebra is a C^*-algebra and a is normal.

Definition 3.2.1 (Normal, unitary, self-adjoint, and positive elements)**.** An element $a \in \mathcal{A}$ is **normal** if $a^*a = aa^*$, **unitary** if $a^*a = aa^* = 1$, **self-adjoint** if $a^* = a$, and **positive** if it is self-adjoint and $\sigma(a) \subseteq [0,\infty) = \mathbb{R}_+$. Write $\mathcal{A}_\nu$, $\mathrm{U}(\mathcal{A})$, $\mathcal{A}_{\mathrm{sa}}$, and $\mathcal{A}_+$ for the sets of normal, unitary, self-adjoint, and positive elements of $\mathcal{A}$, respectively.

Observe that $\mathcal{A}_{\mathrm{sa}} \subseteq \mathcal{A}$ is a closed, real-linear subspace. Thus, $\mathcal{A}_{\mathrm{sa}}$ is a real Banach space.

Lemma 3.2.2 (Spectrum of unitary)**.** *If $u \in \mathrm{U}(\mathcal{A})$, then $\sigma(u) \subseteq S^1 := \{z \in \mathbb{C} : |z| = 1\}$.*

Proof. First, note that $\|u\|^2 = \|u^*u\| = \|1\| = 1$. Thus, $\sigma(u) \subseteq \{z \in \mathbb{C} : |z| \leq 1\}$. Now, since $u^* = u^{-1}$, if $\lambda \in \mathbb{C}$ and $|\lambda| < 1$, then $\lambda - u = -u\big(1 - \lambda u^*\big)$. Since $\|\lambda u^*\| = |\lambda|\,\|u\| = |\lambda| < 1$, we conclude from Theorem 1.4.5(i) that $\lambda - u$ is invertible. Thus, $\sigma(u) \subseteq S^1$, as desired. $\square$

Lemma 3.2.3 (Spectrum of self-adjoint)**.** *If $a \in \mathcal{A}_{\mathrm{sa}}$, then $\sigma(a) \subseteq \mathbb{R}$.*

Proof. First, it is easy to show that if $\mathcal{B}$ is a unital Banach algebra and $b, c \in \mathcal{B}$ satisfy $[b,c] = 0$, then $e^{b+c} = e^b e^c$. Now, if $a \in \mathcal{A}$, then $(e^a)^* = e^{a^*}$. Consequently, if $a \in \mathcal{A}_{\mathrm{sa}}$, then $u := e^{ia}$ is unitary. Next, let $\lambda \in \mathbb{C}$, and write

$$b_\lambda := e^{i\lambda} \sum_{n=1}^{\infty} \frac{i^n}{n!}(a-\lambda)^{n-1}.$$

Then

$$e^{i\lambda} - u = -e^{i\lambda}\big(e^{i(a-\lambda)} - 1\big) = b_\lambda(\lambda - a).$$

Since $[\lambda - a, b_\lambda] = 0$, it is a basic algebra fact that if the product $b_\lambda(\lambda - a)$ is invertible, then both $\lambda - a$ and b_λ are invertible. Consequently, if $\lambda \in \sigma(a)$, i.e., $\lambda - a$ is *not* invertible, then $e^{i\lambda} - u = b_\lambda(\lambda - a)$ is not invertible, i.e., $e^{i\lambda} \in \sigma(u)$. By Lemma 3.2.2, this implies $|e^{i\lambda}| = 1$. Thus, $\lambda \in \mathbb{R}$, as desired. $\square$

If $\mathcal{B}$ is a unital Banach algebra, $\mathcal{C} \subseteq \mathcal{B}$ is a closed unital subalgebra, and $a \in \mathcal{C}$, then it is possible that $\sigma_{\mathcal{B}}(a) \subsetneq \sigma_{\mathcal{C}}(a)$; please see [Con90, Exs. VII.3.2 & VII.5.1]. From Lemma 3.2.3 and the classification of commutative unital C^*-algebras, we get that the analogous pathology cannot occur in a unital C^*-algebra. (Please see [Con90, Prop. VIII.1.14] for another proof.)

Proposition 3.2.4 (Spectrum computed in C^*-subalgebras)**.** *Let $\mathcal{B} \subseteq \mathcal{A}$ be a unital C^*-subalgebra. If $a \in \mathcal{B}$, then $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$.*

Proof. We begin by showing that if $a \in \mathcal{B}_{\text{sa}}$ and a is invertible in $\mathcal{A}$, then $a^{-1} \in \mathcal{B}$. Indeed, if $\mathcal{C}$ is the smallest unital C^*-subalgebra of $\mathcal{A}$ containing a and a^{-1}, then $\mathcal{C}$ is commutative. As mentioned in Example 1.4.8, there exists a compact Hausdorff space X and an isometric $*$-isomorphism $\pi \colon \mathcal{C} \to C(X)$. Write $f := \pi(a) \in C(X)$. Since $a^* = a$, f is real-valued. Since a is invertible in $\mathcal{C}$ (by construction), f never vanishes, and $\pi\big(a^{-1}\big) = 1/f$. By the (real) Stone–Weierstrass theorem applied to the compact set $f(X) \subseteq \mathbb{R} \setminus \{0\}$, there exists a sequence $(q_n(\lambda))_{n\in\mathbb{N}}$ in $\mathbb{R}[\lambda]$ such that $q_n \circ f \to 1/f$ uniformly as $n \to \infty$. If follows that $q_n(a) = \pi^{-1}(q_n \circ f) \to \pi^{-1}(1/f) = a^{-1}$ in $\mathcal{C}$ as $n \to \infty$. Since $q_n(a) \in \mathcal{B}$ for all $n \in \mathbb{N}$, we conclude that $a^{-1} \in \mathcal{B}$, as desired.

Now, we claim that if $a \in \mathcal{B}$ is arbitrary and invertible in $\mathcal{A}$, then $b := a^{-1} \in \mathcal{B}$. Indeed, in this case, a^* is invertible in $\mathcal{A}$ with inverse b^*. Thus, $a^*a \in \mathcal{B}_{\text{sa}}$ is invertible in $\mathcal{A}$ with inverse bb^*. By the previous paragraph, $bb^* = (a^*a)^{-1} \in \mathcal{B}$. Thus, $b = b(a^*)^{-1}a^* = (bb^*)a^* \in \mathcal{B}$, as desired. The result follows. $\square$

There are two key ingredients to the construction of the continuous functional calculus: the spectral mapping theorem for non-holomorphic polynomials (Theorem 3.2.6) and the spectral radius formula for normal elements (Lemma 3.2.7).

Notation 3.2.5. If $\mathcal{B}$ is a unital $\mathbb{C}$-algebra and $P(\boldsymbol{\lambda}) = \sum_{|\alpha|\leq d} c_\alpha \boldsymbol{\lambda}^\alpha \in \mathbb{C}[\boldsymbol{\lambda}] = \mathbb{C}[\lambda_1, \ldots, \lambda_m]$, then we define

$$P(\mathbf{a}) := \sum_{|\alpha|\leq d} c_\alpha\, a_1^{\alpha_1} \cdots a_m^{\alpha_m} \in \mathcal{B}, \qquad \mathbf{a} = (a_1, \ldots, a_m) \in \mathcal{B}^m.$$

This is well defined because $\{\boldsymbol{\lambda}^\alpha : \alpha \in \mathbb{N}_0^m\}$ is a basis for $\mathbb{C}[\boldsymbol{\lambda}]$.

Theorem 3.2.6 (Spectral mapping theorem for $P(a,a^*)$)**.** *If $a \in \mathcal{A}_\nu$ and $P(\lambda,\mu) \in \mathbb{C}[\lambda,\mu]$, then*

$$\sigma(P(a,a^*)) = \{P(\lambda,\bar{\lambda}) : \lambda \in \sigma(a)\}.$$

If $a^* = a$, then Theorem 3.2.6 is a special case of the spectral mapping theorem for the holomorphic functional calculus (Theorem 2.1.22) because $P(a,a^*) = P(a,a) = p(a)$, where $p(\lambda) \coloneqq P(\lambda,\lambda) \in \mathbb{C}[\lambda]$. The general case is substantially more difficult. By the GNS theorem (Theorem 1.4.15) and Proposition 3.2.4, it suffices to treat the case when $\mathcal{A} = B(H)$, where H is a complex Hilbert space. In this case, there are multiple approaches. S. J. Bernau gives a (long) elementary proof in [Ber65]; please see [Ber65, Thm. 2] specifically. R. E. Harte gives a proof in [Har72a, Har72b] based on his notion of the joint spectrum of m-tuples of elements of a unital Banach algebra; please see [Har72b, Eq. (4.3.3)], as well as [Har72b, Thms. 3.4 & 4.3] and the comments at the end of [Har72b, §3], specifically. Finally, B. C. Hall gives a proof in [Hal13] based on "almost eigenvalues" and the spectral theorem for bounded, self-adjoint operators; please see [Hal13, Thm. 10.23] specifically.

Lemma 3.2.7 (Spectral radius of normal)**.** *If $a \in \mathcal{A}_\nu$, then $r(a) = \|a\|$.*

Proof. If $a \in \mathcal{A}_{\text{sa}}$, then $\|a^2\| = \|a^*a\| = \|a\|^2$. By induction,

$$\left\|a^{2^n}\right\| = \|a\|^{2^n}, \qquad n \in \mathbb{N}.$$

Therefore, by Gel'fand's spectral radius formula (Theorem 1.4.5(v)),

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\|a^{2^n}\right\|^{\frac{1}{2^n}} = \|a\|.$$

Consequently, if $a \in \mathcal{A}$ is arbitrary, then $r(a^*a) = \|a^*a\| = \|a\|^2$. To complete the proof, we claim that if $a \in \mathcal{A}_\nu$, then $r(a^*a) = r(a)^2$. Indeed, in this case,

$$\|a^n\|^2 = \|(a^n)^*a^n\| = \|(a^*a)^n\|, \qquad n \in \mathbb{N},$$

so the claim follows from two more applications of Gel'fand's spectral radius formula. □

Theorem 3.2.8 (Continuous functional calculus)**.** *If $a \in \mathcal{A}_\nu$, then there exists a unique unital $*$-homomorphism $\Phi_a \colon C(\sigma(a)) \to \mathcal{A}$ sending $\iota_{\sigma(a)}$ to a. Furthermore, Φ_a is an isometry.*

Proof. Fix a normal element $a \in \mathcal{A}_\nu$, and write $\mathrm{P}^*(\sigma(a)) \subseteq C(\sigma(a))$ for the set of functions of the form $\sigma(a) \ni \lambda \mapsto f_P(\lambda) := P\big(\lambda, \bar{\lambda}\big) \in \mathbb{C}$ for some $P(\lambda, \mu) \in \mathbb{C}[\lambda, \mu]$. If $\Phi, \Psi \colon C(\sigma(a)) \to \mathcal{A}$ are unital $*$-homomorphisms sending $\iota_{\sigma(a)}$ to a, then Φ and Ψ clearly agree on $\mathrm{P}^*(\sigma(a))$. By the Stone–Weierstrass theorem, $\mathrm{P}^*(\sigma(a))$ is dense in $C(\sigma(a))$. By Remark 1.4.10, Φ and Ψ are continuous, so they must agree on all of $C(\sigma(a))$. This takes care of the uniqueness part.

For the existence part, observe first that if $P(\lambda, \mu) \in \mathbb{C}[\lambda, \mu]$, then $P(a, a^*) \in \mathcal{A}_\nu$. Therefore, by Lemma 3.2.7 and Theorem 3.2.6,

$$\|P(a, a^*)\| = \sup\{|\mu| : \mu \in \sigma(P(a, a^*))\} = \sup\big\{\big|P\big(\lambda, \bar{\lambda}\big)\big| : \lambda \in \sigma(a)\big\} = \|f_P\|_{\ell^\infty(\sigma(a))}.$$

Consequently, the map $\mathrm{P}^*(\sigma(a)) \ni f_P \mapsto \pi(f_P) := P(a, a^*) \in \mathcal{A}$ is a well-defined isometry. By an easy calculation, $\pi \colon \mathrm{P}^*(\sigma(a)) \to \mathcal{A}$ is also a unital $*$-homomorphism sending $\iota_{\sigma(a)}$ to a. Since $\mathrm{P}^*(\sigma(a))$ is dense in $C(\sigma(a))$, it follows that π extends to an isometric, unital $*$-homomorphism $\Phi_a \colon C(\sigma(a)) \to \mathcal{A}$ sending $\iota_{\sigma(a)}$ to a. This completes the proof. $\square$

Remark 3.2.9 (Another approach)**.** Another, perhaps more common, approach to the construction of Φ_a proceeds through a finer analysis of the classification of unital, commutative C^*-algebras. Specifically, if $\mathcal{C}$ is the smallest unital C^*-subalgebra of $\mathcal{A}$ containing a, in which case $\mathcal{C}$ is commutative, then one constructs a $*$-isomorphism $\pi \colon \mathcal{C} \to C(\sigma(a))$ from the Gel'fand transform of $\mathcal{C}$ and takes $\Phi_a := \pi^{-1}$. Please see [Rud91, Thm. 11.19] or [Con90, §VIII.2] for this approach. We favor going through Theorem 3.2.6 because doing so leads to a very easy proof in the self-adjoint case, which is the primary case of interest in this dissertation.

We end this section with a few useful consequences.

Corollary 3.2.10 (Agreement with holomorphic functional calculus)**.** *If $a \in \mathcal{A}_\nu$ and $U \subseteq \mathbb{C}$ is an open subset such that $\sigma(a) \subseteq U$, then $\Phi_a\big(f|_{\sigma(a)}\big) = H_a^U(f)$ for all $f \in \mathrm{Hol}(U)$.*

Proof. The map $\mathrm{Hol}(U) \ni f \mapsto \Phi_a\big(f|_{\sigma(a)}\big) \in \mathcal{A}$ is a unital, continuous algebra homomorphism sending ι_U to a, so the result follows from the uniqueness part of Theorem 1.2.14. $\square$

Consequently, the following does not clash with Definition 2.1.10.

Definition 3.2.11 (Continuous functional calculus)**.** The map Φ_a from Theorem 3.2.8 is the **continuous functional calculus** for a, and we write $f(a) \coloneqq \Phi_a(f) \in \mathcal{A}$ for all $f \in C(\sigma(a))$.

Corollary 3.2.12 (Spectral mapping theorem)**.** *$\sigma(f(a)) = f(\sigma(a))$ for all $a \in \mathcal{A}_\nu$ and $f \in C(\sigma(a))$.*

Proof. Let $\mathcal{C}$ be the smallest unital C^*-subalgebra of $\mathcal{A}$ containing a. Since, in the notation of the proof of Theorem 3.2.8, $\mathrm{P}^*(\sigma(a))$ is dense in $C(\sigma(a))$, the map $\Phi_a \colon C(\sigma(a)) \to \mathcal{C}$ is a $*$-isomorphism. Consequently, if $f \in C(\sigma(a))$, then $\sigma(f(a)) = \sigma\big(\Phi_a^{-1}(f(a))\big) = \sigma(f) = f(\sigma(a))$; in the last identity, we used Example 1.4.8. This completes the proof. $\square$

Corollary 3.2.13 (Normal with real spectrum is self-adjoint)**.** *If $a \in \mathcal{A}_\nu$ and $\sigma(a) \subseteq \mathbb{R}$, then $a \in \mathcal{A}_{\mathrm{sa}}$. In particular, if $a \in \mathcal{A}_\nu$ and $\sigma(a) \subseteq \mathbb{R}_+$, then $a \in \mathcal{A}_+$.*

Proof. If $a \in \mathcal{A}_\nu$ and $\sigma(a) \subseteq \mathbb{R}$, then $a^* = \Phi_a(\iota_{\sigma(a)})^* = \Phi_a\big(\overline{\iota_{\sigma(a)}}\big) = \Phi_a(\iota_{\sigma(a)}) = a$. $\square$

3.3 Varopoulos algebra

In this section, we discuss the Varopoulos algebra, a concrete representation of the projective tensor product $C(\Omega_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\Omega_m)$ named after N. Th. Varopoulos [Var67]. Recall that $\Omega_1, \ldots, \Omega_m$ are compact Hausdorff spaces and $\Omega = \Omega_1 \times \cdots \times \Omega_m$

Definition 3.3.1 (Varopoulos algebra)**.** Let $\varphi \in C(\Omega)$, and suppose, for each $i \in \{1, \ldots, m\}$, there exists a sequence $(\varphi_{i,n})_{n\in\mathbb{N}}$ in $C(\Omega_i)$ such that

$$\sum_{n=1}^{\infty}\prod_{i=1}^{m}\|\varphi_{i,n}\|_{\ell^\infty(\Omega_i)} < \infty \;\text{ and }\; \varphi(\boldsymbol{\omega}) = \sum_{n=1}^{\infty}(\varphi_{1,n}\otimes\cdots\otimes\varphi_{m,n})(\boldsymbol{\omega}), \quad \boldsymbol{\omega} \in \Omega. \tag{3.3.2}$$

(Please see Notation 2.4.1(i).) Then we define

$$\|\varphi\|_{V(\Omega_1,\ldots,\Omega_m)} \coloneqq \inf\left\{\sum_{n=1}^{\infty}\prod_{i=1}^{m}\|\varphi_{i,n}\|_{\ell^\infty(\Omega_i)} : (\varphi_{i,n})_{n\in\mathbb{N}} \in C(\Omega_i)^{\mathbb{N}} \text{ satisfy Relation (3.3.2)}\right\}.$$

If no such sequences exist, then $\|\varphi\|_{V(\Omega_1,\ldots,\Omega_m)} \coloneqq \infty$. Finally, the **Varopoulos algebra** is defined to be the set

$$V(\Omega_1, \ldots, \Omega_m) \coloneqq \big\{\varphi \in C(\Omega) : \|\varphi\|_{V(\Omega_1,\ldots,\Omega_m)} < \infty\big\}.$$

In the next proposition, we list the basic properties of $V(\Omega_1, \ldots, \Omega_m)$. The proof is standard and therefore is left to the reader.

Proposition 3.3.3. *The Varopoulos algebra $V(\Omega_1, \ldots, \Omega_m)$ is a unital $*$-subalgebra of $C(\Omega)$, and $\big(V(\Omega_1, \ldots, \Omega_m), \|\cdot\|_{V(\Omega_1,\ldots,\Omega_m)}\big)$ is a unital Banach $*$-algebra. Furthermore,*

$$\|\varphi\|_{\ell^\infty(\Omega)} \le \|\varphi\|_{V(\Omega_1,\ldots,\Omega_m)}, \qquad \varphi \in C(\Omega).$$

In particular, the inclusion $V(\Omega_1, \ldots, \Omega_m) \hookrightarrow C(\Omega)$ is continuous.

Example 3.3.4 (Multivariate polynomials)**.** Let $m \in \mathbb{N}$, and suppose

$$P(\boldsymbol{\lambda}) = \sum_{|\alpha| \le d} c_\alpha \, \boldsymbol{\lambda}^\alpha = \sum_{\alpha \in \mathbb{N}_0^m : |\alpha| \le d} c_\alpha \, \lambda_1^{\alpha_1} \cdots \lambda_m^{\alpha_m} \in \mathbb{C}[\lambda_1, \ldots, \lambda_m] = \mathbb{C}[\boldsymbol{\lambda}].$$

If $r_i > 0$ and $\Omega_i \coloneqq \{z \in \mathbb{C} : |z| \le r_i\}$ for all $i \in \{1, \ldots, m\}$, then

$$\|P|_{\Omega_1 \times \cdots \times \Omega_m}\|_{V(\Omega_1,\ldots,\Omega_m)} \le \sum_{|\alpha| \le d} |c_\alpha| \prod_{i=1}^m \sup_{|\lambda_i| \le r_i} \big|\lambda_i^{\alpha_i}\big| \le \sum_{|\alpha| \le d} |c_\alpha| \, r^{|\alpha|},$$

where $r \coloneqq \max\{r_1, \ldots, r_m\}$. Since $V(\Omega_1, \ldots, \Omega_m)$ is closed under complex conjugation, (the restrictions of) multivariate polynomials in $\boldsymbol{\lambda}$ and $\bar{\boldsymbol{\lambda}}$ belong to $V(\Omega_1, \ldots, \Omega_m)$. Actually, such polynomial functions are dense.

Proposition 3.3.5 (Density of $*$-polynomials)**.** *Suppose $\Omega_i \subseteq \mathbb{C}$ is compact for all $i \in \{1, \ldots, m\}$. The set $\mathrm{P}^*(\Omega_1, \ldots, \Omega_m) \subseteq V(\Omega_1, \ldots, \Omega_m)$ of functions of the form $\Omega \ni \boldsymbol{\lambda} \mapsto P\big(\boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}\big) \in \mathbb{C}$, where $P(\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m) \in \mathbb{C}[\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m]$, is dense in $V(\Omega_1, \ldots, \Omega_m)$.*

Sketch of proof. By definition of $V(\Omega_1, \ldots, \Omega_m)$,

$$T(\Omega_1, \ldots, \Omega_m) \coloneqq \left\{ \sum_{n=1}^N \varphi_{1,n} \otimes \cdots \otimes \varphi_{1,n} : N \in \mathbb{N} \text{ and } (\varphi_{i,n})_{n=1}^N \in C(\Omega_i)^N, \ i \in \{1, \ldots, m\} \right\}$$

is dense in $V(\Omega_1, \ldots, \Omega_m)$. By the Stone–Weierstrass theorem, $\mathrm{P}^*(\Omega_i)$ is dense in $C(\Omega_i)$ for all $i \in \{1, \ldots, m\}$. By approximating the $\varphi_{i,n}$'s by elements of $\mathrm{P}^*(\Omega_i)$, we conclude that $\mathrm{P}^*(\Omega_1, \ldots, \Omega_m)$ is dense in $T(\Omega_1, \ldots, \Omega_m)$. The result follows. □

We now give a description of $V(\Omega_1,\ldots,\Omega_m)$ (with $\Omega_1,\ldots,\Omega_m$ metrizable) inspired by the integral projective tensor products (Definition 5.5.3 below) from the theory of multiple operator integrals (MOIs, Chapter 5).

Lemma 3.3.6. *Suppose $\Omega_1,\ldots,\Omega_m$ are metrizable, and let $(\Sigma,\mathscr{H})$ be a measurable space. If, for all $i\in\{1,\ldots,m\}$, $\varphi_i\colon\Omega_i\times\Sigma\to\mathbb{C}$ is product measurable, i.e., $(\mathcal{B}_{\Omega_i}\otimes\mathscr{H},\mathcal{B}_{\mathbb{C}})$-measurable, and $\varphi_i(\cdot,\sigma)\in C(\Omega_i)$ whenever $\sigma\in\Sigma$, then the map*

$$\Sigma\ni\sigma\mapsto\varphi_1(\cdot,\sigma)\otimes\cdots\otimes\varphi_m(\cdot,\sigma)\in V(\Omega_1,\ldots,\Omega_m)$$

is strongly measurable.

Proof. We first prove the lemma assuming $m=1$, in which case $\Omega_1=\Omega$ and $\varphi\coloneqq\varphi_1$. By the Riesz–Markov theorem, $C(\Omega)^*\cong M(\Omega,\mathcal{B}_\Omega)$ (Notation 1.3.12). Now, if $\mu\in M(\Omega,\mathcal{B}_\Omega)$, then the function $\Sigma\ni\sigma\mapsto\int_\Omega\varphi(\cdot,\sigma)\,\mathrm{d}\mu\in\mathbb{C}$ is measurable by a standard measure theory argument; please see Lemma 5.6.2 below. Therefore, the map $\Sigma\ni\sigma\mapsto\varphi(\cdot,\sigma)\in C(\Omega)$ is weakly measurable. Since Ω is a compact and metrizable, $C(\Omega)$ is a separable Banach space. The strong measurability of $\Sigma\ni\sigma\mapsto\varphi(\cdot,\sigma)\in C(\Omega)$ then follows from Pettis's measurability theorem.

Next, let $m\in\mathbb{N}$ be general, and fix $i\in\{1,\ldots,m\}$. By the previous paragraph, the map $\Sigma\ni\sigma\mapsto F_i(\sigma)\coloneqq\varphi_i(\cdot,\sigma)\in C(\Omega_i)$ is strongly measurable. Let $(s_{i,n})_{n\in\mathbb{N}}$ be a sequence of simple maps $\Sigma\to C(\Omega_i)$ converging pointwise to F_i. Then $(s_{1,n}(\cdot)\otimes\cdots\otimes s_{m,n}(\cdot))_{n\in\mathbb{N}}$ is a sequence of simple maps $\Sigma\to V(\Omega_1,\ldots,\Omega_m)$ converging pointwise to $F(\cdot)\coloneqq F_1(\cdot)\otimes\cdots\otimes F_m(\cdot)$, which shows that F is strongly measurable. □

Theorem 3.3.7 (Integral description of Varopoulos algebra)**.** *Suppose $\Omega_1,\ldots,\Omega_m$ are metrizable. Let $(\Sigma,\mathscr{H},\rho)$ be a measure space, and for all $i\in\{1,\ldots,m\}$, let $\varphi_i\colon\Sigma\times\Omega_i\to\mathbb{C}$ be a product measurable function such that $\varphi_i(\cdot,\sigma)\in C(\Omega_i)$ whenever $\sigma\in\Sigma$. If*

$$\int_\Sigma\prod_{i=1}^m\|\varphi_i(\cdot,\sigma)\|_{\ell^\infty(\Omega_i)}\,\rho(\mathrm{d}\sigma)<\infty\ \text{ and }\ \varphi(\boldsymbol{\omega})\coloneqq\int_\Sigma\prod_{i=1}^m\varphi_i(\omega_i,\sigma)\,\rho(\mathrm{d}\sigma),\qquad\boldsymbol{\omega}\in\Omega,\tag{3.3.8}$$

then

$$\varphi=\int_\Sigma\varphi_1(\cdot,\sigma)\otimes\cdots\otimes\varphi_m(\cdot,\sigma)\,\rho(\mathrm{d}\sigma)\in V(\Omega_1,\ldots,\Omega_m)$$

as a $V(\Omega_1,\dots,\Omega_m)$*-valued Bochner integral, and*

$$\|\varphi\|_{V(\Omega_1,\dots,\Omega_m)} \leq \int_\Sigma \|\varphi_1(\cdot,\sigma)\|_{\ell^\infty(\Omega_1)} \cdots \|\varphi_m(\cdot,\sigma)\|_{\ell^\infty(\Omega_m)}\,\rho(\mathrm{d}\sigma). \tag{3.3.9}$$

Proof. By Lemma 3.3.6, the map

$$\Sigma \ni \sigma \mapsto F(\sigma) \coloneqq \varphi_1(\cdot,\sigma) \otimes \cdots \otimes \varphi_m(\cdot,\sigma) \in V(\Omega_1,\dots,\Omega_m)$$

is strongly measurable. Since

$$\int_\Sigma \|F\|_{V(\Omega_1,\dots,\Omega_m)}\,\mathrm{d}\rho = \int_\Sigma \|\varphi_1(\cdot,\sigma)\|_{\ell^\infty(\Omega_1)} \cdots \|\varphi_m(\cdot,\sigma)\|_{\ell^\infty(\Omega_m)}\,\rho(\mathrm{d}\sigma) < \infty,$$

we get that F is strongly ρ-integrable. The identity $\varphi = \int_\Sigma F\,\mathrm{d}\rho$ then follows by applying the evaluation functionals $\{V(\Omega_1,\dots,\Omega_m) \ni \psi \mapsto \psi(\boldsymbol{\omega}) \in \mathbb{C} : \boldsymbol{\omega} \in \Omega\}$ to $\int_\Sigma F\,\mathrm{d}\rho$. Finally, Inequality (3.3.9) follows from the triangle inequality for Bochner integrals. □

The reason for the name of Theorem 3.3.7 is the following immediate consequence. If $\Omega_1,\dots,\Omega_m$ are metrizable, then $V(\Omega_1,\dots,\Omega_m)$ is precisely the space of functions $\varphi \in C(\Omega)$ such that there exists a measure space $(\Sigma,\mathscr{H},\rho)$ and functions $\varphi_1\colon \Sigma\times\Omega_1 \to \mathbb{C},\dots,\varphi_m\colon \Sigma\times\Omega_m \to \mathbb{C}$ as in Theorem 3.3.7 satisfying

$$\varphi(\boldsymbol{\omega}) = \int_\Sigma \varphi_1(\omega_1,\sigma)\cdots\varphi_m(\omega_m,\sigma)\,\rho(\mathrm{d}\sigma), \qquad \boldsymbol{\omega} \in \Omega.$$

Furthermore,

$$\|\varphi\|_{V(\Omega_1,\dots,\Omega_m)} = \inf\left\{\int_\Sigma \prod_{i=1}^m \|\varphi_i(\cdot,\sigma)\|_{\ell^\infty(\Omega_i)}\,\rho(\mathrm{d}\sigma) : \begin{matrix}(\Sigma,\mathscr{H},\rho) \text{ and } \varphi_1,\dots,\varphi_m \\ \text{are as in the previous sentence}\end{matrix}\right\}. \tag{3.3.10}$$

In the terminology of MOIs, one might say that the Varopoulos algebra $V(\Omega_1,\dots,\Omega_m)$ is the "integral projective tensor product $C(\Omega_1)\hat{\otimes}_i\cdots\hat{\otimes}_i C(\Omega_m)$."

Using Theorem 3.3.7, we provide one more example: "Fourier transforms" of complex measures. First, we set some notation for the case when $\Omega_1 = \cdots = \Omega_m$ is a compact interval in $\mathbb{R}$ since this case plays a special role.

Notation 3.3.11. If $\varphi \in C(\mathbb{R}^m)$, then

$$\beta_{r,m}(\varphi) := \big\|\varphi|_{[-r,r]^m}\big\|_{V([-r,r]_{(m)})} = \big\|\varphi|_{[-r,r]^m}\big\|_{V([-r,r],\ldots,[-r,r])} \in [0,\infty], \qquad r > 0.$$

Example 3.3.12. Let $\nu \in M(\mathbb{R}^m, \mathcal{B}_{\mathbb{R}^m})$ (Notation 1.3.12), and define

$$\varphi(\boldsymbol{\lambda}) := \int_{\mathbb{R}^m} e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\xi}}\,\nu(\mathrm{d}\boldsymbol{\xi}) = \int_{\mathbb{R}^m} e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\xi}}\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}(\boldsymbol{\xi})\,|\nu|(\mathrm{d}\boldsymbol{\xi}), \qquad \boldsymbol{\lambda} \in \mathbb{R}^m.$$

Since $e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\xi}} = e^{i\lambda_1\xi_1}\cdots e^{i\lambda_m\xi_m}$, Theorem 3.3.7 yields that $\varphi|_\Omega \in V(\Omega_1,\ldots,\Omega_m)$ whenever $\Omega_i \subseteq \mathbb{R}$ is a compact set for all $i \in \{1,\ldots,m\}$; furthermore,

$$\sup_{r>0}\beta_{r,m}(\varphi) \le |\nu|(\mathbb{R}^m).$$

Consequently, if $k \in \mathbb{N}$ and $f = \int_{\mathbb{R}} e^{i\cdot\xi}\mu(\mathrm{d}\xi) \in W_k(\mathbb{R})$, then

$$\sup_{r>0}\beta_{r,k+1}\big(f^{[k]}\big) \le \int_{\Delta_k\times\mathbb{R}} |\xi|^k\,(\rho_k \otimes |\mu|)(\mathrm{d}\mathbf{t},\mathrm{d}\xi) = \frac{\mu_{(k)}}{k!} \tag{3.3.13}$$

by Example 1.3.14.

We end this section by proving that $V(\Omega_1,\ldots,\Omega_m) \cong C(\Omega_1)\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi C(\Omega_m)$ whenever $\Omega_1,\ldots,\Omega_m$ are (general) compact Hausdorff spaces. We shall find this result most important in §3.5, where we define $\varphi_\otimes(\mathbf{a})$.

Theorem 3.3.14. *If $\iota_{\Omega_1,\ldots,\Omega_m} \colon C(\Omega_1)\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi C(\Omega_m) \to C(\Omega)$ is the bounded linear map determined via the universal property of $\hat{\otimes}_\pi$ by*

$$\varphi_1 \otimes\cdots\otimes \varphi_m \mapsto ((\omega_1,\ldots,\omega_m) \mapsto \varphi_1(\omega_1)\cdots\varphi_m(\omega_m)),$$

then $\iota_{\Omega_1,\ldots,\Omega_m}$ is an injective, unital $*$-homomorphism.

Proof. The only nontrivial claim is that $\iota_{\Omega_1,\ldots,\Omega_m}$ is injective. We prove this by induction on $m \ge 2$. By [Rya02, Ex. 4.2], $C(\Omega_1)$ has the approximation property. Consequently, the injectivity of $\iota_{\Omega_1,\Omega_2}$ follows from [Rya02, Prop. 4.6]. Now, assume the result is true for $m \ge 2$ spaces, and write $\Xi \coloneqq \Omega_2 \times \cdots \times \Omega_m$. By the $m = 2$ case, the map $\iota_{\Omega_1,\Xi} \colon C(\Omega_1)\hat{\otimes}_\pi C(\Xi) \to C(\Omega_1 \times \Xi) = C(\Omega)$ is injective. By the induction hypothesis, the map $\iota_{\Omega_2,\ldots,\Omega_m} \colon C(\Omega_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\Omega_m) \to C(\Xi)$ is injective. Since $C(\Omega_1)$ has the approximation property, we conclude from [Rya02, Exer. 4.1] that the map

$$\mathrm{id}_{C(\Omega_1)}\hat{\otimes}_\pi \iota_{\Omega_2,\ldots,\Omega_m} \colon \underbrace{C(\Omega_1)\hat{\otimes}_\pi(C(\Omega_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\Omega_m))}_{=C(\Omega_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\Omega_m)} \to C(\Omega_1)\hat{\otimes}_\pi C(\Xi)$$

is injective as well. Since $\iota_{\Omega_1,\ldots,\Omega_m} = \iota_{\Omega_1,\Xi} \circ \big(\mathrm{id}_{C(\Omega_1)}\hat{\otimes}_\pi \iota_{\Omega_2,\ldots,\Omega_m}\big)$, we are done. □

Corollary 3.3.15. *If $\iota_{\Omega_1,\ldots,\Omega_m}$ is as in Theorem 3.3.14, then* $\mathrm{im}\,\iota_{\Omega_1,\ldots,\Omega_m} = V(\Omega_1,\ldots,\Omega_m)$*, and*

$$\big\|\iota_{\Omega_1,\ldots,\Omega_m}(a)\big\|_{V(\Omega_1,\ldots,\Omega_m)} = \|a\|_{C(\Omega_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\Omega_m)}, \qquad a \in C(\Omega_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\Omega_m).$$

In other words, $\iota_{\Omega_1,\ldots,\Omega_m}$ is an isometric $\ast$-isomorphism $C(\Omega_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\Omega_m) \to V(\Omega_1,\ldots,\Omega_m)$.

Proof. Combine Theorem 3.3.14 and Equation (1.5.13). □

3.4 Varopoulos C^k functions

In this section, we introduce the space of "Varopoulos C^k functions" and develop some of its basic properties. First, however, we recall that if $k \in \mathbb{N}_0 \cup \{\infty\}$, then the space $C^k(\mathbb{R})$ is a Fréchet space with respect to the C^k topology; please see Example 1.2.10 and Proposition 1.2.13. By Corollary 1.3.7, the C^k topology is induced by the family

$$\big\{f \mapsto \big\|f^{[i]}\big\|_{\ell^\infty([-r,r]^{i+1})} : 0 \leq i < k+1,\ r > 0\big\}$$

of seminorms. In the space of Varopoulos C^k functions, we shall measure $f^{[i]}$ with the family $\{\beta_{r,i+1} : r > 0\}$ of seminorms.

Definition 3.4.1 (Varopoulos C^k functions)**.** If $m \in \mathbb{N}$, then

$$VC(\mathbb{R}^m) \coloneqq \big\{\varphi \in C(\mathbb{R}^m) : \varphi|_{[-r,r]^m} \in V\big([-r,r]_{(m)}\big) \cong C([-r,r])^{\hat{\otimes}_\pi m} \text{ for all } r > 0\big\}.$$

If $k \in \mathbb{N}$, $f \in C^k(\mathbb{R})$, and $r > 0$, then

$$\|f\|_{VC^k,r} := \sum_{i=0}^{k} \beta_{r,i+1}\big(f^{[i]}\big) \in [0,\infty] \text{ and } VC^k(\mathbb{R}) := \big\{g \in C^k(\mathbb{R}) : \|g\|_{VC^k,s} < \infty \text{ for all } s > 0\big\},$$

i.e., $VC^k(\mathbb{R}) = \big\{g \in C^k(\mathbb{R}) : g^{[i]} \in VC\big(\mathbb{R}^{i+1}\big) \text{ for all } i \in \{0,\ldots,k\}\big\}$. Also, write $VC^\infty(\mathbb{R})$ for $\bigcap_{k\in\mathbb{N}} VC^k(\mathbb{R})$. If $k \in \mathbb{N} \cup \{\infty\}$, the members of $VC^k(\mathbb{R})$ are called **Varopoulos $\boldsymbol{C^k}$ functions**.

Example 3.4.2 (Polynomials)**.** By Examples 1.3.8 and 3.3.4, $\mathbb{C}[\lambda] \subseteq VC^\infty(\mathbb{R})$.

Example 3.4.3 (Wiener space)**.** By Example 3.3.12, $W_k(\mathbb{R}) \subseteq VC^k(\mathbb{R})$ for all $k \in \mathbb{N}$.

If $m \in \mathbb{N}$, then $VC(\mathbb{R}^m) \subseteq C(\mathbb{R}^m)$ is a linear subspace, and $\{\beta_{r,m} : r > 0\}$ is a collection of seminorms on $VC(\mathbb{R}^m)$. Since these seminorms clearly separate points, they make $VC(\mathbb{R}^m)$ into a Hausdorff locally convex topological vector space (HLCTVS). Similarly, if $k \in \mathbb{N}$, then $VC^k(\mathbb{R}) \subseteq C^k(\mathbb{R})$ is a linear subspace, and $VC^k(\mathbb{R})$ is an HLCTVS with the topology induced by the family $\{\|\cdot\|_{VC^k,r} : r > 0\}$ of seminorms. Finally, $VC^\infty(\mathbb{R})$ is an HLCTVS with the topology induced by $\{\|\cdot\|_{VC^k,r} : k \in \mathbb{N},\, r > 0\}$. Here now are the basic properties of the spaces $VC(\mathbb{R}^m)$ and $VC^k(\mathbb{R})$. In the result below, $\hookrightarrow$ indicates continuity of the relevant inclusion map. Also, a **Fréchet $*$-algebra** is a complex Fréchet space with a $*$-algebra structure such that the $*$-operation and product are continuous.

Proposition 3.4.4 (Properties of $VC^k(\mathbb{R})$)**.** *Let $m \in \mathbb{N}$, and let $k \in \mathbb{N} \cup \{\infty\}$.*

(i) $VC(\mathbb{R}^m) \hookrightarrow C(\mathbb{R}^m)$, *and* $VC^k(\mathbb{R}) \hookrightarrow C^k(\mathbb{R})$.

(ii) *For $\mathcal{S} \subseteq \mathbb{C}^{\mathbb{R}^m}$, write $\mathcal{S}_{\text{loc}}$ for the set of functions $\varphi \in \mathbb{C}^{\mathbb{R}^m}$ such that for all $r > 0$, there exists a $\psi \in \mathcal{S}$ such that $\psi|_{[-r,r]^m} = \varphi|_{[-r,r]^m}$. If $\mathcal{S} \subseteq VC(\mathbb{R}^m)$, then $\mathcal{S}_{\text{loc}} \subseteq \overline{\mathcal{S}} \subseteq VC(\mathbb{R}^m)$. (The closure in the previous sentence takes place in $VC(\mathbb{R}^m)$.) If $\mathcal{S} \subseteq VC^k(\mathbb{R})$, then $\mathcal{S}_{\text{loc}} \subseteq \overline{\mathcal{S}} \subseteq VC^k(\mathbb{R})$. (The closure in the previous sentence takes place in $VC^k(\mathbb{R})$.)*

(iii) *If $k < \infty$, $r > 0$, and $f, g \in C^k(\mathbb{R})$, then*

$$\beta_{r,k+1}\big((fg)^{[k]}\big) \le \sum_{i=0}^{k} \beta_{r,i+1}\big(f^{[i]}\big)\, \beta_{r,k-i+1}\big(g^{[k-i]}\big) \;\; \textit{and} \;\; \|fg\|_{VC^k,r} \le \|f\|_{VC^k,r}\|g\|_{VC^k,r}.$$

(iv) $VC(\mathbb{R}^m)$ *and* $VC^k(\mathbb{R})$ *are unital Fréchet $*$-algebras under pointwise operations.*

Proof. We take each item in turn.

(i) The continuity of both inclusions follows from the fact that $\|\cdot\|_{\ell^\infty([-r,r]^m)} \leq \beta_{r,m}$ for all $r > 0$ (Proposition 3.3.3). For the second, we also must appeal to the description of the C^k topology given at the beginning of this section.

(ii) If $\mathcal{S} \subseteq VC(\mathbb{R}^m)$, $\varphi \in \mathcal{S}_{\text{loc}}$, and $n \in \mathbb{N}$, then there exists a $\varphi_n \in \mathcal{S} \subseteq VC(\mathbb{R}^m)$ such that $\varphi_n|_{[-n,n]^m} = \varphi|_{[-n,n]^m}$. If $r > 0$ and $n > r$, then $\beta_{r,m}(\varphi_n - \varphi) = 0$. Thus, $\varphi \in VC(\mathbb{R}^m)$, and $\varphi_n \to \varphi$ in $VC(\mathbb{R}^m)$ as $n \to \infty$. In particular, $\mathcal{S}_{\text{loc}} \subseteq \overline{\mathcal{S}} \subseteq VC(\mathbb{R}^m)$. The second statement may be proven the same way.

(iii) The claimed bound on $\beta_{r,k+1}\big((fg)^{[k]}\big)$ follows easily from Proposition 1.3.3(ii) and the fact that the Varopoulos algebra is a Banach algebra. Consequently,

$$\begin{aligned}\|fg\|_{VC^k,r} &= \sum_{j=0}^{k} \beta_{r,j+1}\big((fg)^{[j]}\big) \leq \sum_{j=0}^{k}\sum_{i=0}^{j} \beta_{r,i+1}\big(f^{[i]}\big)\,\beta_{r,j-i+1}\big(g^{[j-i]}\big) \\ &= \sum_{i=0}^{k} \beta_{r,i+1}\big(f^{[i]}\big) \sum_{j=i}^{k} \beta_{r,j-i+1}\big(g^{[j-i]}\big) \leq \|f\|_{VC^k,r}\|g\|_{VC^k,r}\end{aligned}$$

as well.

(iv) We prove that $VC^k(\mathbb{R})$ is a Fréchet $*$-algebra when $k < \infty$ and leave the proofs for $VC^\infty(\mathbb{R})$ and $VC(\mathbb{R}^m)$ to the reader. First, the topology of $VC^k(\mathbb{R})$ is generated by the countable family $\{\|\cdot\|_{VC^k,N} : N \in \mathbb{N}\}$ of seminorms, so $VC^k(\mathbb{R})$ is metrizable. Next, we prove that $VC^k(\mathbb{R})$ is complete. To this end, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $VC^k(\mathbb{R})$. By the first item, the sequence $(f_n)_{n\in\mathbb{N}}$ is also Cauchy in $C^k(\mathbb{R})$. Since the latter space is complete, there exists an $f \in C^k(\mathbb{R})$ such that $f_n \to f$ in the C^k topology as $n \to \infty$. In particular, if $i \in \{0, \ldots, k\}$, then $f_n^{[i]} \to f^{[i]}$ uniformly on compact sets as $n \to \infty$. Now, if $i \in \{0, \ldots, k\}$ and $r > 0$, then the sequence

$$\Big(f_n^{[i]}|_{[-r,r]^{i+1}}\Big)_{n\in\mathbb{N}}$$

is Cauchy and therefore, by Proposition 3.3.3, convergent in $V\big([-r,r]_{(i+1)}\big)$. Since we already know that $f_n^{[i]} \to f^{[i]}$ pointwise as $n \to \infty$, we conclude that $f^{[i]}|_{[-r,r]^{i+1}} \in V\big([-r,r]_{(i+1)}\big)$ and $f_n^{[i]}|_{[-r,r]^{i+1}} \to f^{[i]}|_{[-r,r]^{i+1}}$ in $V\big([-r,r]_{(i+1)}\big)$ as $n \to \infty$ as well. Thus, $f \in VC^k(\mathbb{R})$, and $f_n \to f$ in $VC^k(\mathbb{R})$ as $n \to \infty$. This completes the proof that $VC^k(\mathbb{R})$ is a Fréchet space.

Finally, the previous item implies that $VC^k(\mathbb{R})$ is an algebra under pointwise multiplication and that pointwise multiplication is a jointly continuous operation. Since it is also clear that $\|\overline{f}\|_{VC^k,r} = \|f\|_{VC^k,r}$ whenever $f \in C^k(\mathbb{R})$ and $r > 0$, complex conjugation is a continuous $*$-operation on $VC^k(\mathbb{R})$. $\square$

Next, we show that C^{k+1} functions are Varopoulos C^k using elementary Fourier analysis.

Notation 3.4.5 (Schwartz functions, distributions, and Fourier transform)**.** If $m \in \mathbb{N}$, then $\mathscr{S}(\mathbb{R}^m)$ is the Fréchet space of Schwartz functions $\mathbb{R}^m \to \mathbb{C}$, and $\mathscr{S}'(\mathbb{R}^m) := \mathscr{S}(\mathbb{R}^m)^*$ is the space of tempered distributions on $\mathbb{R}^m$. Also, if $p \in [1,\infty]$, then $L^p(\mathbb{R}^m) := L^p(\mu)$, where μ is the Lebesgue measure on $\mathbb{R}^m$. Finally, the conventions we use for the Fourier transform and its inverse are, respectively,

$$\widehat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{\mathbb{R}^m} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x \;\text{ and }\; \check{f}(x) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{ix\cdot\xi} f(\xi)\,\mathrm{d}\xi, \qquad f \in L^1(\mathbb{R}^m),$$

with corresponding extensions to $\mathscr{S}'(\mathbb{R}^m)$.

Proposition 3.4.6. *Let $k \in \mathbb{N}$.*

(i) *If $f \in BC(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then*

$$\int_{\mathbb{R}} |\xi|^k \big|\widehat{f}(\xi)\big|\,\mathrm{d}\xi < \infty \iff f \in W_k(\mathbb{R}) \iff f \in C^k(\mathbb{R}) \text{ and } \widehat{f^{(k)}} \in L^1(\mathbb{R}).$$

(ii) *If $f \in C^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $f' \in L^2(\mathbb{R})$, then $\widehat{f} \in L^1(\mathbb{R})$.*

(iii) $C^{k+1}(\mathbb{R}) \subseteq W_k(\mathbb{R})_{\mathrm{loc}}$.

Proof. We take each item in turn.

(i) Suppose $f \in BC(\mathbb{R}) \subseteq \mathscr{S}'(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$. By the Fourier inversion theorem for tempered distributions, the fact that $\widehat{f} \in L^1(\mathbb{R})$, and the continuity of f,

$$f(\lambda) = \mathcal{F}^{-1}\big(\widehat{f}\big)(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda\xi}\widehat{f}(\xi)\,\mathrm{d}\xi, \qquad \lambda \in \mathbb{R}.$$

Since $\mu(\mathrm{d}\xi) \coloneqq \frac{1}{2\pi}\widehat{f}(\xi)\,\mathrm{d}\xi$ is a complex measure with $|\mu|(\mathrm{d}\xi) = \frac{1}{2\pi}\big|\widehat{f}(\xi)\big|\,\mathrm{d}\xi$,

$$\mu_{(k)} = \int_{\mathbb{R}} |\xi|^k \, |\mu|(\mathrm{d}\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} |\xi|^k \big|\widehat{f}(\xi)\big|\,\mathrm{d}\xi.$$

The first equivalence immediately follows from this observation. If $f \in C^k(\mathbb{R})$ as well, then

$$\widehat{f^{(k)}}(\xi) = (i\xi)^k\,\widehat{f}(\xi), \qquad \xi \in \mathbb{R},$$

in the sense of tempered distributions, from which the second equivalence follows.

(ii) If $f \in C^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $f' \in L^2(\mathbb{R})$, then

$$\begin{aligned}\big\|\widehat{f}\big\|_{L^1} &= \int_{\mathbb{R}} \frac{1}{1+|\xi|}(1+|\xi|)\,\big|\widehat{f}(\xi)\big|\,\mathrm{d}\xi = \int_{\mathbb{R}} \frac{1}{1+|\xi|}\big(\big|\widehat{f}(\xi)\big| + \big|\widehat{f'}(\xi)\big|\big)\,\mathrm{d}\xi \\ &\leq \big\|(1+|\cdot|)^{-1}\big\|_{L^2}\big(\big\|\widehat{f}\big\|_{L^2} + \big\|\widehat{f'}\big\|_{L^2}\big) = 2\sqrt{\pi}\big(\|f\|_{L^2} + \big\|f'\big\|_{L^2}\big) < \infty\end{aligned}$$

by the Cauchy–Schwarz inequality and Plancherel's theorem.

(iii) Let $f \in C^{k+1}(\mathbb{R})$, and, for $r > 0$, let $\psi_r \in C_c^\infty(\mathbb{R})$ be such that $\psi_r \equiv 1$ on $[-r,r]$. We claim that $g \coloneqq \psi_r f \in W_k(\mathbb{R})$. Indeed, since $g \in C^{k+1}(\mathbb{R})$ and g has compact support, we have that $g, g^{(k)} \in C^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $g', g^{(k+1)} \in L^2(\mathbb{R})$. Thus, $\mathcal{F}(g), \mathcal{F}\big(g^{(k)}\big) \in L^1(\mathbb{R})$ by the previous item. Since $g \in BC(\mathbb{R})$ as well, the claim then follows from the first item. Since $g|_{[-r,r]} = f|_{[-r,r]}$ and $r > 0$ was arbitrary, $f \in W_k(\mathbb{R})_{\mathrm{loc}}$. □

Corollary 3.4.7. *If $k \in \mathbb{N}$, then $C^{k+1}(\mathbb{R}) \subseteq \overline{W_k(\mathbb{R})} \subseteq VC^k(\mathbb{R})$. (The closure in the previous sentence takes place in $VC^k(\mathbb{R})$.) In particular, $C^\infty(\mathbb{R}) = VC^\infty(\mathbb{R})$.*

Proof. Combine Example 3.4.3, Proposition 3.4.4(ii), and Proposition 3.4.6(iii). □

One also can extract from the proofs that if $f \in C^{k+1}(\mathbb{R})$ has compact support, then

$$\sup_{r>0} \beta_{r,i+1}\big(f^{[i]}\big) \leq \frac{1}{2\pi i!}\int_{\mathbb{R}} |\xi|^i \big|\widehat{f}(\xi)\big|\,\mathrm{d}\xi \leq \frac{1}{i!\sqrt{\pi}}\big(\big\|f^{(i)}\big\|_{L^2} + \big\|f^{(i+1)}\big\|_{L^2}\big), \qquad i \in \{1,\ldots,k\}.$$

We end this section by showing that both $W_k(\mathbb{R})$ and $\mathbb{C}[\lambda]$ are dense in $VC^k(\mathbb{R})$. This takes some effort and may be skipped safely on a first read.

Proposition 3.4.8. *If $k \in \mathbb{N}$, then $W_k(\mathbb{R})$ and $\mathbb{C}[\lambda]$ have the same closures in $VC^k(\mathbb{R})$.*

Proof. We know from Corollary 3.4.7 that $\mathbb{C}[\lambda] \subseteq C^{k+1}(\mathbb{R}) \subseteq \overline{W_k(\mathbb{R})}$. Thus, $\overline{\mathbb{C}[\lambda]} \subseteq \overline{W_k(\mathbb{R})}$. It therefore suffices to prove $W_k(\mathbb{R}) \subseteq \overline{\mathbb{C}[\lambda]}$. To this end, let $f = \int_{\mathbb{R}} e^{i\cdot\xi}\,\mu(\mathrm{d}\xi) \in W_k(\mathbb{R})$.

For $n \in \mathbb{N}$, define $\mu_n(\mathrm{d}\xi) := 1_{[-n,n]}(\xi)\,\mu(\mathrm{d}\xi)$ and

$$f_n(\lambda) := \int_{\mathbb{R}} e^{i\lambda\xi}\,\mu_n(\mathrm{d}\xi) = \int_{\mathbb{R}} e^{i\lambda\xi} 1_{[-n,n]}(\xi)\,\mu(\mathrm{d}\xi), \qquad \lambda \in \mathbb{R}.$$

Then $f_n \in W_k(\mathbb{R})$, and $\operatorname{supp}|\mu_n| \subseteq [-n,n]$ for all $n \in \mathbb{N}$. By Inequality (3.3.13) applied to $f - f_n$ and the dominated convergence theorem,

$$\sup_{r>0} \beta_{r,i+1}\big((f-f_n)^{[i]}\big) \leq \frac{1}{i!}\int_{\mathbb{R}} |\xi|^i (1 - 1_{[-n,n]}(\xi))\,|\mu|(\mathrm{d}\xi) \xrightarrow{n\to\infty} 0, \qquad i \in \{0,\ldots,k\}.$$

In particular, $f_n \to f$ in $VC^k(\mathbb{R})$ as $n \to \infty$. It therefore suffices to assume $\operatorname{supp}|\mu|$ is compact.

Suppose $R > 0$ and $\operatorname{supp}|\mu| \subseteq [-R,R]$. Then $\int_{\mathbb{R}} |f|\,d|\mu| \leq \mu_{(0)}\,\|f\|_{\ell^\infty([-R,R])}$ for all Borel measurable functions $f \colon \mathbb{R} \to \mathbb{C}$. In particular, $\mu_{(m)} \leq R^m \mu_{(0)} < \infty$ for all $m \in \mathbb{N}$. Therefore, we may define

$$q_n(\lambda) := \int_{\mathbb{R}} \sum_{m=0}^{n} \frac{(i\lambda\xi)^m}{m!}\,\mu(\mathrm{d}\xi) = \sum_{m=0}^{n} \frac{(i\lambda)^m}{m!}\int_{\mathbb{R}} \xi^m\,\mu(\mathrm{d}\xi) \in \mathbb{C}[\lambda], \qquad n \in \mathbb{N}.$$

We claim that $q_n \to f$ in $VC^k(\mathbb{R})$ as $n \to \infty$. Indeed, since

$$e^{i\lambda\xi} = \sum_{m=0}^{\infty} \frac{(i\lambda\xi)^m}{m!} \quad \text{and} \quad \int_{\mathbb{R}} e^{|\lambda\xi|}\,|\mu|(\mathrm{d}\xi) \leq e^{|\lambda|R}\mu_{(0)},$$

the dominated convergence theorem gives

$$f(\lambda) - q_n(\lambda) = \int_{\mathbb{R}} \sum_{m=n+1}^{\infty} \frac{(i\lambda\xi)^m}{m!}\,\mu(\mathrm{d}\xi) = \sum_{m=n+1}^{\infty} \frac{(i\lambda)^m}{m!}\int_{\mathbb{R}} \xi^m\,\mu(\mathrm{d}\xi), \qquad \lambda \in \mathbb{R}.$$

Consequently, by Equation (1.3.9) and a simple limiting argument, if $j \in \{0,\ldots,k\}$, then

$$(f-q_n)^{[j]}(\boldsymbol{\lambda}) = \sum_{m=n+1}^{\infty} \frac{i^m}{m!}\int_{\mathbb{R}} \xi^m\,\mu(\mathrm{d}\xi) \sum_{|\alpha|=m-j} \boldsymbol{\lambda}^\alpha, \qquad \boldsymbol{\lambda} \in \mathbb{R}^{j+1}.$$

Therefore, using the fact that $\{\alpha \in \mathbb{N}_0^{j+1} : |\alpha| = m - j\}$ has $\binom{m}{m-j} \leq 2^m$ elements, we get

$$\beta_{r,j+1}\big((f-q_n)^{[j]}\big) \leq \sum_{m=n+1}^{\infty} \binom{m}{m-j} \frac{r^{m-j}}{m!} \mu_{(m)} \leq \frac{\mu_{(0)}}{r^j} \sum_{m=n+1}^{\infty} \frac{(2rR)^m}{m!} \xrightarrow{n\to\infty} 0, \qquad r > 0.$$

In particular, $q_n \to f$ in $VC^k(\mathbb{R})$ as $n \to \infty$. This completes the proof. $\square$

Proposition 3.4.9 (Translation)**.** *The translation operation*

$$\mathbb{R}^m \times VC(\mathbb{R}^m) \ni (\boldsymbol{\mu}, \varphi) \mapsto \tau(\boldsymbol{\mu}, \varphi) = \tau_{\boldsymbol{\mu}}\varphi := \varphi(\cdot + \boldsymbol{\mu}) \in VC(\mathbb{R}^m)$$

is well defined and continuous.

Proof. Write

$$|\boldsymbol{\mu}|_\infty := \max_{1\leq i\leq m} |\mu_i|, \qquad \boldsymbol{\mu} = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m.$$

If $\boldsymbol{\mu} \in \mathbb{R}^m$ and $R := |\boldsymbol{\mu}|_\infty$, then

$$\|\varphi(\cdot + \boldsymbol{\mu})\|_{V([-r,r]_{(m)})} \leq \|\varphi\|_{V([-R-r,r+R]_{(m)})}, \qquad \varphi \in C([-R-r, r+R]^m), \tag{3.4.10}$$

as can be seen from the definition of $\|\cdot\|_{V(\Omega_1,\dots,\Omega_m)}$. It follows from Inequality (3.4.10) that τ is well defined, i.e., it maps $\mathbb{R}^m \times VC(\mathbb{R}^m)$ to $VC(\mathbb{R}^m)$.

Next, we claim that if $\varphi \in VC(\mathbb{R}^m)$ is *fixed*, then the map $\mathbb{R}^m \ni \boldsymbol{\mu} \mapsto \tau_{\boldsymbol{\mu}}\varphi \in VC(\mathbb{R}^m)$ is continuous. Indeed, let $r > 0$, let $(\boldsymbol{\mu}_n)_{n\in\mathbb{N}} = (\mu_{n,1}, \dots, \mu_{n,m})_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}^m$ with limit $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$, and write $R := \sup_{n\in\mathbb{N}} |\boldsymbol{\mu}_n|_\infty < \infty$. By definition of $V(\Omega_1, \dots, \Omega_m)$, there exist sequences $(\varphi_{1,p})_{p\in\mathbb{N}}, \dots, (\varphi_{m,p})_{p\in\mathbb{N}}$ in $C([-R-r, r+R])$ such that

$$\sum_{p=1}^{\infty} \prod_{i=1}^{m} \|\varphi_{i,p}\|_{\ell^\infty([-R-r,r+R])} < \infty \;\text{ and }\; \varphi(\boldsymbol{\lambda}) = \sum_{p=1}^{\infty} \prod_{i=1}^{m} \varphi_{i,p}(\lambda_i), \qquad \boldsymbol{\lambda} \in [-R-r, r+R]^m.$$

Writing $\varphi_p := \varphi_{1,p} \otimes \cdots \otimes \varphi_{m,p}$, we have

$$\tau_{\boldsymbol{\mu}_n}\varphi_p - \tau_{\boldsymbol{\mu}}\varphi_p = \sum_{i=1}^{m} \tau_{\mu_{n,1}}\varphi_{1,p} \otimes \cdots \otimes \tau_{\mu_{n,i-1}}\varphi_{i-1,p} \otimes \big(\tau_{\mu_{n,i}}\varphi_{i,p} - \tau_{\mu_i}\varphi_{i,p}\big) \otimes \tau_{\mu_{i+1}}\varphi_{i+1,p} \otimes \cdots \otimes \tau_{\mu_m}\varphi_{m,p}.$$

It follows that

$$\beta_{r,m}\big(\tau_{\boldsymbol{\mu}_n}\varphi - \tau_{\boldsymbol{\mu}}\varphi\big) = \big\|\tau_{\boldsymbol{\mu}_n}\varphi - \tau_{\boldsymbol{\mu}}\varphi\big\|_{V([-r,r]_{(m)})} \leq \sum_{p=1}^{\infty}\big\|\tau_{\boldsymbol{\mu}_n}\varphi_p - \tau_{\boldsymbol{\mu}}\varphi_p\big\|_{V([-r,r]_{(m)})}$$
$$\leq \sum_{p=1}^{\infty}\sum_{i=1}^{m}\big\|\tau_{\mu_{n,i}}\varphi_{i,p} - \tau_{\mu_i}\varphi_{i,p}\big\|_{\ell^\infty([-r,r])}\prod_{j\neq i}\|\varphi_{j,p}\|_{\ell^\infty([-R-r,r+R])} \xrightarrow{n\to\infty} 0$$

by the uniform continuity of $\varphi_{i,p}$ on $[-R-r, r+R]$ and the dominated convergence theorem. This proves the claim.

Finally, suppose, in addition, that $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence in $VC(\mathbb{R}^m)$ converging to φ. If $r > 0$, then, by Inequality (3.4.10) and the previous paragraph,

$$\beta_{r,m}\big(\tau_{\boldsymbol{\mu}_n}\varphi_n - \tau_{\boldsymbol{\mu}}\varphi\big) \leq \beta_{r,m}\big(\tau_{\boldsymbol{\mu}_n}\varphi_n - \tau_{\boldsymbol{\mu}_n}\varphi\big) + \beta_{r,m}\big(\tau_{\boldsymbol{\mu}_n}\varphi - \tau_{\boldsymbol{\mu}}\varphi\big)$$
$$\leq \beta_{r+R,m}(\varphi_n - \varphi) + \beta_{r,m}\big(\tau_{\boldsymbol{\mu}_n}\varphi - \tau_{\boldsymbol{\mu}}\varphi\big) \xrightarrow{n\to\infty} 0.$$

Since $VC(\mathbb{R}^m)$ is metrizable, this completes the proof. □

Proposition 3.4.11. *Let $\eta \in C_c^\infty(\mathbb{R})$ be such that $\int_{\mathbb{R}}\eta(x)\,\mathrm{d}x = 1$. If $\varphi \in VC(\mathbb{R}^m)$ and*

$$\varphi_\varepsilon(\boldsymbol{\lambda}) := \int_{\mathbb{R}}\eta(x)\,\varphi\big(\boldsymbol{\lambda} - (\varepsilon x)_{(m)}\big)\,\mathrm{d}x = \int_{\mathbb{R}}\eta(x)\,\varphi(\lambda_1 - \varepsilon x, \cdots, \lambda_m - \varepsilon x)\,\mathrm{d}x, \qquad \boldsymbol{\lambda} \in \mathbb{R}^m,\ \varepsilon > 0,$$

then $\varphi_\varepsilon \in VC(\mathbb{R}^m)$, and $\varphi_\varepsilon \to \varphi$ in $VC(\mathbb{R}^m)$ as $\varepsilon \searrow 0$.

Proof. We shall use the fact that $VC(\mathbb{R}^m)$ is a Fréchet space freely in this proof to apply the theory of the Bochner integral reviewed in §1.1. Let $\varepsilon > 0$. By Proposition 3.4.9, the map

$$\mathbb{R} \ni x \mapsto F_\varepsilon(x) := \eta(x)\,\varphi\big(\,\cdot - (\varepsilon x)_{(m)}\big) \in VC(\mathbb{R}^m)$$

is well defined and continuous. Consequently, F_ε is strongly measurable. Proposition 3.4.9 also implies $F_\varepsilon \to F := \eta(\cdot)\,\varphi$ pointwise (as maps $\mathbb{R} \to VC(\mathbb{R}^m)$) as $\varepsilon \searrow 0$. In addition, if $R > 0$, $\operatorname{supp}\eta \subseteq [-R, R]$, and $r > 0$, then

$$\int_{\mathbb{R}}\sup_{0<\delta\leq\varepsilon}\beta_{r,m}(F_\delta(x))\,\mathrm{d}x = \int_{-R}^{R}|\eta(x)|\sup_{0<\delta\leq\varepsilon}\beta_{r,m}\big(\varphi\big(\,\cdot - (\delta x)_{(m)}\big)\big)\,\mathrm{d}x \leq \beta_{r+\varepsilon R,m}(\varphi)\,\|\eta\|_{L^1} < \infty$$

by Inequality (3.4.10). Since $\{\beta_{r,m} : r > 0\}$ generates the topology of $VC(\mathbb{R}^m)$, the inequality above implies that F_ε is strongly integrable and, by the dominated convergence theorem, $\int_{\mathbb{R}} F_\varepsilon(x)\,\mathrm{d}x \to \int_{\mathbb{R}} F(x)\,\mathrm{d}x = \varphi \int_{\mathbb{R}} \eta(x)\,\mathrm{d}x = \varphi$ in $VC(\mathbb{R}^m)$ as $\varepsilon \searrow 0$. Finally, by applying the evaluation functionals $\{VC(\mathbb{R}^m) \ni \psi \mapsto \psi(\boldsymbol{\lambda}) \in \mathbb{C} : \boldsymbol{\lambda} \in \mathbb{R}^m\}$ to the Bochner integral $\int_{\mathbb{R}} F_\varepsilon\,\mathrm{d}x \in VC(\mathbb{R}^m)$, we see that $\varphi_\varepsilon = \int_{\mathbb{R}} F_\varepsilon(x)\,\mathrm{d}x$ for all $\varepsilon > 0$. This completes the proof. □

Theorem 3.4.12 (Density of polynomials and Wiener space)**.** *If $k \in \mathbb{N}$, then both $\mathbb{C}[\lambda]$ and $W_k(\mathbb{R})$ are dense in $VC^k(\mathbb{R})$.*

Proof. By Proposition 3.4.8, it suffices to prove that $W_k(\mathbb{R})$ is dense in $VC^k(\mathbb{R})$. We do so by mollification. Fix $\eta \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \eta(x)\,\mathrm{d}x = 1$, and define $\eta_\varepsilon(x) := \varepsilon^{-1}\eta\big(\varepsilon^{-1}x\big)$ for all $x \in \mathbb{R}$ and $\varepsilon > 0$. If $f \in VC^k(\mathbb{R})$, then $f * \eta_\varepsilon \in C^\infty(\mathbb{R}) \subseteq \overline{W_k(\mathbb{R})} \subseteq VC^k(\mathbb{R})$ by Corollary 3.4.7. To complete the proof, we show that $f * \eta_\varepsilon \to f$ in $VC^k(\mathbb{R})$ as $\varepsilon \searrow 0$. To this end, note that if $g \in C(\mathbb{R})$, $i \in \mathbb{N}$, $\varepsilon > 0$, and $\boldsymbol{\lambda} \in \mathbb{R}^{i+1}$, then

$$\begin{aligned}
\int_{\Delta_i} (g * \eta_\varepsilon)(\mathbf{t} \cdot \boldsymbol{\lambda})\,\rho_i(\mathrm{d}\mathbf{t}) &= \int_{\Delta_i}\int_{\mathbb{R}} g(\mathbf{t} \cdot \boldsymbol{\lambda} - x)\,\eta_\varepsilon(x)\,\mathrm{d}x\,\rho_i(\mathrm{d}\mathbf{t}) \\
&= \int_{\Delta_i}\int_{\mathbb{R}} g\big(\mathbf{t} \cdot \big(\boldsymbol{\lambda} - x_{(i+1)}\big)\big)\,\eta_\varepsilon(x)\,\mathrm{d}x\,\rho_i(\mathrm{d}\mathbf{t}) \\
&= \int_{\mathbb{R}} \eta_\varepsilon(x) \int_{\Delta_i} g\big(\mathbf{t} \cdot \big(\boldsymbol{\lambda} - x_{(i+1)}\big)\big)\,\rho_i(\mathrm{d}\mathbf{t})\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \eta(y) \int_{\Delta_i} g\big(\mathbf{t} \cdot \big(\boldsymbol{\lambda} - (\varepsilon y)_{(i+1)}\big)\big)\,\rho_i(\mathrm{d}\mathbf{t})\,\mathrm{d}y
\end{aligned}$$

by Fubini's theorem and the change of variable $y := \varepsilon^{-1}x$. It follows from Proposition 1.3.3(iii) (twice) that if $i \in \{0, \ldots, k\}$ and $\boldsymbol{\lambda} \in \mathbb{R}^{i+1}$, then

$$\begin{aligned}
(f * \eta_\varepsilon)^{[i]}(\boldsymbol{\lambda}) &= \int_{\Delta_i} (f * \eta_\varepsilon)^{(i)}(\mathbf{t} \cdot \boldsymbol{\lambda})\,\rho_i(\mathrm{d}\mathbf{t}) = \int_{\Delta_i} \big(f^{(i)} * \eta_\varepsilon\big)(\mathbf{t} \cdot \boldsymbol{\lambda})\,\rho_i(\mathrm{d}\mathbf{t}) \\
&= \int_{\mathbb{R}} \eta(y) \int_{\Delta_i} f^{(i)}\big(\mathbf{t} \cdot \big(\boldsymbol{\lambda} - (\varepsilon y)_{(i+1)}\big)\big)\,\rho_i(\mathrm{d}\mathbf{t})\,\mathrm{d}y \\
&= \int_{\mathbb{R}} \eta(y)\, f^{[i]}\big(\boldsymbol{\lambda} - (\varepsilon y)_{(i+1)}\big)\,\mathrm{d}y
\end{aligned}$$

Therefore, by Proposition 3.4.11, $(f * \eta_\varepsilon)^{[i]} \to f^{[i]}$ in $VC\big(\mathbb{R}^{i+1}\big)$ as $\varepsilon \searrow 0$. In other words, $f * \eta_\varepsilon \to f$ in $VC^k(\mathbb{R})$ as $\varepsilon \searrow 0$, as desired. □

Remark 3.4.13 (Jekel's space of noncommutative C^k functions)**.** In [Jek20, Ch. 18], Jekel introduced and briefly studied a space $C^k_{\mathrm{nc}}(\mathbb{R})$ of "noncommutative C^k functions" as an abstract completion of $\mathbb{C}[\lambda]$ with respect to seminorms similar to $\|\cdot\|_{VC^k,r}$ (but defined more algebraically in terms of Voiculescu's free difference quotients). The density of $\mathbb{C}[\lambda]$ in $VC^k(\mathbb{R})$ implies that Jekel's space of noncommutative C^k functions is isomorphic to $VC^k(\mathbb{R})$.

3.5 Two proofs of Theorem 3.1.8

In this section, we provide two proofs of Theorem 3.1.8: one using the method of perturbation formulas explained and demonstrated in §2.3 and one using the "classical" approach of approximation by polynomials [DK56, Hia10]. Throughout this section, we use the identification $V(\Omega_1,\ldots,\Omega_m) = C(\Omega_1)\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi C(\Omega_m)$ from Corollary 3.3.15 without further comment.

Notation 3.5.1 (Projective tensor product functional calculus)**.** Let $\mathcal{A}_1,\ldots,\mathcal{A}_m$ be unital C^*-algebras, and let $\mathbf{a} = (a_1,\ldots,a_m) \in \mathcal{A}_{1,\nu}\times\cdots\times\mathcal{A}_{m,\nu}$. If

$$\varphi \in V(\sigma(a_1),\ldots,\sigma(a_m)) = C(\sigma(a_1))\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi C(\sigma(a_m)),$$

then, in the notation of Corollary 1.5.6,

$$\varphi_\otimes(\mathbf{a}) \coloneqq \big(\Phi_{a_1}\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi\Phi_{a_m}\big)(\varphi) \in \mathcal{A}_1\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi\mathcal{A}_m.$$

where $\Phi_a\colon C(\sigma(a)) \to \mathcal{A}$ is the continuous functional calculus for $a \in \mathcal{A}_\nu$. If $S_i \subseteq \mathbb{C}$ is compact and $\sigma(a_i) \subseteq S_i$, then $\varphi_\otimes(\mathbf{a}) \coloneqq \big(\varphi|_{\sigma(a_1)\times\cdots\times\sigma(a_m)}\big)_\otimes(\mathbf{a})$ for all $\varphi \in C(S_1)\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi C(S_m)$. Also, if $\varphi \in VC(\mathbb{R}^m)$, then $\varphi_{\mathcal{A},\otimes}\colon \mathcal{A}_{\mathrm{sa}}^m \to \mathcal{A}^{\hat{\otimes}_\pi m}$ is the map $\mathbf{a} \mapsto \varphi_\otimes(\mathbf{a})$

Example 3.5.2 (Matrices)**.** Observe that if $\Omega_1,\ldots,\Omega_m$ are finite discrete spaces, then

$$C(\Omega_1)\otimes\cdots\otimes C(\Omega_m) = C(\Omega_1)\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi C(\Omega_m) = V(\Omega_1,\ldots,\Omega_m) = \mathbb{C}^\Omega.$$

Indeed, if $\varphi\colon \Omega \to \mathbb{C}$ is any function, then

$$\varphi = \sum_{\boldsymbol{\omega}\in\Omega} \varphi(\boldsymbol{\omega})\, 1_{\{\omega_1\}}\otimes\cdots\otimes 1_{\{\omega_m\}} \in C(\Omega_1)\otimes\cdots\otimes C(\Omega_m).$$

Consequently, if $n \in \mathbb{N}$, $\mathbf{a} = (a_1, \ldots, a_m) \in \mathrm{M}_n(\mathbb{C})^m_\nu$, and $\varphi\colon \sigma(A_1) \times \cdots \times \sigma(A_m) \to \mathbb{C}$ is any function, then (as we encourage the reader to verify)

$$\varphi_\otimes(\mathbf{a}) = \sum_{\boldsymbol{\lambda} \in \sigma(a_1) \times \cdots \times \sigma(a_m)} \varphi(\boldsymbol{\lambda})\, P^{a_1}_{\lambda_1} \otimes \cdots \otimes P^{a_m}_{\lambda_m},$$

which agrees with Notation 3.1.1.

Here is a nice way to calculate $\varphi_\otimes(\mathbf{a})$ in general.

Proposition 3.5.3. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be unital C^*-algebras, and fix $\mathbf{a} \in \mathcal{A}_{1,\nu} \times \cdots \times \mathcal{A}_{m,\nu}$. Retain the setting of Theorem 3.3.7, but take $\Omega_i = \sigma(a_i)$ for all $i \in \{1, \ldots, m\}$.*

(i) *If φ is as in Relation* (3.3.8), *then*

$$\varphi_\otimes(\mathbf{a}) = \int_\Sigma \varphi_1(a_1, \sigma) \otimes \cdots \otimes \varphi_m(a_m, \sigma)\, \rho(\mathrm{d}\sigma) \in \mathcal{A}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{A}_m,$$

where the right-hand side is a Bochner integral in $\mathcal{A}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{A}_m$.

(ii) *Suppose $\mathcal{A}_1 = \cdots = \mathcal{A}_m = \mathcal{A}$, and let $(\mathcal{I}, \|\cdot\|_\mathcal{I}) \trianglelefteq_{\mathrm{s}} \mathcal{A}$. Also, fix $i \in \{1, \ldots, m-1\}$ and $b = (b_1, \ldots, b_{m-1}) \in \mathcal{A}^{i-1} \times \mathcal{I} \times \mathcal{A}^{m-1-i}$. If φ is as in Relation* (3.3.8), *then*

$$\varphi_\otimes(\mathbf{a}) \#_{m-1} b = \int_\Sigma \varphi_1(a_1, \sigma)\, b_1 \cdots \varphi_{m-1}(a_{m-1}, \sigma)\, b_{m-1}\, \varphi_m(a_m, \sigma)\, \rho(\mathrm{d}\sigma) \in \mathcal{I},$$

where the right-hand side is a Bochner integral in $\mathcal{I}$.

Proof. By Theorem 3.3.7, $\varphi = \int_\Sigma \varphi_1(\cdot, \sigma) \otimes \cdots \otimes \varphi_m(\cdot, \sigma)\, \rho(\mathrm{d}\sigma)$ is a Bochner integral in $V(\sigma(a_1), \ldots, \sigma(a_m)) = C(\sigma(a_1)) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\sigma(a_m))$. Since

$$T \coloneqq \Phi_{a_1} \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \Phi_{a_m} \colon C(\sigma(a_1)) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\sigma(a_m)) \to \mathcal{A}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{A}_m$$

is a bounded linear map, we get

$$\varphi_\otimes(\mathbf{a}) = T(\varphi) = \int_\Sigma T(\varphi_1(\cdot, \sigma) \otimes \cdots \otimes \varphi_m(\cdot, \sigma))\, \rho(\mathrm{d}\sigma) = \int_\Sigma \varphi_1(a_1, \sigma) \otimes \cdots \otimes \varphi_m(a_m, \sigma)\, \rho(\mathrm{d}\sigma),$$

as claimed in the first item.

For the second item, apply the bounded linear map

$$\mathcal{A}^{\hat{\otimes}_\pi m} \ni u \mapsto u\#_{m-1}b \in \mathcal{I}$$

to the Bochner integral $\varphi_\otimes(\mathbf{a}) = \int_\Sigma \varphi_1(a_1,\sigma) \otimes \cdots \otimes \varphi_m(a_m,\sigma)\,\rho(\mathrm{d}\sigma)$. □

Since we now have a formalism for $f_\otimes^{[k]}(a_1,\ldots,a_{k+1})\#_k[b_1,\ldots,b_k]$ whenever $f \in VC^k(\mathbb{R})$, $a_1,\ldots,a_{k+1} \in \mathcal{A}_{\mathrm{sa}}$, and $b_1,\ldots,b_k \in \mathcal{I}$, it should be no surprise—based on the development in §2.3—that we can execute the method of perturbation formulas for Varopoulos C^k functions. We now begin this endeavor.

Notation 3.5.4 (Opposite multiplication operation)**.** Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach algebras. Write $\mathrm{M}_{\mathrm{op}} \colon \big(\mathcal{B}_1\hat{\otimes}_\pi\mathcal{B}_2\big) \times \big(\mathcal{B}_1\hat{\otimes}_\pi\mathcal{B}_2\big) \to \mathcal{B}_1\hat{\otimes}_\pi\mathcal{B}_2$ for the bounded bilinear map determined by

$$\mathrm{M}_{\mathrm{op}}[a \otimes c, b \otimes d] = (ab) \otimes (dc), \qquad a,b \in \mathcal{B}_1,\ c,d \in \mathcal{B}_2.$$

Also, write

$$u \cdot v := \mathrm{M}_{\mathrm{op}}[u,v] \in \mathcal{B}_1\hat{\otimes}_\pi\mathcal{B}_2, \qquad u,v \in \mathcal{B}_1\hat{\otimes}_\pi\mathcal{B}_2.$$

Lemma 3.5.5. *If $\mathcal{B}$ is a Banach algebra, then*

$$(u \cdot v)\#c = u\#[v\#c], \qquad u,v \in \mathcal{B}\hat{\otimes}_\pi\mathcal{B},\ c \in \mathcal{B}.$$

Proof. By a standard argument, it suffices to check the desired identity on pure tensors. If $a_1,b_1,a_2,b_2,c \in \mathcal{B}$, $u := a_1 \otimes b_1$, $v := a_2 \otimes b_2$ then

$$(u \cdot v)\#c = (a_1b_1 \otimes b_2b_1)\#c = a_1b_1cb_2a_2 = a_1(v\#c)a_2 = u\#[v\#c],$$

as desired. □

Proposition 3.5.6 (Perturbation formulas)**.** *If $f \in VC^1(\mathbb{R})$, then*

$$f(a) - f(b) = f_\otimes^{[1]}(a,b)\#[a-b], \qquad a,b \in \mathcal{A}_{\mathrm{sa}}.$$

Now, let $\mathcal{A}_1, \ldots, \mathcal{A}_{k+1}$ be unital C^*-algebras. If $\mathbf{a}, \mathbf{b} \in \mathcal{A}_{1,\mathrm{sa}} \times \cdots \mathcal{A}_{k+1,\mathrm{sa}}$ and $f \in VC^k(\mathbb{R})$, then

$$f_{\otimes}^{[k]}(\mathbf{a}) - f_{\otimes}^{[k]}(\mathbf{b}) = \sum_{i=1}^{k+1} f_{\otimes}^{[k+1]}(a_1, \ldots, a_i, b_i, \ldots, b_{k+1}) \#_{k+1,i}[a_i - b_i].$$

Proof. Let $a, b \in \mathcal{A}_{\mathrm{sa}}$. If $f \in VC^1(\mathbb{R})$, then

$$f(\lambda) - f(\mu) = f^{[1]}(\lambda, \mu)\,(\lambda - \mu), \qquad (\lambda, \mu) \in \sigma(a) \times \sigma(b), \tag{3.5.7}$$

by definition of $f^{[1]}$. By viewing Equation (3.5.7) as an identity in $C(\sigma(a))\hat{\otimes}_\pi C(\sigma(b))$, we may apply the homomorphism $\Phi_a \hat{\otimes}_\pi \Phi_b \colon C(\sigma(a))\hat{\otimes}_\pi C(\sigma(b)) \to \mathcal{A}\hat{\otimes}_\pi\mathcal{A}$ to both sides to obtain

$$f(a) \otimes 1 - 1 \otimes f(b) = f_{\otimes}^{[1]}(a, b)\,(a \otimes 1 - 1 \otimes b).$$

Now, since $\operatorname{im}\Phi_b \subseteq \mathcal{A}$ is commutative, $f_{\otimes}^{[1]}(a, b)\,(a \otimes 1 - 1 \otimes b) = f_{\otimes}^{[1]}(a, b) \cdot (a \otimes 1 - 1 \otimes b)$, as the reader may verify. Therefore, by Lemma 3.5.5,

$$\begin{aligned} f(a) - f(b) &= (f(a) \otimes 1 - 1 \otimes f(b))\#1 = \Big(f_{\otimes}^{[1]}(a, b) \cdot (a \otimes 1 - 1 \otimes b)\Big)\#1 \\ &= f_{\otimes}^{[1]}(a, b)\#[(a \otimes 1 - 1 \otimes b)\#1] = f_{\otimes}^{[1]}(a, b)\#[a - b], \end{aligned}$$

as desired.

Next, let $\mathbf{a}, \mathbf{b} \in \mathcal{A}_{1,\mathrm{sa}} \times \cdots \times \mathcal{A}_{k+1,\mathrm{sa}}$, and write $A \coloneqq \sigma(a_1) \times \cdots \times \sigma(a_{k+1}) \subseteq \mathbb{R}^{k+1}$ and $B \coloneqq \sigma(b_1) \times \cdots \times \sigma(b_{k+1}) \subseteq \mathbb{R}^{k+1}$. If $f \in VC^k(\mathbb{R})$, then

$$f^{[k]}(\boldsymbol{\lambda}) - f^{[k]}(\boldsymbol{\mu}) = \sum_{i=1}^{k+1} \underbrace{f^{[k+1]}(\lambda_1, \ldots, \lambda_i, \mu_i, \ldots, \mu_{k+1})}_{=:\varphi_i(\boldsymbol{\lambda}, \boldsymbol{\mu})}(\lambda_i - \mu_i), \qquad (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in A \times B, \tag{3.5.8}$$

by definition and the symmetry of divided differences (Proposition 1.3.3(i)). By viewing Equation (3.5.8) as an identity in $C(\sigma(a_1))\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\sigma(a_{k+1}))\hat{\otimes}_\pi C(\sigma(b_1))\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\sigma(b_{k+1}))$, we may apply the homomorphism $\Phi_{a_1}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \Phi_{a_{k+1}}\hat{\otimes}_\pi \Phi_{b_1}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \Phi_{b_{k+1}}$ to both sides to obtain

$$f_{\otimes}^{[k]}(\mathbf{a}) \otimes \mathbf{1} - \mathbf{1} \otimes f_{\otimes}^{[k]}(\mathbf{b}) = \sum_{i=1}^{k+1} (\varphi_i)_{\otimes}(\mathbf{a}, \mathbf{b})\,\big(1^{\otimes(i-1)} \otimes a_i \otimes 1^{\otimes(k+1-i)} \otimes \mathbf{1} - \mathbf{1} \otimes 1^{\otimes(i-1)} \otimes b_i \otimes 1^{\otimes(k+1-i)}\big),$$

where $\mathbf{1} = 1^{\otimes(k+1)}$ is the identity in $\mathcal{A}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{A}_{k+1}$. Now, since $\operatorname{im} \Phi_{b_i} \subseteq \mathcal{A}_i$ is commutative,

$$(\varphi_i)_\otimes(\mathbf{a},\mathbf{b})\big(1^{\otimes(i-1)} \otimes a_i \otimes 1^{\otimes(k+1-i)} \otimes \mathbf{1} - \mathbf{1} \otimes 1^{\otimes(i-1)} \otimes b_i \otimes 1^{\otimes(k+1-i)}\big)$$
$$= (\varphi_i)_\otimes(\mathbf{a},\mathbf{b}) \cdot \big(1^{\otimes(i-1)} \otimes a_i \otimes 1^{\otimes(k+1-i)} \otimes \mathbf{1} - \mathbf{1} \otimes 1^{\otimes(i-1)} \otimes b_i \otimes 1^{\otimes(k+1-i)}\big),$$

where we are using Notation 3.5.4 with $\mathcal{B} = \mathcal{A}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{A}_{k+1}$, as the reader may verify. Thus,

$$\begin{aligned} f_\otimes^{[k]}(\mathbf{a}) - f_\otimes^{[k]}(\mathbf{b}) &= \big(f_\otimes^{[k]}(\mathbf{a}) \otimes \mathbf{1} - \mathbf{1} \otimes f_\otimes^{[k]}(\mathbf{b})\big)\#\mathbf{1} \\ &= \sum_{i=1}^{k+1} \Big((\varphi_i)_\otimes(\mathbf{a},\mathbf{b}) \cdot \big(1^{\otimes(i-1)} \otimes a_i \otimes 1^{\otimes(k+1-i)} \otimes \mathbf{1} - \mathbf{1} \otimes 1^{\otimes(i-1)} \otimes b_i \otimes 1^{\otimes(k+1-i)}\big)\Big)\#\mathbf{1} \\ &= \sum_{i=1}^{k+1} (\varphi_i)_\otimes(\mathbf{a},\mathbf{b})\#\Big[\big(1^{\otimes(i-1)} \otimes a_i \otimes 1^{\otimes(k+1-i)} \otimes \mathbf{1} - \mathbf{1} \otimes 1^{\otimes(i-1)} \otimes b_i \otimes 1^{\otimes(k+1-i)}\big)\#\mathbf{1}\Big] \\ &= \sum_{i=1}^{k+1} (\varphi_i)_\otimes(\mathbf{a},\mathbf{b})\#\big[1^{\otimes(i-1)} \otimes (a_i - b_i) \otimes 1^{\otimes(k+1-i)}\big] \\ &= \sum_{i=1}^{k+1} f_\otimes^{[k+1]}(a_1,\ldots,a_i,b_i,\ldots,b_{k+1})\#_{k+1,i}[a_i - b_i] \end{aligned} \tag{3.5.9}$$

by Lemma 3.5.5 and unraveling the notation. (To be clear, the # operation used in the lines before Equation (3.5.9) is the one for the algebra $\mathcal{B} = \mathcal{A}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{A}_{k+1}$.) In Equation (3.5.9), we used the fact that if $\psi \in C(\sigma(a_1)) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(\sigma(a_i)) \hat{\otimes}_\pi C(\sigma(b_i)) \hat{\otimes}_\pi \cdots C(\sigma(b_{k+1}))$ and

$$\varphi(\boldsymbol{\lambda},\boldsymbol{\mu}) \coloneqq \psi(\lambda_1,\ldots,\lambda_i,\mu_i,\ldots,\mu_{k+1}), \qquad (\boldsymbol{\lambda},\boldsymbol{\mu}) \in A \times B,$$

then

$$\varphi_\otimes(\mathbf{a},\mathbf{b})\#\big[1^{\otimes(i-1)} \otimes c \otimes 1^{\otimes(k+1-i)}\big] = \psi_\otimes(a_1,\ldots,a_i,b_i,\ldots,b_{k+1})\#_{k+1,i}c, \qquad c \in \mathcal{A}_i.$$

We leave the verification of this identity to the reader. (Hint: It suffices to treat the case when ψ is a pure tensor function.) This completes the proof. □

Corollary 3.5.10. *If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{A}$, $a, b \in \mathcal{A}_{\mathrm{sa}}$, $f \in VC^1(\mathbb{R})$, and $a - b \in \mathcal{I}$, then $f(a) - f(b) \in \mathcal{I}$.*

Proof. Combine Proposition 2.2.6 with the first formula in Proposition 3.5.6. □

Remark 3.5.11. The second perturbation formula in Proposition 3.5.6 (and its proof) generalizes the first (the $k = 0$ case). Proposition 3.5.6 is stated and proven as it is in order to maximize the first formula's digestibility. Similar comments apply to Proposition 2.3.7.

It is also possible to prove Proposition 3.5.6 using Proposition 3.5.3 by "decomposing" $f^{[k+1]}$ on $[-r,r]^{k+1}$ (with $r > 0$ sufficiently large) as an integral (or series) of pure tensor functions. This is the kind of approach we must take when working with unbounded operators in Chapter 6 below; please see Theorem 6.5.7.

Proposition 3.5.12 (Continuous perturbation property)**.** *If $\varphi \in VC(\mathbb{R}^m)$, then the map $\varphi_{\mathcal{A},\otimes} \colon \mathcal{A}_{\mathrm{sa}}^m \to \mathcal{A}^{\hat{\otimes}_\pi m}$ from Notation 3.5.1 belongs to $C_{\mathrm{bb}}\big(\mathcal{A}_{\mathrm{sa}}^m; \mathcal{A}^{\hat{\otimes}_\pi m}\big)$ (Definition 1.2.9). Moreover, the map $VC(\mathbb{R}^m) \ni \varphi \mapsto \varphi_{\mathcal{A},\otimes} \in C_{\mathrm{bb}}\big(\mathcal{A}_{\mathrm{sa}}^m; \mathcal{A}^{\hat{\otimes}_\pi m}\big)$ is continuous.*

Proof. Write $\mathcal{C}_r \coloneqq \{\mathbf{a} \in \mathcal{A}_{\mathrm{sa}}^m : \|\mathbf{a}\|_\infty \coloneqq \max\{\|a_i\| : i \in \{1,\ldots,m\} \le r\}$ for all $r > 0$. First, observe that if $r > 0$ and $\varphi \in VC(\mathbb{R}^m)$, then

$$\|\varphi_\otimes(\mathbf{a})\|_{\mathcal{A}^{\hat{\otimes}_\pi m}} \le \big\|\varphi|_{\sigma(a_1)\times\cdots\times\sigma(a_m)}\big\|_{V(\sigma(a_1),\ldots,\sigma(a_m))} \le \beta_{r,m}(\varphi), \qquad \mathbf{a} \in \mathcal{C}_r, \tag{3.5.13}$$

because $\Phi_a \colon C(\sigma(a)) \to \mathcal{A}$ is an isometry—in particular, has operator norm equal to one—whenever $a \in \mathcal{A}_\nu$. Next, observe that if

$$P(\boldsymbol{\lambda}) = \sum_{|\alpha|\le d} c_\alpha \, \boldsymbol{\lambda}^\alpha \in \mathbb{C}[\boldsymbol{\lambda}] = \mathbb{C}[\lambda_1,\ldots,\lambda_m],$$

then

$$P_\otimes(\mathbf{a}) = \sum_{|\alpha|\le d} c_\alpha \, a_1^{\alpha_1} \otimes \cdots \otimes a_m^{\alpha_m}, \qquad \mathbf{a} \in \mathcal{A}_{\mathrm{sa}}^m, \tag{3.5.14}$$

from which it is clear that $P_{\mathcal{A},\otimes} \in C_{\mathrm{bb}}\big(\mathcal{A}_{\mathrm{sa}}^m; \mathcal{A}^{\hat{\otimes}_\pi m}\big)$. Finally, Proposition 3.3.5 implies that m-variate polynomial functions $\mathbb{R}^m \to \mathbb{C}$ are dense in $VC(\mathbb{R}^m)$, i.e., if $\varphi \in VC(\mathbb{R}^m)$, then there exists a sequence $(P_n(\boldsymbol{\lambda}))_{n\in\mathbb{N}}$ in $\mathbb{C}[\boldsymbol{\lambda}]$ such that $P_n \to \varphi$ in $VC(\mathbb{R}^m)$ as $n \to \infty$. By Inequality (3.5.13), $(P_n)_{\mathcal{A},\otimes} \to \varphi_{\mathcal{A},\otimes}$ uniformly on bounded sets as $n \to \infty$. Since $(P_n)_{\mathcal{A},\otimes} \in C_{\mathrm{bb}}\big(\mathcal{A}_{\mathrm{sa}}^m; \mathcal{A}^{\hat{\otimes}_\pi m}\big)$ for all $n \in \mathbb{N}$, we conclude that $\varphi_{\mathcal{A},\otimes} \in C_{\mathrm{bb}}\big(\mathcal{A}_{\mathrm{sa}}^m; \mathcal{A}^{\hat{\otimes}_\pi m}\big)$, which is the first part of the proposition. The second follows from another appeal to Inequality (3.5.13). □

Remark 3.5.15 (Different algebras)**.** The same proof shows that if $\mathcal{A}_1,\ldots,\mathcal{A}_m$ are unital C^*-algebras and $\varphi \in VC(\mathbb{R}^m)$, then the map $\mathcal{A}_{1,\mathrm{sa}} \times \cdots \times \mathcal{A}_{m,\mathrm{sa}} \ni \mathbf{a} \mapsto \varphi_{\otimes}(\mathbf{a}) \in \mathcal{A}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{A}_m$ belongs to $C_{\mathrm{bb}}(\mathcal{A}_{1,\mathrm{sa}} \times \cdots \times \mathcal{A}_{m,\mathrm{sa}}; \mathcal{A}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{A}_m)$, and the assignment of $\varphi \in VC(\mathbb{R}^m)$ to $(\mathbf{a} \mapsto \varphi_{\otimes}(\mathbf{a}))$ is continuous as a map $VC(\mathbb{R}^m) \to C_{\mathrm{bb}}(\mathcal{A}_{1,\mathrm{sa}} \times \cdots \times \mathcal{A}_{m,\mathrm{sa}}; \mathcal{A}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{A}_m)$.

We are now prepared for the first proof of Theorem 3.1.8.

First proof of Theorem 3.1.8. Let $b, h \in \mathcal{I}_{\mathrm{sa}}$, and recall that $f_{a,\mathcal{I}}(b) = f(a+b) - f(a)$. We prove the claimed derivative formula by induction on k. For the base case, note that

$$\begin{aligned}
\varepsilon(h) &\coloneqq \frac{1}{\|h\|_{\mathcal{I}}} \big\| f_{a,\mathcal{I}}(b+h) - f_{a,\mathcal{I}}(b) - f_{\otimes}^{[1]}(a+b, a+b) \# h \big\|_{\mathcal{I}} \\
&= \frac{1}{\|h\|_{\mathcal{I}}} \big\| f(a+b+h) - f(a+b) - f_{\otimes}^{[1]}(a+b, a+b) \# h \big\|_{\mathcal{I}} \\
&= \frac{1}{\|h\|_{\mathcal{I}}} \Big\| f_{\otimes}^{[1]}(a+b+h, a+b) \# h - f_{\otimes}^{[1]}(a+b, a+b) \# h \Big\|_{\mathcal{I}} \\
&\leq \Big\| f_{\otimes}^{[1]}(a+b+h, a+b) - f_{\otimes}^{[1]}(a+b, a+b) \Big\|_{\mathcal{A} \hat{\otimes}_\pi \mathcal{A}} \xrightarrow{\|h\|_{\mathcal{I}} \to 0} 0
\end{aligned}$$

by Propositions 3.5.6, 2.2.6, and 3.5.12. Now, assume the claimed derivative formula for the k^{th} derivative. If $b_1, \ldots, b_k \in \mathcal{I}_{\mathrm{sa}}$ and $b_{k+1} \coloneqq h$, then

$$\begin{aligned}
\varepsilon(b_1, \ldots, b_{k+1}) &\coloneqq \frac{1}{\|h\|_{\mathcal{I}}} \Bigg\| \partial_{b_k} \cdots \partial_{b_1} f_{a,\mathcal{I}}(b+h) - \partial_{b_k} \cdots \partial_{b_1} f_{a,\mathcal{I}}(b) \\
&\qquad - \sum_{\sigma \in S_{k+1}} f_{\otimes}^{[k+1]}\big((a+b)_{(k+2)}\big) \#_{k+1} \big[b_{\sigma(1)}, \ldots, b_{\sigma(k+1)}\big] \Bigg\|_{\mathcal{I}} \\
&= \frac{1}{\|h\|_{\mathcal{I}}} \Bigg\| \sum_{\pi \in S_k} \big(f_{\otimes}^{[k]}\big((a+b+h)_{(k+1)}\big) - f_{\otimes}^{[k]}\big((a+b)_{(k+1)}\big)\big) \#_k \big[b_{\pi(1)}, \ldots, b_{\pi(k)}\big] \\
&\qquad - \sum_{\sigma \in S_{k+1}} f_{\otimes}^{[k+1]}\big((a+b)_{(k+2)}\big) \#_{k+1} \big[b_{\sigma(1)}, \ldots, b_{\sigma(k+1)}\big] \Bigg\|_{\mathcal{I}} \\
&= \frac{1}{\|h\|_{\mathcal{I}}} \Bigg\| \sum_{\pi \in S_k} \sum_{i=1}^{k+1} f_{\otimes}^{[k+1]}\big((a+b+h)_{(i)}, (a+b)_{(k+2-i)}\big) \#_{k+1} \big[b_{\pi(1)}, \ldots, b_{\pi(i-1)}, h, b_{\pi(i)}, \ldots, b_{\pi(k)}\big] \\
&\qquad - \sum_{\pi \in S_k} \sum_{i=1}^{k+1} f_{\otimes}^{[k+1]}\big((a+b)_{(k+2)}\big) \#_{k+1} \big[b_{\pi(1)}, \ldots, b_{\pi(i-1)}, h, b_{\pi(i)}, \ldots, b_{\pi(k)}\big] \Bigg\|_{\mathcal{I}} \\
&\leq k!\, C_{\mathcal{I}}^k \|b_1\|_{\mathcal{I}} \cdots \|b_k\|_{\mathcal{I}} \sum_{i=1}^{k+1} \Big\| f_{\otimes}^{[k+1]}\big((a+b+h)_{(i)}, (a+b)_{(k+2-i)}\big) - f_{\otimes}^{[k+1]}\big((a+b)_{(k+2)}\big) \Big\|_{\mathcal{A}^{\hat{\otimes}_\pi (k+2)}}
\end{aligned}$$

by the induction hypothesis and Propositions 3.5.6 and 2.2.6. Writing

$$F(a)[b_1,\ldots,b_{k+1}] \coloneqq \sum_{\sigma\in S_{k+1}} f_{\otimes}^{[k+1]}\big(a_{(k+2)}\big)\#_{k+1}\big[b_{\sigma(1)},\ldots,b_{\sigma(k+1)}\big], \qquad a\in\mathcal{A}_{\mathrm{sa}},\ b_i\in\mathcal{I}_{\mathrm{sa}},$$

we then conclude from Proposition 3.5.12 that

$$\begin{aligned}
&\frac{1}{\|h\|_{\mathcal{I}}}\Big\|D^k f_{a,\mathcal{I}}(b+h)+D^k f_{a,\mathcal{I}}(b)-F(a+b)\Big\|_{B_k(\mathcal{I}^k;\mathcal{I})} \\
&\quad\le k!\,C_{\mathcal{I}}^k\sum_{i=1}^{k+1}\Big\|f_{\otimes}^{[k+1]}\big((a+b+h)_{(i)},(a+b)_{(k+2-i)}\big)-f_{\otimes}^{[k+1]}\big((a+b)_{(k+2)}\big)\Big\|_{\mathcal{A}^{\hat{\otimes}_\pi(k+2)}}\xrightarrow{\|h\|_{\mathcal{I}}\to 0} 0.
\end{aligned}$$

This completes the proof. □

For the second proof, we take the following result as a starting point: If $(\mathcal{I},\|\cdot\|_{\mathcal{I}})\trianglelefteq_{\mathrm{s}}\mathcal{A}$, $n\in\mathbb{N}_0$, and $a\in\mathcal{A}$, then $F_n(b)\coloneqq(a+b)^n-a^n\in\mathcal{I}$ whenever $b\in\mathcal{I}$, $F_n\in\mathrm{Hol}(\mathcal{I};\mathcal{I})$, and

$$\partial_{b_k}\cdots\partial_{b_1}F_n(b)=\sum_{\pi\in S_k}\sum_{|\alpha|=n-k}(a+b)^{\alpha_1}b_{\pi(1)}\cdots(a+b)^{\alpha_k}b_{\pi(k)}(a+b)^{\alpha_{k+1}}, \qquad b,b_i\in\mathcal{I}.$$

This is a special case of Theorem 2.3.10 (via Proposition 2.3.4). It is also not difficult to prove directly by induction on k using a combinatorial version of the method of perturbation formulas; please see [Nik23c, Prop. 4.3.1] for this kind of argument.

Second proof of Theorem 3.1.8. We set some notation. If $f\in VC^k(\mathbb{R})$, then

$$T_kf(b)[b_1,\ldots,b_k]\coloneqq\sum_{\pi\in S_k}f_{\otimes}^{[k]}\big((a+b)_{(k+1)}\big)\#_k\big[b_{\pi(1)},\ldots,b_{\pi(k)}\big], \qquad b,b_i\in\mathcal{I}_{\mathrm{sa}}.$$

We aim to prove that $f_{a,\mathcal{I}}(b)\coloneqq f(a+b)-f(a)\in\mathcal{I}$ whenever $b\in\mathcal{I}_{\mathrm{sa}}$, $f_{a,\mathcal{I}}\in C^k_{\mathrm{bb}}(\mathcal{I}_{\mathrm{sa}};\mathcal{I})$, and $D^kf_{a,\mathcal{I}}=T_kf$. The result of the previous paragraph translates, via Example 1.3.8, to the desired conclusion when $f(\lambda)=p_n(\lambda)=\lambda^n$. Consequently, we have the desired conclusion whenever $f(\lambda)\in\mathbb{C}[\lambda]\subseteq VC^k(\mathbb{R})$.

By Theorem 3.4.12, if $f\in VC^k(\mathbb{R})$ is arbitrary, then there exists a sequence $(q_n)_{n\in\mathbb{N}}$ of polynomials converging to f in $VC^k(\mathbb{R})$. Since $q_n\to f$ uniformly on compact sets, if $c\in\mathcal{A}_{\mathrm{sa}}$, then $q_n(c)\to f(c)$ in $\mathcal{A}$ as $n\to\infty$. Also, observe that $q_n(a+0)-q_n(a)=0=f(a+0)-f(a)$

for all $n \in \mathbb{N}$, so $(q_n)_{a,\mathcal{I}}(0) \to f_{a,\mathcal{I}}(0)$ in $\mathcal{I}$ as $n \to \infty$. Next, for $r > 0$ and $i \in \mathbb{N}$, define

$$\mathcal{I}_{\mathrm{sa},r} := \{b \in \mathcal{I}_{\mathrm{sa}} : \|b\|_{\mathcal{I}} \leq r\} \text{ and } \|\cdot\|_i := \|\cdot\|_{B_i(\mathcal{I}_{\mathrm{sa}}^i;\mathcal{I})}.$$

By Inequality (3.5.13), if $b \in \mathcal{I}_{\mathrm{sa},r}$, $R := \|a\| + C_{\mathcal{I}} r$, and $i \in \{1, \ldots, k\}$, then

$$\begin{aligned}\big\|T_i f(b) - D^i(q_n)_{a,\mathcal{I}}(b)\big\|_i = \|T_i(f - q_n)(b)\|_i &\leq i!\, C_{\mathcal{I}}^{i-1} \Big\|(f - q_n)_{\otimes}^{[i]}\big((a + b)_{(i+1)}\big)\Big\|_{\mathcal{A}^{\hat{\otimes}_\pi (i+1)}} \\ &\leq i!\, C_{\mathcal{I}}^{i-1} \Big\|(f - q_n)^{[i]}\Big\|_{V(\sigma(a+b)_{(i+1)})} \leq i!\, C_{\mathcal{I}}^{i-1} \beta\big((f - q_n)^{[i]}\big)_{R,i+1}.\end{aligned}$$

In the last inequality above, we used that $\sigma(a + b) \subseteq [-\|a + b\|, \|a + b\|] \subseteq [-R, R]$. Thus,

$$\max_{1\leq i\leq k} \sup_{b\in\mathcal{I}_{\mathrm{sa},r}} \big\|T_i f(b) - D^i(q_n)_{a,\mathcal{I}}(b)\big\|_i \lesssim_k \|f - q_n\|_{VC^k,R} \xrightarrow{n\to\infty} 0.$$

Since $r > 0$ was arbitrary, we conclude from Theorem 1.2.12 that $((q_n)_{a,\mathcal{I}})_{n\in\mathbb{N}}$ converges in $C^k_{\mathrm{bb}}(\mathcal{I}_{\mathrm{sa}}; \mathcal{I})$. Furthermore, if $F \in C^k_{\mathrm{bb}}(\mathcal{I}_{\mathrm{sa}}; \mathcal{I})$ is the limit of $((q_n)_{a,\mathcal{I}})_{n\in\mathbb{N}}$, then $DF^i = T_i f$ for all $i \in \{1, \ldots, k\}$, and $F(b) = f(a + b) - f(a)$ for all $b \in \mathcal{I}_{\mathrm{sa}}$. This completes the proof. □

3.6 Examples of Varopoulos C^k functions from Besov spaces

We saw in §3.4 that only elementary methods are required to prove $C^{k+1}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$. However, $VC^k(\mathbb{R})$ is much closer to $C^k(\mathbb{R})$ than that. In this section, we use more advanced harmonic analysis done by V. V. Peller [Pel06] to exhibit two classes of examples of Varopoulos C^k functions that illustrate this point more precisely.

We begin by defining Besov spaces and stating their relevant properties; for (much) more information, please see [Leo17, Pee76, Saw18, Tri83, Tri92].

Definition 3.6.1 (Besov spaces). Let $m \in \mathbb{N}$, and fix $\eta \in C_c^\infty(\mathbb{R}^m)$ such that $0 \leq \eta \leq 1$ everywhere, $\operatorname{supp} \eta \subseteq \{\xi \in \mathbb{R}^m : |\xi|_2 \leq 2\}$, and $\eta \equiv 1$ on $\{\xi \in \mathbb{R}^m : |\xi|_2 \leq 1\}$. (Here and throughout, $|\cdot|_2$ is the Euclidean norm.) Define

$$\eta_i(\xi) := \eta\big(2^{-i}\xi\big) - \eta\big(2^{-i+1}\xi\big), \quad i \in \mathbb{Z},\ \xi \in \mathbb{R}^m.$$

Now, for $(s,p,q) \in \mathbb{R} \times [1,\infty]^2$ and $f \in \mathscr{S}'(\mathbb{R}^m)$, define

$$\|f\|_{\dot{B}^{s,p}_q} := \big\|\big(2^{is}\|\check{\eta}_i * f\|_{L^p}\big)_{i\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})} \in [0,\infty] \quad \text{and}$$
$$\|f\|_{B^{s,p}_q} := \|\check{\eta} * f\|_{L^p} + \big\|\big(2^{is}\|\check{\eta}_i * f\|_{L^p}\big)_{i\in\mathbb{N}}\big\|_{\ell^q(\mathbb{N})} \in [0,\infty].$$

We call

$$\dot{B}^{s,p}_q(\mathbb{R}^m) := \big\{f \in \mathscr{S}'(\mathbb{R}^m) : \|f\|_{\dot{B}^{s,p}_q} < \infty\big\}$$

the **homogeneous (s,p,q)-Besov space** and

$$B^{s,p}_q(\mathbb{R}^m) := \big\{f \in \mathscr{S}'(\mathbb{R}^m) : \|f\|_{B^{s,p}_q} < \infty\big\}$$

the **inhomogeneous (s,p,q)-Besov space**.

Remark 3.6.2. First, note that $\check{\eta} * f$ and $\check{\eta}_i * f$ have compactly supported Fourier transforms and so, by the Paley–Wiener theorem, are smooth; it therefore makes sense to apply the L^p-norm to them. Second, since it is easy to show that $\|f\|_{\dot{B}^{s,p}_q} = 0$ if and only if f is a polynomial, it is often useful to define $\dot{B}^{s,p}_q(\mathbb{R}^m)$ as a quotient space in which all polynomials are zero. The definition of $\dot{B}^{s,p}_q(\mathbb{R}^m)$ above is given in [Pee76, Ch. 3] and [Tri83, §5.1.2 & §5.1.3]. The definition "modulo polynomials" is given in [Saw18, §2.4]. (Please see [Saw18, §1.2.5.3] also.) Finally, beware that the positions of p and q in the notation for $B^{s,p}_q(\mathbb{R}^m)$ and $\dot{B}^{s,p}_q(\mathbb{R}^m)$ vary in the literature.

Here are the properties of Besov spaces that we shall use. Below, the symbol $\hookrightarrow$ indicates (as usual) continuous inclusion, and $\sim$ indicates equivalence of (possibly infinite) norms.

Notation 3.6.3. If $k \in \mathbb{N}$ and $f \in C^k(\mathbb{R}^m)$, then

$$\|f\|_{BC^k} := \sum_{|\alpha|\le k} \|\partial^\alpha f\|_{\ell^\infty(\mathbb{R}^m)} = \sum_{\alpha\in\mathbb{N}_0^m : |\alpha|\le k} \big\|\partial_1^{\alpha_1}\cdots\partial_m^{\alpha_m} f\big\|_{\ell^\infty(\mathbb{R}^m)} \in [0,\infty].$$

Also, $BC^k(\mathbb{R}) := \big\{f \in C^k(\mathbb{R}^m) : \|f\|_{BC^k} < \infty\big\}$.

Theorem 3.6.4 (Properties of Besov spaces)**.** *Fix $s, s_1, s_2 \in \mathbb{R}$ and $p, q, q_1, q_2 \in [1,\infty]$.*

(i) $\big(B^{s,p}_q(\mathbb{R}^m), \|\cdot\|_{B^{s,p}_q}\big)$ *is a Banach space that is independent of the choice of η.*

(ii) *$s_1 > s_2 \Rightarrow B^{s_1,p}_{q_1}(\mathbb{R}^m) \hookrightarrow B^{s_2,p}_{q_2}(\mathbb{R}^m)$, and $q_1 < q_2 \Rightarrow B^{s,p}_{q_1}(\mathbb{R}^m) \hookrightarrow B^{s,p}_{q_2}(\mathbb{R}^m)$.*

(iii) *If $s \geq 0$, then $B^{s,\infty}_1(\mathbb{R}^m) \hookrightarrow BC^{\lfloor s \rfloor}(\mathbb{R}^m)$.*

(iv) *Define*

$$\|f\|_{\mathrm{h},B^{s,p}_q} \coloneqq \|f\|_{L^p} + \|f\|_{\dot{B}^{s,p}_q} \in [0,\infty], \qquad f \in \mathscr{S}'(\mathbb{R}^m).$$

(Of course, we declare $\|f\|_{L^p} \coloneqq \infty$ if f is not induced by a locally integrable function.) If $s > 0$, then $\|\cdot\|_{B^{s,p}_q} \sim \|\cdot\|_{\mathrm{h},B^{s,p}_q}$ on $\mathscr{S}'(\mathbb{R}^m)$. In particular, $B^{s,p}_q(\mathbb{R}^m) = L^p(\mathbb{R}^m) \cap \dot{B}^{s,p}_q(\mathbb{R}^m)$ whenever $s > 0$.

(v) *If V and W are vector spaces, $g\colon V \to W$ is a function, and $x, h \in V$ are vectors, then we define (recursively)*

$$\Delta^1_h g(x) = \Delta_h g(x) \coloneqq g(x+h) - g(x) \quad \text{and}$$
$$\Delta^k_h g(x) \coloneqq \Delta_h\big(\Delta^{k-1}_h g\big)(x) \qquad k \geq 2.$$

Now, suppose $s > 0$. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, define

$$\|f\|_{\mathrm{cl},B^{s,p}_q} \coloneqq \begin{cases} \|f\|_{L^p} + \big(\int_{\mathbb{R}^m} |h|_2^{-sq-m} \|\Delta_h^{\lfloor s \rfloor + 1} f\|^q_{L^p}\, dh\big)^{\frac{1}{q}} & \text{if } q < \infty, \\ \|f\|_{L^p} + \sup_{h \in \mathbb{R}^m \setminus \{0\}} |h|_2^{-s} \|\Delta_h^{\lfloor s \rfloor + 1} f\|_{L^p} & \text{if } q = \infty. \end{cases}$$

If $s > 0$, then $B^{s,p}_q(\mathbb{R}^m) = \big\{f \in L^1_{\mathrm{loc}}(\mathbb{R}^m) : \|f\|_{\mathrm{cl},B^{s,p}_q} < \infty\big\}$, and $\|\cdot\|_{B^{s,p}_q} \sim \|\cdot\|_{\mathrm{cl},B^{s,p}_q}$ on $\mathscr{S}'(\mathbb{R}^m) \cap L^1_{\mathrm{loc}}(\mathbb{R}^m)$.

Here are some references for the proofs of these facts: The first two items are proven in [Tri83, §2.3.2 & §2.3.3], the third item is proven in [Saw18, §2.1.2.4], the fourth item is proven in [Tri92, §2.3.3], and the fifth item is proven in [Tri83, §2.5.12].

Remark 3.6.5. It is also the case that $B^{s,p}_{\min\{p,2\}}(\mathbb{R}^m) \hookrightarrow W^{s,p}(\mathbb{R}^m) \hookrightarrow B^{s,p}_{\max\{p,2\}}(\mathbb{R}^m)$ whenever $s \in \mathbb{R}$ and $1 < p < \infty$, where $W^{s,p}(\mathbb{R}^m) = L^p_s(\mathbb{R}^m) = H^s_p(\mathbb{R}^m)$ is the fractional Sobolev (Bessel potential) space; please see [Tri83, §2.2.2, §2.3.2, & §2.5.6]. Also, in [Leo17, Ch. 17], $B^{s,p}_q(\mathbb{R}^m)$ (with $s > 0$) is defined and studied using $\|\cdot\|_{\mathrm{cl},B^{s,p}_q}$. The equivalence $\|\cdot\|_{\mathrm{cl},B^{s,p}_q} \sim \|\cdot\|_{\mathrm{h},B^{s,p}_q}$ is proven in [Leo17, §17.7].

The most important indices for us are $(s,p,q) = (k \in \mathbb{N}, \infty, 1)$. It turns out in this case that $\dot{B}_1^{k,\infty}(\mathbb{R}) \subseteq C^k(\mathbb{R})$; please see Proposition B.2.7 and the comments thereafter.

Example 3.6.6 (Wiener space)**.** We claim that if $k \in \mathbb{N}$, then $W_k(\mathbb{R}) \subseteq B_1^{k,\infty}(\mathbb{R})$. Indeed, if $f = \int_{\mathbb{R}} e^{i\cdot\xi}\,\mu(\mathrm{d}\xi) \in W_k(\mathbb{R})$, $\chi \in \mathscr{S}(\mathbb{R})$, and $\lambda \in \mathbb{R}$, then

$$(\chi * f)(\lambda) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\lambda-y)\xi}\chi(y)\,\mu(\mathrm{d}\xi)\,\mathrm{d}y = \int_{\mathbb{R}} e^{i\lambda\xi}\int_{\mathbb{R}} e^{-iy\xi}\chi(y)\,\mathrm{d}y\,\mu(\mathrm{d}\xi) = \int_{\mathbb{R}} e^{i\lambda\xi}\widehat{\chi}(\xi)\,\mu(\mathrm{d}\xi)$$

by definition of convolution and Fubini's theorem. In particular,

$$(\check{\eta} * f)(\lambda) = \int_{\mathbb{R}} e^{i\lambda\xi}\eta(\xi)\,\mu(\mathrm{d}\xi) \;\text{ and }\; (\check{\eta}_i * f)(\lambda) = \int_{\mathbb{R}} e^{i\lambda\xi}\eta_i(\xi)\,\mu(\mathrm{d}\xi), \qquad i \in \mathbb{N}.$$

It follows that

$$\begin{aligned}
\|f\|_{B_1^{k,\infty}} &= \|\check{\eta} * f\|_{L^\infty} + \sum_{i=1}^{\infty} 2^{ik}\|\check{\eta}_i * f\|_{L^\infty} \le \int_{\mathbb{R}} |\eta(\xi)|\,|\mu|(\mathrm{d}\xi) + \sum_{i=1}^{\infty} 2^{ik}\int_{\mathbb{R}} |\eta_i(\xi)|\,|\mu|(\mathrm{d}\xi) \\
&= \int_{\{\xi\in\mathbb{R}:|\xi|\le 2\}} |\eta(\xi)|\,|\mu|(\mathrm{d}\xi) + \sum_{i=1}^{\infty}\int_{\{\xi\in\mathbb{R}:2^{i-1}\le|\xi|\le 2^{i+1}\}} 2^{ik}|\eta_i(\xi)|\,|\mu|(\mathrm{d}\xi) \\
&\le \int_{\{\xi\in\mathbb{R}:|\xi|\le 2\}} |\eta(\xi)|\,|\mu|(\mathrm{d}\xi) + 2^k\int_{\mathbb{R}}\sum_{i=1}^{\infty} 1_{\{\xi\in\mathbb{R}:2^{i-1}\le|\xi|\le 2^{i+1}\}}|\xi|^k\,|\eta_i(\xi)|\,|\mu|(\mathrm{d}\xi) \\
&\le \|\eta\|_{L^\infty}\big(|\mu|([-2,2]) + 3\cdot 2^{k+1}\mu_{(k)}\big) < \infty,
\end{aligned}$$

as claimed.

Next, we state an important result of Peller that we shall use to prove $\dot{B}_1^{k,\infty}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$.

Theorem 3.6.7 (Peller)**.** *If $k \in \mathbb{N}$, then there is a constant $a_k < \infty$ such that for all $f \in \dot{B}_1^{k,\infty}(\mathbb{R})$ with $f^{(k)} \in BC(\mathbb{R})$, there exists a σ-finite measure space $(\Sigma, \mathscr{H}, \rho)$ and measurable functions $\varphi_1, \ldots, \varphi_{k+1} \colon \mathbb{R}\times\Sigma \to \mathbb{C}$ satisfying $\varphi_i(\cdot,\sigma) \in BC(\mathbb{R})$ for all $i \in \{1,\ldots,k+1\}$ and $\sigma \in \Sigma$,*

$$\begin{aligned}
&\int_{\Sigma} \|\varphi_1(\cdot,\sigma)\|_{\ell^\infty(\mathbb{R})}\cdots\|\varphi_{k+1}(\cdot,\sigma)\|_{\ell^\infty(\mathbb{R})}\,\rho(\mathrm{d}\sigma) \le a_k\big(\|f^{(k)}\|_{\ell^\infty(\mathbb{R})} + \|f\|_{\dot{B}_1^{k,\infty}}\big) < \infty, \quad \textit{and} \\
&f^{[k]}(\boldsymbol{\lambda}) = \int_{\Sigma} \varphi_1(\lambda_1,\sigma)\cdots\varphi_{k+1}(\lambda_{k+1},\sigma)\,\rho(\mathrm{d}\sigma), \qquad \boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_{k+1}) \in \mathbb{R}^{k+1}.
\end{aligned}$$

In particular, by Theorem 3.6.4(iii)–(iv), there exists a constant $c_k < \infty$ such that

$$\int_\Sigma \|\varphi_1(\cdot,\sigma)\|_{\ell^\infty(\mathbb{R})} \cdots \|\varphi_{k+1}(\cdot,\sigma)\|_{\ell^\infty(\mathbb{R})}\, \rho(\mathrm{d}\sigma) \le c_k \|f\|_{B_1^{k,\infty}}$$

whenever $f \in B_1^{k,\infty}(\mathbb{R})$ as well.

Remark 3.6.8. Since $\varphi_i(\cdot,\sigma) \in C(\mathbb{R})$, we have that $\|\varphi_i(\cdot,\sigma)\|_{\ell^\infty(\mathbb{R})} = \sup_{\lambda\in\mathbb{Q}} |\varphi_i(\lambda,\sigma)|$ for all $\sigma \in \Sigma$ and $i \in \{1,\ldots,k+1\}$. Consequently, all the integrals in the theorem above make sense.

A slightly stronger form of this result is [Pel06, Thm. 5.5] or [Pel16, Thm. 2.2.1]. We state this stronger form as Theorem 6.6.9 below and provide a detailed and mostly self-contained proof in Appendix B. With this result in hand, we now begin the proof that $\dot{B}_1^{k,\infty}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$.

Lemma 3.6.9 (Inhomogeneous Littlewood–Paley decomposition)**.** *Fix $s \in \mathbb{R}$, $p \in [1,\infty]$, and $q \in [1,\infty)$. If $f \in \dot{B}_q^{s,p}(\mathbb{R}^m)$, then*

$$(f_n)_{n\in\mathbb{N}} \coloneqq \left(\check{\eta} * f + \sum_{i=1}^n \check{\eta}_i * f\right)_{n\in\mathbb{N}}$$

is the ***inhomogeneous Littlewood–Paley sequence*** *of f. In this case, $f - f_n \in B_q^{s,p}(\mathbb{R}^m)$ for all $n \in \mathbb{N}$, and $\|f - f_n\|_{B_q^{s,p}} \to 0$ as $n \to \infty$.*

Proof. If $n \in \mathbb{N}$, then $\eta + \sum_{i=1}^n \eta_i = \eta(2^{-n}\cdot)$ by definition, so that

$$f_n = \left(\check{\eta} + \sum_{i=1}^n \check{\eta}_i\right) * f = \widecheck{\eta(2^{-n}\cdot)} * f.$$

Since $\eta(2^{-n}\cdot) \equiv 1$ on $\{\xi \in \mathbb{R}^m : |\xi|_2 \le 2^n\}$, we have that if $i \in \{1,\ldots,n-1\}$, then

$$\check{\eta}_i * f_n = \check{\eta}_i * f \;\text{ and }\; \check{\eta} * f_n = \check{\eta} * f,$$

as can be seen by taking Fourier transforms of both sides and using the fact that η_i is supported in the annulus $\left\{\xi \in \mathbb{R}^m : 2^{i-1} \le |\xi|_2 \le 2^{i+1}\right\}$. Next, note that

$$\widecheck{\eta(2^{-n}\cdot)} = (2^n)^m \check{\eta}(2^n\cdot) \implies \left\|\widecheck{\eta(2^{-n}\cdot)}\right\|_{L^1} = \|\check{\eta}\|_{L^1}.$$

Therefore, by Young's convolution inequality, $\|\chi * f_n\|_{L^p} \leq \|\check{\eta}\|_{L^1}\|\chi * f\|_{L^p}$ for all $\chi \in \mathscr{S}(\mathbb{R}^m)$. Applying this to $\chi = \check{\eta}_i$ and using the definition of $\|\cdot\|_{B_q^{s,p}}$, we get that

$$\|f - f_n\|_{B_q^{s,p}} = \left(\sum_{i=n}^{\infty} 2^{isq}\|\check{\eta}_i * (f - f_n)\|_{L^p}^q\right)^{\frac{1}{q}} \leq (1 + \|\check{\eta}\|_{L^1})\left(\sum_{i=n}^{\infty} 2^{isq}\|\check{\eta}_i * f\|_{L^p}^q\right)^{\frac{1}{q}} \xrightarrow{n\to\infty} 0$$

because $f \in \dot{B}_q^{s,p}(\mathbb{R}^m)$. $\square$

Theorem 3.6.10. *If $k \in \mathbb{N}$, then $\dot{B}_1^{k,\infty}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$. Moreover, if $f \in \dot{B}_1^{k,\infty}(\mathbb{R})$ and $(f_n)_{n\in\mathbb{N}}$ is the inhomogeneous Littlewood–Paley sequence of f, then $f_n \to f$ in $VC^k(\mathbb{R})$ as $n \to \infty$.*

Proof. First, by Theorems 3.6.7 and 3.3.7, if $i \in \{0, \ldots, k\}$, then

$$\sup_{r>0} \beta_{r,i+1}\big(f^{[i]}\big) \leq c_i\|f\|_{B_1^{i,\infty}} \leq c_i\|f\|_{B_1^{k,\infty}} < \infty, \qquad f \in B_1^{k,\infty}(\mathbb{R}). \tag{3.6.11}$$

(Actually, the $i = 0$ case comes from Theorem 3.6.4(iii).) Thus, $B_1^{k,\infty}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$. Next, fix $f \in \dot{B}_1^{k,\infty}(\mathbb{R})$, and let $(f_n)_{n\in\mathbb{N}}$ be the inhomogeneous Littlewood–Paley sequence of f. Note that if $n \in \mathbb{N}$, then f_n has a compactly supported Fourier transform. Therefore, by the Paley–Wiener theorem, $f_n \in C^\infty(\mathbb{R})$. In particular, $f_n \in VC^k(\mathbb{R})$ by Corollary 3.4.7. Now, by Lemma 3.6.9 and Inequality (3.6.11),

$$\sup_{r>0} \|f - f_n\|_{VC^k,r} \lesssim_k \|f - f_n\|_{B_1^{k,\infty}} \xrightarrow{n\to\infty} 0.$$

Thus, $f = f_n + (f - f_n) \in VC^k(\mathbb{R})$, and $f_n \to f$ in $VC^k(\mathbb{R})$ as $n \to \infty$. Since we already observed that $f_n \in VC^k(\mathbb{R})$ for all $n \in \mathbb{N}$, this completes the proof. $\square$

Remark 3.6.12. Via Corollary 3.4.7 (and Proposition 3.4.8), Theorem 3.6.10 demonstrates directly, i.e., without going through Theorem 3.4.12, that $\dot{B}_1^{k,\infty}(\mathbb{R})$ is contained in the closure of $W_k(\mathbb{R})$ (and thus of $\mathbb{C}[\lambda]$) in $VC^k(\mathbb{R})$.

We observe parenthetically that, by Example 3.6.6, the containment $\dot{B}_1^{k,\infty}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$ generalizes the containment $W_k(\mathbb{R}) \subseteq VC^k(\mathbb{R})$. It should be noted, however, that our proof of the former used the latter in a crucial way.

We end this section by defining the Hölder spaces, describing their relationship to the Besov spaces, and proving that $C_{\mathrm{loc}}^{k,\varepsilon}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$. For more information about Hölder spaces, please see [Fio16].

Definition 3.6.13 (Hölder spaces)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. If $\varphi\colon X \to Y$ is a function and $\varepsilon > 0$, then

$$[\varphi]_{C^{0,\varepsilon}(X;Y)} := \sup\left\{\frac{d_Y(\varphi(x),\varphi(y))}{d_X(x,y)^\varepsilon} : x, y \in X,\ x \neq y\right\} \in [0,\infty].$$

If $[\varphi]_{C^{0,\varepsilon}(X;Y)} < \infty$, then φ is $\boldsymbol{\varepsilon}$**-Hölder continuous**, written $\varphi \in C^{0,\varepsilon}(X;Y)$. As usual, we omit Y from the notation when $Y = \mathbb{C}$. Next, if $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$, then

$$[\varphi]_{C^{k,\varepsilon}} := \left(\sum_{|\alpha|=k} \left[\partial^\alpha \varphi\right]_{C^{0,\varepsilon}(\mathbb{R}^m)}^2\right)^{\frac{1}{2}}, \qquad \varphi \in C^k(\mathbb{R}^m),$$

and $C^{k,\varepsilon}(\mathbb{R}^m)$ is the set of $\varphi \in C^k(\mathbb{R}^m)$ such that $[\varphi]_{C^{k,\varepsilon}} < \infty$. Also,

$$\|\varphi\|_{BC^{k,\varepsilon}} := \|\varphi\|_{BC^k} + [\varphi]_{C^{k,\varepsilon}}, \qquad \varphi \in C^k(\mathbb{R}^m),$$

and $BC^{k,\varepsilon}(\mathbb{R}^m)$ is the set of $\varphi \in BC^k(\mathbb{R}^m)$ such that $[\varphi]_{C^{k,\varepsilon}} < \infty$. Finally, $C_{\mathrm{loc}}^{k,\varepsilon}(\mathbb{R}^m)$ is the set of $\varphi \in C^k(\mathbb{R}^m)$ such that $\partial^\alpha\varphi|_{[-r,r]^m} \in C^{0,\varepsilon}([-r,r]^m)$ for all $r > 0$ and $\alpha \in \mathbb{N}_0^m$ with $|\alpha| = k$.

If $\varepsilon > 1$ and $\varphi \in C_{\mathrm{loc}}^{0,\varepsilon}(\mathbb{R}^m)$, then φ is constant. In particular, if $\varepsilon > 1$, $k \in \mathbb{N}$, and $\varphi \in C_{\mathrm{loc}}^{k,\varepsilon}(\mathbb{R}^m)$, then $\varphi \in \mathbb{C}[\lambda_1, \ldots, \lambda_m]$. Also, the use of the ℓ^2-norm (as opposed to the ℓ^1-norm) in the definition of $[\cdot]_{C^{k,\varepsilon}}$ is atypical. We made this choice so that the proof of the following proposition is more pleasant—specifically, so that Inequality (3.6.15) below holds.

Proposition 3.6.14. *If $k \in \mathbb{N}_0$ and $0 < \varepsilon \leq 1$, then $BC^{k,\varepsilon}(\mathbb{R}^m) \hookrightarrow B_\infty^{k+\varepsilon,\infty}(\mathbb{R}^m)$.*

Proof. Suppose $k \in \mathbb{N}_0$ and $0 < \varepsilon \leq 1$, and define

$$[\varphi]_{C^{k,\varepsilon}} := \left(\sum_{i=1}^n [\varphi_i]_{C^{k,\varepsilon}}^2\right)^{\frac{1}{2}}, \qquad \varphi = (\varphi_1, \ldots, \varphi_n) \in C^k(\mathbb{R}^m; \mathbb{C}^n).$$

We claim that if $\varphi \in C^k(\mathbb{R}^m;\mathbb{C}^n)$, then

$$\sup_{x\in\mathbb{R}^m} \left|\Delta_h^{k+1}\varphi(x)\right|_2 \leq [\varphi]_{C^{k,\varepsilon}}|h|_2^{k+\varepsilon}, \qquad h \in \mathbb{R}^m \setminus \{0\}. \tag{3.6.15}$$

First, observe that if $\varphi \in C^k(\mathbb{R}^m;\mathbb{C}^n)$ and $\nabla\varphi \coloneqq (\partial_j\varphi_i)_{1\leq i\leq n, 1\leq j\leq m} \in C^{k-1}(\mathbb{R}^m;\mathbb{C}^{n\times m})$, then $[\nabla\varphi]_{C^{k-1,\varepsilon}} = [\varphi]_{C^{k,\varepsilon}}$. Now, we prove Inequality (3.6.15) by induction. If $k = 0$, then it is immediate from the definition. Now, assume the desired result holds when $k_0 \geq 0$, and let $k \coloneqq k_0 + 1$. If $\varphi \in C^k(\mathbb{R}^m;\mathbb{C}^n)$ and $x, h \in \mathbb{R}^m$, then, by the fundamental theorem of calculus,

$$\Delta_h^{k+1}\varphi(x) = \Delta_h^k\varphi(x+h) - \Delta_h^k\varphi(x) = \int_0^1 \nabla\big(\Delta_h^k\varphi\big)(x+th)h\,\mathrm{d}t = \int_0^1 \Delta_h^k(\nabla\varphi)(x+th)h\,\mathrm{d}t,$$

where the juxtapositions $\nabla\big(\Delta_h^k\varphi\big)(x+th)h$ and $\Delta_h^k(\nabla\varphi)(x+th)h$ above are matrix multiplications. It then follows from the induction hypothesis, the Cauchy–Schwarz inequality, and our initial observation that

$$\left|\Delta_h^{k+1}\varphi(x)\right|_2 \leq \int_0^1 \left|\Delta_h^k(\nabla\varphi)(x+th)\right|_2 |h|_2\,\mathrm{d}t \leq [\nabla\varphi]_{C^{k-1,\varepsilon}}|h|_2^{k-1+\varepsilon}|h|_2 = [\varphi]_{C^{k,\varepsilon}}|h|_2^{k+\varepsilon},$$

as desired. Next, suppose $0 < \varepsilon < 1$. Then $\lfloor k+\varepsilon\rfloor = k$, so Inequality (3.6.15) gives

$$\|f\|_{\mathrm{cl},B_\infty^{k+\varepsilon,\infty}} = \|f\|_{L^\infty} + \sup_{h\neq 0}|h|_2^{-k-\varepsilon}\left\|\Delta_h^{k+1}f\right\|_{L^\infty} \leq \|f\|_{L^\infty} + [f]_{C^{k,\varepsilon}} < \infty, \qquad f \in BC^{k,\varepsilon}(\mathbb{R}^m).$$

Now, if $\varepsilon = 1$, then $\lfloor k+\varepsilon\rfloor = k+1$. Combining Inequality (3.6.15) with the obvious fact that $\left\|\Delta_h^{k+2}f\right\|_{L^\infty} \leq 2\left\|\Delta_h^{k+1}f\right\|_{L^\infty}$ then gives

$$\|f\|_{\mathrm{cl},B_\infty^{k+1,\infty}} \leq \|f\|_{L^\infty} + 2\sup_{h\neq 0}|h|_2^{-k-1}\left\|\Delta_h^{k+1}f\right\|_{L^\infty} \leq \|f\|_{L^\infty} + 2[f]_{C^{k,1}} < \infty, \qquad f \in BC^{k,1}(\mathbb{R}^m).$$

An appeal to Theorem 3.6.4(v) completes the proof. □

Remark 3.6.16. In fact, $BC^{k,\varepsilon}(\mathbb{R}^m) = B_\infty^{k+\varepsilon,\infty}(\mathbb{R}^m)$ whenever $0 < \varepsilon < 1$ and $k \in \mathbb{N}_0$. In general, $B_\infty^{s,\infty}(\mathbb{R}^m)$ is the Hölder–Zygmund space $\mathcal{C}^s(\mathbb{R}^m)$ whenever $s > 0$. For more information, please see [Tri83, §2.2.2, §2.3.5, §2.5.7, & §2.5.12], [Tri92, §1.2.2, §1.5.1, & §2.6.5], or [Saw18, §2.2.2].

As a consequence, we obtain the inclusion $C^{k,\varepsilon}_{\mathrm{loc}}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$.

Theorem 3.6.17. *If $k \in \mathbb{N}$ and $\varepsilon > 0$, then $C^{k,\varepsilon}_{\mathrm{loc}}(\mathbb{R}) \subseteq VC^k(\mathbb{R})$.*

Proof. Fix $\varepsilon, r > 0$, $k \in \mathbb{N}$, and $f \in C^{k,\varepsilon}_{\mathrm{loc}}(\mathbb{R})$. If $\varepsilon > 1$, then $f(\lambda) \in \mathbb{C}[\lambda] \subseteq VC^k(\mathbb{R})$, so we assume $0 < \varepsilon \leq 1$. Now, if $\psi_r \in C_c^\infty(\mathbb{R})$ is such that $\psi_r \equiv 1$ on $[-r, r]$, then

$$\psi_r f \in BC^{k,\varepsilon}(\mathbb{R}) \subseteq B^{k+\varepsilon,\infty}_\infty(\mathbb{R}) \subseteq B^{k,\infty}_1(\mathbb{R}) \subseteq \dot{B}^{k,\infty}_1(\mathbb{R})$$

by Proposition 3.6.14 and Theorem 3.6.4(ii),(iv). Since $(\psi_r f)|_{[-r,r]} = f|_{[-r,r]}$ and $r > 0$ was arbitrary, we get

$$f \in B^{k,\infty}_1(\mathbb{R})_{\mathrm{loc}} \subseteq \dot{B}^{k,\infty}_1(\mathbb{R})_{\mathrm{loc}} \subseteq \overline{\dot{B}^{k,\infty}_1(\mathbb{R})} \subseteq VC^k(\mathbb{R})$$

from Theorem 3.6.10 and Proposition 3.4.4(ii). $\square$

3.7 Demonstration that $W_k(\mathbb{R})_{\mathrm{loc}} \subsetneq VC^k(\mathbb{R})$

The formula (1.3.16) for the divided differences of a function in $W_k(\mathbb{R})$ is quite easy to work with, so it is reasonable to ask whether all examples of interest can be dealt with by "localizing" $W_k(\mathbb{R})$. We already saw (Theorem 3.4.12) that $W_k(\mathbb{R})$ is dense in $VC^k(\mathbb{R})$, but what we are really asking is whether the stronger statement $W_k(\mathbb{R})_{\mathrm{loc}} = VC^k(\mathbb{R})$ holds as well. The goal of this section is to prove that this is not the case.

Theorem 3.7.1. *If $k \in \mathbb{N}$, then $W_k(\mathbb{R})_{\mathrm{loc}} \subsetneq VC^k(\mathbb{R})$. Specifically, we have the following counterexample. Fix $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on $[-1, 1]$ and $\operatorname{supp}\psi \subseteq [-2, 2]$, and define*

$$\kappa(x) := 1_{(0,\infty)}(x)\, \psi(x) \sqrt{x}\, e^{-\frac{i}{x}}, \qquad x \in \mathbb{R}.$$

If $f \in C^k(\mathbb{R})$ and $f^{(k)} = \kappa$, then $f \in C^{k,1/4}(\mathbb{R}) \setminus W_k(\mathbb{R})_{\mathrm{loc}} \subseteq VC^k(\mathbb{R}) \setminus W_k(\mathbb{R})_{\mathrm{loc}}$.

We break the proof into a few lemmas.

Lemma 3.7.2. $W_k(\mathbb{R})_{\mathrm{loc}} = \big\{ f \in C^k(\mathbb{R}) : \eta f \in W_k(\mathbb{R})$ *for all* $\eta \in C_c^\infty(\mathbb{R}) \big\}$.

Proof. We first observe that if $f = \int_{\mathbb{R}} e^{i\cdot\xi}\,\mu(\mathrm{d}\xi) \in W_k(\mathbb{R})$ and $\eta \in C_c^\infty(\mathbb{R})$, then $\eta f \in W_k(\mathbb{R})$. Indeed, note that

$$\mathcal{F}(\eta f)(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\eta(x) f(x)\,\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-ix(\xi-y)}\eta(x)\,\mu(\mathrm{d}y)\,\mathrm{d}x = \int_{\mathbb{R}} \widehat{\eta}(\xi - y)\,\mu(\mathrm{d}y), \qquad \xi \in \mathbb{R},$$

by Fubini's theorem. Consequently, if $i \in \{0, \ldots, k\}$, then

$$\begin{aligned}\int_{\mathbb{R}} |\xi|^i |\mathcal{F}(\eta f)(\xi)|\,\mathrm{d}\xi &\le \int_{\mathbb{R}}\int_{\mathbb{R}} |\xi|^i |\widehat{\eta}(\xi - y)|\,|\mu|(\mathrm{d}y)\,\mathrm{d}\xi = \int_{\mathbb{R}}\int_{\mathbb{R}} |\xi|^i |\widehat{\eta}(\xi - y)|\,\mathrm{d}\xi\,|\mu|(\mathrm{d}y) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} |\zeta + y|^i |\widehat{\eta}(\zeta)|\,\mathrm{d}\zeta\,|\mu|(\mathrm{d}y) \le 2^{i-1}\int_{\mathbb{R}}\int_{\mathbb{R}} (|\zeta|^i + |y|^i)\,|\widehat{\eta}(\zeta)|\,\mathrm{d}\zeta\,|\mu|(\mathrm{d}y) \\ &= 2^{i-1}\big(\mu_{(0)}\,\big\|\mathcal{F}\big(\eta^{(i)}\big)\big\|_{L^1} + \mu_{(i)}\,\|\widehat{\eta}\|_{L^1}\big) < \infty\end{aligned}$$

by Tonelli's theorem. It follows from Proposition 3.4.6(i) that $\eta f \in W_k(\mathbb{R})$.

Next, fix $f \in W_k(\mathbb{R})_{\mathrm{loc}}$ and $\eta \in C_c^\infty(\mathbb{R})$. Suppose $\operatorname{supp}\eta \subseteq [-r, r]$. By definition of $W_k(\mathbb{R})_{\mathrm{loc}}$, there exists a $g \in W_k(\mathbb{R})$ such that $g|_{[-r,r]} = f|_{[-r,r]}$. But then $\eta f = \eta g \in W_k(\mathbb{R})$ by the previous paragraph. This proves $W_k(\mathbb{R})_{\mathrm{loc}} \subseteq \big\{f \in C^k(\mathbb{R}) : \eta f \in W_k(\mathbb{R}) \text{ for all } \eta \in C_c^\infty(\mathbb{R})\big\}$.

Finally, suppose $f \in C^k(\mathbb{R})$ is such that $\eta f \in W_k(\mathbb{R})$ for all $\eta \in C_c^\infty(\mathbb{R})$. For $r > 0$, let $\eta \in C_c^\infty(\mathbb{R})$ be such that $\eta \equiv 1$ on $[-r, r]$. Taking $g \coloneqq \eta f \in W_k(\mathbb{R})$, we have that $g|_{[-r,r]} = (\eta f)|_{[-r,r]} = f|_{[-r,r]}$. We conclude that $f \in W_k(\mathbb{R})_{\mathrm{loc}}$, which completes the proof. □

Lemma 3.7.3. *Fix $g \in C_c(\mathbb{R})$ and $h \in C^k(\mathbb{R})$ such that $h^{(k)} = g$. Then*

$$h \in W_k(\mathbb{R})_{\mathrm{loc}} \iff \widehat{g} \in L^1(\mathbb{R}).$$

Proof. Let $\eta \in C_c^\infty(\mathbb{R})$ be arbitrary. Then $f_\eta \coloneqq \eta h \in C_c^k(\mathbb{R})$. In particular, by Proposition 3.4.6(ii), $\widehat{f_\eta} \in L^1(\mathbb{R})$. Now, by the product rule,

$$f_\eta^{(k)} = \sum_{i=0}^k \binom{k}{i}\eta^{(i)}h^{(k-i)} = \eta\,h^{(k)} + \underbrace{\sum_{i=1}^k \binom{k}{i}\eta^{(i)}h^{(k-i)}}_{\coloneqq \chi} = \eta\,g + \chi.$$

Since no more than $k-1$ derivatives fall on h in the definition of χ, we have that $\chi \in C^1(\mathbb{R})$.

Since χ has compact support, Proposition 3.4.6(ii) yields that $\widehat{\chi} \in L^1(\mathbb{R})$. It then follows from Proposition 3.4.6(i) that

$$f_\eta = \eta h \in W_k(\mathbb{R}) \iff \mathcal{F}\big(f_\eta^{(k)}\big) \in L^1(\mathbb{R}) \iff \mathcal{F}(\eta g) \in L^1(\mathbb{R}).$$

We combine this observation with the characterization of $W_k(\mathbb{R})_{\mathrm{loc}}$ in Lemma 3.7.2 to finish the proof. Suppose $h \in W_k(\mathbb{R})_{\mathrm{loc}}$, and choose $\eta \in C_c^\infty(\mathbb{R})$ such that $\eta \equiv 1$ on $\operatorname{supp} g$. Then $\widehat{g} = \mathcal{F}(\eta g) \in L^1(\mathbb{R})$. Now, suppose $\widehat{g} \in L^1(\mathbb{R})$, and let $\eta \in C_c^\infty(\mathbb{R})$ be arbitrary. Then $\mathcal{F}(\eta g) = \frac{1}{2\pi}\widehat{\eta} * \widehat{g} \in L^1(\mathbb{R})$ because $\widehat{\eta} \in L^1(\mathbb{R})$ and $\widehat{g} \in L^1(\mathbb{R})$. Thus, $\eta h \in W_k(\mathbb{R})$. Since $\eta \in C_c^\infty(\mathbb{R})$ was arbitrary, we conclude that $h \in W_k(\mathbb{R})_{\mathrm{loc}}$. □

Lemma 3.7.4. *If $\kappa \in C_c(\mathbb{R})$ is as in Theorem 3.7.1, then $\widehat{\kappa} \notin L^1(\mathbb{R})$.*

Proof. Let $\xi > 0$. Then

$$\widehat{\kappa}(\xi) = \int_0^\infty e^{-i(x\xi + x^{-1})}\psi(x)\sqrt{x}\,\mathrm{d}x = \xi^{-\frac{3}{4}}\int_0^\infty e^{-i\sqrt{\xi}(y+y^{-1})}\psi\big(\xi^{-\frac{1}{2}}y\big)\sqrt{y}\,\mathrm{d}y \tag{3.7.5}$$

by the change of variable $y := \sqrt{\xi}\,x$. We use the method of stationary phase to analyze the oscillatory integral on the right-hand side of Equation (3.7.5). First, note that the phase $\phi(y) := y + y^{-1}$ $(y > 0)$ has a unique critical ("stationary") point at $y = 1$, and this critical point is non-degenerate because $\phi''(1) = 2 \neq 0$. Next, let $\chi \in C_c^\infty(\mathbb{R})$ be such that $\chi \equiv 1$ on $[3/4, 3/2]$ and $\operatorname{supp}\chi \subseteq [1/2, 2]$. Then

$$I_1(\zeta) := \int_0^\infty e^{-i\zeta\phi(y)}\chi(y)\,\psi\big(\zeta^{-1}y\big)\sqrt{y}\,\mathrm{d}y = \int_0^\infty e^{-i\zeta\phi(y)}\chi(y)\sqrt{y}\,\mathrm{d}y, \qquad \zeta \geq 2,$$

because $\psi \equiv 1$ on $[0,1]$. Therefore, by [Hör83, Thm. 7.7.5 & Eq. 3.4.6],

$$\begin{aligned} I_1(\zeta) &= \chi(1)\sqrt{1}\,e^{-i\zeta\phi(1) - i\,\mathrm{sgn}(\phi''(1))\frac{\pi}{4}}\sqrt{\frac{2\pi}{\zeta|\phi''(1)|}} + O(\zeta^{-1}) \\ &= \sqrt{\pi}\,e^{-i(2\zeta + \frac{\pi}{4})}\zeta^{-\frac{1}{2}} + O(\zeta^{-1}) \ \text{ as } \ \zeta \to \infty. \end{aligned} \tag{3.7.6}$$

Second, note that $\phi'(y) \neq 0$ whenever $0 < y \in \operatorname{supp}(1-\chi)$. One can therefore apply the "method of nonstationary phase" (integration by parts) to prove that

$$I_2(\zeta) := \int_0^\infty e^{-i\zeta\phi(y)}(1-\chi(y))\,\psi\big(\zeta^{-1}y\big)\sqrt{y}\,\mathrm{d}y = O\big(\zeta^{-1}\big) \;\text{ as }\; \zeta\to\infty. \tag{3.7.7}$$

Due to the singularities of ϕ and the square root function at zero, standard theorems do not apply directly, so we need to prove this by hand. The calculations necessary to do so are elementary but rather tedious, so we relegate them to the end of the section. In the end, combining Equations (3.7.5)–(3.7.7) gives

$$\widehat{\kappa}(\xi) = \xi^{-\frac{3}{4}}\big(I_1\big(\xi^{\frac{1}{2}}\big) + I_2\big(\xi^{\frac{1}{2}}\big)\big) = \sqrt{\pi}\,e^{-i(2\sqrt{\xi}+\frac{\pi}{4})}\xi^{-1} + O\big(\xi^{-\frac{5}{4}}\big) \;\text{ as }\; \xi\to\infty.$$

It follows that $\widehat{\kappa}\notin L^1(\mathbb{R})$, as claimed. □

Proof of Theorem 3.7.1. It is an elementary exercise to show that $\kappa\in C^{0,1/4}(\mathbb{R})$. (For instance, one can adapt the argument from [Fio16, Ex. 1.1.8].) In particular, if $f\in C^k(\mathbb{R})$ and $f^{(k)}=\kappa$, then $f\in C^{k,1/4}(\mathbb{R})$. Thus, $f\in VC^k(\mathbb{R})$ by Theorem 3.6.17. But $f\notin W_k(\mathbb{R})_{\mathrm{loc}}$ by Lemmas 3.7.3 and 3.7.4. □

The above development provides a recipe for constructing functions in $VC^k(\mathbb{R})\setminus W_k(\mathbb{R})_{\mathrm{loc}}$. Indeed, any compactly supported $g\in C^{0,\varepsilon}(\mathbb{R})$ with $\widehat{g}\notin L^1(\mathbb{R})$ can be used to produce a function in $VC^k(\mathbb{R})\setminus W_k(\mathbb{R})_{\mathrm{loc}}$ via Lemma 3.7.3; J. Sterbenz suggested $g=\kappa$ as an example. (In general, for such a g to exist, one must have $\varepsilon\le 1/2$. This can be proven using Remark 3.6.5 and an argument like the one in the proof of Proposition 3.4.6(ii).)

Proof of Equation (3.7.7). Fix $\psi,\chi\in C_c^\infty(\mathbb{R})$ such that $\psi\equiv 1$ on $[-1,1]$, $\operatorname{supp}\psi\subseteq[-2,2]$, $\chi\equiv 1$ on $[3/4,3/2]$, and $\operatorname{supp}\chi\subseteq[1/2,2]$. Define

$$\phi(y) := y + y^{-1} \;\text{ and }\; g_\zeta(y) := (1-\chi(y))\,\psi\big(\zeta^{-1}y\big) \qquad y,\zeta>0.$$

We aim to show that

$$I_2(\zeta) := \int_0^\infty y^{\frac{1}{2}}\,g_\zeta(y)\,e^{-i\zeta\phi(y)}\,\mathrm{d}y = O\big(\zeta^{-1}\big) \;\text{ as }\; \zeta\to\infty.$$

To do so, we shall need to integrate by parts three times. We record a few derivatives for this purpose. First, $\phi'(y) = 1 - y^{-2} = y^{-2}(y^2-1)$ and $\phi''(y) = 2y^{-3}$. Second,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y}\big(y^{\frac12} g_\zeta(y)\big) &= \frac12 y^{-\frac12} g_\zeta(y) + y^{\frac12} g_\zeta'(y),\\
\frac{\mathrm{d}^2}{\mathrm{d}y^2}\big(y^{\frac12} g_\zeta(y)\big) &= -\frac14 y^{-\frac32} g_\zeta(y) + y^{-\frac12} g_\zeta'(y) + y^{\frac12} g_\zeta''(y), \quad \text{and}\\
\frac{\mathrm{d}^3}{\mathrm{d}y^3}\big(y^{\frac12} g_\zeta(y)\big) &= \frac38 y^{-\frac52} g_\zeta(y) - \frac34 y^{-\frac32} g_\zeta'(y) + \frac32 y^{-\frac12} g_\zeta''(y) + y^{\frac12}\, g_\zeta'''(y).
\end{aligned}$$

Recall now that $\phi'(y) \neq 0$ for $y \in \operatorname{supp} g_\zeta$ (since $g_\zeta \equiv 0$ near 1), and note that

$$\sup_{\zeta \geq 1} \big\| g_\zeta^{(k)} \big\|_{\ell^\infty(\mathbb{R})} < \infty, \qquad k \in \mathbb{N}_0.$$

Therefore, as $\zeta \to \infty$, we have

$$\begin{aligned}
I_2(\zeta) &= \int_0^\infty y^{\frac12}\, g_\zeta(y)\, e^{-i\zeta\phi(y)}\,\mathrm{d}y = \frac{1}{i\zeta}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y^{\frac12}\, g_\zeta(y)}{\phi'(y)}\right) e^{-i\zeta\phi(y)}\,\mathrm{d}y\\
&= \frac{1}{i\zeta}\int_0^\infty \left(\frac{-\phi''(y)}{\phi'(y)^2} y^{\frac12}\, g_\zeta(y) + \frac{1}{\phi'(y)}\frac{\mathrm{d}}{\mathrm{d}y}\big(y^{\frac12}\, g_\zeta(y)\big)\right) e^{-i\zeta\phi(y)}\,\mathrm{d}y\\
&= -\frac{2}{i\zeta}\int_0^\infty \frac{g_\zeta(y)\, y^{\frac32}}{(y^2-1)^2}\, e^{-i\zeta\phi(y)}\,\mathrm{d}y + \frac{1}{i\zeta}\int_0^\infty \frac{1}{\phi'(y)}\frac{\mathrm{d}}{\mathrm{d}y}\big(y^{\frac12}\, g_\zeta(y)\big)\, e^{-i\zeta\phi(y)}\,\mathrm{d}y\\
&= O(\zeta^{-1}) - \frac{1}{\zeta^2}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{\phi'(y)^2}\frac{\mathrm{d}}{\mathrm{d}y}\big(y^{\frac12}\, g_\zeta(y)\big)\right) e^{-i\zeta\phi(y)}\,\mathrm{d}y\\
&= O(\zeta^{-1}) - \frac{1}{\zeta^2}\int_0^\infty \left(\frac{-2\phi''(y)}{\phi'(y)^3}\frac{\mathrm{d}}{\mathrm{d}y}\big(y^{\frac12}\, g_\zeta(y)\big) + \frac{1}{\phi'(y)^2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\big(y^{\frac12}\, g_\zeta(y)\big)\right) e^{-i\zeta\phi(y)}\,\mathrm{d}y\\
&= O(\zeta^{-1}) + \frac{2}{\zeta^2}\int_0^\infty \frac{y^{\frac52} g_\zeta(y) + 2y^{\frac72} g_\zeta'(y)}{(y^2-1)^3}\, e^{-i\zeta\phi(y)}\,\mathrm{d}y - \frac{1}{\zeta^2}\int_0^\infty \frac{1}{\phi'(y)^2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\big(y^{\frac12}\, g_\zeta(y)\big)\, e^{-i\zeta\phi(y)}\,\mathrm{d}y\\
&= O(\zeta^{-1}) + O(\zeta^{-2}) - \frac{1}{i\zeta^3}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{\phi'(y)^3}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\big(y^{\frac12}\, g_\zeta(y)\big)\right) e^{-i\zeta\phi(y)}\,\mathrm{d}y\\
&= O(\zeta^{-1}) - \frac{1}{i\zeta^3}\int_0^\infty \left(\frac{-3\phi''(y)}{\phi'(y)^4}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\big(y^{\frac12}\, g_\zeta(y)\big) + \frac{1}{\phi'(y)^3}\frac{\mathrm{d}^3}{\mathrm{d}y^3}\big(y^{\frac12}\, g_\zeta(y)\big)\right) e^{-i\zeta\phi(y)}\,\mathrm{d}y\\
&= O(\zeta^{-1}) - \frac{1}{i\zeta^3}\int_0^\infty \left(\frac{\frac32 y^{\frac72} g_\zeta(y) - 6y^{\frac92} g_\zeta'(y) - 6y^{\frac{11}{2}} g_\zeta''(y)}{(y^2-1)^4} + \frac{1}{\phi'(y)^3}\frac{\mathrm{d}^3}{\mathrm{d}y^3}\big(y^{\frac12}\, g_\zeta(y)\big)\right) e^{-i\zeta\phi(y)}\,\mathrm{d}y\\
&= O(\zeta^{-1}) + O(\zeta^{-3}) - \frac{1}{i\zeta^3}\int_0^\infty \frac{1}{\phi'(y)^3}\frac{\mathrm{d}^3}{\mathrm{d}y^3}\big(y^{\frac12}\, g_\zeta(y)\big)\, e^{-i\zeta\phi(y)}\,\mathrm{d}y.
\end{aligned}$$

(We leave it to the reader to confirm that there are no boundary terms at zero.) But

$$\frac{1}{\phi'(y)^3}\frac{\mathrm{d}^3}{\mathrm{d}y^3}\big(y^{\frac{1}{2}}\,g_\zeta(y)\big) = \frac{y^6}{(y^2-1)^3}\left(\frac{3}{8}y^{-\frac{5}{2}}g_\zeta(y) - \frac{3}{4}y^{-\frac{3}{2}}g_\zeta'(y) + \frac{3}{2}y^{-\frac{1}{2}}g_\zeta''(y) + y^{\frac{1}{2}}\,g_\zeta'''(y)\right),$$

and $g_\zeta(y) = 0$ whenever $y \geq 2\zeta$. It follows, due to the dominant $y^{1/2}$ term, that

$$\int_0^\infty \frac{1}{\phi'(y)^3}\frac{\mathrm{d}^3}{\mathrm{d}y^3}\big(y^{\frac{1}{2}}\,g_\zeta(y)\big)\,e^{-i\zeta\phi(y)}\,\mathrm{d}y = O(\zeta^{\frac{3}{2}}) \;\text{ as }\; \zeta \to \infty.$$

Thus, $I_2(\zeta) = O(\zeta^{-1}) + \zeta^{-3}O(\zeta^{\frac{3}{2}}) = O(\zeta^{-1})$ as $\zeta \to \infty$, as desired. □

3.8 The space $NC^k(\mathbb{R})$

By Corollary 3.1.10, if $f \in VC^k(\mathbb{R})$, then $f_{\mathcal{A}} \in C^k(\mathcal{A}_{\mathrm{sa}};\mathcal{A})$, and the k^{th} derivative of $f_{\mathcal{A}}$ may be computed, via Proposition 3.5.3, in terms of local decompositions of $f^{[k]}$ as an integral (over $\sigma \in \Sigma$) of pure tenor functions $\varphi_1(\cdot,\sigma)\otimes\cdots\otimes\varphi_{k+1}(\cdot,\sigma)$, where $\varphi_i(\cdot,\sigma)$ is continuous. In this section, we show, loosely speaking, that we can take $\varphi_i(\cdot,\sigma)$ to be measurable. Doing so will require background material covered later, e.g., the (bounded) Borel functional calculus from Definition 4.2.13, and results from the theory of multiple operator integrals (MOIs) covered in Chapter 5; we provide "forward references" as necessary. To begin, we define ℓ^∞-projective tensor product $\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m})$, the idea for which is due to Peller [Pel06].

Lemma 3.8.1 (Measurability). *Let Ξ be a Polish space, and let $(\Sigma,\mathscr{H},\rho)$ be a σ-finite measure space. If $\varphi\colon \Xi\times\Sigma\to\mathbb{C}$ is product measurable, then the function $\Sigma \ni \sigma \mapsto \|\varphi(\cdot,\sigma)\|_{\ell^\infty(\Xi)} \in [0,\infty]$ is $\big(\overline{\mathscr{H}}^\rho,\mathcal{B}_{[0,\infty]}\big)$-measurable, where $\overline{\mathscr{H}}^\rho$ is the ρ-completion of $\mathscr{H}$.*

Proof. Since every σ-finite measure is equivalent to (i.e., has the same null sets as) a finite measure, we may assume ρ is finite. By [Cra02, Cor. 2.13], which relies on the measurable projection theorem [CV77, Thm. III.23], if $\varphi\colon \Xi\times\Sigma\to\mathbb{C}$ is product measurable and $C\colon\Sigma\to 2^\Xi$ is such that $\{(\sigma,\xi) : \sigma\in\Sigma,\, \xi\in C(\sigma)\} \in \mathscr{H}\otimes\mathcal{B}_\Xi$, then the function

$$\Sigma \ni \sigma \mapsto \sup_{\xi\in C(\sigma)} |\varphi(\xi,\sigma)| \in [0,\infty]$$

is $\big(\overline{\mathscr{H}}^\rho,\mathcal{B}_{[0,\infty]}\big)$-measurable. Applying this to the map $C \equiv \Xi$ yields the desired result. □

This measurability lemma ensures that the integral in Inequality (3.8.3) below makes sense as a Lebesgue integral with respect to the completion of ρ.

Definition 3.8.2 (IPTPs)**.** An **ℓ^∞-integral projective decomposition** (IPD) of a function $\varphi\colon \Omega \to \mathbb{C}$ is a choice $(\Sigma, \rho, \varphi_1, \ldots, \varphi_m)$ of a σ-finite measure space $(\Sigma, \mathscr{H}, \rho)$ and, for each $i \in \{1, \ldots, m\}$, a product measurable function $\varphi_i\colon \Omega_i \times \Sigma \to \mathbb{C}$ such that $\varphi_i(\cdot, \sigma) \in \ell^\infty(\Omega_i, \mathcal{B}_{\Omega_i})$ whenever $\sigma \in \Sigma$,

$$\int_\Sigma \|\varphi_1(\cdot,\sigma)\|_{\ell^\infty(\Omega_1)} \cdots \|\varphi_m(\cdot,\sigma)\|_{\ell^\infty(\Omega_m)}\, \rho(\mathrm{d}\sigma) < \infty, \quad \text{and} \tag{3.8.3}$$
$$\varphi(\boldsymbol{\omega}) = \int_\Sigma \varphi_1(\omega_1,\sigma)\cdots\varphi_m(\omega_m,\sigma)\,\rho(\mathrm{d}\sigma), \qquad \boldsymbol{\omega} \in \Omega.$$

Also, for any function $\varphi\colon \Omega \to \mathbb{C}$, define $\|\varphi\|_{\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m})}$ to be

$$\inf\left\{\int_\Sigma \prod_{i=1}^m \|\varphi_i(\cdot,\sigma)\|_{\ell^\infty(\Omega_i)}\,\rho(\mathrm{d}\sigma) : (\Sigma,\rho,\varphi_1,\ldots,\varphi_m) \text{ is a } \ell^\infty\text{-IPD of } \varphi\right\},$$

where $\inf\emptyset \coloneqq \infty$. Finally, define the **integral projective tensor product**

$$\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m}) \coloneqq \left\{\varphi \in \ell^\infty(\Omega,\mathcal{B}_\Omega) : \|\varphi\|_{\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m})} < \infty\right\}.$$

Proposition 3.8.4 (Properties of IPTPs)**.** *The following hold.*

(i) *If $\varphi\colon \Omega \to \mathbb{C}$ is a function, then $\|\varphi\|_{\ell^\infty(\Omega)} \leq \|\varphi\|_{\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m})}$.*

(ii) *$\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m}) \subseteq \ell^\infty(\Omega,\mathcal{B}_\Omega)$ is a unital $*$-subalgebra, and*

$$\left(\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m}), \|\cdot\|_{\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m})}\right)$$

is a unital Banach $*$-algebra under pointwise operations.

(iii) *Suppose $i, j \in \{1, \ldots, m\}$. If $\varphi\colon \Omega_1 \times \cdots \times \Omega_i \to \mathbb{C}$ and $\psi\colon \Omega_k \times \cdots \times \Omega_m \to \mathbb{C}$ are functions and $\chi(\boldsymbol{\omega}) \coloneqq \varphi(\omega_1,\ldots,\omega_i)\,\psi(\omega_j,\ldots,\omega_m)$ for all $\boldsymbol{\omega} \in \Omega$, then*

$$\|\chi\|_{\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m})} \leq \|\varphi\|_{\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_i,\mathcal{B}_{\Omega_i})}\|\psi\|_{\ell^\infty(\Omega_j,\mathcal{B}_{\Omega_j})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m})}.$$

Proof. By Example 5.5.7 below, the first and second items are special cases of Proposition 5.5.5, but we provide proofs from scratch for the reader's benefit. For ease of notation, write

$$(\mathscr{B}, \|\cdot\|_{\mathscr{B}}) \coloneqq \big(\ell^\infty(\Omega_1, \mathcal{B}_{\Omega_1})\hat{\otimes}_i \cdots \hat{\otimes}_i \ell^\infty(\Omega_m, \mathcal{B}_{\Omega_m}), \|\cdot\|_{\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m})}\big).$$

We take each item in turn.

(i) Let $(\Sigma, \rho, \varphi_1, \ldots, \varphi_m)$ be an ℓ^∞-IPD of φ. If $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m) \in \Omega$, then

$$|\varphi(\boldsymbol{\omega})| \leq \int_\Sigma |\varphi_1(\omega_1, \sigma) \cdots \varphi_m(\omega_m, \sigma)|\, \rho(\mathrm{d}\sigma) \leq \int_\Sigma \|\varphi_1(\cdot, \sigma)\|_{\ell^\infty(\Omega_1)} \cdots \|\varphi_m(\cdot, \sigma)\|_{\ell^\infty(\Omega_m)}\, \rho(\mathrm{d}\sigma).$$

Taking the supremum over $\boldsymbol{\omega} \in \Omega$ and then the infimum over ℓ^∞-IPDs $(\Sigma, \rho, \varphi_1, \ldots, \varphi_m)$ gives the desired inequality. Note that this inequality implies that $\|\varphi\|_{\mathscr{B}} = 0$ and only if $\varphi \equiv 0$ on Ω.

(ii) We leave it to the reader to prove $\|c\,\varphi\|_{\mathscr{B}} = |c|\, \|\varphi\|_{\mathscr{B}} = |c|\, \|\overline{\varphi}\|_{\mathscr{B}}$ whenever $c \in \mathbb{C}$ and $\varphi \in \mathscr{B}$. Next, suppose $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{B}$ such that $\sum_{n=1}^\infty \|\varphi_n\|_{\mathscr{B}} < \infty$. By the previous item, we have $\sum_{n=1}^\infty \|\varphi_n\|_{\ell^\infty(\Omega)} \leq \sum_{n=1}^\infty \|\varphi_n\|_{\mathscr{B}} < \infty$, so the series $\varphi \coloneqq \sum_{n=1}^\infty \varphi_n$ converges in $\ell^\infty(\Omega, \mathcal{B}_\Omega)$. We claim that $\|\varphi\|_{\mathscr{B}} \leq \sum_{n=1}^\infty \|\varphi_n\|_{\mathscr{B}}$. To see this, fix $\varepsilon > 0$ and $n \in \mathbb{N}$. By definition of $\|\cdot\|_{\mathscr{B}}$, there exists an ℓ^∞-IPD $(\Sigma_n, \rho_n, \varphi_{n,1}, \ldots, \varphi_{n,m})$ of φ_n such that

$$\int_{\Sigma_n} \|\varphi_{n,1}(\cdot, \sigma_n)\|_{\ell^\infty(\Omega_1)} \cdots \|\varphi_{n,m}(\cdot, \sigma_n)\|_{\ell^\infty(\Omega_m)}\, \rho_n(\mathrm{d}\sigma_n) < \|\varphi_n\|_{\mathscr{B}} + \frac{\varepsilon}{2^n}.$$

Define $(\Sigma, \mathscr{H}, \rho)$ to be the disjoint union of the measure spaces $\{(\Sigma_n, \mathscr{H}_n, \rho_n) : n \in \mathbb{N}\}$. Also, for $i \in \{1, \ldots, m\}$, define $\chi_i \colon \Omega_i \times \Sigma \to \mathbb{C}$ to be the unique measurable function satisfying $\chi_i|_{\Omega_i \times \Sigma_n} = \varphi_{n,i}$, for all $n \in \mathbb{N}$. It is easy to see that $(\Sigma, \rho, \chi_1, \ldots, \chi_m)$ is an ℓ^∞-IPD of φ, so that

$$\begin{aligned}
\|\varphi\|_{\mathscr{B}} &\leq \int_\Sigma \|\chi_1(\cdot, \sigma)\|_{\ell^\infty(\Omega_1)} \cdots \|\chi_m(\cdot, \sigma)\|_{\ell^\infty(\Omega_m)}\, \rho(\mathrm{d}\sigma) \\
&= \sum_{n=1}^\infty \int_{\Sigma_n} \|\varphi_{n,1}(\cdot, \sigma_n)\|_{\ell^\infty(\Omega_1)} \cdots \|\varphi_{n,m}(\cdot, \sigma_n)\|_{\ell^\infty(\Omega_m)}\, \rho_n(\mathrm{d}\sigma_n) \\
&\leq \sum_{n=1}^\infty \left(\|\varphi_n\|_{\mathscr{B}} + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^\infty \|\varphi_n\|_{\mathscr{B}} + \varepsilon.
\end{aligned}$$

Taking $\varepsilon \searrow 0$ results in the desired estimate. Taking $\varphi_n \equiv 0$ for $n \geq 3$, we conclude that $\mathscr{B}$ is closed under addition and that $\|\cdot\|_{\mathscr{B}}$ satisfies the triangle inequality. Applying the inequality we just proved to the sequence $(\varphi_{n+N})_{n\in\mathbb{N}}$ for fixed $N\in\mathbb{N}$ yields

$$\left\|\varphi-\sum_{n=1}^{N}\varphi_n\right\|_{\mathscr{B}}\leq\sum_{n=N+1}^{\infty}\|\varphi_n\|_{\mathscr{B}}\xrightarrow{N\to\infty}0.$$

Combining this with the observation from the end of the proof of the previous item, we conclude that $\mathscr{B}$ is a Banach space.

Finally, we prove that if $\varphi,\psi\in\mathscr{B}$, then $\|\varphi\,\psi\|_{\mathscr{B}}\leq\|\varphi\|_{\mathscr{B}}\|\psi\|_{\mathscr{B}}$. To do so, fix ℓ^∞-IPDs $(\Sigma_1,\rho_1,\varphi_1,\ldots,\varphi_m)$ and $(\Sigma_2,\rho_2,\psi_1,\ldots,\psi_m)$ of φ and ψ, respectively. Next, redefine $(\Sigma,\mathscr{H},\rho)$ to be the product $(\Sigma_1\times\Sigma_2,\mathscr{H}_1\otimes\mathscr{H}_2,\rho_1\otimes\rho_2)$ and $\chi_i(\omega_i,\sigma):=\varphi_i(\omega_i,\sigma_1)\,\psi_i(\omega_i,\sigma_2)$ whenever $i\in\{1,\ldots,m\}$, $\omega_i\in\Omega_i$, and $\sigma=(\sigma_1,\sigma_2)\in\Sigma$. We claim that $(\Sigma,\rho,\chi_1,\ldots,\chi_m)$ is an ℓ^∞-IPD of $\varphi\,\psi$. Indeed, by Tonelli's theorem,

$$\int_\Sigma\prod_{i=1}^{m}\|\chi_i(\cdot,\sigma)\|_{\ell^\infty(\Omega_i)}\,\rho(\mathrm{d}\sigma)\leq\int_{\Sigma_1}\prod_{i=1}^{m}\|\varphi_i(\cdot,\sigma_1)\|_{\ell^\infty(\Omega_i)}\,\rho_1(\mathrm{d}\sigma_1)\int_{\Sigma_2}\prod_{i=1}^{m}\|\psi_i(\cdot,\sigma_2)\|_{\ell^\infty(\Omega_i)}\,\rho_2(\mathrm{d}\sigma_2),$$

which is finite. Now, by Fubini's Theorem,

$$\int_\Sigma\prod_{i=1}^{m}\chi_i(\omega_i,\sigma)\,\rho(\mathrm{d}\sigma)=\int_{\Sigma_1}\prod_{i=1}^{m}\varphi_i(\omega_i,\sigma_1)\,\rho_1(\mathrm{d}\sigma_1)\int_{\Sigma_2}\prod_{i=1}^{m}\psi_i(\omega_i,\sigma_2)\,\rho_2(\mathrm{d}\sigma_2)=\varphi(\boldsymbol{\omega})\,\psi(\boldsymbol{\omega})$$

whenever $\boldsymbol{\omega}=(\omega_1,\ldots,\omega_m)\in\Omega$. It follows that

$$\|\varphi\,\psi\|_{\mathscr{B}}\leq\int_{\Sigma_1}\prod_{i=1}^{m}\|\varphi_i(\cdot,\sigma_1)\|_{\ell^\infty(\Omega_i)}\,\rho_1(\mathrm{d}\sigma_1)\int_{\Sigma_2}\prod_{i=1}^{m}\|\psi_i(\cdot,\sigma_2)\|_{\ell^\infty(\Omega_i)}\,\rho_2(\mathrm{d}\sigma_2).$$

Taking the infimum over ℓ^∞-IPDs of φ and ψ gives the desired result.

(iii) By the previous item, it suffices to consider the cases $\varphi\equiv 1$ and $\psi\equiv 1$. We leave these cases to the reader, as they are easy consequences of the definitions. □

Remark 3.8.5. Since $\ell^\infty(\Omega_i,\mathcal{B}_{\Omega_i})$ is a unital, commutative C^*-algebra, $\ell^\infty(\Omega_i,\mathcal{B}_{\Omega_i})\cong C(X_i)$ for some compact Hausdorff space X_i. In view of Theorem 3.3.7, one might hope that the ℓ^∞-integral projective tensor product $\ell^\infty(\Omega_1,\mathcal{B}_{\Omega_1})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_m,\mathcal{B}_{\Omega_m})$ is (isometrically) isomorphic to the Varopoulos algebra $V(X_1,\ldots,X_m)$. However, the spaces $X_1,\ldots,X_m$ are not generally metrizable, and the hypothesis of metrizability was used very strongly in the proof Theorem

3.3.7. Specifically, for a compact Hausdorff space X, the metrizability of X is equivalent to the separability of $C(X)$, and such separability was used in the proof of Lemma 3.3.6 to establish the strong measurability of a key map. This complication means that we cannot invoke the universal property of the projective tensor product when working with integral projective tensor products of ℓ^∞-spaces. This is precisely what makes proving results about MOIs, e.g., Theorem 3.8.15(ii) below, so difficult.

We shall work mainly with the case $\Omega_1 = \cdots = \Omega_m$ is a compact interval in $\mathbb{R}$, for which we use the following notation.

Notation 3.8.6. If $\varphi\colon \mathbb{R}^m \to \mathbb{C}$ is a function, then

$$\|\varphi\|_{r,m} := \big\|\varphi|_{[-r,r]^m}\big\|_{\ell^\infty([-r,r],\mathcal{B}_{[-r,r]})^{\hat{\otimes}_i m}} \in [0,\infty], \qquad r > 0.$$

Example 3.8.7. If $\varphi\colon \mathbb{R}^m \to \mathbb{C}$ is a function, then $\|\varphi\|_{r,m} \leq \beta_{r,m}(\varphi)$ for all $r > 0$. Consequently, if $\varphi \in VC(\mathbb{R}^m)$, then $\|\varphi\|_{r,m} < \infty$ for all $r > 0$.

Now, we introduce a new space of C^k functions containing $VC^k(\mathbb{R})$.

Notation 3.8.8 (The space $\mathcal{C}^{[k]}(\mathbb{R})$)**.** If $k \in \mathbb{N}$, $f \in C^k(\mathbb{R})$, and $r > 0$, then

$$\|f\|_{\mathcal{C}^{[k]},r} := \sum_{i=0}^{k} \Big\|f^{[i]}\Big\|_{r,i+1} \in [0,\infty] \ \text{ and } \mathcal{C}^{[k]}(\mathbb{R}) := \big\{g \in C^k(\mathbb{R}) : \|g\|_{\mathcal{C}^{[k]},s} < \infty \text{ for all } s > 0\big\}.$$

In other words, $\mathcal{C}^{[k]}(\mathbb{R})$ is the set of $f \in C^k(\mathbb{R})$ such that $f^{[i]} \in \ell^\infty\big([-r,r],\mathcal{B}_{[-r,r]}\big)^{\hat{\otimes}_i (i+1)}$ for all $i \in \{0,\ldots,k\}$. Also, let $\mathcal{C}^{[\infty]}(\mathbb{R}) := \bigcap_{k\in\mathbb{N}} \mathcal{C}^{[k]}(\mathbb{R})$.

Example 3.8.9. By Example 3.8.7, $VC^k(\mathbb{R}) \subseteq \mathcal{C}^{[k]}(\mathbb{R})$. In particular, all the examples of Varopoulos C^k functions from §3.4 and §3.6 belong to $\mathcal{C}^{[k]}(\mathbb{R})$.

By Proposition 3.8.4, $\mathcal{C}^{[k]}(\mathbb{R}) \subseteq C^k(\mathbb{R})$ is a linear subspace and $\{\|\cdot\|_{\mathcal{C}^{[k]},r} : r > 0\}$ is a collection of seminorms on $\mathcal{C}^{[k]}(\mathbb{R})$. Since these seminorms clearly separate points, they make $\mathcal{C}^{[k]}(\mathbb{R})$ into an HLCTVS. Similarly, $\mathcal{C}^{[\infty]}(\mathbb{R})$ is an HLCTVS with the topology induced by the family $\{\|\cdot\|_{\mathcal{C}^{[k]},r} : k \in \mathbb{N},\, r > 0\}$ of seminorms. Here now are the basic properties of the space $\mathcal{C}^{[k]}(\mathbb{R})$ $(k \in \mathbb{N} \cup \{\infty\})$. Recall that $\hookrightarrow$ indicates continuous inclusion.

Proposition 3.8.10 (Properties of $\mathcal{C}^{[k]}(\mathbb{R})$). *Let $k \in \mathbb{N} \cup \{\infty\}$.*

(i) $VC^k(\mathbb{R}) \hookrightarrow \mathcal{C}^{[k]}(\mathbb{R}) \hookrightarrow C^k(\mathbb{R})$.

(ii) *If $\mathcal{S} \subseteq \mathcal{C}^{[k]}(\mathbb{R})$, then $\mathcal{S}_{\text{loc}} \subseteq \overline{\mathcal{S}} \subseteq \mathcal{C}^{[k]}(\mathbb{R})$.*

(iii) *If $k < \infty$, $r > 0$, and $f, g \in C^k(\mathbb{R})$, then*

$$\left\|(fg)^{[k]}\right\|_{r,k+1} \leq \sum_{i=0}^{k} \left\|f^{[i]}\right\|_{r,i+1} \left\|g^{[k-i]}\right\|_{r,k-i+1} \quad \textit{and} \quad \|fg\|_{\mathcal{C}^{[k]},r} \leq \|f\|_{\mathcal{C}^{[k]},r} \|g\|_{\mathcal{C}^{[k]},r}.$$

(iv) *$\mathcal{C}^{[k]}(\mathbb{R})$ is a unital Fréchet $*$-algebra under pointwise operations.*

Proof. We address each item in turn.

(i) Since $\|\cdot\|_{\mathcal{C}^{[k]},r} \leq \|\cdot\|_{VC^k,r}$ for all $r > 0$, it is clear that $VC^k(\mathbb{R}) \hookrightarrow \mathcal{C}^{[k]}(\mathbb{R})$. Now, let $f \in C^k(\mathbb{R})$. If $0 \leq i < k+1$ and $r > 0$, then

$$\left\|f^{[i]}\right\|_{\ell^\infty([-r,r]^{i+1})} \leq \left\|f^{[i]}\right\|_{r,i+1}$$

by Proposition 3.8.4(i). Thus, $\mathcal{C}^{[k]}(\mathbb{R}) \hookrightarrow C^k(\mathbb{R})$.

(ii) Let $\mathcal{S} \subseteq \mathcal{C}^{[k]}(\mathbb{R})$. If $f \in \mathcal{S}_{\text{loc}}$ and $n \in \mathbb{N}$, then there exists a $g_n \in \mathcal{S} \subseteq \mathcal{C}^{[k]}(\mathbb{R})$ such that $g_n|_{[-n,n]} = f|_{[-n,n]}$. If $r > 0$, $n > r$, and $0 \leq i < k+1$, then $\|g_n - f\|_{\mathcal{C}^{[i]},r} = 0$. Thus, $f \in \mathcal{C}^{[k]}(\mathbb{R})$, and $g_n \to f$ in $\mathcal{C}^{[k]}(\mathbb{R})$ as $n \to \infty$. In other words, $\mathcal{S}_{\text{loc}} \subseteq \overline{\mathcal{S}} \subseteq \mathcal{C}^{[k]}(\mathbb{R})$.

(iii) The claimed bound on $\left\|(fg)^{[k]}\right\|_{r,k+1}$ follows immediately from Propositions 1.3.3(ii) and 3.8.4(iii). Consequently,

$$\begin{aligned}
\|fg\|_{\mathcal{C}^{[k]},r} &= \sum_{j=0}^{k} \left\|(fg)^{[j]}\right\|_{r,j+1} \leq \sum_{j=0}^{k} \sum_{i=0}^{j} \left\|f^{[i]}\right\|_{r,i+1} \left\|g^{[j-i]}\right\|_{r,j-i+1} \\
&= \sum_{i=0}^{k} \left\|f^{[i]}\right\|_{r,i+1} \sum_{j=i}^{k} \left\|g^{[j-i]}\right\|_{r,j-i+1} \leq \|f\|_{\mathcal{C}^{[k]},r} \|g\|_{\mathcal{C}^{[k]},r}
\end{aligned}$$

as well.

(iv) We prove that $\mathcal{C}^{[k]}(\mathbb{R})$ is a Fréchet $*$-algebra when $k < \infty$ and leave the $k = \infty$ case to the reader. First, the topology of $\mathcal{C}^{[k]}(\mathbb{R})$ is generated by the countable family $\{\|\cdot\|_{\mathcal{C}^{[k]},N} : N \in \mathbb{N}\}$ of seminorms, so $\mathcal{C}^{[k]}(\mathbb{R})$ is metrizable. Next, we prove that $\mathcal{C}^{[k]}(\mathbb{R})$ is complete. To this end, let

$(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}^{[k]}(\mathbb{R})$. By the first item, the sequence $(f_n)_{n\in\mathbb{N}}$ is also Cauchy in $C^k(\mathbb{R})$. Since the latter space is complete, there exists an $f \in C^k(\mathbb{R})$ such that $f_n \to f$ in the C^k topology as $n \to \infty$. In particular, if $i \in \{0,\ldots,k\}$, then $f_n^{[i]} \to f^{[i]}$ uniformly on compact sets as $n \to \infty$. Now, if $i \in \{0,\ldots,k\}$ and $r > 0$, then the sequence

$$\left(f_n^{[i]}\big|_{[-r,r]^{i+1}}\right)_{n\in\mathbb{N}}$$

is Cauchy and therefore, by Proposition 3.8.4(ii), convergent in $\ell^\infty\big([-r,r],\mathcal{B}_{[-r,r]}\big)^{\hat{\otimes}_i(i+1)}$. Since we already know that $f_n^{[i]} \to f^{[i]}$ pointwise as $n \to \infty$, we conclude that

$$f^{[i]}\big|_{[-r,r]^{i+1}} \in \ell^\infty\big([-r,r],\mathcal{B}_{[-r,r]}\big)^{\hat{\otimes}_i(i+1)}$$

and $f_n^{[i]}|_{[-r,r]^{i+1}} \to f^{[i]}|_{[-r,r]^{i+1}}$ in $\ell^\infty\big([-r,r],\mathcal{B}_{[-r,r]}\big)^{\hat{\otimes}_i(i+1)}$ as $n \to \infty$ as well. Thus, $f \in \mathcal{C}^{[k]}(\mathbb{R})$, and $f_n \to f$ in $\mathcal{C}^{[k]}(\mathbb{R})$ as $n \to \infty$. This completes the proof that $\mathcal{C}^{[k]}(\mathbb{R})$ is a Fréchet space.

Finally, the previous item implies that $\mathcal{C}^{[k]}(\mathbb{R})$ is an algebra under pointwise multiplication and that pointwise multiplication is continuous. Since it is also clear that $\|\overline{f}\|_{\mathcal{C}^{[k]},r} = \|f\|_{\mathcal{C}^{[k]},r}$ whenever $f \in C^k(\mathbb{R})$ and $r > 0$, complex conjugation is a continuous $*$-operation on $\mathcal{C}^{[k]}(\mathbb{R})$. $\square$

This and Example 3.8.9 bring us to our main space of interest in this section. Inspired by the proof of Theorem 3.1.2 in §3.10 (specifically, Lemma 3.10.8), we make the following definition.

Definition 3.8.11 (Noncommutative C^k functions)**.** For $k \in \mathbb{N} \cup \{\infty\}$, define $NC^k(\mathbb{R})$ to be the closure of $\mathbb{C}[\lambda]$ in $\mathcal{C}^{[k]}(\mathbb{R})$. The members of $NC^k(\mathbb{R})$ are **noncommutative $\boldsymbol{C^k}$ functions**.

The idea for the name of $NC^k(\mathbb{R})$ comes from the parallel work by Jekel, mentioned in Remark 3.4.13, on $C^k_{\mathrm{nc}}(\mathbb{R})$.

Theorem 3.8.12. *With the topology of $\mathcal{C}^{[k]}(\mathbb{R})$, $NC^k(\mathbb{R})$ is a unital Fréchet $*$-algebra under pointwise operations. Moreover, $VC^k(\mathbb{R}) \subseteq NC^k(\mathbb{R})$, and $W_k(\mathbb{R})$ is dense in $NC^k(\mathbb{R})$.*

Proof. Since $\mathbb{C}[\lambda] \subseteq \mathcal{C}^{[k]}(\mathbb{R})$ is a $*$-subalgebra, $NC^k(\mathbb{R})$ is a closed $*$-subalgebra of the Fréchet $*$-algebra $\mathcal{C}^{[k]}(\mathbb{R})$. Thus, $NC^k(\mathbb{R})$ is a Fréchet $*$-algebra in its own right. Since $VC^k(\mathbb{R}) \hookrightarrow \mathcal{C}^{[k]}(\mathbb{R})$ by Proposition 3.8.10(i), the containment $VC^k(\mathbb{R}) \subseteq NC^k(\mathbb{R})$ follows from the density of $\mathbb{C}[\lambda]$

in $VC^k(\mathbb{R})$ (Theorem 3.4.12). Finally, by Proposition 3.4.8, if $p(\lambda) \in \mathbb{C}[\lambda]$, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $W_k(\mathbb{R})$ converging to p in $VC^k(\mathbb{R})$ to p. Since $VC^k(\mathbb{R}) \hookrightarrow NC^k(\mathbb{R})$, the sequence $(f_n)_{n\in\mathbb{N}}$ converges in $NC^k(\mathbb{R})$ to p. Thus, $\mathbb{C}[\lambda]$ is contained in the closure of $W_k(\mathbb{R})$ in $NC^k(\mathbb{R})$. Since $\mathbb{C}[\lambda]$ is dense in $NC^k(\mathbb{R})$ by definition, this completes the proof. □

Next, we describe a very special case of the "separation of variables" approach to defining MOIs, which is covered in detail in Chapter 5. Recall that H is a complex Hilbert space and $\mathcal{M} \subseteq B(H)$ is a von Neumann algebra. Given $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in \mathcal{M}_\nu^{k+1}$ and $\varphi \in \ell^\infty\big(\sigma(a_1), \mathcal{B}_{\sigma(a_1)}\big) \hat{\otimes}_i \cdots \hat{\otimes}_i \ell^\infty\big(\sigma(a_{k+1}), \mathcal{B}_{\sigma(a_{k+1})}\big)$, the goal is to make sense of

$$\big(I^{\mathbf{a}}\varphi\big)[b] := \int_{\sigma(a_{k+1})} \cdots \int_{\sigma(a_1)} \varphi(\boldsymbol{\lambda})\, P^{a_1}(\mathrm{d}\lambda_1)\, b_1 \cdots P^{a_k}(\mathrm{d}\lambda_k)\, b_k\, P^{a_{k+1}}(\mathrm{d}\lambda_{k+1}) = \text{“}\varphi_\otimes(\mathbf{a}) \#_k b\text{”}$$

in $\mathcal{M}$ for all $b = (b_1, \ldots, b_k) \in \mathcal{M}^k$. Above, $P^a \colon \mathcal{B}_{\sigma(a)} \to \mathcal{M}$ is the projection-valued spectral measure of $a \in \mathcal{M}_\nu$ (Definition 4.2.13). Heuristically, if $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ is an ℓ^∞-IPD of φ, then we should have

$$\begin{aligned}
\big(I^{\mathbf{a}}\varphi\big)[b] &= \int_{\sigma(a_{k+1})} \cdots \int_{\sigma(a_1)} \int_\Sigma \prod_{i=1}^{k+1} \varphi_i(\lambda_i, \sigma)\, \rho(\mathrm{d}\sigma)\, P^{a_1}(\mathrm{d}\lambda_1)\, b_1 \cdots P^{a_k}(\mathrm{d}\lambda_k)\, b_k\, P^{a_{k+1}}(\mathrm{d}\lambda_{k+1}) \\
&= \int_\Sigma \int_{\sigma(a_{k+1})} \cdots \int_{\sigma(a_1)} \prod_{i=1}^{k+1} \varphi_i(\lambda_i, \sigma)\, P^{a_1}(\mathrm{d}\lambda_1)\, b_1 \cdots P^{a_k}(\mathrm{d}\lambda_k)\, b_k\, P^{a_{k+1}}(\mathrm{d}\lambda_{k+1})\, \rho(\mathrm{d}\sigma) \\
&= \int_\Sigma \left(\int_{\sigma(a_1)} \varphi_1(\cdot, \sigma)\, \mathrm{d}P^{a_1} \right) b_1 \cdots \left(\int_{\sigma(a_k)} \varphi_k(\cdot, \sigma)\, \mathrm{d}P^{a_k} \right) b_k \left(\int_{\sigma(a_{k+1})} \varphi_{k+1}(\cdot, \sigma)\, \mathrm{d}P^{a_{k+1}} \right) \rho(\mathrm{d}\sigma) \\
&= \int_\Sigma \varphi_1(a_1, \sigma)\, b_1 \cdots \varphi_k(a_k, \sigma)\, b_k\, \varphi_{k+1}(a_{k+1}, \sigma)\, \rho(\mathrm{d}\sigma) \qquad (3.8.13)
\end{aligned}$$

in analogy with Proposition 3.5.3. Accordingly, we shall use Equation (3.8.13) as a definition. To do so, one must address exactly what kind of integral $\int_\Sigma \cdot \, \mathrm{d}\rho$ is being used above and whether this integral depends on the chosen ℓ^∞-IPD of φ. We address these now.

Let $(\Xi, \mathscr{F}, \mu)$ be a measure space. A map $F \colon \Xi \to B(H)$ is **pointwise weakly measurable** if $\langle F(\cdot)h_1, h_2\rangle \colon \Xi \to \mathbb{C}$ is measurable whenever $h_1, h_2 \in H$. If, in addition,

$$\int_\Xi |\langle F(\xi)h_1, h_2\rangle|\, \mu(\mathrm{d}\xi) < \infty, \qquad h_1, h_2 \in H,$$

then F is **pointwise Pettis ($\boldsymbol{\mu}$-)integrable**. In this case, $F(\cdot)h\colon \Xi \to H$ is weakly integrable for all $h \in H$, and the linear map

$$H \ni h \mapsto Th := \int_\Xi F(\xi)h\,\mu(\mathrm{d}\xi) \in H$$

is bounded. Furthermore, if $W^*(S) \subseteq B(H)$ is the smallest von Neumann algebra containing $S \subseteq B(H)$, then $T \in W^*(F(\xi) : \xi \in \Xi)$. The operator T is the **pointwise Pettis ($\boldsymbol{\mu}$-)integral** of F, and we write $\int_\Xi F\,\mathrm{d}\mu = \int_\Xi F(\xi)\,\mu(\mathrm{d}\xi) := T$. Please see Lemma 5.4.1 and Remark 5.4.4 for proofs of the assertions in this paragraph.

Lemma 3.8.14. *Suppose $(\Sigma, \mathscr{H})$ and $(\Xi, \mathscr{F})$ are measurable spaces, $P\colon \mathscr{F} \to B(H)$ is a projection-valued measure (Definition 4.2.8), and $\varphi\colon \Xi \times \Sigma \to \mathbb{C}$ is product measurable. If $\varphi(\cdot, \sigma) \in \ell^\infty(\Xi, \mathscr{F})$ for all $\sigma \in \Sigma$ and $F\colon \Sigma \to B(H)$ is pointwise weakly measurable, then so are $F(\cdot)\int_\Xi \varphi(\xi, \cdot)\,P(\mathrm{d}\xi)\colon \Sigma \to B(H)$ and $\int_\Xi \varphi(\xi, \cdot)\,P(\mathrm{d}\xi)\,F(\cdot)\colon \Sigma \to B(H)$.*

The lemma above is a special case of Proposition 5.6.3 below. Here now is the main result that allows us to make sense of the MOI of interest.

Theorem 3.8.15 (Definition of MOIs)**.** *Fix $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in \mathcal{M}_\nu^{k+1}$,*

$$\varphi \in \ell^\infty\big(\sigma(a_1), \mathcal{B}_{\sigma(a_1)}\big)\hat{\otimes}_i \cdots \hat{\otimes}_i \ell^\infty\big(\sigma(a_{k+1}), \mathcal{B}_{\sigma(a_{k+1})}\big),$$

and $b = (b_1, \ldots, b_k) \in \mathcal{M}^k$.

(i) *If $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ is an ℓ^∞-IPD of φ, then the map*

$$\Sigma \ni \sigma \mapsto F(\sigma) := \varphi_1(a_1, \sigma)\,b_1 \cdots \varphi_k(a_k, \sigma)\,b_k\,\varphi_{k+1}(a_{k+1}, \sigma) \in \mathcal{M}$$

is pointwise Pettis integrable.

(ii) *If F is as in the previous item, then the pointwise Pettis integral $\int_\Sigma F\,\mathrm{d}\rho \in \mathcal{M}$ is independent of the chosen ℓ^∞-IPD of φ; in this case, we write*

$$\big(I^{\mathbf{a}}\varphi\big)[b] = \int_{\sigma(a_{k+1})} \cdots \int_{\sigma(a_1)} \varphi(\boldsymbol{\lambda})\,P^{a_1}(\mathrm{d}\lambda_1)\,b_1 \cdots P^{a_k}(\mathrm{d}\lambda_k)\,b_k\,P^{a_{k+1}}(\mathrm{d}\lambda_{k+1}) := \int_\Sigma F\,\mathrm{d}\rho.$$

(iii) *The assignment* $\mathcal{M}^k \ni b \mapsto (I^{\mathbf{a}}\varphi)[b] \in \mathcal{M}$ *is* k*-linear and bounded. Furthermore, the assignment* $\ell^\infty\big(\sigma(a_1), \mathcal{B}_{\sigma(a_1)}\big)\hat{\otimes}_i \cdots \hat{\otimes}_i \ell^\infty\big(\sigma(a_{k+1}), \mathcal{B}_{\sigma(a_{k+1})}\big) \ni \varphi \mapsto I^{\mathbf{a}}\varphi \in B_k\big(\mathcal{M}^k; \mathcal{M}\big)$ *is linear and has operator norm at most one.*

Sketch of proof. We take each item in turn. Write $\|\cdot\| = \|\cdot\|_{H\to H}$.

(i) By Lemma 3.8.14 and induction, $F \colon \Sigma \to B(H)$ is pointwise weakly measurable. To prove the integrability, fix $h_1, h_2 \in H$ and $\sigma \in \Sigma$. Then

$$\begin{aligned}|\langle F(\sigma)h_1, h_2\rangle| &\le \|h_1\|\,\|h_2\|\left(\prod_{j=1}^{k}\|b_j\|\right)\prod_{i=1}^{k+1}\|\varphi_i(a_i,\sigma)\| \\ &\le \|h_1\|\,\|h_2\|\left(\prod_{j=1}^{k}\|b_j\|\right)\prod_{i=1}^{k+1}\|\varphi_i(\cdot,\sigma)\|_{\ell^\infty(\sigma(a_i))}.\end{aligned}$$

Therefore,

$$\int_\Sigma |\langle F(\sigma)h_1, h_2\rangle|\,\rho(\mathrm{d}\sigma) \le \|h_1\|\,\|h_2\|\left(\prod_{j=1}^{k}\|b_j\|\right)\int_\Sigma \prod_{i=1}^{k+1}\|\varphi_i(\cdot,\sigma)\|_{\ell^\infty(\sigma(a_i))}\,\rho(\mathrm{d}\sigma) < \infty, \quad (3.8.16)$$

so F is pointwise Pettis integrable.

(ii) For this item, it suffices to assume $\mathcal{M} = B(H)$. First, suppose $h_1, \tilde{h}_1, \ldots, h_k, \tilde{h}_k \in H$ and $b_i = \langle\cdot, h_i\rangle\tilde{h}_i$ for all $i \in \{1, \ldots, k\}$. Also, for $h_0, \tilde{h}_{k+1} \in H$, define

$$\nu := P^{a_1}_{\tilde{h}_1, h_0} \otimes \cdots \otimes P^{a_{k+1}}_{\tilde{h}_{k+1}, h_k}.$$

(This is a product of complex measures.) Then one can show without much difficulty that

$$\left\langle \left(\int_\Sigma F\,\mathrm{d}\rho\right)\tilde{h}_{k+1}, h_0\right\rangle = \int_{\sigma(a_1)\times\cdots\times\sigma(a_{k+1})} \varphi\,\mathrm{d}\nu. \quad (3.8.17)$$

For this calculation or similar ones, please see the proof of Theorem 5.6.11 below, the proof of [ACDS09, Lem. 4.3], or the proof [Pel16, Thm. 2.1.1]. From Equation (3.8.17) and k-linearity, we conclude that $\int_\Sigma F\,\mathrm{d}\rho$ is independent of the chosen ℓ^∞-IPD of φ when $b_1, \ldots, b_k$ are finite-rank operators. Now, if H is separable and $c \in B(H)$, then there exists a sequence of finite-rank operators converging to c in the strong operator topology (SOT, Definition 4.1.1(ii)). This allows one to use an operator-valued dominated convergence theorem to extend the claimed independence to arbitrary $b_1,\ldots,b_k \in B(H)$. This is what is done in [ACDS09, Pel16]. When H is not separable, much more care is required. The claim is again extended from finite-rank to arbitrary bounded operators by density but in a different topology: the ultraweak topology (aka the σ-WOT—please see Definition 4.1.1(iv) and Theorem 4.3.3(vi)). Indeed, one proves that, for fixed $i \in \{1,\ldots,k\}$ and $b_1,\ldots,b_{i-1},b_{i+1},\ldots,b_k \in B(H)$, the assignment

$$B(H) \ni c \mapsto \int_\Sigma \varphi_1(a_1,\sigma)\Bigg(\prod_{j=1}^{i-1} b_j\,\varphi_{j+1}(a_{j+1},\sigma)\Bigg)c\Bigg(\prod_{j=i+1}^{k} \varphi_j(a_j,\sigma)\,b_j\Bigg)\varphi_{k+1}(a_{k+1},\sigma)\,\rho(\mathrm{d}\sigma) \in B(H)$$

is ultraweakly continuous. (Above, empty products are declared to be 1.) Proving this is quite technical. Please see Corollary 5.6.10 and its lead-up for the details.

In the present setting, which is less general than that of Chapter 5, we can employ a different argument to deduce the non-separable case from the separable case. First, suppose that $\mathcal{A} \subseteq B(H)$ is a unital subalgebra. We claim that if $\mathcal{A}$ is SOT-separable and $h_1,\ldots,h_n \in H$, then there exists a closed, separable linear subspace $K \subseteq H$ such that $h_1,\ldots,h_n \in K$ and $\mathcal{A}K \subseteq K$, i.e., K is **$\mathcal{A}$-invariant**. Indeed, if

$$K \coloneqq \overline{\operatorname{span}}\big(\mathcal{A}h_1 \cup \cdots \cup \mathcal{A}h_n\big) \subseteq H,$$

then K is a closed linear subspace of H containing $h_1,\ldots,h_n$. Also, K is separable because if $\mathcal{A}_0 \subseteq \mathcal{A}$ is a countable SOT-dense subset, then the $\mathbb{Q}[i]$-span of $\mathcal{A}_0h_1 \cup \cdots \cup \mathcal{A}_0h_n$ is dense in K. Finally, K is $\mathcal{A}$-invariant because $\mathcal{A}$ is a subalgebra and closed linear spans of $\mathcal{A}$-invariant subsets are $\mathcal{A}$-invariant. Next, fix $h_1,h_2 \in H$, and apply this result to $\mathcal{A} \coloneqq W^*(a_1,\ldots,a_{k+1},b_1,\ldots,b_k)$ to obtain a closed, separable, $\mathcal{A}$-invariant linear subspace $K \subseteq H$ that contains h_1 and h_2. (Note that $\mathcal{A}$ is separable in the SOT because the $\mathbb{Q}[i]$-span of noncommutative monomials in $a_1,a_1^*,\ldots,a_{k+1},a_{k+1}^*,b_1,b_1^*,\ldots,b_k,b_k^*$ is SOT-dense in $\mathcal{A}$.) If we write $\pi_K\colon H \to K$ for the orthogonal projection onto K, $\iota_K\colon K \to H$ for the inclusion of K into H, $\tilde{a}_j \coloneqq \pi_K a_i \iota_K \in B(K)_\nu$ for all $i \in \{1,\ldots,k+1\}$, and $\tilde{b}_j \coloneqq \pi_K b_j \iota_K \in B(K)$ for all $j \in \{1,\ldots,k\}$, then

$$F(\sigma)h = \varphi_1\big(\tilde{a}_1,\sigma\big)\,\tilde{b}_1\cdots\varphi_k\big(\tilde{a}_k,\sigma\big)\,\tilde{b}_k\,\varphi_{k+1}\big(\tilde{a}_{k+1},\sigma\big)h, \qquad \sigma \in \Sigma,\ h \in K,$$

as we encourage the reader to verify using the $\mathcal{A}$-invariance of K. Therefore,

$$\begin{aligned}\left\langle\left(\int_\Sigma F\,\mathrm{d}\rho\right)h_1,h_2\right\rangle_H &= \int_\Sigma \langle F(\sigma)h_1,h_2\rangle_H\,\rho(\mathrm{d}\sigma)\\ &= \int_\Sigma \big\langle\varphi_1\big(\tilde a_1,\sigma\big)\,\tilde b_1\cdots\varphi_k\big(\tilde a_k,\sigma\big)\,\tilde b_k\,\varphi_{k+1}\big(\tilde a_{k+1},\sigma\big)h_1,h_2\big\rangle_K\,\rho(\mathrm{d}\sigma)\\ &= \left\langle\left(\int_\Sigma \varphi_1\big(\tilde a_1,\sigma\big)\,\tilde b_1\cdots\varphi_k\big(\tilde a_k,\sigma\big)\,\tilde b_k\,\varphi_{k+1}\big(\tilde a_{k+1},\sigma\big)\,\rho(\mathrm{d}\sigma)\right)h_1,h_2\right\rangle_K.\end{aligned}$$

By the separable case, the last quantity is independent of the chosen ℓ^∞-IPD. Since $h_1,h_2\in H$ were arbitrary, this completes the proof of this item.[1]

(iii) First, the k-linearity of $b\mapsto(I^{\mathbf{a}}\varphi)[b]$ is clear from the linearity of pointwise Pettis integrals. Second, if $(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})$ is an ℓ^∞-IPD of φ, then Equation (3.8.16) gives

$$\begin{aligned}\big\|\big(I^{\mathbf{a}}\varphi\big)[b]\big\| = \left\|\int_\Sigma F\,\mathrm{d}\rho\right\| &= \sup\left\{\left|\left\langle\left(\int_\Sigma F\,\mathrm{d}\rho\right)h_1,h_2\right\rangle\right| : \|h_1\|,\|h_2\|\le 1\right\}\\ &\le \sup\left\{\int_\Sigma |\langle F(\sigma)h_1,h_2\rangle|\,\rho(\mathrm{d}\sigma) : \|h_1\|,\|h_2\|\le 1\right\}\\ &\le \|b_1\|\cdots\|b_k\|\int_\Sigma \|\varphi_1(\cdot,\sigma)\|_{\ell^\infty(\sigma(a_1))}\cdots\|\varphi_{k+1}(\cdot,\sigma)\|_{\ell^\infty(\sigma(a_{k+1}))}\,\rho(\mathrm{d}\sigma).\end{aligned}$$

Taking the infimum over ℓ^∞-IPDs of φ therefore gives

$$\big\|I^{\mathbf{a}}\varphi\big\|_{B_k(\mathcal{M}^k;\mathcal{M})} \le \|\varphi\|_{\ell^\infty(\sigma(a_1),\mathcal{B}_{\sigma(a_1)})\hat\otimes_i\cdots\hat\otimes_i\ell^\infty(\sigma(a_{k+1}),\mathcal{B}_{\sigma(a_{k+1})})}.$$

We leave the proof that $\varphi\mapsto I^{\mathbf{a}}\varphi$ is linear to the reader. Alternatively, please see Proposition 5.7.1(i), which is more general. □

Example 3.8.18. If $\mathbf{a}\in\mathcal{M}_\nu^{k+1}$ and $\varphi\in V(\sigma(a_1),\ldots,\sigma(a_{k+1}))$, then

$$\int_{\sigma(a_{k+1})}\cdots\int_{\sigma(a_1)}\varphi(\boldsymbol{\lambda})\,P^{a_1}(\mathrm{d}\lambda_1)\,b_1\cdots P^{a_k}(\mathrm{d}\lambda_k)\,b_k\,P^{a_{k+1}}(\mathrm{d}\lambda_{k+1}) = \varphi_\otimes(\mathbf{a})\#_k b,\qquad b\in\mathcal{M}^k,$$

by definition of $I^{\mathbf{a}}\varphi$ and Proposition 3.5.3(ii). Consequently, it is at least conceptually justified to write $\varphi_\otimes(\mathbf{a})\#_k b \coloneqq \big(I^{\mathbf{a}}\varphi\big)[b]$ when $\varphi\in\ell^\infty\big(\sigma(a_1),\mathcal{B}_{\sigma(a_1)}\big)\hat\otimes_i\cdots\hat\otimes_i\ell^\infty\big(\sigma(a_{k+1}),\mathcal{B}_{\sigma(a_{k+1})}\big)$.

[1] This argument can be made to work when a_i is not necessarily bounded, i.e., when $a_i\;\eta\;\mathcal{M}_{\mathrm{sa}}$ (Definition 4.2.16). We encourage the interested reader to ponder this.

With this under our belts, we are almost ready for the main result of this section.

Lemma 3.8.19. *If $f \in NC^k(\mathbb{R})$, $\mathbf{a} \in \mathcal{M}_{\mathrm{sa}}^{k+1}$, and $b \in \mathcal{M}^k$, then*

$$\big(I^{\mathbf{a}} f^{[k]}\big)[b] \in C^*(1, a_1, \ldots, a_{k+1}, b_1, \ldots, b_k) \subseteq \mathcal{M},$$

where $C^*(S) \subseteq \mathcal{M}$ denotes the smallest C^*-subalgebra of $\mathcal{M}$ containing $S \subseteq \mathcal{M}$.

Proof. Write $\|\cdot\| \coloneqq \|\cdot\|_{H\to H}$. Let $(q_n(\lambda))_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}[\lambda]$ converging to f in $NC^k(\mathbb{R})$. First, it is clear from Examples 3.8.18 and 1.3.8 (and Equation (3.5.14)) that

$$\big(I^{\mathbf{a}} q_n^{[k]}\big)[b] \in C^*(1, a_1, \ldots, a_{k+1}, b_1, \ldots, b_k), \qquad n \in \mathbb{N}.$$

Now, if $r \coloneqq \max\{\|a_i\| : i \in \{1, \ldots, k+1\}\}$, then

$$\begin{aligned}\big\|\big(I^{\mathbf{a}} f^{[k]}\big)[b] - \big(I^{\mathbf{a}} q_n^{[k]}\big)[b]\big\| &= \big\|\big(I^{\mathbf{a}}\big((f-q_n)^{[k]}\big)\big)[b]\big\| \\ &\leq \big\|(f-q_n)^{[k]}\big\|_{\ell^\infty(\sigma(a_1),\mathcal{B}_{\sigma(a_1)})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\sigma(a_{k+1}),\mathcal{B}_{\sigma(a_{k+1})})}\|b_1\|\cdots\|b_k\| \\ &\leq \big\|(f-q_n)^{[k]}\big\|_{r,k+1}\|b_1\|\cdots\|b_k\| \leq \|f-q_n\|_{\mathcal{C}^{[k]},r}\|b_1\|\cdots\|b_k\| \xrightarrow{n\to\infty} 0\end{aligned}$$

by Theorem 3.8.15(iii) and the fact that $q_n \to f$ in $NC^k(\mathbb{R})$ as $n \to \infty$. The result follows. $\square$

Here now is the main result. Recall that $\mathcal{A}$ is a fixed, unital C^*-algebra. Also, if $f \in C(\mathbb{R})$, then $f_{\mathcal{A}} \colon \mathcal{A}_{\mathrm{sa}} \to \mathcal{A}$ is defined via the continuous functional calculus by $a \mapsto f(a)$.

Theorem 3.8.20 (Derivatives of operator functions in terms of MOIs)**.** *Suppose the von Neumann algebra $\mathcal{M} \subseteq B(H)$ contains $\mathcal{A}$ as a unital C^*-subalgebra. If $f \in NC^k(\mathbb{R})$, then $f_{\mathcal{A}} \in C_{\mathrm{bb}}^k(\mathcal{A}_{\mathrm{sa}}; \mathcal{A})$. Furthermore, we have*

$$\partial_{b_k}\cdots\partial_{b_1} f_{\mathcal{A}}(a) = \sum_{\pi\in S_k} \underbrace{\int_{\sigma(a)}\cdots\int_{\sigma(a)}}_{k+1\,\text{times}} f^{[k]}(\boldsymbol{\lambda})\, P^a(\mathrm{d}\lambda_1)\, b_{\pi(1)}\cdots P^a(\mathrm{d}\lambda_k)\, b_{\pi(k)}\, P^a(\mathrm{d}\lambda_{k+1}) \quad (3.8.21)$$

for all $a, b_1, \ldots, b_k \in \mathcal{A}_{\mathrm{sa}}$.

Proof. Equation (3.8.21) rewrites to

$$D^k f_{\mathcal{A}}(a) = k!\, S\big(\big(I^{a,\ldots,a} f^{[k]}\big)\big|_{\mathcal{A}_{\mathrm{sa}}^k}\big), \qquad a \in \mathcal{A}_{\mathrm{sa}}. \tag{3.8.22}$$

Recall from Notation 1.2.5(iv) that $S(T)[v_1,\ldots,v_k] = (k!)^{-1}\sum_{\pi\in S_k} T[v_{\pi(1)},\ldots,v_{\pi(k)}]$ is the symmetrization of the k-linear map T.

Now, let $n \in \mathbb{N}_0$, and define $p_n(\lambda) := \lambda^n$ as usual. Then $(p_n)_{\mathcal{A}} \colon \mathcal{A}_{\mathrm{sa}} \to \mathcal{A}$ is the restriction of the homogeneous polynomial $\mathcal{A} \ni a \mapsto a^n \in \mathcal{A}$. Therefore, Equation (3.8.22) holds when $f = p_n$ by Proposition 1.2.6, Examples 1.3.8 and 3.8.18, and Equation (3.5.14). Consequently, by linearity, Equation (3.8.22) holds whenever $f(\lambda) \in \mathbb{C}[\lambda]$.

Finally, fix $f \in NC^k(\mathbb{R})$ and a sequence $(q_n(\lambda))_{n\in\mathbb{N}}$ in $\mathbb{C}[\lambda]$ converging to f in $NC^k(\mathbb{R})$. By Lemma 3.8.19, if $\mathbf{a} \in \mathcal{A}_{\mathrm{sa}}^{k+1}$ and $b \in \mathcal{A}^k$, then $(I^{\mathbf{a}} f^{[k]})[b] \in \mathcal{A}$. We shall take this for granted in our notation. Now, fix $r > 0$ and $i \in \{1,\ldots,k\}$, and define

$$\mathcal{A}_{\mathrm{sa},r} := \{a \in \mathcal{A}_{\mathrm{sa}} : \|a\| \le r\} \;\text{ and }\; \|\cdot\|_i := \|\cdot\|_{B_i(\mathcal{A}_{\mathrm{sa}}^i;\mathcal{A})}.$$

Then

$$\sup_{a\in\mathcal{A}_{\mathrm{sa},r}} \|f(a) - q_n(a)\| = \|f - q_n\|_{\ell^\infty([-r,r])} \xrightarrow{n\to\infty} 0.$$

Also, by Theorem 3.8.15(iii) and the previous paragraph, if $a \in \mathcal{A}_{\mathrm{sa},r}$, then

$$\begin{aligned}\Big\|i!\, S\big(I^{a,\ldots,a} f^{[i]}\big) - D^i(q_n)_{\mathcal{A}}(a)\Big\|_i &= i!\Big\|S\big(I^{a,\ldots,a}\big((f-q_n)^{[i]}\big)\big)\Big\|_i \le i!\Big\|I^{a,\ldots,a}\big((f-q_n)^{[i]}\big)\Big\|_i \\ &\le i!\big\|(f-q_n)^{[i]}\big\|_{\ell^\infty(\sigma(a),\mathcal{B}_{\sigma(a)})^{\hat{\otimes}_i(i+1)}} \le i!\big\|(f-q_n)^{[i]}\big\|_{r,i+1}.\end{aligned}$$

In particular,

$$\max_{1\le i\le k}\sup_{a\in\mathcal{A}_{\mathrm{sa},r}} \Big\|i!\, S\big(I^{a,\ldots,a} f^{[i]}\big) - D^i(q_n)_{\mathcal{A}}(a)\Big\|_i \le k!\|f - q_n\|_{\mathcal{C}^{[k]},r} \xrightarrow{n\to\infty} 0.$$

Since $r > 0$ was arbitrary, we conclude from Proposition 1.2.13 that $f_{\mathcal{A}} \in C_{\mathrm{bb}}^k(\mathcal{A}_{\mathrm{sa}};\mathcal{A})$ and that Equation (3.8.22) holds. This completes the proof. □

The purpose of $\mathcal{M}$ in Theorem 3.8.20 is, of course, to allow us to make sense of the right-hand side of Equation (3.8.21), since, *a priori*, MOIs are defined only in von Neumann algebras (though Lemma 3.8.19 morally says that MOIs like the ones in Equation (3.8.21) make sense in unital C^*-algebras.) Of course, if $\mathcal{A}$ happens to be a von Neumann algebra, then we may take $\mathcal{M} = \mathcal{A}$. For an arbitrary (abstract) unital C^*-algebra $\mathcal{A}$, a reasonable choice of $\mathcal{M}$ is the double dual $\mathcal{A}^{**}$ of $\mathcal{A}$, which has a von Neumann algebra structure with respect to which the natural embedding $\mathcal{A} \hookrightarrow \mathcal{A}^{**}$ is a unital $*$-homomorphism [Sak71, Thm. 1.17.2]. Consequently, if $a \in \mathcal{A}_{\text{sa}}$, then one may always interpret the projection-valued spectral measure P^a in Equation (3.8.21) as taking values in $\mathcal{A}^{**}$, even when it does not make sense in $\mathcal{A}$. However, we highlight that the double dual $\mathcal{A}^{**}$ of a C^*-algebra $\mathcal{A}$ is frequently quite large; specifically, it is frequently not representable on a separable Hilbert space. This is why we must understand MOIs on non-separable Hilbert spaces.

3.9 Demonstration that $\mathcal{C}^{[k]}(\mathbb{R}) \subsetneq C^k(\mathbb{R})$

In §3.6, we saw that $VC^k(\mathbb{R})$ is "close" to $C^k(\mathbb{R})$ in the sense that a function only has to be "slightly better than C^k" to belong to $VC^k(\mathbb{R})$. In particular, by Theorem 3.8.12, a function only has to be "slightly better than C^k" to belong to $\mathcal{C}^{[k]}(\mathbb{R})$. The goal of this section is to show that nevertheless $\mathcal{C}^{[k]}(\mathbb{R}) \subsetneq C^k(\mathbb{R})$, for all $k \in \mathbb{N}$. Specifically, we combine Schatten estimates for Taylor remainders of operator functions (Proposition 3.9.7) with a construction of D. Potapov et al. from [PSST17] (Theorem 3.9.3) to prove the following.

Theorem 3.9.1. *If $k \in \mathbb{N}$, then $\mathcal{C}^{[k]}(\mathbb{R}) \subsetneq C^k(\mathbb{R})$. Specifically, we have the following counterexample. Fix $\eta \in C_c^\infty(\mathbb{R})$ such that $\eta \equiv 1$ on $[-1/2-1/e, 1/e+1/2]$ and $\operatorname{supp}\eta \subseteq [-3/5-1/e, 1/e+3/5]$, and define*

$$h(x) := 1_{(0,1)}(|x|)\frac{\eta(x)\,|x|}{\sqrt{\log|\log|x|-1|}}, \qquad x \in \mathbb{R}.$$

If

$$f_k(x) := x^{k-1}h(x), \qquad x \in \mathbb{R},$$

then $f_k \in C^k(\mathbb{R}) \setminus \mathcal{C}^{[k]}(\mathbb{R})$.

To begin, we set some notation for Taylor remainders.

Definition 3.9.2 (Taylor remainder)**.** Let V and W be normed vector spaces. If $F\colon V \to W$ is $(k-1)$-times Fréchet differentiable and $p \in V$, then

$$R_{k,F,p}(h) \coloneqq F(p+h) - \sum_{i=0}^{k-1} \frac{1}{i!}\partial_h^i F(p) = F(p+h) - F(p) - \sum_{i=1}^{k-1} \frac{1}{i!} D^j F(p)[\underbrace{h,\ldots,h}_{i \text{ times}}] \in W, \qquad h \in V,$$

is the $\boldsymbol{k}^{\mathbf{th}}$ **Taylor remainder of $\boldsymbol{F}$ at $\boldsymbol{p}$**.

Recall that if $f \in C^k(\mathbb{R})$, then $f \in VC^{k-1}(\mathbb{R})$ (if $VC^0(\mathbb{R}) \coloneqq C(\mathbb{R})$) by Corollary 3.4.7. Consequently, if $\mathcal{A}$ is a unital C^*-algebra, then $f_{\mathcal{A}} \in C^{k-1}(\mathcal{A}_{\text{sa}};\mathcal{A})$ by Corollary 3.1.10. In particular, $R_{k,f_{\mathcal{A}},a}(b) \in \mathcal{A}$ makes sense whenever $f \in C^k(\mathbb{R})$ and $a,b \in \mathcal{A}_{\text{sa}}$. Now, we state one of the key ingredients of the proof of Theorem 3.9.1. If H is a Hilbert space and $1 \leq p < \infty$, then $(\mathcal{S}_p(H), \|\cdot\|_{\mathcal{S}_p})$ is the ideal of Schatten p-class operators on H (Definition 4.3.1 below), and $(\mathcal{S}_\infty(H), \|\cdot\|_{\mathcal{S}_\infty}) \coloneqq (B(H), \|\cdot\|_{H\to H})$.

Theorem 3.9.3 (Potapov–Skripka–Sukochev–Tomskova [PSST17, Thm. 5.1])**.** *If $f_k\colon \mathbb{R} \to \mathbb{C}$ is as in Theorem 3.9.1, then there exist a separable complex Hilbert space H and operators $a,b \in B(H)_{\text{sa}}$ such that $b \in \mathcal{S}_k(H)$ and $R_{k,(f_k)_{B(H)},a}(b) \notin \mathcal{S}_1(H)$.*

Next, we work toward the Taylor remainder estimates that will help to disqualify f_k from belonging to $\mathcal{C}^{[k]}(\mathbb{R})$. Please see [ST19, §5.4] for more information about the applications of MOI theory to the analysis of Taylor remainders of maps induced by functional calculus.

Lemma 3.9.4 (Schatten estimates)**.** *Suppose $p,p_1,\ldots,p_k \in [1,\infty]$ satisfy $1/p = 1/p_1 + \cdots + 1/p_k$. If H is a complex Hilbert space, $\mathbf{a} = (a_1,\ldots,a_{k+1}) \in B(H)_\nu^{k+1}$, and*

$$\varphi \in \ell^\infty(\sigma(a_1),\mathcal{B}_{\sigma(a_1)})\hat{\otimes}_i \cdots \hat{\otimes}_i \ell^\infty(\sigma(a_{k+1}),\mathcal{B}_{\sigma(a_{k+1})}),$$

then

$$\big\|\big(I^{\mathbf{a}}\varphi\big)[b]\big\|_{\mathcal{S}_p} \leq \|\varphi\|_{\ell^\infty(\sigma(a_1),\mathcal{B}_{\sigma(a_1)})\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\sigma(a_{k+1}),\mathcal{B}_{\sigma(a_{k+1})})}\|b_1\|_{\mathcal{S}_{p_1}}\cdots\|b_k\|_{\mathcal{S}_{p_k}}$$

for all $b = (b_1,\ldots,b_k) \in B(H)^k$. (As usual, $0\cdot\infty \coloneqq 0$.)

Proof. This lemma is a special case of Proposition 5.7.2 below, but we supply a proof for the reader's convenience. Fix $b = (b_1,\ldots,b_k) \in B(H)^k$ and an ℓ^∞-IPD $(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})$

of φ. Also, if $\sigma \in \Sigma$, then we define $F(\sigma) \coloneqq \varphi_1(a_1,\sigma)\, b_1 \cdots \varphi_k(a_k,\sigma)\, b_k\, \varphi_{k+1}(a_{k+1},\sigma)$. If $p_1,\ldots,p_k,p \in [1,\infty]$ satisfy $1/p = 1/p_1 + \cdots + 1/p_k$, then

$$\begin{aligned}\|F(\sigma)\|_{\mathcal{S}_p} &\leq \|\varphi_1(a_1,\sigma)\|_{\mathcal{S}_\infty}\|b_1\|_{\mathcal{S}_{p_1}} \cdots \|\varphi_k(a_k,\sigma)\|_{\mathcal{S}_\infty}\|b_k\|_{\mathcal{S}_{p_k}}\|\varphi_{k+1}(a_{k+1},\sigma)\|_{\mathcal{S}_\infty} \\ &\leq \|\varphi_1(\cdot,\sigma)\|_{\ell^\infty(\sigma(a_1))}\,\|b_1\|_{\mathcal{S}_{p_1}} \cdots \|\varphi_k(\cdot,\sigma)\|_{\ell^\infty(\sigma(a_k))}\,\|b_k\|_{\mathcal{S}_{p_k}}\|\varphi_{k+1}(\cdot,\sigma)\|_{\ell^\infty(\sigma(a_{k+1}))}\end{aligned}$$

by Hölder's inequality for the Schatten norms (Theorem 4.3.3(iii) below). Therefore, by the Schatten norm Minkowski's integral inequality (Theorem 5.4.12 below),

$$\big\|\big(I^{\mathbf{a}}\varphi\big)[b]\big\|_{\mathcal{S}_p} = \left\|\int_\Sigma F\,\mathrm{d}\rho\right\|_{\mathcal{S}_p} \leq \|b_1\|_{\mathcal{S}_{p_1}} \cdots \|b_k\|_{\mathcal{S}_{p_k}} \int_\Sigma \prod_{i=1}^{k+1} \|\varphi_i(\cdot,\sigma)\|_{\ell^\infty(\sigma(a_i))}\,\rho(\mathrm{d}\sigma).$$

Taking the infimum over ℓ^∞-IPDs of φ then gives the desired estimate. □

Proposition 3.9.5 (Taylor remainder formula)**.** *Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{M}$ be a von Neumann algebra containing $\mathcal{A}$ as a unital C^*-subalgebra. If $f \in C^{[k]}(\mathbb{R})$, then*

$$R_{k,f_{\mathcal{A}},a}(b) = \underbrace{\int_{\sigma(a)} \cdots \int_{\sigma(a)}}_{k \text{ times}} \int_{\sigma(a+b)} f^{[k]}(\boldsymbol{\lambda})\, P^{a+b}(\mathrm{d}\lambda_1)\, b\, P^a(\mathrm{d}\lambda_2) \cdots b\, P^a(\mathrm{d}\lambda_{k+1}), \qquad a,b \in \mathcal{A}_{\mathrm{sa}}, \tag{3.9.6}$$

for all $a,b \in \mathcal{A}_{\mathrm{sa}}$, where the right-hand side of Equation (3.9.6) *is an MOI in $\mathcal{M}$.*

Proof. In this proof, we shall use perturbation formulas from Chapter 6 that were avoided in the previous section by using polynomial approximation arguments. First, by a smooth cutoff argument, it suffices to assume $f \in C^{[k]}(\mathbb{R})$ is compactly supported, in which case $f^{[i]} \in \ell^\infty(\mathbb{R},\mathcal{B}_{\mathbb{R}})^{\hat{\otimes}_i(i+1)}$ for all $i \in \{0,\ldots,k+1\}$. (This will ensure that we can apply the aforementioned perturbation formulas.) Under this assumption, we prove Equation (3.9.6) by induction on $k \geq 1$.

To begin, we have

$$R_{1,f_{\mathcal{A}},a}(b) = f(a+b) - f(a) = \int_{\sigma(a)} \int_{\sigma(a+b)} f^{[1]}(\lambda_1,\lambda_2)\, P^{a+b}(\mathrm{d}\lambda_1)\, b\, P^a(\mathrm{d}\lambda_2)$$

by Equation (6.5.8) in Theorem 6.5.7. Now, assume Equation (3.9.6) holds. Then

$$\begin{aligned} R_{k+1,f_{\mathcal{A}},a}(b) &= R_{k,f_{\mathcal{A}},a}(b) - \frac{1}{k!}\partial_b^k f_{\mathcal{A}}(a) = \big(I^{a+b,a,\ldots,a} f^{[k]}\big)[b,\ldots,b] - \big(I^{a,\ldots,a} f^{[k]}\big)[b,\ldots,b] \\ &= \int_{\sigma(a)} \cdots \int_{\sigma(a)} \int_{\sigma(a+b)} f^{[k+1]}(\lambda_1,\ldots,\lambda_{k+2})\, P^{a+b}(\mathrm{d}\lambda_1)\, b\, P^a(\mathrm{d}\lambda_2) \cdots b\, P^a(\mathrm{d}\lambda_{k+2}) \end{aligned}$$

by the induction hypothesis, Corollaries 3.1.10 and 3.4.7, and Equation (6.5.9) in Theorem 6.5.7. This completes the proof. □

Proposition 3.9.7 (Taylor remainder estimates)**.** *If H is a complex Hilbert space H, $f \in \mathcal{C}^{[k]}(\mathbb{R})$, $a, b \in B(H)_{\mathrm{sa}}$, and $p \in [1,\infty]$, then*

$$\big\|R_{k,f_{B(H)},a}(b)\big\|_{\mathcal{S}_p} \le \big\|f^{[k]}\big\|_{\ell^\infty(\sigma(a+b),\mathcal{B}_{\sigma(a+b)})\hat{\otimes}_i \ell^\infty(\sigma(a),\mathcal{B}_{\sigma(a)})^{\hat{\otimes}_i k}} \|b\|_{\mathcal{S}_{kp}}^k.$$

In particular, if $b \in \mathcal{S}_{kp}(H)$ as well, then $R_{k,f_{B(H)},a}(b) \in \mathcal{S}_p(H)$.

Proof. Combine Proposition 3.9.5 (with $\mathcal{A} = \mathcal{M} = B(H)$) and Lemma 3.9.4. □

Proof of a weaker result without Proposition 3.9.5. We prove the following weaker result without Proposition 3.9.5 (thus avoiding the use of the perturbation formulas from Chapter 6): If H is a complex Hilbert space H, $f \in NC^k(\mathbb{R})$, $a, b \in B(H)_{\mathrm{sa}}$, $r := \max\{\|a + tb\|_{H\to H} : t \in [0,1]\}$, and $p \in [1,\infty]$, then

$$\big\|R_{k,f_{B(H)},a}(b)\big\|_{\mathcal{S}_p} \le \big\|f^{[k]}\big\|_{r,k+1} \|b\|_{\mathcal{S}_{kp}}^k, \tag{3.9.8}$$

In particular, if $b \in \mathcal{S}_{kp}(H)$ as well, then $R_{k,f_{B(H)},a}(b) \in \mathcal{S}_p(H)$.

To begin, we recall one form of Taylor's theorem [HJ14, Thm. 1.107]: If V is a normed vector space, W is a Banach space, and $F \in C^k(V;W)$, then

$$R_{k,F,p}(h) = \frac{1}{(k-1)!}\int_0^1 (1-t)^{k-1}\partial_h^k F(p+th)\,\mathrm{d}t, \qquad p, h \in V.$$

In particular, if $f \in NC^k(\mathbb{R})$, then, by Theorem 3.8.20 (with $\mathcal{A} = \mathcal{M} = B(H)$), we have

$$R_{k,f_{B(H)},a}(b) = k\int_0^1 (1-t)^{k-1}\big(I^{a+tb,\ldots,a+tb} f^{[k]}\big)[b,\ldots,b]\,\mathrm{d}t. \tag{3.9.9}$$

In this case, the integral above is a pointwise Pettis integral, as we urge the reader to check. Now, if $t \in [0,1]$, then $\sigma(a+tb) \subseteq [-r,r]$. Therefore, if $p \in [1,\infty]$, then Lemma 3.9.4 gives

$$\big\|\big(I^{a+tb,\ldots,a+tb}f^{[k]}\big)[b,\ldots,b]\big\|_{\mathcal{S}_p} \leq \big\|f^{[k]}\big\|_{\ell^\infty(\sigma(a+tb),\mathcal{B}_{\sigma(a+tb)})^{\hat{\otimes}_i(k+1)}}\|b\|_{\mathcal{S}_{kp}}^k \leq \big\|f^{[k]}\big\|_{r,k+1}\|b\|_{\mathcal{S}_{kp}}^k.$$

Thus,

$$\big\|R_{k,f_{B(H)},a}(b)\big\|_{\mathcal{S}_p} \leq k\big\|f^{[k]}\big\|_{r,k+1}\|b\|_{\mathcal{S}_{kp}}^k\int_0^1(1-t)^{k-1}\,\mathrm{d}t = \big\|f^{[k]}\big\|_{r,k+1}\|b\|_{\mathcal{S}_{kp}}^k$$

by Equation (3.9.9) and the Schatten norm Minkowski's integral inequality (Theorem 5.4.12 below). This completes the proof. □

Proof of Theorem 3.9.1. It is shown in [PSST17, App. A] that $f_k \in C^k(\mathbb{R})$. Now, let H be a complex Hilbert space. If $f \in \mathcal{C}^{[k]}(\mathbb{R})$, $a,b \in B(H)_{\mathrm{sa}}$, and $b \in \mathcal{S}_k(H)$, then $R_{k,f_{B(H)},a}(b) \in \mathcal{S}_1(H)$ by Proposition 3.9.7. Consequently, $f_k \notin \mathcal{C}^{[k]}(\mathbb{R})$ by Theorem 3.9.3. □

Proof that $f_k \notin NC^k(\mathbb{R})$ without Proposition 3.9.5. Let H be a complex Hilbert space. If $f \in NC^k(\mathbb{R})$, $a,b \in B(H)_{\mathrm{sa}}$, and $b \in \mathcal{S}_k(H)$, then $R_{k,f_{B(H)},a}(b) \in \mathcal{S}_1(H)$ by Inequality (3.9.9). Consequently, $f_k \notin NC^k(\mathbb{R})$ by Theorem 3.9.3. □

In view of Theorem 3.8.20, there is another possible approach to proving $f_k \notin NC^k(\mathbb{R})$.

Conjecture 3.9.10. *If f_k is as in Theorem 3.9.1, then $(f_k)_{B(\ell^2(\mathbb{N}))}\colon B\big(\ell^2(\mathbb{N})\big)_{\mathrm{sa}} \to B\big(\ell^2(\mathbb{N})\big)$ is not k-times Fréchet differentiable.*

If this conjecture is correct, then we would immediately conclude $f_k \notin NC^k(\mathbb{R})$ from Theorem 3.8.20. Based partly on private correspondence with E. McDonald and F. A. Sukochev, it seems possible that ideas from [PSST17] and [AP16] could be adapted to prove Conjecture 3.9.10, but to the author's knowledge, this has never been carried out. Interestingly, for $k \geq 2$, it even seems to be the case that the literature lacks explicit examples of functions $f \in C^k(\mathbb{R})$ such that $f_{B(\ell^2(\mathbb{N}))}\colon B\big(\ell^2(\mathbb{N})\big)_{\mathrm{sa}} \to B\big(\ell^2(\mathbb{N})\big)$ has been confirmed not to be k-times Fréchet differentiable, though it is widely accepted that such functions should exist. (By results of Peller [Pel85], any $f \in C^1(\mathbb{R}) \setminus \dot{B}_1^{1,1}(\mathbb{R})_{\mathrm{loc}}$ would do for the $k=1$ case.)

3.10 Proof of Theorem 3.1.2 by polynomial approximation

In this section, we prove Theorem 3.1.2 using the polynomial approximation approach of Daletskii–Krein [DK56] (and Hiai [Hia10]). At this time, the reader should review Notation 3.1.1 and recall that if $a \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}$ and $f\colon \sigma(a) \to \mathbb{C}$ is a function, then $f(a) = \sum_{\lambda \in \sigma(a)} f(\lambda)\, P_\lambda^a$.

Lemma 3.10.1. *Let $\mathcal{A} \subseteq \mathrm{M}_n(\mathbb{C})$ be a unital $*$-subalgebra, and let $\mathcal{I} \trianglelefteq \mathcal{A}$ be an ideal. If $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in \mathcal{A}_{\mathrm{sa}}^{k+1}$, $\varphi\colon \sigma(a_1) \times \cdots \times \sigma(a_{k+1}) \to \mathbb{C}$ is a function, $i \in \{1, \ldots, k\}$, and $b = (b_1, \ldots, b_k) \in \mathcal{A}^{i-1} \times \mathcal{I} \times \mathcal{A}^{k-i}$, then $\varphi_\otimes(\mathbf{a})\#_k b \in \mathcal{I}$.*

Proof. Let $a \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}$, and write $\mathcal{A}_0 \subseteq \mathrm{M}_n(\mathbb{C})$ for the unital ($*$-)subalgebra generated by a. Since $\sigma(a) \subseteq \mathbb{R}$ is finite, if $f\colon \sigma(a) \to \mathbb{C}$ is a function, there exists a polynomial $p(\lambda) \in \mathbb{C}[\lambda]$ such that $p|_{\sigma(a)} = f$. Thus, $f(a) = p(a) \in \mathcal{A}_0$. In particular, $P_\lambda^a = 1_{\{\lambda\}}(a) \in \mathcal{A}_0$ for all $\lambda \in \sigma(a)$. Since $\mathcal{I} \trianglelefteq \mathcal{A}$, it follows that if $\mathbf{a}$ and b are as in the statement, then $P_{\lambda_1}^{a_1} b_1 \cdots P_{\lambda_k}^{a_k} b_k P_{\lambda_{k+1}}^{a_{k+1}} \in \mathcal{I}$ whenever $\lambda_i \in \sigma(a_i)$ for all $i \in \{1, \ldots, k+1\}$. The result then follows from the definition of $\varphi_\otimes(\mathbf{a})\#_k b$. $\square$

Next, we present Löwner's formula, one of the first perturbation formulas.

Lemma 3.10.2 (Löwner's formula [Löw34])**.** *Let $a, b \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}$. If $f\colon \sigma(a) \cup \sigma(b) \to \mathbb{C}$ is a function, then*

$$f(a) - f(b) = f_\otimes^{[1]}(a,b)\#[a-b] = \sum_{\lambda \in \sigma(a)} \sum_{\mu \in \sigma(b)} f^{[1]}(\lambda,\mu)\, P_\lambda^a (a-b) P_\mu^b,$$

where $f^{[1]}(\lambda,\mu)$ may be assigned any value when $\lambda = \mu$.

Proof. We have

$$\begin{aligned} f(a) - f(b) &= \sum_{\lambda \in \sigma(a)} f(\lambda)\, P_\lambda^a - \sum_{\mu \in \sigma(b)} f(\mu)\, P_\mu^b = \sum_{\lambda \in \sigma(a)} \sum_{\mu \in \sigma(b)} (f(\lambda) - f(\mu)) P_\lambda^a P_\mu^b \\ &= \sum_{\lambda \in \sigma(a)} \sum_{\mu \in \sigma(b)} f^{[1]}(\lambda,\mu) P_\lambda^a (\lambda - \mu) P_\mu^b = \sum_{\lambda \in \sigma(a)} \sum_{\mu \in \sigma(b)} f^{[1]}(\lambda,\mu)\, P_\lambda^a (a-b) P_\mu^b. \end{aligned}$$

In the first line, we used that $\sum_{\lambda \in \sigma(a)} P_\lambda^a = I_n = \sum_{\mu \in \sigma(b)} P_\mu^b$. In the second line, we used that $P_\lambda^a P_\mu^a = \delta_{\lambda\mu} P_\mu^a$, $P_\lambda^b P_\mu^b = \delta_{\lambda\mu} P_\mu^b$, $a = \sum_{\lambda \in \sigma(a)} \lambda\, P_\lambda^a$, and $b = \sum_{\lambda \in \sigma(b)} \lambda\, P_\lambda^b$. $\square$

Combining Lemmas 3.10.1 and 3.10.2, we see that if $\mathcal{A} \subseteq \mathrm{M}_n(\mathbb{C})$ is a unital $*$-subalgebra, $\mathcal{I} \trianglelefteq \mathcal{A}$ is an ideal, $a, b \in \mathcal{A}_{\mathrm{sa}}$, $f\colon \sigma(a) \cup \sigma(b) \to \mathbb{C}$ is a function, and $a - b \in \mathcal{I}$, then $f(a) - f(b) \in \mathcal{I}$, i.e., the perturbed matrix function $f_{a,\mathcal{I}}\colon \mathcal{I}_{\mathrm{sa}} \to \mathcal{I}$ in Theorem 3.1.2 is well defined. By another appeal to Lemma 3.10.1, to prove Theorem 3.1.2, it suffices to show that if $f \in C^k(\mathbb{R})$ and $f_{\mathrm{M}_n(\mathbb{C})}\colon \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}} \to \mathrm{M}_n(\mathbb{C})$ is defined by $a \mapsto f(a)$, then $f_{\mathrm{M}_n(\mathbb{C})} \in C^k(\mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}; \mathrm{M}_n(\mathbb{C}))$, and

$$\partial_{b_k} \cdots \partial_{b_1} f_{\mathrm{M}_n(\mathbb{C})}(a) = \sum_{\pi \in S_k} f_{\otimes}^{[k]}\big(a_{(k+1)}\big) \#_k \big[b_{\pi(1)}, \ldots, b_{\pi(k)}\big], \qquad a, b_1, \ldots, b_k \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}. \quad (3.10.3)$$

We encourage the reader to think about why.

We now set our sights on Equation (3.10.3).

Lemma 3.10.4 (Pure tensor functions)**.** *Let* $\mathbf{a} = (a_1, \ldots, a_m) \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}^m$. *If* $\varphi_i\colon \sigma(a_i) \to \mathbb{C}$ *is a function for all* $i \in \{1, \ldots, m\}$ *and* $\varphi \coloneqq \varphi_1 \otimes \cdots \otimes \varphi_m$, *then* $\varphi_{\otimes}(\mathbf{a}) = \varphi_1(a_1) \otimes \cdots \otimes \varphi_m(a_m)$. *In particular, if* $P(\boldsymbol{\lambda}) = \sum_{|\alpha| \le d} c_\alpha \boldsymbol{\lambda}^\alpha \in \mathbb{C}[\boldsymbol{\lambda}] = \mathbb{C}[\lambda_1, \ldots, \lambda_m]$ *is an* m*-variate polynomial, then*

$$P_{\otimes}(\mathbf{a}) = \sum_{|\alpha| \le d} c_\alpha \, a_1^{\alpha_1} \otimes \cdots \otimes a_m^{\alpha_m}.$$

Proof. We have

$$\begin{aligned}
\varphi_{\otimes}(\mathbf{a}) &= \sum_{\boldsymbol{\lambda} \in \sigma(a_1) \times \cdots \times \sigma(a_m)} \varphi_1(\lambda_1) \cdots \varphi_m(\lambda_m) \, P_{\lambda_1}^{a_1} \otimes \cdots \otimes P_{\lambda_m}^{a_m} \\
&= \left(\sum_{\lambda_1 \in \sigma(a_1)} \varphi_1(\lambda_1) \, P_{\lambda_1}^{a_1} \right) \otimes \cdots \otimes \left(\sum_{\lambda_m \in \sigma(a_m)} \varphi_m(\lambda_m) P_{\lambda_m}^{a_m} \right) \\
&= \varphi_1(a_1) \otimes \cdots \otimes \varphi_m(a_m),
\end{aligned}$$

as desired. □

Lemma 3.10.5. *If* $\mathbf{a} \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}^{k+1}$ *and* $\varphi \in \sigma(a_1) \times \cdots \times \sigma(a_{k+1}) \to \mathbb{C}$ *is a function, then*

$$\|\#_k \varphi_{\otimes}(\mathbf{a})\|_{B_k(\mathrm{M}_n(\mathbb{C})^k; \mathrm{M}_n(\mathbb{C}))} \le n^k \max\{|\varphi(\boldsymbol{\lambda})| : \boldsymbol{\lambda} \in \sigma(a_1) \times \cdots \times \sigma(a_{k+1})\}, \quad (3.10.6)$$

where $\mathrm{M}_n(\mathbb{C})$ *is given the operator norm* $\|\cdot\| = \|\cdot\|_{\mathbb{C}^n \to \mathbb{C}^n}$.

Proof. If $b = (b_1, \ldots, b_k) \in \mathrm{M}_n(\mathbb{C})^k$, then

$$\begin{aligned}
\|\varphi_\otimes(\mathbf{a})\#_k b\| &= \left\| \sum_{\lambda_2\in\sigma(a_2),\ldots,\lambda_{k+1}\in\sigma(a_{k+1})} \varphi(a_1,\lambda_2,\ldots,\lambda_{k+1})\, b_1 P^{a_2}_{\lambda_2}\cdots b_k P^{a_{k+1}}_{\lambda_{k+1}} \right\| \\
&\leq \sum_{\lambda_2\in\sigma(a_2),\ldots,\lambda_{k+1}\in\sigma(a_{k+1})} \|\varphi(a_1,\lambda_2,\ldots,\lambda_{k+1})\|\, \|b_1\| \left\|P^{a_2}_{\lambda_2}\right\| \cdots \|b_k\| \left\|P^{a_{k+1}}_{\lambda_{k+1}}\right\| \\
&= \sum_{\lambda_2\in\sigma(a_2),\ldots,\lambda_{k+1}\in\sigma(a_{k+1})} \max_{\lambda_1\in\sigma(a_1)} |\varphi(\lambda_1,\ldots,\lambda_{k+1})|\, \|b_1\|\cdots\|b_k\| \\
&\leq n^k \max\{|\varphi(\boldsymbol{\lambda})| : \boldsymbol{\lambda} \in \sigma(a_1)\times\cdots\times\sigma(a_{k+1})\}\, \|b_1\|\cdots\|b_k\|
\end{aligned}$$

because a_i has at most n distinct eigenvalues whenever $i \in \{1, \ldots, k+1\}$. $\square$

Remark 3.10.7. It turns out that

$$\|\#_k\varphi_\otimes(\mathbf{a})\|_{B_k((\mathrm{M}_n(\mathbb{C}),\|\cdot\|_{\mathrm{HS}})^k;(\mathrm{M}_n(\mathbb{C}),\|\cdot\|_{\mathrm{HS}}))} = \max\{|\varphi(\boldsymbol{\lambda})| : \boldsymbol{\lambda} \in \sigma(a_1)\times\cdots\times\sigma(a_{k+1})\},$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert–Schmidt norm; please see [ST19, Prop. 4.1.3]. Due to the inequality $\|\cdot\| \leq \|\cdot\|_{\mathrm{HS}} \leq n^{1/2}\|\cdot\|$, we therefore may replace the n^k in Inequality (3.10.6) with $n^{k/2}$. Note that even this sharper estimate depends on the dimension n in an unbounded way, which suggests difficulties with the infinite-dimensional case.

Lemma 3.10.8. *If $k \in \mathbb{N}$, then $\mathbb{C}[\lambda]$ is dense in $C^k(\mathbb{R})$ with the C^k topology.*

Proof. We first prove that if $r > 0$ and $f \in C^k(\mathbb{R})$, then there exists a sequence $(q_n)_{n\in\mathbb{N}}$ of polynomials such that for all $i \in \{0, \ldots, k\}$, $q_n^{(i)} \to f^{(i)}$ uniformly on $[-r, r]$ as $n \to \infty$. To this end, use the classical Weierstrass approximation theorem to find a sequence $(q_{0,n})_{n\in\mathbb{N}}$ of polynomials such that $q_{0,n} \to f^{(k)}$ uniformly on $[-r, r]$ as $n \to \infty$. Now, for $i \in \{1, \ldots, k\}$ and $n \in \mathbb{N}$, recursively define $q_{i,n}(\lambda) \coloneqq f^{(k-i)}(0) + \int_0^\lambda q_{i-1,n}(t)\,dt$ for all $\lambda \in \mathbb{R}$. Note that $q_{i,n}(\lambda) \in \mathbb{C}[\lambda]$. By an induction argument using the dominated convergence theorem and the fundamental theorem of calculus, the sequence $(q_n)_{n\in\mathbb{N}} \coloneqq (q_{k,n})_{n\in\mathbb{N}}$ accomplishes the stated goal. Next, let $f \in C^k(\mathbb{R})$. By what we just proved, if $N \in \mathbb{N}$, then there exists a $q_N(\lambda) \in \mathbb{C}[\lambda]$ such that $\left\|(f - q_N)^{(i)}\right\|_{\ell^\infty([-N,N])} < 1/N$ for all $i \in \{0, \ldots, k\}$. The sequence $(q_N)_{N\in\mathbb{N}}$ of polynomials converges to f in the C^k topology. $\square$

Proof of Equation (3.10.3). First, let $m \in \mathbb{N}_0$, and define $p_m(\lambda) \coloneqq \lambda^m \in \mathbb{C}[\lambda]$ as usual. Then the map $(p_m)_{\mathrm{M}_n(\mathbb{C})}\colon \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}} \to \mathrm{M}_n(\mathbb{C})$ is the restriction of the homogeneous polynomial $\mathrm{M}_n(\mathbb{C}) \ni a \mapsto a^n \in \mathrm{M}_n(\mathbb{C})$. Therefore, Equation (3.10.3) holds when $f = p_n$ by Proposition 1.2.6, Example 1.3.8, and Lemma 3.10.4. Consequently, by linearity, Equation (3.10.3) holds whenever $f(\lambda) \in \mathbb{C}[\lambda]$.

Next, recall that if V and W are normed vector spaces and $T \in B_k(V^k; W)$, then

$$S(T)[v_1, \ldots, v_k] \coloneqq \frac{1}{k!} \sum_{\pi \in S_k} T[v_{\pi(1)}, \ldots, v_{\pi(k)}], \qquad v_i \in V.$$

In this notation, Equation (3.10.3) rewrites to

$$D^k f_{\mathrm{M}_n(\mathbb{C})}(a) = k!\, S\Big(\#_k f_{\otimes}^{[k]}\big(a_{(k+1)}\big)\Big)\Big|_{\mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}^k}, \qquad a \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}. \tag{3.10.9}$$

To prove this equation, let $f \in C^k(\mathbb{R})$. By Lemma 3.10.8, there exists a sequence $(q_N(\lambda))_{n\in\mathbb{N}}$ in $\mathbb{C}[\lambda]$ converging to f in $C^k(\mathbb{R})$. Fix $r > 0$ and $i \in \{1, \ldots, k\}$, and define

$$\mathrm{M}_n(\mathbb{C})_{\mathrm{sa},r} \coloneqq \{a \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa}} : \|a\| \le r\} \ \text{ and } \ \|\cdot\|_i \coloneqq \|\cdot\|_{B_i(\mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}^i; \mathrm{M}_n(\mathbb{C}))}.$$

Then $\sup_{a \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa},r}} \|f(a) - q_N(a)\| = \|f - q_N\|_{\ell^\infty([-r,r])} \to 0$ as $N \to \infty$. Also, by the previous paragraph, Lemma 3.10.5, and Corollary 1.3.7, if $a \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa},r}$, then

$$\begin{aligned}\Big\|i!\, S\Big(\#_i f_{\otimes}^{[i]}\big(a_{(i+1)}\big)\Big) - D^i(q_N)_{\mathrm{M}_n(\mathbb{C})}(a)\Big\|_i &= i!\Big\|S\Big(\#_i (f - q_N)_{\otimes}^{[i]}\big(a_{(i+1)}\big)\Big)\Big\|_i \\ &\le i!\Big\|\#_i (f - q_N)_{\otimes}^{[i]}\big(a_{(i+1)}\big)\Big\|_i \le i!\, n^i \big\|(f - q_N)^{[i]}\big\|_{\ell^\infty(\sigma(a)^{i+1})} \le n^i \big\|(f - q_N)^{(i)}\big\|_{\ell^\infty([-r,r])}.\end{aligned}$$

In particular,

$$\sup_{a \in \mathrm{M}_n(\mathbb{C})_{\mathrm{sa},r}} \Big\|i!\, S\Big(\#_i f_{\otimes}^{[i]}\big(a_{(i+1)}\big)\Big) - D^i(q_N)_{\mathrm{M}_n(\mathbb{C})}(a)\Big\|_i \le n^i \big\|(f - q_N)^{(i)}\big\|_{\ell^\infty([-r,r])} \xrightarrow{N\to\infty} 0.$$

Since $r > 0$ and $i \in \{1, \ldots, k\}$ were arbitrary, Theorem 1.2.12 gives $f_{\mathrm{M}_n(\mathbb{C})} \in C^k(\mathrm{M}_n(\mathbb{C})_{\mathrm{sa}}; \mathrm{M}_n(\mathbb{C}))$ and Equation (3.10.9). This completes the proof. $\square$

The reason this proof works is that, in the finite-dimensional case, the map $b \mapsto f_{\otimes}^{[k]}(\mathbf{a})\#_k b$ satisfies a (dimension-dependent) operator norm estimate involving the uniform norm of $f^{[k]}$. In the infinite-dimensional case, the uniform norm is too weak for this operator norm estimate. However, there are stronger norms, e.g., the (ℓ^∞-integral) projective tensor norm, that work.

3.11 Acknowledgment

Chapter 3, in part, is a reprint of the material as it appears in "Noncommutative C^k functions and Fréchet derivatives of operator functions" (2023). Nikitopoulos, Evangelos A. *Expositiones Mathematicae*, 41, 115–163. Also, Chapter 3, in part, is being prepared for submission for publication of the material. Nikitopoulos, Evangelos A.

Chapter 4

Background II

In this chapter, we briefly review some additional background necessary for Chapters 5 and 6. Specifically, we discuss several standard topologies on spaces of operators (§4.1); projection-valued measures, the spectral theorem, and operators affiliated with von Neumann algebras (§4.2); and Schatten-class operators and noncommutative L^p-spaces of semifinite von Neumann algebras (§4.3).

Standing assumptions. Throughout, H and K are complex Hilbert spaces, and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_H$. Also, recall that if $S \subseteq B(H)$, then $S' = \{a \in B(H) : ab = ba \text{ for all } b \in S\}$.

4.1 Operator topologies

In this section, we record facts that we need about some standard locally convex topologies on $B(H;K)$. We assume the reader is quite familiar with these in the $H = K$ case, which is covered in [Tak79, Ch. II] and [Dix81, Pt. I, Ch. 3]. When $H \neq K$, all the basic properties still hold with essentially the same proofs.

Definition 4.1.1 (Operator topologies)**.** Recall that $\mathbb{R}_+ := [0, \infty)$.

(i) The **weak operator topology** (**WOT**) on $B(H;K)$ is the one generated by the seminorms

$$B(H;K) \ni A \mapsto |\langle Ah, k \rangle_K| \in \mathbb{R}_+, \quad h \in H, \ k \in K.$$

(ii) The **strong operator topology** (**SOT**) is generated by the seminorms

$$B(H;K) \ni A \mapsto \|Ah\|_K \in \mathbb{R}_+, \quad h \in H.$$

(iii) The **strong* operator topology** (**S*OT**) is generated by the seminorms

$$B(H;K) \ni A \mapsto \|Ah\|_K + \|A^*k\|_H \in \mathbb{R}_+, \qquad h \in H,\ k \in K.$$

Next, define

$$\ell^2(\mathbb{N};H) := \left\{ (h_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}} : \sum_{n=1}^{\infty} \|h_n\|_H^2 < \infty \right\}$$

with the inner product

$$\langle (h_n)_{n\in\mathbb{N}}, (k_n)_{n\in\mathbb{N}} \rangle_{\ell^2(\mathbb{N};H)} := \sum_{n=1}^{\infty} \langle h_n, k_n \rangle_H, \qquad (h_n)_{n\in\mathbb{N}}, (k_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};H).$$

(iv) The ***σ*-weak operator topology** (***σ*-WOT**) is generated by the seminorms

$$B(H;K) \ni A \mapsto |\langle (Ah_n)_{n\in\mathbb{N}}, (k_n)_{n\in\mathbb{N}} \rangle_{\ell^2(\mathbb{N};K)}| \in \mathbb{R}_+,$$

where $(h_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};H)$ and $(k_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};K)$.

(v) The ***σ*-strong operator topology** (***σ*-SOT**) is generated by the seminorms

$$B(H;K) \ni A \mapsto \|(Ah_n)_{n\in\mathbb{N}}\|_{\ell^2(\mathbb{N};K)} \in \mathbb{R}_+, \qquad (h_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};H).$$

(vi) The ***σ*-strong* operator topology** (***σ*-S*OT**) is generated by the seminorms

$$B(H;K) \ni A \mapsto \|(Ah_n)_{n\in\mathbb{N}}\|_{\ell^2(\mathbb{N};K)} + \|(A^*k_n)_{n\in\mathbb{N}}\|_{\ell^2(\mathbb{N};H)} \in \mathbb{R}_+,$$

where $(h_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};H)$ and $(k_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};K)$.

When referring to these topologies, we shall often omit the term "operator." Also, if $V \subseteq B(H;K)$, then the subspace topologies inherited by V from the above-defined topologies on $B(H;K)$ are given the same names as above. For example, the σ-weak topology on V is the subspace topology V inherits from the σ-WOT on $B(H;K)$.

Here are all the facts we need about these topologies.

Theorem 4.1.2 (Properties of operator topologies)**.** *Let $V \subseteq B(H;K)$ be a linear subspace, let $\ell\colon V \to \mathbb{C}$ be a linear functional, and fix $\mathcal{T} \in \{\text{WOT, SOT, S}^*\text{OT}\}$.*

(i) *The topology $\mathcal{T}$ agrees with the topology σ-$\mathcal{T}$ on norm-bounded subsets of $B(H;K)$. In particular, since the net of finite-rank orthogonal projections on K converges in the WOT to $1 = \mathrm{id}_K$, the finite-rank linear operators $H \to K$ are σ-weakly dense in $B(H;K)$.*

(ii) *ℓ is $\mathcal{T}$-continuous if and only if there exist $h_1, \ldots, h_n \in H$ and $k_1, \ldots, k_n \in K$ such that*

$$\ell(A) = \sum_{i=1}^{n} \langle Ah_i, k_i \rangle_K, \qquad A \in V.$$

(iii) *ℓ is σ-$\mathcal{T}$-continuous if and only if there exist $(h_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};H)$ and $(k_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};K)$ such that*

$$\ell(A) = \langle (Ah_n)_{n\in\mathbb{N}}, (k_n)_{n\in\mathbb{N}} \rangle_{\ell^2(\mathbb{N};K)} = \sum_{n=1}^{\infty} \langle Ah_n, k_n \rangle_K, \qquad A \in V.$$

Suppose now that $V \subseteq B(H;K)$ is also σ-weakly closed.

(iv) *If $V_* := \{\sigma\text{-weakly continuous linear functionals } V \to \mathbb{C}\} = (V, \sigma\text{-WOT})^*$, then $V_* \subseteq V^*$ is a (norm-)closed linear subspace, and the map $\mathrm{ev}_V\colon V \to {V_*}^*$ defined by $A \mapsto (\ell \mapsto \ell(A))$ is an isometric isomorphism. We therefore call V_* the* ***predual*** *of V.*

(v) *The map ev_V from the previous part is a homeomorphism with respect to the σ-weak topology on V and the weak* topology on ${V_*}^*$. The σ-weak topology on V is therefore also called the* ***weak* topology****.*

Finally, suppose $\mathcal{M} \subseteq B(H)$ and $\mathcal{N} \subseteq B(K)$ are von Neumann algebras.

(vi) *If $\pi\colon \mathcal{M} \to \mathcal{N}$ is a unital $*$-isomorphism in the algebraic sense, then π is a homeomorphism with respect to the σ-weak topologies on $\mathcal{M}$ and $\mathcal{N}$.*

If $H = K$, then the first five items are proven in [Dix81, Pt. I, §3.1] and [Tak79, §II.2]. The proofs of these statements when $H \neq K$ are slight notational modifications of the proofs in the aforementioned references. The final item is part of [Dix81, Pt. I, Cor. 4.1].

4.2 Unbounded operators and the spectral theorem

Here, we provide information about unbounded operators, projection-valued measures, and the spectral theorem that is necessary for our purposes. Please see [BS80, Chs. 3–6] or [Con90, Chs. IX & X] for more information and proofs of the facts that are stated without proof.

Definition 4.2.1 (Unbounded operator)**.** An (**unbounded linear**) **operator** A from H to K or $H \to K$ ("on H" if $H = K$) is a linear subspace $\operatorname{dom}(A) \subseteq H$—the **domain** of A—and a linear map $A\colon \operatorname{dom}(A) \to K$. The operator A is

(i) **densely defined** if $\operatorname{dom}(A) \subseteq H$ is dense;

(ii) **closable** if the closure in $H \times K$ of its **graph** $\Gamma(A) \coloneqq \{(h, Ah) : h \in \operatorname{dom}(A)\}$ is the graph of an operator $\overline{A}$ from H to K, called the **closure** of A; and

(iii) **closed** if $\Gamma(A)$ is closed in $H \times K$, i.e., $\overline{A} = A$.

$C(H; K)$ is the set of closed, densely defined operators $H \to K$, and $C(H) \coloneqq C(H; H)$.

Notation 4.2.2 (Sum, product, and containment)**.** If A and B are operators $H \to K$, then $A+B$ is the operator $H \to K$ defined by $\operatorname{dom}(A+B) \coloneqq \operatorname{dom}(A) \cap \operatorname{dom}(B)$ and $(A+B)h \coloneqq Ah + Bh$ for $h \in \operatorname{dom}(A+B)$. Also, we write $A \subseteq B$ if $\Gamma(A) \subseteq \Gamma(B)$, i.e., $\operatorname{dom}(A) \subseteq \operatorname{dom}(B)$ and $Ah = Bh$ for all $h \in \operatorname{dom}(A)$, and $A = B$ if $A \subseteq B$ and $B \subseteq A$. Finally, if $\mathcal{L}$ is another complex Hilbert space and C is an operator $K \to \mathcal{L}$, then CA is the operator $H \to \mathcal{L}$ defined by $\operatorname{dom}(CA) \coloneqq A^{-1}(\operatorname{dom}(C))$ and $(CA)h \coloneqq C(Ah)$ for $h \in \operatorname{dom}(CA)$.

Definition 4.2.3 (Adjoint)**.** If A is a densely defined operator $H \to K$, then its **adjoint** A^* is the operator $K \to H$ defined by: $\operatorname{dom}(A^*)$ is the set of $k \in K$ such that the linear functional $\operatorname{dom}(A) \ni h \mapsto \langle Ah, k\rangle_K \in \mathbb{C}$ is bounded, and for $k \in \operatorname{dom}(A^*)$, $A^*k \in H$ is the unique vector in H such that $\langle Ah, k\rangle_K = \langle h, A^*k\rangle_H$ for all $h \in \operatorname{dom}(A)$. An operator $A \in C(H)$ is

(i) **normal** (written $A \in C(H)_\nu$) if $A^*A = AA^*$,

(ii) **self-adjoint** (written $A \in C(H)_{\mathrm{sa}}$) if $A^* = A$, and

(iii) **positive** (written $A \in C(H)_+$) if A is self-adjoint and $\langle Ah, h\rangle \geq 0$ whenever $h \in \operatorname{dom}(A)$.

Proposition 4.2.4 (Properties of the adjoint [Con90, Prop. IX.1.6])**.** *If A is a densely defined operator from H to K, then A^* is a closed operator from K to H. Moreover, A^* is densely defined if and only if A is closable, in which case $\overline{A} = A^{**} := (A^*)^*$.*

We now extend the notion of spectrum to unbounded operators.

Definition 4.2.5 (Resolvent and spectrum)**.** If A is on operator on H, then the **resolvent set** $\rho(A) \subseteq \mathbb{C}$ of A is the set of $\lambda \in \mathbb{C}$ such that $\lambda - A = \lambda \, \mathrm{id}_H - A \colon \mathrm{dom}(A) \to H$ is a linear isomorphism with bounded inverse; in this case, we view $(\lambda - A)^{-1}$ as a member of $B(H)$. The **spectrum** of A is the set $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

Proposition 4.2.6 (Properties of the resolvent and spectrum [Con90, Props. X.1.15 & X.1.17])**.** *If A is an operator on H, then $\rho(A) \subseteq \mathbb{C}$ is open (empty if A is not closed); thus, $\sigma(A) \subseteq \mathbb{C}$ is closed. Moreover, the **resolvent** $\rho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in B(H)$ is holomorphic.*

Next, we move on to basic definitions and facts about projection-valued measure theory.

Notation 4.2.7 (Projections)**.** If $\mathcal{M} \subseteq B(H)$ is a von Neumann algebra, then

$$\mathrm{Proj}(\mathcal{M}) := \{ p \in \mathcal{M} : p^2 = p = p^* \}$$

is the lattice of (orthogonal) projections in $\mathcal{M}$.

Definition 4.2.8 (Projection-valued measure)**.** Let $(\Omega, \mathscr{F})$ be a measurable space. A map $P \colon \mathscr{F} \to B(H)$ is a **projection-valued measure** if it is a vector measure (Definition A.2.1(v)) with respect to the WOT such that $P(\Omega) = 1 = \mathrm{id}_H$ and $P(G) \in \mathrm{Proj}(B(H))$ whenever $G \in \mathscr{F}$. In this case, $(\Omega, \mathscr{F}, H, P)$ is a **projection-valued measure space**. Also, a property holds **P-almost everywhere** if there exists a $G \in \mathscr{F}$ with $P(\Omega \setminus G) = 0$ on which the property holds.

It is common to include the requirement that

$$P(G_1 \cap G_2) = P(G_1)\, P(G_2), \qquad G_1, G_2 \in \mathscr{F},$$

in the definition of a projection-valued measure. However, by [BS80, Thm. 5.1.1], the definition given above implies this property.

Of course, a measure's purpose in life is usually to integrate functions. This certainly is true for projection-valued measures, so we now turn to the construction of integrals of scalar functions with respect to projection-valued measures. (Please review Notation 1.3.12 at this time.)

Notation 4.2.9. If $(\Omega, \mathscr{F}, H, P)$ is a projection-valued measure space, then $\sim_P$ denotes the P-almost everywhere equivalence relation on $\ell^0(\Omega, \mathscr{F})$, and $L^0(P) := \ell^0(\Omega, \mathscr{F})/{\sim_P}$. Also,

$$\|f\|_{L^\infty(P)} := P\text{-}\operatorname*{ess\,sup}_{\omega \in \Omega} |f(\omega)| = \inf\{c \geq 0 : P(\{\omega \in \Omega : |f(\omega)| > c\}) = 0\}, \qquad f \in L^0(P),$$

and $L^\infty(P) := \{f \in L^0(P) : \|f\|_{L^\infty(P)} < \infty\}$.

By repeating the arguments from the scalar case, it is easy to see that $L^0(P)$ is a $*$-algebra and $\left(L^\infty(P), \|\cdot\|_{L^\infty(P)}\right)$ is a C^*-algebra under pointwise P-almost everywhere operations.

The result below summarizes much of the development in [BS80, Ch. 5].

Proposition 4.2.10 (Integration with respect to a projection-valued measure)**.** *Let* $(\Omega, \mathscr{F}, H, P)$ *be a projection-valued measure space, and fix* $f, g \in L^0(P)$*.*

(i) *Fix* $h, k \in H$*. If*

$$P_{h,k}(G) := \langle P(G)h, k\rangle, \qquad G \in \mathscr{F},$$

then $P_{h,k}$ *is a complex measure such that* $\|P_{h,k}\| \leq \|h\|\,\|k\|$*. Also,* $P_{h,k} \ll P$ *in the sense that if* $P(G) = 0$*, then* $P_{h,k}(G_0) = 0$ *whenever* $\mathscr{F} \ni G_0 \subseteq G$*, i.e.,* $|P_{h,k}|(G) = 0$*. Finally,* $P_{h,h}$ *is a (finite) positive measure.*

(ii) *If*

$$\operatorname{dom}(P(f)) := \left\{h \in H : \int_\Omega |f|^2 \, \mathrm{d}P_{h,h} < \infty\right\},$$

then $\operatorname{dom}(P(f)) \subseteq H$ *is a dense linear subspace. If* $h \in \operatorname{dom}(P(f))$*, then* $f \in L^1(|P_{h,k}|)$ *for all* $k \in H$*, and there exists a unique vector* $P(f)h \in H$ *satisfying*

$$\langle P(f)h, k\rangle = \int_\Omega f \, \mathrm{d}P_{h,k}, \qquad k \in H.$$

Furthermore, if we define $P(f)\colon \operatorname{dom}(P(f)) \to H$ *by* $h \mapsto P(f)h$*, then* $P(f) \in C(H)_\nu$*. The operator* $\int_\Omega f \, \mathrm{d}P = \int_\Omega f(\omega)\, P(\mathrm{d}\omega) := P(f)$ *is the* ***integral*** *of* f *with respect to* P*.*

(iii) $P(f)^* = P(\overline{f})$, $\operatorname{dom}(P(f)P(g)) = \operatorname{dom}(P(g)) \cap \operatorname{dom}(P(fg))$, $\overline{P(f)P(g)} = P(fg)$, *and* $\overline{P(f) + P(g)} = P(f+g)$. *In particular,* $P(f)^*P(f) = P(|f|^2)$; *and if* $g \in L^\infty(P)$, *then* $P(f)P(g) = P(fg)$, *and* $P(f+g) = P(f) + P(g)$.

(iv) *The map* $L^\infty(P) \ni f \mapsto P(f) \in B(H)$ *is an isometric, unital* $*$*-homomorphism.*

(v) *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* $L^\infty(P)$. *If* $\sup_{n\in\mathbb{N}} \|f_n\|_{L^\infty(P)} < \infty$ *and* $f_n \to f$ *pointwise* P*-almost everywhere, then* $P(f_n) \to P(f)$ *in the strong* operator topology as* $n \to \infty$.

The reason projection-valued measures are relevant for us is the spectral theorem.

Theorem 4.2.11 (Spectral theorem [Con90, Thm. X.4.11])**.** *If* $A \in C(H)_\nu$, *then there exists a unique projection-valued measure* $P^A \colon \mathcal{B}_\mathbb{C} \to B(H)$ *such that* $A = \int_\mathbb{C} \lambda \, P^A(\mathrm{d}\lambda)$. *Furthermore,* $P^A(\mathbb{C} \setminus \sigma(A)) = P^A(\rho(A)) = 0$, *and* $P^A(U) \neq 0$ *whenever* $U \subseteq \sigma(A)$ *is a nonempty open set.*

Proposition 4.2.12 (Agreement with continuous functional calculus)**.** *If* $A \in B(H)_\nu$ *and* P^A *is as in Theorem 4.2.11, then* $\int_{\sigma(A)} f \, \mathrm{d}P^A = \Phi_A(f)$ *for all* $f \in C(\sigma(A))$.

Proof. The map $C(\sigma(A)) \ni f \mapsto \int_\mathbb{C} f \, \mathrm{d}P^A = \int_{\sigma(A)} f \, \mathrm{d}P^A \in B(H)$ is a unital $*$-homomorphism sending $\iota_{\sigma(A)}$ to A by Proposition 4.2.10(iv) and the definition of P^A, so the result follows from the uniqueness part of Theorem 3.2.8. □

Consequently, the following does not clash with Definition 3.2.11.

Definition 4.2.13 (Projection-valued spectral measure and functional calculus)**.** Let $A \in C(H)_\nu$. The projection-valued measure P^A given by the spectral theorem is the **projection-valued spectral measure** of A; we frequently consider P^A to be a map $\mathcal{B}_{\sigma(A)} \to B(H)$. Also, define

$$f(A) \coloneqq P^A(f) = \int_{\sigma(A)} f \, \mathrm{d}P^A \in C(H)_\nu, \qquad f \in \ell^0\big(\sigma(A), \mathcal{B}_{\sigma(A)}\big).$$

The map $\ell^0\big(\sigma(A), \mathcal{B}_{\sigma(A)}\big) \ni f \mapsto f(A) \in C(H)_\nu$ is the (**Borel**) **functional calculus** of A.

The spectral theorem enables the construction of the absolute value of an arbitrary closed, densely defined operator on H. First, we comment that if $A \in C(H)_\nu$, then $A \in C(H)_{\mathrm{sa}}$ if and only if $\sigma(A) \subseteq \mathbb{R}$, and $A \in C(H)_+$ if and only if $\sigma(A) \subseteq [0, \infty)$.

Theorem 4.2.14 (Von Neumann's theorem [Con90, Prop. X.4.2(d)])**.** *If A is a closed, densely defined operator from H to K, then A^*A is a positive operator on H, i.e., $A^*A \in C(H)_+$. In particular, $\sigma(A^*A) \subseteq [0,\infty)$.*

Consequently, we can make the following definition via the functional calculus.

Definition 4.2.15 (Absolute value)**.** If $A \in C(H)$ is arbitrary, then $|A| \coloneqq (A^*A)^{\frac{1}{2}} \in C(H)_+$ is the **absolute value** of A.

Also, there exists a unique partial isometry $U \in B(H)$ with initial space $\overline{\operatorname{im}|A|} = \overline{\operatorname{im}(A^*)}$ and final space $\overline{\operatorname{im} A}$ such that $A = U|A|$. (In particular, $\operatorname{dom}(A) = \operatorname{dom}(|A|)$.) This is called the **polar decomposition** of A; please see [BS80, Thms. 8.1.2 & 8.1.3].

We end this section with the concept of an unbounded operator affiliated with $\mathcal{M}$. (This is the closet an unbounded operator can come to "being in" $\mathcal{M}$.)

Definition 4.2.16 (Affiliated operators)**.** An operator $a \in C(H)$ is **affiliated with $\mathcal{M}$** if $u^*au = a$ for all unitaries u belonging to the commutant $\mathcal{M}'$. In this case, we write $a \,\eta\, \mathcal{M}$. If, in addition, a is normal (respectively, self-adjoint), then we write $a \,\eta\, \mathcal{M}_\nu$ (respectively, $a \,\eta\, \mathcal{M}_{\mathrm{sa}}$).

Here are some properties of affiliated operators.

Proposition 4.2.17. *Let $(\Omega, \mathscr{F}, H, P)$ be a projection-valued measure space.*

(i) *If $a \in B(H)$, then $a \,\eta\, \mathcal{M}$ if and only if $a \in \mathcal{M}$.*

(ii) *If $P(G) \in \mathcal{M}$ for all $G \in \mathscr{F}$, then $P(f) \,\eta\, \mathcal{M}$ for all $f \in L^0(P)$. In particular, by the previous item, $P(f) \in \mathcal{M}$ for all $f \in L^\infty(P)$.*

(iii) *If $a \in C(H)_\nu$, then $a \,\eta\, \mathcal{M}$ if and only if $P^a(G) \in \mathcal{M}$ for all $G \in \mathcal{B}_{\sigma(a)}$, in which case $f(a) \,\eta\, \mathcal{M}$ for all $f \in L^0(P^a)$. In particular, $f(a) \in \mathcal{M}$ for all $f \in L^\infty(P^a)$.*

(iv) *If $a \in C(H)$ and $a = u|a|$ is its polar decomposition, then $a \,\eta\, \mathcal{M}$ if and only if $u \in \mathcal{M}$ and $P^{|a|}(G) \in \mathcal{M}$ for all $G \in \mathcal{B}_{\sigma(|a|)}$.*

We sketch the proofs for the reader's convenience. As we shall see, the first three properties follow without much difficulty from the definitions, the bicommutant theorem, and the spectral theorem. For the difficult part of the fourth item, please see also [MvN36, Lem. 4.4.1].

Sketch of proof. We take each item in turn.

(i) Fix $a \in B(H)$ and a unitary $u \in \mathcal{M}'$. If $a \in \mathcal{M}$, then, of course, $u^*au = u^*ua = a$. Now, if $a \,\eta\, \mathcal{M}$, then $au = uu^*au = ua$. Since all C^*-algebras are spanned by their unitaries, we conclude that $ab = ba$ for all $b \in \mathcal{M}'$. Thus, $a \in \mathcal{M}'' = \mathcal{M}$ by the bicommutant theorem.

(ii) Suppose $P(G) \in \mathcal{M}$ for all $G \in \mathscr{F}$. If $h, k \in H$ and $u \in \mathcal{M}'$ is a unitary, then it is easy to see that $P_{uh,uk} = P_{h,k}$. Unraveling the definition of $P(f)$ then gives $u^*P(f)u = P(f)$. Thus, $P(f) \,\eta\, \mathcal{M}$. It is worth mentioning that one can prove much more directly—without knowing anything about unbounded operators or the bicommutant theorem—that if $P(G) \in \mathcal{M}$ for all $G \in \mathscr{F}$, then $P(f) \in \mathcal{M}$ for all $f \in \ell^\infty(\Omega, \mathscr{F})$. Indeed, $\mathbb{H} := \{f \in \ell^\infty(\Omega, \mathscr{F}) : P(f) \in \mathcal{M}\}$ is a unital $*$-subalgebra of $\ell^\infty(\Omega, \mathscr{F})$ by Proposition 4.2.10(iv), $\mathbb{H}$ is closed under bounded convergence by Proposition 4.2.10(v), and $\{1_G : G \in \mathscr{F}\} \subseteq \mathbb{H}$ by assumption. By the multiplicative system theorem (Theorem 5.2.5), $\mathbb{H} = \ell^\infty(\Omega, \mathscr{F})$.

(iii) If $P^a(G) \in \mathcal{M}$ for all $G \in \mathcal{B}_{\sigma(a)}$, then $a = \int_{\sigma(a)} \lambda \, P^a(d\lambda) \,\eta\, \mathcal{M}$ by the previous item. Now, suppose $a \,\eta\, \mathcal{M}_\nu$, and let $u \in \mathcal{M}'$ be a unitary. Note that $Q^a := u^*P^a(\cdot)u \colon \mathcal{B}_{\sigma(a)} \to B(H)$ is a projection-valued measure, and it is easy to see from the spectral theorem and the definition of Q^a that $u^*au = \int_{\sigma(a)} \lambda \, Q^a(d\lambda)$. But $u^*au = a$ by assumption, so the uniqueness part of the spectral theorem forces $P^a = Q^a = u^*P^a(\cdot)u$. In other words, $P^a(G) \,\eta\, \mathcal{M}$ and thus, by the first item, $P^a(G) \in \mathcal{M}$ for all $G \in \mathcal{B}_{\sigma(a)}$.

(iv) Let $a \in C(H)$, let $a = u|a|$ be the polar decomposition of a, and let $v \in \mathcal{M}'$ be a unitary. If $P^{|a|}(G) \in \mathcal{M}$ for all $G \in \mathcal{B}_{\sigma(|a|)}$, then $|a| \,\eta\, \mathcal{M}$ by the previous item. If, in addition, $u \in \mathcal{M}$, then we have that

$$v^*av = v^*u|a|v = v^*uvv^*|a|v = u|a| = a.$$

Thus, $a \,\eta\, \mathcal{M}$. Conversely, if $a \,\eta\, \mathcal{M}$, then $|a| = |v^*av| = v^*|a|v$. Thus, $|a| \,\eta\, \mathcal{M}$, and by the previous item, $P^{|a|}(G) \in \mathcal{M}$ for all $G \in \mathcal{B}_{\sigma(|a|)}$. Next, notice that v^*uv is a partial isometry, and $a = v^*av = v^*u|a|v = v^*uv|a|$ by what just proved. Finally, $|a| \,\eta\, \mathcal{M}$ implies that v^*uv has initial space $\overline{\operatorname{im}|a|}$, and $a \,\eta\, \mathcal{M}$ implies that v^*uv has final space $\overline{\operatorname{im} a}$. We conclude that $v^*uv = u$ by the uniqueness of the polar decomposition. Thus, $u \,\eta\, \mathcal{M}$ and so, by the first item, $u \in \mathcal{M}$. $\square$

4.3 Schatten classes and noncommutative L^p-spaces

We now record some standard facts about Schatten p-class operators $H \to K$ that will be of use to us. Please see [Rin71, Ch. 2] for the proofs of these basics (and more) in the $H = K$ case. For just the cases $p \in \{1, 2, \infty\}$, please see [Con00, §§18–20] as well. As with the material in §4.1, all the basic properties in the $H \neq K$ case have essentially the same proofs; the main tools this time are the singular value and polar decompositions.

Definition 4.3.1 (Schatten classes)**.** If $p \in [1, \infty)$, $\mathcal{E} \subseteq H$ is an orthonormal basis, and $A \in B(H; K)$, then we define

$$\|A\|_{\mathcal{S}_p(H;K)} = \|A\|_{\mathcal{S}_p} := \left(\sum_{e \in \mathcal{E}} \langle |A|^p e, e \rangle_H \right)^{\frac{1}{p}} \in [0, \infty]$$

and $\mathcal{S}_p(H; K) := \{A \in B(H; K) : \|A\|_{\mathcal{S}_p} < \infty\}$. Also, we define $\mathcal{S}_\infty(H; K) := B(H; K)$ with the operator norm

$$\|A\|_{\mathcal{S}_\infty(H;K)} = \|A\|_{\mathcal{S}_\infty} := \|A\| = \|A\|_{H \to K}.$$

For $p \in [1, \infty]$, $\mathcal{S}_p(H; K)$ is the set of **Schatten** (or **Schatten–von Neumann**) ***p*-class operators** from H to K. Also, we write $\mathcal{K}(H; K) := \{\text{compact linear operators } H \to K\}$, $\mathcal{K}(H) := \mathcal{K}(H; H)$, and $\mathcal{S}_p(H) := \mathcal{S}_p(H; H)$.

Remark 4.3.2. We caution the reader that $\mathcal{S}_\infty(H; K)$ sometimes is taken to be the space of compact operators $H \to K$, and often the letter $\mathcal{C}$ is used instead of $\mathcal{S}$.

Theorem 4.3.3 (Properties of Schatten classes)**.** *Let* $p \in [1, \infty]$.

(i) $(\mathcal{S}_p(H; K), \|\cdot\|_{\mathcal{S}_p})$ *is a Banach space,* $\|\cdot\|_{\mathcal{S}_p}$ *is independent of the chosen orthonormal basis, and when* $p < \infty$*, the set of finite-rank linear operators* $H \to K$ *is dense in* $\mathcal{S}_p(H; K)$*. Also,* $(\mathcal{K}(H; K), \|\cdot\|)$ *is a Banach space with the set of finite-rank linear operators* $H \to K$ *as a dense linear subspace. Finally, if* $1 \leq p \leq q < \infty$*, then* $\mathcal{S}_p(H; K) \subseteq \mathcal{S}_q(H; K) \subseteq \mathcal{K}(H; K)$*, and the inclusions* $\mathcal{S}_p \hookrightarrow \mathcal{S}_q \hookrightarrow K$ *each have operator norm at most one.*

(ii) *If $A \in B(H)$ and $\mathcal{E} \subseteq H$ is an orthonormal basis, then*

$$\sum_{e \in \mathcal{E}} |\langle Ae, e\rangle_H| \le \|A\|_{\mathcal{S}_1}.$$

If $A \in \mathcal{S}_1(H)$, then

$$\mathrm{Tr}(A) \coloneqq \sum_{e \in \mathcal{E}} \langle Ae, e\rangle_H \in \mathbb{C}$$

is the ***trace*** *of A and is independent of the choice of $\mathcal{E}$. Furthermore, $\|A^*\|_{\mathcal{S}_1} = \|A\|_{\mathcal{S}_1}$ and $\mathrm{Tr}(A^*) = \overline{\mathrm{Tr}(A)}$ for all $A \in \mathcal{S}_1(H)$.*

(iii) (Hölder's inequality) *If $H_1, \ldots, H_{k+1}$ are complex Hilbert spaces and $p_1, \ldots, p_k \in [1, \infty]$ satisfy $1/p = 1/p_1 + \cdots + 1/p_k$, then*

$$\|A_1 \cdots A_k\|_{\mathcal{S}_p(H_{k+1};H_1)} \le \|A_1\|_{\mathcal{S}_{p_1}(H_2;H_1)} \cdots \|A_k\|_{\mathcal{S}_{p_k}(H_{k+1};H_k)}$$

for all $A_1 \in B(H_2; H_1), \ldots, A_k \in B(H_{k+1}; H_k)$. (As usual, $0 \cdot \infty \coloneqq 0$.)

(iv) *If $q \in [1, \infty]$, $1/p + 1/q = 1$, and $A \in \mathcal{S}_p(H; K)$, $B \in \mathcal{S}_q(K; H)$, then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.*

(v) *If $p \in [1, \infty), q \in (1, \infty]$, and $1/p + 1/q = 1$, then $\mathcal{S}_q(H; K) \cong \mathcal{S}_p(K; H)^*$ isometrically via $A \mapsto (B \mapsto \mathrm{Tr}(AB))$. Also, $\mathcal{S}_1(H; K) \cong \mathcal{K}(K; H)^*$ isometrically via the same map.*

(vi) *The weak* topology on $B(H; K)$ induced by the identification*

$$B(H; K) = \mathcal{S}_\infty(H; K) \cong \mathcal{S}_1(K; H)^*$$

is called the ***ultraweak topology****, and it agrees with the σ-WOT. In particular, finite-rank linear operators $H \to K$ are ultraweakly dense in $B(H; K)$.*

Next, we review some basics of semifinite von Neumann algebras and noncommutative L^p-spaces. (The reader who is uninterested in semifinite von Neumann algebras may skip at this time to Chapter 5.)

Notation 4.3.4. If $a, b \in B(H)$, then we write $a \le b$ or $b \ge a$ to mean that $b - a \in B(H)_+$. If $\mathcal{M} \subseteq B(H)$ is a von Neumann algebra, then $\mathcal{M}_+ \coloneqq \{a \in \mathcal{M} : a \ge 0\} = B(H)_+ \cap \mathcal{M}$.

It is easy to see that $\mathcal{M}_+$ is closed in the WOT. We also have the following.

Proposition 4.3.5 (Vigier's theorem [Con00, Prop. 43.1])**.** *Let $\mathcal{M} \subseteq B(H)$ be a von Neumann algebra. If $(a_i)_{i \in I}$ is a net in $\mathcal{M}_{\mathrm{sa}}$ that is bounded and increasing ($i \leq j \Rightarrow a_i \leq a_j$), then $a := \sup_{i \in I} a_I$ exists in $B(H)_{\mathrm{sa}}$, and $\lim_{i \in I} a_i = a$ in the WOT. In particular, $a \in \mathcal{M}_{\mathrm{sa}}$, and $a \in \mathcal{M}_+$ whenever $\{a_i : i \in I\} \subseteq \mathcal{M}_+$.*

Definition 4.3.6 (Trace)**.** Let $\mathcal{M} \subseteq B(H)$ be a von Neumann algebra. A map $\tau \colon \mathcal{M}_+ \to [0, \infty]$ is a **trace** if

(i) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in \mathcal{M}_+$,

(ii) $\tau(\lambda a) = \lambda\, \tau(a)$ for all $a \in \mathcal{M}_+$ and $\lambda \in \mathbb{R}_+$, and

(iii) $\tau(c^* c) = \tau(c c^*)$ for all $c \in \mathcal{M}$.

A trace $\tau \colon \mathcal{M} \to [0, \infty]$ on $\mathcal{M}$ is

(iv) **normal** if

$$\tau\Big(\sup_{i \in I} a_i\Big) = \sup_{i \in I} \tau(a_i)$$

whenever $(a_i)_{i \in I}$ is a bounded and increasing net in $\mathcal{M}_+$,

(v) **faithful** if $a \in \mathcal{M}_+$ and $\tau(a) = 0$ imply $a = 0$, and

(vi) **semifinite** if $\tau(a) = \sup\{\tau(b) : a \geq b \in \mathcal{M}_+, \tau(b) < \infty\}$ for all $a \in \mathcal{M}_+$.

If τ is a normal, faithful, semifinite trace on $\mathcal{M}$, then $(\mathcal{M}, \tau)$ is called a **semifinite von Neumann algebra**.

Remark 4.3.7. In the presence of Conditions (i) and (ii), Condition (iii) is equivalent to $\tau(u^* a u) = \tau(a)$ for all $a \in \mathcal{M}_+$ and $u \in \mathrm{U}(\mathcal{M})$. This is [Dix81, Pt. I, Corollary 6.1].

For basic properties of traces on von Neumann algebras, please see [Dix81, Pt. I, Ch. 6] or [Tak79, §V.2]. Motivating examples of semifinite von Neumann algebras are $(B(H), \mathrm{Tr})$ and $(L^\infty(\Omega, \mu), \int_\Omega \cdot \, \mathrm{d}\mu)$, where $(\Omega, \mathscr{F}, \mu)$ is a localizable (e.g., σ-finite) measure space, and $L^\infty(\Omega, \mu)$ is represented as multiplication operators on $L^2(\Omega, \mu)$. We now record some basics of L^p-spaces associated to a normal, faithful, semifinite trace. We shall mostly draw results from [dS18, Dix53]. For more information and different perspectives, please see [FK86, Nel74, Ter81, Yea75].

Notation 4.3.8. Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra. If $a \in \mathcal{M}$, then

$$\|a\|_{L^p(\tau)} \coloneqq \tau(|a|^p)^{\frac{1}{p}} \in [0,\infty] \text{ and } \mathcal{L}^p(\tau) \coloneqq \{b \in \mathcal{M} : \|b\|_{L^p(\tau)}^p = \tau(|b|^p) < \infty\}, \quad p \in [1,\infty).$$

For the $p=\infty$ case, we take $\mathcal{L}^\infty(\tau) \coloneqq \mathcal{M}$ with $\|\cdot\|_{L^\infty(\tau)} \coloneqq \|\cdot\|_{\mathcal{M}}$.

It turns out that $\mathcal{L}^1(\tau) \subseteq \mathcal{M}$ is an ideal of $\mathcal{M}$ spanned by $\mathcal{L}^1(\tau)_+ = \mathcal{L}^1(\tau) \cap \mathcal{M}_+$. Furthermore, there exists a unique linear extension of $\tau|_{\mathcal{L}_1(\tau)_+} : \mathcal{L}_1(\tau)_+ \to \mathbb{C}$ to $\mathcal{L}^1(\tau)$, which we notate the same way, and this extension satisfies

$$\tau(ab) = \tau(ba) \quad a \in \mathcal{M},\ b \in \mathcal{L}^1(\tau).$$

Finally, if $b \in \mathcal{L}^1(\tau)$, then the map $\mathcal{M} \ni a \mapsto \tau(ab) \in \mathbb{C}$ is σ-weakly continuous. These facts are proven as [Dix81, Pt. I, Prop. 6.1], together with the sentence before [Dix81, Pt. I, Prop. 1.9].

Theorem 4.3.9 (Properties)**.** *Fix a semifinite von Neumann algebra $(\mathcal{M},\tau)$ and $p \in [1,\infty]$.*

(i) *$(\mathcal{L}^p(\tau), \|\cdot\|_{L^p(\tau)})$ is a normed vector space. Its completion, denoted by $(L^p(\tau), \|\cdot\|_{L^p(\tau)})$, is the **noncommutative L^p-space** associated to $(\mathcal{M},\tau)$.*

(ii) *If $a \in \mathcal{L}^1(\tau)$, then $|\tau(a)| \le \tau(|a|) = \|a\|_{L^1(\tau)}$. Thus, $\tau\colon \mathcal{L}^1(\tau) \to \mathbb{C}$ extends uniquely to a bounded linear map, notated the same way, $L^1(\tau) \to \mathbb{C}$.*

(iii) (Nonommutative Hölder's inequality) *If $p_1,\ldots,p_k \in [1,\infty]$ satisfy $1/p = 1/p_1 + \cdots + 1/p_k$,*

$$\|a_1 \cdots a_k\|_{L^p(\tau)} \le \|a_1\|_{L^{p_1}(\tau)} \cdots \|a_k\|_{L^{p_k}(\tau)}, \quad a_1,\ldots,a_k \in \mathcal{M}.$$

(iv) *If $q \in [1,\infty]$ is such that $1/p + 1/q = 1$, then*

$$\|a\|_{L^p(\tau)} = \sup\{\|ab\|_{L^1(\tau)} : b \in \mathcal{L}^q(\tau),\ \|b\|_{L^q(\tau)} \le 1\}, \quad a \in \mathcal{M}.$$

If $(\mathcal{M},\tau) = (B(H),\mathrm{Tr})$, then $\mathcal{L}^p(\tau) = L^p(\tau) = \mathcal{S}_p(H)$ and $\|\cdot\|_{L^p(\tau)} = \|\cdot\|_{\mathcal{S}_p}$. Therefore, Theorem 4.3.9 generalizes parts of Theorem 4.3.3 in the case $H = K$.

Chapter 5

Multiple operator integrals

A multiple operator integral (MOI) is an indispensable tool in several branches of noncommutative analysis. However, there are substantial technical issues with the existing literature on the "separation of variables" approach to defining MOIs, especially when the underlying Hilbert spaces are not separable. In this chapter, we provide a detailed development of this approach in a very general setting that resolves existing technical issues. Along the way, we characterize several kinds of "weak" operator-valued integrals in terms of easily checkable conditions and prove a useful Minkowski-type integral inequality for maps with values in a semifinite von Neumann algebra.

Standing assumptions. Throughout, $k \in \mathbb{N}$; $H_1, \ldots, H_{k+1}, K, H$ are complex Hilbert spaces; and $\langle \cdot, \cdot \rangle_H = \langle \cdot, \cdot \rangle$. In §5.3, we retain the standing assumptions from §1.1; please see the beginning of Chapter 1. In §§5.5–5.8, $(\Omega_i, \mathscr{F}_i, H_i, P_i)$ is a projection-valued measure space for all $i \in \{1, \ldots, k+1\}$, and

$$(\Omega, \mathscr{F}, H, P) \coloneqq (\Omega_1 \times \cdots \times \Omega_{k+1}, \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_{k+1}, H_1 \otimes_2 \cdots \otimes_2 H_{k+1}, P_1 \otimes \cdots \otimes P_{k+1})$$

is their tensor product (Theorem 5.1.4). In §5.9, Ω is a set.

5.1 Introduction

Let $(\Omega, \mathscr{F}, H, P)$ be a projection-valued measure space. In Proposition 4.2.10, we described the construction of P-integrals of *scalar-valued* functions. However, there are instances where it seems necessary to define a notion of $\int_\Omega \Phi \, \mathrm{d}P$ for *operator-valued* functions $\Phi \colon \Omega \to B(H)$.

For example, when one studies the smoothness properties of scalar functions of operators [DK56, dPS04, Pel06, ACDS09, AP16, Pel16, CLMSS19, LMS20, LMM21, Nik23a, Nik23c] or spectral shift [AP11, DS14, PSUZ15, Skr18], one must consider integrals of the form

$$\int_{\Omega_{k+1}} \cdots \int_{\Omega_1} \varphi(\omega_1, \ldots, \omega_{k+1}) \, P_1(\mathrm{d}\omega_1) \, b_1 \cdots P_k(\mathrm{d}\omega_k) \, b_k \, P_{k+1}(\mathrm{d}\omega_{k+1}), \tag{5.1.1}$$

where $(\Omega_i, \mathscr{F}_i, H_i, P_i)$ is a projection-valued measure space, $\varphi \colon \Omega = \Omega_1 \times \cdots \times \Omega_{k+1} \to \mathbb{C}$ is a scalar function, and b_i is a bounded operator on H. The innermost integral $\int_{\Omega_1} \varphi(\cdot, \omega_2, \ldots, \omega_{k+1}) \, \mathrm{d}P_1$ makes sense using the standard theory from Proposition 4.2.10, but it is already unclear how to integrate the map $\omega_2 \mapsto \int_{\Omega_1} \varphi(\cdot, \omega_2, \ldots, \omega_{k+1}) \, \mathrm{d}P_1 \, b_1$ with respect to P_2. Yu. L. Daletskii and S. G. Krein made the first attempt at doing so in their seminal paper [DK56], wherein they used a Riemann–Stieltjes-type construction to define $\int_s^t \Phi(r) \, P(\mathrm{d}r)$ for certain Borel projection-valued measures on compact intervals $[s,t] \subseteq \mathbb{R}$ and maps $\Phi \colon [s,t] \to B(H)$. This approach, which requires rather stringent regularity assumptions on Φ, allows one to make sense of (5.1.1) as an iterated operator-valued integral, i.e., a *multiple operator integral*, for certain (highly regular) φ.

In general, an object that gives a rigorous meaning to (5.1.1) is called a **multiple operator integral** (MOI). Under the assumption that H is separable, these have been studied and applied extensively to various branches of noncommutative analysis. Please see A. Skripka and A. Tomskova's book [ST19] for an excellent survey of the MOI literature and its applications. In this chapter, we shall concern ourselves with the "separation of variables" approach to defining MOIs that is useful for differentiating operator functions; please see, e.g., [ACDS09, Nik23a, Pel06] and Chapter 6. Loosely speaking, this means that one assumes φ admits a decomposition

$$\varphi(\boldsymbol{\omega}) = \int_{\Sigma} \varphi_1(\omega_1, \sigma) \cdots \varphi_{k+1}(\omega_{k+1}, \sigma) \, \rho(\mathrm{d}\sigma), \qquad \boldsymbol{\omega} = (\omega_1, \ldots, \omega_{k+1}) \in \Omega, \tag{5.1.2}$$

where $(\Sigma, \mathscr{H}, \rho)$ is a measure space and $\varphi_i \colon \Omega_i \times \Sigma \to \mathbb{C}$ is a (product) measurable function, and then one defines (5.1.1) to be the "weak" operator-valued integral

$$\int_{\Sigma} P_1(\varphi_1(\cdot, \sigma)) \, b_1 \cdots P_k(\varphi_k(\cdot, \sigma)) \, b_k \, P_{k+1}(\varphi_{k+1}(\cdot, \sigma)) \, \rho(\mathrm{d}\sigma). \tag{5.1.3}$$

When taking this approach, there are at least three questions to be addressed.

(Q.1) Exactly which decompositions does one allow in Equation (5.1.2)?

(Q.2) Exactly what kind of operator-valued integral is (5.1.3)?

(Q.3) Assuming satisfactory answers to (Q.1) and (Q.2), does (5.1.3) depend on the decomposition chosen in Equation (5.1.2)?

There are various answers to these questions available in the literature, but the existing answers are inadequate to cover the case when H is not separable, and some of them have issues even when H is separable. (Please see, e.g., the comments in [DDSZ20, §4.6] and §6.7.) In this chapter, we provide detailed, very general answers to all three questions above without assuming that H is separable.

(A.1) We consider *integral projective decompositions* (Definition 5.5.3) of φ. In other words, we take φ in the *integral projective tensor product* $L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$, the idea for which is due to V. V. Peller [Pel06]. There are substantial "measurability issues," discussed in Remark 5.5.4, with existing definitions of this object. We resolve these in §5.5.

(A.2) Let $V \subseteq B(H;K)$ be a linear subspace. In Theorem 5.4.5, we characterize weak integrability of maps $\Sigma \to V$ in the weak, strong, strong*, σ-weak, σ-strong, and σ-strong* operator topologies on V. As an application of this independently interesting characterization, we prove in §5.6 that if $V = \mathcal{M} \subseteq B(H)$ is a von Neumann algebra and $P_i(G) \in \mathcal{M}$ for all $i \in \{1,\ldots,k+1\}$ and $G \in \mathscr{F}_i$, then the integrand in (5.1.3) is weakly integrable in the σ-weak operator (aka weak*) topology on $\mathcal{M}$ whenever $b_1,\ldots,b_k \in \mathcal{M}$.

(A.3) The independence of (5.1.3) of the chosen integral projective decomposition (5.1.2) of φ is highly nontrivial and has not yet been proven for non-separable H. In §5.6, we present a robust new argument that proves this fact for general H. The key ingredient to the argument, which we discuss in §5.2, is a basic fact from measure theory: the multiplicative system theorem (Theorem 5.2.5).

We also prove in §5.8 that the above-described approach to defining (5.1.1) agrees with another commonly used approach, due to B. S. Pavlov [Pav69], when both apply. Finally, even with all of (Q.1)–(Q.3) answered, applications often demand answers to an additional question.

(Q.4) What kinds of quantitative norm estimates for (5.1.1) are available?

Our development gives us some answers to this question as well.

(A.4) In §5.4, we prove Minkowski-type integral inequalities for Schatten p-norms and noncommutative L^p-norms of operator-valued integrals that seem to be new in the non-separable case and are of independent interest. These inequalities allow us to prove Schatten p-norm and noncommutative L^p-norm estimates for (5.1.1) in §5.7.

Actually, the aforementioned Minkowski-type integral inequalities can be combined with the theory of symmetric operator spaces to give a *much* more general answer to (Q.4). We carry this out in Chapter 6 and use it to prove new results about higher derivatives of operator functions in ideals of von Neumann algebras.

With this high-level overview under our belts, we give a precise summary of our main results on MOIs. For reasons explained at the beginning of §5.3 and in Remark 5.5.4, we shall be forced to integrate non-measurable functions. For this purpose, we use upper (and lower) integrals. If $(\Sigma, \mathscr{H}, \rho)$ is a measure space and $f\colon \Sigma \to [0,\infty]$ is any function, then

$$\overline{\int_\Sigma} f(\sigma)\,\rho(d\sigma) = \overline{\int_\Sigma} f\,d\rho := \inf\left\{\int_\Sigma \tilde{f}\,d\rho : f \leq \tilde{f} \;\; \rho\text{-a.e., } \tilde{f}\colon \Sigma \to [0,\infty] \text{ measurable}\right\}$$

is the **upper integral** of f. Proposition 5.3.2 lists the properties of this upper integral (and its lower counterpart) that we need.

Next, we state the precise definition of $L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$. To do so, we need the notion of the tensor product of projection-valued measures. We write $\otimes_2$ for the Hilbert space tensor product; please see the beginning of §5.9.

Theorem 5.1.4 (Birman–Solomyak [BS96]). *Let $(\Omega_1, \mathscr{F}_1, H_1, P_1), \ldots, (\Omega_{k+1}, \mathscr{F}_{k+1}, H_{k+1}, P_{k+1})$ be projection-valued measure spaces, and write $(\Omega, \mathscr{F}) := (\Omega_1 \times \cdots \times \Omega_{k+1}, \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_{k+1})$. There exists a unique projection-valued measure $P\colon \mathscr{F} \to B(H_1 \otimes_2 \cdots \otimes_2 H_{k+1})$ such that*

$$P(G_1 \times \cdots \times G_{k+1}) = P_1(G_1) \otimes \cdots \otimes P_{k+1}(G_{k+1}), \qquad G_1 \in \mathscr{F}_1, \ldots, G_{k+1} \in \mathscr{F}_{k+1}.$$

*We call P the **tensor product** of $P_1, \ldots, P_{k+1}$ and write $P_1 \otimes \cdots \otimes P_{k+1} := P$.*

For completeness, we supply a proof in §5.9. Now, retain the setup of Theorem 5.1.4, and let $\varphi\colon \Omega \to \mathbb{C}$ be a function. A $\boldsymbol{L_P^\infty}$**-integral projective decomposition** of φ is a choice $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ of a σ-finite measure space $(\Sigma, \mathscr{H}, \rho)$ and measurable functions $\varphi_i\colon \Omega_i \times \Sigma \to \mathbb{C}$ such that $\varphi_i(\cdot, \sigma) \in L^\infty(P_i)$ for all $\sigma \in \Sigma$,

$$\overline{\int_\Sigma} \|\varphi_1(\cdot,\sigma)\|_{L^\infty(P_1)} \cdots \|\varphi_{k+1}(\cdot,\sigma)\|_{L^\infty(P_{k+1})}\, \rho(\mathrm{d}\sigma) < \infty, \tag{5.1.5}$$

and Equation (5.1.2) holds P-almost everywhere. (The integral on the right-hand side of Equation (5.1.2) makes sense P-almost everywhere by Lemma 5.5.1.) Now, define $\|\varphi\|_{L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})}$ to be the infimum of the set of numbers (5.1.5) as $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ ranges over all L_P^∞-integral projective decompositions of φ. In §5.5, we prove that if $L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$ is the space of P-almost everywhere equivalence classes of functions φ admitting L_P^∞-integral projective decompositions, then $L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$ is a unital Banach $*$-algebra under P-almost everywhere operations and the norm $\|\cdot\|_{L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})}$.

Theorem 5.1.6 (Well-definition of MOIs)**.** *Let $\mathcal{M} \subseteq B(H)$ be a von Neumann algebra. Suppose, for each $i \in \{1, \ldots, k+1\}$, that $(\Omega_i, \mathscr{F}_i, H, P_i)$ is a projection-valued measure space such that $P_i(G) \in \mathcal{M}$ whenever $G \in \mathscr{F}_i$. If $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ is an L_P^∞-integral projective decomposition of a function $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$ and $b = (b_1, \ldots, b_k) \in \mathcal{M}^k$, then the map*

$$\Sigma \ni \sigma \mapsto P_1(\varphi_1(\cdot,\sigma))\, b_1 \cdots P_k(\varphi_k(\cdot,\sigma))\, b_k\, P_{k+1}(\varphi_{k+1}(\cdot,\sigma)) \in \mathcal{M}$$

is weakly integrable in the σ-weak operator topology on $\mathcal{M}$, and the weak integral

$$\big(I^{P_1,\ldots,P_{k+1}}\varphi\big)[b] := \int_\Sigma P_1(\varphi_1(\cdot,\sigma))\, b_1 \cdots P_k(\varphi_k(\cdot,\sigma))\, b_k\, P_{k+1}(\varphi_{k+1}(\cdot,\sigma))\, \rho(\mathrm{d}\sigma) \in \mathcal{M}$$

is independent of the chosen decomposition $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ and the representation of $\mathcal{M}$.

Proof. Combine Corollary 5.6.4, Theorem 5.6.11, and Theorem 5.6.20. □

We also prove in Proposition 5.7.1 that the assignment $\varphi \mapsto I^{P_1,\ldots,P_{k+1}}\varphi$ is linear and multiplicative in a certain sense. Finally, when $(\mathcal{M}, \tau)$ is a semifinite von Neumann algebra, we also prove (Proposition 5.7.3) that if $p, p_1, \ldots, p_k \in [1, \infty]$ are such that $1/p_1 + \cdots + 1/p_k = 1/p$, then $\left\|\left(I^{P_1,\ldots,P_{k+1}}\varphi\right)[b_1,\ldots,b_k]\right\|_{L^p(\tau)} \leq \|\varphi\|_{L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})}\|b_1\|_{L^{p_1}(\tau)}\cdots\|b_k\|_{L^{p_k}(\tau)}$ for all $b_1,\ldots,b_k \in \mathcal{M}$. This allows for an "extension" of the MOI $I^{P_1,\ldots,P_{k+1}}\varphi\colon \mathcal{M}^k \to \mathcal{M}$ to a bounded k-linear map $L^{p_1}(\tau)\times\cdots\times L^{p_k}(\tau)\to L^p(\tau)$.

5.2 Discussion of the well-definition argument

Retain the setup of Theorem 5.1.6 with $\mathcal{M} = B(H)$. In this section, we discuss the essential elements of the proof that the integral (5.1.3) is independent of the chosen L_P^∞-integral projective decomposition of φ and why this argument is delicate when H is not separable. To maximize readability, we stick to the case of a *double operator integral* (DOI), i.e., the case $k = 1$.

Let $b \in B(H)$. The goal is to show that if $(\Sigma,\rho,\varphi_1,\varphi_2)$ is a $L^\infty_{P_1\otimes P_2}$-integral projective decomposition of $\varphi \in L^\infty(P_1)\hat{\otimes}_i L^\infty(P_2)$, then $\int_\Sigma P_1(\varphi_1(\cdot,\sigma))\,b\,P_2(\varphi_2(\cdot,\sigma))\,\rho(\mathrm{d}\sigma)$ does not depend on $(\Sigma,\rho,\varphi_1,\varphi_2)$. This is actually not difficult to prove, as is done in [ACDS09, Pel16], when b has finite rank, so the proof is complete if we can somehow reduce to this case. In [Pel16], it is stated that this reduction is "easy to see." This is certainly not the case when H is not separable. When H is separable (as is assumed in [ACDS09]), every $b \in B(H)$ is a strong operator limit of a *sequence* of finite-rank operators. One can then use a vector-valued dominated convergence theorem to finish the proof. But this argument does not work when H is not separable because, for instance, id_H is not a strong operator limit of a sequence of finite-rank operators.

We opt instead to work with a different topology on $B(H)$ with respect to which finite-rank operators are dense: the ultraweak (aka σ-weak) topology. If we can show that the map

$$B(H) \ni b \mapsto I^{P_1,P_2}(\Sigma,\rho,\varphi_1,\varphi_2)[b] \coloneqq \int_\Sigma P_1(\varphi_1(\cdot,\sigma))\,b\,P_2(\varphi_2(\cdot,\sigma))\,\rho(\mathrm{d}\sigma) \in B(H)$$

is ultraweakly continuous, then the proof will be complete. This ultraweak continuity is asserted in [PS10] without proof or reference. When H is not separable, it is not at all obvious and, to the author's knowledge, has remained unproven until now. To prove it, we must show that for all $a \in \mathcal{S}_1(H)$, there exists a $Ta \in \mathcal{S}_1(H)$ such that

$$\operatorname{Tr}\left(I^{P_1,P_2}(\Sigma,\rho,\varphi_1,\varphi_2)[b]\,a\right) = \operatorname{Tr}(b\,Ta), \qquad b \in B(H). \tag{5.2.1}$$

To motivate what Ta should be, fix $a, b \in \mathcal{S}_1(H)$. Then the maps $c \mapsto \mathrm{Tr}(ca)$ and $c \mapsto \mathrm{Tr}(bc)$ are ultraweakly continuous. Therefore, by definition of the weak integral and basic properties of Tr,

$$\begin{aligned}
\mathrm{Tr}\left(I^{P_1,P_2}(\Sigma,\rho,\varphi_1,\varphi_2)[b]\,a\right) &= \int_\Sigma \mathrm{Tr}(P_1(\varphi_1(\cdot,\sigma))\,b\,P_2(\varphi_2(\cdot,\sigma))\,a)\,\rho(\mathrm{d}\sigma) \\
&= \int_\Sigma \mathrm{Tr}(b\,P_2(\varphi_2(\cdot,\sigma))\,a\,P_1(\varphi_1(\cdot,\sigma)))\,\rho(\mathrm{d}\sigma) \\
&= \mathrm{Tr}\left(b\int_\Sigma P_2(\varphi_2(\cdot,\sigma))\,a\,P_1(\varphi_1(\cdot,\sigma))\,\rho(\mathrm{d}\sigma)\right). \qquad (5.2.2)
\end{aligned}$$

We therefore should take $Ta = \int_\Sigma P_2(\varphi_2(\cdot,\sigma))\,a\,P_1(\varphi_1(\cdot,\sigma))\,\rho(\mathrm{d}\sigma)$ in Equation (5.2.1). (Those familiar with the subject will recognize this as related to the Birman–Solomyak [BS66] definition of a DOI. We elaborate on this in §5.8.) For this to have any chance of making sense, we need to know at the very least that

$$a \in \mathcal{S}_1(H) \implies \int_\Sigma P_2(\varphi_2(\cdot,\sigma))\,a\,P_1(\varphi_1(\cdot,\sigma))\,\rho(\mathrm{d}\sigma) \in \mathcal{S}_1(H). \qquad (5.2.3)$$

Even *this* is not obvious when H is not separable! It follows, however, from one of our Minkowski-type integral inequalities (Theorem 5.4.12) or Theorem 5.2.7 at the end of this section.

Assuming we know Relation (5.2.3), we still must verify Equation (5.2.2) for *all* $b \in B(H)$, not just for $b \in \mathcal{S}_1(H)$. If $b \in B(H)$ is arbitrary, then the map $\mathcal{S}_1(H) \ni c \mapsto \mathrm{Tr}(bc) \in \mathbb{C}$ is bounded with respect to $\|\cdot\|_{\mathcal{S}_1(H)}$. Therefore, we could reverse the calculation that led to Equation (5.2.2) if we knew that $\Sigma \ni \sigma \mapsto P_2(\varphi_2(\cdot,\sigma))\,a\,P_1(\varphi_1(\cdot,\sigma)) \in \mathcal{S}_1(H)$ were weakly integrable as a map $\Sigma \to \left(\mathcal{S}_1(H), \|\cdot\|_{\mathcal{S}_1(H)}\right)$, not just as a map $\Sigma \to (B(H), \sigma\text{-WOT})$, whenever $a \in \mathcal{S}_1(H)$. This is not automatic. Furthermore, if H is not separable, then $\mathcal{S}_1(H)$ is not separable, so strong (aka Bochner) integral techniques do not automatically apply either. We tiptoe around these difficulties using our key ingredient: the multiplicative system theorem, a "functional form" of the Dynkin system theorem. To state it, we recall the notion of bounded convergence.

Definition 5.2.4 (Bounded convergence)**.** Let S be a set. A sequence $(\varphi_n)_{n\in\mathbb{N}}$ of functions $S \to \mathbb{C}$ **converges boundedly** to $\varphi \in \mathbb{C}^S$ if $\varphi_n \to \varphi$ pointwise as $n \to \infty$ and $\sup_{n\in\mathbb{N}} \|\varphi_n\|_{\ell^\infty(S)} < \infty$. A set $\mathscr{S} \subseteq \mathbb{C}^S$ is **closed under bounded convergence** if whenever $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{S}$ converging boundedly to φ, we have that $\varphi \in \mathscr{S}$.

Theorem 5.2.5 (Multiplicative system theorem [DM75, Thm. I.21])**.** *Let S be a set. Suppose $\mathbb{H} \subseteq \mathbb{C}^S$ is a (complex) linear subspace containing the constant function 1 that is closed under complex conjugation and bounded convergence. If $\mathbb{M} \subseteq \mathbb{H}$ is closed under multiplication and complex conjugation, then $\ell^\infty(S, \sigma(\mathbb{M})) \subseteq \mathbb{H}$.*

The corollary most relevant to the argument presently under discussion is as follows.

Corollary 5.2.6. *Let $(\Omega, \mathscr{F})$ and $(\Sigma, \mathscr{H})$ be measurable spaces, and suppose $\mathbb{H}$ is a (complex) linear subspace of $\ell^\infty(\Omega \times \Sigma)$ that is closed under complex conjugation and bounded convergence. If $\{1_{G\times S} : G \in \mathscr{F}, S \in \mathscr{H}\} \subseteq \mathbb{H}$, then $\ell^\infty(\Omega \times \Sigma, \mathscr{F} \otimes \mathscr{H}) \subseteq \mathbb{H}$.*

Proof. If $\mathbb{M} \coloneqq \{1_{G\times S} : G \in \mathscr{F}, S \in \mathscr{H}\}$, then $\mathbb{M}$ is closed under complex conjugation and pointwise multiplication (because $\{G \times S : G \in \mathscr{F}, S \in \mathscr{H}\}$ is a π-system). Since $1 \in \mathbb{M} \subseteq \mathbb{H}$ and $\sigma(\mathbb{M}) = \mathscr{F} \otimes \mathscr{H}$, the conclusion follows from the multiplicative system theorem. $\square$

By carefully using this consequence of the multiplicative system theorem and a truncation argument, we are able to prove (in §5.6) the following key result.

Theorem 5.2.7 (Strong measurability in $\mathcal{S}_1$)**.** *Let $(\Omega, \mathscr{F}, H, P)$ and $(\Xi, \mathscr{G}, K, Q)$ be projection-valued measure spaces, and let $(\Sigma, \mathscr{H})$ be a measurable space. Suppose $\varphi\colon \Omega \times \Sigma \to \mathbb{C}$ and $\psi\colon \Xi \times \Sigma \to \mathbb{C}$ are measurable functions such that $\varphi(\cdot, \sigma) \in L^\infty(P)$ and $\psi(\cdot, \sigma) \in L^\infty(Q)$ for all $\sigma \in \Sigma$. If $A\colon \Sigma \to \mathcal{S}_1(H; K)$ is strongly measurable, then the map*

$$\Sigma \ni \sigma \mapsto Q(\psi(\cdot, \sigma))\, A(\sigma)\, P(\varphi(\cdot, \sigma)) \in \mathcal{S}_1(H; K)$$

is strongly measurable as well.

This result yields the desired $\mathcal{S}_1(H)$-valued weak (in fact, strong) integrability of the map $\Sigma \ni \sigma \mapsto P_2(\varphi_2(\cdot, \sigma))\, a\, P_1(\varphi_1(\cdot, \sigma)) \in \mathcal{S}_1(H)$ whenever $a \in \mathcal{S}_1(H)$. The relevant results are Theorem 5.6.9 and Corollary 5.6.10. Please see Remark 5.6.23 as well.

5.3 More on vector-valued integrals

In this section, we wrap up the general discussion of vector-valued integrals started in §1.1. Specifically, we establish more general versions of the triangle inequality and the dominated convergence theorem and study a useful situation in which weak integrals always exist (while strong integrals may not). For the former, we must overcome the technical difficulty that if α is a continuous seminorm on V, then the weak measurability of a map $F\colon \Omega \to V$ generally is not sufficient to guarantee the measurability of $\alpha(F)\colon \Omega \to \mathbb{R}_+$ even if α is a norm. Therefore, we are forced to integrate non-measurable scalar functions using upper and lower integrals.

Definition 5.3.1 (Upper and lower integrals)**.** If $f\colon \Omega \to [0,\infty]$ is an arbitrary function, then

$$\overline{\int_\Omega} f(\omega)\,\mu(\mathrm{d}\omega) = \overline{\int_\Omega} f\,\mathrm{d}\mu := \inf\left\{\int_\Omega \tilde{f}\,\mathrm{d}\mu : f \le \tilde{f}\ \ \mu\text{-a.e.},\ \tilde{f}\colon \Omega \to [0,\infty] \text{ measurable}\right\} \text{ and}$$
$$\underline{\int_\Omega} f(\omega)\,\mu(\mathrm{d}\omega) = \underline{\int_\Omega} f\,\mathrm{d}\mu := \sup\left\{\int_\Omega \tilde{f}\,\mathrm{d}\mu : \tilde{f} \le f\ \ \mu\text{-a.e.},\ \tilde{f}\colon \Omega \to [0,\infty] \text{ measurable}\right\}$$

are, respectively, the **upper (μ-)integral** and **lower (μ-)integral** of f.

Of course, if f is $\big(\overline{\mathscr{F}}^\mu, \mathcal{B}_{[0,\infty]}\big)$-measurable, where $\overline{\mathscr{F}}^\mu$ is the μ-completion of $\mathscr{F}$, then $\underline{\int_\Omega} f\,\mathrm{d}\mu = \overline{\int_\Omega} f\,\mathrm{d}\mu$. Here are the properties of upper and lower integrals relevant to this dissertation.

Proposition 5.3.2 (Properties of upper and lower integrals)**.** *Let $f, f_1, f_2\colon \Omega \to [0,\infty]$ be arbitrary functions, and let $c \ge 0$.*

(i) $\underline{\int_\Omega} f\,\mathrm{d}\mu \le \overline{\int_\Omega} f\,\mathrm{d}\mu$, $\overline{\int_\Omega} c\,f\,\mathrm{d}\mu = c\overline{\int_\Omega} f\,\mathrm{d}\mu$, *and* $\overline{\int_\Omega}(f_1+f_2)\,\mathrm{d}\mu \le \overline{\int_\Omega} f_1\,\mathrm{d}\mu + \overline{\int_\Omega} f_2\,\mathrm{d}\mu$.

(ii) *If $f_1 \le f_2$ μ-almost everywhere, then* $\underline{\int_\Omega} f_1\,\mathrm{d}\mu \le \underline{\int_\Omega} f_2\,\mathrm{d}\mu$, *and* $\overline{\int_\Omega} f_1\,\mathrm{d}\mu \le \overline{\int_\Omega} f_2\,\mathrm{d}\mu$.

(iii) *If $S \in \mathscr{F}$, then* $\underline{\int_S} f|_S\,\mathrm{d}\mu = \underline{\int_\Omega} 1_S f\,\mathrm{d}\mu$, *and* $\overline{\int_S} f|_S\,\mathrm{d}\mu = \overline{\int_\Omega} 1_S f\,\mathrm{d}\mu$.

(iv) (Dominated convergence theorem) *If $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions $\Omega \to [0,\infty]$ such that $f_n \to 0$ pointwise μ-almost everywhere as $n \to \infty$, then*

$$\overline{\int_\Omega} \sup_{n\in\mathbb{N}} f_n\,\mathrm{d}\mu < \infty \implies \lim_{n\to\infty} \underline{\int_\Omega} f_n\,\mathrm{d}\mu = 0.$$

(v) *If $(\Omega_n, \mathscr{F}_n, \mu_n)_{n\in\mathbb{N}}$ is a sequence of measure spaces and $(\Omega, \mathscr{F}, \mu)$ is their disjoint union, i.e., $(\Omega, \mathscr{F}, \mu) = \big(\coprod_{n\in\mathbb{N}} \Omega_n, \coprod_{n\in\mathbb{N}} \mathscr{F}_n, \sum_{n=1}^\infty \mu_n\big)$, then*

$$\underline{\int_\Omega} f\,\mathrm{d}\mu = \sum_{n=1}^\infty \underline{\int_{\Omega_n}} f|_{\Omega_n}\,\mathrm{d}\mu_n, \quad \textit{and} \quad \overline{\int_\Omega} f\,\mathrm{d}\mu = \sum_{n=1}^\infty \overline{\int_{\Omega_n}} f|_{\Omega_n}\,\mathrm{d}\mu_n.$$

Proof. The first three items are easy consequences of the definitions, so we leave them to the reader. We take the remaining items in turn.

(iv) By definition of the upper integral, there exists a measurable function $f\colon \Omega \to [0,\infty]$ such that $\int_\Omega f \,\mathrm{d}\mu < \infty$ and $\sup_{n\in\mathbb{N}} f_n \le f$ μ-almost everywhere. By definition of the lower integral, if $n \in \mathbb{N}$, then there exists a measurable function $\tilde{f}_n\colon \Omega \to [0,\infty]$ such that $0 \le \tilde{f}_n \le f_n$ μ-almost everywhere and

$$\underline{\int_\Omega} f_n \,\mathrm{d}\mu - \frac{1}{n} < \int_\Omega \tilde{f}_n \,\mathrm{d}\mu.$$

Since $f_n \to 0$ μ-almost everywhere as $n \to \infty$ and $0 \le \tilde{f}_n \le f_n$ μ-almost everywhere, we have that $\tilde{f}_n \to 0$ μ-almost everywhere as $n \to \infty$. Also, $\tilde{f}_n \le f_n \le f$ μ-almost everywhere. Therefore, by the dominated convergence theorem,

$$\limsup_{n\to\infty} \underline{\int_\Omega} f_n \,\mathrm{d}\mu = \limsup_{n\to\infty} \left(\underline{\int_\Omega} f_n \,\mathrm{d}\mu - \frac{1}{n} \right) \le \limsup_{n\to\infty} \int_\Omega \tilde{f}_n \,\mathrm{d}\mu = 0,$$

as desired.

(v) We prove the claimed identity for upper integrals and leave the proof of the identity for lower integrals to the reader. First, the definition of the disjoint union measure space and a standard application of the monotone convergence theorem give the desired identity when $f\colon \Omega \to [0,\infty]$ is measurable.

Next, for general f, suppose $\tilde{f}\colon \Omega \to [0,\infty]$ is measurable and $f \le \tilde{f}$ μ-almost everywhere. Then, for all $n \in \mathbb{N}$, $\tilde{f}|_{\Omega_n}\colon \Omega_n \to [0,\infty]$ is measurable and $f|_{\Omega_n} \le \tilde{f}|_{\Omega_n}$ μ_n-almost everywhere. Therefore, by definition of the upper integral and our initial observation,

$$\sum_{n=1}^{\infty} \overline{\int_{\Omega_n}} f|_{\Omega_n} \,\mathrm{d}\mu_n \le \sum_{n=1}^{\infty} \int_{\Omega_n} \tilde{f}|_{\Omega_n} \,\mathrm{d}\mu_n = \int_\Omega \tilde{f} \,\mathrm{d}\mu.$$

Taking the infimum over $\tilde{f}$ then yields $\sum_{n=1}^{\infty} \overline{\int_{\Omega_n}} f|_{\Omega_n} \,\mathrm{d}\mu_n \le \overline{\int_\Omega} f \,\mathrm{d}\mu$.

Finally, let $\varepsilon > 0$. By definition of the upper integral, if $n \in \mathbb{N}$, then there exists a measurable function $\tilde{f}_n\colon \Omega_n \to [0,\infty]$ such that $f|_{\Omega_n} \le \tilde{f}_n$ μ_n-almost everywhere and

$$\int_{\Omega_n} \tilde{f}_n \,\mathrm{d}\mu_n \le \overline{\int_{\Omega_n}} f|_{\Omega_n} \,\mathrm{d}\mu_n + \frac{\varepsilon}{2^n}.$$

Letting $\tilde{f}\colon \Omega \to [0,\infty]$ be the unique measurable function such that $\tilde{f}|_{\Omega_n} = \tilde{f}_n$ for all $n \in \mathbb{N}$, we have that $f \le \tilde{f}$ μ-almost everywhere and

$$\overline{\int_\Omega} f \, \mathrm{d}\mu \le \int_\Omega \tilde{f} \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{\Omega_n} \tilde{f}_n \, \mathrm{d}\mu_n \le \sum_{n=1}^{\infty} \overline{\int_{\Omega_n}} f|_{\Omega_n} \, \mathrm{d}\mu_n + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get that $\overline{\int_\Omega} f \, \mathrm{d}\mu \le \sum_{n=1}^{\infty} \overline{\int_{\Omega_n}} f|_{\Omega_n} \, \mathrm{d}\mu_n$ as well. □

Knowing now what upper and lower integrals are, we can state the triangle inequality and dominated convergence theorem properly.

Proposition 5.3.3 (Triangle inequality)**.** *If $F\colon \Omega \to V$ is weakly integrable and α is a continuous seminorm on V, then*

$$\alpha\left(\int_\Omega F \, \mathrm{d}\mu \right) \le \underline{\int_\Omega} \alpha(F) \, \mathrm{d}\mu.$$

In particular, if V is normed, then

$$\left\| \int_\Omega F \, \mathrm{d}\mu \right\|_V \le \underline{\int_\Omega} \|F\|_V \, \mathrm{d}\mu.$$

Proof. Let $v \coloneqq \int_\Omega F \, \mathrm{d}\mu$. By the Hahn–Banach theorem, there is some linear $\ell\colon V \to \mathbb{F}$ such that $\ell(v) = \alpha(v)$ and $|\ell(w)| \le \alpha(w)$ for all $w \in V$. Since α is continuous, $\ell \in V^*$. We then get

$$\alpha\left(\int_\Omega F \, \mathrm{d}\mu \right) = \ell\left(\int_\Omega F \, \mathrm{d}\mu \right) = \left| \int_\Omega (\ell \circ F) \, \mathrm{d}\mu \right| \le \int_\Omega |\ell \circ F| \, \mathrm{d}\mu \le \underline{\int_\Omega} \alpha(F) \, \mathrm{d}\mu$$

from the definition of the weak integral and the lower integral. □

Proposition 5.3.4 (Dominated convergence theorem)**.** *Suppose V is sequentially complete, and let $(F_n)_{n\in\mathbb{N}}$ be a sequence of weakly integrable maps $\Omega \to V$ converging pointwise to $F\colon \Omega \to V$. If*

$$\overline{\int_\Omega} \sup_{n\in\mathbb{N}} \alpha(F_n) \, \mathrm{d}\mu < \infty \tag{5.3.5}$$

whenever α is a continuous seminorm on V, then F is weakly integrable, and

$$\lim_{n\to\infty} \int_\Omega F_n \, \mathrm{d}\mu = \int_\Omega F \, \mathrm{d}\mu.$$

Proof. Let α be a continuous seminorm on V. Observe that

$$\overline{\int_\Omega} \sup_{n,m\in\mathbb{N}} \alpha(F_n - F_m)\,\mathrm{d}\mu \leq 2\overline{\int_\Omega} \sup_{n\in\mathbb{N}} \alpha(F_n)\,\mathrm{d}\mu < \infty.$$

Therefore, by the triangle inequality and Proposition 5.3.2(iv),

$$\alpha\bigg(\int_\Omega F_n\,\mathrm{d}\mu - \int_\Omega F_m\,\mathrm{d}\mu\bigg) = \alpha\bigg(\int_\Omega (F_n - F_m)\,\mathrm{d}\mu\bigg) \leq \underline{\int_\Omega} \alpha(F_n - F_m)\,\mathrm{d}\mu \xrightarrow{n,m\to\infty} 0.$$

Consequently, the sequence $\big(\int_\Omega F_n\,\mathrm{d}\mu\big)_{n\in\mathbb{N}}$ is Cauchy in V. Since V is sequentially complete, $\big(\int_\Omega F_n\,\mathrm{d}\mu\big)_{n\in\mathbb{N}}$ converges in V; write $v \in V$ for its limit. Now, let $\ell \in V^*$. Since $|\ell|$ is a continuous seminorm, $|\ell \circ F| \leq \sup_{n\in\mathbb{N}} |\ell \circ F_n|$, and $\ell \circ F_n \to \ell \circ F$ pointwise as $n \to \infty$, Inequality (5.3.5) and the scalar-valued dominated convergence theorem yield that $\int_\Omega |\ell \circ F|\,\mathrm{d}\mu < \infty$ and

$$\int_\Sigma (\ell \circ F)\,\mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega (\ell \circ F_n)\,\mathrm{d}\mu = \lim_{n\to\infty} \ell\bigg(\int_\Omega F_n\,\mathrm{d}\mu\bigg) = \ell(v).$$

Thus, F is weakly integrable, and $v = \int_\Omega F\,\mathrm{d}\mu$. □

Remark 5.3.6. Let $\mathscr{S} \subseteq \mathbb{R}_+^V$ be a collection of continuous seminorms on V that generate the topology of V. For every continuous seminorm α on V, there exist a $C \geq 0$ and $\alpha_1, \ldots, \alpha_m \in \mathscr{S}$ such that $\alpha \leq C \sum_{i=1}^m \alpha_i$. Consequently, Inequality (5.3.5) holds for all continuous seminorms α on V if and only if it holds for all $\alpha \in \mathscr{S}$.

Next, show that weak* and Dunford integrals exist.

Proposition 5.3.7 (Weak* integrals)**.** *Suppose V is a Fréchet space. A map $F\colon \Omega \to V^*$ is weakly measurable in the weak* topology on V^* if and only if it is **weak* measurable**, i.e., $F(\cdot)(v)\colon \Omega \to \mathbb{C}$ is measurable whenever $v \in V$. A weak* measurable map $F\colon \Omega \to V^*$ is weakly μ-integrable in the weak* topology if and only if*

$$\int_\Omega |F(\omega)(v)|\,\mu(\mathrm{d}\omega) < \infty, \qquad v \in V. \tag{5.3.8}$$

*In this case, F is **weak* (μ-)integrable**, and $\int_\Omega F\,\mathrm{d}\mu \in V^*$ is the **weak* (μ-)integral** of F.*

Proof. By [Rud91, §3.14], the map $V \ni v \mapsto (\ell \mapsto \ell(v)) \in (V^*, \text{weak}^*)^*$ is a linear isomorphism, from which the first statement and the "only if" part of the second statement of the proposition follow. It remains to prove that if F is weak* measurable and Inequality (5.3.8) holds, then F is weakly integrable in the weak* topology. To this end, define $T\colon V \to L^1(\mu)$ by $v \mapsto F(\cdot)(v)$. Certainly, T is linear. Also, if $(v_n)_{n\in\mathbb{N}}$ is a sequence in V converging to $v \in V$, then

$$\lim_{n\to\infty}(Tv_n)(\omega) = \lim_{n\to\infty} F(\omega)(v_n) = F(\omega)(v) = (Tv)(\omega), \qquad \omega \in \Omega.$$

Consequently, if $(Tv_n)_{n\in\mathbb{N}}$ converges in $L^1(\mu)$, then its limit must be Tv. In other words, T is closed. By the closed graph theorem, T is continuous. Finally, if $I_F\colon V \to \mathbb{F}$ is defined by $v \mapsto \int_\Omega F(\omega)(v)\,\mu(\mathrm{d}\omega) = \int_\Omega (Tv)(\omega)\,\mu(\mathrm{d}\omega)$, then $I_F \in V^*$ because T and $\int_\Omega \boldsymbol{\cdot}\,\mathrm{d}\mu\colon L^1(\mu) \to \mathbb{C}$ are continuous. Unraveling the definitions and appealing again to the first sentence of the proof, we conclude that F is weakly integrable in the weak* topology with $I_F = \int_\Omega F\,\mathrm{d}\mu$. □

Corollary 5.3.9 (Dunford integrals)**.** *Suppose V is a Banach space, and write* $\mathrm{ev}\colon V \hookrightarrow V^{**}$ *for the natural inclusion. If $F\colon \Omega \to V$ is weakly measurable and $\int_\Omega |\ell \circ F|\,\mathrm{d}\mu < \infty$ whenever $\ell \in V^*$, then* $\mathrm{ev} \circ F\colon \Omega \to V^{**} = (V^*)^*$ *is weak* integrable, and* $\int_\Omega (\mathrm{ev} \circ F)\,\mathrm{d}\mu \in V^{**}$ *is called the* ***Dunford (μ-)integral*** *of F. If, in addition, V is reflexive, then F is weakly integrable, and*

$$\mathrm{ev}\int_\Omega F\,\mathrm{d}\mu = \int_\Omega (\mathrm{ev} \circ F)\,\mathrm{d}\mu. \tag{5.3.10}$$

Proof. Since $(\mathrm{ev} \circ F)(\omega)(\ell) = (\ell \circ F)(\omega)$ for all $\ell \in V^*$ and $\omega \in \Omega$, the assumptions on F translate to the weak* integrability of $\mathrm{ev} \circ F\colon \Omega \to V^{**} = (V^*)^*$. If V is reflexive, then there exists a unique $v \in V$ such that $\mathrm{ev}(v) = \int_\Omega (\mathrm{ev} \circ F)\,\mathrm{d}\mu$, where the latter is the Dunford integral of F. Unraveling the definitions yields that $v = \int_\Omega F\,\mathrm{d}\mu$, i.e., Equation (5.3.10) holds. □

Example 5.3.11 (Hilbert space)**.** Let H be a Hilbert space. By the Riesz representation theorem, $\ell \in H^*$ if and only if there exists a $k \in H$ such that $\ell(h) = \langle h, k\rangle$ for all $h \in H$. Therefore, $F\colon \Omega \to H$ is weakly measurable if and only if $\langle F(\cdot), k\rangle\colon \Omega \to \mathbb{C}$ is measurable whenever $k \in H$. Also, since H is reflexive, Corollary 5.3.9 yields that $F\colon \Omega \to H$ is weakly integrable if and only if $\langle F(\cdot), k\rangle \in L^1(\mu)$ whenever $k \in H$, e.g., if F is weakly measurable and $\underline{\int_\Omega}\|F\|\,\mathrm{d}\mu < \infty$.

We have now collected all the general properties of vector-valued integrals needed in this dissertation. In the next section, we specialize to the case when V is a (σ-weakly closed) linear subspace of $B(H;K)$ with various topologies.

Remark 5.3.12 (Integrability of continuous maps)**.** Though we shall not use it, we would be remiss if we did not mention the fact that continuous maps are frequently weakly integrable. For $S \subseteq V$, write $\overline{\operatorname{conv}}(S) \subseteq V$ for the closure of the convex hull of S. It can be shown that if X is a compact Hausdorff space, ν is a finite Borel measure on X, $F\colon X \to V$ is a continuous map, and $\overline{\operatorname{conv}}(F(X))$ is compact, then F is weakly ν-integrable, and $\int_X F\,\mathrm{d}\nu \in \nu(X)\,\overline{\operatorname{conv}}(F(X))$. Furthermore, the hypothesis that $\overline{\operatorname{conv}}(F(X))$ is compact is superfluous when V is a Fréchet space. Please see [Rud91, Thms. 3.20(c) & 3.27] for details.

5.4 Operator-valued integrals

To define MOIs, we need to integrate maps $\Sigma \to V$, where $V \subseteq B(H;K)$ is a (σ-weakly closed) linear subspace. Given the number of interesting topologies on $B(H;K)$, there are potentially many notions of weak integrability of a map $\Sigma \to V \subseteq B(H;K)$. It turns out the choice of topology (from §4.1) does not matter in most reasonable circumstances. We now introduce a notion of integrability—*pointwise Pettis integrability*—in this setting that is, in practice, quite easy to check. Then we describe the relationship between pointwise Pettis integrability and weak integrability in various operator topologies.

Lemma 5.4.1. *If $F\colon \Omega \to B(H;K)$ is such that $\langle F(\cdot)h,k\rangle_K\colon \Omega \to \mathbb{C}$ is measurable and $\int_\Omega |\langle F(\omega)h,k\rangle_K|\,\mu(\mathrm{d}\omega) < \infty$ for all $h \in H$ and $k \in K$, then $F(\cdot)h\colon \Omega \to K$ is weakly integrable for all $h \in H$, and the map $\mu(F)\colon H \to K$ defined by $h \mapsto \int_\Omega F(\omega)h\,\mu(\mathrm{d}\omega)$ belongs to $B(H;K)$.*

Proof. If $B\colon H \times K \to \mathbb{C}$ is defined by $(h,k) \mapsto \int_\Omega \langle F(\omega)h,k\rangle_K\,\mu(\mathrm{d}\omega)$, then B is sesquilinear. We claim that B is bounded. Indeed, fix $h \in H$ and $k \in K$. By the characterization in Example 5.3.11, both $F(\cdot)h\colon \Omega \to K$ and $F(\cdot)^*k\colon \Omega \to H$ are weakly integrable. In particular, $\big\langle \int_\Omega F(\omega)h\,\mu(\mathrm{d}\omega),k\big\rangle_K = \int_\Omega \langle F(\omega)h,k\rangle_K\,\mu(\mathrm{d}\omega) = \int_\Omega \langle h,F(\omega)^*k\rangle_H\,\mu(\mathrm{d}\omega) = \big\langle h,\int_\Omega F(\omega)^*k\,\mu(\mathrm{d}\omega)\big\rangle_H$. Thus, B is bounded in each argument separately. By [Rud91, Thm. 2.17], B is bounded. Since $\langle \mu(F)h,k\rangle_K = B(h,k)$ for all $h \in H$ and $k \in K$, we conclude that $\mu(F) \in B(H;K)$. □

Definition 5.4.2 (Pointwise Pettis measurability and integrability)**.** A map $F\colon \Omega \to B(H;K)$ is **pointwise weakly measurable** if $\langle F(\cdot)h, k\rangle_K \colon \Omega \to \mathbb{C}$ is measurable whenever $h \in H$ and $k \in K$. If, in addition, $\int_\Omega |\langle F(\omega)h, k\rangle_K|\, \mu(\mathrm{d}\omega) < \infty$ for all $h \in H$ and $k \in K$, then F is **pointwise Pettis ($\boldsymbol{\mu}$-)integrable**. In this case, the operator $\mu(F) \in B(H;K)$ from Lemma 5.4.1 is called the **pointwise Pettis ($\boldsymbol{\mu}$-)integral** of F. Finally, if also $V \subseteq B(H;K)$ is a linear subspace, $F(\Omega) \subseteq V$, and $\mu(F) \in V$, then is F **pointwise Pettis ($\boldsymbol{\mu}$-)integrable in $\boldsymbol{V}$**.

Remark 5.4.3 (Nonstandard terminology)**.** The use of the term "pointwise" is not standard at all; we have chosen it to avoid overusing or abusing the terms "weak" and "strong." The pointwise Pettis integral above is often called a "weak integral" in contrast to the "stronger" Bochner integral. However, we shall see in Theorem 5.4.5 that the pointwise Pettis integrability of $F\colon \Omega \to B(H;K)$ is equivalent to the weak integrability of F as a map with values in $(B(H;K), \mathrm{WOT})$ *or* $(B(H;K), \mathrm{SOT})$. It therefore is arguably just as appropriate to apply the term "strong" to the pointwise Pettis integral.

Remark 5.4.4 (Von Neumann algebras)**.** If $H = K$ and $V = \mathcal{M} \subseteq B(H)$ is a von Neumann algebra, then any pointwise Pettis integrable map $F\colon \Sigma \to \mathcal{M} \subseteq B(H)$ is actually pointwise Pettis integrable in $\mathcal{M}$. Indeed, if $a \in \mathcal{M}'$, then it is easy to see from the definition that

$$a\,\mu(F) = \mu(a\,F) = \mu(F\,a) = \mu(F)\,a,$$

i.e., $\mu(F) \in \mathcal{M}'' = \mathcal{M}$ by the bicommutant theorem.

We now compare the notion of pointwise Pettis integrability to various notions of weak integrability. To this end, we recall (Theorem 4.1.2(iv)–(v)) that if $V \subseteq B(H;K)$ is a σ-weakly closed linear subspace, e.g., a von Neumann algebra, then $V_* \coloneqq (V, \sigma\text{-WOT})^*$ is the predual of V. More precisely, V_* is a Banach space with the operator norm, and the map

$$V \ni A \mapsto (\ell \mapsto \ell(A)) \in {V_*}^*$$

is an isometric isomorphism that is also a homeomorphism with respect to the σ-WOT on V and weak* topology on ${V_*}^*$.

Theorem 5.4.5 (Integrals in $V \subseteq B(H;K)$)**.** *Let $V \subseteq B(H;K)$ be a linear subspace, and fix a map $F\colon \Omega \to V$ and a choice $\mathcal{T} \in \{\text{WOT, SOT, S}^*\text{OT}\}$. (Here, we view $\mathcal{T}$ as a topology on V.)*

(i) *If*

$$\mathscr{F} := \sigma(\{V \ni A \mapsto \operatorname{Tr}(AB) \in \mathbb{C} : B \in \mathcal{S}_1(K;H)\}) \subseteq 2^V \quad \text{and}$$
$$\mathscr{G} := \sigma(\{V \ni A \mapsto \langle Ah, k\rangle_K \in \mathbb{C} : h \in H,\ k \in K\}) \subseteq 2^V,$$

then $\mathscr{F} = \mathscr{G} = \sigma\big((V, \sigma\text{-}\mathcal{T})^*\big) = \sigma\big((V,\mathcal{T})^*\big)$. In particular, F is pointwise weakly measurable if and only if it is weakly measurable in $\mathcal{T}$, if and only if it is weakly measurable in σ-$\mathcal{T}$.

(ii) *F is pointwise Pettis integrable in V if and only if it is weakly integrable in $\mathcal{T}$, in which case the pointwise Pettis and weak integrals of F agree. In particular, we may write $\mu(F) = \int_\Omega F \,\mathrm{d}\mu$ with no chance of confusion. Also, writing $\|\cdot\| := \|\cdot\|_{H\to K}$, the following triangle inequality holds in this case:*

$$\left\| \int_\Omega F \,\mathrm{d}\mu \right\| \leq \underline{\int_\Omega} \|F\| \,\mathrm{d}\mu.$$

(iii) *Suppose $V \subseteq B(H;K)$ is σ-weakly closed. Then F is weak integrable in σ-$\mathcal{T}$ if and only if it is weak* integrable under the usual identification $V \cong {V_*}^*$, if and only if F is pointwise weakly measurable and*

$$\int_\Omega |\langle (F(\omega)h_n)_{n\in\mathbb{N}}, (k_n)_{n\in\mathbb{N}}\rangle_{\ell^2(\mathbb{N};K)}| \,\mu(\mathrm{d}\omega) < \infty \tag{5.4.6}$$

for all $(h_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};H)$ and $(k_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};K)$, in which case the weak, weak, and pointwise Pettis integrals of F all agree.*

(iv) *If $H = K$ and $V = \mathcal{M} \subseteq B(H)$ is a von Neumann algebra, then the notions of pointwise weak measurability and weak integrability in σ-$\mathcal{T}$ are independent of the representation of $\mathcal{M}$. More precisely, if $\mathcal{N}$ is another von Neumann algebra and $\pi\colon \mathcal{M} \to \mathcal{N}$ is a $*$-isomorphism in the algebraic sense, then F is pointwise weakly measurable (respectively, weakly integrable in σ-$\mathcal{T}$) if and only if $\pi \circ F$ is pointwise weakly measurable (respectively, weakly integrable in σ-$\mathcal{T}$, in which case $\pi\big(\int_\Omega F \,\mathrm{d}\mu\big) = \int_\Omega (\pi \circ F)\,\mathrm{d}\mu$).*

Proof. We take each item in turn.

(i) By Theorem 4.1.2(ii),

$$(V, \mathcal{T})^* = \operatorname{span}\{V \ni A \mapsto \langle Ah, k\rangle_K \in \mathbb{C} : h \in H,\ k \in K\}. \tag{5.4.7}$$

Thus, $\mathscr{G} = \sigma\big((V, \mathcal{T})^*\big)$. By Theorem 4.1.2(iii), $(V, \sigma\text{-}\mathcal{T})^* = (V, \sigma\text{-WOT})^*$. By the Hahn–Banach theorem and Theorem 4.3.3(vi),

$$(V, \sigma\text{-WOT})^* = \{V \ni A \mapsto \operatorname{Tr}(AB) \in \mathbb{C} : B \in \mathcal{S}_1(K; H)\}. \tag{5.4.8}$$

Thus, $\mathscr{F} = \sigma\big((V, \sigma\text{-}\mathcal{T})^*\big)$. Since $(V, \mathcal{T})^* \subseteq (V, \sigma\text{-}\mathcal{T})^*$,

$$\mathscr{F} = \sigma\big((V, \sigma\text{-}\mathcal{T})^*\big) \subseteq \sigma\big((V, \mathcal{T})^*\big) = \mathscr{G},$$

so it suffices to prove that any element of $(V, \sigma\text{-}\mathcal{T})^*$ is $\mathscr{G}$-measurable. To this end, $\ell \in (V, \sigma\text{-}\mathcal{T})^*$. By Theorem 4.1.2(iii), there are $(h_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}; H)$ and $(k_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}; K)$ such that

$$\ell(A) = \langle (Ah_n)_{n\in\mathbb{N}}, (k_n)_{n\in\mathbb{N}}\rangle_{\ell^2(\mathbb{N};K)} = \sum_{n=1}^{\infty} \langle Ah_n, k_n\rangle_K, \qquad A \in V.$$

This exhibits ℓ a pointwise limit of elements of $\operatorname{span}\{V \ni A \mapsto \langle Ah, k\rangle_K \in \mathbb{C} : h \in H, k \in K\}$. Thus, ℓ is $\mathscr{G}$-measurable.

(ii) The equivalence of weak integrability in $\mathcal{T}$ and pointwise Pettis integrability in V (with the agreement of weak and pointwise Pettis integrals) follows directly from the definitions and Equation (5.4.7). For the triangle inequality, note that if $F : \Omega \to B(H; K)$ is pointwise Pettis integrable and $h \in H$, then the K-valued triangle inequality gives

$$\left\| \int_\Omega F(\omega)h\, \mu(\mathrm{d}\omega) \right\|_K \leq \underline{\int_\Omega} \|F(\omega)h\|_K\, \mu(\mathrm{d}\omega) \leq \|h\|_H \underline{\int_\Omega} \|F\|\, \mathrm{d}\mu.$$

Taking the supremum over $h \in H$ with $\|h\|_H \leq 1$ gives the desired result.

(iii) Since $(V, \sigma\text{-}\mathcal{T})^* = (V, \sigma\text{-WOT})^* = V_*$, we may and do assume $\mathcal{T} = \text{WOT}$. Under the usual identification ${V_*}^* \cong V$, the weak* topology on ${V_*}^*$ corresponds to the σ-WOT on V.

This implies the first equivalence. By the first item, the pointwise weak measurability of F is equivalent to the weak measurability of F in the σ-WOT on V and therefore to the weak measurability of F in the weak* topology on ${V_*}^*$. By Theorem 4.1.2(iii), Inequality (5.4.6) holds for all $(h_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};H)$ and $(k_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};K)$ if and only if

$$\int_\Omega |\ell \circ F| \, \mathrm{d}\mu < \infty, \qquad \ell \in V_* = (V, \sigma\text{-WOT})^*.$$

Proposition 5.3.7 and the form of the identification $V \cong {V_*}^*$ then give the second equivalence.

(iv) Again, we may and do assume $\mathcal{T} = \mathrm{WOT}$. This item follows from the fact that $*$-isomorphisms are automatically σ-WOT-homeomorphisms (Theorem 4.1.2(vi)), pointwise weak measurability is equivalent to weak measurability in the σ-WOT (the first item), and Proposition 1.1.7(ii) applied to π and π^{-1}. □

Let $V \subseteq B(H;K)$ be a σ-weakly closed linear subspace, and let $F\colon \Omega \to V$ be a map. In view of Theorem 5.4.5(iii) and its proof, we have the following. First, the weak measurability of F in the σ-WOT is equivalent to the weak measurability of F in the weak* topology when we identify $V \cong {V_*}^*$ in the usual way, which, in turn, is equivalent to the pointwise weak measurability of F. We therefore are justified in using the term **weak* measurable** in place of pointwise weakly measurable. Second, the weak integrability of F in the σ-WOT is equivalent to the weak* integrability of F when we identify $V \cong {V_*}^*$ in the usual way, which, in turn, is equivalent to the weak* measurability of F and the requirement that Inequality (5.4.6) holds for all $(h_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};H)$ and $(k_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};K)$. We therefore are justified in using the term **weak* integrable** in place of (any of) the terms in the previous sentence. We end this discussion by isolating an important takeaway from this possibly confusing development.

Corollary 5.4.9 (Criterion for weak* integrability)**.** *Let $V \subseteq B(H;K)$ be a σ-weakly closed linear subspace. If $F\colon \Omega \to V \cong {V_*}^*$ is pointwise weakly measurable and $\underline{\int_\Omega} \|F\| \, \mathrm{d}\mu < \infty$, then F is weak* integrable, and the weak* integral $\int_\Omega F \, \mathrm{d}\mu \in V$ is uniquely determined by*

$$\left\langle \left(\int_\Omega F \, \mathrm{d}\mu \right) h, k \right\rangle_K = \int_\Omega \langle F(\omega)h, k \rangle_K \, \mu(\mathrm{d}\omega), \qquad h \in H, \ k \in K.$$

Proof. If $(h_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};H)$ and $(k_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};K)$, then

$$\begin{aligned}\int_\Omega |\langle (F(\omega)h_n)_{n\in\mathbb{N}}, (k_n)_{n\in\mathbb{N}}\rangle_{\ell^2(\mathbb{N};K)}|\,\mu(\mathrm{d}\omega) &\leq \underline{\int_\Omega} \|(F(\omega)h_n)_{n\in\mathbb{N}}\|_{\ell^2(\mathbb{N};K)}\|(k_n)_{n\in\mathbb{N}}\|_{\ell^2(\mathbb{N};K)}\,\mu(\mathrm{d}\omega) \\ &\leq \|(h_n)_{n\in\mathbb{N}}\|_{\ell^2(\mathbb{N};H)}\|(k_n)_{n\in\mathbb{N}}\|_{\ell^2(\mathbb{N};K)} \underline{\int_\Omega} \|F\|\,\mathrm{d}\mu\end{aligned}$$

by the Cauchy–Schwarz inequality. Consequently, if the right-hand side is finite, then Theorem 5.4.5(iii) yields that F is weak* integrable, and $\mu(F)$ is the weak* integral of F. □

Our last order of business concerning operator-valued integrals is to prove a Schatten p-norm Minkowski's integral inequality for pointwise Pettis integrals. After doing so, we use a similar technique to prove a noncommutative L^p-norm Minkowski's integral inequality for weak* integrals in a semifinite von Neumann algebra. We begin by proving a well-known and useful recharacterization of $\|\cdot\|_{\mathcal{S}_1} = \|\cdot\|_{\mathcal{S}_1(H;K)}$. When $H = K$, this recharacterization is the $p = 1$ case of [Rin71, Lem. 2.3.4].

Definition 5.4.10 (Orthonormal frames)**.** If $n \in \mathbb{N}_0$, then

$$O_n(H) \coloneqq \{\mathbf{e} = (e_1,\ldots,e_n) \in H^n : e_1,\ldots,e_n \text{ is orthonormal}\}$$

is the set of **orthonormal frames of length *n***. Note that $O_0(H) = \emptyset$.

Lemma 5.4.11. *If $A \in B(H;K)$, then*

$$\|A\|_{\mathcal{S}_1} = \sup\left\{\sum_{i=1}^n |\langle Ae_i, f_i\rangle_K| : n \in \mathbb{N}_0,\ \mathbf{e} \in O_n(H),\ \mathbf{f} \in O_n(K)\right\}.$$

where, as usual, empty sums are zero. In particular, $A \in \mathcal{S}_1(H;K)$ if and only if the supremum on the right-hand side above is finite.

Proof. Let $A = U|A|$ be the polar decomposition of A, and let $\mathcal{E}_1$ be an orthonormal basis of $\ker|A| = \ker A$. Recall that the polar decomposition of A is the (unique) product decomposition $A = U|A|$, where $U \in B(H;K)$ is a partial isometry with initial space $(\ker A)^\perp = (\ker|A|)^\perp$ and final space $\overline{\operatorname{im} A}$. Note that $|A| = U^*A$.

First, by definition, if $e \in \mathcal{E}_1$, then $|A|e = 0$ and therefore $\langle |A|e, e\rangle_H = 0$. Next, complete $\mathcal{E}_1$ to an orthonormal basis $\mathcal{E} \supseteq \mathcal{E}_1$ of H. Then

$$\|A\|_{\mathcal{S}_1} = \sum_{e \in \mathcal{E}} \langle |A|e, e\rangle_H = \sum_{e \in \mathcal{E}\setminus\mathcal{E}_1} \langle |A|e, e\rangle_H = \sum_{e \in \mathcal{E}\setminus\mathcal{E}_1} \langle U^*Ae, e\rangle_H = \sum_{e \in \mathcal{E}\setminus\mathcal{E}_1} \langle Ae, Ue\rangle_K.$$

Of course, $\mathcal{E} \setminus \mathcal{E}_1$ is an orthonormal basis of $(\ker|A|)^\perp$, the initial space of U, on which U is an isometry by definition. Consequently, if we define $f_e \coloneqq Ue$ for all $e \in \mathcal{E} \setminus \mathcal{E}_1$, then we have that $\langle f_e, f_{\tilde{e}}\rangle_K = \langle Ue, U\tilde{e}\rangle_K = \langle e, \tilde{e}\rangle_H = \delta_{e\tilde{e}}$ whenever $e, \tilde{e} \in \mathcal{E} \setminus \mathcal{E}_1$, i.e., $(f_e)_{e\in\mathcal{E}\setminus\mathcal{E}_1}$ is orthonormal. It follows (by taking finite subsets $E \subseteq \mathcal{E} \setminus \mathcal{E}_1$) that

$$\|A\|_{\mathcal{S}_1} \le \sup\left\{\sum_{i=1}^n |\langle Ae_i, f_i\rangle_K| : n \in \mathbb{N}_0,\ \mathbf{e} \in O_n(H),\ \mathbf{f} \in O_n(K)\right\}.$$

For the other inequality, suppose $\|A\|_{\mathcal{S}_1} < \infty$, and fix $n \in \mathbb{N}$, $\mathbf{e} \in O_n(H)$, and $\mathbf{f} \in O_n(K)$. Let $V\colon H \to K$ be the unique partial isometry such that $Ve_i = f_i$ for all $i \in \{1, \ldots, n\}$ and $V \equiv 0$ on $(\operatorname{span}\{e_1, \ldots, e_n\})^\perp$. If we complete $\{e_1, \ldots, e_n\}$ to an orthonormal basis $\mathcal{E}$ of H, then

$$\begin{aligned}\sum_{i=1}^n |\langle Ae_i, f_i\rangle_K| = \sum_{i=1}^n |\langle Ae_i, Ve_i\rangle_K| &= \sum_{i=1}^n |\langle V^*Ae_i, e_i\rangle_H| \\ &\le \sum_{e\in\mathcal{E}} |\langle V^*Ae, e\rangle_H| \le \|V^*A\|_{\mathcal{S}_1} \le \|A\|_{\mathcal{S}_1}\end{aligned}$$

because $\|V^*\|_{K\to H} = \|V\|_{H\to K} = 1$. Thus,

$$\sup\left\{\sum_{i=1}^n |\langle Ae_i, f_i\rangle_K| : n \in \mathbb{N}_0,\ \mathbf{e} \in O_n(H),\ \mathbf{f} \in O_n(K)\right\} \le \|A\|_{\mathcal{S}_1},$$

as desired. □

Theorem 5.4.12 (Schatten norm Minkowski's integral inequality)**.** *If $F\colon \Omega \to B(H;K)$ is pointwise Pettis integrable, then*

$$\left\|\int_\Omega F\,\mathrm{d}\mu\right\|_{\mathcal{S}_p} \le \underline{\int_\Omega} \|F\|_{\mathcal{S}_p}\,\mathrm{d}\mu, \qquad p \in [1,\infty].$$

In particular, if the right-hand side is finite for some $p \in [1,\infty]$, then $\int_\Omega F\,\mathrm{d}\mu \in \mathcal{S}_p(H;K)$.

Proof. The case $p = \infty$ is contained in Theorem 5.4.5(ii). We first prove the $p = 1$ case, from which the remaining cases will follow. Define

$$A := \int_\Omega F \,\mathrm{d}\mu \in B(H;K),$$

and fix $n \in \mathbb{N}_0$, $\mathbf{e} \in O_n(H)$, and $\mathbf{f} \in O_n(K)$. By definition of the pointwise Pettis integral and Lemma 5.4.11, we have

$$\sum_{i=1}^n |\langle Ae_i, f_i\rangle_K| = \sum_{i=1}^n \left| \int_\Omega \langle F(\omega)e_i, f_i\rangle_K \,\mu(\mathrm{d}\omega)\right| \leq \int_\Omega \underbrace{\sum_{i=1}^n |\langle F(\omega)e_i, f_i\rangle_K|}_{\leq \|F(\omega)\|_{\mathcal{S}_1}} \,\mu(\mathrm{d}\omega) \leq \underline{\int_\Omega} \|F\|_{\mathcal{S}_1} \,\mathrm{d}\mu.$$

Taking the supremum over $n \in \mathbb{N}_0$, $\mathbf{e} \in O_n(H)$, and $\mathbf{f} \in O_n(K)$ and applying Lemma 5.4.11 a second times gives

$$\left\| \int_\Omega F \,\mathrm{d}\mu \right\|_{\mathcal{S}_1} = \|A\|_{\mathcal{S}_1} \leq \underline{\int_\Omega} \|F\|_{\mathcal{S}_1} d\mu.$$

Next, let $p, q \in (1,\infty)$ be such that $1/p + 1/q = 1$. If $B \in B(K;H)$, then, by what we just proved and Hölder's inequality for the Schatten norms,

$$\|AB\|_{\mathcal{S}_1} = \left\| \int_\Omega F(\omega)B \,\mu(\mathrm{d}\omega)\right\|_{\mathcal{S}_1} \leq \underline{\int_\Omega} \|F(\omega)B\|_{\mathcal{S}_1} \,\mu(\mathrm{d}\omega) \leq \|B\|_{\mathcal{S}_q} \underline{\int_\Omega} \|F\|_{\mathcal{S}_p} \,\mathrm{d}\mu.$$

Consequently, if $\underline{\int_\Omega} \|F\|_{\mathcal{S}_p} \,\mathrm{d}\mu < \infty$, then $A \in \mathcal{S}_p(H;K)$, and

$$\left\| \int_\Omega F \,\mathrm{d}\mu \right\|_{\mathcal{S}_p} = \|A\|_{\mathcal{S}_p} = \sup\{\underbrace{|\operatorname{Tr}(AB)|}_{\leq \|AB\|_{\mathcal{S}_1}} : B \in B(K;H),\ \|B\|_{\mathcal{S}_q} \leq 1\} \leq \underline{\int_\Omega} \|F\|_{\mathcal{S}_p} \,d\mu$$

by duality for the Schatten classes (Theorem 4.3.3(v)). □

As we just saw, the case $p = 1$ is the key to Theorem 5.4.12. We therefore offer a few more words about it. The proof presented above is "from first principles" in the sense that it did not use any technology from the theory of vector-valued integrals; we only used Lemma 5.4.11 and the definition of the pointwise Pettis integral. There is, however, an interesting alternative proof that uses Proposition 5.3.7 instead of Lemma 5.4.11.

Second proof of Theorem 5.4.12 when $p = 1$**.** If $\underline{\int_\Omega}\|F\|_{\mathcal{S}_1}\,\mathrm{d}\mu = \infty$, then the conclusion is clear, so we assume $\underline{\int_\Omega}\|F\|_{\mathcal{S}_1}\,\mathrm{d}\mu < \infty$. In this case, $\|F\|_{\mathcal{S}_1} < \infty$ μ-almost everywhere (exercise). Since neither $\int_\Omega F\,\mathrm{d}\mu$ nor $\underline{\int_\Omega}\|F\|_{\mathcal{S}_1}\,\mathrm{d}\mu$ changes if we modify F on a set of measure zero, we may and do assume $\|F(\omega)\|_{\mathcal{S}_1} < \infty$ for all $\omega \in \Omega$, i.e., $F(\Omega) \subseteq \mathcal{S}_1(H;K)$. We claim in this case that $F\colon \Omega \to \mathcal{S}_1(H;K)$ is weak* integrable when we identify $\mathcal{S}_1(H;K) \cong \mathcal{K}(K;H)^*$ as in Theorem 4.3.3(v). Indeed, if $B\colon K \to H$ is a finite-rank operator, then $\operatorname{Tr}(F(\cdot)B)\colon \Omega \to \mathbb{C}$ is measurable by Theorem 5.4.5(i). Now, if $B \in \mathcal{K}(K;H)$ is arbitrary, then there is a sequence $(B_n)_{n\in\mathbb{N}}$ of finite-rank operators $K \to H$ such that $\|B - B_n\| \to 0$ as $n \to \infty$. This gives

$$|\operatorname{Tr}(F(\omega)B) - \operatorname{Tr}(F(\omega)B_n)| = |\operatorname{Tr}(F(\omega)(B - B_n))| \le \|F(\omega)\|_{\mathcal{S}_1}\|B - B_n\| \xrightarrow{n\to\infty} 0, \qquad \omega \in \Omega.$$

Thus, $\operatorname{Tr}(F(\cdot)B)\colon \Omega \to \mathbb{C}$ is measurable. Also,

$$\int_\Omega |\operatorname{Tr}(F(\omega)B)|\,\mu(\mathrm{d}\omega) \le \underline{\int_\Omega}\|F(\omega)B\|_{\mathcal{S}_1}\,\mu(\mathrm{d}\omega) \le \|B\|\underline{\int_\Omega}\|F\|_{\mathcal{S}_1}\,\mathrm{d}\mu < \infty.$$

Therefore, by Proposition 5.3.7, $F\colon \Omega \to \mathcal{K}(K;H)^*$ is weak* integrable, and

$$\left\|\int_\Omega F\,\mathrm{d}\mu\right\|_{\mathcal{S}_1} = \left\|\int_\Omega F\,\mathrm{d}\mu\right\|_{\mathcal{K}(K;H)^*} \le \underline{\int_\Omega}\|F\|_{\mathcal{K}(K;H)^*}\,\mathrm{d}\mu = \underline{\int_\Omega}\|F\|_{\mathcal{S}_1}\,\mathrm{d}\mu.$$

Modulo the detail, which we leave to the reader, that the weak* integral of F agrees with its pointwise Pettis integral, this completes the proof. □

Remark 5.4.13 (Separable case)**.** It is worth mentioning that when H and K are separable, it is possible to prove Theorem 5.4.12 using the basic theory of the Bochner integral because $\mathcal{S}_p(H;K)$ is separable (when $p < \infty$) in this case. Since we dealt with the general case, additional care—in the form of either Lemma 5.4.11 or Proposition 5.3.7—was required.

Finally, we generalize Theorem 5.4.12 (with $H = K$) to noncommutative L^p-norms of a semifinite von Neumann algebra. (The uninterested reader may skip at this time to the next section.) For this purpose, we first prove a version of Lemma 5.4.11 appropriate for this setting; this is rather standard, but we supply a transparent proof for the reader's convenience.

Lemma 5.4.14. *If* $(\mathcal{M}, \tau)$ *is a semifinite von Neumann algebra, then*

$$\|a\|_{L^1(\tau)} = \tau(|a|) = \sup\{|\tau(ab)| : b \in \mathcal{L}^1(\tau),\ \|b\| \le 1\}, \qquad a \in \mathcal{M}.$$

Proof. Let $a \in \mathcal{M}$. If $b \in \mathcal{L}^1(\tau)$, then, by Theorem 4.3.9(ii)–(iii),

$$|\tau(ab)| \le \|ab\|_{L^1(\tau)} \le \|a\|_{L^1(\tau)} \|b\|_{L^\infty(\tau)} = \tau(|a|)\,\|b\|.$$

Thus,

$$\sup\{|\tau(ab)| : b \in \mathcal{L}^1(\tau),\ \|b\| \le 1\} \le \tau(|a|).$$

Now, let $a = u|a|$ be the polar decomposition of a. Suppose $p \in \mathcal{M}$ is a τ-finite projection, i.e., $p \in \mathrm{Proj}(\mathcal{M})$ and $\tau(p) < \infty$. If $b \coloneqq pu^*$, then $b \in \mathcal{L}^1(\tau)$, $\|b\| \le 1$, and

$$\tau(ab) = \tau(apu^*) = \tau(u^*ap) = \tau(|a|p) = \tau\Big(|a|^{\frac{1}{2}}|a|^{\frac{1}{2}}p\Big) = \tau\Big(|a|^{\frac{1}{2}}p|a|^{\frac{1}{2}}\Big). \tag{5.4.15}$$

If we could show that the net of τ-finite projections (directed by $\le$) increases to the identity, then the normality of τ would give

$$\tau(|a|) = \sup\Big\{\tau\Big(|a|^{\frac{1}{2}}p|a|^{\frac{1}{2}}\Big) : p \in L^1(\tau) \cap \mathrm{Proj}(\mathcal{M})\Big\}.$$

Using Equation (5.4.15), we would then conclude that

$$\tau(|a|) \le \sup\{|\tau(ab)| : b \in \mathcal{L}^1(\tau),\ \|b\| \le 1\},$$

as desired.

To complete the proof, we must show that $\mathrm{Proj}\left(L^1(\tau)\right) \coloneqq L^1(\tau) \cap \mathrm{Proj}(\mathcal{M})$ increases to the identity, i.e., $\sup \mathrm{Proj}\left(L^1(\tau)\right) = 1$. (A priori, this supremum exists and belongs to $\mathrm{Proj}(\mathcal{M})$ by [Tak79, Prop. V.1.1].) To this end, suppose $0 \ne q \in \mathrm{Proj}(\mathcal{M})$ is arbitrary. We claim that there exists a nonzero $p \in \mathrm{Proj}\left(L^1(\tau)\right)$ such that $p \le q$. Indeed, by the faithfulness and semifiniteness of τ, there is some $x \in \mathcal{M}_+$ such that $0 \ne x \le p$ and $\tau(x) < \infty$. Since x is positive, it is self-adjoint, and $\sigma(x) \subseteq [0,\infty)$. Recalling $P^x\colon \mathcal{B}_{\sigma(x)} \to \mathcal{M}$ is its projection-valued spectral measure, we have that if $\varepsilon > 0$ and $G_\varepsilon \coloneqq \sigma(x) \cap [\varepsilon,\infty)$, then

$$\varepsilon\, P^x(G_\varepsilon) = \int_{\sigma(x)} \varepsilon\, 1_{G_\varepsilon}\, \mathrm{d}P^x \le \int_{\sigma(x)} \lambda\, P^x(\mathrm{d}\lambda) = x.$$

Since $x \neq 0$ and x is normal, $\sigma(x) \neq \{0\}$. Therefore, there is some $\varepsilon > 0$ such that $P^x(G_\varepsilon) \neq 0$. For this choice of ε, let $p \coloneqq P^x(G_\varepsilon)$. Then $0 \neq \varepsilon\, p \le x$, so that $\tau(p) \le \varepsilon^{-1}\tau(x) < \infty$, i.e., $p \in \operatorname{Proj}\big(L^1(\tau)\big)$. But also, $\varepsilon\, p \le x \le q$. Since p and q are both projections, this implies $p \le q$. This proves the claim. But now, by definition of $\sup\operatorname{Proj}\big(L^1(\tau)\big)$, there can be no nonzero τ-finite projection $\le 1 - \sup\operatorname{Proj}\big(L^1(\tau)\big)$, so $1 - \sup\operatorname{Proj}\big(L^1(\tau)\big) = 0$ by what we just proved. $\square$

Theorem 5.4.16 (Noncommutative Minkowski's integral inequality)**.** *IF $(\mathcal{M},\tau)$ is a semifinite von Neumann algebra and $F\colon \Omega \to \mathcal{M}$ is weak* integrable, then*

$$\left\| \int_\Omega F\, \mathrm{d}\mu \right\|_{L^p(\tau)} \le \underline{\int_\Omega} \|F\|_{L^p(\tau)}\, \mathrm{d}\mu, \qquad p \in [1,\infty].$$

In particular, if the right-hand side is finite for some $p \in [1,\infty]$, then $\int_\Omega F\, \mathrm{d}\mu \in \mathcal{L}^p(\tau)$.

Proof. Define $a \coloneqq \int_\Omega F\, \mathrm{d}\mu$. The case $p = \infty$ is contained in Theorem 5.4.5(ii). We first prove the $p = 1$ case, from which the remaining cases will follow. Since F is weak* integrable (i.e., weakly integrable in the σ-weak topology) and the map $\mathcal{M} \ni c \mapsto \tau(cb) \in \mathbb{C}$ is σ-weakly continuous whenever $b \in \mathcal{L}^1(\tau)$, we have that

$$\tau(ab) = \int_\Omega \tau(F(\omega)\, b)\, \mu(\mathrm{d}\omega), \qquad b \in \mathcal{L}^1(\tau).$$

Lemma 5.4.14 (twice) then gives

$$\begin{aligned} \|a\|_{L^1(\tau)} &= \sup\left\{ |\tau(ab)| = \left| \int_\Omega \tau(F(\omega)\, b)\, \mu(\mathrm{d}\omega) \right| : b \in \mathcal{L}^1(\tau),\ \|b\| \le 1 \right\} \\ &\le \sup\left\{ \int_\Omega |\tau(F(\omega)\, b)|\, \mu(\mathrm{d}\omega) : b \in \mathcal{L}^1(\tau),\ \|b\| \le 1 \right\} \le \underline{\int_\Omega} \|F\|_{L^1(\tau)}\, \mathrm{d}\mu, \end{aligned}$$

as desired.

Now, let $p, q \in (1, \infty)$ be such that $1/p + 1/q = 1$. If $b \in \mathcal{L}^q(\tau)$, then, by what we just proved and noncommutative Hölder's inequality,

$$\|ab\|_{L^1(\tau)} = \left\| \int_\Omega F(\omega)\, b\, \mu(\mathrm{d}\omega) \right\|_{L^1(\tau)} \leq \underline{\int_\Omega} \|F(\omega)\, b\|_{L^1(\tau)}\, \mu(\mathrm{d}\omega) \leq \|b\|_{L^q(\tau)} \underline{\int_\Omega} \|F\|_{L^p(\tau)}\, \mathrm{d}\mu.$$

Consequently, if $\underline{\int_\Omega} \|F\|_{L^p(\tau)}\, \mathrm{d}\mu < \infty$, then $\int_\Omega F\, \mathrm{d}\mu = a \in \mathcal{L}^p(\tau)$, and

$$\left\| \int_\Omega F\, \mathrm{d}\mu \right\|_{L^p(\tau)} = \|a\|_{L^p(\tau)} = \sup\{\|ab\|_{L^1(\tau)} : b \in \mathcal{M},\ \|b\|_{L^q(\tau)} \leq 1\} \leq \underline{\int_\Omega} \|F\|_{L^p(\tau)}\, d\mu$$

by the dual characterization of the noncommutative L^p-norm (Theorem 4.3.9(iv)). □

The motivation for the name is the classical Minkowski integral inequality [Fol99, 6.19]. In view of Proposition 5.3.3, it would be just as reasonable to call Theorems 5.4.12 and 5.4.16 the Schatten p-norm and noncommutative L^p-norm (integral) triangle inequalities, respectively.

5.5 Integral projective tensor products of L^∞-spaces

We now discuss integral projective tensor products of L^∞-spaces. Formally, the idea is to replace the countable sum in the decomposition (1.5.11) of elements of the classical projective tensor product with an integral over a σ-finite measure space. To make this rigorous, we first observe that Minkowski's integral inequality with $p = \infty$ holds for projection-valued measures.

Lemma 5.5.1. *Let $(\Xi, \mathscr{G}, K, Q)$ be a projection-valued measure space, and let $(\Sigma, \mathscr{H}, \rho)$ be a σ-finite measure space. If $\Phi \colon \Xi \times \Sigma \to [0, \infty]$ is measurable, then*

$$\left\| \int_\Sigma \Phi(\cdot, \sigma)\, \rho(\mathrm{d}\sigma) \right\|_{L^\infty(Q)} \leq \underline{\int_\Sigma} \|\Phi(\cdot, \sigma)\|_{L^\infty(Q)}\, \rho(\mathrm{d}\sigma), \tag{5.5.2}$$

i.e., $\int_\Sigma \Phi(\xi, \sigma)\, \rho(\mathrm{d}\sigma) \leq \underline{\int_\Sigma} \|\Phi(\cdot, \sigma)\|_{L^\infty(Q)}\, \rho(\mathrm{d}\sigma)$ for Q-almost every $\xi \in \Xi$.

Proof. If $\underline{\int_\Sigma} \|\Phi(\cdot, \sigma)\|_{L^\infty(Q)}\, \rho(\mathrm{d}\sigma) = \infty$, then the conclusion is obvious. We therefore suppose

$$c \coloneqq \underline{\int_\Sigma} \|\Phi(\cdot, \sigma)\|_{L^\infty(Q)}\, \rho(\mathrm{d}\sigma) < \infty.$$

Next, by (the proof of) Tonelli's theorem, the function

$$\Xi \ni \xi \mapsto \int_\Sigma \Phi(\xi,\sigma)\,\rho(\mathrm{d}\sigma) \in [0,\infty]$$

is measurable. Thus,

$$G \coloneqq \left\{\xi \in \Xi : \int_\Sigma \Phi(\xi,\sigma)\,\rho(\mathrm{d}\sigma) > c\right\} \in \mathscr{G}.$$

Now, let $h \in K$. Since $Q_{h,h} = \langle Q(\cdot)h,h\rangle_K$ is a finite measure, the classical Minkowski integral inequality gives

$$\left\|\int_\Sigma \Phi(\cdot,\sigma)\,\rho(\mathrm{d}\sigma)\right\|_{L^\infty(Q_{h,h})} \leq \int_\Sigma \|\Phi(\cdot,\sigma)\|_{L^\infty(Q_{h,h})}\,\rho(\mathrm{d}\sigma) \leq \underline{\int_\Sigma}\|\Phi(\cdot,\sigma)\|_{L^\infty(Q)}\,\rho(\mathrm{d}\sigma) = c.$$

(Part of what we are using from Minkowski's integral inequality is the measurability of the function $\Sigma \ni \sigma \mapsto \|\Phi(\cdot,\sigma)\|_{L^\infty(Q_{h,h})} \in [0,\infty]$.) In other words, $\langle Q(G)h,h\rangle_K = Q_{h,h}(G) = 0$. Since $h \in K$ was arbitrary, we conclude that $Q(G) = 0$. $\square$

Definition 5.5.3 (Integral projective tensor products)**.** An $\boldsymbol{L^\infty_P}$**-integral projective decomposition** (L^∞_P-IPD) of a function $\varphi\colon \Omega \to \mathbb{C}$ is a choice $(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})$ of a σ-finite measure space $(\Sigma,\mathscr{H},\rho)$ and measurable functions $\varphi_1\colon \Omega_1\times\Sigma \to \mathbb{C},\ldots,\varphi_{k+1}\colon \Omega_{k+1}\times\Sigma\to\mathbb{C}$ such that

(i) $\varphi_i(\cdot,\sigma) \in L^\infty(P_i)$ for all $i \in \{1,\ldots,k+1\}$ and $\sigma \in \Sigma$,

(ii) $\overline{\int_\Sigma}\|\varphi_1(\cdot,\sigma)\|_{L^\infty(P_1)}\cdots\|\varphi_{k+1}(\cdot,\sigma)\|_{L^\infty(P_{k+1})}\,\rho(\mathrm{d}\sigma) < \infty$, and

(iii) $\varphi(\boldsymbol{\omega}) = \int_\Sigma \varphi_1(\omega_1,\sigma)\cdots\varphi_{k+1}(\omega_{k+1},\sigma)\,\rho(\mathrm{d}\sigma)$ for P-almost every $\boldsymbol{\omega} = (\omega_1,\ldots,\omega_{k+1}) \in \Omega$.

(The integral in the third requirement is defined for P-almost every $\boldsymbol{\omega} = (\omega_1,\ldots,\omega_{k+1}) \in \Omega$ by Lemma 5.5.1 and the second requirement.) Now, define

$$\|\varphi\|_{L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})} \coloneqq \inf\left\{\overline{\int_\Sigma}\prod_{i=1}^{k+1}\|\varphi_i(\cdot,\sigma)\|_{L^\infty(P_i)}\,\rho(\mathrm{d}\sigma) : \substack{(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1}) \\ \text{is an } L^\infty_P\text{-IPD of } \varphi}\right\},$$

where $\inf\emptyset \coloneqq \infty$. Noting that $\|\cdot\|_{L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})}$ is well defined on $L^\infty(P)$, the **integral projective tensor product** $L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})$ is defined to be the set of $\varphi \in L^\infty(P)$ such that $\|\varphi\|_{L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})} < \infty$.

Remark 5.5.4 (Measurability issues)**.** The literature is rather cavalier with the definition of the IPTP above. Indeed, if $(\Xi, \mathscr{G}, K, Q)$ is a projection-valued measure space, $(\Sigma, \mathscr{H}, \rho)$ is a σ-finite measure space, and $\Phi\colon \Xi \times \Sigma \to \mathbb{C}$ is a measurable function, then the function $\Sigma \ni \sigma \mapsto \|\Phi(\cdot, \sigma)\|_{L^\infty(Q)} \in [0, \infty]$ is *not* necessarily measurable. In particular, it is important to specify which integral (upper or lower) is being used in Inequality (5.5.2) and the second item in Definition 5.5.3. This detail, which is important in arguments to come, has been sidestepped in the existing literature.

It is worth discussing "how non-measurable" $\sigma \mapsto \Phi_Q(\sigma) := \|\Phi(\cdot, \sigma)\|_{L^\infty(Q)}$ can be in various situations. We proceed from least to most pathological. First, if Q is equivalent to a σ-finite scalar measure—as is always the case when K is separable—then Φ_Q is measurable. Now, for the remainder of the remark, assume $Q(G) = 0$ if and only if $G = \emptyset$. (Please see Example 5.5.7.) For the second example, suppose X is a complete, separable metric space and $(\Xi, \mathscr{G}) = (X, \mathcal{B}_X)$. Then Φ_Q is "almost measurable," i.e., Φ_Q is measurable with respect to the ρ-completion of $\mathscr{H}$, by [Cra02, Cor. 2.13], which relies on the (highly nontrivial) measurable projection theorem [CV77, Thm. III.23]. Because Φ_Q is "almost measurable" in this case, the upper and lower integrals of Φ_Q agree. This is used implicitly—and perhaps unknowingly—in the literature (e.g., [ACDS09, dPS04, DDSZ20]) but is never proven or cited as it should be. Finally, let $Y \subseteq [0, 1]$ be a non-Lebesgue-measurable set, $(\Xi, \mathscr{G}) = (Y, \mathcal{B}_Y)$, and $(\Sigma, \mathscr{H}, \rho) = ([0, 1], \mathcal{B}_{[0,1]}, \text{Lebesgue})$. If $\Phi := 1_{\Delta \cap (Y \times [0,1])}$, where $\Delta := \{(x, x) : x \in [0, 1]\}$ is the diagonal, then $\Phi_Q(\sigma) = \|\Phi(\cdot, \sigma)\|_{\ell^\infty(Y)} = 1_Y(\sigma)$ for all $\sigma \in [0, 1]$. Thus, Φ_Q is not even Lebesgue measurable in this case.

The proposition below gives the basic properties of $L^\infty(P_1) \hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$. Special cases have been stated in the literature, e.g., [dPS04, Lem. 4.6], but no proofs are written down. For the sake of completeness, especially in view of the measurability issues discussed in Remark 5.5.4, we provide a full proof here. In the statement below, a recall that a Banach $*$-algebra is a Banach algebra endowed with an isometric $*$-operation.

Proposition 5.5.5 (Basic properties of IPTPs)**.** *If* $\varphi\colon \Omega \to \mathbb{C}$ *is a function, then*

$$\|\varphi\|_{L^\infty(P)} \le \|\varphi\|_{L^\infty(P_1) \hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})}. \tag{5.5.6}$$

Also, $L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1}) \subseteq L^\infty(P)$ *is a* $*$*-subalgebra, and*

$$\big(L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1}), \|\cdot\|_{L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})}\big)$$

is a unital Banach $*$*-algebra.*

Proof. Write $\mathscr{B} \coloneqq L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$ and $\|\cdot\|_{\mathscr{B}} \coloneqq \|\cdot\|_{L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})}$. If $\varphi \in \mathscr{B}$, $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ is an L_P^∞-IPD of φ, and

$$\Phi(\boldsymbol{\omega}, \sigma) \coloneqq \varphi_1(\omega_1, \sigma) \cdots \varphi_{k+1}(\omega_{k+1}, \sigma), \qquad (\boldsymbol{\omega}, \sigma) \in \Omega \times \Sigma,$$

then

$$\|\varphi\|_{L^\infty(P)} \leq \left\| \int_\Sigma |\Phi(\cdot,\sigma)|\, \rho(\mathrm{d}\sigma) \right\|_{L^\infty(P)} \leq \underline{\int_\Sigma} \|\Phi(\cdot,\sigma)\|_{L^\infty(P)}\, \rho(\mathrm{d}\sigma) \leq \underline{\int_\Sigma} \prod_{i=1}^{k+1} \|\varphi_i(\cdot,\sigma)\|_{L^\infty(P_i)}\, \rho(\mathrm{d}\sigma)$$

by definition of Φ, the third item in Definition 5.5.3, and Lemma 5.5.1. Using the fact that $\underline{\int_\Sigma} \cdot \, \mathrm{d}\rho \leq \overline{\int_\Sigma} \cdot \, \mathrm{d}\rho$ and taking the infimum over the decompositions $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ gives Inequality (5.5.6). In particular, $\|\varphi\|_{\mathscr{B}} = 0$ if and only if $\varphi \equiv 0$ P-almost everywhere.

Now, we begin the proof that $\mathscr{B} \subseteq L^\infty(P)$ is a $*$-subalgebra and that $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ is a Banach $*$-algebra. First, it is clear from the definition that $\mathscr{B} \subseteq L^\infty(P)$ is closed under scalar multiplication and complex conjugation and that $\|\alpha\varphi\|_{\mathscr{B}} = |\alpha|\, \|\varphi\|_{\mathscr{B}} = |\alpha|\, \|\overline{\varphi}\|_{\mathscr{B}}$ for all $\alpha \in \mathbb{C}$ and $\varphi \in \mathscr{B}$. Also, $1 \in \mathscr{B}$.

Second, let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{B}$ such that $\sum_{n=1}^\infty \|\varphi_n\|_{\mathscr{B}} < \infty$. Then

$$\sum_{n=1}^\infty \|\varphi_n\|_{L^\infty(P)} \leq \sum_{n=1}^\infty \|\varphi_n\|_{\mathscr{B}} < \infty,$$

so that $\varphi \coloneqq \sum_{n=1}^\infty \varphi_n$ converges in $L^\infty(P)$. We claim that $\|\varphi\|_{\mathscr{B}} \leq \sum_{n=1}^\infty \|\varphi_n\|_{\mathscr{B}}$, from which it follows that $\mathscr{B} \subseteq L^\infty(P)$ is a linear subspace and $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ is a Banach space. To see this, fix $\varepsilon > 0$ and $n \in \mathbb{N}$. By definition of $\|\cdot\|_{\mathscr{B}}$, there exists an L_P^∞-IPD $(\Sigma_n, \rho_n, \varphi_{1,n}, \ldots, \varphi_{k+1,n})$ of φ_n such that

$$\overline{\int_{\Sigma_n}} \prod_{i=1}^{k+1} \|\varphi_{i,n}(\cdot,\sigma_n)\|_{L^\infty(P_i)}\, \rho_n(\mathrm{d}\sigma_n) < \|\varphi_n\|_{\mathscr{B}} + \frac{\varepsilon}{2^n}.$$

This gives

$$\sum_{n=1}^{\infty} \overline{\int_{\Sigma_n}} \prod_{i=1}^{k+1} \|\varphi_{i,n}(\cdot,\sigma_n)\|_{L^\infty(P_i)}\,\rho_n(\mathrm{d}\sigma_n) \le \sum_{n=1}^{\infty} \|\varphi_n\|_{\mathscr{B}} + \varepsilon < \infty.$$

Redefine $(\Sigma, \mathscr{H}, \rho)$ to be the disjoint union of the measure spaces $\{(\Sigma_n, \mathscr{H}_n, \rho_n) : n \in \mathbb{N}\}$ and

$$\chi_i(\omega_i, \sigma) := \varphi_{i,n}(\omega_i, \sigma), \qquad \omega_i \in \Omega_i,\ \sigma \in \Sigma_n \subseteq \coprod_{m \in \mathbb{N}} \Sigma_m = \Sigma,\ i \in \{1, \ldots, k+1\}.$$

Then $(\Sigma, \rho, \chi_1, \ldots, \chi_{k+1})$ is an L_P^∞-IPD of φ. Indeed, the first item in Definition 5.5.3 is clear. Next, by Proposition 5.3.2(v),

$$\overline{\int_{\Sigma}} \prod_{i=1}^{k+1} \|\chi_i(\cdot,\sigma)\|_{L^\infty(P_i)}\,\rho(\mathrm{d}\sigma) = \sum_{n=1}^{\infty} \overline{\int_{\Sigma_n}} \prod_{i=1}^{k+1} \|\varphi_{i,n}(\cdot,\sigma_n)\|_{L^\infty(P_i)}\,\rho_n(\mathrm{d}\sigma_n) < \infty.$$

Finally, for P-almost every $\boldsymbol{\omega} \in \Omega$,

$$\int_{\Sigma} \prod_{i=1}^{k+1} \chi_i(\omega_i, \sigma)\,\rho(\mathrm{d}\sigma) = \sum_{n=1}^{\infty} \int_{\Sigma_n} \prod_{i=1}^{k+1} \varphi_{i,n}(\omega_i, \sigma_n)\,\rho_n(\mathrm{d}\sigma_n) = \sum_{n=1}^{\infty} \varphi_n(\boldsymbol{\omega}) = \varphi(\boldsymbol{\omega}).$$

From this, we conclude that $\varphi \in \mathscr{B}$ and

$$\|\varphi\|_{\mathscr{B}} \le \sum_{n=1}^{\infty} \overline{\int_{\Sigma_n}} \prod_{i=1}^{k+1} \|\varphi_{i,n}(\cdot,\sigma_n)\|_{L^\infty(P_i)}\,\rho_n(\mathrm{d}\sigma_n) \le \sum_{n=1}^{\infty} \|\varphi_n\|_{\mathscr{B}} + \varepsilon.$$

Taking $\varepsilon \searrow 0$ completes the proof of the claim.

Third, we show that if $\varphi, \psi \in \mathscr{B}$, then $\|\varphi\psi\|_{\mathscr{B}} \le \|\varphi\|_{\mathscr{B}}\|\psi\|_{\mathscr{B}}$, which will complete the proof of the proposition. To this end, suppose $(\Sigma_1, \rho_1, \varphi_1, \ldots, \varphi_{k+1})$ and $(\Sigma_2, \rho_2, \psi_1, \ldots, \psi_{k+1})$ are L_P^∞-IPDs of φ and ψ, respectively. Redefine $(\Sigma, \mathscr{H}, \rho) := (\Sigma_1 \times \Sigma_2, \mathscr{H}_1 \otimes \mathscr{H}_2, \rho_1 \otimes \rho_2)$ and

$$\chi_i(\omega_i, \sigma) := \varphi_i(\omega_i, \sigma_1)\,\psi_i(\omega_i, \sigma_2), \qquad (\omega_i, \sigma) = (\omega_i, \sigma_1, \sigma_2) \in \Omega_i \times \Sigma,\ i \in \{1, \ldots, k+1\}.$$

We claim $(\Sigma, \rho, \chi_1, \ldots, \chi_{k+1})$ is an L_P^∞-IPD of $\varphi\psi$. Once again, the first item of Definition 5.5.3 is clear. Now, by Tonelli's theorem and the definition of the upper integral,

$$\overline{\int_{\Sigma}} \prod_{i=1}^{k+1} \|\chi_i(\cdot,\sigma)\|_{L^\infty(P_i)}\,\rho(\mathrm{d}\sigma) \le \overline{\int_{\Sigma_1}} \prod_{i=1}^{k+1} \|\varphi_i(\cdot,\sigma_1)\|_{L^\infty(P_i)}\,\rho_1(\mathrm{d}\sigma_1) \overline{\int_{\Sigma_2}} \prod_{i=1}^{k+1} \|\psi_i(\cdot,\sigma_2)\|_{L^\infty(P_i)}\,\rho_2(\mathrm{d}\sigma_2),$$

which is finite. Finally, for P-almost every $\boldsymbol{\omega} \in \Omega$,

$$\varphi(\boldsymbol{\omega})\,\psi(\boldsymbol{\omega}) = \int_{\Sigma_1} \prod_{i=1}^{k+1} \varphi_i(\omega_i, \sigma_1)\,\rho_1(\mathrm{d}\sigma_1) \int_{\Sigma_2} \prod_{i=1}^{k+1} \psi_i(\omega_i, \sigma_2)\,\rho_2(\mathrm{d}\sigma_2) = \int_{\Sigma} \prod_{i=1}^{k+1} \chi_i(\omega_i, \sigma)\,\rho(\mathrm{d}\sigma)$$

by Fubini's theorem. This proves $\varphi\psi \in \mathscr{B}$ and, after taking infima, $\|\varphi\psi\|_{\mathscr{B}} \leq \|\varphi\|_{\mathscr{B}}\|\psi\|_{\mathscr{B}}$. □

Example 5.5.7 (ℓ^∞-IPTPs)**.** Let $(\Xi, \mathscr{G})$ be a measurable space, and write $\ell^2(\Xi) \coloneqq L^2\big(\Xi, 2^\Xi, \kappa\big)$, where κ is the counting measure on Ξ. For $G \in \mathscr{G}$, let $Q(G) \in B\big(\ell^2(\Xi)\big)$ be multiplication by 1_G. Then we call $Q \colon \mathscr{G} \to B\big(\ell^2(\Xi)\big)$ the **projection-valued counting measure on $(\Xi, \mathscr{G})$**. Note that $L^\infty(Q) = \ell^\infty(\Xi, \mathscr{G})$ with $\|\cdot\|_{L^\infty(Q)} = \|\cdot\|_{\ell^\infty(\Xi)}$ because $Q(G) = 0$ if and only if $G = \emptyset$.

We define

$$\ell^\infty(\Omega_1, \mathscr{F}_1)\hat{\otimes}_i \cdots \hat{\otimes}_i \ell^\infty(\Omega_{k+1}, \mathscr{F}_{k+1}) \coloneqq L^\infty(Q_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(Q_{k+1}) \text{ and}$$
$$\|\cdot\|_{\ell^\infty(\Omega_1,\mathscr{F}_1)\hat{\otimes}_i\cdots\hat{\otimes}_i\ell^\infty(\Omega_{k+1},\mathscr{F}_{k+1})} \coloneqq \|\cdot\|_{L^\infty(Q_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(Q_{k+1})},$$

where Q_i is the projection-valued counting measure on $(\Omega_i, \mathscr{F}_i)$ for all $i \in \{1, \ldots, k+1\}$. It is easy to see that $Q \coloneqq Q_1 \otimes \cdots \otimes Q_{k+1}$ is the projection-valued counting measure on $(\Omega, \mathscr{F})$ when we identify

$$\ell^2(\Omega_1) \otimes_2 \cdots \otimes_2 \ell^2(\Omega_{k+1}) \cong \ell^2(\Omega_1 \times \cdots \times \Omega_{k+1}) = \ell^2(\Omega).$$

Thus,

$$\ell^\infty(\Omega_1, \mathscr{F}_1)\hat{\otimes}_i \cdots \hat{\otimes}_i \ell^\infty(\Omega_{k+1}, \mathscr{F}_{k+1}) \subseteq L^\infty(Q) = \ell^\infty(\Omega, \mathscr{F}).$$

This space is the **integral projective tensor product** of $\ell^\infty(\Omega_1, \mathscr{F}_1), \ldots, \ell^\infty(\Omega_{k+1}, \mathscr{F}_{k+1})$, and L^∞_Q-integral projective decompositions are called **$\boldsymbol{\ell^\infty}$-integral projective decompositions**.

Variants of the ℓ^∞-integral projective tensor product are often used in the literature (e.g., [ACDS09, dPS04, DDSZ20]). As the above example shows, ℓ^∞-integral projective tensor products are special cases of L^∞-integral projective tensor products if one allows non-separable Hilbert spaces.

5.6 Well-definition of MOIs

The primary goal of this section is to show that if $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$ and $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ is an L^∞_P-IPD of φ, then the object

$$\int_\Sigma P_1(\varphi_1(\cdot,\sigma))\, b_1 \cdots P_k(\varphi_k(\cdot,\sigma))\, b_k\, P_{k+1}(\varphi_{k+1}(\cdot,\sigma))\, \rho(\mathrm{d}\sigma) \in B(H_{k+1}; H_1)$$

makes sense as a weak* integral and is independent of the chosen L^∞_P-IPD $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ of φ whenever $b_i \in B(H_{i+1}; H_i)$ for all $i \in \{1, \ldots, k\}$.

Definition 5.6.1 (Complex Markov kernel)**.** Let $(\Xi, \mathscr{G})$ and $(\Sigma, \mathscr{H})$ be measurable spaces. A **complex Markov kernel** (with **source** Σ and **target** Ξ) is a map $\nu \colon \Sigma \to M(\Xi, \mathscr{G})$ such that the function $\Sigma \ni \sigma \mapsto \nu_\sigma(G) := \nu(\sigma)(G) \in \mathbb{C}$ is measurable whenever $G \in \mathscr{G}$.

Lemma 5.6.2. *Let $(\Xi, \mathscr{G})$ and $(\Sigma, \mathscr{H})$ be measurable spaces, and let $\nu \colon \Sigma \to M(\Xi, \mathscr{G})$ be a complex Markov kernel. If $\varphi \colon \Xi \times \Sigma \to \mathbb{C}$ is measurable and $\varphi(\cdot, \sigma) \in L^1(\nu_\sigma) = L^1(|\nu_\sigma|)$ for all $\sigma \in \Sigma$, then the function*

$$\Sigma \ni \sigma \mapsto \int_\Xi \varphi(\xi, \sigma)\, \nu_\sigma(\mathrm{d}\xi) \in \mathbb{C}$$

is measurable.

Sketch of proof. By a truncation argument, it suffices to prove the claim when φ is bounded. To this end, let

$$\mathbb{H} := \left\{ \varphi \in \ell^\infty(\Xi \times \Sigma, \mathscr{G} \otimes \mathscr{H}) : \sigma \mapsto \int_\Xi \varphi(\xi, \sigma)\, \nu_\sigma(\mathrm{d}\xi) \text{ is measurable} \right\}.$$

Clearly, $\mathbb{H}$ is a vector space that is closed under complex conjugation. It is closed under bounded convergence by the dominated convergence theorem. Now, if $G \in \mathscr{G}$ and $S \in \mathscr{H}$, then

$$\int_\Xi 1_{G \times S}(\xi, \sigma)\, \nu_\sigma(\mathrm{d}\xi) = 1_S(\sigma)\, \nu_\sigma(G), \qquad \sigma \in \Sigma.$$

By definition of a complex Markov kernel, $1_{G\times S} \in \mathbb{H}$. By the multiplicative system theorem (in the form of Corollary 5.2.6), we conclude that $\ell^\infty(\Xi \times \Sigma, \mathscr{G} \otimes \mathscr{H}) \subseteq \mathbb{H}$, as desired. □

Proposition 5.6.3. *Let* $(\Xi, \mathscr{G}, K, Q)$ *be a projection-valued measure space, let* L *be another complex Hilbert space, and let* $(\Sigma, \mathscr{H}, \rho)$ *be a measure space. Suppose* $\varphi\colon \Xi\times\Sigma \to \mathbb{C}$ *is a measurable function such that* $\varphi(\cdot,\sigma) \in L^\infty(Q)$ *for all* $\sigma \in \Sigma$*. If* $A\colon \Sigma \to B(K;L)$ *and* $B\colon \Sigma \to B(L;K)$ *are weak* measurable, then the maps*

$$\Sigma \ni \sigma \mapsto A(\sigma)\, Q(\varphi(\cdot,\sigma)) \in B(K;L) \quad \textit{and} \quad \Sigma \ni \sigma \mapsto Q(\varphi(\cdot,\sigma))\, B(\sigma) \in B(L;K)$$

are weak measurable as well. If, in addition,*

$$\underline{\int_\Sigma} \|A(\sigma)\|_{B(K;L)} \|\varphi(\cdot,\sigma)\|_{L^\infty(Q)}\, \rho(\mathrm{d}\sigma) + \underline{\int_\Sigma} \|\varphi(\cdot,\sigma)\|_{L^\infty(Q)} \|B(\sigma)\|_{B(L;K)}\, \rho(\mathrm{d}\sigma) < \infty,$$

then the aforementioned maps are weak integrable.*

Proof. To prove the first part, it suffices, by Theorem 5.4.5(i), to show that $\sigma \mapsto A(\sigma)\, Q(\varphi(\cdot,\sigma))$ and $\sigma \mapsto Q(\varphi(\cdot,\sigma))\, B(\sigma)$ are pointwise weakly measurable. If $k \in K$ and $l \in L$, then

$$\big\langle A(\sigma)\, Q(\varphi(\cdot,\sigma))\, k, l \big\rangle_L = \big\langle Q(\varphi(\cdot,\sigma))\, k, A(\sigma)^* l \big\rangle_K = \int_\Xi \varphi(\xi,\sigma)\, Q_{k,A(\sigma)^*l}(\mathrm{d}\xi), \qquad \sigma \in \Sigma.$$

By the pointwise weak measurability of A,

$$\nu_\sigma^A(G) \coloneqq Q_{k,A(\sigma)^*l}(G) = \langle Q(G)k, A(\sigma)^* l\rangle_K = \langle A(\sigma) Q(G) k, l\rangle_L, \qquad \sigma \in \Sigma,\ G \in \mathscr{G},$$

defines a complex Markov kernel $\Sigma \to M(\Xi, \mathscr{G})$. Therefore, it follows from Lemma 5.6.2 that the map $\sigma \mapsto A(\sigma)\, Q(\varphi(\cdot,\sigma))$ is pointwise weakly measurable. A similar argument establishes the pointwise weak measurability of the map $\sigma \mapsto Q(\varphi(\cdot,\sigma))\, B(\sigma)$. The second part then follows from Corollary 5.4.9 because

$$\|A(\sigma)\, Q(\varphi(\cdot,\sigma))\|_{B(K;L)} \le \|A(\sigma)\|_{B(K;L)} \|Q(\varphi(\cdot,\sigma))\|_{B(K)} = \|A(\sigma)\|_{B(K;L)} \|\varphi(\cdot,\sigma)\|_{L^\infty(Q)} \text{ and}$$
$$\|Q(\varphi(\cdot,\sigma))\, B(\sigma)\|_{B(L;K)} \le \|Q(\varphi(\cdot,\sigma))\|_{B(K)} \|B(\sigma)\|_{B(L;K)} = \|\varphi(\cdot,\sigma)\|_{L^\infty(Q)} \|B(\sigma)\|_{B(L;K)}$$

whenever $\sigma \in \Sigma$. $\square$

Corollary 5.6.4. *Let* $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$*, let* $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ *be an* L^∞_P*-IPD of* φ*, and let* $b_1 \in B(H_2; H_1), \ldots, b_k \in B(H_{k+1}; H_k)$*. If* $F \colon \Sigma \to B(H_{k+1}; H_k)$ *is defined by*

$$F(\sigma) \coloneqq P_1(\varphi_1(\cdot,\sigma))\, b_1 \cdots P_k(\varphi_k(\cdot,\sigma))\, b_k\, P_{k+1}(\varphi_{k+1}(\cdot,\sigma)) \in B(H_{k+1}; H_1), \qquad \sigma \in \Sigma,$$

then F *is weak* integrable, and*

$$\left\| \int_\Sigma F \, \mathrm{d}\rho \right\|_{B(H_{k+1};H_1)} \leq \left(\prod_{j=1}^{k} \|b_j\|_{B(H_{j+1};H_j)} \right) \underline{\int_\Sigma} \prod_{i=1}^{k+1} \|\varphi_i(\cdot,\sigma)\|_{L^\infty(P_i)}\, \rho(\mathrm{d}\sigma). \tag{5.6.5}$$

Proof. By Proposition 5.6.3 and induction, F is weak* integrable. Inequality (5.6.5) then follows from the triangle inequality in Theorem 5.4.5(ii) and the fact that

$$\|F(\sigma)\|_{B(H_{k+1};H_1)} \leq \left(\prod_{j=1}^{k} \|b_j\|_{B(H_{j+1};H_j)} \right) \prod_{i=1}^{k+1} \|\varphi_i(\cdot,\sigma)\|_{L^\infty(P_i)}$$

whenever $\sigma \in \Sigma$. □

Notation 5.6.6 (MOI, take I)**.** Let $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$, let $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ be an L^∞_P-IPD of φ, and write $\mathbf{P} \coloneqq (P_1, \ldots, P_{k+1})$. Define

$$I^{\mathbf{P}}(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})[b] \coloneqq \int_\Sigma P_1(\varphi_1(\cdot,\sigma))\, b_1 \cdots P_k(\varphi_k(\cdot,\sigma))\, b_k\, P_{k+1}(\varphi_{k+1}(\cdot,\sigma))\, \rho(\mathrm{d}\sigma)$$

whenever $b = (b_1, \ldots, b_k) \in B(H_2; H_1) \times \cdots \times B(H_{k+1}; H_k)$.

Of course, the definition of $I^{\mathbf{P}}(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})[b]$ makes sense as a weak* integral in $B(H_{k+1}; H_1)$ by Corollary 5.6.4. By the linearity of the integral and Inequality (5.6.5), the map

$$I^{\mathbf{P}}(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1}) \colon B(H_2; H_1) \times \cdots \times B(H_{k+1}; H_k) \to B(H_{k+1}; H_1)$$

is k-linear and bounded. Our next and most important task is to prove that this map is ultraweakly continuous in each argument. As described in §5.2, this is rather delicate when $H_1, \ldots, H_{k+1}$ are not separable.

Lemma 5.6.7. *Let H, K, and L be Hilbert spaces, and fix $Q \in B(K;L)$, $C \in \mathcal{S}_1(L;H)$, $D \in \mathcal{S}_1(H;K)$, and $(Q_n)_{n\in\mathbb{N}} \in B(K;L)^{\mathbb{N}}$. If $Q_n \to Q$ in the SOT, then $Q_nD \to QD$ in $\mathcal{S}_1(H;L)$ as $n \to \infty$. If $Q_n^* \to Q^*$ in the SOT, then $CQ_n \to CQ$ in $\mathcal{S}_1(K;H)$ as $n \to \infty$. In particular, if $Q_n \to Q$ in the S^*OT, then $Q_nD \to QD$ in $\mathcal{S}_1(H;L)$, and $CQ_n \to CQ$ in $\mathcal{S}_1(K;H)$ as $n \to \infty$.*

Proof. Without loss of generality, we can take $Q = 0$. Assume $Q_n \to 0$ in the SOT as $n \to \infty$, let $E \in \mathcal{S}_2(H;K)$, and fix an orthonormal basis $\mathcal{E} \subseteq H$. Then $\|Q_nE\|_{\mathcal{S}_2}^2 = \sum_{e\in\mathcal{E}} \|Q_nEe\|_L^2 \to 0$ as $n \to \infty$ by the dominated convergence theorem. Explicitly, $\lim_{n\to\infty} \|Q_nEe\|_L = 0$ whenever $e \in \mathcal{E}$, and $\|Q_nEe\|_L^2 \leq \|Ee\|_K^2 \sup_{m\in\mathbb{N}} \|Q_m\|_{B(K;L)}^2 \in L^1(\mathcal{E}, \text{counting})$ because $E \in \mathcal{S}_2(H;K)$. Next, assume $Q_n^* \to 0$ in the SOT as $n \to \infty$, and let $E \in \mathcal{S}_2(L;H)$. By what we just proved, $\|EQ_n\|_{\mathcal{S}_2} = \|Q_n^*E^*\|_{\mathcal{S}_2} \to 0$ as $n \to \infty$.

Finally, let $C = U|C|$ and $D = V|D|$ be the polar decompositions of $C \in \mathcal{S}_1(L;H)$ and $D \in \mathcal{S}_1(H;K)$, respectively. Then $|D|^{1/2} \in \mathcal{S}_2(H)$, $V|D|^{1/2} \in \mathcal{S}_2(H;K)$, $|C|^{1/2} \in \mathcal{S}_2(L)$, and $U|C|^{1/2} \in \mathcal{S}_2(L;H)$. Consequently, if $Q_n \to 0$ or $Q_n^* \to 0$ in the SOT as $n \to \infty$, then

$$\begin{aligned} \|Q_nD\|_{\mathcal{S}_1} &= \left\|Q_nV|D|^{\frac{1}{2}}|D|^{\frac{1}{2}}\right\|_{\mathcal{S}_1} \leq \left\|Q_nV|D|^{\frac{1}{2}}\right\|_{\mathcal{S}_2}\left\||D|^{\frac{1}{2}}\right\|_{\mathcal{S}_2} \xrightarrow{n\to\infty} 0 \text{ or} \\ \|CQ_n\|_{\mathcal{S}_1} &= \left\|U|C|^{\frac{1}{2}}|C|^{\frac{1}{2}}Q_n\right\|_{\mathcal{S}_1} \leq \left\|U|C|^{\frac{1}{2}}\right\|_{\mathcal{S}_2}\left\||C|^{\frac{1}{2}}Q_n\right\|_{\mathcal{S}_2} \xrightarrow{n\to\infty} 0, \end{aligned}$$

respectively, by Hölder's inequality for the Schatten norms and the previous paragraph. □

Now, we are prepared to prove Theorem 5.2.7.

Proof of Theorem 5.2.7. By Pettis's measurability theorem, if V is a metrizable, locally convex topological vector space, then the pointwise limit of a sequence of strongly measurable maps $\Sigma \to V$ is strongly measurable. We shall use this fact several times without further comment.

To begin, it suffices to treat the $\varphi \equiv 1$ and $\psi \equiv 1$ cases. Indeed, suppose we know that $\Sigma \ni \sigma \mapsto Q(\psi(\cdot,\sigma))\,A(\sigma) \in \mathcal{S}_1(H;K)$ and $\Sigma \ni \sigma \mapsto A(\sigma)\,P(\varphi(\cdot,\sigma)) \in \mathcal{S}_1(H;K)$ are strongly measurable whenever $A\colon \Sigma \to \mathcal{S}_1(H;K)$ is strongly measurable. If $A\colon \Sigma \to \mathcal{S}_1(H;K)$ is strongly measurable and $B\colon \Sigma \to \mathcal{S}_1(H;K)$ is defined by $\sigma \mapsto Q(\psi(\cdot,\sigma))\,A(\sigma)$, then B is strongly measurable, so that $\Sigma \ni \sigma \mapsto B(\sigma)\,P(\varphi(\cdot,\sigma)) = Q(\psi(\cdot,\sigma))\,A(\sigma)\,P(\varphi(\cdot,\sigma)) \in \mathcal{S}_1(H;K)$ is strongly measurable as well. This is the desired result.

We treat the $\psi \equiv 1$ case; the $\varphi \equiv 1$ case is nearly identical. Let $A \colon \Sigma \to \mathcal{S}_1(H;K)$ be strongly measurable, and define $\mathbb{H}$ to be the set of $\varphi \in \ell^\infty(\Omega \times \Sigma, \mathscr{F} \otimes \mathscr{H})$ such that the maps

$$\Sigma \ni \sigma \mapsto A(\sigma)\, P(\varphi(\cdot,\sigma)) \in \mathcal{S}_1(H;K) \;\text{ and }\; \Sigma \ni \sigma \mapsto P(\varphi(\cdot,\sigma))\, A(\sigma)^* \in \mathcal{S}_1(K;H)$$

are strongly measurable. We show that $\mathbb{H} = \ell^\infty(\Omega \times \Sigma, \mathscr{F} \otimes \mathscr{H})$ using the multiplicative system theorem (in the form of Corollary 5.2.6). It is clear that $\mathbb{H}$ is a linear subspace of $\ell^\infty(\Omega \times \Sigma)$, and $\mathbb{H}$ is closed under complex conjugation because $L^\infty(P) \ni f \mapsto P(f) \in B(H)$ respects complex conjugation (Proposition 4.2.10(iv)) and $\mathcal{S}_1(H;K) \ni A \mapsto A^* \in \mathcal{S}_1(K;H)$ is an isometric, conjugate-linear isomorphism (Theorem 4.3.3(ii)). Also, if $G \in \mathscr{F}$, $S \in \mathscr{H}$, and $\sigma \in \Sigma$, then

$$\begin{aligned} A(\sigma)\, P(1_{G\times S}(\cdot,\sigma)) &= 1_S(\sigma)\, A(\sigma)\, P(G), \quad \text{and} \\ P(1_{G\times S}(\cdot,\sigma))\, A(\sigma)^* &= 1_S(\sigma)\, P(G)\, A(\sigma)^*. \end{aligned}$$

Since A and A^* are strongly measurable, it follows that $1_{G\times S} \in \mathbb{H}$.

It remains to show $\mathbb{H}$ is closed under bounded convergence. In fact, we claim something more general: If $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions $\Omega \times \Sigma \to \mathbb{C}$ converging pointwise to $\varphi \colon \Omega \times \Sigma \to \mathbb{C}$ and

$$\sup_{n\in\mathbb{N}} \|\varphi_n(\cdot,\sigma)\|_{L^\infty(P)} < \infty, \qquad \sigma \in \Sigma,$$

then the sequences of maps

$$(\Sigma \ni \sigma \mapsto A(\sigma)\, P(\varphi_n(\cdot,\sigma)) \in \mathcal{S}_1(H;K))_{n\in\mathbb{N}} \;\text{ and }\; (\Sigma \ni \sigma \mapsto P(\varphi_n(\cdot,\sigma))\, A(\sigma)^* \in \mathcal{S}_1(K;H))_{n\in\mathbb{N}}$$

converge pointwise to the maps

$$\Sigma \ni \sigma \mapsto A(\sigma)\, P(\varphi(\cdot,\sigma)) \in \mathcal{S}_1(H;K) \;\text{ and }\; \Sigma \ni \sigma \mapsto P(\varphi(\cdot,\sigma))\, A(\sigma)^* \in \mathcal{S}_1(K;H),$$

respectively. Indeed, in this case, if $\sigma \in \Sigma$, then $P(\varphi_n(\cdot,\sigma)) \to P(\varphi(\cdot,\sigma))$ in the S*OT as $n \to \infty$ by Proposition 4.2.10(iv). Consequently, $A(\sigma)\, P(\varphi_n(\cdot,\sigma)) \to A(\sigma)\, P(\varphi(\cdot,\sigma))$ in $\mathcal{S}_1(H;K)$, and $P(\varphi_n(\cdot,\sigma))\, A(\sigma)^* \to P(\varphi(\cdot,\sigma))\, A(\sigma)^*$ in $\mathcal{S}_1(K;H)$ as $n \to \infty$ by Lemma 5.6.7, as claimed.

Finally, let φ be as in the statement of the theorem, and define

$$\varphi_n \coloneqq \varphi\, 1_{\{(\omega,\sigma)\in\Omega\times\Sigma : |\varphi(\omega,\sigma)|\le n\}}, \qquad n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$, $|\varphi_n| \le \max\{n, |\varphi|\}$, so φ_n is bounded, and

$$\sup_{m\in\mathbb{N}} \|\varphi_m(\cdot,\sigma)\|_{L^\infty(P)} \le \|\varphi(\cdot,\sigma)\|_{L^\infty(P)} < \infty, \qquad \sigma \in \Sigma.$$

Also, $(\varphi_n)_{n\in\mathbb{N}}$ converges pointwise to φ. By the previous paragraph, the sequence of maps $(\Sigma \ni \sigma \mapsto A(\sigma)\, P(\varphi_n(\cdot,\sigma)) \in \mathcal{S}_1(H;K))_{n\in\mathbb{N}}$ converges pointwise to $\sigma \mapsto A(\sigma)\, P(\varphi(\cdot,\sigma))$. Since we know from the last two paragraphs that the map $\Sigma \ni \sigma \mapsto A(\sigma)\, P(\varphi_n(\cdot,\sigma)) \in \mathcal{S}_1(H;K)$ is strongly measurable for all $n \in \mathbb{N}$, we conclude that $\Sigma \ni \sigma \mapsto A(\sigma)\, P(\varphi(\cdot,\sigma)) \in \mathcal{S}_1(H;K)$ is strongly measurable, as desired. □

Remark 5.6.8 (Separable case)**.** If H and K are separable, then there is an easy argument for the following more general result: If $F\colon \Sigma \to B(H;K)$ is weak* (i.e., pointwise weakly) measurable and $F(\Sigma) \subseteq \mathcal{S}_1(H;K)$, then F is strongly measurable as a map $\Sigma \to \big(\mathcal{S}_1(H;K), \|\cdot\|_{\mathcal{S}_1}\big)$. Indeed, $\mathcal{S}_1(H;K)$ is separable in this case, so Pettis's measurability theorem says we only need to verify that $F\colon \Sigma \to \big(\mathcal{S}_1(H;K), \|\cdot\|_{\mathcal{S}_1}\big)$ is weakly measurable. By Theorem 4.3.3(v), if $\ell \in \mathcal{S}_1(H;K)^*$, then there exists a $B \in B(K;H)$ such that $\ell(A) = \operatorname{Tr}(AB)$ for all $A \in \mathcal{S}_1(H;K)$. Since K is separable, any orthonormal basis is countable, from which it is easy to see that ℓ is the pointwise limit of a *sequence* of elements of $\operatorname{span}\{\mathcal{S}_1(H;K) \ni A \mapsto \langle Ah, k\rangle_K : h \in H,\, k \in K\}$. The result follows. Therefore, in the separable case, we obtain Theorem 5.2.7 immediately from the first part of Proposition 5.6.3.

Theorem 5.6.9. *Suppose $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$, $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ is an L^∞_P-integral projective decomposition of φ, and $b = (b_1, \ldots, b_k) \in B(H_2;H_1) \times \cdots \times B(H_{k+1};H_k)$. If $b_i \in \mathcal{S}_1(H_{i+1};H_i)$ for some $i \in \{1, \ldots, k\}$, then the map*

$$\Sigma \ni \sigma \mapsto F(\sigma) \coloneqq P_1(\varphi_1(\cdot,\sigma))\, b_1 \cdots P_k(\varphi_k(\cdot,\sigma))\, b_k\, P_{k+1}(\varphi_{k+1}(\cdot,\sigma)) \in (\mathcal{S}_1(H_{k+1};H_1), \|\cdot\|_{\mathcal{S}_1})$$

is strongly ρ-integrable, and its Bochner ρ-integral is $I^{\mathbf{P}}(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})[b]$.

Proof. Since the constant map $\Sigma \ni \sigma \mapsto b_i \in \mathcal{S}_1(H_{i+1};H_i)$ is strongly measurable, the map $F\colon \Sigma \to \mathcal{S}_1(H_{k+1};H_k)$ from the statement of the theorem is strongly measurable by Theorem 5.2.7 and induction. Since

$$\int_\Sigma \|F(\sigma)\|_{\mathcal{S}_1}\,\rho(\mathrm{d}\sigma) \leq \|b_i\|_{\mathcal{S}_1(H_{i+1};H_i)} \prod_{p\neq i} \|b_p\|_{B(H_{p+1};H_p)} \underline{\int_\Sigma} \prod_{j=1}^{k+1} \|\varphi_j(\cdot,\sigma)\|_{L^\infty(P_j)}\,\rho(\mathrm{d}\sigma) < \infty$$

as well, (the second part of) Pettis's measurability theorem yields the strong integrability of $F\colon \Sigma \to \mathcal{S}_1(H_{k+1};H_1)$. The final statement follows from Proposition 1.1.7(ii), the definition of $I^{\mathbf{P}}(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})[b]$ as a weak* integral (i.e., a weak integral in the σ-WOT), and the continuity of the inclusion $(\mathcal{S}_1(H_{k+1};H_k), \|\cdot\|_{\mathcal{S}_1}) \hookrightarrow (B(H_{k+1};H_k), \sigma\text{-WOT})$. □

Corollary 5.6.10 (Ultraweak continuity of MOI)**.** *Suppose* $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$, $(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})$ *is an* L^∞_P*-IPD of* φ, *and* $b = (b_1,\ldots,b_k) \in B(H_2;H_1)\times\cdots\times B(H_{k+1};H_k)$. *If* $b_{k+1} \in \mathcal{S}_1(H_1;H_{k+1})$ *and* $\pi \in S_{k+1}$ *is a cyclic permutation of* $\{1,\ldots,k+1\}$, *then*

$$\mathrm{Tr}\left(I^{\mathbf{P}}(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})[b]\,b_{k+1}\right) = \mathrm{Tr}\left(I^{\mathbf{P}_\pi}(\Sigma,\rho,\varphi_{\pi(1)},\ldots,\varphi_{\pi(k+1)})[b_\pi]\,b_{\pi(k+1)}\right),$$

where $\mathbf{P}_\pi = (P_{\pi(1)},\ldots,P_{\pi(k+1)})$ *and* $b_\pi = (b_{\pi(1)},\ldots,b_{\pi(k)})$. *In particular, the bounded* k*-linear map* $I^{\mathbf{P}}(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})\colon B(H_2;H_1)\times\cdots\times B(H_{k+1};H_k) \to B(H_{k+1};H_1)$ *is argumentwise ultraweakly continuous, i.e., ultraweakly continuous in each argument separately.*

Proof. If we define $F(\sigma) \coloneqq P_1(\varphi_1(\cdot,\sigma))\,b_1\cdots P_k(\varphi_k(\cdot,\sigma))\,b_k\,P_{k+1}(\varphi_{k+1}(\cdot,\sigma))$ and

$$F_\pi(\sigma) \coloneqq P_{\pi(1)}\big(\varphi_{\pi(1)}(\cdot,\sigma)\big)\,b_{\pi(1)}\cdots P_{\pi(k)}\big(\varphi_{\pi(k)}(\cdot,\sigma)\big)\,b_{\pi(k)}\,P_{\pi(k+1)}\big(\varphi_{\pi(k+1)}(\cdot,\sigma)\big)$$

for all $\sigma \in \Sigma$, then

$$\begin{aligned}\mathrm{Tr}\left(I^{\mathbf{P}}(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})[b]\,b_{k+1}\,b_{k+1}\right) &= \int_\Sigma \mathrm{Tr}(F(\sigma)\,b_{k+1})\,\rho(\mathrm{d}\sigma) = \int_\Sigma \mathrm{Tr}\left(F_\pi(\sigma)\,b_{\pi(k+1)}\right)\rho(\mathrm{d}\sigma)\\ &= \mathrm{Tr}\left(\left(\int_\Sigma F_\pi\,\mathrm{d}\rho\right)b_{\pi(k+1)}\right) = \mathrm{Tr}\left(I^{\mathbf{P}_\pi}(\Sigma,\rho,\varphi_{\pi(1)},\ldots,\varphi_{\pi(k+1)})[b_\pi]\,b_{\pi(k+1)}\right)\end{aligned}$$

by the σ-weak continuity of $c \mapsto \mathrm{Tr}(cb_{k+1})$, Theorem 4.3.3(iv), and Theorem 5.6.9 (plus the fact that the map $\mathcal{S}_1 \ni c \mapsto \mathrm{Tr}\left(cb_{\pi(k+1)}\right) \in \mathbb{C}$ is bounded linear). □

We now reap the benefits of this technical work: The ultraweak continuity we just proved allows us to show that $I^{\mathbf{P}}(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})$ does not depend on the chosen L_P^∞-IPD of φ and is therefore a reasonable definition of the multiple operator integral (5.1.1).

Theorem 5.6.11 (Well-definition of MOI)**.** *If $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$, then*

$$I^{\mathbf{P}}(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1}) = I^{\mathbf{P}}\big(\tilde{\Sigma},\tilde{\rho},\tilde{\varphi}_1,\ldots,\tilde{\varphi}_{k+1}\big)$$

whenever $(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})$ and $(\tilde{\Sigma},\tilde{\rho},\tilde{\varphi}_1,\ldots,\tilde{\varphi}_{k+1})$ are L_P^∞-IPDs of φ.

Proof. By the argumentwise ultraweak continuity of the k-linear maps $I^{\mathbf{P}}(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})$ and $I^{\mathbf{P}}(\tilde{\Sigma},\tilde{\rho},\tilde{\varphi}_1,\ldots,\tilde{\varphi}_{k+1})$ (Corollary 5.6.10) and the ultraweak density of finite-rank operators (Theorem 4.3.3(i)), it suffices to prove that

$$I^{\mathbf{P}}(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})[b] = I^{\mathbf{P}}\big(\tilde{\Sigma},\tilde{\rho},\tilde{\varphi}_1,\ldots,\tilde{\varphi}_{k+1}\big)[b]$$

whenever $b=(b_1,\ldots,b_k) \in B(H_2;H_1)\times\cdots\times B(H_{k+1};H_k)$ is such that b_i has rank at most one for all $i \in \{1,\ldots,k\}$.

To this end, write $m \coloneqq k+1$, and for all $i \in \{1,\ldots,m-1\}$, let $b_i = \langle\cdot,h_i\rangle_{H_{i+1}}k_i$, where $k_i \in H_i$ and $h_i \in H_{i+1}$. Then

$$P_i(\varphi_i(\cdot,\sigma))\,b_i = \langle\cdot,h_i\rangle_{H_{i+1}}P_i(\varphi_i(\cdot,\sigma))\,k_i, \qquad \sigma\in\Sigma.$$

If, in addition, $k_m \in H_m$ and

$$F(\sigma) \coloneqq P_1(\varphi_1(\cdot,\sigma))\,b_1\cdots P_{m-1}(\varphi_{m-1}(\cdot,\sigma))\,b_{m-1}\,P_m(\varphi_m(\cdot,\sigma)), \qquad \sigma\in\Sigma,$$

then

$$\begin{aligned} F(\sigma)k_m &= \prod_{i=2}^m \big\langle P_i(\varphi_i(\cdot,\sigma))k_i,h_{i-1}\big\rangle_{H_i} P_1(\varphi_1(\cdot,\sigma))k_1 \\ &= \left(\prod_{i=2}^m \int_{\Omega_i}\varphi_i(\cdot,\sigma)\,\mathrm{d}(P_i)_{k_i,h_{i-1}}\right)P_1(\varphi_1(\cdot,\sigma))k_1, \qquad \sigma\in\Sigma. \end{aligned} \tag{5.6.12}$$

Next, if $h_0 \in H_1$ and

$$\nu \coloneqq (P_1)_{k_1,h_0} \otimes (P_2)_{k_2,h_1} \otimes \cdots \otimes (P_m)_{k_m,h_{m-1}} = P_{k_1\otimes\cdots\otimes k_m, h_0\otimes\cdots\otimes h_{m-1}} \in M(\Omega, \mathscr{F}),$$

then $\nu \ll P$ in the sense that $P(G) = 0$ implies $|\nu|(G) = 0$. Now, note that

$$\int_\Omega \prod_{i=1}^m |\varphi_i(\omega_i, \sigma)| \, |\nu|(\mathrm{d}\boldsymbol{\omega}) \leq \prod_{i=1}^m \|\varphi_i(\cdot, \sigma)\|_{L^\infty(P_i)} \|k_i\|_{H_i} \|h_{i-1}\|_{H_i} < \infty, \qquad \sigma \in \Sigma. \tag{5.6.13}$$

Consequently, by Equation (5.6.12) and Fubini's theorem,

$$\begin{aligned} \langle F(\sigma)k_m, h_0\rangle_{H_1} &= \left(\prod_{i=2}^m \int_{\Omega_i} \varphi_i(\cdot, \sigma) \, \mathrm{d}(P_i)_{k_i,h_{i-1}}\right) \big\langle P_1(\varphi_1(\cdot, \sigma))k_1, h_0\big\rangle_{H_1} \\ &= \prod_{i=1}^m \int_{\Omega_i} \varphi_i(\cdot, \sigma) \, \mathrm{d}(P_i)_{k_i,h_{i-1}} = \int_\Omega \varphi_1(\omega_1, \sigma) \cdots \varphi_m(\omega_m, \sigma) \, \nu(\mathrm{d}\boldsymbol{\omega}) \end{aligned} \tag{5.6.14}$$

whenever $\sigma \in \Sigma$. Now, by Inequality (5.6.13),

$$\int_\Sigma \int_\Omega \prod_{i=1}^m |\varphi_i(\omega_i, \sigma)| \, |\nu|(\mathrm{d}\boldsymbol{\omega}) \, \rho(\mathrm{d}\sigma) \leq \left(\prod_{i=1}^m \|k_i\|_{H_i} \|h_{i-1}\|_{H_i}\right) \underline{\int_\Sigma} \prod_{i=1}^m \|\varphi_i(\cdot, \sigma)\|_{L^\infty(P_i)} \, \rho(\mathrm{d}\sigma) < \infty.$$

Thus, by definition of pointwise Pettis integrals, Equation (5.6.14), and Fubini's theorem,

$$\begin{aligned} \big\langle I^{\mathbf{P}}(\Sigma, \rho, \varphi_1, \ldots, \varphi_m)[b]k_m, h_0\big\rangle_{H_1} &= \int_\Sigma \int_\Omega \varphi_1(\omega_1, \sigma) \cdots \varphi_m(\omega_m, \sigma) \, \nu(\mathrm{d}\boldsymbol{\omega}) \, \rho(\mathrm{d}\sigma) \\ &= \int_\Omega \int_\Sigma \varphi_1(\omega_1, \sigma) \cdots \varphi_m(\omega_m, \sigma) \, \rho(\mathrm{d}\sigma) \, \nu(\mathrm{d}\boldsymbol{\omega}). \end{aligned} \tag{5.6.15}$$

Since $\nu \ll P$, the definition of L_P^∞-IPD implies

$$\varphi(\boldsymbol{\omega}) = \int_\Sigma \varphi_1(\omega_1, \sigma) \cdots \varphi_m(\omega_m, \sigma) \, \rho(\mathrm{d}\sigma), \qquad |\nu|\text{-a.e. } \boldsymbol{\omega} \in \Omega.$$

Therefore, Equation (5.6.15) becomes

$$\big\langle I^{\mathbf{P}}(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})[b]k_m, h_0\big\rangle_{H_1} = \int_\Omega \varphi \, \mathrm{d}\nu = \langle P(\varphi)(k_1 \otimes \cdots \otimes k_m), h_0 \otimes \cdots \otimes h_{m-1}\rangle_{H_1 \otimes_2 \cdots \otimes_2 H_m}.$$

Since the right-hand side is independent of the chosen L_P^∞-IPD, we are done. □

Remark 5.6.16. By carefully inspecting the proofs above, we see that if Definition 5.5.3(ii) were changed to the requirement that $\underline{\int_\Sigma}\|\varphi_1(\cdot,\sigma)\|_{L^\infty(P_1)}\cdots\|\varphi_{k+1}(\cdot,\sigma)\|_{L^\infty(P_{k+1})}\,\rho(\mathrm{d}\sigma)<\infty$, then Theorem 5.6.11 (and the results in this section leading up to it) would still hold. We use the upper integral in Definition 5.5.3 so that $\|\cdot\|_{L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})}$ is a norm.

We are finally allowed to make the following long-awaited definition.

Definition 5.6.17 (MOI, take II)**.** If $\varphi\in L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})$, then we define

$$\begin{aligned}\big(I^{\mathbf{P}}\varphi\big)[b]&=\int_{\Omega_{k+1}}\cdots\int_{\Omega_1}\varphi(\omega_1,\ldots,\omega_{k+1})\,P_1(\mathrm{d}\omega_1)\,b_1\cdots P_k(\mathrm{d}\omega_k)\,b_k\,P_{k+1}(\mathrm{d}\omega_{k+1})\\&\coloneqq\int_\Sigma P_1(\varphi_1(\cdot,\sigma))\,b_1\cdots P_k(\varphi_k(\cdot,\sigma))\,b_k\,P_{k+1}(\varphi_{k+1}(\cdot,\sigma))\,\rho(\mathrm{d}\sigma)\in B(H_{k+1};H_1)\end{aligned}$$

for all $b=(b_1,\ldots,b_k)\in B(H_2;H_1)\times\cdots\times B(H_{k+1};H_k)$ and any L^∞_P-IPD $(\Sigma,\rho,\varphi_1,\ldots,\varphi_{k+1})$ of φ. The map $I^{\mathbf{P}}\varphi\colon B(H_2;H_1)\times\cdots\times B(H_{k+1};H_k)\to B(H_{k+1};H_1)$ is the **multiple operator integral** (MOI) of φ with respect to $\mathbf{P}=(P_1,\ldots,P_{k+1})$. We also write

$$(P_1\otimes\cdots\otimes P_{k+1})(\varphi)\#[b_1,\ldots,b_k]=P(\varphi)\#b\coloneqq\big(I^{\mathbf{P}}\varphi\big)[b].$$

Remark 5.6.18 (# operation)**.** For vector spaces V and W, write $\operatorname{Hom}(V;W)$ for the set of linear maps $V\to W$. The # in the definition above *formally* stands for the algebraic operation

$$\#=\#_k\colon B(H_1)\otimes\cdots\otimes B(H_{k+1})\to\operatorname{Hom}(B(H_2;H_1)\otimes\cdots\otimes B(H_{k+1};H_k);B(H_{k+1};H_1))$$

determined (linearly) by

$$(a_1\otimes\cdots\otimes a_{k+1})\#[b_1\otimes\cdots\otimes b_k]\coloneqq a_1b_1\cdots a_1b_ka_{k+1},\qquad a_i\in B(H_i),\ b_j\in B(H_{j+1};H_j).$$

Now, the von Neumann algebra tensor product $B(H_1)\bar{\otimes}\cdots\bar{\otimes}B(H_{k+1})$ is naturally isomorphic to $B(H)=B(H_1\otimes_2\cdots\otimes_2 H_{k+1})$. Morally speaking, "the multiple operator integral $(I^{P_1,\ldots,P_{k+1}}\varphi)[b_1,\ldots,b_k]$ is $P(\varphi)=\int_\Omega\varphi\,\mathrm{d}(P_1\otimes\cdots\otimes P_{k+1})\in B(H)=B(H_1)\bar{\otimes}\cdots\bar{\otimes}B(H_{k+1})$ acting on $b_1\otimes\cdots\otimes b_k$ via #," even though this may not make sense (i.e., # may not extend to the von Neumann algebra tensor product). We continue this discussion in Remark 5.8.2.

We end this section by restricting the MOI we just defined to a von Neumann algebra. Notice first that Theorem 5.6.11 and Corollary 5.6.4 give

$$\big\|(I^{\mathbf{P}}\varphi)[b]\big\|_{B(H_{k+1};H_1)} \leq \|\varphi\|_{L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})} \prod_{i=1}^k \|b_i\|_{B(H_{i+1};H_i)} \tag{5.6.19}$$

for all $b = (b_1, \ldots, b_k) \in B(H_2; H_1) \times \cdots \times B(H_{k+1}; H_k)$.

Theorem 5.6.20 (MOIs in $\mathcal{M}$). *Suppose $H_1 = \cdots = H_{k+1} = K$, $\mathcal{M} \subseteq B(K)$ is a von Neumann algebra, and P_i takes values in $\mathcal{M}$ for all $i \in \{1, \ldots, k+1\}$. If $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$, $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ is an L^∞_P-IPD of φ, and $b = (b_1, \ldots, b_k) \in \mathcal{M}^k$, then*

$$\big(I^{\mathbf{P}}\varphi\big)[b] = \int_\Sigma P_1(\varphi_1(\cdot,\sigma))\, b_1 \cdots P_k(\varphi_k(\cdot,\sigma))\, b_k\, P_{k+1}(\varphi_{k+1}(\cdot,\sigma))\, \rho(\mathrm{d}\sigma) \tag{5.6.21}$$

is a weak integral in $\mathcal{M}$. Furthermore, $I^{\mathbf{P}}\varphi \colon B(K)^k \to B(K)$ restricts to an argumentwise σ-weakly continuous k-linear map $\mathcal{M}^k \to \mathcal{M}$ satisfying*

$$\big\|I^{\mathbf{P}}\varphi\big\|_{B_k(\mathcal{M}^k;\mathcal{M})} \leq \|\varphi\|_{L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})}. \tag{5.6.22}$$

Finally, $I^{\mathbf{P}}\varphi$ is independent of the representation of $\mathcal{M}$ in the sense that if $\mathcal{N}$ is another von Neumann algebra, and $\pi \colon \mathcal{M} \to \mathcal{N}$ is an algebraic $*$-isomorphism, then

$$\pi\Big(\big(I^{P_1,\ldots,P_{k+1}}\varphi\big)[b_1, \ldots, b_k]\Big) = \big(I^{\pi\circ P_1,\ldots,\pi\circ P_{k+1}}\varphi\big)[\pi(b_1), \ldots, \pi(b_k)], \quad b_1, \ldots, b_k \in \mathcal{M}.$$

Proof. By Proposition 4.2.17(ii), the definition of $I^{\mathbf{P}}\varphi$, and Corollary 5.4.9, the right-hand side of Equation (5.6.21) is a weak* integral in $\mathcal{M}$ whenever $b \in \mathcal{M}^k$. We know from Inequality (5.6.19) that the restriction $I^{\mathbf{P}}\varphi \colon \mathcal{M}^k \to \mathcal{M}$ satisfies Inequality (5.6.22). Corollary 5.6.10 implies the restriction $I^{\mathbf{P}}\varphi \colon \mathcal{M}^k \to \mathcal{M}$ is argumentwise σ-weakly continuous because the σ-weak operator topology on $\mathcal{M}$ is the subspace topology induced by the σ-weak topology on $B(K)$ and the latter is the same as the ultraweak topology. The final claim follows from Theorem 5.4.5(iv) and the fact that $\pi(P_i(f)) = (\pi \circ P_i)(f)$ for all $f \in L^\infty(P_i)$ by another multiplicative system theorem argument, which we leave to the reader. $\square$

Remark 5.6.23 (General semifinite case)**.** Let $(\mathcal{M} \subseteq B(K), \tau)$ be a semifinite von Neumann algebra. The arguments in this section are robust in the sense that they can be used to prove the following generalizations (in the $H_1 = \cdots = H_{k+1} = K$ case) of Theorems 5.2.7 and 5.6.9.

(i) Let $(\Omega, \mathscr{F}, K, P)$ and $(\Xi, \mathscr{G}, K, Q)$ be projection-valued measure spaces such that P and Q take values in $\mathcal{M}$, and let $(\Sigma, \mathscr{H})$ be a measurable space. Suppose $\varphi\colon \Omega \times \Sigma \to \mathbb{C}$ and $\psi\colon \Xi \times \Sigma \to \mathbb{C}$ are measurable functions such that $\varphi(\cdot, \sigma) \in L^\infty(P)$ and $\psi(\cdot, \sigma) \in L^\infty(Q)$ for all $\sigma \in \Sigma$. If $A\colon \Sigma \to L^1(\tau)$ is strongly measurable, then the map

$$\Sigma \ni \sigma \mapsto Q(\psi(\cdot, \sigma))\, A(\sigma)\, P(\varphi(\cdot, \sigma)) \in L^1(\tau)$$

is strongly measurable as well.

(ii) Suppose we are in the setup of Theorem 5.6.20. If $i \in \{1, \ldots, k\}$ and $b_i \in \mathcal{L}^1(\tau)$, then the map $\Sigma \ni \sigma \mapsto P_1(\varphi_1(\cdot, \sigma))\, b_1 \cdots P_k(\varphi_k(\cdot, \sigma))\, b_k\, P_{k+1}(\varphi_{k+1}(\cdot, \sigma)) \in \big(L^1(\tau), \|\cdot\|_{L^1(\tau)}\big)$ is strongly ρ-integrable, and its Bochner ρ-integral is the multiple operator integral (5.6.21).

To prove these, one uses the same arguments with [ACDS09, Lem. 2.5] instead of Lemma 5.6.7 and basic properties of L^1 instead of $\mathcal{S}_1$. Facts such as the two above can be useful when proving trace formulas; please see [ST19, §5.5] for a survey of some existing results on trace formulas.

5.7 Algebraic properties and noncommutative L^p estimates

In this section, we prove linearity and multiplicativity properties of the MOI defined in the previous section. Then we prove Schatten p-norm and, in the case of a semifinite von Neumann algebra, noncommutative L^p-norm estimates for MOIs.

Proposition 5.7.1 (Algebraic properties of MOIs)**.** *Let $m \in \{1, \ldots, k\}$.*

(i) *If $\varphi, \psi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$ and $\alpha \in \mathbb{C}$, then $I^{\mathbf{P}}(\varphi + \alpha\, \psi) = I^{\mathbf{P}}\varphi + \alpha\, I^{\mathbf{P}}\psi$.*

(ii) *If $\psi_1 \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_m)$, $\psi_2 \in L^\infty(P_{m+1})\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$, and*

$$(\psi_1 \otimes \psi_2)(\boldsymbol{\omega}) := \psi_1(\omega_1, \ldots, \omega_m)\, \psi_2(\omega_{m+1}, \ldots, \omega_{k+1}), \qquad \boldsymbol{\omega} \in \Omega,$$

then $\psi_1 \otimes \psi_2 \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$, *and*

$$\big(I^{\mathbf{P}}(\psi_1 \otimes \psi_2)\big)[b] = \big(I^{P_1,\ldots,P_m}\psi_1\big)[b_1,\ldots,b_{m-1}]\, b_m \left(I^{P_{m+1},\ldots,P_{k+1}}\psi_2\right)[b_{m+1},\ldots,b_k]$$

for all $b = (b_1,\ldots,b_k) \in B(H_2;H_1) \times \cdots \times B(H_{k+1};H_k)$.

(iii) *If* $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$, $\psi \in L^\infty(P_m)\hat{\otimes}_i L^\infty(P_{m+1})$, *and*

$$\tilde{\psi}(\boldsymbol{\omega}) \coloneqq \psi(\omega_m, \omega_{m+1}) \qquad \boldsymbol{\omega} \in \Omega$$

then

$$\big(I^{\mathbf{P}}\big(\varphi\tilde{\psi}\big)\big)[b] = \big(I^{\mathbf{P}}\varphi\big)\big[b_1,\ldots,b_{m-1},\big(I^{P_m,P_{m+1}}\psi\big)[b_m], b_{m+1},\ldots,b_k\big],$$

for all $b = (b_1,\ldots,b_k) \in B(H_2;H_1) \times \cdots \times B(H_{k+1};H_k)$.

Proof. We take each item in turn.

(i) It is easy to see that

$$I^{\mathbf{P}}(\alpha\varphi) = \alpha\, I^{\mathbf{P}}\varphi.$$

To prove that $I^{\mathbf{P}}$ is additive, let $(\Sigma_1, \rho_1, \varphi_1, \ldots, \varphi_{k+1})$ and $(\Sigma_2, \rho_2, \psi_1, \ldots, \psi_{k+1})$ be L^∞_P-IPDs of φ and ψ, respectively. Take $(\Sigma, \mathscr{H}, \rho)$ to be the disjoint union of the measure spaces $(\Sigma_1, \mathscr{H}_1, \rho_1)$ and $(\Sigma_2, \mathscr{H}_2, \rho_2)$, and for $i \in \{1,\ldots,k+1\}$, define

$$\chi_i(\omega_i,\sigma) \coloneqq \begin{cases} \varphi_i(\omega_i,\sigma) & \text{if } (\omega_i,\sigma) \in \Omega_i \times \Sigma_1 \subseteq \Omega_i \times \Sigma, \\ \psi_i(\omega_i,\sigma) & \text{if } (\omega_i,\sigma) \in \Omega_i \times \Sigma_2 \subseteq \Omega_i \times \Sigma. \end{cases}$$

As is argued in the proof of Proposition 5.5.5, $(\Sigma, \rho, \chi_1, \ldots, \chi_{k+1})$ is a L^∞_P-IPD of $\varphi + \psi$. Thus, by definition of the disjoint union measure space and pointwise Pettis integrals,

$$\begin{aligned} I^{\mathbf{P}}(\varphi+\psi) &= I^{\mathbf{P}}(\Sigma,\rho,\chi_1,\ldots,\chi_{k+1}) \\ &= I^{\mathbf{P}}(\Sigma_1,\rho_1,\varphi_1,\ldots,\varphi_{k+1}) + I^{\mathbf{P}}(\Sigma_2,\rho_2,\psi_1,\ldots,\psi_{k+1}) = I^{\mathbf{P}}\varphi + I^{\mathbf{P}}\psi. \end{aligned}$$

Thus, $\varphi \mapsto I^{\mathbf{P}}\varphi$ is linear.

(ii) If $(\Sigma_1, \rho_1, \varphi_1, \ldots, \varphi_m)$ and $(\Sigma_2, \rho_2, \varphi_{m+1}, \ldots, \varphi_{k+1})$ are, respectively, $L^\infty_{P_1 \otimes \cdots \otimes P_m}$- and $L^\infty_{P_{m+1} \otimes \cdots \otimes P_{k+1}}$-IPDs of ψ_1 and ψ_2, then

$$(\Sigma_1, \rho_1, \varphi_1, \ldots, \varphi_m, \underbrace{1, \cdots, 1}_{k+1-m}) \text{ and } (\Sigma_2, \rho_2, \underbrace{1, \ldots, 1}_{m}, \varphi_{m+1}, \ldots, \varphi_{k+1})$$

are, respectively, $L^\infty_{\mathbf{P}}$-IPDs of $\psi_1 \otimes 1$ and $1 \otimes \psi_2$. But then $\psi_1 \otimes \psi_2 = (\psi_1 \otimes 1)(1 \otimes \psi_2)$ belongs to $L^\infty(P_1) \hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$ because $L^\infty(P_1) \hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$ is an algebra. Furthermore, by the arguments from the proof of Proposition 5.5.5, if $(\Sigma, \mathscr{H}, \rho) := (\Sigma_1 \times \Sigma_2, \mathscr{H}_1 \otimes \mathscr{H}_2, \rho_1 \otimes \rho_2)$ and

$$\chi_i(\omega_i, \sigma) := \begin{cases} \varphi_i(\omega_i, \sigma_1) & \text{if } 1 \leq i \leq m, \\ \varphi_i(\omega_i, \sigma_2) & \text{if } m+1 \leq i \leq k+1, \end{cases}$$

for all $(\omega_i, \sigma) = (\omega_i, \sigma_1, \sigma_2) \in \Omega_i \times \Sigma_1 \times \Sigma_2 = \Omega_i \times \Sigma$, then $(\Sigma, \rho, \chi_1, \ldots, \chi_{k+1})$ is a $L^\infty_{\mathbf{P}}$-IPD of $\psi_1 \otimes \psi_2$. This observation implies the result. Indeed, fix $h_1 \in H_{k+1}$ and $h_2 \in H_1$, and define

$$\begin{aligned} h_3 &:= (I^{P_{m+1}, \ldots, P_{k+1}} \psi_2)[b_{m+1}, \ldots, b_k] h_1 \text{ and} \\ T &:= \big(I^{P_1, \ldots, P_m} \psi_1\big)[b_1, \ldots, b_{m-1}]\, b_m \, \big(I^{P_{m+1}, \ldots, P_{k+1}} \psi_2\big)[b_{m+1}, \ldots, b_k]. \end{aligned}$$

Then

$$\begin{aligned} \langle T h_1, h_2 \rangle_{H_1} &= \big\langle \big(I^{P_1, \ldots, P_m} \psi_1\big)[b_1, \ldots, b_{m-1}] b_m h_3, h_2 \big\rangle_{H_1} \\ &= \int_{\Sigma_1} \left\langle \left(\prod_{i=1}^m P_i(\varphi_i(\cdot, \sigma_1))\, b_i \right) h_3, h_2 \right\rangle_{H_1} \rho_1(\mathrm{d}\sigma_1) \\ &= \int_{\Sigma_1} \int_{\Sigma_2} \Bigg\langle \left(\prod_{i=1}^m P_i(\varphi_i(\cdot, \sigma_1))\, b_i \right) \left(\prod_{j=m+1}^{k} P_j(\varphi_j(\cdot, \sigma_2))\, b_j \right) \\ &\qquad\qquad \times P_{k+1}(\varphi_{k+1}(\cdot, \sigma_2)) h_1, h_2 \Bigg\rangle_{H_1} \rho_2(\mathrm{d}\sigma_2)\, \rho_1(\mathrm{d}\sigma_1) \\ &= \int_\Sigma \big\langle P_1(\chi_1(\cdot, \sigma))\, b_1 \cdots P_k(\chi_k(\cdot, \sigma))\, b_k\, P_{k+1}(\chi_{k+1}(\cdot, \sigma)) h_1, h_2 \big\rangle_{H_1} \rho(\mathrm{d}\sigma) \\ &= \big\langle \big(I^{\mathbf{P}}(\psi_1 \otimes \psi_2)\big)[b_1, \ldots, b_k] h_1, h_2 \big\rangle_{H_1} \end{aligned}$$

by definition and Fubini's theorem. This completes the proof of the first multiplicativity claim.

(iii) Let $(\Sigma_1, \rho_1, \varphi_1, \ldots, \varphi_{k+1})$ be an L^∞_P-IPD of φ, and let $(\Sigma_2, \rho_2, \psi_m, \varphi_{m+1})$ be a $L^\infty_{P_m \otimes P_{m+1}}$-IPD of ψ. Then

$$(\Sigma_2, \rho_2, \underbrace{1, \ldots, 1}_{m-1}, \psi_m, \psi_{m+1}, \underbrace{1, \cdots, 1}_{k-m})$$

is an L^∞_P-IPD of $\tilde{\psi}$. Once again, by the arguments from the proof of Proposition 5.5.5, if $(\Sigma, \mathscr{H}, \rho) \coloneqq (\Sigma_1 \times \Sigma_2, \mathscr{H}_1 \otimes \mathscr{H}_2, \rho_1 \otimes \rho_2)$ and for all $(\omega_i, \sigma) = (\omega_i, \sigma_1, \sigma_2) \in \Omega_i \times \Sigma$,

$$\chi_i(\omega_i, \sigma) \coloneqq \begin{cases} \varphi_i(\omega_i, \sigma_1) & \text{if } 1 \le i \le m-1, \\ \varphi_i(\omega_i, \sigma_1)\, \psi_i(\omega_i, \sigma_2) & \text{if } m \le i \le m+1, \\ \varphi_i(\omega_i, \sigma_1) & \text{if } m+2 \le i \le k+1, \end{cases}$$

then $(\Sigma, \rho, \chi_1, \ldots, \chi_{k+1})$ is an L^∞_P-IPD of $\varphi\tilde{\psi}$. Now, if $h_1 \in H_{k+1}$, $h_2 \in H_1$, $b_m^\psi \coloneqq \big(I^{P_m, P_{m+1}}\psi\big)[b_m]$, and $T \coloneqq \big(I^{P_1, \ldots, P_{k+1}}\varphi\big)\big[b_1, \ldots, b_{m-1}, b_m^\psi, b_{m+1}, \ldots, b_k\big]$, then

$$\begin{aligned}
\langle Th_1, h_2\rangle_{H_1} &= \big\langle \big(I^{P_1,\ldots,P_{k+1}}\varphi\big)\big[b_1, \ldots, b_{m-1}, b_m^\psi, b_{m+1}, \ldots, b_k\big]h_1, h_2\big\rangle_{H_1} \\
&= \int_{\Sigma_1} \Bigg\langle \left(\prod_{i=1}^{m-1} P_i(\varphi_i(\cdot, \sigma_1))\, b_i\right) P_m(\varphi_m(\cdot, \sigma_1))\, b_m^\psi \\
&\qquad \times \left(\prod_{i=m+1}^{k} P_i(\varphi_i(\cdot, \sigma_1))\, b_i\right) P_{k+1}(\varphi_{k+1}(\cdot, \sigma_1)) h_1, h_2 \Bigg\rangle_{H_1} \rho_1(\mathrm{d}\sigma_1) \\
&= \int_{\Sigma_1}\int_{\Sigma_2} \Bigg\langle \left(\prod_{i=1}^{m-1} P_i(\varphi_i(\cdot, \sigma_1))\, b_i\right) P_m(\varphi_m(\cdot, \sigma_1)) \\
&\qquad \times P_m(\psi_m(\cdot, \sigma_2))\, b_m\, P_{m+1}(\psi_{m+1}(\cdot, \sigma_2))\, P_{m+1}(\varphi_{m+1}(\cdot, \sigma_1))\, b_{m+1} \\
&\qquad \times \left(\prod_{i=m+2}^{k} P_i(\varphi_i(\cdot, \sigma_1))\, b_i\right) P_{k+1}(\varphi_{k+1}(\cdot, \sigma_1)) h_1, h_2 \Bigg\rangle_{H_1} \rho_2(\mathrm{d}\sigma_2)\, \rho_1(\mathrm{d}\sigma_1) \\
&= \int_\Sigma \big\langle P_1(\chi_1(\cdot, \sigma))\, b_1 \cdots P_k(\chi_k(\cdot, \sigma))\, b_k\, P_{k+1}(\chi_{k+1}(\cdot, \sigma)) h_1, h_2\big\rangle_{H_1} \rho(\mathrm{d}\sigma) \\
&= \big\langle \big(I^{\mathbf{P}}(\psi_1 \otimes \psi_2)\big)[b_1, \ldots, b_k] h_1, h_2\big\rangle_{H_1}
\end{aligned}$$

by the multiplicativity of integration with respect to a projection-valued measure and Fubini's theorem. This completes the proof of the second multiplicativity claim. □

Proposition 5.7.2 (Schatten estimates on MOIs)**.** *If* $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$ *and* $p, p_1, \ldots, p_k \in [1,\infty]$ *satisfy* $1/p = 1/p_1 + \cdots + 1/p_k$*, then*

$$\big\|(I^{\mathbf{P}}\varphi)[b]\big\|_{\mathcal{S}_p} \leq \|\varphi\|_{L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})}\|b_1\|_{\mathcal{S}_{p_1}}\cdots\|b_k\|_{\mathcal{S}_{p_k}}$$

for all $b = (b_1, \ldots, b_k) \in B(H_2; H_1) \times \cdots \times B(H_{k+1}; H_k)$*. (As usual,* $0 \cdot \infty := 0$*.)*

Proof. Let $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ be a L_P^∞-IPD of φ. By definition, Theorem 5.4.12, and Hölder's inequality for the Schatten norms,

$$\begin{aligned}
\big\|\big(I^{\mathbf{P}}\varphi\big)[b]\big\|_{\mathcal{S}_p} &= \left\|\int_\Sigma P_1(\varphi_1(\cdot,\sigma))\, b_1 \cdots P_k(\varphi_k(\cdot,\sigma))\, b_k\, P_{k+1}(\varphi_{k+1}(\cdot,\sigma))\, \rho(\mathrm{d}\sigma)\right\|_{\mathcal{S}_p} \\
&\leq \underline{\int_\Sigma} \big\|P_1(\varphi_1(\cdot,\sigma))\, b_1 \cdots P_k(\varphi_k(\cdot,\sigma))\, b_k\, P_{k+1}(\varphi_{k+1}(\cdot,\sigma))\big\|_{\mathcal{S}_p}\, \rho(\mathrm{d}\sigma) \\
&\leq \|b_1\|_{\mathcal{S}_{p_1}} \cdots \|b_k\|_{\mathcal{S}_{p_k}} \underline{\int_\Sigma} \prod_{i=1}^{k+1} \|P_i(\varphi_i(\cdot,\sigma))\|_{\mathcal{S}_\infty}\, \rho(\mathrm{d}\sigma) \\
&= \|b_1\|_{\mathcal{S}_{p_1}} \cdots \|b_k\|_{\mathcal{S}_{p_k}} \underline{\int_\Sigma} \prod_{i=1}^{k+1} \|\varphi_i(\cdot,\sigma)\|_{L^\infty(P_i)}\, \rho(\mathrm{d}\sigma).
\end{aligned}$$

Using that $\underline{\int_\Sigma} \cdot \, \mathrm{d}\rho \leq \overline{\int_\Sigma} \cdot \, \mathrm{d}\rho$ and then taking the infimum over all L_P^∞-IPDs $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ of φ gives the desired result. □

By the same proof, using Theorem 5.4.16 in place of Theorem 5.4.12 and noncommutative Hölder's inequality in place of Hölder's inequality for the Schatten norms, we get the following.

Proposition 5.7.3 (Noncommutative L^p estimates on MOIs)**.** *Suppose* $H_1 = \cdots = H_{k+1} = K$*,* $(\mathcal{M} \subseteq B(K), \tau)$ *is a semifinite von Neumann algebra, and* $P_1, \ldots, P_{k+1}$ *take values in* $\mathcal{M}$*. If* $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$ *and* $p, p_1, \ldots, p_k \in [1,\infty]$ *satisfy* $1/p = 1/p_1 + \cdots + 1/p_k$*, then*

$$\big\|\big(I^{\mathbf{P}}\varphi\big)[b]\big\|_{L^p(\tau)} \leq \|\varphi\|_{L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})}\|b_1\|_{L^{p_1}(\tau)}\cdots\|b_k\|_{L^{p_k}(\tau)}, \qquad b = (b_1, \ldots, b_k) \in \mathcal{M}^k.$$

for all $b = (b_1, \ldots, b_k) \in \mathcal{M}^k$*. (As usual,* $0 \cdot \infty := 0$*.) In particular,* $I^{\mathbf{P}}\varphi$ *extends to a bounded* k*-linear map* $L^{p_1}(\tau) \times \cdots \times L^{p_k}(\tau) \to L^p(\tau)$ *with operator norm at most* $\|\varphi\|_{L^\infty(P_1)\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty(P_{k+1})}$*.*

5.8 Relation to other definitions

For completeness, we now review a common alternative definition, due to Pavlov [Pav69] and Birman–Solomyak [BS96], of (5.1.1) and prove that it agrees with the definition from the previous section when both definitions apply. This alternative definition requires the construction of a certain vector measure; please see §A.2 for the relevant background and notation.

Theorem 5.8.1. *If* $b = (b_1, \ldots, b_k) \in \mathcal{S}_2(H_2; H_1) \times \cdots \times \mathcal{S}_2(H_{k+1}; H_k)$*, then there exists a unique vector measure* $P\#b \colon \mathscr{F} \to \mathcal{S}_2(H_{k+1}; H_1)$ *such that*

$$(P\#b)(G_1 \times \cdots \times G_{k+1}) = P_1(G_1)\, b_1 \cdots P_k(G_k)\, b_k\, P_{k+1}(G_{k+1}), \qquad G_i \in \mathscr{F}_i.$$

The semivariation $\|P\#b\|_{\mathrm{svar}}$ *of* $P\#b$ *is at most* $\|b_1\|_{\mathcal{S}_2} \cdots \|b_k\|_{\mathcal{S}_2}$*, and* $P\#b \ll P$ *in the sense that* $\{G \in \mathscr{F} : P(G) = 0\} \subseteq \{G \in \mathscr{F} : (P\#b)(\tilde{G}) = 0$ *whenever* $\mathscr{F} \ni \tilde{G} \subseteq G\}$.

Remark 5.8.2. The notation for the vector measure in Theorem 5.8.1 is not standard. It is inspired by the # operation discussed in Remark 5.6.18. As the notation suggests, morally speaking, "$P\#b$ is the projection-valued measure $P = P_1 \otimes \cdots \otimes P_{k+1}$ acting on $b_1 \otimes \cdots \otimes b_k$ via #." Indeed, the condition uniquely characterizing $P\#b$ can be rewritten genuinely as $(P\#b)(G) = P(G)\#b$ for all $G = G_1 \times \cdots \times G_{k+1}$ with $G_1 \in \mathscr{F}_1, \ldots, G_{k+1} \in \mathscr{F}_{k+1}$ because in this case $P(G) \in B(H_1) \otimes \cdots \otimes B(H_{k+1}) \subseteq B(H)$. Therefore, morally speaking, integrating a function φ with respect to $P\#b$ may also be viewed as "$\int_\Omega \varphi \, \mathrm{d}P$ acting on $b_1 \otimes \cdots \otimes b_k$ via #," which matches the interpretation discussed in Remark 5.6.18.

Pavlov's original proof of Theorem 5.8.1 (from [Pav69]) has an error. Birman–Solomyak pointed it out and sketched a correction in [BS96]. For the reader's benefit, we provide a complete proof in §5.10. In any case, following Pavlov, Theorem 5.8.1 allows us to define (5.1.1) as $\int_\Omega \varphi \, \mathrm{d}(P\#b) \in \mathcal{S}_2(H_{k+1}; H_1)$ for *all* $\varphi \in L^\infty(P)$ but only $b \in \mathcal{S}_2(H_2; H_1) \times \cdots \times \mathcal{S}_2(H_{k+1}; H_k)$. (Please see [DU77, pp. 5–6] for the definition of this integral.) In this case,

$$\left\| \int_\Omega \varphi \, \mathrm{d}(P\#b) \right\|_{\mathcal{S}_2} \leq \|\varphi\|_{L^\infty(P\#b)} \|P\#b\|_{\mathrm{svar}} \leq \|\varphi\|_{L^\infty(P)} \|b_1\|_{\mathcal{S}_2} \cdots \|b_k\|_{\mathcal{S}_2}. \tag{5.8.3}$$

We now show this definition agrees with the one we developed in §5.6 when they both apply.

Theorem 5.8.4 (Agreement with Pavlov MOI). *If* $\varphi \in L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})$, *then*

$$\big(I^{\mathbf{P}}\varphi\big)[b] = \int_\Omega \varphi \, \mathrm{d}(P\#b), \qquad b \in \mathcal{S}_2(H_2;H_1) \times \cdots \times \mathcal{S}_2(H_{k+1};H_k).$$

Proof. By Inequality (5.8.3) and the k-linearity of the condition uniquely characterizing $P\#b$,

$$\mathcal{S}_2(H_2;H_1) \times \cdots \times \mathcal{S}_2(H_{k+1};H_k) \ni b \mapsto \int_\Omega \varphi \, \mathrm{d}(P\#b) \in \mathcal{S}_2(H_{k+1};H_1)$$

is a bounded k-linear map with operator norm at most $\|\varphi\|_{L^\infty(P)}$. By Proposition 5.7.2,

$$\mathcal{S}_2(H_2;H_1) \times \cdots \times \mathcal{S}_2(H_{k+1};H_k) \ni b \mapsto \big(I^{\mathbf{P}}\varphi\big)[b] \in \mathcal{S}_2(H_{k+1};H_1)$$

is a bounded k-linear map with operator norm at most $\|\varphi\|_{L^\infty(P_1)\hat{\otimes}_i \cdots \hat{\otimes}_i L^\infty(P_{k+1})}$. Since finite-rank operators are dense in $\mathcal{S}_2$, it therefore suffices to prove that $\big(I^{\mathbf{P}}\varphi\big)[b] = \int_\Omega \varphi \, \mathrm{d}(P\#b)$ for all $b = (b_1,\ldots,b_k) \in \mathcal{S}_2(H_2;H_1) \times \cdots \times \mathcal{S}_2(H_{k+1};H_k)$ such that $b_1,\ldots,b_k$ all have rank at most one.

Now, recall $\mathcal{S}_2(H_1;H_{k+1}) \cong \mathcal{S}_2(H_{k+1};H_1)^*$ via the map $B \mapsto (A \mapsto \mathrm{Tr}(AB))$. Therefore, $\int_\Omega \varphi \, \mathrm{d}(P\#b)$ is determined by the requirement

$$\mathrm{Tr}\left(\int_\Omega \varphi \, \mathrm{d}(P\#b)\, b_{k+1}\right) = \int_\Omega \varphi(\boldsymbol{\omega}) \; \mathrm{Tr}((P\#b)(\mathrm{d}\boldsymbol{\omega})\, b_{k+1}), \qquad b_{k+1} \in \mathcal{S}_2(H_1;H_{k+1}).$$

Once again, since finite-rank operators are dense in $\mathcal{S}_2$ and the above equation is bounded linear in b_{k+1}, it suffices to take $b_{k+1} \colon H_1 \to H_{k+1}$ with rank at most one. It therefore suffices to prove

$$\mathrm{Tr}\big(\big(I^{\mathbf{P}}\varphi\big)[b]\, b_{k+1}\big) = \int_\Omega \varphi(\boldsymbol{\omega}) \; \mathrm{Tr}((P\#b)(\mathrm{d}\boldsymbol{\omega})\, b_{k+1})$$

for all $b = (b_1,\ldots,b_k) \in \mathcal{S}_2(H_2;H_1) \times \cdots \times \mathcal{S}_2(H_{k+1};H_k)$ and $b_{k+1} \in \mathcal{S}_2(H_1;H_{k+1})$ such that $b_1,\ldots,b_{k+1}$ all have rank at most one. Now, write $m \coloneqq k+1$, $T \coloneqq (I^{\mathbf{P}}\varphi)[b]$, $b_i \coloneqq \langle \cdot, h_i\rangle_{H_{i+1}} k_i$ for $i \in \{1,\ldots,m-1\}$, and $b_m \coloneqq \langle \cdot, h_0\rangle_{H_1} k_m$. By the calculation done in the proof of Theorem 5.6.11, if $\nu = P_{k_1\otimes\cdots\otimes k_m, h_0\otimes\cdots\otimes h_{m-1}} = (P_1)_{k_1,h_0} \otimes (P_2)_{k_2,h_1} \otimes \cdots \otimes (P_m)_{k_m,h_{m-1}}$, then

$$\mathrm{Tr}\big(\big(I^{\mathbf{P}}\varphi\big)[b]\, b_m\big) = \mathrm{Tr}(T \circ (\langle \cdot, h_0\rangle_{H_1} k_m)) = \langle T k_m, h_0\rangle_{H_1} = \int_\Omega \varphi \, \mathrm{d}\nu.$$

But now, by definition of the vector measure $P\#b\colon \mathscr{F} \to \mathcal{S}_2(H_{k+1};H_1)$, if $G = G_1 \times \cdots \times G_{k+1}$ with $G_1 \in \mathscr{F}_1, \ldots, G_{k+1} \in \mathscr{F}_{k+1}$, then

$$\begin{aligned}\operatorname{Tr}((P\#b)(G)\, b_{k+1}) &= \operatorname{Tr}(P_1(G_1)\, b_1 \cdots P_{k+1}(G_{k+1})\, b_{k+1}) \\ &= \prod_{i=1}^{m} \langle P_i(G_i)k_i, h_{i-1}\rangle_{H_i} = \prod_{i=1}^{m} (P_i)_{k_i,h_{i-1}}(G_i) = \nu(G).\end{aligned}$$

(This is a special case of the calculation resulting in Equations (5.6.12) and (5.6.14) from the proof of Theorem 5.6.11.) It follows that $\operatorname{Tr}((P\#b)(\cdot)\, b_m) = \nu$ as complex measures on $(\Omega, \mathscr{F})$. This completes the proof. □

We end this section by discussing Birman–Solomyak's original definition of DOIs, i.e., the $k = 1$ case of MOIs. Before doing so, we make an observation. Redefine $H \coloneqq H_1$ and $K \coloneqq H_2$. It is well known that $H \otimes_2 K^* \cong \mathcal{S}_2(K;H)$ isometrically via the bounded linear map determined by $h \otimes \ell \mapsto (k \mapsto \ell(k)\, h)$. This identification gives us a natural isometric isomorphism $\#\colon B(H \otimes_2 K^*) \to B(\mathcal{S}_2(K;H))$ that is a homeomorphism with respect to all the usual topologies—in particular, the WOT. Viewing $B(H) \otimes B(K^*)$ as a subset of $B(H \otimes_2 K^*)$, one can show this map is the unique WOT-continuous linear extension of the linear map determined by $B(H) \otimes B(K^*) \ni a \otimes b^{\mathrm{t}} \mapsto (c \mapsto acb) \in B(\mathcal{S}_2(K;H))$, where, for $b \in B(K)$, the **transpose** $b^{\mathrm{t}} \in B(K^*)$ is defined by $\ell \mapsto \ell \circ b$, i.e., the adjoint of b without identifying K^* with K via the Riesz representation theorem.[1]

Now, note that both $P_2^{\mathrm{t}}\colon \mathscr{F}_2 \to B(K^*)$ and $\tilde{P} \coloneqq \#\big(P_1 \otimes P_2^{\mathrm{t}}\big)\colon \mathscr{F}_1 \otimes \mathscr{F}_2 \to B(\mathcal{S}_2(K;H))$ are projection-valued measures. We therefore may define, following Birman–Solomyak [BS66],

$$T_\varphi^{P_1,P_2}(b) \coloneqq \#\left(\int_{\Omega_1 \times \Omega_2} \varphi \, \mathrm{d}(P_1 \otimes P_2^{\mathrm{t}})\right) b = \tilde{P}(\varphi) b \in \mathcal{S}_2(K;H)$$

for all $\varphi \in L^\infty(P_1 \otimes P_2) = L^\infty\big(P_1 \otimes P_2^{\mathrm{t}}\big) = L^\infty\big(\tilde{P}\big)$ and $b \in \mathcal{S}_2(K;H)$. One can show (e.g., by starting with finite-rank b and then approximating in $\mathcal{S}_2$) that $T_\varphi^{P_1,P_2}(b) = \int_{\Omega_1\times\Omega_2} \varphi \, \mathrm{d}((P_1 \otimes P_2)\#b)$, i.e., this agrees with Pavlov's definition. Now, Birman–Solomyak define $T_\varphi^{P_1,P_2}(b) \in B(K;H)$ for $b \in B(K;H)$ as follows. Recall that $B(H;K) \cong \mathcal{S}_1(K;H)^*$ isometrically via $B \mapsto (A \mapsto \operatorname{Tr}(AB))$.

[1] What is being said here is that the operation $\# = \#_1$ from Remark 5.6.18 *does* extend to the von Neumann algebra tensor product $B(H)\bar{\otimes}B(K^*) = B(H \otimes_2 K^*)$ when the codomain is taken to be $B(\mathcal{S}_2(K;H))$.

Therefore, $B(K;H)$ is isometrically conjugate-isomorphic to $\mathcal{S}_1(K;H)^*$ via $C \mapsto (A \mapsto \operatorname{Tr}(AC^*))$. Consequently, if $T\colon \mathcal{S}_1(K;H) \to \mathcal{S}_1(K;H)$ is a bounded linear map, then we may speak of its adjoint $T^*\colon B(K;H) \to B(K;H)$, which is characterized by

$$\operatorname{Tr}(T(A)\,C^*) = \operatorname{Tr}(A\,T^*(C)^*), \qquad A \in \mathcal{S}_1(K;H),\ C \in B(K;H).$$

Now, if $\varphi \in L^\infty(P_1 \otimes P_2)$ satisfies

$$T_\varphi^{P_1,P_2}(\mathcal{S}_1(K;H)) \subseteq \mathcal{S}_1(K;H) \subseteq \mathcal{S}_2(K;H),$$

e.g., if $\varphi \in L^\infty(P_1)\hat{\otimes}_i L^\infty(P_2)$ by Theorem 5.8.4 and Proposition 5.7.2, then it is easy to show

$$T_{\overline{\varphi}}^{P_1,P_2}(\mathcal{S}_1(K;H)) \subseteq \mathcal{S}_1(K;H) \ \text{ and } \ \big\|T_{\overline{\varphi}}^{P_1,P_2}\big\|_{B(\mathcal{S}_1(K;H))} = \big\|T_{\varphi}^{P_1,P_2}\big\|_{B(\mathcal{S}_1(K;H))} < \infty.$$

In this situation, Birman–Solomyak define

$$T_\varphi^{P_1,P_2}(b) := \Big(T_{\overline{\varphi}}^{P_1,P_2}\big|_{\mathcal{S}_1(K;H)}\Big)^*(b) \in B(K;H), \qquad b \in B(K;H).$$

Now, let $\varphi \in L^\infty(P_1)\hat{\otimes}_i L^\infty(P_2)$. By Corollary 5.6.10, if $b_1 \in B(K;H)$ and $b_2 \in \mathcal{S}_1(K;H)$, then

$$\begin{aligned}\operatorname{Tr}\left(\int_{\Omega_2}\int_{\Omega_1} \varphi(\omega_1,\omega_2)\,P_1(d\omega_1)\,b_1\,P_2(d\omega_2)\,b_2^*\right) &= \operatorname{Tr}\left(\int_{\Omega_1}\int_{\Omega_2} \varphi(\omega_1,\omega_2)\,P_2(d\omega_2)\,b_2^*\,P_1(d\omega_1)\,b_1\right)\\ &= \operatorname{Tr}\left(b_1\left(\int_{\Omega_2}\int_{\Omega_1} \overline{\varphi(\omega_1,\omega_2)}\,P_1(d\omega_1)\,b_2\,P_2(d\omega_2)\right)^*\right).\end{aligned}$$

This says precisely that

$$\Big(I^{P_1,P_2}\overline{\varphi}|_{\mathcal{S}_1(K;H)}\Big)^* = I^{P_1,P_2}\varphi.$$

Since we already know our definition of the MOI agrees with that of Pavlov when they both apply and thus $(I^{P_1,P_2}\varphi)[b] = T_\varphi^{P_1,P_2}(b)$ whenever $b \in \mathcal{S}_2(K;H)$, we obtain the following theorem.

Theorem 5.8.5 (Agreement with Birman–Solomyak DOI)**.** *If $\varphi \in L^\infty(P_1)\,\hat{\otimes}_i\,L^\infty(P_2)$, then $I^{P_1,P_2}\varphi = T_\varphi^{P_1,P_2}$ on all of $B(K;H)$.*

5.9 Proof of Theorem 5.1.4

In this section, we present the proof of Theorem 5.1.4. To begin, we recall the definition and basic properties of the Hilbert space tensor product; please see [BO08, §3.2] or [KR97a, §2.6] for information. Let $H_1, \ldots, H_m$ be complex Hilbert spaces. There exists a unique inner product $\langle \cdot, \cdot \rangle_{H_1 \otimes \cdots \otimes H_m}$ on $H_1 \otimes \cdots \otimes H_m$ such that

$$\langle h_1 \otimes \cdots \otimes h_m, k_1 \otimes \cdots \otimes k_m \rangle_{H_1 \otimes \cdots \otimes H_m} = \langle h_1, k_1 \rangle_{H_1} \cdots \langle h_m, k_m \rangle_{H_m}, \qquad h_i, k_i \in H_i.$$

The **Hilbert space tensor product** $(H_1 \otimes_2 \cdots \otimes_2 H_m, \langle \cdot, \cdot \rangle_{H_1 \otimes_2 \cdots \otimes_2 H_m})$ is defined to be the completion of $H_1 \otimes \cdots \otimes H_m$ with respect to $\langle \cdot, \cdot \rangle_{H_1 \otimes \cdots \otimes H_m}$. If $A_i \in B(H_i)$ for all $i \in \{1, \ldots, m\}$, then there exists a unique bounded linear map $A_1 \otimes_2 \cdots \otimes_2 A_m \in B(H_1 \otimes_2 \cdots \otimes_2 H_m)$ such that

$$(A_1 \otimes_2 \cdots \otimes_2 A_m)(h_1 \otimes \cdots \otimes h_m) = A_1 h_1 \otimes \cdots \otimes A_m h_m, \qquad h_i \in H_i.$$

Furthermore, $\|A_1 \otimes_2 \cdots \otimes_2 A_m\|_{B(H_1 \otimes_2 \cdots \otimes_2 H_m)} = \|A_1\|_{B(H_1)} \cdots \|A_m\|_{B(H_m)}$, and the linear map $B(H_1) \otimes \cdots \otimes B(H_m) \to B(H_1 \otimes_2 \cdots \otimes_2 H_m)$ determined by $A_1 \otimes \cdots \otimes A_m \mapsto A_1 \otimes_2 \cdots \otimes_2 A_m$ is an injective $*$-homomorphism when $B(H_1) \otimes \cdots \otimes B(H_m)$ is given the tensor product $*$-algebra structure. This allows us to view $B(H_1) \otimes \cdots \otimes B(H_m)$ as a $*$-subalgebra of $B(H_1 \otimes_2 \cdots \otimes_2 H_m)$ and justifies writing, as we shall, $A_1 \otimes \cdots \otimes A_m$ instead of $A_1 \otimes_2 \cdots \otimes_2 A_m$.

The proof of Theorem 5.1.4 goes through an extension theorem for projection-valued measures: Theorem 5.9.4 below. Notably, we shall not need to use the sledgehammer that is the Carathéodory–Hahn–Kluvánek extension theorem.

Definition 5.9.1. Suppose $\mathscr{E} \subseteq 2^{\Omega}$ contains $\emptyset$ and Ω. A function $P^0 \colon \mathscr{E} \to B(H)$ is

(i) **projection-valued** if $P^0(\Omega) = \mathrm{id}_H = 1$ and $P^0(G)^2 = P^0(G) = P^0(G)^*$ for all $G \in \mathscr{E}$,

(ii) a **projection-valued protomeasure** if $\mathscr{E}$ is an elementary family (as in [Fol99, §1.2]) and P^0 is projection-valued and WOT-countably additive, and

(iii) a **projection-valued premeasure** if $\mathscr{E}$ is an algebra and P^0 is projection-valued and WOT-countably additive.

Remark 5.9.2. By the proof of [BS80, Thm. 5.1.1], if $\mathscr{E} \subseteq 2^\Omega$ is a ring of sets and $P^0 \colon \mathscr{E} \to P(H)$ is projection-valued and finitely additive, then $P^0(G_1 \cap G_2) = P^0(G_1)\, P^0(G_2)$ for all $G_1, G_2 \in \mathscr{E}$.

As in classical measure theory, a protomeasure extends to a measure.

Lemma 5.9.3 (Extending to an algebra)**.** *Suppose $\mathscr{E} \subseteq 2^\Omega$ is an elementary family containing Ω. If $P^{00} \colon \mathscr{E} \to B(H)$ is a projection-valued protomeasure such that $P^{00}(G_1)\, P^{00}(G_2) = 0$ whenever $G_1, G_2 \in \mathscr{E}$ and $G_1 \cap G_2 = \emptyset$, then P^{00} extends uniquely to a projection-valued premeasure $P^0 \colon \mathrm{alg}(\mathscr{E}) \to B(H)$.*

Proof. By Lemma A.2.3, $P^{00} \colon \mathscr{E} \to B(H)$ extends uniquely to a WOT-countably additive function $P^0 \colon \mathrm{alg}(\mathscr{E}) \to B(H)$, so we only need to show that P^0 is projection-valued. To this end, let $G \in \mathrm{alg}(\mathscr{E})$. Then there exist disjoint sets $G_1, \ldots, G_n \in \mathscr{E}$ such that $G = \bigcup_{i=1}^n G_i$, in which case $P^0(G) = \sum_{i=1}^n P^{00}(G_i)$. By assumption, this exhibits $P^0(G)$ as the sum of pairwise orthogonal projections. Thus, $P^0(G)$ is an orthogonal projection. Since $P^0(\Omega) = P^{00}(\Omega) = 1$ as well, we are done. □

Theorem 5.9.4 (Projection-valued Carathéodory's theorem [BS80, Thms. 5.2.3 & 5.2.4(2)])**.** *Suppose $\mathscr{A} \subseteq 2^\Omega$ is an algebra. If $P^0 \colon \mathscr{A} \to B(H)$ is a projection-valued premeasure, then P^0 extends uniquely to a projection-valued measure $P \colon \sigma(\mathscr{A}) \to B(H)$.*

The proof of Theorem 5.9.4 proceeds as in classical measure theory, using a projection-valued analog of Carathéodory's theorem, which concerns itself with the **projection-valued outer measure**

$$P^*(G) \coloneqq \inf \left\{ P^0(G_1) : G_1 \supseteq G,\, G_1 \in \mathscr{A} \right\}, \qquad G \subseteq \Omega.$$

In fact, the whole proof amounts to transferring the result of Carathéodory's theorem for the outer measures $\mu_{h,h}^*(G) \coloneqq \inf \left\{ \left\langle P^0(G_1)h, h \right\rangle_H : G_1 \supseteq G,\, G_1 \in \mathscr{A} \right\}$ to a result about P^*.

Proof of Theorem 5.1.4. Write $m \coloneqq k+1$ and $\Omega \coloneqq \Omega_1 \times \cdots \times \Omega_m$, and define

$$\mathscr{E} \coloneqq \{ G_1 \times \cdots \times G_m \subseteq \Omega : G_1 \in \mathscr{F}_1, \ldots, G_m \in \mathscr{F}_m \}$$

to be the set of measurable rectangles. Now, define

$$P^{00}(G) := P_1(G_1) \otimes \cdots \otimes P_m(G_m) \in B(H_1 \otimes_2 \cdots \otimes_2 H_m), \qquad G = G_1 \times \cdots \times G_m \in \mathscr{E}.$$

Recall that $\mathscr{E}$ is an elementary family. We claim that P^{00} is a projection-valued protomeasure such that $P^{00}(G \cap \tilde{G}) = P^{00}(G)\, P^{00}(\tilde{G})$ for all $G, \tilde{G} \in \mathscr{E}$. If so, then an appeal to Lemma 5.9.3 and Theorem 5.9.4 completes the proof because $\sigma(\mathrm{alg}(\mathscr{E})) = \sigma(\mathscr{E}) = \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_m$.

To prove the claim, let $G := G_1 \times \cdots \times G_m, \tilde{G} := \tilde{G}_1 \times \cdots \times \tilde{G}_m \in \mathscr{E}$. Then

$$\begin{aligned} P^{00}(G \cap \tilde{G}) &= P^{00}((G_1 \cap \tilde{G}_1) \times \cdots \times (G_m \cap \tilde{G}_m)) \\ &= P_1(G_1 \cap \tilde{G}_1) \otimes \cdots \otimes P_m(G_m \cap \tilde{G}_m) \\ &= (P_1(G_1)\, P_1(\tilde{G}_1)) \otimes \cdots \otimes (P_m(G_m)\, P_m(\tilde{G}_m)) \\ &= (P_1(G_1) \otimes \cdots \otimes P_m(G_m))(P_1(\tilde{G}_1) \otimes \cdots \otimes P_m(\tilde{G}_m)) = P^{00}(G)\, P^{00}(\tilde{G}). \end{aligned}$$

Also,

$$\begin{aligned} P^{00}(G)^* &= (P_1(G_1) \otimes \cdots \otimes P_m(G_m))^* = P_1(G_1)^* \otimes \cdots \otimes P_m(G_m)^* \\ &= P_1(G_1) \otimes \cdots \otimes P_m(G_m) = P^{00}(G). \end{aligned}$$

Since it is clear that $P^{00}(\emptyset) = 0$ and $P^{00}(\Omega) = 1$, we only have the WOT-countable additivity of P^{00} left to prove. To this end, write $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{H_1 \otimes_2 \cdots \otimes_2 H_m}$ for the tensor inner product. By definition, we need to show that the assignment $\mathscr{E} \ni G \mapsto P^{00}_{h,k}(G) := \left\langle P^{00}(G)h, k \right\rangle \in \mathbb{C}$ is countably additive for all $h, k \in H_1 \otimes_2 \cdots \otimes_2 H_m$. Taking first pure tensors $h = h_1 \otimes \cdots \otimes h_m$, $k = k_1 \otimes \cdots \otimes k_m$, we have that if $G = G_1 \times \cdots \times G_m \in \mathscr{E}$, then

$$\begin{aligned} P^{00}_{h,k}(G) &= \langle (P_1(G_1) \otimes \cdots \otimes P_m(G_m))(h_1 \otimes \cdots \otimes h_m), k_1 \otimes \cdots \otimes k_m \rangle \\ &= \langle P_1(G_1)h_1, k_1 \rangle_{H_1} \cdots \langle P_m(G_m)h_m, k_m \rangle_{H_m} \\ &= (P_1)_{h_1,k_1}(G_1) \cdots (P_m)_{h_m,k_m}(G_m) \\ &= ((P_1)_{h_1,k_1} \otimes \cdots \otimes (P_m)_{h_m,k_m})(G). \end{aligned}$$

It follows that $P^{00}_{h,k}$ is countably additive whenever h and k are pure tensors and therefore also whenever $h, k \in H_1 \otimes \cdots \otimes H_m$. Now, let $(G_n)_{n\in\mathbb{N}} \in \mathscr{E}^{\mathbb{N}}$ be a disjoint sequence such that the union $G := \bigcup_{n\in\mathbb{N}} G_n$ belongs to $\mathscr{E}$, and let $h, k \in H_1 \otimes_2 \cdots \otimes_2 H_m$ be arbitrary. First, we show that $\sum_{n=1}^{\infty} \langle P^{00}(G_n)h, k\rangle$ is absolutely convergent. Indeed, $(P^{00}(G_n))_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal projections, so Bessel's inequality implies

$$\left(\sum_{n=1}^{\infty} \left\|P^{00}(G_n)h\right\|^2\right)^{\frac{1}{2}} \le \|h\|.$$

Therefore, by the Cauchy–Schwarz inequality (twice),

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|\left\langle P^{00}(G_n)h, k\right\rangle\right| &= \sum_{n=1}^{\infty} \left|\left\langle P^{00}(G_n)h, P^{00}(G_n)k\right\rangle\right| \le \sum_{n=1}^{\infty} \left\|P^{00}(G_n)h\right\| \left\|P^{00}(G_n)k\right\| \\
&\le \left(\sum_{n=1}^{\infty} \left\|P^{00}(G_n)h\right\|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left\|P^{00}(G_n)k\right\|^2\right)^{\frac{1}{2}} \le \|h\| \, \|k\| < \infty.
\end{aligned}$$

Next, choose sequences $(h_j)_{j\in\mathbb{N}}, (k_j)_{j\in\mathbb{N}}$ in $H_1 \otimes \cdots \otimes H_m$ such that $h_j \to h$ and $k_j \to k$ in $H_1 \otimes_2 \cdots \otimes_2 H_m$ as $j \to \infty$. Then

$$\begin{aligned}
\left|\left\langle P^{00}(G)h_j, k_j\right\rangle - \sum_{n=1}^{\infty} \left\langle P^{00}(G_n)h, k\right\rangle\right| &= \left|\sum_{n=1}^{\infty} \left\langle P^{00}(G_n)h_j, k_j\right\rangle - \sum_{n=1}^{\infty} \left\langle P^{00}(G_n)h, k\right\rangle\right| \\
&= \left|\sum_{n=1}^{\infty} \left(\left\langle P^{00}(G_n)(h_j - h), k_j\right\rangle + \left\langle P^{00}(G_n)h, k_j - k\right\rangle\right)\right| \\
&\le \|h_j - h\| \, \|k_j\| + \|h\| \, \|k_j - k\| \xrightarrow{j\to\infty} 0.
\end{aligned}$$

Since it is also the case that

$$\lim_{j\to\infty} \langle P^{00}(G)h_j, k_j\rangle = \langle P^{00}(G)h, k\rangle,$$

we conclude that

$$\left\langle P^{00}(G)h, k\right\rangle = \sum_{n=1}^{\infty} \left\langle P^{00}(G_n)h, k\right\rangle,$$

as desired. □

5.10 Proof of Theorem 5.8.1

In this section, we prove Theorem 5.8.1 using the approach of Pavlov [Pav69] and Birman–Solomyak [BS96]. The construction is similar in spirit to that of the tensor product of projection-valued measures, but the technical details are complicated substantially by the fact that we need the Carathéodory–Hahn–Kluvánek extension theorem (§A.2) instead of the (much easier) Carathéodory's theorem for projection-valued measures. Before beginning in earnest, we take care of a combinatorial detail that arises in the proof.

Lemma 5.10.1. *Let $m \in \mathbb{N}$, and write $[m] := \{1, \ldots, m\}$. Suppose $\mathscr{A}_i$ is an algebra of subsets of the set Ω_i for all $i \in [m]$, and write $\mathscr{E} := \{G_1 \times \cdots \times G_m : G_1 \in \mathscr{A}_1, \ldots, G_m \in \mathscr{A}_m\}$ for the elementary family of rectangles in $\Omega := \Omega_1 \times \cdots \times \Omega_m$. If $\nu \colon \mathscr{E} \to \mathbb{C}$ is finitely additive and $R_1, \ldots, R_n \in \mathscr{E}$ is a partition of Ω by rectangles, then there exists a partition*

$$\left\{G^{\boldsymbol{\ell}} := G_1^{\ell_1} \times \cdots \times G_m^{\ell_m} : \boldsymbol{\ell} = (\ell_1, \ldots, \ell_m) \in [n_1] \times \cdots \times [n_m] =: [\boldsymbol{n}]\right\}$$

of Ω, where $G_i^1, \ldots, G_i^{n_j} \in \mathscr{A}_i$ is a partition of Ω_i for all $i \in [m]$, such that

$$\sum_{i=1}^{n} |\nu(R_i)| \leq \sum_{\boldsymbol{\ell} \in [\boldsymbol{n}]} \left|\nu\big(G^{\boldsymbol{\ell}}\big)\right|.$$

Proof. The key observation is that if $\tilde{G}^1, \ldots, \tilde{G}^n \in \mathscr{A}_1$ is a cover (*not* necessarily a partition) of Ω_1, then there exists a partition $G^1, \ldots, G^N \in \mathscr{A}_1$ of Ω_1 such that for all $i \in [n]$, $\tilde{G}^i$ is a disjoint union of some of the G's. We prove this by induction on $n \geq 1$. The $n = 1$ case is trivial. Now, assume the result for all sets, all algebras, and all covers of length less than $n \in \mathbb{N}$. Then, given a cover $\tilde{G}^1, \ldots, \tilde{G}^n \in \mathscr{A}_1$ of Ω_1, we get a partition $G_0^1, \ldots, G_0^{N_0} \in \mathscr{A}_1$ of $\bigcup_{i=2}^n \tilde{G}^i$ with the property that $\tilde{G}^i$ is a disjoint union of G_0's for all $i \in \{2, \ldots, n\}$. Let $\mathscr{P} := \big\{G_0^1, \ldots, G_0^{N_0}\big\}$. Then the desired partition of Ω_1 is

$$\big\{P \in \mathscr{P} : \tilde{G}^1 \cap P = \emptyset\big\} \cup \big\{\tilde{G}^1 \cap P : P \in \mathscr{P} \text{ and } \tilde{G}^1 \cap P \neq \emptyset\big\} \cup \left\{\tilde{G}^1 \setminus \left(\bigcup_{P \in \mathscr{P}} P\right)\right\}.$$

Enumerating the above family as $G^1, \ldots, G^N \in \mathscr{A}_1$ completes the proof of this initial observation.

Now, writing $R_i = \tilde{G}_1^i \times \cdots \times \tilde{G}_m^i$, apply the observation from the previous paragraph to $\tilde{G}_j^1, \ldots, \tilde{G}_j^n \in \mathscr{A}_j$ to obtain a partition $G_j^1, \ldots, G_j^{n_j} \in \mathscr{A}_j$ of Ω_j such that for all $i \in [n]$, $\tilde{G}_j^i$ is a disjoint union of some G_j's. By the finite additivity assumption, we then get

$$\nu(R_i) = \sum_{\ell_1 : G_1^{\ell_1} \subseteq \tilde{G}_1^i} \cdots \sum_{\ell_m : G_m^{\ell_m} \subseteq \tilde{G}_m^i} \nu\big(G_1^{\ell_1} \times \cdots \times G_m^{\ell_m}\big).$$

Because $R_1, \ldots, R_n$ are disjoint, if $(\ell_1, \ldots, \ell_m) \in [n_1] \times \cdots \times [n_m]$ is such that

$$G_1^{\ell_1} \subseteq \tilde{G}_1^i, \ldots, G_m^{\ell_m} \subseteq \tilde{G}_m^i,$$

then it cannot be that $G_1^{\ell_1} \subseteq \tilde{G}_1^j, \ldots, G_m^{\ell_m} \subseteq \tilde{G}_m^j$ for some $j \neq i$ unless $G_1^{\ell_1} \times \cdots \times G_m^{\ell_m}$ is empty, in which case $\nu\big(G_1^{\ell_1} \times \cdots \times G_m^{\ell_m}\big) = 0$. This "no double-counting" observation and the above identity together imply that

$$\sum_{i=1}^n |\nu(R_i)| \leq \sum_{i=1}^n \sum_{\ell_1 : G_1^{\ell_1} \subseteq \tilde{G}_1^i} \cdots \sum_{\ell_m : G_m^{\ell_m} \subseteq \tilde{G}_m^i} \big|\nu\big(G_1^{\ell_1} \times \cdots \times G_m^{\ell_m}\big)\big| \leq \sum_{\boldsymbol{\ell} \in [\boldsymbol{n}]} \big|\nu\big(G^{\boldsymbol{\ell}}\big)\big|,$$

as desired. □

Proof of Theorem 5.8.1. Let $b = (b_1, \ldots, b_k) \in \mathcal{S}_2(H_2; H_1) \times \cdots \times \mathcal{S}_2(H_{k+1}; H_k)$, and write

$$\mathscr{E} := \{G_1 \times \cdots \times G_{k+1} : G_1 \in \mathscr{F}_1, \ldots, G_{k+1} \in \mathscr{F}_{k+1}\}$$

for the set of rectangles. For $G_1 \times \cdots \times G_{k+1} \in \mathscr{E}$, define

$$\begin{aligned} \mu_b^{00}(G_1 \times \cdots \times G_{k+1}) &:= P_1(G_1)\, b_1 \cdots P_k(G_k)\, b_k\, P_{k+1}(G_{k+1}) \in \mathcal{S}_2(H_{k+1}; H_1) \\ &= (P_1(G_1) \otimes \cdots \otimes P_{k+1}(G_{k+1})) \# [b_1 \otimes \cdots \otimes b_k] \end{aligned}$$

in the notation of Remark 5.6.18. We break up the proof into five steps.

Step 1. Prove that μ_b^{00} is finitely additive and therefore, by Lemma A.2.3, extends to a finitely additive vector measure $\mu_b^0 \colon \mathrm{alg}(\mathscr{E}) \to \mathcal{S}_2(H_{k+1}; H_1)$.

Step 2. Prove that for any partition $G_i^1, \dots, G_i^{n_i} \in \mathscr{F}_i$ of Ω_i (for each $i \in \{1, \dots, k+1\}$) and any $b_{k+1} \in \mathcal{S}_2(H_1; H_{k+1})$, we have that

$$\sum_{(\ell_1,\dots,\ell_{k+1})\in[n_1]\times\cdots\times[n_{k+1}]} \left| \operatorname{Tr}\left(\mu_b^{00}\left(G_1^{\ell_1}\times\cdots\times G_{k+1}^{\ell_{k+1}}\right) b_{k+1}\right)\right| \le \|b_1\|_{\mathcal{S}_2}\cdots\|b_{k+1}\|_{\mathcal{S}_2}.$$

Step 3. Conclude $\left\|\mu_b^0\right\|_{\mathrm{svar}} \le \|b_1\|_{\mathcal{S}_2}\cdots\|b_k\|_{\mathcal{S}_2}$.

Step 4. Prove that μ_b^{00} is weakly countably additive, which, again by Lemma A.2.3, means that μ_b^0 is also weakly countably additive. Then apply Theorem A.2.7 and Proposition A.2.8 to get $P\#b$ from μ_b^0.

Step 5. Prove $P\#b \ll P$.

Recall that we write

$$(\Omega, \mathscr{F}, H, P) \coloneqq (\Omega_1\times\cdots\times\Omega_{k+1}, \mathscr{F}_1\otimes\cdots\otimes\mathscr{F}_{k+1} H_1\otimes_2\cdots\otimes_2 H_{k+1}, P_1\otimes\cdots\otimes P_{k+1}).$$

Let us begin.

Step 1. There are a number of direct ways to see that μ_b^{00} is finitely additive. We provide a cute proof using Theorem 5.1.4 and the $\#$ operation from Remark 5.6.18. By definition of P, if $G \in \mathscr{E}$, then $P(G) \in B(H_1)\otimes\cdots\otimes B(H_{k+1}) \subseteq B(H)$. By definition, $\mu_b^{00}(G) = P(G)\#[b_1\otimes\cdots\otimes b_k]$ for all $G \in \mathscr{E}$. Since we know that $P\colon \sigma(\mathscr{E}) = \mathscr{F} \to B(H)$ is finitely additive, we conclude from the linearity of $\#$ that μ_b^{00} is finitely additive on $\mathscr{E}$. Furthermore, the finitely additive extension $\mu_b^0\colon \mathrm{alg}(\mathscr{E}) \to \mathcal{S}_2(H_{k+1}; H_1)$ is also given by the formula

$$\mu_b^0(G) = P(G)\#[b_1\otimes\cdots\otimes b_k], \qquad G \in \mathrm{alg}(\mathscr{E}).$$

This formula makes sense because $\mathrm{alg}(\mathscr{E})$ is the set of *finite* disjoint unions of elements of $\mathscr{E}$, so $P(G)$ is a *finite* sum of pure tensors and thus lies in $B(H_1)\otimes\cdots\otimes B(H_{k+1})$ whenever $G \in \mathrm{alg}(\mathscr{E})$.

Step 2. Let

$$\Delta \coloneqq \left\{ G^{\boldsymbol{\ell}} \coloneqq G_1^{\ell_1}\times\cdots\times G_{k+1}^{\ell_{k+1}} : \boldsymbol{\ell} = (\ell_1,\dots,\ell_{k+1}) \in [n_1]\times\cdots\times[n_{k+1}] =: [\boldsymbol{n}] \right\}$$

be the partition of $\Omega = \Omega_1 \times \cdots \times \Omega_{k+1}$ obtained from the fixed partitions of $\Omega_1, \ldots, \Omega_{k+1}$. For ease of notation, write $T^{\boldsymbol{\ell}} \coloneqq \operatorname{Tr}\left(\mu_b^{00}(G^{\boldsymbol{\ell}})\, b_{k+1}\right)$ and $|\Delta| \coloneqq \sum_{\boldsymbol{\ell}\in[\boldsymbol{n}]} \left|T^{\boldsymbol{\ell}}\right|$. The goal of this step is the estimate $|\Delta| \leq \|b_1\|_{\mathcal{S}_2} \cdots \|b_{k+1}\|_{\mathcal{S}_2}$.

To begin, note that if $\boldsymbol{\ell} \in [\boldsymbol{n}]$, then

$$\begin{aligned} T^{\boldsymbol{\ell}} &= \operatorname{Tr}\Big(P_1\big(G_1^{\ell_1}\big)\, b_1 \cdots P_k\big(G_k^{\ell_k}\big)\, b_k\, P_{k+1}\big(G_{k+1}^{\ell_{k+1}}\big)\, b_{k+1}\Big) \\ &= \operatorname{Tr}\Big(P_1\big(G_1^{\ell_1}\big)^2 b_1 \cdots P_k\big(G_k^{\ell_k}\big)^2 b_k P_{k+1}\big(G_{k+1}^{\ell_{k+1}}\big)^2 b_{k+1}\Big) \\ &= \operatorname{Tr}\Big(\Big[P_1\big(G_1^{\ell_1}\big)\, b_1\, P_2\big(G_2^{\ell_2}\big)\Big] \cdots \Big[P_k\big(G_k^{\ell_k}\big)\, b_k\, P_{k+1}\big(G_{k+1}^{\ell_{k+1}}\big)\Big]\Big[P_{k+1}\big(G_{k+1}^{\ell_{k+1}}\big)\, b_{k+1}\, P_1\big(G_1^{\ell_1}\big)\Big]\Big). \end{aligned}$$

For $i \in [k+1]$, $\ell_i \leq n_i$, and $\ell_{i+1} \leq n_{i+1}$ (adding mod $k+1$), define $\Pi_i^{\ell_i,\ell_{i+1}} \in B(\mathcal{S}_2(H_{i+1}; H_i))$ by

$$\Pi_i^{\ell_i,\ell_{i+1}}(c) \coloneqq P_i\big(G_i^{\ell_i}\big)\, c\, P_{i+1}\big(G_{i+1}^{\ell_{i+1}}\big) = \big(P_i\big(G_i^{\ell_i}\big) \otimes P_{i+1}\big(G_{i+1}^{\ell_{i+1}}\big)\big)\#c, \qquad c \in \mathcal{S}_2(H_{i+1}; H_i).$$

Then $T^{\boldsymbol{\ell}} = \operatorname{Tr}\left(\Pi_1^{\ell_1,\ell_2}(b_1) \cdots \Pi_k^{\ell_k,\ell_{k+1}}(b_k)\, \Pi_{k+1}^{\ell_{k+1},\ell_1}(b_{k+1})\right)$, so

$$\left|T^{\boldsymbol{\ell}}\right| \leq \left\|\Pi_1^{\ell_1,\ell_2}(b_1) \cdots \Pi_k^{\ell_k,\ell_{k+1}}(b_k)\, \Pi_{k+1}^{\ell_{k+1},\ell_1}(b_{k+1})\right\|_{\mathcal{S}_1} \leq \prod_{i=1}^{k+1} \left\|\Pi_i^{\ell_i,\ell_{i+1}}(b_i)\right\|_{\mathcal{S}_2}$$

(remembering to reduce mod $k+1$) for all $\boldsymbol{\ell} \in [\boldsymbol{n}]$.

Next, since $\left\{G_i^{\ell_i} \times G_{i+1}^{\ell_{i+1}} : \ell_i \in [n_i],\, \ell_{i+1} \in [n_{i+1}]\right\}$ is a partition of $\Omega_i \times \Omega_{i+1}$ by rectangles, it is easy to see that $\left\{\Pi_i^{\ell_i,\ell_{i+1}} : \ell_i \in [n_i],\, \ell_{i+1} \in [n_{i+1}]\right\}$ is a collection of mutually orthogonal projections in $B(\mathcal{S}_2(H_{i+1}; H_i))$ such that

$$\begin{aligned} \sum_{\ell_i=1}^{n_i} \sum_{\ell_{i+1}=1}^{n_{i+1}} \Pi_i^{\ell_i,\ell_{i+1}}(c) &= \left(\sum_{\ell_i=1}^{n_i} \sum_{\ell_{i+1}=1}^{n_{i+1}} P_i\big(G_i^{\ell_i}\big) \otimes P_{i+1}\big(G_{i+1}^{\ell_{i+1}}\big)\right)\#c \\ &= \left(\left(\sum_{\ell_i=1}^{n_i} P_i\big(G_i^{\ell_i}\big)\right) \otimes \left(\sum_{\ell_{i+1}=1}^{n_{i+1}} P_{i+1}\big(G_{i+1}^{\ell_{i+1}}\big)\right)\right)\#c \\ &= (P_i(\Omega_i) \otimes P_{i+1}(\Omega_{i+1}))\#c = c, \end{aligned}$$

so that whenever $c \in \mathcal{S}_2(H_{i+1}; H_i)$, we have

$$\sum_{\ell_i=1}^{n_i} \sum_{\ell_{i+1}=1}^{n_{i+1}} \left\|\Pi_i^{\ell_i,\ell_{i+1}}(c)\right\|_{\mathcal{S}_2}^2 = \|c\|_{\mathcal{S}_2}^2.$$

Consequently, if $k+1$ is even, then, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
|\Delta| &\leq \sum_{\boldsymbol{\ell}\in[\boldsymbol{n}]} \prod_{i=1}^{k+1} \left\|\Pi_i^{\ell_i,\ell_{i+1}}(b_i)\right\|_{\mathcal{S}_2} \\
&= \sum_{\boldsymbol{\ell}\in[\boldsymbol{n}]} \prod_{p=1}^{(k+1)/2} \left\|\Pi_{2p-1}^{\ell_{2p-1},\ell_{2p}}(b_{2p-1})\right\|_{\mathcal{S}_2} \prod_{q=1}^{(k+1)/2} \left\|\Pi_{2q}^{\ell_{2q},\ell_{2q+1}}(b_{2q})\right\|_{\mathcal{S}_2} \\
&\leq \left(\sum_{\boldsymbol{\ell}\in[\boldsymbol{n}]} \prod_{p=1}^{(k+1)/2} \left\|\Pi_{2p-1}^{\ell_{2p-1},\ell_{2p}}(b_{2p-1})\right\|_{\mathcal{S}_2}^2 \right)^{\frac{1}{2}} \left(\sum_{\boldsymbol{\ell}\in[\boldsymbol{n}]} \prod_{q=1}^{(k+1)/2} \left\|\Pi_{2q}^{\ell_{2q},\ell_{2q+1}}(b_{2q})\right\|_{\mathcal{S}_2}^2 \right)^{\frac{1}{2}} \\
&= \left(\prod_{p=1}^{(k+1)/2} \|b_{2p-1}\|_{\mathcal{S}_2}^2 \right)^{\frac{1}{2}} \left(\prod_{q=1}^{(k+1)/2} \|b_{2q}\|_{\mathcal{S}_2}^2 \right)^{\frac{1}{2}} = \|b_1\|_{\mathcal{S}_2} \cdots \|b_{k+1}\|_{\mathcal{S}_2},
\end{aligned}$$

as desired.

If $k+1$ is odd, then we estimate in a slightly different way. Just as we write

$$\boldsymbol{\ell} = (\ell_1, \ldots, \ell_{k+1}) \in [n_1] \times \cdots \times [n_{k+1}] = [\boldsymbol{n}],$$

we shall use the shorthand

$$\tilde{\boldsymbol{\ell}} = (\ell_2, \ldots, \ell_k) \in [n_2] \times \cdots \times [n_k] = [\tilde{\boldsymbol{n}}].$$

By the Cauchy–Schwarz inequality and above,

$$\begin{aligned}
|\Delta| &\leq \sum_{\boldsymbol{\ell}\in[\boldsymbol{n}]} \prod_{i=1}^{k+1} \left\|\Pi_i^{\ell_i,\ell_{i+1}}(b_i)\right\|_{\mathcal{S}_2} \\
&= \sum_{\ell_1=1}^{n_1} \sum_{\ell_{k+1}=1}^{n_{k+1}} \left\|\Pi_{k+1}^{\ell_{k+1},\ell_1}(b_{k+1})\right\|_{\mathcal{S}_2} \sum_{\tilde{\boldsymbol{\ell}}\in[\tilde{\boldsymbol{n}}]} \prod_{i=1}^{k} \left\|\Pi_i^{\ell_i,\ell_{i+1}}(b_i)\right\|_{\mathcal{S}_2} \\
&\leq \left(\sum_{\ell_1=1}^{n_1} \sum_{\ell_{k+1}=1}^{n_{k+1}} \left\|\Pi_{k+1}^{\ell_{k+1},\ell_1}(b_{k+1})\right\|_{\mathcal{S}_2}^2 \right)^{\frac{1}{2}} \left(\sum_{\ell_1=1}^{n_1} \sum_{\ell_{k+1}=1}^{n_{k+1}} \left(\sum_{\tilde{\boldsymbol{\ell}}\in[\tilde{\boldsymbol{n}}]} \prod_{i=1}^{k} \left\|\Pi_i^{\ell_i,\ell_{i+1}}(b_i)\right\|_{\mathcal{S}_2} \right)^2 \right)^{\frac{1}{2}} \\
&= \|b_{k+1}\|_{\mathcal{S}_2} \left(\sum_{\ell_1=1}^{n_1} \sum_{\ell_{k+1}=1}^{n_{k+1}} \Bigg(\underbrace{\sum_{\tilde{\boldsymbol{\ell}}\in[\tilde{\boldsymbol{n}}]} \prod_{i=1}^{k} \left\|\Pi_i^{\ell_i,\ell_{i+1}}(b_i)\right\|_{\mathcal{S}_2}}_{=: S^{\ell_1,\ell_{k+1}}} \Bigg)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Since k is even, we may now estimate as in the even case. If $\ell_1 \in [n_1]$ and $\ell_{k+1} \in [n_{k+1}]$, then

$$\begin{aligned} S^{\ell_1,\ell_{k+1}} &= \sum_{\tilde{\boldsymbol{\ell}} \in [\tilde{\boldsymbol{n}}]} \prod_{p=1}^{k/2} \left\| \Pi_{2p-1}^{\ell_{2p-1},\ell_{2p}}(b_{2p-1}) \right\|_{\mathcal{S}_2} \prod_{q=1}^{k/2} \left\| \Pi_{2q}^{\ell_{2q},\ell_{2q+1}}(b_{2q}) \right\|_{\mathcal{S}_2} \\ &\leq \left(\sum_{\tilde{\boldsymbol{\ell}} \in [\tilde{\boldsymbol{n}}]} \prod_{p=1}^{k/2} \left\| \Pi_{2p-1}^{\ell_{2p-1},\ell_{2p}}(b_{2p-1}) \right\|_{\mathcal{S}_2}^2 \right)^{\frac{1}{2}} \left(\sum_{\tilde{\boldsymbol{\ell}} \in [\tilde{\boldsymbol{n}}]} \prod_{q=1}^{k/2} \left\| \Pi_{2q}^{\ell_{2q},\ell_{2q+1}}(b_{2q}) \right\|_{\mathcal{S}_2}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Also,

$$\begin{aligned} \sum_{\tilde{\boldsymbol{\ell}} \in [\tilde{\boldsymbol{n}}]} \prod_{p=1}^{k/2} \left\| \Pi_{2p-1}^{\ell_{2p-1},\ell_{2p}}(b_{2p-1}) \right\|_{\mathcal{S}_2}^2 &= \sum_{\ell_2=1}^{n_2} \left\| \Pi_1^{\ell_1,\ell_2}(b_1) \right\|_{\mathcal{S}_2}^2 \prod_{p=2}^{k/2} \|b_{2p-1}\|_{\mathcal{S}_2}^2 \quad \text{and} \\ \sum_{\tilde{\boldsymbol{\ell}} \in [\tilde{\boldsymbol{n}}]} \prod_{q=1}^{k/2} \left\| \Pi_{2q}^{\ell_{2q},\ell_{2q+1}}(b_{2q}) \right\|_{\mathcal{S}_2}^2 &= \prod_{q=1}^{(k-1)/2} \|b_{2q}\|_{\mathcal{S}_2} \sum_{\ell_k=1}^{n_k} \left\| \Pi_k^{\ell_k,\ell_{k+1}}(b_k) \right\|_{\mathcal{S}_2}^2. \end{aligned}$$

Therefore,

$$S^{\ell_1,\ell_{k+1}} \leq \|b_2\|_{\mathcal{S}_2} \cdots \|b_{k-1}\|_{\mathcal{S}_2} \left(\sum_{\ell_2=1}^{n_2} \left\| \Pi_1^{\ell_1,\ell_2}(b_1) \right\|_{\mathcal{S}_2}^2 \right)^{\frac{1}{2}} \left(\sum_{\ell_k=1}^{n_k} \left\| \Pi_k^{\ell_k,\ell_{k+1}}(b_k) \right\|_{\mathcal{S}_2}^2 \right)^{\frac{1}{2}},$$

whence it follows that

$$\begin{aligned} |\Delta| &\leq \|b_{k+1}\|_{\mathcal{S}_2} \prod_{i=2}^{k-1} \|b_i\|_{\mathcal{S}_2} \left(\sum_{\ell_1=1}^{n_1} \sum_{\ell_2=1}^{n_2} \left\| \Pi_1^{\ell_1,\ell_2}(b_1) \right\|_{\mathcal{S}_2}^2 \sum_{\ell_k=1}^{n_k} \sum_{\ell_{k+1}=1}^{n_{k+1}} \left\| \Pi_k^{\ell_k,\ell_{k+1}}(b_k) \right\|_{\mathcal{S}_2}^2 \right)^{\frac{1}{2}} \\ &= \|b_{k+1}\|_{\mathcal{S}_2} \left(\prod_{i=2}^{k-1} \|b_i\|_{\mathcal{S}_2} \right) \|b_1\|_{\mathcal{S}_2} \|b_k\|_{\mathcal{S}_2} = \|b_1\|_{\mathcal{S}_2} \cdots \|b_{k+1}\|_{\mathcal{S}_2}, \end{aligned}$$

as desired. This completes Step 2.

Step 3. By the Riesz representation theorem and the definition of the inner product on $\mathcal{S}_2(H_{k+1}; H_1)$, if $\ell \in \mathcal{S}_2(H_{k+1}; H_1)^*$, then there exists a unique $B \in \mathcal{S}_2(H_{k+1}; H_1)$ such that $\ell(A) = \langle A, B \rangle_{\mathcal{S}_2} = \operatorname{Tr}(B^*A) = \operatorname{Tr}(AB^*)$ for all $A \in \mathcal{S}_2(H_{k+1}; H_1)$. Writing

$$b_{k+1} \coloneqq B^* \in \mathcal{S}_2(H_1; H_{k+1}) \quad \text{and} \quad \nu_{b,b_{k+1}}(G) = \nu_{b_1,\ldots,b_{k+1}}(G) \coloneqq \operatorname{Tr}\left(\mu_b^0(G)\, b_{k+1}\right), \qquad G \in \operatorname{alg}(\mathscr{E}),$$

this tells us the goal of this step is to prove $\|\nu_{b,b_{k+1}}\| = |\nu_{b,b_{k+1}}|(\Omega) \leq \|b_1\|_{\mathcal{S}_2} \cdots \|b_{k+1}\|_{\mathcal{S}_2}$.

To begin, we make the simple observation that if $\mathscr{A} \subseteq 2^{\Omega}$ is an algebra, $\nu \colon \mathscr{A} \to \mathbb{C}$ is a finitely additive complex measure, and $\mathscr{E}_0 \subseteq \mathscr{A}$ is an elementary family generating $\mathscr{A}$ as an algebra, then

$$|\nu|(G) = \sup\left\{\sum_{i=1}^{n} |\nu(R_i)| : R_1, \ldots, R_n \in \mathscr{E}_0 \text{ is a partition of } G\right\}, \quad G \in \mathscr{A}.$$

Applying this observation to our case, we have

$$\left|\nu_{b_1,\ldots,b_{k+1}}\right|(G) = \sup\left\{\sum_{i=1}^{n} \left|\operatorname{Tr}\left(\mu_b^{00}(R_i)\, b_{k+1}\right)\right| : R_1, \ldots, R_n \in \mathscr{E} \text{ is a partition of } G\right\}$$

for all $G \in \operatorname{alg}(\mathscr{E})$. Therefore, by Lemma 5.10.1,

$$\left\|\nu_{b_1,\ldots,b_{k+1}}\right\| = \sup\left\{\sum_{\boldsymbol{\ell}\in[\boldsymbol{n}]} \left|\operatorname{Tr}\left(\mu_b^{00}(G^{\boldsymbol{\ell}})\, b_{k+1}\right)\right| : \Delta = \left\{G^{\boldsymbol{\ell}} : \boldsymbol{\ell} \in [\boldsymbol{n}]\right\} \text{ as in Step 2}\right\}.$$

It then follows from Step 2 that $\|\nu_{b_1,\ldots,b_{k+1}}\| \leq \|b_1\|_{\mathcal{S}_2} \cdots \|b_{k+1}\|_{\mathcal{S}_2}$. This completes Step 3.

Step 4. According to the comments at the beginning of and notation in Step 3, the goal of this step is to prove that $\nu_{b_1,\ldots,b_{k+1}}$ is countably additive for all $b_i \in \mathcal{S}_2(H_{i+1}; H_i)$, $i \in [k]$, and $b_{k+1} \in \mathcal{S}_2(H_1; H_{k+1})$. As mentioned in the outline of the proof, Lemma A.2.3 tells us we only need to check the countable additivity of $\nu_{b_1,\ldots,b_{k+1}}$ on $\mathscr{E}$. Henceforth, write $m \coloneqq k+1$.

First, suppose $b_i = \langle \cdot, h_i\rangle_{H_{i+1}} k_i$, where $k_i \in H_i$ and $h_i \in H_{i+1}$, for all $i \in [m-1]$, and $b_m = \langle \cdot, h_0\rangle_{H_1} k_m$, where $h_0 \in H_1$ and $k_m \in H_m$. If $G = G_1 \times \cdots \times G_m \in \mathscr{E}$, then

$$\mu_b^{00}(G)\, b_m = \left(\prod_{i=2}^{m} \langle P_i(G_i)k_i, h_{i-1}\rangle_{H_i}\right) \langle \cdot, h_0\rangle_{H_1} P_1(G_1)k_1,$$

so that

$$\begin{aligned}
\nu_{b_1,\ldots,b_m}(G) = \operatorname{Tr}\left(\mu_b^{00}(G)\, b_m\right) &= \langle P_1(G_1)k_1, h_0\rangle_{H_1} \prod_{i=2}^{m} \langle P_i(G_i)k_i, h_{i-1}\rangle_{H_i} \\
&= ((P_1)_{k_1,h_0} \otimes (P_2)_{k_2,h_1} \otimes \cdots \otimes (P_m)_{k_m,h_{m-1}})(G) \\
&= P_{k_1\otimes\cdots\otimes k_m, h_0\otimes\cdots\otimes h_{m-1}}(G). \qquad (5.10.2)
\end{aligned}$$

Since this formula is the restriction to $\mathscr{E} \subseteq \mathscr{F}$ of a complex measure, we get that $\nu_{b_1,\ldots,b_m}$ is countably additive. Since $\nu_{b_1,\ldots,b_m}$ is clearly m-linear in $(b_1,\ldots,b_m)$, we then conclude that $\nu_{b_1,\ldots,b_m}$ is countably additive for all finite-rank operators $b_1 \in \mathcal{S}_2(H_2;H_1),\ldots,b_{m-1} \in \mathcal{S}_2(H_m;H_{m-1})$ and $b_m \in \mathcal{S}_2(H_1;H_m)$. To finish this step, we approximate arbitrary b's by finite-rank ones.

Let $b_1 \in \mathcal{S}_2(H_2;H_1),\ldots,b_{m-1} \in \mathcal{S}_2(H_m;H_{m-1})$, and $b_m \in \mathcal{S}_2(H_1;H_m)$ be arbitrary. If $(G_p)_{p\in\mathbb{N}} \in \mathscr{E}^{\mathbb{N}}$ is a pairwise disjoint sequence with $G \coloneqq \bigcup_{p\in\mathbb{N}} G_p \in \mathscr{E}$, then we must show

$$\delta_N \coloneqq \left|\nu_{b_1,\ldots,b_m}(G) - \sum_{p=1}^N \nu_{b_1,\ldots,b_m}(G_p)\right| \xrightarrow{N\to\infty} 0.$$

To this end, let $(b_1^n)_{n\in\mathbb{N}},\ldots,(b_m^n)_{n\in\mathbb{N}}$ be sequences of finite-rank operators such that $b_i^n \to b_i$ in $\mathcal{S}_2$ as $n\to\infty$ for all $i\in[m]$. Then, by the previous paragraph,

$$\begin{aligned}\delta_N &= \left|\nu_{b_1,\ldots,b_m}(G) - \nu_{b_1^n,\ldots,b_m^n}(G) + \sum_{p=1}^\infty \nu_{b_1^n,\ldots,b_m^n}(G_p) - \sum_{p=1}^N \nu_{b_1,\ldots,b_m}(G_p)\right| \\ &\leq |\nu_{b_1,\ldots,b_m}(G) - \nu_{b_1^n,\ldots,b_m^n}(G)| + \sum_{p=1}^N |\nu_{b_1,\ldots,b_m}(G_p) - \nu_{b_1^n,\ldots,b_m^n}(G_p)| + \sum_{p>N} |\nu_{b_1^n,\ldots,b_m^n}(G_p)|\end{aligned}$$

for all $n,N\in\mathbb{N}$, where the last term—for fixed $n\in\mathbb{N}$—goes to zero as $N\to\infty$. But now, notice that the m-linearity gives us that

$$\nu_{b_1,\ldots,b_m} - \nu_{b_1^n,\ldots,b_m^n} = \sum_{i=1}^m \nu_{b_1^n,\ldots,b_{i-1}^n,b_i-b_i^n,b_{i+1},\ldots,b_m}. \tag{5.10.3}$$

This observation and Step 3 then imply

$$\begin{aligned}\limsup_{N\to\infty} \delta_N &\leq \sum_{i=1}^m \left(\left|\nu_{b_1^n,\ldots,b_{i-1}^n,b_i-b_i^n,b_{i+1},\ldots,b_m}(G)\right| + \sum_{p=1}^\infty \left|\nu_{b_1^n,\ldots,b_{i-1}^n,b_i-b_i^n,b_{i+1},\ldots,b_m}(G_p)\right| \right) \\ &\leq 2\sum_{i=1}^m \left|\nu_{b_1^n,\ldots,b_{i-1}^n,b_i-b_i^n,b_{i+1},\ldots,b_m}\right|(G) \leq 2\sum_{i=1}^m \|\nu_{b_1^n,\ldots,b_{i-1}^n,b_i-b_i^n,b_{i+1},\ldots,b_m}\|_{\mathrm{var}} \\ &\leq 2\sum_{i=1}^m \|b_i^n\|_{\mathcal{S}_2}\cdots\|b_{i-1}^n\|_{\mathcal{S}_2}\|b_i-b_i^n\|_{\mathcal{S}_2}\|b_{i+1}\|_{\mathcal{S}_2}\cdots\|b_m\|_{\mathcal{S}_2} \xrightarrow{n\to\infty} 0.\end{aligned}$$

We conclude that $\lim_{N\to\infty}\delta_N = 0$. Thus, μ_b^0 is weakly countably additive. Since $\mathcal{S}_2(H_{k+1};H_1)$ is a Hilbert space and therefore reflexive, Step 3, what we just proved, Theorem A.2.7, and

Proposition A.2.8 yield that μ_b^0 extends uniquely to a $\mathcal{S}_2(H_{k+1}; H_1)$-valued vector measure $\mu_b = P\#b$ on $\sigma(\mathscr{E}) = \mathscr{F}$ with $\|P\#b\|_{\text{svar}} = \|\mu_b^0\|_{\text{svar}} \leq \|b_1\|_{\mathcal{S}_2} \cdots \|b_k\|_{\mathcal{S}_2}$. This completes Step 4 and the construction of $P\#b$.

Step 5. We use the approximation argument from Step 4. Suppose $G \in \mathscr{F}$ is such that $P(G) = 0$ (which implies $P(\tilde{G}) = 0$ when $\mathscr{F} \ni \tilde{G} \subseteq G$ because P is a projection-valued measure). If $\mathscr{F} \ni \tilde{G} \subseteq G$ and $b_1 \in \mathcal{S}_2(H_2; H_1), \ldots, b_k \in \mathcal{S}_2(H_{k+1}; H_k)$, and $b_{k+1} \in \mathcal{S}_2(H_1; H_{k+1})$ have rank at most one, then Equation (5.10.2) implies that $\operatorname{Tr}((P\#b)(\tilde{G})\, b_{k+1}) = 0$. By multilinearity, this implies $\operatorname{Tr}((P\#b)(\tilde{G})\, b_{k+1}) = 0$ for all finite-rank $b_1, \ldots, b_{k+1}$. Now, approximating in $\mathcal{S}_2$ arbitrary $b_1, \ldots, b_{k+1}$ by finite-rank operators gives, using Equation (5.10.3) and the semivariation bound, that $\operatorname{Tr}((P\#b)(\tilde{G})\, b_{k+1}) = 0$. We conclude $(P\#b)(\tilde{G}) = 0$. This completes the proof. $\square$

5.11 Acknowledgment

Chapter 5, in part, is a reprint of the material as it appears in "Multiple operator integrals in non-separable von Neumann algebras" (2023). Nikitopoulos, Evangelos A. *Journal of Operator Theory*, 89, 361–427.

Chapter 6

Differentiating at unbounded operators

Let $\mathcal{M}$ be a von Neumann algebra, and let a be a self-adjoint operator affiliated with $\mathcal{M}$. We define the notion of an "integral symmetrically normed ideal" of $\mathcal{M}$ and introduce a space $OC^{[k]}(\mathbb{R}) \subseteq C^k(\mathbb{R})$ of functions $\mathbb{R} \to \mathbb{C}$ such that the following result holds: For any integral symmetrically normed ideal $\mathcal{I}$ of $\mathcal{M}$ and any $f \in OC^{[k]}(\mathbb{R})$, the operator function $\mathcal{I}_{\mathrm{sa}} \ni b \mapsto f(a+b) - f(a) \in \mathcal{I}$ is k-times continuously Fréchet differentiable, and the formula for its derivatives may be written in terms of multiple operator integrals. Furthermore, we prove that if $f \in \dot{B}_1^{1,\infty}(\mathbb{R}) \cap \dot{B}_1^{k,\infty}(\mathbb{R})$ and f' is bounded, then $f \in OC^{[k]}(\mathbb{R})$. Finally, we prove that all the following ideals are integral symmetrically normed: $\mathcal{M}$ itself, separable symmetrically normed ideals, Schatten p-ideals, the ideal of compact operators, and—when $\mathcal{M}$ is semifinite—ideals induced by fully symmetric spaces of measurable operators.

Standing assumptions. Throughout, H is a complex Hilbert space, $\mathcal{M} \subseteq B(H)$ is a von Neumann algebra, and $\|\cdot\|_{H\to H} = \|\cdot\|$. In §6.3 and §6.4, $\mathcal{M}$ is (semifinite and) equipped with a trace $\tau \colon \mathcal{M}_+ \to [0,\infty]$. In §6.5, $k \in \mathbb{N}$. In §6.6, $k \in \mathbb{N}$, $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq \mathcal{M}$, and $\mathcal{I}_{\mathrm{sa}} \coloneqq \mathcal{I} \cap \mathcal{M}_{\mathrm{sa}}$.

6.1 Introduction

Given an appropriately regular scalar function $f \colon \mathbb{R} \to \mathbb{C}$, one of the goals of perturbation theory is to Taylor expand, i.e., differentiate many times, the "operator function" that takes a(n unbounded) self-adjoint operator A on H and maps it to the operator $f(A)$ constructed via the functional calculus for A. This delicate problem has its beginnings in [DK56], which initiated the subject of multiple operator integration (Chapter 5).

Let us quote the best-known general results on higher derivatives of operator functions. If $\dot{B}^{s,p}_q(\mathbb{R}^m)$ is the homogeneous Besov space (Definition 3.6.1), then we write

$$PB^k(\mathbb{R}) \coloneqq \dot{B}^{k,\infty}_1(\mathbb{R}) \cap \big\{f \in C^k(\mathbb{R}) : f^{(k)} \text{ is bounded}\big\} \tag{6.1.1}$$

for the $\boldsymbol{k}^{\mathbf{th}}$ **Peller–Besov space**. It turns out that $PB^1(\mathbb{R}) \cap PB^k(\mathbb{R}) = PB^1(\mathbb{R}) \cap \dot{B}^{k,\infty}_1(\mathbb{R})$. (Please see the paragraph containing Equation (B.2.10) at the end of §B.2.)

Theorem 6.1.2 (Peller [Pel06, Thm. 5.6])**.** *Suppose H is separable. If A is a self-adjoint operator on H, $B \in B(H)_{\mathrm{sa}}$, and $f \in PB^1(\mathbb{R}) \cap PB^k(\mathbb{R})$, then the map $\mathbb{R} \ni t \mapsto f(A+tB) - f(A) \in B(H)$ is k-times differentiable in the operator norm, and*

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big|_{t=0} f(A+tB) = k! \underbrace{\int_{\sigma(A)} \cdots \int_{\sigma(A)}}_{k+1 \text{ times}} f^{[k]}(\lambda_1, \ldots, \lambda_{k+1})\, P^A(\mathrm{d}\lambda_1)\, B \cdots P^A(\mathrm{d}\lambda_k)\, B\, P^A(\mathrm{d}\lambda_{k+1}),$$

where the MOI above is interpreted in accordance with Chapter 5.

We also quote a result from [ACDS09]. To do so, we define *property (F)*. A symmetrically normed ideal $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ of $\mathcal{M}$ (Definition 2.2.1) has **property (F)** if whenever $(a_i)_{i \in I}$ is a net in $\mathcal{I}$ such that $\sup_{i \in I} \|a_i\|_{\mathcal{I}} < \infty$ and $a_i \to a \in \mathcal{M}$ in the strong* operator topology, we get $a \in \mathcal{I}$ and $\|a\|_{\mathcal{I}} \le \sup_{i \in I} \|a_i\|_{\mathcal{I}}$. Also, recall $W_k(\mathbb{R}) \subseteq C^k(\mathbb{R})$ is the k^{th} *Wiener space* (Definition 1.3.13).

Theorem 6.1.3 (Azamov–Carey–Dodds–Sukochev [ACDS09, Thm. 5.7])**.** *Suppose H is separable, and let $a \;\eta\; \mathcal{M}_{\mathrm{sa}}$ (Definition 4.2.16). If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{M}$ has property (F), $\mathcal{I}_{\mathrm{sa}} \coloneqq \{b \in \mathcal{I} : b^* = b\}$, and $f \in W_{k+1}(\mathbb{R})$, then the map*

$$\mathcal{I}_{\mathrm{sa}} \ni b \mapsto f_{a,\mathcal{I}}(b) \coloneqq f(a+b) - f(a) \in \mathcal{I}$$

is well defined and k-times Fréchet differentiable with respect to $\|\cdot\|_{\mathcal{I}}$, and

$$\partial_{b_k} \cdots \partial_{b_1} f_a(0) = \sum_{\pi \in S_k} \underbrace{\int_{\sigma(a)} \cdots \int_{\sigma(a)}}_{k+1 \text{ times}} f^{[k]}(\lambda_1, \ldots, \lambda_{k+1})\, P^a(\mathrm{d}\lambda_1)\, b_{\pi(1)} \cdots P^a(\mathrm{d}\lambda_k)\, b_{\pi(k)}\, P^a(\mathrm{d}\lambda_{k+1})$$

for all $b_1, \ldots, b_k \in \mathcal{I}_{\mathrm{sa}}$.

As is noted in [ACDS09], the motivating example of a symmetrically normed ideal with property (F) comes from the theory of *symmetric operator spaces.* (Please see §6.3 for the meanings of the terms to follow.) Indeed, if $(E, \|\cdot\|_E)$ is a symmetric Banach function space with the Fatou property, $(\mathcal{M}, \tau)$ is a semifinite von Neumann algebra, and $(E(\tau), \|\cdot\|_{E(\tau)})$ is the symmetric space of τ-measurable operators induced by E, then

$$(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \coloneqq (E(\tau) \cap \mathcal{M}, \|\cdot\|_{E(\tau)\cap\mathcal{M}}) = (E(\tau) \cap \mathcal{M}, \max\{\|\cdot\|_{E(\tau)}, \|\cdot\|_{\mathcal{M}}\}) \tag{6.1.4}$$

is a symmetrically normed ideal of $\mathcal{M}$ with property (F). Though Theorem 6.1.3 applies to this interesting general setting, much more regularity is demanded of f than in Theorem 6.1.2. (Indeed, $W_k(\mathbb{R}) \subsetneq PB^1(\mathbb{R}) \cap PB^k(\mathbb{R})$.) It is an open problem [ST19, Prob. 5.3.22] to find less restrictive conditions for the higher Fréchet differentiability of maps induced by functional calculus ("operator functions") in the symmetric operator space ideals described above. This chapter makes substantial progress on this problem: A corollary of our main results is that if E is fully symmetric (a weaker condition than the Fatou property), then the result of Theorem 6.1.3 holds for $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ as in Equation (6.1.4) with $f \in PB^1(\mathbb{R}) \cap PB^k(\mathbb{R})$. In other words, we are able to close the regularity gap between Theorems 6.1.2 and 6.1.3 in the (fully) symmetric operator space context. Moreover, we are able, for the first time in the literature on higher derivatives of operator functions, to remove the separability assumption on H by using the MOI development from Chapter 5.

Remark 6.1.5 (Related work)**.** The Schatten p-ideals have property (F), so Theorem 6.1.3 applies to them when the underlying Hilbert space is separable. There are, however, much sharper results known about the differentiability of operator functions in the Schatten p-ideals (again, when the underlying Hilbert space is separable); please see [LMS20, LMM21].

Also, there is a seminal paper of de Pagter and Sukochev [dPS04] that studies the (once) Gateaux differentiability of operator functions in certain symmetric operator spaces at measurable operators; we discuss its relation to the results in this paper in Remark 6.6.17.

We now summarize our main results. The ideals we introduce are the *integral symmetrically normed ideals* (ISNIs). The definition of integral symmetrically normed is an "integrated" version of the symmetrically normed condition $\|arb\|_{\mathcal{I}} \leq \|a\|\,\|r\|_{\mathcal{I}}\|b\|$. Loosely speaking, a Banach ideal $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq$ is integral symmetrically normed if

$$\left\| \int_\Omega A(\omega)\, r\, B(\omega)\, \mu(\mathrm{d}\omega) \right\|_{\mathcal{I}} \leq \|r\|_{\mathcal{I}} \int_\Omega \|A\|\,\|B\|\,\mathrm{d}\mu, \qquad r \in \mathcal{I}.$$

The precise definition (Definition 6.2.2(ii)) is slightly technical, so we omit it for now. Our first main result comes in the form of a list of interesting examples of ISNIs.

Theorem 6.1.6 (Examples of ISNIs)**.** *Suppose H is arbitrary, i.e., not necessarily separable.*

(i) *The trivial ideal $(\mathcal{M}, \|\cdot\|)$ is integral symmetrically normed.*

(ii) *If $\mathcal{I}$ is a symmetrically normed ideal of $\mathcal{M}$ such that $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is separable, then $\mathcal{I}$ is integral symmetrically normed.*

(iii) *The ideal $\mathcal{K}(H) \trianglelefteq B(H)$ of compact operators is integral symmetrically normed.*

(iv) *If $1 \leq p \leq \infty$, then the ideal of $\mathcal{S}_p(H) \trianglelefteq B(H)$ of Schatten p-class operators is integral symmetrically normed.*

(v) *Suppose $(\mathcal{M}, \tau)$ is a semifinite von Neumann algebra. If $(E, \|\cdot\|_E)$ is a fully symmetric space of τ-measurable operators (Definition 6.3.1(iii)) and $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \coloneqq (E \cap \mathcal{M}, \|\cdot\|_{E\cap\mathcal{M}})$, then $\mathcal{I}$ is an integral symmetrically normed ideal of $\mathcal{M}$.*

Proof. The first item is Example 6.2.6, the second is part of Proposition 6.2.8, the third follows from Proposition 6.2.10 (or Remark 6.2.11), the fourth is a special case of Example 6.2.7, and the fifth is Theorem 6.4.1. □

With these in mind, we state our second main result.

Theorem 6.1.7 (Derivatives of operator functions in ISNIs)**.** *Let H be arbitrary, i.e., not necessarily separable, and let $a\;\eta\;\mathcal{M}_{\mathrm{sa}}$. If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is an integral symmetrically normed ideal of $\mathcal{M}$ and $f \in PB^1(\mathbb{R}) \cap PB^k(\mathbb{R})$, then the map*

$$\mathcal{I}_{\mathrm{sa}} \ni b \mapsto f_{a,\mathcal{I}}(b) \coloneqq f(a+b) - f(a) \in \mathcal{I}$$

is well defined and k-times continuously Fréchet differentiable with respect to $\|\cdot\|_{\mathcal{I}}$*, and*

$$\partial_{b_k}\cdots\partial_{b_1} f_a(0) = \sum_{\pi\in S_k} \underbrace{\int_{\sigma(a)}\cdots\int_{\sigma(a)}}_{k+1 \text{ times}} f^{[k]}(\lambda_1,\ldots,\lambda_{k+1})\, P^a(\mathrm{d}\lambda_1)\, b_{\pi(1)}\cdots P^a(\mathrm{d}\lambda_k)\, b_{\pi(k)}\, P^a(\mathrm{d}\lambda_{k+1})$$

for all $b_1,\ldots,b_k \in \mathcal{I}_{\mathrm{sa}}$*.*

Proof. Combine Theorem 6.6.16 and Corollary 6.6.10. □

Theorems 6.1.7 and 6.1.6(iv) generalize the best known results, from [LMS20], on the differentiability of operator functions in the ideal $(\mathcal{I},\|\cdot\|_{\mathcal{I}}) = (\mathcal{S}_p(H),\|\cdot\|_{\mathcal{S}_p})$ to the non-separable case when $p=1$. We do not, however, recover the optimal regularity on f, established in [LMM21], when $p\in(1,\infty)$. Also, to the author's knowledge, the present paper's result on the ideal of compact operators (i.e., Theorems 6.1.7 and 6.1.6(iii)) is new even when H is separable. Finally, as promised at the end of the previous section, Theorems 6.1.7 and 6.1.6(v) (together with Fact 6.3.2) make substantial progress on the open problem [ST19, Prob. 5.3.22] of finding general conditions for the higher Fréchet differentiability of operator functions in ideals of semifinite von Neumann algebras induced by (fully) symmetric Banach function spaces.

6.2 Integral symmetrically normed ideals

In this section, we introduce some abstract properties of ideals of $\mathcal{M}$ that are useful in the study of MOIs and their applications to the differentiation of operator functions. We also give several classes of examples of ideal satisfying these properties. In §6.4, we give a large class of additional examples using the theory of symmetric operator spaces.

To begin, we prove some basic properties of ideals of von Neumann algebra.

Proposition 6.2.1 (Ideals of von Neumann algebras)**.** *Let* $\mathcal{I}\subseteq\mathcal{M}$ *be an ideal of* $\mathcal{M}$ *(i.e., a linear subspace such that* $atb\in\mathcal{I}$ *whenever* $a,b\in\mathcal{M}$ *and* $t\in\mathcal{I}$*), and fix* $r,s\in\mathcal{M}$*.*

(i) $r\in\mathcal{I} \iff r^*\in\mathcal{I} \iff |r|\in\mathcal{I}$. *In particular,* $\mathcal{I}$ *is a* $\boldsymbol{*}$***-ideal*** *of* $\mathcal{M}$*.*

(ii) *If* $s\in\mathcal{I}$ *and* $|r|\le|s|$*, then* $r\in\mathcal{I}$*.*

Suppose, in addition, that $\|\cdot\|_{\mathcal{I}}$ *is a norm on* $\mathcal{I}$ *such that* $\|atb\|_{\mathcal{I}} \le \|a\|\,\|t\|_{\mathcal{I}}\|b\|$ *whenever* $t \in \mathcal{I}$ *and* $a, b \in \mathcal{M}$.

(iii) *If* $r \in \mathcal{I}$, *then* $\|r\|_{\mathcal{I}} = \|r^*\|_{\mathcal{I}} = \||r|\|_{\mathcal{I}}$.

(iv) *If* $s \in \mathcal{I}$ *and* $|r| \le |s|$, *then* $\|r\|_{\mathcal{I}} \le \|s\|_{\mathcal{I}}$.

Proof. For the first and third items, let $r = u|r|$ be the polar decomposition of r, and recall that $|r| = u^*r$ as well. Since $r \in \mathcal{M}$, we have that $u, |r| \in \mathcal{M}$. Consequently, if $r \in \mathcal{I}$, then $r^* = |r|u^* = u^*ru^* \in \mathcal{I}$ because $\mathcal{I}$ is an ideal. Now, if $r^* \in \mathcal{I}$, then $|r| = |r|^* = (u^*r)^* = r^*u \in \mathcal{I}$ because $\mathcal{I}$ is an ideal. Finally, if $|r| \in \mathcal{I}$, then $r = u|r| \in \mathcal{I}$ because $\mathcal{I}$ is an ideal. This takes care of the first item. For the third, note that if $r \in \mathcal{I}$, then

$$\begin{aligned}\|r^*\|_{\mathcal{I}} &= \|u^*ru^*\|_{\mathcal{I}} \le \|u^*\|\,\|r\|_{\mathcal{I}}\|u^*\| = \|r\|_{\mathcal{I}} = \|u|r|\|_{\mathcal{I}} \\ &\le \|u\|\,\||r|\|_{\mathcal{I}} = \||r|\|_{\mathcal{I}} = \|r^*u\|_{\mathcal{I}} \le \|r^*\|_{\mathcal{I}}\|u\| = \|r^*\|_{\mathcal{I}},\end{aligned}$$

which yields the desired result.

For second and fourth items, note that it suffices (by the other items) to assume $r, s \ge 0$, so that $r = |r|$ and $s = |s|$. By (the proof of) [Dix81, Pt. I, Lem. 1.2], if $0 \le r \le s$, then there exists a $c \in \mathcal{M}$ such that $\|c\| \le 1$ and $\sqrt{r} = c\sqrt{s}$. In particular, if $s \in \mathcal{I}$, then

$$r = \sqrt{r}\big(\sqrt{r}\big)^* = c\sqrt{s}\big(c\sqrt{s}\big)^* = csc^* \in \mathcal{I}$$

because $\mathcal{I}$ is an ideal. This takes care of the second item. Continuing for the fourth item, we get

$$\|r\|_{\mathcal{I}} = \|csc^*\|_{\mathcal{I}} \le \|c\|\,\|s\|_{\mathcal{I}}\|c^*\| \le \|s\|_{\mathcal{I}},$$

as desired. □

Consequently, the definitions of an invariant operator ideal of $\mathcal{M}$ in [ACDS09] and a symmetrically normed ideal of $\mathcal{M}$ in [ST19] are equivalent, up to a constant multiple of the ideal's norm, to our definition of a symmetrically normed ideal of $\mathcal{M}$ (Definition 2.2.1).

Next, we make an observation. (At this time, the reader should review §5.4.) Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq \mathcal{M}$ be a Banach ideal, and let $F \colon \Omega \to \mathcal{I} \subseteq \mathcal{M}$ be a weak* measurable map. By definition,

$$\underline{\int_\Omega} \|F\| \, \mathrm{d}\mu \leq C_{\mathcal{I}} \underline{\int_\Omega} \|F\|_{\mathcal{I}} \, \mathrm{d}\mu.$$

In particular, if $\underline{\int_\Omega} \|F\|_{\mathcal{I}} \, \mathrm{d}\mu < \infty$, then Corollary 5.4.9 says that $F \colon \Omega \to \mathcal{M}$ is weak* integrable. We now define three additional properties one can demand of Banach or symmetrically normed ideals of a von Neumann algebra.

Definition 6.2.2 (Properties of Banach ideals of $\mathcal{M}$)**.** Fix $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq \mathcal{M}$.

(i) $\mathcal{I}$ has the **Minkowski integral inequality property**—or **property (M)** for short—if whenever $(\Omega, \mathscr{F}, \mu)$ is a measure space and $F \colon \Omega \to \mathcal{I} \subseteq \mathcal{M}$ is weak* measurable with $\underline{\int_\Omega} \|F\|_{\mathcal{I}} \, \mathrm{d}\mu < \infty$, we have

$$\int_\Omega F \, \mathrm{d}\mu \in \mathcal{I} \ \text{ and } \ \left\| \int_\Omega F \, \mathrm{d}\mu \right\|_{\mathcal{I}} \leq \underline{\int_\Omega} \|F\|_{\mathcal{I}} \, \mathrm{d}\mu.$$

(ii) $\mathcal{I}$ is **integral symmetrically normed** if whenever $(\Omega, \mathscr{F}, \mu)$ is a measure space, $A, B \colon \Omega \to \mathcal{M}$ are weak* measurable, $A(\cdot)\, c\, B(\cdot) \colon \Omega \to \mathcal{M}$ is weak* measurable whenever $c \in \mathcal{M}$, and $\underline{\int_\Omega} \|A\| \, \|B\| \, \mathrm{d}\mu < \infty$, it follows that, for all $r \in \mathcal{I}$,

$$\int_\Omega A(\omega)\, r\, B(\omega)\, \mu(\mathrm{d}\omega) \in \mathcal{I} \ \text{ and } \ \left\| \int_\Omega A(\omega)\, r\, B(\omega)\, \mu(\mathrm{d}\omega) \right\|_{\mathcal{I}} \leq \|r\|_{\mathcal{I}} \underline{\int_\Omega} \|A\| \, \|B\| \, \mathrm{d}\mu.$$

(iii) $\mathcal{I}$ is **MOI-friendly** if whenever we are in the setup of Theorem 5.6.20, $i \in \{1, \ldots, k\}$, and $\varphi \in \ell^\infty(\Omega_1, \mathscr{F}_1) \hat{\otimes}_i \cdots \hat{\otimes}_i \ell^\infty(\Omega_{k+1}, \mathscr{F}_{k+1})$, the MOI $I^{\mathbf{P}}\varphi \colon \mathcal{M}^k \to \mathcal{M}$ restricts to a bounded k-linear map $(\mathcal{M}, \|\cdot\|)^{i-1} \times (\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \times (\mathcal{M}, \|\cdot\|)^{k-i} \to (\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ with operator norm at most $\|\varphi\|_{\ell^\infty(\Omega_1, \mathscr{F}_1) \hat{\otimes}_i \cdots \hat{\otimes}_i \ell^\infty(\Omega_{k+1}, \mathscr{F}_{k+1})}$. Of course, in this case, $I^{\mathbf{P}}\varphi$ also restricts to a bounded k-linear map $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})^k \to (\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ with operator norm at most $C_{\mathcal{I}}^{k-1} \|\varphi\|_{\ell^\infty(\Omega_1, \mathscr{F}_1) \hat{\otimes}_i \cdots \hat{\otimes}_i \ell^\infty(\Omega_{k+1}, \mathscr{F}_{k+1})}$.

Remark 6.2.3. First, the name for property (M) is inspired by Theorem 5.4.16. However, inequalities like the one required in Definition 6.2.2(i) are called "triangle inequalities" in the theory of vector-valued integrals. Therefore, it would also be appropriate to name Definition 6.2.2(i) the "integral triangle inequality property." However, this leads naturally to the abbreviation "property (T)," which is decidedly taken. Second, if H is separable, then one can show that the pointwise product of weak* measurable maps $\Omega \to \mathcal{M}$ is weak* measurable. In particular, the requirement in Definition 6.2.2(ii) that "$A(\cdot)\, c\, B(\cdot) \colon \Omega \to \mathcal{M}$ is weak* measurable whenever $c \in \mathcal{M}$" is redundant when H is separable.

By testing the definition on the one-point probability space, we see that an integral symmetrically normed ideal is symmetrically normed. We also have the following.

Proposition 6.2.4. *If a symmetrically normed ideal $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{M}$ has property (M), then $\mathcal{I}$ is integral symmetrically normed.*

Proof. Suppose $\mathcal{I} \trianglelefteq_{\mathrm{s}} \mathcal{M}$ has property (M). Let $A, B \colon \Omega \to \mathcal{M}$ be as in Definition 6.2.2(ii), and fix $r \in \mathcal{I}$. Since $\mathcal{I}$ is symmetrically normed,

$$\|A(\omega)\, r\, B(\omega)\|_{\mathcal{I}} \leq \|r\|_{\mathcal{I}} \|A(\omega)\|\, \|B(\omega)\|, \qquad \omega \in \Omega.$$

Applying Definition 6.2.2(i) to $F \coloneqq A(\cdot)\, r\, B(\cdot)$, we conclude that $\int_\Omega A(\omega)\, r\, B(\omega)\, \mu(\mathrm{d}\omega) \in \mathcal{I}$ and

$$\left\| \int_\Omega A(\omega)\, r\, B(\omega)\, \mu(\mathrm{d}\omega) \right\|_{\mathcal{I}} \leq \underline{\int_\Omega} \|A(\omega)\, r\, B(\omega)\|_{\mathcal{I}}\, \mu(\mathrm{d}\omega) \leq \|r\|_{\mathcal{I}} \underline{\int_\Omega} \|A\|\, \|B\|\, \mathrm{d}\mu.$$

Thus, $\mathcal{I}$ is integral symmetrically normed. □

Proposition 6.2.5. *If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq \mathcal{M}$ is integral symmetrically normed, then $\mathcal{I}$ is MOI-friendly.*

Proof. Suppose $\mathcal{I}$ is integral symmetrically normed and we are in the setup of Theorem 5.6.20. Fix $i \in \{1, \ldots, k\}$, $b = (b_1, \ldots, b_k) \in \mathcal{M}^{i-1} \times \mathcal{I} \times \mathcal{M}^{k-i}$, and an ℓ^∞-IPD $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ of

$$\varphi \in \ell^\infty(\Omega_1, \mathscr{F}_1) \hat{\otimes}_i \cdots \hat{\otimes}_i \ell^\infty(\Omega_{k+1}, \mathscr{F}_{k+1}).$$

Now, apply the definition of integral symmetrically normed with the maps

$$A(\sigma) := \left(\prod_{j=1}^{i-1} P_j(\varphi_j(\cdot,\sigma))\, b_j\right) P_i(\varphi_i(\cdot,\sigma)) \text{ and}$$

$$B(\sigma) := P_{i+1}(\varphi_{i+1}(\cdot,\sigma)) \prod_{j=i+2}^{k+1} b_{j-1} P_j(\varphi_j(\cdot,\sigma)),$$

where empty products are, as usual, 1. This yields $\big(I^{\mathbf{P}}\varphi\big)[b] = \int_\Sigma A(\sigma)\, b_i\, B(\sigma)\, \rho(\mathrm{d}\sigma) \in \mathcal{I}$ and

$$\big\|\big(I^{\mathbf{P}}\varphi\big)[b]\big\|_{\mathcal{I}} \leq \|b_i\|_{\mathcal{I}} \underline{\int_\Sigma} \|A\|\,\|B\|\,\mathrm{d}\rho \leq \|b_i\|_{\mathcal{I}} \prod_{p\neq p} \|b_p\| \underline{\int_\Sigma} \prod_{i=1}^{k+1} \|\varphi_i(\cdot,\sigma)\|_{\ell^\infty(\Omega_i)}\, \rho(\mathrm{d}\sigma).$$

Using that $\underline{\int_\Sigma} \cdot \,\mathrm{d}\rho \leq \overline{\int_\Sigma} \cdot \,\mathrm{d}\rho$ and taking the infimum over all ℓ^∞-IPDs $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ of φ gives the desired result. □

Finally, here are the promised examples.

Example 6.2.6 (Trivial ideals)**.** The trivial symmetrically normed ideals $\mathcal{I} = \{0\}$ and $\mathcal{I} = \mathcal{M}$ both have property (M). The latter follows, of course, from Theorem 5.4.5(ii).

Example 6.2.7 (Noncommutative L^p ideals)**.** Suppose $\mathcal{M}$ is semifinite with normal, faithful, semifinite trace τ. If $1 \leq p < \infty$ and $\mathcal{L}^p(\tau)$ is given the norm $\|\cdot\|_{\mathcal{L}^p(\tau)} := \max\{\|\cdot\|_{L^p(\tau)}, \|\cdot\|\}$, then $\big(\mathcal{L}^p(\tau), \|\cdot\|_{\mathcal{L}^p(\tau)}\big) \trianglelefteq_{\mathrm{s}} \mathcal{M}$ by noncommutative Hölder's inequality (Theorem 4.3.9(iii)) and the completeness of $\big(L^p(\tau), \|\cdot\|_{L^p(\tau)}\big)$ and $\big(\mathcal{M}, \|\cdot\|\big)$. If we combine Example 6.2.6 with Theorem 5.4.16, then we conclude that $\mathcal{L}^p(\tau)$ has property (M) and is therefore integral symmetrically normed (Proposition 6.2.4). Note that if $(\mathcal{M}, \tau) = (B(H), \mathrm{Tr})$, then $\big(\mathcal{L}^p(\mathrm{Tr}), \|\cdot\|_{\mathcal{L}^p(\mathrm{Tr})}\big) = (\mathcal{S}_p(H), \|\cdot\|_{\mathcal{S}_p})$ is the ideal of Schatten p-class operators on H.

The ideal of compact operators is left out of Example 6.2.7. To include it in the mix, we first prove that separable ideals have property (M).

Proposition 6.2.8 (Separable ideals)**.** *If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq \mathcal{M}$ is a Banach ideal such that $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is separable, then $\mathcal{I}$ has property (M). In particular, if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq_{\mathrm{s}} \mathcal{M}$ is separable, then $\mathcal{I}$ is integral symmetrically normed.*

Proof. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, let $F \colon \Omega \to \mathcal{I} \subseteq \mathcal{M}$ be a weak* measurable map, and let $h, k \in H$. Now, define $\ell_{h,k} \colon \mathcal{I} \to \mathbb{C}$ by $r \mapsto \langle rh, k\rangle$. Since the inclusion $\iota_{\mathcal{I}} \colon (\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \hookrightarrow (\mathcal{M}, \|\cdot\|)$ is bounded, $\ell_{h,k}$ is a continuous function $\mathcal{I} \to \mathbb{C}$. Also, $\ell_{h,k} \circ F = \langle F(\cdot)h, k\rangle \colon \Omega \to \mathbb{C}$ is measurable by assumption. Since the collection $\{\ell_{h,k} : h, k \in H\}$ clearly separates points, we conclude from the completeness and separability of $\mathcal{I}$ and [VTC87, Prop. I.1.10] that $F \colon \Omega \to (\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is Borel measurable. Using again the separability of $\mathcal{I}$, this implies $F \colon \Omega \to (\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is strongly measurable. Consequently, if, in addition, $\underline{\int_\Omega} \|F\|_{\mathcal{I}} \, \mathrm{d}\mu = \int_\Omega \|F\|_{\mathcal{I}} \, \mathrm{d}\mu < \infty$, then $F \colon \Omega \to (\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is also Bochner integrable, and—by applying $\ell_{h,k}$ to the Bochner integral—the Bochner and weak* integrals of F agree. Thus, $\int_\Omega F \, \mathrm{d}\mu \in \mathcal{I}$ and $\big\| \int_\Omega F \, \mathrm{d}\mu \big\|_{\mathcal{I}} \leq \int_\Omega \|F\|_{\mathcal{I}} \, \mathrm{d}\mu$ by the triangle inequality for Bochner integrals. This completes the proof. □

In particular, if H is separable, then the ideal $\mathcal{K}(H) \trianglelefteq_{\mathrm{s}} B(H)$ of compact operators $H \to H$ has property (M). Actually, this also implies the non-separable case by an argument suggested by J. Jeon.

Lemma 6.2.9. *For a closed linear subspace $K \subseteq H$, write $\iota_K \colon K \to H$ and $\pi_K \colon H \to K$ for the inclusion of and orthogonal projection onto K, respectively. Let $A \in B(H)$. Then $A \in \mathcal{K}(H)$ if and only if $A_K \coloneqq \pi_K A \iota_K \in \mathcal{K}(K)$ for all closed, separable linear subspaces $K \subseteq H$.*

Proof. The "only if" direction is clear. For the "if" direction, suppose $A_K = \pi_K A \iota_K \in \mathcal{K}(K)$ for all closed, separable linear subspaces $K \subseteq H$. If $(h_n)_{n\in\mathbb{N}}$ is a bounded sequence in H, then set

$$K \coloneqq \overline{\operatorname{span}}\big\{A^k h_n : k \in \mathbb{N}_0,\ n \in \mathbb{N}\big\}.$$

Of course, K is a closed, separable linear subspace of H that contains $\{h_n : n \in \mathbb{N}\}$ and is invariant under A. Since A_K is compact, there is a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ such that $\big(A_K h_{n_k}\big)_{k\in\mathbb{N}}$ converges. But

$$A_K h_{n_k} = \pi_K A h_{n_k} = A h_{n_k}, \qquad k \in \mathbb{N},$$

because K is A-invariant. We conclude that $A \in \mathcal{K}(H)$. □

Proposition 6.2.10 (Compact operators)**.** *$(\mathcal{K}(H), \|\cdot\|) \trianglelefteq_{\mathrm{s}} B(H)$ has property (M).*

Proof. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Suppose $F\colon \Omega \to \mathcal{K}(H) \subseteq B(H)$ is weak* measurable and $\underline{\int_\Omega} \|F\| \, \mathrm{d}\mu < \infty$. Since we already know the triangle inequality for the operator norm, it suffices to prove $\int_\Omega F \, \mathrm{d}\mu \in \mathcal{K}(H)$. To this end, let $K \subseteq H$ be a closed, separable linear subspace. In the notation of Lemma 6.2.9, $F_K = \pi_K F(\cdot)\iota_K \colon \Omega \to \mathcal{K}(K) \subseteq B(K)$ is weak* measurable, and

$$\underline{\int_\Omega} \|F_K\| \, \mathrm{d}\mu \leq \underline{\int_\Omega} \|F\| \, \mathrm{d}\mu < \infty.$$

Since $\mathcal{K}(K)$ is separable, Proposition 6.2.8 gives $\int_\Omega F_K \, \mathrm{d}\mu \in \mathcal{K}(K)$. Since

$$\left(\int_\Omega F \, \mathrm{d}\mu \right)_K = \pi_K \left(\int_\Omega F \, \mathrm{d}\mu \right) \iota_K = \int_\Omega \pi_K F(\omega) \, \iota_K \, \mu(\mathrm{d}\omega) = \int_\Omega F_K \, \mathrm{d}\mu \in \mathcal{K}(K),$$

we conclude from Lemma 6.2.9 that $\int_\Omega F \, \mathrm{d}\mu \in \mathcal{K}(H)$. □

Remark 6.2.11. In case one only wants to know that $\mathcal{K}(H)$ is integral symmetrically normed, there is a different proof available that does not go through the separable case first. Indeed, let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and suppose $A, B \colon \Omega \to B(H)$ are as in Definition 6.2.2(ii). It suffices to show that if $c \in \mathcal{K}(H)$, then $\int_\Omega A(\omega) \, c \, B(\omega) \, \mu(\mathrm{d}\omega) \in \mathcal{K}(H)$. To this end, suppose first that c has finite rank. Then $c \in \mathcal{S}_1(H)$. Since $(\mathcal{S}_1(H), \| \cdot \|_{\mathcal{S}_1}) \trianglelefteq B(H)$ is integral symmetrically normed, $\int_\Omega A(\omega) \, c \, B(\omega) \, \mu(\mathrm{d}\omega) \in \mathcal{S}_1(H) \subseteq \mathcal{K}(H)$. Now, if $c \in \mathcal{K}(H)$ is arbitrary, then—using, e.g., the singular value decomposition—there exists a sequence $(c_n)_{n \in \mathbb{N}}$ of finite-rank linear operators $H \to H$ such that $\|c_n - c\| \to 0$ as $n \to \infty$. But then, by the operator norm triangle inequality, $\int_\Omega A(\omega) \, c_n \, B(\omega) \, \mu(\mathrm{d}\omega) \to \int_\Omega A(\omega) \, c \, B(\omega) \, \mu(\mathrm{d}\omega)$ in the operator norm topology as $n \to \infty$. Since this exhibits $\int_\Omega A(\omega) \, c \, B(\omega) \, \mu(\mathrm{d}\omega)$ as the limit in the operator norm topology of a sequence of compact operators, $\int_\Omega A(\omega) \, c \, B(\omega) \, \mu(\mathrm{d}\omega)$ is compact, as desired.

6.3 Interlude: Symmetric operator spaces

In the next section, we make use of the theory of *symmetric operator spaces*. In the present section, we review the notation, terminology, and results from this theory that are necessary for our purposes. We refer the reader to [DdP14] for extra exposition, examples, and references. (The reader who is uninterested in the next section may safely skip the present section.)

Recall that $(\mathcal{M}, \tau)$ is a semifinite von Neumann algebra and $\mathrm{Proj}(\mathcal{M})$ is the lattice of (orthogonal) projections in $\mathcal{M}$. An operator $a \,\eta\, \mathcal{M}$ is called **τ-measurable** if there exists some $s \geq 0$ such that $\tau(P^{|a|}((s,\infty))) < \infty$. Write

$$S(\tau) \coloneqq \{a \,\eta\, \mathcal{M} : a \text{ is } \tau\text{-measurable}\},$$

and let $a, b \in S(\tau)$. Then $a + b$ is closable, and $\overline{a+b} \in S(\tau)$; ab is closable, and $\overline{ab} \in S(\tau)$; and $a^*, |a| \in S(\tau)$. Furthermore, $S(\tau)$ is a $*$-algebra under the adjoint, strong sum (closure of sum), and strong product (closure of product) operations; we shall therefore omit the closures from strong sums and products in the future. Please see [Nel74, Ter81] for proofs of the preceding facts (and more) about τ-measurable operators.

Let $a \in S(\tau)$, and define

$$d_s(a) \coloneqq \tau\big(P^{|a|}((s,\infty))\big) \in [0,\infty], \qquad s \geq 0.$$

By definition of τ-measurability, $d_s(a) < \infty$ for sufficiently large s. The function $d(a) = d_\cdot(a)$ is the (**noncommutative**) **distribution function** of a. Now, define

$$\mu_t(a) \coloneqq \inf\{s \geq 0 : d_s(a) \leq t\} \in [0,\infty), \qquad t > 0.$$

The function $\mu(a) = \mu_\cdot(a)$ is the (**generalized**) **singular value function** or (**noncommutative**) **decreasing rearrangement** of a, and $\mu(a)$ is decreasing and right-continuous. For properties of $d(a)$ and $\mu(a)$, please see [FK86]. Now, let

$$S(\tau)_+ \coloneqq S(\tau) \cap C(H)_+.$$

If $a \in \mathcal{M}_+ = S(\tau)_+ \cap \mathcal{M}$, then we have the identity

$$\tau(a) = \int_0^\infty \mu_t(a)\,\mathrm{d}t.$$

We therefore extend τ to all of $S(\tau)_+$ via the formula above; this extension is still denoted by

$\tau\colon S(\tau)_+ \to [0,\infty]$. Finally, if $a, b \in S(\tau)$, then we write

$$a \prec\!\prec b \ \text{ if } \int_0^t \mu_s(a)\,\mathrm{d}s \le \int_0^t \mu_s(b)\,\mathrm{d}s, \text{ for all } t \ge 0.$$

In this case, we say that a is **submajorized by** b or that b **submarjorizes** a (in the "noncommutative" sense of Hardy–Littlewood–Pólya). We now define symmetric operator spaces.

Definition 6.3.1 (Symmetric operator spaces)**.** Let $E \subseteq S(\tau)$ be a linear subspace, and let $\|\cdot\|_E$ be a norm on E such that $(E, \|\cdot\|_E)$ is a Banach space.

(i) $(E, \|\cdot\|_E)$ is a **symmetric** (or **rearrangement-invariant**) **space of τ-measurable operators**—a **symmetric space**[1] for short—if $a \in S(\tau)$, $b \in E$, and $\mu(a) \le \mu(b)$ imply that $a \in E$ and $\|a\|_E \le \|b\|_E$.

(ii) $(E, \|\cdot\|_E)$ is a **strongly symmetric space of τ-measurable operators**—a **strongly symmetric space** for short—if it is a symmetric space, and $a, b \in E$ and $a \prec\!\prec b$ imply that $\|a\|_E \le \|b\|_E$.

(iii) $(E, \|\cdot\|_E)$ is a **fully symmetric space of τ-measurable operators**—a **fully symmetric space** for short—if $a \in S(\tau)$, $b \in E$, and $a \prec\!\prec b$ imply that $a \in E$ and $\|a\|_E \le \|b\|_E$.

If $(E, \|\cdot\|_E)$ is a symmetric space, then we define

$$\mathrm{Proj}(E) \coloneqq E \cap \mathrm{Proj}(\mathcal{M}) \ \text{ and } \ c_E \coloneqq \sup \mathrm{Proj}(E) \in \mathrm{Proj}(\mathcal{M}).$$

The projection c_E is the **carrier projection** of E.

Next, we describe a large class of examples of symmetric spaces. Let m be the Lebesgue measure on $(0,\infty)$, and let $(\mathcal{N}, \eta) \coloneqq (L^\infty(m), \int_0^\infty \cdot \, \mathrm{d}m)$, where $L^\infty(m)$ is represented as multiplication operators on $L^2(m)$. Then the set of densely defined, closed operators affiliated with $\mathcal{N}$ is precisely $L^0(m)$, i.e., the space of m-almost everywhere equivalence classes of measurable functions $(0,\infty) \to \mathbb{C}$, viewed as unbounded multiplication operators on $L^2(m)$; and

$$S(\eta) = \big\{ f \in L^0(m) : d_s(f) = m(\{x \in (0,\infty) : |f(x)| > s\}) < \infty \text{ for some } s \ge 0 \big\}.$$

[1]Beware: This has nothing to do with the notion of a (Riemannian) symmetric space from geometry.

Please see [CK17, §2.3] for proofs of these facts. A (**strongly**, **fully**) **symmetric Banach function space** is a (strongly, fully) symmetric space of η-measurable operators is; please see [KPS82, Ch. II] for the classical theory of such spaces.

Fact 6.3.2. *Let $(E \subseteq L^0(m), \|\cdot\|_E)$ be a (strongly, fully) symmetric Banach function space. If*

$$E(\tau) \coloneqq \{a \in S(\tau) : \mu(a) \in E\} \text{ and } \|a\|_{E(\tau)} \coloneqq \|\mu(a)\|_E, \qquad a \in E(\tau),$$

then $(E(\tau), \|\cdot\|_{E(\tau)})$ is a (strongly, fully) symmetric space of τ-measurable operators.

For the strongly/fully symmetric cases, please see [DdP14, §9.1]. For the (highly nontrivial) case of an arbitrary symmetric space, please see [KS08]. When $1 \leq p \leq \infty$ and $E = L^p \coloneqq L^p(m)$, $L^p(\tau)$ as defined using the construction in Fact 6.3.2 is a concrete description of the abstract (completion-based) definition from §4.3. When $p = \infty$, this follows from [FK86, Lem. 2.5(i)]; when $p < \infty$, it follows from [FK86, Lem. 2.5(iv)] and [DDdP93, Prop. 2.8]. Furthermore,

$$\big(L^p(\tau), \|\cdot\|_{L^p(\tau)}\big) = \Big(\{a \in S(\tau) : \tau(|a|^p) < \infty\}, \tau(|\cdot|^p)^{\frac{1}{p}}\Big), \qquad 1 \leq p < \infty.$$

As a result, $(L^p \cap L^\infty)(\tau) = L^p(\tau) \cap L^\infty(\tau) = L^p(\tau) \cap \mathcal{M} = \mathcal{L}^p(\tau)$ with equality of norms (if we give $\mathcal{L}^p(\tau)$ the norm $\max\{\|\cdot\|_{L^p(\tau)}, \|\cdot\|\}$). It is also true that $(L^1 + L^\infty)(\tau) = L^1(\tau) + \mathcal{M}$ with equality of norms. (This follows from [DDdP93, Prop. 2.5].) To be clear, if Z is a vector space and $X, Y \subseteq Z$ are normed linear subspaces with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, then the subspace $X \cap Y \subseteq Z$ is given the norm $\|\cdot\|_{X\cap Y} \coloneqq \max\{\|\cdot\|_X, \|\cdot\|_Y\}$, and the subspace $X + Y \subseteq Z$ is given the norm $\|z\|_{X+Y} \coloneqq \inf\{\|x\|_X + \|y\|_Y : x \in X, \, y \in Y, \, z = x + y\}$.

In general, if $(E, \|\cdot\|_E)$ is a strongly symmetric space of τ-measurable operators, then $E \subseteq L^1(\tau) + \mathcal{M}$ with continuous inclusion, and $c_E = 1$ if and only if $L^1(\tau) \cap \mathcal{M} \subseteq E$ with continuous inclusion. This is [DdP14, Lem. 25] (combined with the last paragraph of the proof of Lemma 5.4.14). By [KPS82, Thm. II.4.1], if $(\tilde{E}, \|\cdot\|_{\tilde{E}})$ is a nonzero symmetric Banach function space, then

$$L^1 \cap L^\infty \subseteq \tilde{E} \subseteq L^1 + L^\infty$$

with continuous inclusions, i.e., $c_{\tilde{E}} = 1$.

Finally, we discuss Köthe duals. For a symmetric space $(E, \|\cdot\|_E)$, define

$$E^\times \coloneqq \{a \in S(\tau) : ab \in L^1(\tau), \text{ for all } b \in E\} \text{ and}$$
$$\|a\|_{E^\times} \coloneqq \sup\{\tau(|ab|) : b \in E, \ \|b\|_E \leq 1\}, \qquad a \in S(\tau).$$

Of course, $\|a\|_{E^\times}$ could be infinite.

Fact 6.3.3 (Köthe dual)**.** *If $(E, \|\cdot\|_E)$ be a strongly symmetric space of τ-measurable operators with $c_E = 1$, then*

$$\|a\|_{E^\times} = \sup\left\{\tau(|ab|) : b \in \mathcal{L}^1(\tau) = L^1(\tau) \cap \mathcal{M}, \ \|b\|_E \leq 1\right\}, \qquad a \in S(\tau).$$

*Furthermore, $a \in E^\times$ if and only if $\|a\|_{E^\times} < \infty$. Finally, $\|\cdot\|_{E^\times}$ is a norm on $E^\times$ such that $(E^\times, \|\cdot\|_{E^\times})$ is a fully symmetric space with $c_{E^\times} = 1$. We call $E^\times$ the **Köthe dual** of E.*

Remark 6.3.4. In the classical case of symmetric Banach function spaces, the Köthe dual of E is called the *associate space of* E or the *space associated with* E.

Please see [DDdP93, §5] or [DdP14, §5.2 & §6] for a proof of this fact. Now, let $(E, \|\cdot\|_E)$ be a strongly symmetric space of τ-measurable operators with $c_E = 1$. Since $E^\times$ is fully symmetric and $c_{E^\times} = 1$, we can consider the **Köthe bidual** $(E^{\times\times}, \|\cdot\|_{E^{\times\times}}) = ((E^\times)^\times, \|\cdot\|_{(E^\times)^\times})$ of E as a (fully) symmetric space. It is always the case that $E \subseteq E^{\times\times}$ and $\|\cdot\|_{E^{\times\times}} \leq \|\cdot\|_E$ on E. If $E = E^{\times\times}$ and $\|\cdot\|_E = \|\cdot\|_{E^{\times\times}}$ on E, then is E **Köthe reflexive**. (This term is not standard; a more common term is *maximal*.) Note that, by Fact 6.3.3, if E is Köthe reflexive, then E is automatically fully symmetric.

The following is a celebrated equivalent characterization of Köthe reflexivity. It is stated and proven as [DDdP93, Prop. 5.14] and [DdP14, Thm. 32].

Theorem 6.3.5 (Noncommutative Lorentz–Luxemburg theorem)**.** *Let $(E, \|\cdot\|_E)$ be a strongly symmetric space of τ-measurable operators with $c_E = 1$. Then E is Köthe reflexive if and only if E has the **Fatou property**: Whenever $(a_i)_{i \in I}$ is an increasing net in $E \cap S(\tau)_+$ (i.e., $i \leq j \Rightarrow a_j - a_i \in S(\tau)_+$) with $\sup_{i \in I} \|a_i\|_E < \infty$, we have that $\sup_{i \in I} a_i$ exists in $E \cap S(\tau)_+$, and $\|\sup_{i \in I} a_i\|_E = \sup_{i \in I} \|a_i\|_E$.*

The definition of the Fatou property involves rather arbitrary nets. It is therefore reasonable to be concerned that verifying the Fatou property in classical situations might be quite difficult. However, as we explain shortly, the sequence formulation of the Fatou property is equivalent in classical situations. Let $\big(E \subseteq L^0(m), \|\cdot\|_E\big)$ be a symmetric Banach function space. We say that E has the **classical Fatou property** if whenever $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence of nonnegative functions in E such that $\sup_{n\in\mathbb{N}} \|f_n\|_E < \infty$, we have that $\sup_{n\in\mathbb{N}} f_n \in E$ and $\|\sup_{n\in\mathbb{N}} f_n\|_E = \sup_{n\in\mathbb{N}} \|f_n\|_E$. It turns out that if E has the classical Fatou property, then E is fully symmetric [BS88, Theorem 2.4.6], so we may speak of its Köthe dual as a (fully) symmetric Banach function space when E is nonzero. The classical Lorentz–Luxemburg theorem [Zaa67, Thm. 71.1] says that a nonzero symmetric Banach function space has the classical Fatou property if and only if it is (strongly symmetric and) Köthe reflexive. In particular, by the noncommutative Lorentz–Luxemburg theorem, a symmetric Banach function space has the Fatou property if and only if it has the classical Fatou property.

Example 6.3.6. Let $(E, \|\cdot\|_E)$ be a nonzero strongly symmetric Banach function space (which implies $c_E = 1$ as noted above). By [DDdP93, Thm. 5.6],

$$\big(E(\tau)^\times, \|\cdot\|_{E(\tau)^\times}\big) = \big(E^\times(\tau), \|\cdot\|_{E^\times(\tau)}\big).$$

In particular, if E is Köthe reflexive (i.e., has the classical Fatou property), then $E(\tau)$ is Köthe reflexive (i.e., has the Fatou property) as well.

Remark 6.3.7. Let E be a symmetric Banach function space. By [Zaa67, Thm. 65.3], E has the classical Fatou property if and only if whenever $(f_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative functions in E with $\liminf_{n\to\infty} \|f_n\|_E < \infty$, we have that $\liminf_{n\to\infty} f_n \in E$ and

$$\Big\|\liminf_{n\to\infty} f_n\Big\|_E \leq \liminf_{n\to\infty} \|f_n\|_E,$$

i.e, Fatou's lemma holds for $\|\cdot\|_E$. Hence the property's name.

6.4 Examples of ideals II

In this section, we provide examples of integral symmetrically normed ideals of $\mathcal{M}$ using the theory of symmetric operator spaces. To begin, we note that if $(E, \|\cdot\|_E)$ is a symmetric space of τ-measurable operators, then

$$(\mathcal{E}, \|\cdot\|_{\mathcal{E}}) \coloneqq (E \cap \mathcal{M}, \|\cdot\|_{E\cap\mathcal{M}}) = (E \cap \mathcal{M}, \max\{\|\cdot\|_E, \|\cdot\|\}) \trianglelefteq_{\mathrm{s}} \mathcal{M}.$$

This follows from [DdP14, Prop. 17]. We call $\mathcal{E}$ the **ideal induced by $\boldsymbol{E}$**. We prove that ideals induced by fully symmetric spaces are integral symmetrically normed and ideals induced by symmetric spaces with the Fatou property have property (M).

Theorem 6.4.1 (Fully symmetric $\Rightarrow$ integral symmetrically normed)**.** *If $(E, \|\cdot\|_E)$ is a fully symmetric space, then $(\mathcal{E}, \|\cdot\|_{\mathcal{E}}) \coloneqq (E \cap \mathcal{M}, \|\cdot\|_{E\cap\mathcal{M}}) \trianglelefteq_{\mathrm{s}} \mathcal{M}$ is integral symmetrically normed.*

Proof. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and suppose $A, B \colon \Omega \to \mathcal{M}$ are as in Definition 6.2.2(ii). Define $T_\infty \colon \mathcal{M} \to \mathcal{M}$ by $\mathcal{M} \ni c \mapsto \int_\Omega A(\omega)\, c\, B(\omega)\, \mu(\mathrm{d}\omega) \in \mathcal{M}$. Then

$$\|T_\infty\|_{\mathcal{M}\to\mathcal{M}} \leq \underline{\int_\Omega} \|A\|\, \|B\|\, \mathrm{d}\mu$$

by the operator norm triangle inequality. Also, if $c \in L^1(\tau) \cap \mathcal{M}$, then

$$\|T_\infty c\|_{L^1(\tau)} \leq \underline{\int_\Omega} \|A(\omega)\, c\, B(\omega)\|_{L^1(\tau)}\, \mu(\mathrm{d}\omega) \leq \|c\|_{L^1(\tau)} \underline{\int_\Omega} \|A\|\, \|B\|\, \mathrm{d}\mu$$

by Theorem 5.4.16. Since $L^1(\tau) \cap \mathcal{M}$ is dense in $L^1(\tau)$ [DDdP93, Prop. 2.8], we get that $T_\infty|_{L^1(\tau)\cap\mathcal{M}}$ extends uniquely to a bounded linear map $T_1 \colon L^1(\tau) \to L^1(\tau)$ with

$$\|T_1\|_{L^1(\tau)\to L^1(\tau)} \leq \underline{\int_\Omega} \|A\|\, \|B\|\, \mathrm{d}\mu.$$

Since T_∞ and T_1 agree on $L^1(\tau) \cap \mathcal{M}$,

$$T(x+y) \coloneqq T_1 x + T_\infty y, \qquad x \in L^1(\tau),\ y \in \mathcal{M},$$

is a well-defined linear map $T\colon L^1(\tau)+\mathcal{M}\to L^1(\tau)+\mathcal{M}$. Furthermore,

$$\|T\|_{L^1(\tau)+\mathcal{M}\to L^1(\tau)+\mathcal{M}} \le \max\{\|T_1\|_{L^1(\tau)\to L^1(\tau)}, \|T_\infty\|_{\mathcal{M}\to\mathcal{M}}\} \le \underline{\int_\Omega} \|A\|\,\|B\|\,\mathrm{d}\mu.$$

By [DDdP93, Prop. 4.1], this implies

$$Tc \prec\!\prec \left(\underline{\int_\Omega}\|A\|\,\|B\|\,\mathrm{d}\mu\right)c, \qquad c\in L^1(\tau)+\mathcal{M}.$$

In particular, if $c\in E\subseteq L^1(\tau)+\mathcal{M}$, then

$$Tc\in E \;\text{ and }\; \|Tc\|_E \le \|c\|_E \underline{\int_\Omega}\|A\|\,\|B\|\,\mathrm{d}\mu$$

because E is fully symmetric, i.e., T restricts to a bounded linear map $T_E\colon E\to E$ with

$$\|T_E\|_{E\to E} \le \underline{\int_\Omega}\|A\|\,\|B\|\,\mathrm{d}\mu.$$

We conclude that if $c\in\mathcal{E}=E\cap\mathcal{M}$, then

$$\int_\Omega A(\omega)\,c\,B(\omega)\,\mu(\mathrm{d}\omega) = T_\infty c = T_E c\in\mathcal{E} \;\text{ and }\; \left\|\int_\Omega A(\omega)\,c\,B(\omega)\,\mu(\mathrm{d}\omega)\right\|_{\mathcal{E}} \le \|c\|_{\mathcal{E}}\underline{\int_\Omega}\|A\|\,\|B\|\,\mathrm{d}\mu.$$

Thus, $\mathcal{E}$ is integral symmetrically normed. □

Remark 6.4.2. The argument above is inspired in part by [DDSZ20, §4.4].

The second main result of this section upgrades Theorem 6.4.1 when the symmetric space in question is a Köthe dual. (It also generalizes Theorem 5.4.16.)

Theorem 6.4.3 (Köthe duals and property (M))**.** *Let $(E,\|\cdot\|_E)$ be a strongly symmetric space with $c_E=1$. If $(\Omega,\mathscr{F},\mu)$ is a measure space and $F\colon\Omega\to\mathcal{M}$ is weak* integrable, then*

$$\left\|\int_\Omega F\,\mathrm{d}\mu\right\|_{E^\times} \le \underline{\int_\Omega}\|F\|_{E^\times}\,\mathrm{d}\mu.$$

In particular, $(\mathcal{E}^\times,\|\cdot\|_{\mathcal{E}^\times}) \coloneqq (E^\times\cap\mathcal{M},\|\cdot\|_{E^\times\cap\mathcal{M}}) \trianglelefteq_{\mathrm{s}} \mathcal{M}$ has property (M).

Proof. Let $a \coloneqq \int_\Omega F\,\mathrm{d}\mu \in \mathcal{M}$. By Fact 6.3.3 (twice) and Theorem 5.4.16,

$$\begin{aligned}\left\|\int_\Omega F\,\mathrm{d}\mu\right\|_{E^\times} &= \|a\|_{E^\times} = \sup\left\{\tau(|ab|) : b \in \mathcal{L}^1(\tau),\ \|b\|_E \le 1\right\} \\ &= \sup\left\{\left\|\int_\Omega F(\omega)\,b\,\mu(\mathrm{d}\omega)\right\|_{L^1(\tau)} : b \in \mathcal{L}^1(\tau),\ \|b\|_E \le 1\right\} \\ &\le \sup\left\{\underline{\int_\Omega}\|F(\omega)\,b\|_{L^1(\tau)}\,\mu(\mathrm{d}\omega) : b \in \mathcal{L}^1(\tau),\ \|b\|_E \le 1\right\} \le \underline{\int_\Omega}\|F\|_{E^\times}\,\mathrm{d}\mu,\end{aligned}$$

as desired. □

Corollary 6.4.4. *Let $(E, \|\cdot\|_E)$ be a strongly symmetric space with $c_E = 1$. If $(\Omega, \mathscr{F}, \mu)$ is a measure space, $F\colon \Omega \to \mathcal{M}$ is weak* integrable, and $F(\Omega) \subseteq E \cap \mathcal{M}$, then*

$$\left\|\int_\Omega F\,\mathrm{d}\mu\right\|_{E^{\times\times}} \le \underline{\int_\Omega}\|F\|_E\,\mathrm{d}\mu.$$

In particular, by Fact 6.3.3, if the right-hand side is finite, then $\int_\Omega F\,\mathrm{d}\mu \in E^{\times\times}$.

Proof. Applying Theorem 6.4.3 to the space $E^{\times\times} = (E^\times)^\times$ and using that $\|\cdot\|_{E^{\times\times}} \le \|\cdot\|_E$ on E, we get that $\left\|\int_\Omega F\,\mathrm{d}\mu\right\|_{E^{\times\times}} \le \underline{\int_\Omega}\|F\|_{E^{\times\times}}\,\mathrm{d}\mu \le \underline{\int_\Omega}\|F\|_E\,\mathrm{d}\mu$, as desired. □

Remark 6.4.5. Please see [KPS82, Ineq. (II.0.5)] for a classical analog of this Minkowski-type integral inequality.

Combining the noncommutative Lorentz–Luxemburg theorem (Theorem 6.3.5) with Corollary 6.4.4, we obtain the following result.

Theorem 6.4.6 (Fatou property ⇒ property (M))**.** *Let $(E, \|\cdot\|_E)$ be a strongly symmetric space with $c_E = 1$. Suppose $(\Omega, \mathscr{F}, \mu)$ is a measure space, $F\colon \Omega \to \mathcal{M}$ is a weak* integrable map, and $F(\Omega) \subseteq E \cap \mathcal{M}$. If E has the Fatou property and $\underline{\int_\Omega}\|F\|_E\,\mathrm{d}\mu < \infty$, then*

$$\int_\Omega F\,\mathrm{d}\mu \in E, \quad \textit{and} \quad \left\|\int_\Omega F\,\mathrm{d}\mu\right\|_E \le \underline{\int_\Omega}\|F\|_E\,\mathrm{d}\mu.$$

In particular, $(\mathcal{E}, \|\cdot\|_{\mathcal{E}}) \coloneqq (E \cap \mathcal{M}, \|\cdot\|_{E\cap\mathcal{M}}) \trianglelefteq_{\mathrm{s}} \mathcal{M}$ has property (M).

Proof. By the noncommutative Lorentz–Luxemburg theorem, $(E, \|\cdot\|_E) = (E^{\times\times}, \|\cdot\|_{E^{\times\times}})$. Therefore, by Corollary 6.4.4, we have that $\int_\Omega F\,\mathrm{d}\mu \in E^{\times\times} = E$ and

$$\left\| \int_\Omega F\,\mathrm{d}\mu \right\|_E = \left\| \int_\Omega F\,\mathrm{d}\mu \right\|_{E^{\times\times}} \leq \underline{\int_\Omega} \|F\|_E\,\mathrm{d}\mu,$$

as desired. □

6.5 Perturbation formulas

In this and the following section, we differentiate maps induced by functional calculus that have been perturbed by an unbounded self-adjoint operator. As before, we shall use the method of perturbation formulas. Due to the complicated nature of MOIs, it will take substantial technical effort to implement the method in this case. This section's goal is to establish perturbation formulas using a generalization of the argument from the proof of [Pel16, Thm. 1.2.3].

Lemma 6.5.1 (Operator-valued dominated convergence theorem). *Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, let $(F_n)_{n\in\mathbb{N}}$ be a sequence of weak* integrable maps $\Omega \to \mathcal{M}$, and suppose $F\colon \Omega \to \mathcal{M}$ is such that $F_n \to F$ pointwise in the weak, strong, or strong* operator topology as $n \to \infty$. If*

$$\overline{\int_\Omega} \sup_{n\in\mathbb{N}} \|F_n\|\,\mathrm{d}\mu < \infty, \tag{6.5.2}$$

then $F\colon \Omega \to \mathcal{M}$ is weak* integrable, and

$$\lim_{n\to\infty} \int_\Omega F_n\,\mathrm{d}\mu = \int_\Omega F\,\mathrm{d}\mu,$$

in the weak, strong, or strong* operator topology, respectively.

Proof. Let $h, k \in H$. In all cases, $F_n \to F$ pointwise in the WOT as $n \to \infty$, so F is weak* measurable. Also, $\|F\| \leq \sup_{n\in\mathbb{N}} \|F_n\|$, so F is weak* integrable by Inequality (6.5.2) and Corollary 5.4.9. Now, Inequality (6.5.2) also gives

$$\int_\Omega \sup_{n\in\mathbb{N}} |\langle F_n(\omega)h, k\rangle|\,\mu(\mathrm{d}\omega) \leq \|h\|\,\|k\| \underline{\int_\Omega} \sup_{n\in\mathbb{N}} \|F_n\|\,\mathrm{d}\mu < \infty.$$

Therefore, by the dominated convergence theorem,

$$\left\langle\left(\int_\Omega F_n\,\mathrm{d}\mu\right)h,k\right\rangle=\int_\Omega\langle F_n(\omega)h,k\rangle\,\mu(\mathrm{d}\omega)\xrightarrow{n\to\infty}\int_\Omega\langle F(\omega)h,k\rangle\,\mu(\mathrm{d}\omega)=\left\langle\left(\int_\Omega F\,\mathrm{d}\mu\right)h,k\right\rangle.$$

Thus, $\int_\Omega F_n\,\mathrm{d}\mu\to\int_\Omega F\,\mathrm{d}\mu$ in the WOT as $n\to\infty$. Now, assume $F_n\to F$ pointwise in the SOT as $n\to\infty$, and write $T_n:=\int_\Omega F_n\,\mathrm{d}\mu$ and $T:=\int_\Omega F\,\mathrm{d}\mu$. Then

$$\|T_nh-Th\|=\left\|\left(\int_\Omega(F_n-F)\,\mathrm{d}\mu\right)h\right\|\leq\underline{\int_\Omega}\|(F_n(\omega)-F(\omega))h\|\,\mu(\mathrm{d}\omega)\xrightarrow{n\to\infty}0$$

by the triangle inequality for weak integrals and Proposition 5.3.2(iv), which applies because of Inequality (6.5.2) and the fact that $\sup_{n\in\mathbb{N}}\|(F_n-F)h\|\leq 2\|h\|\sup_{n\in\mathbb{N}}\|F_n\|$. Finally, the S*OT case follows from the SOT case because $(F_n^*)_{n\in\mathbb{N}}$ and F^* satisfy the same hypotheses as $(F_n)_{n\in\mathbb{N}}$ and F, and the adjoint commutes with the weak* integral. □

Notation 6.5.3. Let $a_1,\ldots,a_{k+1}\;\eta\;\mathcal{M}_\nu$. If we are in the setting of Theorem 5.6.20 with $\mathbf{P}=(P^{a_1},\ldots,P^{a_{k+1}})$, then we shall write

$$\varphi(a_1,\ldots,a_{k+1})\#b:=\big(I^{\mathbf{P}}\varphi\big)[b],\qquad b\in\mathcal{M}^k.$$

in analogy with the development from Chapter 3.

Lemma 6.5.4. *Let $a_1,\ldots,a_{k+1}\in C(H)_{\mathrm{sa}}$. Define $\chi_n(t):=t\,1_{[-n,n]}(t)$ for all $t\in\mathbb{R}$ and $n\in\mathbb{N}$ Also, if $i\in\{1,\ldots,k+1\}$, then define*

$$a_{i,n}:=a_iP^{a_i}([-n,n])=\chi_n(a_i)\in B(H)_{\mathrm{sa}},\qquad n\in\mathbb{N}.$$

If $\varphi\in\ell^\infty(\mathbb{R},\mathcal{B}_\mathbb{R})^{\hat{\otimes}_i(k+1)}$ and $(b_{\cdot,n})_{n\in\mathbb{N}}=(b_{1,n},\ldots,b_{k,n})_{n\in\mathbb{N}}$ is a sequence in $B(H)^k$ converging in the (product) SOT to $b=(b_1,\ldots,b_k)\in B(H)^k$, then

$$\lim_{n\to\infty}\varphi(a_{1,n},\ldots,a_{k+1,n})\#b_{\cdot,n}=\varphi(a_1,\ldots,a_{k+1})\#b$$

in the SOT.

Proof. First, fix $i \in \{1, \ldots, k+1\}$ and $n \in \mathbb{N}$. If $f\colon \mathbb{R} \to \mathbb{C}$ is Borel measurable, then $f(a_{i,n}) = f(\chi_n(a_i)) = (f \circ \chi_n)(a_i)$ by [KR97a, Cor. 5.6.29]. Now, if f is also bounded, then $\sup_{n\in\mathbb{N}} \|f \circ \chi_n\|_{\ell^\infty(\mathbb{R})} \le \|f\|_{\ell^\infty(\mathbb{R})} < \infty$ and $f \circ \chi_n \to f \circ \mathrm{id}_{\mathbb{R}} = f$ pointwise as $n \to \infty$. Therefore, by Proposition 4.2.10(v), $f(a_{i,n}) \to f(a_i)$ in the S*OT as $n \to \infty$.

Next, let $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ be a ℓ^∞-IPD of φ. By definition,

$$\varphi(a_{1,n}, \ldots, a_{k+1,n})\#b = \int_\Sigma \varphi_1(a_{1,n}, \sigma)\, b_1 \cdots \varphi_k(a_{k,n}, \sigma)\, b_k\, \varphi_{k+1}(a_{k+1,n}, \sigma)\, \rho(\mathrm{d}\sigma), \qquad b \in B(H)^k.$$

By the previous paragraph's observations, for all $\sigma \in \Sigma$,

$$\varphi_1(a_{1,n}, \sigma)\, b_{1,n} \cdots \varphi_k(a_{k,n}, \sigma)\, b_{k,n}\varphi_{k+1}(a_{k+1,n}, \sigma) \xrightarrow{n\to\infty} \varphi_1(a_1, \sigma)\, b_1 \cdots \varphi_k(a_k, \sigma)\, b_k\varphi_{k+1}(a_{k+1}, \sigma)$$

in the SOT. Since

$$\begin{aligned}\overline{\int_\Sigma} \sup_{n\in\mathbb{N}} &\|\varphi_1(a_{1,n}, \sigma)\, b_{1,n} \cdots \varphi_k(a_{k,n}, \sigma)\, b_{k,n}\, \varphi_{k+1}(a_{k+1,n}, \sigma)\|\, \rho(\mathrm{d}\sigma) \\ &\le \sup_{n\in\mathbb{N}}(\|b_{1,n}\| \cdots \|b_{k,n}\|) \overline{\int_\Sigma} \|\varphi_1(\cdot, \sigma)\|_{\ell^\infty(\mathbb{R})} \cdots \|\varphi_{k+1}(\cdot, \sigma)\|_{\ell^\infty(\mathbb{R})}\, \rho(\mathrm{d}\sigma) < \infty,\end{aligned}$$

the desired result follows Lemma 6.5.1 and the definition of MOIs. □

Before stating and proving our perturbation formulas, we make a useful observation. If $f\colon \mathbb{R} \to \mathbb{C}$ is Lipschitz, then there exist constants $C_1, C_2 \ge 0$ such that $|f(\lambda)| \le C_1|\lambda| + C_2$ for all $\lambda \in \mathbb{R}$. In particular,

$$\mathrm{dom}(a) \subseteq \mathrm{dom}(f(a)), \qquad a \in C(H)_{\mathrm{sa}}, \tag{6.5.5}$$

by definition of the Borel functional calculus and the spectral theorem.

Notation 6.5.6. Fix a set S, $m \in \mathbb{N}$, and $s = (s_1, \ldots, s_m) \in S^m$. If $i \in \{1, \ldots, m+1\}$, then

$$s_{i-} \coloneqq (s_1, \ldots, s_{i-1}) \in S^{i-1} \;\text{ and }\; s_{i+} \coloneqq (s_i, \ldots, s_m) \in S^{m+1-i},$$

where s_{1-} and $s_{(m+1)+}$ are both the empty list.

Theorem 6.5.7 (Perturbation formulas)**.** *Let $a \in C(H)_{\mathrm{sa}}$. If $f \colon \mathbb{R} \to \mathbb{C}$ is a C^1 function such that $f^{[1]} \in \ell^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \hat{\otimes}_i \ell^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, then*

$$f(a+c) - f(a) = f^{[1]}(a+c, a) \# c, \qquad c \in B(H)_{\mathrm{sa}}. \tag{6.5.8}$$

More precisely, $f(a+c) - f(a)$ is densely defined and bounded, and $f^{[1]}(a+c,a)\#c$ is its unique bounded linear extension.

Now, suppose $k \geq 2$ and $\vec{a} = (a_1, \ldots, a_{k-1}) \in C(H)_{\mathrm{sa}}^{k-1}$. If $f \in C^k(\mathbb{R})$ is such that $f^{[k-1]} \in \ell^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}})^{\hat{\otimes}_i k}$ and $f^{[k]} \in \ell^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}})^{\hat{\otimes}_i (k+1)}$, then

$$f^{[k-1]}\big(\vec{a}_{i-}, a+c, \vec{a}_{i+}\big)\#b - f^{[k-1]}\big(\vec{a}_{i-}, a, \vec{a}_{i+}\big)\#b = f^{[k]}\big(\vec{a}_{i-}, a+c, a, \vec{a}_{i+}\big)\#[b_{i-}, c, b_{i+}] \tag{6.5.9}$$

for all $b = (b_1, \ldots, b_{k-1}) \in B(H)_{\mathrm{sa}}^{k-1}$ and $i \in \{1, \ldots, k\}$.

Proof. We first make an important observation. Fix $a \in C(H)_{\mathrm{sa}}$, $c \in B(H)_{\mathrm{sa}}$, and $n \in \mathbb{N}$. Now, define $p_n \coloneqq P^a([-n,n])$, $q_n \coloneqq P^{a+c}([-n,n])$, $a_n \coloneqq a\, p_n = \chi_n(a)$, and $d_n \coloneqq (a+c)\, q_n = \chi_n(a+c)$ in the notation of Lemma 6.5.4. If $\psi_1, \psi_2 \in \ell^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, then

$$\begin{aligned} q_n\, \psi_1(d_n)(d_n - a_n)\psi_2(a_n)\, p_n &= 1_{[-n,n]}(a+c)\,(\psi_1 \circ \chi_n)(a+c)\,(d_n - a_n)\,(\psi_2 \circ \chi_n)(a)\,1_{[-n,n]}(a) \\ &= ((\psi_1 \circ \chi_n)1_{[-n,n]})(a+c)\,(d_n - a_n)\,((\psi_2 \circ \chi_n)1_{[-n,n]})(a) \\ &= (\psi_1\, 1_{[-n,n]})(a+c)\,(d_n - a_n)\,(\psi_2\, 1_{[-n,n]})(a) \\ &= \psi_1(a+c)\, q_n\,(d_n - a_n)\, p_n\, \psi_2(a) = \psi_1(a+c)\, q_n\, c\, p_n\, \psi_2(a), \end{aligned}$$

where

$$q_n(d_n - a_n)p_n = q_n d_n p_n - q_n a_n p_n = q_n(a+c)p_n - q_n a p_n = q_n c p_n$$

because $\operatorname{im} p_n \subseteq \operatorname{dom}(a) = \operatorname{dom}(a+c)$.

We now begin in earnest. If $f \in C^1(\mathbb{R})$ is such that $f^{[1]} \in \ell^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \hat{\otimes}_i \ell^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, then

$$f(\lambda) - f(\mu) = f^{[1]}(\lambda, \mu)(\lambda - \mu), \qquad (\lambda, \mu) \in \mathbb{R} \times \mathbb{R}. \tag{6.5.10}$$

Now, let $a, c \in B(H)_{\mathrm{sa}}$. Since $\sigma(a)$ and $\sigma(a+c)$ are compact and $f \in C(\mathbb{R})$, the functions

$$\sigma(a+c)\times\sigma(a) \ni (\lambda,\mu) \mapsto \psi(\lambda,\mu) = \lambda-\mu \in \mathbb{C} \text{ and}$$
$$\sigma(a+c)\times\sigma(a) \ni (\lambda,\mu) \mapsto \varphi(\lambda,\mu) = f(\lambda)-f(\mu) \in \mathbb{C}$$

belong to $\ell^\infty(\sigma(a+c),\mathcal{B}_{\sigma(a+c)})\hat{\otimes}_i\ell^\infty(\sigma(a),\mathcal{B}_{\sigma(a)})$. By Proposition 5.7.1(iii) and Equation (6.5.10),

$$I^{a+c,a}\varphi = \big(I^{a+c,a}f^{[1]}\big)\circ\big(I^{a+c,a}\psi\big).$$

Applying this to the identity $1 = \mathrm{id}_H \in B(H)$, we conclude that

$$f(a+c)-f(a) = \big(I^{a+c,a}\varphi\big)[1] = \big(I^{a+c,a}f^{[1]}\big)\big[\big(I^{a+c,a}\psi\big)[1]\big] = \big(I^{a+c,a}f^{[1]}\big)[c] = f^{[1]}(a+c,a)\#c,$$

as desired.

For general $a \in C(H)_{\mathrm{sa}}$, we begin by showing that $f(a+c)-f(a)$ is densely defined; specifically, we show $\mathrm{dom}(a) \subseteq \mathrm{dom}(f(a+c)-f(a))$. Indeed, since $f^{[1]} \in \ell^\infty(\mathbb{R}^2,\mathcal{B}_{\mathbb{R}^2})$, f is Lipschitz on $\mathbb{R}$. By Relation (6.5.5), we have that $\mathrm{dom}(a_0) \subseteq \mathrm{dom}(f(a_0))$ for all $a_0 \in C(H)_{\mathrm{sa}}$. In particular, since $\mathrm{dom}(a) = \mathrm{dom}(a+c)$, we get

$$\mathrm{dom}(a) = \mathrm{dom}(a)\cap\mathrm{dom}(a+c) \subseteq \mathrm{dom}(f(a+c))\cap\mathrm{dom}(f(a)) = \mathrm{dom}(f(a+c)-f(a)),$$

as desired. Next, let p_n, q_n, a_n, and d_n be as in the first paragraph. If $(\Sigma,\rho,\varphi_1,\varphi_2)$ is a ℓ^∞-IPD of $f^{[1]}$, then the results of the previous two paragraphs and Lemma 6.5.1 give

$$\begin{aligned} q_n(f(d_n)-f(a_n))p_n &= q_n\, f^{[1]}(d_n,a_n)\#(d_n-a_n)\,p_n = \int_\Sigma q_n\,\varphi(d_n,\sigma)(d_n-a_n)\varphi_2(a_n,\sigma)\,p_n\,\rho(\mathrm{d}\sigma) \\ &= \int_\Sigma \varphi(a+c,\sigma)\,q_n\,c\,p_n\,\varphi_2(a,\sigma)\,\rho(\mathrm{d}\sigma) = f^{[1]}(a+c,c)\#[q_n\,c\,p_n] \xrightarrow{n\to\infty} f^{[1]}(a+c,a)\#c \end{aligned}$$

in the SOT, since $q_n \to 1$ and $p_n \to 1$ in the SOT as $n\to\infty$. But now, notice

$$f(a_n)\,p_n = (f\circ\chi_n)(a)\,1_{[-n,n]}(a) = ((f\circ\chi_n)\,1_{[-n,n]})(a) = (f\,1_{[-n,n]})(a) = f(a)\,p_n,$$

and similarly, $q_n f(d_n)p_n = q_n f(a+c)p_n$. (For the latter, use $\operatorname{im} p_n \subseteq \operatorname{dom}(a) \subseteq \operatorname{dom}(f(a+c))$.) It follows that if $m \in \mathbb{N}$, $h \in \operatorname{im} p_m$, and $n \geq m$, then

$$\begin{aligned} q_n(f(d_n)-f(a_n))p_n h &= q_n(f(a+c)-f(a))p_n h = q_n(f(a+c)-f(a))p_n p_m h \\ &= q_n(f(a+c)-f(a))p_m h \xrightarrow{n\to\infty} (f(a+c)-f(a))p_m h = (f(a+c)-f(a))h \end{aligned}$$

in H. We have now proven that $(f(a+c)-f(a))h = (f^{[1]}(a+c,c)\#c)h$ for all $h \in \operatorname{im} p_m$. Since $\bigcup_{m\in\mathbb{N}} \operatorname{im} p_m \subseteq H$ is a dense linear subspace, we are done with the first part.

Next, let $k \geq 2$, and suppose $f \in C^k(\mathbb{R})$ is such that $f^{[k-1]} \in \ell^\infty(\mathbb{R},\mathcal{B}_\mathbb{R})^{\hat{\otimes}_i k}$ and $f^{[k]} \in \ell^\infty(\mathbb{R},\mathcal{B}_\mathbb{R})^{\hat{\otimes}_i (k+1)}$. By definition and symmetry of divided differences, if $i \in \{1,\ldots,k\}$, $\lambda,\mu \in \mathbb{R}$, and $\vec{\lambda} = (\lambda_1,\ldots,\lambda_{k-1}) \in \mathbb{R}^{k-1}$, then

$$f^{[k-1]}\big(\vec{\lambda}_{i-},\lambda,\vec{\lambda}_{i+}\big) - f^{[k-1]}\big(\vec{\lambda}_{i-},\mu,\vec{\lambda}_{i+}\big) = f^{[k]}\big(\vec{\lambda}_{i-},\lambda,\mu,\vec{\lambda}_{i+}\big)(\lambda-\mu). \tag{6.5.11}$$

Now, fix $\vec{a} = (a_1,\ldots,a_{k-1}) \in C(H)_{\mathrm{sa}}^{k-1}$, $b = (b_1,\ldots,b_{k-1}) \in B(H)^{k-1}$, and $a,c \in B(H)_{\mathrm{sa}}$. Since $\sigma(a)$ and $\sigma(a+c)$ are compact and $f^{[k-1]} \in \ell^\infty(\mathbb{R},\mathcal{B}_\mathbb{R})^{\hat{\otimes}_i k}$, both of the functions

$$\begin{aligned} &\mathbb{R}^{i-1} \times \sigma(a+c) \times \sigma(a) \times \mathbb{R}^{k-i} \ni (u,\lambda,\mu,v) \overset{\psi}{\mapsto} \lambda-\mu \in \mathbb{C} \ \text{ and} \\ &\mathbb{R}^{i-1} \times \sigma(a+c) \times \sigma(a) \times \mathbb{R}^{k-i} \ni (u,\lambda,\mu,v) \overset{\varphi}{\mapsto} f^{[k-1]}(u,\lambda,v) - f^{[k-1]}(u,\mu,v) \in \mathbb{C} \end{aligned}$$

belong to $\ell^\infty(\mathbb{R},\mathcal{B}_\mathbb{R})^{\hat{\otimes}_i(i-1)} \hat{\otimes}_i \ell^\infty(\sigma(a+c),\mathcal{B}_{\sigma(a+c)}) \hat{\otimes}_i \ell^\infty(\sigma(a),\mathcal{B}_{\sigma(a)}) \hat{\otimes}_i \ell^\infty(\mathbb{R},\mathcal{B}_\mathbb{R})^{\hat{\otimes}_i(k-i)}$. This allows us to apply Equation $I^{\vec{a}_{i-},a+c,a,\vec{a}_{i+}}$ to (6.5.11), which may be rewritten $\varphi = f^{[k]}\psi$. If we do so and plug $(b_{i-},1,b_{i+})$ into the result, then we get

$$\begin{aligned} f^{[k-1]}\big(\vec{a}_{i-},a+c,\vec{a}_{i+}\big)\#b - f^{[k-1]}\big(\vec{a}_{i-},a,\vec{a}_{i+}\big)\#b &= \varphi\big(\vec{a}_{i-},a+c,a,\vec{a}_{i+}\big)\#[b_{i-},1,b_{i+}] \\ &= \big(f^{[k]}\psi\big)\big(\vec{a}_{i-},a+c,a,\vec{a}_{i+}\big)\#[b_{i-},1,b_{i+}] \\ &= f^{[k]}\big(\vec{a}_{i-},a+c,a,\vec{a}_{i+}\big)\#[b_{i-},c,b_{i+}], \end{aligned}$$

where Proposition 5.7.1(iii) and the definition of ψ were used in the last line.

Finally, for general $a \in C(H)_{\mathrm{sa}}$, let p_n, q_n, a_n, and d_n be as in the first paragraph. If

$1 < i < k$, then we also let $b_{\cdot,n} \coloneqq (b_{(i-1)-}, b_{i-1}q_n, p_n b_i, b_{(i+1)+})$. Since $p_n \to 1$ and $q_n \to 1$ in the SOT as $n \to \infty$, Lemma 6.5.4 gives

$$f^{[k-1]}\big(\vec{a}_{i-}, d_n, \vec{a}_{i+}\big)\#b_{\cdot,n} \xrightarrow{n\to\infty} f^{[k-1]}\big(\vec{a}_{i-}, a+c, \vec{a}_{i+}\big)\#b \text{ and}$$
$$f^{[k-1]}\big(\vec{a}_{i-}, a_n, \vec{a}_{i+}\big)\#b_{\cdot,n} \xrightarrow{n\to\infty} f^{[k-1]}\big(\vec{a}_{i-}, a, \vec{a}_{i+}\big)\#b$$

in the SOT. Now, let $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ be a ℓ^∞-IPD of $f^{[k]}$. Then

$$\begin{aligned}
T_{i,n} &\coloneqq f^{[k]}\big(\vec{a}_{i-}, d_n, a_n, \vec{a}_{i+}\big)\#[(b_{\cdot,n})_{i-}, d_n - a_n, (b_{\cdot,n})_{i+}] \\
&= \int_\Sigma \left(\prod_{j=1}^{i-1} \varphi(a_j,\sigma)\, b_j\right) q_n\, \varphi_i(d_n,\sigma)(d_n - a_n)\varphi_{i+1}(a_n,\sigma)\, p_n \left(\prod_{j=i}^{k-1} b_j\, \varphi(a_{j+2},\sigma)\right) \rho(\mathrm{d}\sigma) \\
&= \int_\Sigma \left(\prod_{j=1}^{i-1} \varphi(a_j,\sigma)\, b_j\right) \varphi_i(a+c,\sigma)\, q_n\, c\, p_n\, \varphi_{i+1}(a,\sigma) \left(\prod_{j=i}^{k-1} b_j\, \varphi(a_{j+2},\sigma)\right) \rho(\mathrm{d}\sigma) \\
&= f^{[k]}\big(\vec{a}_{i-}, a+c, a, \vec{a}_{i+}\big)\#[b_{i-}, q_n\, c\, p_n, b_{i+}] \\
&\xrightarrow{n\to\infty} f^{[k]}\big(\vec{a}_{i-}, a+c, a, \vec{a}_{i+}\big)\#[b_{i-}, c, b_{i+}]
\end{aligned}$$

in the SOT by the observation from the first paragraph and Lemma 6.5.4. Since we already know from the previous paragraph that

$$\begin{aligned}
&f^{[k-1]}\big(\vec{a}_{i-}, d_n, \vec{a}_{i+}\big)\#b_{\cdot,n} - f^{[k-1]}\big(\vec{a}_{i-}, a_n, \vec{a}_{i+}\big)\#b_{\cdot,n} \\
&\qquad\qquad = f^{[k]}\big(\vec{a}_{i-}, d_n, a_n, \vec{a}_{i+}\big)\#[(b_{\cdot,n})_{i-}, d_n - a_n, (b_{\cdot,n})_{i+}], \qquad n \in \mathbb{N},
\end{aligned}$$

this completes the proof when $1 < i < k$. For the cases $i \in \{1, k\}$, we redefine

$$b_{\cdot,n} \coloneqq (p_n b_1, b_{2+}) \text{ and } \tilde{b}_{\cdot,n} \coloneqq (b_{(k-1)-}, b_{k-1}q_n).$$

Then we use an argument similar to the one above to see that

$$\begin{aligned}
q_n(f^{[k-1]}(d_n, \vec{a})\#b_{\cdot,n} - f^{[k-1]}(a_n, \vec{a})\#b_{\cdot,n}) &= q_n f^{[k]}(d_n, a_n, \vec{a})\#[d_n - a_n, b_{\cdot,n}] \\
&= f^{[k]}(a+c, a, \vec{a})\#[q_n\, c\, p_n, b]
\end{aligned}$$

and

$$\big(f^{[k-1]}(\vec{a},d_n)\#\tilde{b}_{\cdot,n} - f^{[k-1]}(\vec{a},a_n)\#\tilde{b}_{\cdot,n}\big)p_n = \big(f^{[k]}(\vec{a},d_n,a_n)\#\big[\tilde{b}_{\cdot,n}, d_n - a_n\big]\big)p_n$$
$$= f^{[k]}(\vec{a},a+c,a)\#[b,q_n\,c\,p_n].$$

Then we use Lemma 6.5.1 to take $n \to \infty$. This completes the proof. □

Corollary 6.5.12. *Let $a\,\eta\,\mathcal{M}_{\mathrm{sa}}$, and suppose $(\mathcal{I},\|\cdot\|_{\mathcal{I}}) \trianglelefteq \mathcal{M}$ is MOI-friendly. If $f \in C^1(\mathbb{R})$ is such that $f^{[1]} \in \ell^\infty(\mathbb{R},\mathcal{B}_{\mathbb{R}})\hat{\otimes}_i\ell^\infty(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ and $c \in \mathcal{I}_{\mathrm{sa}} = \mathcal{I}\cap\mathcal{M}_{\mathrm{sa}}$, then $f(a+c)-f(a) \in \mathcal{I}$, and*

$$\|f(a+c)-f(a)\|_{\mathcal{I}} \le \big\|f^{[1]}\big\|_{\ell^\infty(\sigma(a+c),\mathcal{B}_{\sigma(a+c)})\hat{\otimes}_i\ell^\infty(\sigma(a),\mathcal{B}_{\sigma(a)})}\|c\|_{\mathcal{I}}.$$

Proof. Since $a\,\eta\,\mathcal{M}_{\mathrm{sa}}$ and $c \in \mathcal{I}_{\mathrm{sa}} \subseteq \mathcal{M}_{\mathrm{sa}}$, $a+c\,\eta\,\mathcal{M}_{\mathrm{sa}}$ as well. In particular, P^a and P^{a+c} take values in $\mathcal{M}$. It then follows from Equation (6.5.8) and the definition of an MOI-friendly ideal that $f(a+c)-f(a) \in \mathcal{I}$ and $\|f(a+c)-f(a)\|_{\mathcal{I}} \le \|f^{[1]}\|_{\ell^\infty(\sigma(a+c),\mathcal{B}_{\sigma(a+c)})\hat{\otimes}_i\ell^\infty(\sigma(a),\mathcal{B}_{\sigma(a)})}\|c\|_{\mathcal{I}}$. □

Remark 6.5.13 (Quasicommutators)**.** Let $f \in C^1(\mathbb{R})$ be such that $f^{[1]} \in \ell^\infty(\mathbb{R},\mathcal{B}_{\mathbb{R}})\hat{\otimes}_i\ell^\infty(\mathbb{R},\mathcal{B}_{\mathbb{R}})$. One can show using essentially the same proofs that if $a,b \in C(H)_{\mathrm{sa}}$ and $q \in B(H)$ are such that $aq-qb \in B(H)$ (i.e., $aq-qb$ is densely defined and bounded), then $f(a)q-qf(b) \in B(H)$, and

$$f(a)q-qf(b) = f^{[1]}(a,b)\#[aq-qb].$$

As a result, we get a quasicommutator estimate in MOI-friendly ideals. Suppose also that $(\mathcal{I},\|\cdot\|_{\mathcal{I}}) \trianglelefteq \mathcal{M}$ is MOI-friendly. If $a,b\,\eta\,\mathcal{M}_{\mathrm{sa}}$ and $q \in B(H)$ are such that $aq-qb \in \mathcal{I}$, then $f(a)q-qf(b) \in \mathcal{I}$ and

$$\|f(a)q-qf(b)\|_{\mathcal{I}} \le \|f^{[1]}\|_{\ell^\infty(\sigma(a),\mathcal{B}_{\sigma(a)})\hat{\otimes}_i\ell^\infty(\sigma(b),\mathcal{B}_{\sigma(b)})}\|aq-qb\|_{\mathcal{I}}.$$

Such quasicommutator estimates are of interest in the study of *operator Lipschitz functions*. Please see [AP16, Pel16] for more information.

6.6 Derivative formulas

In this section, we compute the derivative formulas of interest. To begin, we introduce the functions whose (perturbed) functional calculi we shall be differentiating. Then we use Peller's work from [Pel06], which we review in detail in Appendix B, to give a large class of examples of such functions.

Definition 6.6.1 (Operator continuous)**.** A Borel measurable function $f\colon \mathbb{R} \to \mathbb{C}$ is **operator continuous** if

(i) for every complex Hilbert space H, $a \in C(H)_{\mathrm{sa}}$, and $c \in B(H)_{\mathrm{sa}}$, $f(a+c) - f(a)$ is densely defined and bounded; and

(ii) for every complex Hilbert space H and $a \in C(H)_{\mathrm{sa}}$, $f(a+c) - f(a) \to 0$ in $B(H)$ as $c \to 0$ in $B(H)_{\mathrm{sa}}$. (More precisely, for every $a \in C(H)_{\mathrm{sa}}$ and $\varepsilon > 0$, there is some $\delta > 0$ such that $\|f(a+c) - f(a)\| < \varepsilon$ whenever $c \in B(H)_{\mathrm{sa}}$ and $\|c\| < \delta$.)

In this case, we write $f \in OC(\mathbb{R})$. If, in addition, f is bounded, then we write $f \in BOC(\mathbb{R})$.

Taking $H = \mathbb{C}$ in the definition, it is clear that operator continuous functions are continuous. Also, we observe that if $f, g \in BOC(\mathbb{R})$, $a \in C(H)_{\mathrm{sa}}$, and $c \in B(H)_{\mathrm{sa}}$, then

$$(fg)(a+c) - (fg)(a) = (f(a+c) - f(a))g(a+c) + f(a)(g(a+c) - g(a)).$$

But then

$$\|(fg)(a+c) - (fg)(a)\| \le \|g\|_{\ell^\infty(\mathbb{R})}\|f(a+c) - f(a)\| + \|f\|_{\ell^\infty(\mathbb{R})}\|g(a+c) - g(a)\| \xrightarrow{\|c\| \to 0} 0.$$

Thus, $fg \in BOC(\mathbb{R})$. It is even easier to see that $f + g \in BOC(\mathbb{R})$ and $\overline{f} \in BOC(\mathbb{R})$.

Next, if $(\Sigma, \mathscr{H})$ is a measurable space, $\psi\colon \mathbb{R} \times \Sigma \to \mathbb{C}$ is measurable, and $\psi(\cdot, \sigma) \in C(\mathbb{R})$ for all $\sigma \in \Sigma$, then

$$\|\psi(\cdot,\sigma)\|_{\ell^\infty(\mathbb{R})} = \sup_{t \in \mathbb{Q}} |\psi(t,\sigma)|, \qquad \sigma \in \Sigma.$$

In particular, the function $\Sigma \ni \sigma \mapsto \|\psi(\cdot,\sigma)\|_{\ell^\infty(\mathbb{R})} \in \mathbb{R}$ is measurable. Thus, the following definition makes sense (without needing to use upper or lower integrals).

Definition 6.6.2 (Integral projective tensor products II)**.** Let $\varphi\colon \mathbb{R}^{k+1} \to \mathbb{C}$ be a function. A ***BOC*-integral projective decomposition** (BOCIPD) of φ is a choice $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ of a σ-finite measure space $(\Sigma, \mathscr{H}, \rho)$ and measurable functions $\varphi_1, \ldots, \varphi_{k+1}\colon \mathbb{R} \times \Sigma \to \mathbb{C}$ such that

(i) $\varphi_i(\cdot, \sigma) \in BOC(\mathbb{R})$ for all $i \in \{1, \ldots, k+1\}$ and $\sigma \in \Sigma$,

(ii) $\int_\Sigma \|\varphi_1(\cdot,\sigma)\|_{\ell^\infty(\mathbb{R})} \cdots \|\varphi_{k+1}(\cdot,\sigma)\|_{\ell^\infty(\mathbb{R})}\, \rho(\mathrm{d}\sigma) < \infty$, and

(iii) $\varphi(\boldsymbol{\lambda}) = \int_\Sigma \varphi_1(\lambda_1,\sigma) \cdots \varphi_{k+1}(\lambda_{k+1},\sigma)\, \rho(\mathrm{d}\sigma)$ for all $\boldsymbol{\lambda} \in \mathbb{R}^{k+1}$.

Now, define

$$\|\varphi\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}} \coloneqq \inf\left\{\int_\Sigma \prod_{i=1}^{k+1} \|\varphi_i(\cdot,\sigma)\|_{\ell^\infty(\mathbb{R})}\, \rho(\mathrm{d}\sigma) : (\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1}) \text{ is a BOCIPD of } \varphi\right\},$$

where $\inf \emptyset \coloneqq \infty$. Finally, we define

$$BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)} \coloneqq \left\{\varphi : \|\varphi\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}} < \infty\right\}$$

to be the $\boldsymbol{(k+1)^{\mathrm{st}}}$ **integral projective tensor power of** $\boldsymbol{BOC}(\mathbb{R})$.

Proposition 6.6.3. $BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)} \subseteq BC\big(\mathbb{R}^{k+1}\big)$ *is a unital $*$-subalgebra, and*

$$\big(BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}, \|\cdot\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}}\big)$$

is a unital Banach $*$-algebra under pointwise operations.

Sketch of proof. The containment $BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)} \subseteq BC\big(\mathbb{R}^{k+1}\big)$ follows from the definitions and an application of the dominated convergence theorem. The rest of the statement follows from the observation above that $BOC(\mathbb{R})$ is a $*$-algebra and arguments similar to (but easier than) those in the proof of Proposition 5.5.5. □

Next come the functions of interest.

Notation 6.6.4. If $f \in C^k(\mathbb{R})$, then

$$[f]_{OC^{[k]}(\mathbb{R})} \coloneqq \sum_{i=1}^{k} \big\|f^{[i]}\big\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(i+1)}} \in [0,\infty] \text{ and } OC^{[k]}(\mathbb{R}) \coloneqq \big\{g \in C^k(\mathbb{R}) : [g]_{OC^{[k]}(\mathbb{R})} < \infty\big\}.$$

Notice that if $f \in C^1(\mathbb{R})$ and $[f]_{OC^1(\mathbb{R})} = \|f^{[1]}\|_{BOC(\mathbb{R})\hat{\otimes}_i BOC(\mathbb{R})} = 0$, then $f^{[1]} \equiv 0$, so f must be constant. In particular, $[\cdot]_{OC^{[k]}(\mathbb{R})}$ is a seminorm but not quite a norm. If we define

$$\|f\|_{OC^{[k]},r} := \|f\|_{\ell^\infty([-r,r])} + [f]_{OC^{[k]}(\mathbb{R})}, \qquad r > 0,$$

then it can be shown, using standard arguments and Proposition 6.6.3, that $OC^{[k]}(\mathbb{R})$ is a Fréchet $*$-algebra with the topology induced by the collection $\{\|\cdot\|_{OC^{[k]},r} : r > 0\}$ of seminorms and pointwise operations. Since we shall not need these facts, we shall not dwell on them. Instead, we turn to examples.

Lemma 6.6.5. *If $\xi \in \mathbb{R}$ and $f(\lambda) := e^{i\lambda\xi}$ for all $\lambda \in \mathbb{R}$, then $f \in BOC(\mathbb{R})$.*

Proof. Of course, f is bounded and continuous. Now, if $\lambda, \mu \in \mathbb{R}$, then

$$f^{[1]}(\lambda, \mu) = \int_0^1 f'(t\lambda + (1-t)\mu)\,\mathrm{d}t = i\xi \int_0^1 e^{it\lambda\xi} e^{i(1-t)\xi\mu}\,\mathrm{d}t$$

by Proposition 1.3.3(iii). This is clearly a ℓ^∞-integral projective decomposition of $f^{[1]}$ that yields

$$\|f^{[1]}\|_{\ell^\infty(\mathbb{R},\mathcal{B}_\mathbb{R})\hat{\otimes}_i \ell^\infty(\mathbb{R},\mathcal{B}_\mathbb{R})} \le |\xi|.$$

In particular, if $a \in C(H)_{\mathrm{sa}}$ and $c \in B(H)_{\mathrm{sa}}$, then $\|f(a+c) - f(a)\| \le |\xi|\,\|c\|$ by Corollary 6.5.12. Thus, f is operator continuous. $\square$

Proposition 6.6.6. $W_k(\mathbb{R}) \subseteq OC^{[k]}(\mathbb{R})$. *Specifically, if $f(\lambda) = \int_\mathbb{R} e^{i\cdot\xi}\,\mu(\mathrm{d}\xi) \in W_k(\mathbb{R})$, then*

$$[f]_{OC^{[k]}(\mathbb{R})} \le \sum_{i=1}^k \frac{\mu_{(i)}}{i!}.$$

Proof. If f is as in the statement and $j \in \{1, \ldots, k\}$, then

$$f^{[j]}(\boldsymbol{\lambda}) = \int_{\Delta_j \times \mathbb{R}} e^{it_1\lambda_1\xi} \cdots e^{it_{j+1}\lambda_{j+1}\xi}\,(i\xi)^j (\rho_j \otimes \mu)(\mathrm{d}\mathbf{t}, \mathrm{d}\xi), \qquad \boldsymbol{\lambda} \in \mathbb{R}^{j+1},$$

by Example 1.3.14. By Lemma 6.6.5, this is, after writing $d\mu = \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\mathrm{d}|\mu|$ to match the definition, a BOCIPD of $f^{[j]}$ that yields

$$\|f^{[j]}\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(j+1)}} \leq \int_{\Delta_j\times\mathbb{R}} |\xi|^j\,(\rho_j\otimes|\mu|)(\mathrm{d}\mathbf{t},\mathrm{d}\xi) = \rho_j(\Delta_j)\int_{\mathbb{R}}|\xi|^j\,|\mu|(\mathrm{d}\xi) = \frac{\mu_{(j)}}{j!}.$$

Summing over $j\in\{1,\ldots,k\}$ gives the desired bound. □

Remark 6.6.7. For similar reasons, if $f\in C^k(\mathbb{R})$ and for all $i\in\{1,\ldots,k\}$, $f^{(i)}$ and the Fourier transform of $f^{(i)}$ belong to $L^1(\mathbb{R})$, then $f\in OC^{[k]}(\mathbb{R})$.

Now, we use more serious harmonic analysis done by Peller [Pel06] to exhibit a large class—containing $W_k(\mathbb{R})$ strictly—of functions belonging to $OC^{[k]}(\mathbb{R})$.

Definition 6.6.8 (Peller–Besov spaces)**.** If $k\in\mathbb{N}$, then we define

$$PB^k(\mathbb{R}) := \dot{B}_1^{k,\infty}(\mathbb{R})\cap\big\{f\in C^k(\mathbb{R}) : f^{(k)} \text{ is bounded}\big\}$$

to be the $\boldsymbol{k}^{\text{th}}$ **Peller–Besov space**.

The following result is a slight upgrade of [Pel06, Thm. 5.5] or [Pel16, Thm. 2.2.1].

Theorem 6.6.9 (Peller)**.** *There exists a constant $c_k<\infty$ such that*

$$\|f^{[k]}\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}} \leq \frac{1}{k!}\inf_{x\in\mathbb{R}}\big|f^{(k)}(x)\big| + c_k\|f\|_{\dot{B}_1^{k,\infty}}, \qquad f\in PB^k(\mathbb{R}),$$

and if $k\geq 2$, then

$$\|f^{[k]}\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}} \leq c_k\|f\|_{\dot{B}_1^{k,\infty}}$$

for all $f\in PB^1(\mathbb{R})\cap\dot{B}_1^{k,\infty}(\mathbb{R}) = PB^1(\mathbb{R})\cap PB^k(\mathbb{R}) = \bigcap_{i=1}^k PB^i(\mathbb{R})$.

The proof given in [Pel06] is not very detailed and is only explicit in the cases $k\in\{1,2\}$, so we present a full proof of Theorem 6.6.9 in Appendix B. As a result, we obtain the following.

Corollary 6.6.10. $PB^1(\mathbb{R})\cap PB^k(\mathbb{R}) = PB^1(\mathbb{R})\cap\dot{B}_1^{k,\infty}(\mathbb{R})\subseteq OC^{[k]}(\mathbb{R})$. *Specifically,*

$$[f]_{OC^{[k]}(\mathbb{R})} \leq \inf_{x\in\mathbb{R}}|f'(x)| + \sum_{i=1}^k c_i\|f\|_{\dot{B}^{i,\infty}}, \qquad f\in PB^1(\mathbb{R})\cap PB^k(\mathbb{R}).$$

Since $W_k(\mathbb{R}) \subsetneq PB^1(\mathbb{R}) \cap \dot{B}_1^{k,\infty}(\mathbb{R})$, Corollary 6.6.10 generalizes Proposition 6.6.6.

We now launch into the proof of this section's main results. Seeing as we already have perturbation formulas, we need the second ingredient in the method of perturbation formulas: a continuous perturbation property. This will be Lemma 6.6.11; it is the main reason *integral* symmetrically normed ideals are considered in this chapter.

Lemma 6.6.11 (Continuous perturbation property)**.** *If $\mathcal{I}$ is integral symmetrically normed, $a_1, \ldots, a_{k+1} \, \eta \, \mathcal{M}_{\text{sa}}$, and $\varphi \in BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}$, then the map*

$$\mathcal{I}_{\text{sa}}^{k+1} \ni (c_1, \ldots, c_{k+1}) \mapsto I^{a_1+c_1,\ldots,a_{k+1}+c_{k+1}}\varphi \in B_k\big(\mathcal{I}^k; \mathcal{I}\big)$$

is continuous. (To be clear, $\mathcal{I}$ and $\mathcal{I}_{\text{sa}}$ are always endowed with the norm $\|\cdot\|_{\mathcal{I}}$.)

Remark 6.6.12. Recall from Proposition 6.2.4 that integral symmetrically normed ideals are MOI-friendly. In particular, the map under consideration in Lemma 6.6.11 does actually make sense by definition of MOI-friendly and the fact that $a + c \, \eta \, \mathcal{M}_{\text{sa}}$ whenever $a \, \eta \, \mathcal{M}_{\text{sa}}$ and $c \in \mathcal{I}_{\text{sa}}$. (As in the proof of Corollary 6.5.12, the latter imply that P^a and P^{a+c} take values in $\mathcal{M}$.)

Proof. Write $\varphi_a \colon \mathcal{I}_{\text{sa}}^{k+1} \to B_k\big(\mathcal{I}^k; \mathcal{I}\big)$ for the map in question. Now, let $c = (c_1, \ldots, c_{k+1}) \in \mathcal{I}_{\text{sa}}^{k+1}$, and let $(c_{\cdot,n})_{n\in\mathbb{N}} = (c_{1,n}, \ldots, c_{k+1,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{I}_{\text{sa}}^{k+1}$ converging to c. Then

$$\varphi_a(c_{\cdot,n}) - \varphi_a(c) = \sum_{i=1}^{k+1} \underbrace{(\varphi_a(c_{1,n}, \ldots, c_{i,n}, c_{i+1}, \ldots, c_{k+1}) - \varphi_a(c_{1,n}, \ldots, c_{i-1,n}, c_i, \ldots, c_{k+1}))}_{\coloneqq T_{i,n}}.$$

Now, fix a BOCIPD $(\Sigma, \rho, \varphi_1, \ldots, \varphi_{k+1})$ of φ and $b_1, \ldots, b_k \in \mathcal{I}$, and write $b_{k+1} \coloneqq 1$. By definition of the multiple operator integral, $T_{i,n}(b_1, \ldots, b_k)$ is precisely

$$\int_\Sigma \left(\prod_{j=1}^{i-1} \varphi_j(a_j + c_{j,n}, \sigma)\, b_j\right)(\varphi_i(a_i + c_{i,n}, \sigma) - \varphi_i(a_i + c_i, \sigma))\, b_i \left(\prod_{j=i+1}^{k+1} \varphi_j(a_j + c_j, \sigma)\, b_j\right) \rho(\mathrm{d}\sigma),$$

where empty products are the identity. Now, if $1 \leq i < k+1$,

$$A_n(\sigma) \coloneqq \left(\prod_{j=1}^{i-1} \varphi_j(a_j + c_{j,n}, \sigma)\, b_j\right)(\varphi_i(a_i + c_{i,n}, \sigma) - \varphi_i(a_i + c_i, \sigma)),$$

and $B(\sigma) \coloneqq \prod_{j=i+1}^{k+1} \varphi_j(a_j + c_j, \sigma))\, b_j$, then

$$T_{i,n}(b_1, \ldots, b_k) = \int_\Sigma A_n(\sigma)\, b_i\, B(\sigma)\, \rho(\mathrm{d}\sigma).$$

But

$$\underline{\int_\Sigma} \|A_n\|\, \|B\|\, \mathrm{d}\rho \le \prod_{p\neq i} \|b_p\| \underline{\int_\Sigma} \big\|\varphi_i(a_i + c_{i,n}, \sigma) - \varphi_i(a_i + c_i, \sigma)\big\| \prod_{j\neq i} \|\varphi_j(\cdot, \sigma)\|_{\ell^\infty(\mathbb{R})}\, \rho(\mathrm{d}\sigma) < \infty.$$

Therefore, the definition of integral symmetrically normed gives $T_{i,n}(b_1, \ldots, b_k) \in \mathcal{I}$ and

$$\begin{aligned} \|T_{i,n}(b_1, \ldots, b_k)\|_{\mathcal{I}} &\le \|b_i\|_{\mathcal{I}} \prod_{p\neq i} \|b_p\| \underline{\int_\Sigma} \big\|\varphi_i(a_i + c_{i,n}, \sigma) - \varphi_i(a_i + c_i, \sigma)\big\| \prod_{j\neq i} \|\varphi_j(\cdot, \sigma)\|_{\ell^\infty(\mathbb{R})}\, \rho(\mathrm{d}\sigma) \\ &\le C_{\mathcal{I}}^{k-1} \|b_1\|_{\mathcal{I}} \cdots \|b_k\|_{\mathcal{I}} \underline{\int_\Sigma} \big\|\varphi_i(a_i + c_{i,n}, \sigma) - \varphi_i(a_i + c_i, \sigma)\big\| \prod_{j\neq i} \|\varphi_j(\cdot, \sigma)\|_{\ell^\infty(\mathbb{R})}\, \rho(\mathrm{d}\sigma). \end{aligned}$$

Thus,

$$\|T_{i,n}\|_{B_k(\mathcal{I}^k;\mathcal{I})} \le C_{\mathcal{I}}^{k-1} \underline{\int_\Sigma} \big\|\varphi_i(a_i + c_{i,n}, \sigma) - \varphi_i(a_i + c_i, \sigma)\big\| \prod_{j\neq i} \|\varphi_j(\cdot, \sigma)\|_{\ell^\infty(\mathbb{R})}\, \rho(\mathrm{d}\sigma). \tag{6.6.13}$$

Next, let $\sigma \in \Sigma$. Since $\|c_{i,n} - c_i\| \le C_{\mathcal{I}} \|c_{i,n} - c_i\|_{\mathcal{I}} \to 0$ as $n \to \infty$, the operator continuity of $\varphi_i(\cdot, \sigma)$ gives that $\|\varphi_i(a_i + c_{i,n}, \sigma) - \varphi_i(a_i + c_i, \sigma)\| \to 0$ as $n \to \infty$. Since

$$\begin{aligned} &\overline{\int_\Sigma} \sup_{n\in\mathbb{N}} \left(\big\|\varphi_i(a_i + c_{i,n}, \sigma) - \varphi_i(a_i + c_i,\sigma)\big\| \prod_{m\neq i} \|\varphi_m(\cdot, \sigma)\|_{\ell^\infty(\mathbb{R})} \right) \rho(\mathrm{d}\sigma) \\ &\qquad\qquad \le 2 \int_\Sigma \prod_{j=1}^{k+1} \|\varphi_j(\cdot, \sigma)\|_{\ell^\infty(\mathbb{R})}\, \rho(\mathrm{d}\sigma) < \infty, \end{aligned}$$

we conclude from Inequality (6.6.13) and Proposition 5.3.2(iv) that $\|T_{i,n}\|_{B_k(\mathcal{I}^k;\mathcal{I})} \to 0$ as $n \to \infty$. If $i = k+1$, then we run the same argument with

$$A_n(\sigma) \coloneqq \left(\prod_{j=1}^{k-1} \varphi_j(a_j + c_{j,n}, \sigma)\, b_j \right) \varphi_k(a_k + c_{k,n}, \sigma) \;\; \text{and}$$

$$B_n(\sigma) \coloneqq \varphi_{k+1}(a_{k+1} + c_{k+1,n}, \sigma) - \varphi_{k+1}(a_{k+1} + c_{k+1}, \sigma)$$

to prove that $\|T_{k+1,n}\|_{B_k(\mathcal{I}^k;\mathcal{I})} \to 0$ as $n \to \infty$. We conclude that

$$\|\varphi_a(c_{\cdot,n}) - \varphi_a(c)\|_{B_k(\mathcal{I}^k;\mathcal{I})} \leq \sum_{i=1}^{k+1} \|T_{i,n}\|_{B_k(\mathcal{I}^k;\mathcal{I})} \xrightarrow{n\to\infty} 0,$$

as claimed. □

Definition 6.6.14 ($\mathcal{I}$-differentiability)**.** Let $a \,\eta\, \mathcal{M}_{\text{sa}}$. A Borel measurable function $f\colon \mathbb{R} \to \mathbb{C}$ is ***k*-times (Fréchet) $\boldsymbol{\mathcal{I}}$-differentiable at *a*** if there is an open set $U \subseteq \mathcal{I}_{\text{sa}}$ with $0 \in U$ such that

(i) $f(a+b) - f(a) \in \mathcal{I}$ for all $b \in U$ (i.e., if $b \in U$, then $f(a+b) - f(a)$ is densely defined and bounded, and its unique bounded linear extension belongs to $\mathcal{I}$), and

(ii) the map $U \ni b \mapsto f_{a,\mathcal{I}}(b) \coloneqq f(a+b) - f(a) \in \mathcal{I}$ is k-times Fréchet differentiable (with respect to $\|\cdot\|_{\mathcal{I}}$) at $0 \in U \subseteq \mathcal{I}_{\text{sa}}$.

In this case, we write

$$D^k_{\mathcal{I}} f(a) \coloneqq D^k f_{a,\mathcal{I}}(0) \in B_k\big(\mathcal{I}^k_{\text{sa}};\mathcal{I}\big)$$

for the k^{th} Fréchet derivative of $f_{a,\mathcal{I}}\colon U \to \mathcal{I}$ at $0 \in U$. If f is k-times $\mathcal{I}$-differentiable at a for all $a \,\eta\, \mathcal{M}_{\text{sa}}$, then f is ***k*-times $\boldsymbol{\mathcal{I}}$-differentiable**.

Suppose $f\colon \mathbb{R} \to \mathbb{C}$ is Lipschitz and $f(a+c) - f(a) \in \mathcal{I}$ for all $a \,\eta\, \mathcal{M}_{\text{sa}}$ and $c \in \mathcal{I}_{\text{sa}}$ (i.e., $f_{a,\mathcal{I}}\colon \mathcal{I}_{\text{sa}} \to \mathcal{I}$ is defined everywhere). We claim that if f is k-times $\mathcal{I}$-differentiable and $a \,\eta\, \mathcal{M}_{\text{sa}}$, then $f_{a,\mathcal{I}}$ is k-times Fréchet differentiable everywhere, not just at $0 \in \mathcal{I}_{\text{sa}}$. Indeed, let $b, c \in \mathcal{I}_{\text{sa}}$, and note that

$$f_{a,\mathcal{I}}(b+c) - f_{a,\mathcal{I}}(b) = f(a+b+c) - f(a+b) = f_{a+b,\mathcal{I}}(c). \tag{6.6.15}$$

This is the case because Equation (6.6.15) is immediate from the definition on

$$\operatorname{dom}(a) = \operatorname{dom}(a) \cap \operatorname{dom}(a+b+c) \cap \operatorname{dom}(a+b) \subseteq \operatorname{dom}(f(a)) \cap \operatorname{dom}(f(a+b+c)) \cap \operatorname{dom}(f(a+b)),$$

whic is dense in H. (Note that we used Relation (6.5.5).) In other words,

$$f_{a,\mathcal{I}}(b+c) = f_{a+b,\mathcal{I}}(c) + f_{a,\mathcal{I}}(b), \qquad c \in \mathcal{I}_{\text{sa}}.$$

Since $c \mapsto f_{a+b,\mathcal{I}}(c)$ is k-times differentiable at $0 \in \mathcal{I}_{\mathrm{sa}}$, we conclude that $f_{a,\mathcal{I}}$ is k-times differentiable at b with

$$D^k f_{a,\mathcal{I}}(b) = D^k f_{a+b,\mathcal{I}}(0) = D^k_{\mathcal{I}} f(a+b).$$

With this in mind, here is the main result of this section.

Theorem 6.6.16 (Derivatives of operator functions in ISNIs)**.** *Suppose* $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq \mathcal{M}$ *is integral symmetrically normed, and fix* $a \,\eta\, \mathcal{M}_{\mathrm{sa}}$*. If* $f \in OC^{[k]}(\mathbb{R})$*, then* $f_{a,\mathcal{I}} \colon \mathcal{I}_{\mathrm{sa}} \to \mathcal{I}$ *is defined everywhere, and* $f_{a,\mathcal{I}} \in C^k_{\mathrm{bb}}(\mathcal{I}_{\mathrm{sa}}; \mathcal{I})$*. In particular,* f *is* k*-times* $\mathcal{I}$*-differentiable. Furthermore,*

$$D^k_{\mathcal{I}} f(a)[b_1, \ldots, b_k] = \sum_{\pi \in S_k} f^{[k]}\big(a_{(k+1)}\big) \# [b_{\pi(1)}, \ldots, b_{\pi(k)}], \qquad b_1, \ldots, b_k \in \mathcal{I}_{\mathrm{sa}}.$$

(Please review Notations 1.2.5(i) and 6.5.3.)

Proof. Let $a \,\eta\, \mathcal{M}_{\mathrm{sa}}$. Note that if $f \in OC^{[k]}(\mathbb{R}) \subseteq C^k(\mathbb{R})$, then $f^{[1]} \in \ell^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \hat{\otimes}_i \ell^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Consequently, by Corollary 6.5.12, $f_{a,\mathcal{I}}(c) = f(a+c) - f(a) \in \mathcal{I}$ for all $c \in \mathcal{I}_{\mathrm{sa}}$. In addition, observe that if $f \in OC^{[k]}(\mathbb{R})$, then the map

$$\mathcal{I}_{\mathrm{sa}} \ni c \mapsto I^{a+c,\ldots,a+c} f^{[k]} \in B_k\big(\mathcal{I}^k; \mathcal{I}\big)$$

is continuous by Lemma 6.6.11. Therefore, the claimed k^{th} derivative map is, in fact, continuous. (We encourage the reader to notice that it is also uniformly bounded.) Thus, to prove the theorem, it suffices to prove the claimed formula for $D^k_{\mathcal{I}} f(a)$. We do so by induction on $k \geq 1$.

Let $c \in \mathcal{I}_{\mathrm{sa}}$. By Theorem 6.5.7,

$$\begin{aligned} f_{a,\mathcal{I}}(c) - f_{a,\mathcal{I}}(0) - f^{[1]}(a,a) \# c &= f(a+c) - f(a) - f^{[1]}(a,a) \# c \\ &= f^{[1]}(a+c, a) \# c - f^{[1]}(a,a) \# c \\ &= \big(I^{a+c,a} f^{[1]} - I^{a,a} f^{[1]}\big)[c]. \end{aligned}$$

Therefore, by Lemma 6.6.11,

$$\frac{1}{\|c\|_{\mathcal{I}}} \big\| f_{a,\mathcal{I}}(c) - f_{a,\mathcal{I}}(0) - f^{[1]}(a,a) \# c \big\|_{\mathcal{I}} \leq \big\| I^{a+c,a} f^{[1]} - I^{a,a} f^{[1]} \big\|_{B(\mathcal{I})} \xrightarrow{\|c\|_{\mathcal{I}}} 0$$

This completes the proof when $k = 1$. Next, suppose $k \geq 2$ and that we have proven the claimed derivative formula when $f \in OC^{[k-1]}(\mathbb{R})$. To prove the formula for $f \in OC^{[k]}(\mathbb{R})$, we make some preliminary observations. Fix $b = (b_1, \ldots, b_{k-1}) \in \mathcal{I}^{k-1}$ and $f \in OC^{[k]}(\mathbb{R}) \subseteq OC^{[k-1]}(\mathbb{R})$. By Theorem 6.5.7,

$$\begin{aligned}
\delta(b, c) &\coloneqq f^{[k-1]}\big((a+c)_{(k)}\big)\#b - f^{[k-1]}\big(a_{(k)}\big)\#b \\
&= \sum_{i=1}^{k} \big(f^{[k-1]}\big((a+c)_{(i)}, a_{(k-i)}\big)\#b - f^{[k-1]}\big((a+c)_{(i-1)}, a_{(k-i+1)}\big)\#b\big) \\
&= \sum_{i=1}^{k} f^{[k]}\big((a+c)_{(i)}, a_{(k+1-i)}\big)\#[b_{i-}, c, b_{i+}],
\end{aligned}$$

using Notation 6.5.6. Next, by the induction hypothesis,

$$D^{k-1} f_{a,\mathcal{I}}(c_0)[b] = D^{k-1}_{\mathcal{I}} f(a + c_0)[b] = \sum_{\tau \in S_{k-1}} f^{[k-1]}\big((a+c_0)_{(k)}\big)\#b^{\tau}, \qquad c_0 \in \mathcal{I}_{\mathrm{sa}},$$

where

$$b^{\tau} = (b_{\tau(1)}, \ldots, b_{\tau(k-1)}), \qquad \tau \in S_{k-1}.$$

Combining this induction hypothesis with the expression for $\delta(b, c)$ above gives

$$\begin{aligned}
\varepsilon(b, c) &\coloneqq D^{k-1} f_{a,\mathcal{I}}(c)[b] - D^{k-1} f_{a,\mathcal{I}}(0)[b] - \sum_{\tau \in S_{k-1}} \sum_{i=1}^{k} f^{[k]}\big(a_{(k+1)}\big)\#[b^{\tau}_{i-}, c, b^{\tau}_{i+}] \\
&= \sum_{\tau \in S_{k-1}} \Big(f^{[k-1]}\big((a+c)_{(k)}\big)\#b^{\tau} - f^{[k-1]}\big(a_{(k)}\big)\#b^{\tau}\Big) - \sum_{\tau \in S_{k-1}} \sum_{i=1}^{k} f^{[k]}\big(a_{(k+1)}\big)\#[b^{\tau}_{i-}, c, b^{\tau}_{i+}] \\
&= \sum_{\tau \in S_{k-1}} \left(\delta(b^{\tau}, c) - \sum_{i=1}^{k} f^{[k]}\big(a_{(k+1)}\big)\#[b^{\tau}_{i-}, c, b^{\tau}_{i+}]\right) \\
&= \sum_{\tau \in S_{k-1}} \sum_{i=1}^{k} \Big(f^{[k]}\big((a+c)_{(i)}, a_{(k+1-i)}\big)\#[b^{\tau}_{i-}, c, b^{\tau}_{i+}] - f^{[k]}\big(a_{(k+1)}\big)\#[b^{\tau}_{i-}, c, b^{\tau}_{i+}]\Big).
\end{aligned}$$

It follows from Lemma 6.6.11 that

$$\frac{\|\varepsilon(\cdot, c)\|_{B_{k-1}(\mathcal{I}^{k-1}_{\mathrm{sa}};\mathcal{I})}}{\|c\|_{\mathcal{I}}} \leq (k-1)! \sum_{i=1}^{k} \big\|I^{(a+c)_{(i)}, a_{(k+1-i)}} f^{[k]} - I^{a_{(k+1)}} f^{[k]}\big\|_{B_k(\mathcal{I}^k;\mathcal{I})} \xrightarrow{\|c\|_{\mathcal{I}}} 0.$$

Writing $\tilde{b} \coloneqq (b_0, b_1, \ldots, b_{k-1})$, this proves

$$D^k_{\mathcal{I}} f(a)\big[\tilde{b}\big] = D^k f_{a,\mathcal{I}}(0)\big[\tilde{b}\big] = \sum_{\tau \in S_{k-1}} \sum_{i=1}^{k} f^{[k]}\big(a_{(k+1)}\big) \# [b^{\tau}_{i-}, b_0, b^{\tau}_{i+}] = \sum_{\pi \in S_k} f^{[k]}\big(a_{(k+1)}\big) \# \tilde{b}^{\pi},$$

as claimed. This completes the proof. □

Remark 6.6.17. Let H be a separable complex Hilbert space, let $(\mathcal{M} \subseteq B(H), \tau)$ be a semifinite von Neumann algebra, and let $(E, \|\cdot\|_E)$ be a separable symmetric Banach function space. In [dPS04, Thm. 5.16], it is proven that if $f \colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f^{[1]}$ admits a decomposition as in Definition 6.6.2 with only $\varphi_1(\cdot, \sigma), \varphi_2(\cdot, \sigma) \in BC(\mathbb{R})$ (i.e., these functions are not assumed to be operator continuous) and if $a \in S(\tau)_{\mathrm{sa}}$, then the map $E(\tau)_{\mathrm{sa}} \ni b \mapsto f(a+b) - f(a) \in E(\tau)_{\mathrm{sa}}$ is well defined and Gateaux differentiable at 0 with Gateaux derivatives expressible as double operator integrals involving $f^{[1]}$. In particular, this result applies when $E = L^p$ with $1 \leq p < \infty$. It is noted, however, in [dPS04, §1] that Fréchet differentiability does not generally hold in this setting. This is why we must work in the space $\big(\mathcal{E}(\tau), \|\cdot\|_{\mathcal{E}(\tau)}\big) = (E(\tau) \cap \mathcal{M}, \|\cdot\|_{E(\tau)\cap\mathcal{M}})$, e.g., $\mathcal{L}^p(\tau)$, instead of the space $(E(\tau), \|\cdot\|_{E(\tau)})$, e.g., $L^p(\tau)$, to prove positive results about Fréchet differentiability in this setting. (Also, our method, particularly the extra assumption of operator continuity in our decompositions, allows us to assume only that $a \;\eta\; \mathcal{M}_{\mathrm{sa}}$, i.e., we need not assume a is τ-measurable.) In short, the results in [dPS04] are, for good reason, of a different flavor than the results in the present paper.

6.7 Comments about property (F)

A Banach ideal $(\mathcal{I}, \|\cdot\|_{\mathcal{I}}) \trianglelefteq \mathcal{M}$ has (the **sequential**) **property (F)** if whenever $a \in \mathcal{M}$ and $(a_i)_{i \in I}$ is a net (sequence) in $\mathcal{I}$ such that $\sup_{i \in I} \|a_i\|_{\mathcal{I}} < \infty$ and $a_i \to a$ in the S*OT, we have that $a \in \mathcal{I}$ and $\|a\|_{\mathcal{I}} \leq \sup_{i \in I} \|a_i\|_{\mathcal{I}}$. In [ACDS09], certain MOIs in invariant operator ideals with property (F) are considered. We now take some time to discuss the relationship between properties (M) and (F). First, there are certainly ideals with property (M) that do not have property (F), e.g., the ideal of compact operators (Proposition 6.2.10). Second, as mentioned in [ACDS09], the motivating example of an invariant operator ideal with property (F) is an ideal induced via Fact 6.3.2 by a (nonzero) symmetric Banach function space with the Fatou property. By Theorem 6.4.6 and Example 6.3.6, such ideals have property (M). Third, the author is unaware of an example of a symmetrically normed ideal with property (F) that does not have property (M). It would be interesting to know if such an ideal exists.

In this context, it is worth discussing a technical issue in [ACDS09] with its treatment of operator-valued integrals. For the rest of this section, assume H is separable. It is implicitly assumed in the proof of (the second sentence of) [ACDS09, Lem. 4.6] that at least some form of the integral triangle inequality holds for $\|\cdot\|_{\mathcal{I}}$ when $\mathcal{I}$ has property (F). Specifically, it seems to be assumed that if $(\Omega, \mathscr{F}, \mu)$ is a finite measure space and $F \colon \Omega \to \mathcal{I} \subseteq \mathcal{M}$ is $\|\cdot\|_{\mathcal{I}}$-bounded and weak* measurable, then

$$\int_\Omega F \,\mathrm{d}\mu \in \mathcal{I} \quad\text{and}\quad \left\|\int_\Omega F\,\mathrm{d}\mu\right\|_{\mathcal{I}} \leq \int_\Omega \|F\|_{\mathcal{I}}\,\mathrm{d}\mu$$

(ignoring that $\|F\|_{\mathcal{I}}$ may not be measurable). Let us call this the **finite property (M)**. Then we may rephrase the implicit claim as "property (F) implies the finite property (M)." As far as the author can tell, the arguments in [ACDS09] are only sufficient to prove

$$\int_\Omega F \,\mathrm{d}\mu \in \mathcal{I} \quad\text{and}\quad \left\|\int_\Omega F\,\mathrm{d}\mu\right\|_{\mathcal{I}} \leq \mu(\Omega) \sup_{\omega\in\Omega} \|F(\omega)\|_{\mathcal{I}}.$$

Indeed, the authors of [ACDS09] prove that $\mathcal{I}$ has property (F) if and only if $\{r \in \mathcal{I} : \|r\|_{\mathcal{I}} \leq 1\}$ is a Polish space in the S*OT and then apply [VTC87, Props. I.1.9 & I.1.10] to approximate F by simple maps in the S*OT. Crucially, [VTC87, Props. I.1.9 & I.1.10] only guarantee the existence of a sequence $(F_n)_{n\in\mathbb{N}}$ of simple maps $\Omega \to \mathcal{I}$ such that

$$\sup_{\omega\in\Omega} \|F_n(\omega)\|_{\mathcal{I}} \leq \sup_{\omega\in\Omega} \|F(\omega)\|_{\mathcal{I}}, \qquad n \in \mathbb{N},$$

and $F_n \to F$ pointwise in the S*OT as $n \to \infty$. Now, by the dominated convergence theorem (Lemma 6.5.1), $\int_\Omega F_n\,\mathrm{d}\mu \to \int_\Omega F\,\mathrm{d}\mu$ in the S*OT as $n \to \infty$. Also, by the (obvious) triangle inequality for integrals of simple maps,

$$\sup_{n\geq k}\left\|\int_\Omega F_n\,\mathrm{d}\mu\right\|_{\mathcal{I}} \leq \sup_{n\geq k}\int_\Omega \|F_n\|_{\mathcal{I}}\,\mathrm{d}\mu \leq \int_\Omega \sup_{n\geq k}\|F_n\|_{\mathcal{I}}\,\mathrm{d}\mu \leq \mu(\Omega)\sup_{\omega\in\Omega}\|F(\omega)\|_{\mathcal{I}}, \qquad k \in \mathbb{N}.$$

Thus, (the sequential) property (F) and the dominated convergence theorem give

$$\int_\Omega F\,\mathrm{d}\mu \in \mathcal{I} \quad\text{and}\quad \left\|\int_\Omega F\,\mathrm{d}\mu\right\|_{\mathcal{I}} \le \int_\Omega \limsup_{n\to\infty} \|F_n\|_{\mathcal{I}}\,\mathrm{d}\mu \le \mu(\Omega) \sup_{\omega\in\Omega} \|F(\omega)\|_{\mathcal{I}}. \tag{6.7.1}$$

The definition of property (F) does *not* guarantee that

$$\lim_{n\to\infty} \|F_n(\omega)\|_{\mathcal{I}} = \|F(\omega)\|_{\mathcal{I}},$$

so we cannot evaluate the limit superior above much further without an upgraded version of property (F). (Interestingly, this does not damage the applications in [ACDS09], since it seems only Inequality (6.7.1) is used seriously.) It therefore seems that property (F) *almost* implies some weaker form of property (M)—but perhaps not quite.

Remark 6.7.2. Though we centered the discussion above on the "finite property (M)," it is worth pointing out that, in order to prove [ACDS09, Lem. 4.6], it would actually suffice to know the following "finite integral symmetrically normed" condition: For every finite measure space $(\Omega, \mathscr{F}, \mu)$ and $\|\cdot\|$-bounded, weak* measurable $A, B\colon \Omega \to \mathcal{M}$, we have

$$\int_\Omega A(\omega)\,r\,B(\omega)\,\mu(\mathrm{d}\omega) \in \mathcal{I} \quad\text{and}\quad \left\|\int_\Omega A(\omega)\,r\,B(\omega)\,\mu(\mathrm{d}\omega)\right\|_{\mathcal{I}} \le \|r\|_{\mathcal{I}} \underline{\int_\Omega} \|A\|\,\|B\|\,\mathrm{d}\mu, \qquad r \in \mathcal{I}.$$

As mentioned, in the presence of property (F), we would already know $\int_\Omega A(\omega)\,r\,B(\omega)\,\mu(\mathrm{d}\omega) \in \mathcal{I}$, so, as was the case above, it is only the integral triangle inequality that is potentially missing.

6.8 Acknowledgment

Chapter 6, in part, is a reprint of the material as it appears in "Higher derivatives of operator functions in ideals of von Neumann algebras" (2023). Nikitopoulos, Evangelos A. *Journal of Mathematical Analysis and Applications*, 519, 126705.

Chapter 7

Application: Functional free Itô formula

In §3.8, we introduced a rich class $NC^k(\mathbb{R})$ of "noncommutative C^k" functions $\mathbb{R} \to \mathbb{C}$ whose operator functional calculus is k-times differentiable and has derivatives expressible in terms of multiple operator integrals (MOIs). In this chapter, we explore a connection between free stochastic calculus and the theory of MOIs by proving an Itô formula for noncommutative C^2 functions of self-adjoint free Itô processes. To do this, we first extend P. Biane and R. Speicher's theory of free stochastic calculus, including their free Itô formula for polynomials, to allow free Itô processes driven by multidimensional semicircular Brownian motions. Then, in the self-adjoint case, we reinterpret the objects appearing in the free Itô formula for polynomials in terms of MOIs. This allows us to enlarge the class of functions for which one can formulate and prove a free Itô formula from the space originally considered by Biane and Speicher (the Wiener space $W_2(\mathbb{R})$) to the strictly larger space $NC^2(\mathbb{R})$. Along the way, we also obtain a useful "traced" Itô formula for arbitrary C^2 scalar functions of self-adjoint free Itô processes. Finally, as motivation, we study an Itô formula for C^2 scalar functions of $N \times N$ Hermitian matrix Itô processes.

Standing assumptions. Throughout, H is a complex Hilbert space, and $(\mathcal{M} \subseteq B(H), (\mathcal{M}_t)_{t \geq 0}, \tau)$ is a filtered W^*-probability space (§7.2). Unless otherwise specified, all vector spaces are complex.

7.1 Introduction

In [BS98], P. Biane and R. Speicher developed a theory of free stochastic calculus with respect to semicircular Brownian motion that has yielded many fruitful applications, e.g., to free SDEs [CDM05, Dem08, Gao06, Kar11], free entropy and transport [Voi99, BS01, Shl09, Dab14, DGS21, JLS22], analysis on Wigner space [BS98, KNPS12], and the calculation of Brown measures [DHK22, DH22, HH22, Ho22, HZ23, HH23]. In this chapter, we present an extension and reinterpretation of this free stochastic calculus that naturally connects the Itô-type formulas thereof to the theory of multiple operator integrals (MOIs, Chapter 5) via the class $NC^k(\mathbb{R})$ of noncommutative C^k functions (Definition 3.8.11) introduced in §3.8.

The chapter's main results (Theorems 7.6.6 and 7.7.9) are "free Itô formulas" for scalar functions of self-adjoint "free Itô processes" driven by an n-dimensional semicircular Brownian motion $(x_1, \ldots, x_n)$. As a consequence of the work of D. Voiculescu [Voi91], $(x_1, \ldots, x_n)$ is in a precise sense the large-N limit of an n-tuple $\big(X_1^{(N)}, \ldots, X_n^{(N)}\big)$ of independent Brownian motions on the space of $N \times N$ Hermitian matrices. Therefore, interesting formulas involving $(x_1, \ldots, x_n)$ are often best motivated by studying formulas involving $\big(X_1^{(N)}, \ldots, X_n^{(N)}\big)$ and then (formally or rigorously) taking $N \to \infty$. This certainly is true for our formulas. In §7.8, we study some independently interesting matrix stochastic calculus formulas that motivate the chapter's main results. To explain the appearance of MOIs, we discuss a special case of one of these formulas.

In this preliminary discussion and §7.8, we assume familiarity with the theory of continuous-time stochastic processes and stochastic integration, though these subjects are not used elsewhere in the chapter. Please see [CW90, KS91] for some relevant background. Fix a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, P)$, with filtration satisfying the usual conditions, to which all processes we discuss will be adapted.

We begin by recalling the statement of Itô's formula from classical stochastic analysis. Let V and W be finite-dimensional inner product spaces, and let $M = (M(t))_{t \geq 0}$ be a continuous V-valued semimartingale. Itô's formula says that if $F \in C^2(V; W)$, then

$$\mathrm{d}F(M(t)) = DF(M(t))[\mathrm{d}M(t)] + \frac{1}{2} D^2 F(M(t))[\mathrm{d}M(t), \mathrm{d}M(t)], \tag{7.1.1}$$

where $D^k F$ is the k^{th} Fréchet derivative of F. The $DF(M)[\mathrm{d}M]$ term in Equation (7.1.1) is the differential notation for the stochastic integral against M of the process $DF(M)$, which takes values in $\mathrm{Hom}(V; W) = \{\text{linear maps } V \to W\}$. The notation for the second term (the "Itô correction term") in Equation (7.1.1) is to be understood as follows. Let $e_1, \ldots, e_n \in V$ be a basis for V, and write $M = \sum_{i=1}^n M_i e_i$. Then

$$\int_0^t D^2F(M(s))[\mathrm{d}M(s), \mathrm{d}M(s)] = \sum_{i,j=1}^n \int_0^t \underbrace{D^2F(M(s))[e_i, e_j]}_{\partial_{e_j}\partial_{e_i}F(M(s))} \mathrm{d}M_i(s)\,\mathrm{d}M_j(s),$$

where $\mathrm{d}M_i(s)\,\mathrm{d}M_j(s) = d\langle\langle M_i, M_j\rangle\rangle(s)$ denotes Stieltjes integration against the quadratic covariation $\langle\langle M_i, M_j\rangle\rangle$ of M_i and M_j. Our present motivation is an application of Equation (7.1.1) to matrix-valued processes M and maps F arising from scalar functional calculus.

Notation 7.1.2. Define $\langle A, B\rangle_N \coloneqq N\,\mathrm{Tr}(B^*A) = N^2\,\mathrm{tr}(B^*A)$ for all $A, B \in \mathrm{M}_N(\mathbb{C})$, where $\mathrm{tr} = N^{-1}\,\mathrm{Tr}$ is the normalized trace. Also, if $A \in \mathrm{M}_N(\mathbb{C})$ and $\lambda \in \sigma(A)$, then $P_\lambda^A \in \mathrm{M}_N(\mathbb{C})$ is the orthogonal projection onto the λ-eigenspace of A.

Note that $\langle\cdot,\cdot\rangle_N$ restricts to a real inner product on the real vector space $\mathrm{M}_N(\mathbb{C})_{\mathrm{sa}}$. Now, let $\big(X_1^{(N)}, \ldots, X_n^{(N)}\big) = (X_1, \ldots, X_n)$ be an n-tuple of independent standard $(\mathrm{M}_N(\mathbb{C})_{\mathrm{sa}}, \langle\cdot,\cdot\rangle_N)$-valued Brownian motions, and let M be a $\mathrm{M}_N(\mathbb{C})$-valued stochastic process satisfying

$$\mathrm{d}M(t) = \sum_{i=1}^n \sum_{j=1}^\ell A_{ij}(t)\,\mathrm{d}X_i(t)\,B_{ij}(t) + K(t)\,\mathrm{d}t \tag{7.1.3}$$

for some continuous adapted $\mathrm{M}_N(\mathbb{C})$-valued processes A_{ij}, B_{ij}, K. The term $A_{ij}(t)\,\mathrm{d}X_i(t)\,B_{ij}(t)$ above is the differential notation for the stochastic integral against X_i of the process

$$[0,\infty) \times \Omega \ni (t,\omega) \mapsto (E \mapsto A_{ij}(t,\omega)\,E\,B_{ij}(t,\omega)) \in \mathrm{End}(\mathrm{M}_N(\mathbb{C})) = \mathrm{Hom}(\mathrm{M}_N(\mathbb{C}); \mathrm{M}_N(\mathbb{C})).$$

Such processes M are special kinds of $N \times N$ "matrix Itô processes" (Definition 7.8.3).

Theorem 7.1.4. *If M is as in Equation* (7.1.3), *$M^* = M$, and $f \in C^2(\mathbb{R})$, then*

$$\mathrm{d}f(M(t)) = \sum_{\lambda,\mu\in\sigma(M(t))} f^{[1]}(\lambda,\mu)\,P_\lambda^{M(t)}\,\mathrm{d}M(t)\,P_\mu^{M(t)} + \sum_{i=1}^n C_i(t)\,\mathrm{d}t,$$

where the process C_i above is given by

$$C_i = \sum_{j,k=1}^\ell \sum_{\lambda,\mu,\nu\in\sigma(M)} f^{[2]}(\lambda,\mu,\nu)\big(P_\lambda^M A_{ij}\,\mathrm{tr}(B_{ij}P_\mu^M A_{ik})\,B_{ik}P_\nu^M + P_\lambda^M A_{ik}\,\mathrm{tr}(B_{ik}P_\mu^M A_{ij})\,B_{ij}P_\nu^M\big).$$

Remark 7.1.5. This is the special case of Theorem 7.8.13 with $U_i = \sum_{j=1}^{\ell} A_{ij} \otimes B_{ij}$.

This result is proven from Itô's formula using the quadratic covariation rules

$$A(t)\,\mathrm{d}X_i(t)\,B(t)\,\mathrm{d}X_j(t)\,C(t) = \delta_{ij}\,A(t)\,\operatorname{tr}(B(t))\,C(t)\,\mathrm{d}t \quad \text{and} \tag{7.1.6}$$

$$A(t)\,\mathrm{d}X_i(t)\,B(t)\,\mathrm{d}t\,C(t) = A(t)\,\mathrm{d}t\,B(t)\,\mathrm{d}X_i(t)\,C(t) = A(t)\,\mathrm{d}t\,B(t)\,\mathrm{d}t\,C(t) = 0 \tag{7.1.7}$$

and the identity (of Daletskii–Krein [DK56])

$$\partial_{B_k}\cdots\partial_{B_1} f_{\mathrm{M}_N(\mathbb{C})}(M) = \sum_{\pi\in S_k}\sum_{\boldsymbol{\lambda}\in\sigma(M)^{k+1}} f^{[k]}(\boldsymbol{\lambda})\,P_{\lambda_1}^M B_{\pi(1)}\cdots P_{\lambda_k}^M B_{\pi(k)} P_{\lambda_{k+1}}^M \tag{7.1.8}$$

at least for $k \in \{1,2\}$. One of the main results of this chapter is the formal large-N limit of (a generalization of) Theorem 7.1.4 that arises—at least heuristically—by taking $N \to \infty$ in Equations (7.1.6)–(7.1.8).

Biane noticed in [Bia97] that Voiculescu's results from [Voi91] imply that there exists a von Neumann algebra $\mathcal{M}$ with a (finite) trace $\tau\colon \mathcal{M} \to \mathbb{C}$ and "freely independent processes" $x_1,\ldots,x_n\colon [0,\infty) = \mathbb{R}_+ \to \mathcal{M}_{\mathrm{sa}}$ called semicircular Brownian motions such that

$$\operatorname{tr}\left(P\Big(X_{i_1}^{(N)}(t_1),\ldots,X_{i_r}^{(N)}(t_r)\Big)\right) \xrightarrow{N\to\infty} \tau(P(x_{i_1}(t_1),\ldots,x_{i_r}(t_r)))$$

almost surely (and in expectation) for all indices $i_1,\ldots,i_r \in \{1,\ldots,n\}$, times $t_1,\ldots,t_r \geq 0$, and polynomials P in r noncommuting indeterminates. Now, using Biane and Speicher's work from [BS98], one can make sense of stochastic differentials

$$a(t)\,\mathrm{d}x_i(t)\,b(t)$$

when $a,b\colon \mathbb{R}_+ \to \mathcal{M}$ are "continuous adapted processes." Imagining then a situation in which

$$(A,B,C) = \big(A^{(N)},B^{(N)},C^{(N)}\big) \text{ “} \xrightarrow{N\to\infty} \text{” } (a,b,c),$$

we might expect to be able to take $N \to \infty$ in Equations (7.1.6) and (7.1.7) and thereby to get quadratic covariation rules

$$a(t)\,\mathrm{d}x_i(t)\,b(t)\,\mathrm{d}x_j(t)\,c(t) = \delta_{ij}\,a(t)\,\tau(b(t))\,c(t)\,\mathrm{d}t \quad \text{and} \tag{7.1.9}$$

$$a(t)\,\mathrm{d}x_i(t)\,b(t)\,\mathrm{d}t\,c(t) = a(t)\,\mathrm{d}t\,b(t)\,\mathrm{d}x_i(t)\,c(t) = a(t)\,\mathrm{d}t\,b(t)\,\mathrm{d}t\,c(t) = 0. \tag{7.1.10}$$

Interpreted appropriately, these rules do hold (Theorem 7.4.9). How about Equation (7.1.8), as least with $k \in \{1, 2\}$? In this operator algebraic setting, we would be working with the **operator function** $f_{\mathcal{M}} \colon \mathcal{M}_{\mathrm{sa}} \to \mathcal{M}$ defined via the functional calculus by

$$\mathcal{M}_{\mathrm{sa}} \ni m \mapsto f(m) = \int_{\sigma(m)} f\,\mathrm{d}P^m \in \mathcal{M},$$

where P^m is the projection-valued spectral measure of m (§7.2). Therefore, it would be appropriate to guess that we should replace the sums $\sum_{\lambda\in\sigma(M)} \cdot\, P^M_\lambda$ in Equation (7.1.8) with integrals $\int_{\sigma(m)} \cdot\, \mathrm{d}P^m$. Explicitly, we might expect that if $f \in C^k(\mathbb{R})$, then $f_{\mathcal{M}} \in C^k(\mathcal{M}_{\mathrm{sa}}; \mathcal{M})$, and

$$D^k f_{\mathcal{M}}(m)[b] = \sum_{\pi\in S_k} \underbrace{\int_{\sigma(m)} \cdots \int_{\sigma(m)}}_{k+1 \text{ times}} f^{[k]}(\boldsymbol{\lambda})\,P^m(\mathrm{d}\lambda_1)\,b_{\pi(1)} \cdots P^m(\mathrm{d}\lambda_k)\,b_{\pi(k)}\,P^m(\mathrm{d}\lambda_{k+1}), \tag{7.1.11}$$

where $b = (b_1, \ldots, b_k)$ above. (These integrals actually do not make sense with standard projection-valued measure theory. We shall ignore this subtlety for now.) Finally, consider a process $m \colon \mathbb{R}_+ \to \mathcal{M}$ satisfying

$$\mathrm{d}m(t) = \sum_{i=1}^{n}\sum_{j=1}^{\ell} a_{ij}(t)\,\mathrm{d}x_i(t)\,b_{ij}(t) + k(t)\,\mathrm{d}t \tag{7.1.12}$$

for some continuous adapted processes $a_{ij}, b_{ij}, k \colon \mathbb{R}_+ \to \mathcal{M}$. Such processes m are special kinds of "free Itô processes" (Definition 7.4.1). Formally combining Equations (7.1.9)–(7.1.11) and applying the hypothetical Itô formula

$$\mathrm{d}f_{\mathcal{M}}(m(t)) = Df_{\mathcal{M}}(m(t))[\mathrm{d}m(t)] + \frac{1}{2}D^2 f_{\mathcal{M}}(m(t))[\mathrm{d}m(t), \mathrm{d}m(t)]$$

then gives the following guess.

Pseudotheorem 7.1.13. *If m is as in Equation* (7.1.12)*, $m^* = m$, and $f \in C^2(\mathbb{R})$, then*

$$\mathrm{d}f(m(t)) = \int_{\sigma(m(t))}\int_{\sigma(m(t))} f^{[1]}(\lambda,\mu)\, P^{m(t)}(\mathrm{d}\lambda)\,\mathrm{d}m(t)\, P^{m(t)}(\mathrm{d}\mu) + \sum_{i=1}^{n} c_i(t)\,\mathrm{d}t,$$

where

$$c_i = \sum_{j,k=1}^{\ell} \int_{\sigma(m)}\int_{\sigma(m)}\int_{\sigma(m)} f^{[2]}(\lambda,\mu,\nu)\,\big(P^m(\mathrm{d}\lambda)\,a_{ij}\tau(b_{ij}\,P^m(\mathrm{d}\mu)\,a_{ik})\,b_{ik}\,P^m(d\nu) + P^m(\mathrm{d}\lambda)\,a_{ik}\tau(b_{ik}\,P^m(\mathrm{d}\mu)\,a_{ij})\,b_{ij}\,P^m(d\nu)\big).$$

As we hinted above, the integrals in Equation (7.1.11) and the pseudotheorem above are purely formal: *A priori*, it doesn't make sense to integrate operator-valued functions against projection-valued measures. In fact, this is precisely the (nontrivial) problem multiple operator integrals (MOIs) were invented to solve. However, even with the realization that an MOI is the right object to consider when interpreting Pseudotheorem 7.1.13, the relevant MOIs do not necessarily make sense for arbitrary $f \in C^2(\mathbb{R})$. This is where noncommutative C^2 functions come in. The space $NC^2(\mathbb{R}) \subseteq C^2(\mathbb{R})$ is essentially tailor-made to ensure that MOI expressions such as the ones above make sense and are well behaved. (For example, the derivative formula (7.1.11) is proven rigorously in §3.8 for $f \in NC^k(\mathbb{R})$.) The result is that we are able to turn Pseudotheorem 7.1.13 into a (special case of a) rigorous statement—Theorem 7.7.9—if we take $f \in NC^2(\mathbb{R})$. Moreover, we demonstrate in Example 7.7.17 that Theorem 7.7.9 generalizes and conceptually clarifies [BS98, Prop. 4.3.4], Biane and Speicher's free Itô formula for scalar functions in the Wiener space $W_2(\mathbb{R})$ (Definition 1.3.13).

We end this section by describing the structure of the chapter and summarizing our results. All the results are proven both for n-dimensional semicircular Brownian motions and n-dimensional circular Brownian motions. To ease the present exposition, we summarize only the statements in the semicircular case.

In §7.2, we review some terminology and relevant results from free probability theory, e.g., the concepts of filtered W^*-probability spaces and (semi)circular Brownian motions. In §7.3, we review some material on tensor products—most importantly, the von Neumann algebra tensor product $\bar{\otimes}$—and the construction from [BS98] of the free stochastic integral of certain "biprocesses" against semicircular Brownian motion. More specifically, if $(\mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \tau)$ is a filtered W^*-probability space and $x\colon \mathbb{R}_+ \to \mathcal{M}_{\mathrm{sa}}$ is a semicircular Brownian motion, then $\int_0^t u(s)\#\mathrm{d}x(s) \in \mathcal{M}_t$ is defined for certain maps $u\colon \mathbb{R}_+ \to \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, where $\mathcal{M}^{\mathrm{op}}$ is the opposite of $\mathcal{M}$. The # stands for the operation determined by $(a \otimes b)\#c = acb$, and the free stochastic integral $\int_0^t u(s)\#\mathrm{d}x(s)$ is determined in an appropriate sense by

$$\int_0^t (1_{[r_1,r_2)} a \otimes b)(s)\#\mathrm{d}x(s) = (a \otimes b)\#[x(r_1 \wedge t) - x(r_2 \wedge t)] = a(x(r_1 \wedge t) - x(r_2 \wedge t))b$$

whenever $r_1 \leq r_2$ and $a, b \in \mathcal{M}_{r_1}$. Now, fix an n-dimensional semicircular Brownian motion $(x_1, \ldots, x_n)\colon \mathbb{R}_+ \to \mathcal{M}_{\mathrm{sa}}^n$. In §7.4, we define a free Itô process (Definition 7.4.1) as a process $m\colon \mathbb{R}_+ \to \mathcal{M}$ that satisfies (the integral form of) an equation

$$\mathrm{d}m(t) = \sum_{i=1}^n u_i(t)\#\mathrm{d}x_i(t) + k(t)\,\mathrm{d}t \tag{7.1.14}$$

for biprocesses $u_1, \ldots, u_n\colon \mathbb{R}_+ \to \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ and a process $k\colon \mathbb{R}_+ \to \mathcal{M}$. Then we prove a product rule for free Itô processes (Theorem 7.4.9) that makes the quadratic covariation rules (7.1.9) and (7.1.10) rigorous. This product rule is a "well-known" generalization of Biane and Speicher's product formula (the $n = 1$ case, [BS98, Thm. 4.1.2]). It is "well known" in the sense that it is used regularly in the literature, and it was proven in the "concrete" setting (the Cuntz algebra) as [KS92, Thm. 5]. However, it seems that, until now, the literature lacks a full proof of this formula in the present "abstract Wigner space" setting.

In §7.5, we define noncommutative derivatives $\partial^k p$ of polynomials; $\partial^1 p$ corresponds to Voiculescu's free difference quotient from [Voi00]. Then we use the free Itô product rule to prove a "functional" Itô formula for polynomials of free Itô processes (Theorem 7.5.7), which says that if m is a free Itô process satisfying Equation (7.1.14), then

$$\mathrm{d}p(m(t)) = \partial p(m(t))\#\mathrm{d}m(t) + \frac{1}{2}\sum_{i=1}^n \Delta_{u_i(t)} p(m(t))\,\mathrm{d}t,$$

where $\Delta_u p(m)$ is defined (Notation 7.5.3 and Definition 7.5.5) in terms of $\partial^2 p$. This formula generalizes [BS98, Prop. 4.3.2] (the $n = 1$ case). Our first main result then comes in §7.6, where we use the free Itô formula for polynomials, some beautiful symmetry properties of the objects in the formula, and an approximation argument to prove a "traced" Itô formula (Theorem 7.6.6) for *all* C^2 functions of self-adjoint free Itô processes. (The aforementioned symmetry properties allow one to avoid the MOI-related complications mentioned earlier.) The formula says that if 1) m is a free Itô process satisfying Equation (7.1.14), 2) $m^* = m$, and 3) $f \colon \mathbb{R} \to \mathbb{C}$ is a function that is C^2 on a neighborhood of the closure of $\bigcup_{t \geq 0} \sigma(m(t))$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau(f(m(t))) = \tau\big(f'(m(t))\,k(t)\big) + \frac{1}{2}\sum_{i=1}^{n}\int_{\mathbb{R}^2}\frac{f'(\lambda)-f'(\mu)}{\lambda-\mu}\,\rho_{m(t),u_i(t)}(\mathrm{d}\lambda,\mathrm{d}\mu),$$

where ρ_{m,u_i} is the finite Borel measure on $\mathbb{R}^2$ determined by

$$\int_{\mathbb{R}^2}\lambda^{j_1}\mu^{j_2}\,\rho_{m,u_i}(\mathrm{d}\lambda,\mathrm{d}\mu) = \langle (m^{j_1}\otimes m^{j_2})u_i, u_i\rangle_{L^2(\tau\bar{\otimes}\tau^{\mathrm{op}})} = (\tau\bar{\otimes}\tau^{\mathrm{op}})(u_i^*(m^{j_1}\otimes m^{j_2})u_i), \quad j_1, j_2 \in \mathbb{N}_0.$$

The result is not stated in exactly this way, but this interpretation is derived in Remark 7.6.9. As an application, we demonstrate in Example 7.6.10 how to use Theorem 7.6.6 to give simple, computationally transparent (re-)proofs of some key identities from [DHK22, HZ23, DH22, HH22] that are used in the computation of Brown measures of solutions to various free SDEs. The original proofs of these identities proceeded via rather unintuitive power series arguments, and understanding what was really happening in these arguments was the original motivation for the present study of functional free Itô formulas. We note that Theorem 7.6.6 is also motivated in the §7.8; the corresponding matrix stochastic calculus formula is given in Corollary 7.8.15.

Finally, we arrive to §7.7, which contains our second main result: the functional free Itô formula for noncommutative C^2 functions (Theorem 7.7.9), a generalization of the rigorous version of Pseudotheorem 7.1.13 and an extension—in the self-adjoint case—of the free Itô formula for polynomials to functions in $NC^2(\mathbb{R})$. It says that if 1) m is a free Itô process satisfying Equation (7.1.14), 2) $m^* = m$, and 3) $f \in NC^2(\mathbb{R})$, then

$$\mathrm{d}f(m(t)) = \partial f(m(t))\#\mathrm{d}m(t) + \frac{1}{2}\sum_{i=1}^{n}\Delta_{u_i(t)}f(m(t))\,\mathrm{d}t, \tag{7.1.15}$$

where

$$\partial f(m(t))\#\mathrm{d}m(t) = \int_{\sigma(m(t))}\int_{\sigma(m(t))} f^{[1]}(\lambda,\mu)\, P^{m(t)}(\mathrm{d}\lambda)\,\mathrm{d}m(t)\, P^{m(t)}(\mathrm{d}\mu)$$

and $\Delta_u f(m)$ (defined officially in Definition 7.7.6) is determined, in a certain sense (Corollary 7.7.16 and Remark 7.7.18), as a quadratic form by

$$\frac{1}{2}\Delta_{a\otimes b} f(m) = \int_{\sigma(m)}\int_{\sigma(m)}\int_{\sigma(m)} f^{[2]}(\lambda_1,\lambda_2,\lambda_3)\, P^m(\mathrm{d}\lambda_1)\, a\, \tau(b\, P^m(\mathrm{d}\lambda_2)\, a)\, b\, P^m(\mathrm{d}\lambda_3), \quad a,b\in\mathcal{M}.$$

for $a,b\in\mathcal{M}$. Now, Biane and Speicher also established a formula [BS98, Prop. 4.3.4] for $f(m)$ when $f\in W_2(\mathbb{R})$ and m is a self-adjoint free Itô process driven by a single semicircular Brownian motion. In Example 7.7.17, we show that when $n=1$ and $f\in W_2(\mathbb{R})$, Equation (7.1.15) recovers Biane and Speicher's formula. Owing to the strict containment $W_k(\mathbb{R})_{\mathrm{loc}}\subsetneq NC^k(\mathbb{R})$ (Theorem 3.7.1), this means that not only have we extended Biane and Speicher's formula to the case $n>1$, but we have also, through the use of MOIs, meaningfully enlarged the class of functions for which it can be formulated.

7.2 Free probability

In this section, we discuss some basic definitions and facts about free probability, noncommutative L^p-spaces, noncommutative martingales, and free Brownian motions. We assume the reader is familiar with these, and we recall only what is necessary for the present application. For a proper treatment of the basics of free probability, please see [NS06, MS17].

A **$*$-probability space** is a pair $(\mathcal{A},\varphi)$, where $\mathcal{A}$ is a unital $*$-algebra and $\varphi\colon\mathcal{A}\to\mathbb{C}$ is a **state**, i.e., φ is complex linear, unital ($\varphi(1)=1$), and positive ($\varphi(a^*a)\geq 0$ whenever $a\in\mathcal{A}$). A collection $(\mathcal{A}_i)_{i\in I}$ of (not necessarily $*$-)subalgebras of $\mathcal{A}$ is **freely independent** if $\varphi(a_1\cdots a_n)=0$ whenever $\varphi(a_1)=\cdots=\varphi(a_n)=0$ and $a_1\in\mathcal{A}_{i_1},\ldots,a_n\in\mathcal{A}_{i_n}$ with $i_1\neq i_2, i_2\neq i_3,\ \ldots, i_{n-2}\neq i_{n-1}, i_{n-1}\neq i_n$. When applied to elements or subsets of $\mathcal{A}$, the term "($*$-)freely independent" refers to the ($*$-)subalgebras these elements or subsets generate, e.g., $a\in\mathcal{A}$ and $S\subseteq\mathcal{A}$ are ($*$-)freely independent if the ($*$-)subalgebra generated by a is freely independent from the ($*$-)subalgebra generated by S.

A pair $(\mathcal{M}, \tau)$ is a **W^*-probability space** if $\mathcal{M}$ is a von Neumann algebra and $\tau\colon \mathcal{M} \to \mathbb{C}$ a **trace**, i.e., τ is a state that is tracial ($\tau(ab) = \tau(ba)$ for $a, b \in \mathcal{M}$), faithful ($\tau(a^*a) = 0$ implies $a = 0$), and normal (σ-WOT continuous). All $*$-probability spaces considered in this chapter are W^*-probability spaces. Please see [Dix81] for more information about von Neumann algebras.

Fix now a W^*-probability space $(\mathcal{M}, \tau)$. If $a \in \mathcal{M}$ is normal, i.e., $a^*a = aa^*$, then the **$*$-distribution** of a is the Borel probability measure $\mu_a(\mathrm{d}\lambda) := \tau(P^a(\mathrm{d}\lambda))$ on the spectrum $\sigma(a) \subseteq \mathbb{C}$ of a, where $P^a\colon \mathcal{B}_{\sigma(a)} \to \mathcal{M}$ is the projection-valued spectral measure of a. Recall $f(a) = \int_{\sigma(a)} f(\lambda)\, P^a(\mathrm{d}\lambda) = \int_{\sigma(a)} f\, \mathrm{d}P^a \in \mathcal{M}$ for all $f \in \ell^\infty(\sigma(a), \mathcal{B}_{\sigma(a)})$.

Write $\mu_0^{\mathrm{sc}} := \delta_0$ and

$$\mu_t^{\mathrm{sc}}(\mathrm{d}s) := \frac{1}{2\pi t}\sqrt{(4t - s^2)_+}\,\mathrm{d}s, \qquad t > 0,$$

for the semicircle distribution of variance t. Notice that if $t \geq 0$, then $\operatorname{supp}\mu_t^{\mathrm{sc}}$ is equal to $[-2\sqrt{t}, 2\sqrt{t}] \subseteq \mathbb{R}$, so that if $a \in \mathcal{M}$ is normal and has $*$-distribution μ_t^{sc}, then $a \in \mathcal{M}_{\mathrm{sa}}$. Such an element a is called a **semicircular element of variance t**. We call $b \in \mathcal{M}$ a **circular element of variance t** if $b = 2^{-1/2}(a_1 + ia_2)$ for two freely independent semicircular elements $a_1, a_2 \in \mathcal{M}_{\mathrm{sa}}$ of variance t. Since $-a_2$ is still semicircular, we have that if $b \in \mathcal{M}$ is a circular element of variance t, then b^* is as well.

It is worth mentioning that there is a more general algebraic/combinatorial definition of $*$-distribution, and one may define (semi)circular elements in a $*$-probability space in a more "intrinsic" way using the notion of free cumulants. Please see [NS06] for this approach. Since we do not need this combinatorial machinery, we content ourselves with the analytic definition.

Next, we turn to noncommutative L^p-spaces. Please see [dS18] for a detailed development of the basic properties of noncommutative L^p-spaces.

Notation 7.2.1 (Noncommutative L^p-spaces)**.** Let $(\mathcal{M}, \tau)$ be a W^*-probability space. Define $L^\infty(\mathcal{M}, \tau) := \mathcal{M}$ and $\|\cdot\|_{L^\infty(\tau)} := \|\cdot\|_{\mathcal{M}} = \|\cdot\|$. If $p \in [1, \infty)$, then we define

$$\|a\|_{L^p(\tau)} := \tau(|a|^p)^{\frac{1}{p}} = \tau\big((a^*a)^{\frac{p}{2}}\big)^{\frac{1}{p}}, \qquad a \in \mathcal{M},$$

and $L^p(\mathcal{M}, \tau)$ to be the completion of $\mathcal{M}$ with respect to the norm $\|\cdot\|_{L^p(\tau)}$.

Similar to the classical case, we have noncommutative Hölder's inequality:

$$\|a_1 \cdots a_n\|_{L^p(\tau)} \leq \|a_1\|_{L^{p_1}(\tau)} \cdots \|a_n\|_{L^{p_n}(\tau)}$$

whenever $a_1, \ldots, a_n \in \mathcal{M}$ and $p_1, \ldots, p_n, p \in [1, \infty]$ and $1/p_1 + \cdots + 1/p_n = 1/p$. This allows us to extend multiplication to a bounded n-linear map $L^{p_1}(\mathcal{M}, \tau) \times \cdots \times L^{p_n}(\mathcal{M}, \tau) \to L^p(\mathcal{M}, \tau)$. In addition, there is a dual characterization of the noncommutative L^p-norm:

$$\|a\|_{L^p(\tau)} = \sup\{\tau(ab) : b \in \mathcal{M}, \|b\|_{L^q(\tau)} \leq 1\}, \qquad a \in \mathcal{M},$$

whenever $1/p + 1/q = 1$. This leads to the duality relationship $L^q(\mathcal{M}, \tau) \cong L^p(\mathcal{M}, \tau)^*$, via the map $a \mapsto (b \mapsto \tau(ab))$, when $1/p + 1/q = 1$ and $p \neq \infty$, as in the classical case. Moreover, the σ-WOT on $\mathcal{M}$ coincides with the weak* topology on $L^1(\mathcal{M}, \tau)^* \cong L^\infty(\mathcal{M}, \tau) = \mathcal{M}$.

Finally, we briefly discuss noncommutative martingales and free Brownian motions. For this, we recall that if $\mathcal{N} \subseteq \mathcal{M}$ is a $\boldsymbol{W^*}$**-subalgebra**, i.e., a WOT-closed $*$-subalgebra, then there exists a unique positive linear map $\tau[\,\cdot \mid \mathcal{N}] \colon \mathcal{M} \to \mathcal{N}$ such that $\tau[b_1 a b_2 \mid \mathcal{N}] = b_1 \tau[a \mid \mathcal{N}] b_2$ for all $a \in \mathcal{M}$ and $b_1, b_2 \in \mathcal{N}$. We call $\tau[\,\cdot \mid \mathcal{N}]$ the **conditional expectation onto** $\boldsymbol{\mathcal{N}}$. It was introduced in [Tak72]. It extends to a (weak) contraction $L^p(\mathcal{M}, \tau) \to L^p(\mathcal{N}, \tau)$ for all $p \in [1, \infty]$. When $p = 2$, we get the orthogonal projection of $L^2(\mathcal{M}, \tau)$ onto $L^2(\mathcal{N}, \tau) \subseteq L^2(\mathcal{M}, \tau)$. In particular, as it is often useful to remember, if $a \in \mathcal{M}$ and $b \in \mathcal{N}$, then $b = \tau[a \mid \mathcal{N}]$ if and only if $\tau(b_0 a) = \tau(b_0 b)$ for all $b_0 \in \mathcal{N}$. This implies, for instance, that if a is freely independent from $\mathcal{N}$, then $\tau[a \mid \mathcal{N}] = \tau(a)1 = \tau(a)$.

Now, an increasing collection $(\mathcal{M}_t)_{t \geq 0}$ of W^*-subalgebras of $\mathcal{M}$ is called a **filtration** of $\mathcal{M}$, and the triple $(\mathcal{M}, (\mathcal{M}_t)_{t \geq 0}, \tau)$ is called a **filtered** $\boldsymbol{W^*}$**-probability space**. Fix a filtration $(\mathcal{M}_t)_{t \geq 0}$ of $\mathcal{M}$ and $p \in [1, \infty]$. A $\boldsymbol{L^p}$**-process** $a = (a(t))_{t \geq 0} \colon \mathbb{R}_+ \to L^p(\mathcal{M}, \tau)$ is **adapted (to** $\boldsymbol{(\mathcal{M}_t)_{t \geq 0}}$**)** if $a(t) \in L^p(\mathcal{M}_t, \tau) \subseteq L^p(\mathcal{M}, \tau)$, for every $t \geq 0$. An adapted L^p-process $m \colon \mathbb{R}_+ \to L^p(\mathcal{M}, \tau)$ is called a **noncommutative** $\boldsymbol{L^p}$**-martingale (with respect to** $\boldsymbol{((\mathcal{M}_t)_{t \geq 0}, \tau)}$**)** if $\tau[m(t) \mid \mathcal{M}_s] = m(s)$ whenever $0 \leq s \leq t < \infty$. If $p = \infty$, then we shall omit the "L^p" from these terms.

Let $n \in \mathbb{N}$. An n-tuple $m = (m_1, \ldots, m_n) \colon \mathbb{R}_+ \to \mathcal{M}^n$ of adapted processes is called an **n-dimensional (semi)circular Brownian motion (in $(\mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \tau)$)** if $m(0) = 0$ and $\{m_i(t) - m_i(s) : 1 \leq i \leq n\}$ is a $*$-freely independent collection of (semi)circular elements of variance $t - s$ that is $*$-freely independent from $\mathcal{M}_s$ when $0 \leq s < t < \infty$. More concisely, $m(0) = 0$ and m has "jointly $*$-free (semi)circular increments." It follows from the comments about conditional expectation and the free increments property that (semi)circular Brownian motion is a noncommutative martingale. Also, if m is an n-dimensional circular Brownian motion, then the process $\sqrt{2}(\operatorname{Re} m, \operatorname{Im} m) = 2^{-1/2}(m + m^*, -i(m - m^*))$ is a $2n$-dimensional semicircular Brownian motion.

7.3 Free stochastic integrals

In this section, we review Biane and Speicher's construction from [BS98] of the free stochastic integral against (semi)circular Brownian motion. We begin by reviewing some information about the minimal C^*-tensor product $\otimes_{\min}$ and von Neumann algebra tensor product $\bar{\otimes}$. Recall that $\otimes_2$ is the Hilbert space tensor product (§5.9).

Though we assume the reader has some familiarity with $\otimes_{\min}$ and $\bar{\otimes}$, we recall their definitions—at least for two tensorands—for convenience. Let H and K be complex Hilbert spaces. Recall that the natural map $B(H) \otimes B(K) \to B(H \otimes_2 K)$ is an injective, unital $*$-homomorphism when $B(H) \otimes B(K)$ is given the tensor product $*$-algebra structure, so we view $B(H) \otimes B(K)$ as a $*$-subalgebra of $B(H \otimes_2 K)$. In particular, if $\mathcal{A} \subseteq B(H)$ and $\mathcal{B} \subseteq B(K)$ are C^*-algebras, then we may view $\mathcal{A} \otimes \mathcal{B}$ as a $*$-subalgebra of $B(H \otimes_2 K)$. The **minimal C^*-tensor product** $\mathcal{A} \otimes_{\min} \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is the operator norm closure of $\mathcal{A} \otimes \mathcal{B}$ in $B(H \otimes_2 K)$. If, in addition, $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, then the **von Neumann algebra tensor product** $\mathcal{A} \bar{\otimes} \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is the WOT closure—equivalently, by Kaplansky's density theorem, the σ-WOT closure—of $\mathcal{A} \otimes \mathcal{B}$ in $B(H \otimes_2 K)$. If $\tau_1 \colon \mathcal{A} \to \mathbb{C}$ and $\tau_2 \colon \mathcal{B} \to \mathbb{C}$ are traces, then we write $\tau_1 \bar{\otimes} \tau_2 \colon \mathcal{A} \bar{\otimes} \mathcal{B} \to \mathbb{C}$ for the **tensor product trace**, which is uniquely determined by

$$(\tau_1 \bar{\otimes} \tau_2)(a \otimes b) = \tau_1(a)\, \tau_2(b), \qquad a \in \mathcal{A},\ b \in \mathcal{B}.$$

For more information on $\otimes_{\min}$ and $\bar{\otimes}$, please see [BO08, Ch. 3] or [KR97b, Ch. 11].

Proposition 7.3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. If $\iota_{\min}\colon \mathcal{A}\hat{\otimes}_\pi\mathcal{B} \to \mathcal{A}\otimes_{\min}\mathcal{B}$ is the map induced via the universal property of $\hat{\otimes}_\pi$ by the inclusion $\mathcal{A}\otimes\mathcal{B}\hookrightarrow\mathcal{A}\otimes_{\min}\mathcal{B}$, then $\iota_{\min}$ is injective.*

This follows from [Haa85, Prop. 2.2] and the remark following it. From Proposition 7.3.1 and Theorem 1.5.10, we see that if $\mathcal{A}\subseteq B(H)$ and $\mathcal{B}\subseteq B(K)$ are C^*-algebras, then $\mathcal{A}\hat{\otimes}_\pi\mathcal{B}$ can be represented as the subalgebra of $B(H\otimes_2 K)$ of elements $u\in B(H\otimes_2 K)$ admitting a decomposition $u=\sum_{n=1}^\infty a_n\otimes b_n\in B(H\otimes_2 K)$ such that $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$, $(b_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{B}$, and $\sum_{n=1}^\infty \|a_n\|_{B(H)}\|b_n\|_{B(K)}<\infty$. In particular, we have the chain of inclusions $\mathcal{A}\otimes\mathcal{B}\subseteq\mathcal{A}\hat{\otimes}_\pi\mathcal{B}\subseteq\mathcal{A}\otimes_{\min}\mathcal{B}\subseteq B(H\otimes_2 K)$.

Next, we set notation for a few useful algebraic operations.

Notation 7.3.2 (Algebraic operations)**.** Recall that H is a complex Hilbert space and $\mathcal{M}\subseteq B(H)$ is a von Neumann algebra.

(i) $\mathcal{M}^{\mathrm{op}}$ is the opposite von Neumann algebra of $\mathcal{M}$, i.e., the von Neumann algebra with the same addition, $*$-operation, and topological structure as $\mathcal{M}$ but the opposite multiplication operation $a\cdot b\coloneqq ba$. If $\tau\colon\mathcal{M}\to\mathbb{C}$ is a trace, then $\tau^{\mathrm{op}}\colon\mathcal{M}^{\mathrm{op}}\to\mathbb{C}$ is the induced trace on $\mathcal{M}^{\mathrm{op}}$ induced by τ.

(ii) $(\cdot)^{\mathrm{flip}}\colon\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}\to\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ is the unique σ-WOT continuous (and isometric) linear map determined by $(a\otimes b)^{\mathrm{flip}}=b\otimes a$. Also, $u^\star\coloneqq(u^*)^{\mathrm{flip}}$ for all $u\in\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, where $(\cdot)^*$ denotes the standard tensor product $*$-operation on $\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ (e.g., $(a\otimes b)^*=a^*\otimes b^*$).

(iii) $\#\colon\mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}}\to B(\mathcal{M})$ is the bounded linear map—actually, algebra homomorphism—determined by $\#(a\otimes b)c=acb$. Write $u\#c\coloneqq\#(u)c$ for all $u\in\mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}}$ and $c\in\mathcal{M}$. Note that if $u\in\mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}}\subseteq\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, then $u^*,u^{\mathrm{flip}},u^\star\in\mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}}$ and $(u\#c)^*=(u^\star)\#c^*$ for all $c\in\mathcal{M}$.

(iv) (Not used until §7.5) $\#_2^\otimes\colon(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})^{\hat{\otimes}_\pi 3}\to B_2((\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})^2;\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})$ for the bounded linear map determined by

$$\#_2^\otimes(u_1\otimes u_2\otimes u_3)[v_1,v_2]=u_1v_1u_2v_2u_3,\qquad u_1,u_2,u_3,v_1,v_2\in\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}.$$

If $A\in(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})^{\hat{\otimes}_\pi 3}$ and $u,v\in\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, then $A\#_2^\otimes[u,v]\coloneqq\#_2^\otimes(A)[u,v]$.

Remark 7.3.3. If H is finite-dimensional and $\mathcal{M} = B(H)$, then one can use elementary linear algebra to show that $\#\colon \mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}} = \mathcal{M}\otimes\mathcal{M}^{\mathrm{op}} \to B(\mathcal{M})$ is a linear isomorphism. Furthermore, $\#$ is a $*$-homomorphism when $\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}}$ is given the tensor product $*$-operation and $B(\mathcal{M})$ is given the adjoint operation associated to the Hilbert–Schmidt inner product on $\mathcal{M} = B(H)$. This is why we have chosen to write $(\cdot)^*$ for the tensor product $*$-operation on $\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$; in [BS98], the symbol $(\cdot)^*$ is used for the operation $(\cdot)^\star$ from (ii).

Some justification is in order for what is written in the first two items above. First, we observe that $\mathcal{M}^{\mathrm{op}}$ is, indeed, a von Neumann algebra. Abstractly, $\mathcal{M}^{\mathrm{op}}$ is clearly a C^*-algebra with a predual (the same predual as $\mathcal{M}$). Concretely, $\mathcal{M}^{\mathrm{op}}$ can be represented on the dual H^* of H via the transpose map $B(H) \ni a \mapsto (H^* \ni \ell \mapsto \ell\circ a \in H^*) \in B(H^*)$. This map is a $*$-anti-homomorphism that is a homeomorphism with respect to the WOT and the σ-WOT, so the image of $\mathcal{M}$ under the transpose map is a von Neumann algebra isomorphic to $\mathcal{M}^{\mathrm{op}}$. Next, using this representation of $\mathcal{M}^{\mathrm{op}}$, we confirm that $(\cdot)^{\mathrm{flip}}$ is well defined. Certainly, the condition in the definition determines a linear map $(\cdot)^{\mathrm{flip}}\colon \mathcal{M}\otimes\mathcal{M}^{\mathrm{op}} \to \mathcal{M}\otimes\mathcal{M}^{\mathrm{op}}$. What remains to be confirmed is that the latter linear map is σ-WOT continuous and isometric. To see this, write $(\cdot)^{\mathfrak{f}}\colon H\otimes_2 H^* \to H\otimes_2 H^*$ for the conjugate-linear surjective isometry determined by $h\otimes\langle\cdot,k\rangle \mapsto k\otimes\langle\cdot,h\rangle$. Then it is easy to show that

$$\langle u^{\mathrm{flip}}\xi,\eta\rangle_{H\otimes_2 H^*} = \left\langle u\eta^{\mathfrak{f}},\xi^{\mathfrak{f}}\right\rangle_{H\otimes_2 H^*}, \qquad u\in\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}} \subseteq B(H\otimes_2 H^*),\ \xi,\eta\in H\otimes_2 H^*.$$

This implies both desired conclusions.

Next, we define simple biprocesses and their integrals against arbitrary functions. Recall that $(\mathcal{M},(\mathcal{M}_t)_{t\geq 0},\tau)$ is a fixed filtered W^*-probability space.

Definition 7.3.4 (Biprocesses)**.** A **biprocess** is a map $u\colon \mathbb{R}_+ \to \mathcal{M}\otimes\mathcal{M}^{\mathrm{op}}$. If $u(t)\in\mathcal{M}_t\otimes\mathcal{M}_t^{\mathrm{op}}$ for all $t\geq 0$, then u is **adapted**. If there exists a finite partition $0 = t_0 < t_1 < \cdots < t_n < \infty$ of $\mathbb{R}_+$ such that for all $i\in\{1,\ldots,n\}$, u is constant on $[t_{i-1},t_i)$, and $u(t) = 0$ whenever $t\geq t_n$, then u is **simple**. We write $\mathbb{S}$ for the space of simple biprocesses and $\mathbb{S}_{\mathrm{a}}\subseteq\mathbb{S}$ for the subspace of simple adapted biprocesses.

Notation 7.3.5 (Integrals of simple biprocesses)**.** If $u \in \mathbb{S}$, then

$$u = \sum_{i=1}^{n} 1_{[t_{i-1},t_i)} u(t_{i-1})$$

for some partition $0 = t_0 < \cdots < t_n < \infty$. If $m\colon \mathbb{R}_+ \to \mathcal{M}$ is any function, then we define

$$\int_0^\infty u(t)\#\mathrm{d}m(t) = \int_0^\infty u\#\mathrm{d}m \coloneqq \sum_{i=1}^{n} u(t_{i-1})\#[m(t_i) - m(t_{i-1})] \in \mathcal{M}.$$

By standard arguments (from scratch or using the basic theory of finitely additive vector measures), $\int_0^\infty u\#\mathrm{d}m \in \mathcal{M}$ does not depend on the chosen decomposition of u, and the map $\mathbb{S} \ni u \mapsto \int_0^\infty u\#\mathrm{d}m \in \mathcal{M}$ is linear.

Note that if $u \in \mathbb{S}$ and $0 \leq r \leq s$, then $1_{[r,s)}u \in \mathbb{S}$ and $u^\star \in \mathbb{S}$. Thus, the statement of the lemma below makes sense. Its proof is left to the reader,

Lemma 7.3.6 (Properties of integrals of simple biprocesses)**.** *Let $m\colon \mathbb{R}_+ \to \mathcal{M}$ be any function, let $u \in \mathbb{S}$, and suppose $0 \leq r \leq s$. Define*

$$\int_r^s u(t)\#\mathrm{d}m(t) = \int_r^s u\#\mathrm{d}m \coloneqq \int_0^\infty (1_{[r,s)}u)\#\mathrm{d}m \in \mathcal{M}.$$

Then

(i) *the map $\mathbb{S} \ni u \mapsto \int_r^s u(t)\#\mathrm{d}m(t) \in \mathcal{M}$ is linear;*

(ii) $\big(\int_r^s u\#\mathrm{d}m\big)^* = \int_r^s u^\star\#\mathrm{d}m^*$*;*

(iii) *if $u \in \mathbb{S}_{\mathrm{a}}$ and m is adapted, then $\int_0^\cdot u\#\mathrm{d}m \coloneqq \big(\int_0^t u\#\mathrm{d}m\big)_{t\geq 0}$ is adapted;*

(iv) *if $u(t) = 0$ for all $t \geq s$, then $\int_r^{s_1} u(t)\#\mathrm{d}m(t) = \int_r^{s_2} u(t)\#\mathrm{d}m(t)$ for all $s_1, s_2 \geq s$; and*

(v) $\int_r^s u(t)\#\mathrm{d}m(t) = \int_0^s u(t)\#\mathrm{d}m(t) - \int_0^r u(t)\#\mathrm{d}m(t)$*.*

Next, we introduce a larger space of integrands for the case when m is a (semi)circular Brownian motion. Notice that a simple biprocess $u\colon \mathbb{R}_+ \to \mathcal{M}\otimes\mathcal{M}^{\mathrm{op}} \subseteq L^p(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}, \tau\bar{\otimes}\tau^{\mathrm{op}})$ is a compactly supported simple—in particular, strongly integrable—map $\mathbb{R}_+ \to L^p(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}, \tau\bar{\otimes}\tau^{\mathrm{op}})$ for all $p \in [1,\infty]$.

Notation 7.3.7. Fix $p, q \in [1, \infty]$, and let $(\mathcal{N}, \eta)$ be a W^*-probability space.

(i) If $u \in L^q_{\text{loc}}(\mathbb{R}_+; L^p(\mathcal{N}, \eta)) = L^q_{\text{loc}}(\mathbb{R}_+, \text{Lebesgue}; L^p(\mathcal{N}, \eta))$ and $t \geq 0$, then

$$\|u\|_{L^q_t L^p(\eta)} := \left(\int_0^t \|u(s)\|^q_{L^p(\eta)} \, \mathrm{d}s \right)^{\frac{1}{q}} \text{ and } \|u\|_{L^q L^p(\eta)} := \left(\int_0^\infty \|u(s)\|^q_{L^p(\eta)} \, \mathrm{d}s \right)^{\frac{1}{q}}$$

with the obvious modification for $q = \infty$. Of course, $\|\cdot\|_{L^2_t L^2(\eta)}$ comes from the "inner product" $\langle u, v \rangle_{L^2_t L^2(\eta)} = \int_0^t \langle u(s), v(s) \rangle_{L^2(\eta)} \, \mathrm{d}s$.

(ii) Define

$$\mathcal{L}^{2,p} := \overline{\mathbb{S}_{\text{a}}} \subseteq L^2(\mathbb{R}_+; L^p(\mathcal{M} \bar{\otimes} \mathcal{M}^{\text{op}}, \tau \bar{\otimes} \tau^{\text{op}})) \text{ and}$$
$$\Lambda^{2,p} := \overline{\mathbb{S}_{\text{a}}} \subseteq L^2_{\text{loc}}(\mathbb{R}_+; L^p(\mathcal{M} \bar{\otimes} \mathcal{M}^{\text{op}}, \tau \bar{\otimes} \tau^{\text{op}})),$$

where the first closure above takes place in the Banach space $L^2(\mathbb{R}_+; L^p(\mathcal{M} \bar{\otimes} \mathcal{M}^{\text{op}}, \tau \bar{\otimes} \tau^{\text{op}}))$ and the second takes place in the Fréchet space $L^2_{\text{loc}}(\mathbb{R}_+; L^p(\mathcal{M} \bar{\otimes} \mathcal{M}^{\text{op}}, \tau \bar{\otimes} \tau^{\text{op}}))$. We write

$$\mathcal{L}^2 := \mathcal{L}^{2,\infty} \subseteq L^2(\mathbb{R}_+; \mathcal{M} \bar{\otimes} \mathcal{M}^{\text{op}}) \text{ and}$$
$$\Lambda^2 := \Lambda^{2,\infty} \subseteq L^2_{\text{loc}}(\mathbb{R}_+; \mathcal{M} \bar{\otimes} \mathcal{M}^{\text{op}})$$

for the $p = \infty$ case.

To be clear, the L^q- and L^q_{loc}-spaces above are the Bochner L^q- and L^q_{loc}-spaces.

Remark 7.3.8. The use of $\mathcal{L}$ and Λ above is inspired by the notation used in [CW90] for the classical case. Biane and Speicher use the notation $\mathscr{B}^a_p$ in [BS98] for the space $\mathcal{L}^{2,p}$, though their definition is stated as an abstract completion of $\mathbb{S}_{\text{a}}$. Also, we note that simple biprocesses take values in $\mathcal{M} \otimes \mathcal{M}^{\text{op}} \subseteq \mathcal{M} \otimes_{\min} \mathcal{M}^{\text{op}}$, and $\mathcal{M} \otimes_{\min} \mathcal{M}^{\text{op}} \subseteq \mathcal{M} \bar{\otimes} \mathcal{M}^{\text{op}}$ is a norm-closed subspace. In particular, all the elements of Λ^2 actually take values (almost everywhere) in $\mathcal{M} \otimes_{\min} \mathcal{M}^{\text{op}}$. In other words, $\Lambda^2 \subseteq L^2_{\text{loc}}(\mathbb{R}_+; \mathcal{M} \otimes_{\min} \mathcal{M}^{\text{op}})$.

Only the case $p = \infty$ will matter to us in later sections. However, in the case $p = 2$, there is an Itô isometry, just as in the classical case. It says that if $x \colon \mathbb{R}_+ \to \mathcal{M}$ is a semicircular

Brownian motion (or, in fact, a circular Brownian motion), then

$$\left\langle \int_0^t u\#\mathrm{d}x, \int_0^t v\#\mathrm{d}x \right\rangle_{L^2(\tau)} = \langle u,v\rangle_{L_t^2L^2(\tau\bar{\otimes}\tau^{\mathrm{op}})}, \qquad u,v\in\mathbb{S}_{\mathrm{a}},\ t\ge 0.$$

Please see [BS98, Prop. 3.1.1]. We now focus on the $p=\infty$ case.

Theorem 7.3.9 (Biane–Speicher [BS98])**.** *Let $x\colon\mathbb{R}_+\to\mathcal{M}_{\mathrm{sa}}$ be a semicircular Brownian motion, and let $z\colon\mathbb{R}_+\to\mathcal{M}$ be a circular Brownian motion. Fix $u\in\mathbb{S}_{\mathrm{a}}$ and $m\in\{x,z,z^*\}$.*

(i) $\int_0^{\cdot} u\#\mathrm{d}m$ *is a noncommutative martingale.*

(ii) (L^∞-Burkholder–Davis–Gundy (BDG) inequality) *We have*

$$\left\|\int_0^\infty u(t)\#\mathrm{d}x(t)\right\| \le 2\sqrt{2}\|u\|_{L^2L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})}.$$

It follows that the map $\{(r_1,r_2):0\le r_1\le r_2\}\ni(s,t)\mapsto\int_s^t u\#\mathrm{d}m\in\mathcal{M}$ is continuous.

Proof. If $m=x$, then the first item is [BS98, Prop. 2.2.2]. The inequality in the second item is [BS98, Thm. 3.2.1]. The remainder of the claims in the theorem (i.e., those for $m\in\{z,z^*\}$) follow from the corresponding claims for $m=x$ because $z=2^{-1/2}(x_1+ix_2)$ and $z^*=2^{-1/2}(x_1-ix_2)$, where $x_1=\sqrt{2}\,\mathrm{Re}\,z$ and $x_2=\sqrt{2}\,\mathrm{Im}\,z$ are semicircular Brownian motions. □

Corollary 7.3.10. *Retain the setup of Theorem 7.3.9, and fix $s\ge 0$. The linear map $\int_s^{\cdot}\boldsymbol{\cdot}\#\mathrm{d}m\colon\mathbb{S}_{\mathrm{a}}\to C([s,\infty);\mathcal{M})$ extends uniquely to a continuous linear map $\Lambda^2\to C([s,\infty);\mathcal{M})$, which we notate the same way. If $u\in\Lambda^2$, then $\int_0^{\cdot}u\#\mathrm{d}m$ is a continuous noncommutative martingale that satisfies the identities*

$$\int_s^t u\#\mathrm{d}m=\int_0^t u\#\mathrm{d}m-\int_0^s u\#\mathrm{d}m \ \ \textit{and}\ \ \left(\int_s^t u\#\mathrm{d}m\right)^*=\int_s^t u^\star\#\mathrm{d}m^*,$$

and the bounds

$$\left\|\int_s^t u\#\mathrm{d}x\right\|\le 2\sqrt{2}\left(\int_s^t\|u(r)\|^2_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})}\,\mathrm{d}r\right)^{\frac{1}{2}},\ \left\|\int_s^t u\#\mathrm{d}z^\varepsilon\right\|\le 4\left(\int_s^t\|u(r)\|^2_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})}\,\mathrm{d}r\right)^{\frac{1}{2}}$$

for $t\ge s$ and $\varepsilon\in\{1,*\}$. Similar comments apply to $\int_0^\infty u\#\mathrm{d}m$ for $u\in\mathcal{L}^2$.

Definition 7.3.11 (Free stochastic integral)**.** For every $u \in \Lambda^2$ and $m \in \{x, z, z^*\}$ as above, the process $\int_0^\cdot u\#\mathrm{d}m$ from Corollary 7.3.10 is called the **free stochastic integral** of u against m.

We end this section by giving a large class of examples of members of $\Lambda^{2,p}$. Note that $u \in \Lambda^{2,p}$ if and only if $1_{[0,t)}u \in \mathcal{L}^{2,p}$ for all $t > 0$. We shall use this freely below.

Proposition 7.3.12. *Suppose $u\colon \mathbb{R}_+ \to \mathcal{M} \otimes_{\min} \mathcal{M}^{\mathrm{op}}$ is (norm) right-continuous, locally bounded, and adapted, i.e., $u(t) \in \mathcal{M}_t \otimes_{\min} \mathcal{M}_t^{\mathrm{op}}$ for all $t \geq 0$. If $p \in [1,\infty]$ and $v \in \Lambda^{2,p}$, then $u\,v \in \Lambda^{2,p}$. The latter juxtaposition is the (pointwise) usual action of $\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ on $L^p(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}, \tau\bar{\otimes}\tau^{\mathrm{op}})$.*

Proof. First, note that if $0 \leq s \leq t < \infty$ and $w \in \mathcal{M}_s \otimes \mathcal{M}_s^{\mathrm{op}}$, then $1_{[s,t)}w\,v \in \mathcal{L}^{2,p}$. (Approximate v by simple adapted biprocesses to see this.) We claim this holds for $w \in \mathcal{M}_s \otimes_{\min} \mathcal{M}_s^{\mathrm{op}}$ as well. Indeed, let $(w_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_s \otimes \mathcal{M}_s^{\mathrm{op}}$ converging in the norm topology to w. By noncommutative Hölder's inequality,

$$\|1_{[s,t)}w_n\,v - 1_{[s,t)}w\,v\|_{L^2L^p(\tau\bar{\otimes}\tau^{\mathrm{op}})} \leq \|w_n - w\|_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})}\|v\|_{L_t^2L^p(\tau\bar{\otimes}\tau^{\mathrm{op}})} \xrightarrow{n\to\infty} 0,$$

i.e., $1_{[s,t)}w_n\,v \to 1_{[s,t)}w\,v$ in $L^2(\mathbb{R}_+; L^p(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}, \tau\bar{\otimes}\tau^{\mathrm{op}}))$ as $n \to \infty$. Thus, $1_{[s,t)}w\,v \in \mathcal{L}^{2,p}$.

Now, let $t > 0$, and define $u^n := \sum_{i=1}^n 1_{[\frac{i-1}{n}t,\,\frac{i}{n}t)}u\big(\frac{i-1}{n}t\big)$ for all $n \in \mathbb{N}$. By the previous paragraph, $u^n\,v \in \mathcal{L}^2$ for all $n \in \mathbb{N}$. Since u is right-continuous, $u^n \to 1_{[0,t)}u$ pointwise in $\mathcal{M} \otimes_{\min} \mathcal{M}^{\mathrm{op}} \subseteq \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ as $n \to \infty$. In particular, $u^n\,v \to 1_{[0,t)}u\,v$ pointwise in $L^p(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}, \tau\bar{\otimes}\tau^{\mathrm{op}})$ as $n \to \infty$. Also,

$$\sup_{n\in\mathbb{N}}\|u^n\,v\|_{L^p(\tau\bar{\otimes}\tau^{\mathrm{op}})} \leq 1_{[0,t)}\|v\|_{L^p(\tau\bar{\otimes}\tau^{\mathrm{op}})}\sup_{0\leq r<t}\|u(r)\|_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})} \in L^2(\mathbb{R}_+).$$

Therefore, by the dominated convergence theorem,

$$\|u^n\,v - 1_{[0,t)}u\,v\|_{L^2L^p(\tau\bar{\otimes}\tau^{\mathrm{op}})} = \|u^n\,v - u\,v\|_{L_t^2L^p(\tau\bar{\otimes}\tau^{\mathrm{op}})} \xrightarrow{n\to\infty} 0$$

Thus, $u^n\,v \to 1_{[0,t)}u\,v$ in $L^2(\mathbb{R}_+; L^p(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}, \tau\bar{\otimes}\tau^{\mathrm{op}}))$ as $n \to \infty$. We conclude $1_{[0,t)}u\,v \in \mathcal{L}^{2,p}$, and therefore, since $t > 0$ was arbitrary, $u\,v \in \Lambda^{2,p}$, as desired. □

The most useful consequence is as follows.

Corollary 7.3.13. *Suppose* $u\colon \mathbb{R}_+ \to \mathcal{M} \otimes_{\min} \mathcal{M}^{\mathrm{op}}$ *is RCLL, i.e., u is (norm) right-continuous and the left limit* $u(t-) \coloneqq \lim_{s\nearrow t} u(s) \in \mathcal{M} \otimes_{\min} \mathcal{M}^{\mathrm{op}}$ *exists for each* $t \geq 0$. *If u is adapted,* $p \in [1,\infty]$, *and* $v \in \Lambda^{2,p}$, *then* $u\,v \in \Lambda^{2,p}$.

Proof. RCLL implies right-continuous and locally bounded, so Proposition 7.3.12 applies. $\square$

Example 7.3.14. Suppose $u\colon \mathbb{R}_+ \to \mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}}$ is continuous with respect to $\|\cdot\|_{\mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}}}$ and $u(t) \in \mathcal{M}_t\hat{\otimes}_\pi\mathcal{M}_t^{\mathrm{op}}$ for all $t \geq 0$. Since $\mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}} \hookrightarrow \mathcal{M} \otimes_{\min} \mathcal{M}^{\mathrm{op}}$, u satisfies the hypotheses of Corollary 7.3.13. A common example of this form is

$$u = \sum_{i=1}^n a_i \otimes b_i = \left(\sum_{i=1}^n a_i(t) \otimes b_i(t)\right)_{t\geq 0},$$

where $a_i, b_i\colon \mathbb{R}_+ \to \mathcal{M}$ are continuous adapted processes for all $i \in \{1,\ldots,n\}$.

7.4 Free Itô product rule

In this section, we set up and prove an Itô product rule for free Itô processes (Theorem 7.4.9). We begin by defining free Itô processes. Recall that $(\mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \tau)$ is our fixed W^*-probability space.

Definition 7.4.1 (Free Itô process)**.** Fix $n \in \mathbb{N}$ and an n-dimensional semicircular Brownian motion $(x_1,\ldots,x_n)\colon \mathbb{R}_+ \to \mathcal{M}_{\mathrm{sa}}^n$. A **free Itô process** is a process $m\colon \mathbb{R}_+ \to \mathcal{M}$ satisfying

$$\begin{aligned} \mathrm{d}m(t) &= \sum_{i=1}^n u_i(t)\#\mathrm{d}x_i(t) + k(t)\,\mathrm{d}t, \quad \text{i.e.,} \qquad (7.4.2)\\ m &= m(0) + \sum_{i=1}^n \int_0^\cdot u_i(t)\#\mathrm{d}x_i(t) + \int_0^\cdot k(t)\,\mathrm{d}t, \end{aligned}$$

where $m(0) \in \mathcal{M}_0$, $u_i \in \Lambda^2$ for all $i \in \{1,\ldots,n\}$, and $k\colon \mathbb{R}_+ \to \mathcal{M}$ is adapted and locally strongly integrable. If $w\colon \mathbb{R}_+ \to \mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}}$ is continuous and adapted as in Example 7.3.14 and $m_1\colon \mathbb{R}_+ \to \mathcal{M}$ is a process, then we shall write $\mathrm{d}m_1(t) = w(t)\#\mathrm{d}m(t)$ to mean

$$m_1 = m_1(0) + \int_0^\cdot w(t)\#\mathrm{d}m(t) \coloneqq m_1(0) + \sum_{i=1}^n \int_0^\cdot (w(t)\,u_i(t))\#\mathrm{d}x_i(t) + \int_0^\cdot w(t)\#k(t)\,\mathrm{d}t,$$

where the multiplication $w\,u_i$ occurs in $\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$.

Note that if k is as above, then $\int_0^{\cdot} k(t)\,\mathrm{d}t \colon \mathbb{R}_+ \to \mathcal{M}$ is adapted because $\mathcal{M}_t \subseteq \mathcal{M}$ is norm-closed for all $t \geq 0$. In particular, free Itô processes are continuous and adapted. Also, if m and w are as above, then $w\,u_i \in \Lambda^2$ by Corollary 7.3.13, and $w\#k \colon \mathbb{R}_+ \to \mathcal{M}$ is locally strongly integrable because k is locally strongly integrable and $\mathbb{R}_+ \ni t \mapsto \#w(t) \in B(\mathcal{M})$ is continuous. In particular, both the free stochastic integrals and the Bochner integrals in the second part of the definition above make sense.

Now, suppose $(z_1, \ldots, z_n) \colon \mathbb{R}_+ \to \mathcal{M}^n$ is an n-dimensional circular Brownian motion. If $k \colon \mathbb{R}_+ \to \mathcal{M}$ is locally strongly integrable and adapted, $u_i, v_i \in \Lambda^2$ for all $i \in \{1, \ldots, n\}$, and $m \colon \mathbb{R}_+ \to \mathcal{M}$ is an adapted process satisfying

$$\mathrm{d}m(t) = \sum_{i=1}^{n} \big(u_i(t)\#\mathrm{d}z_i(t) + v_i(t)\#\mathrm{d}z_i^*(t)\big) + k(t)\,\mathrm{d}t, \tag{7.4.3}$$

then m is a free Itô process driven by a $2n$-dimensional semicircular Brownian motion. Indeed, if $x_i \coloneqq \sqrt{2}\,\mathrm{Re}\,z_i$ and $y_i \coloneqq \sqrt{2}\,\mathrm{Im}\,z_i$, then $(x_1, y_1 \ldots, x_n, y_n) \colon \mathbb{R}_+ \to \mathcal{M}_{\mathrm{sa}}^{2n}$ is a $2n$-dimensional semicircular Brownian motions, and m satisfies

$$\mathrm{d}m(t) = \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \big((u_j(t) + v_j(t))\#\mathrm{d}x_j(t) + i(u_j(t) - v_j(t))\#\mathrm{d}y_j(t)\big) + k(t)\,\mathrm{d}t.$$

Next, we introduce the operations that show up in the free Itô product rule.

Notation 7.4.4. Let $\mathfrak{m}_{\mathcal{M}} \colon \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ be the linear map induced by multiplication, and let

$$\mathcal{M}_\tau \coloneqq \mathfrak{m}_{\mathcal{M}} \circ (\mathrm{id}_{\mathcal{M}} \otimes \tau \otimes \mathrm{id}_{\mathcal{M}}) \colon \mathcal{M} \otimes \mathcal{M} \otimes \mathcal{M} \to \mathcal{M},$$

Also, let

$$Q_\tau(u, v) \coloneqq \mathcal{M}_\tau((1 \otimes v) \cdot (u \otimes 1)), \qquad u, v \in \mathcal{M} \otimes \mathcal{M}^{\mathrm{op}}, \tag{7.4.5}$$

where $\cdot$ is multiplication in $\mathcal{M} \otimes \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$. In other words, $\mathcal{M}_\tau$ and Q_τ are determined, respectively, as linear and bilinear maps by

$$\mathcal{M}_\tau(a \otimes b \otimes c) = a\,\tau(b)\,c = \tau(b)\,ac \ \text{ and } \ Q_\tau(a \otimes b, c \otimes d) = a\,\tau(bc)\,d, \qquad a, b, c, d \in \mathcal{M}.$$

In [BS98], $\mathcal{M}_\tau$ is written as η, and Q_τ is written as $\langle\langle\cdot,\cdot\rangle\rangle$. Note that, using the universal property of the projective tensor product, $\mathcal{M}_\tau$ extends to a bounded linear map $\mathcal{M}^{\hat{\otimes}_\pi 3} \to \mathcal{M}$, and Q_τ extends to a bounded bilinear map $(\mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}})^2 \to \mathcal{M}$. Unfortunately, the multiplication map $\mathfrak{m}_{\mathcal{M}} \colon \mathcal{M}\otimes\mathcal{M} \to \mathcal{M}$ is *not* bounded with respect to $\|\cdot\|_{L^\infty(\tau\bar{\otimes}\tau)}$ [DS13, Prop. 3.6], so there is no hope of extending $\mathcal{M}_\tau$ to a bounded linear map $\mathcal{M}\otimes_{\min}\mathcal{M}\otimes_{\min}\mathcal{M} \to \mathcal{M}$, let alone $\mathcal{M}\bar{\otimes}\mathcal{M}\bar{\otimes}\mathcal{M} \to \mathcal{M}$. Nevertheless, using the following elementary but crucial algebraic observation, we learn that the "tracing out the middle" in the definition implies that Q_τ can be extended sensibly to a bounded bilinear map $(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})^2 \to \mathcal{M}$.

Lemma 7.4.6. *If $u,v \in \mathcal{M}\otimes\mathcal{M}^{\mathrm{op}}$ and $a,b,c,d \in \mathcal{M}$, then*

$$\tau(a\,\mathcal{M}_\tau((1\otimes v)\cdot(b\otimes c\otimes d)\cdot(u\otimes 1))) = (\tau\otimes\tau^{\mathrm{op}})((a\otimes 1)(b\otimes 1)uv^{\mathrm{flip}}(1\otimes c)(d\otimes 1)),$$

where the juxtapositions on the right-hand side are multiplications in $\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}}$.

Proof. It suffices to assume $u = a_1\otimes b_1$ and $v = c_1\otimes d_1$ are pure tensors. In this case,

$$\begin{aligned}
\tau(a\,\mathcal{M}_\tau((1\otimes v)\cdot(b\otimes c\otimes d)\cdot(u\otimes 1))) &= \tau(aba_1\tau(b_1cc_1)d_1d) \\
&= (\tau\otimes\tau^{\mathrm{op}})((aba_1d_1d)\otimes(c_1\cdot c\cdot b_1)) \\
&= (\tau\otimes\tau^{\mathrm{op}})((a\otimes 1)(b\otimes 1)(a_1\otimes 1)(d_1\otimes c_1)(d\otimes c)(1\otimes b_1)) \\
&= (\tau\otimes\tau^{\mathrm{op}})((1\otimes b_1)(a\otimes 1)(b\otimes 1)(a_1\otimes 1)(d_1\otimes c_1)(1\otimes c)(d\otimes 1)) \\
&= (\tau\otimes\tau^{\mathrm{op}})((a\otimes 1)(b\otimes 1)(a_1\otimes b_1)(d_1\otimes c_1)(1\otimes c)(d\otimes 1)).
\end{aligned}$$

In the second-to-last equality, we used the traciality of $\tau\otimes\tau^{\mathrm{op}}$. ☐

In particular, if $u,v \in \mathcal{M}\otimes\mathcal{M}^{\mathrm{op}}$, then

$$\tau(a\,Q_\tau(u,v)) = (\tau\bar{\otimes}\tau^{\mathrm{op}})((a\otimes 1)uv^{\mathrm{flip}}), \qquad a\in\mathcal{M}.$$

Now, note that the right-hand side of the identity above makes sense for arbitrary $u,v\in\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ and $a\in L^1(\mathcal{M},\tau)$. Consequently, we may use the relationship $L^1(\mathcal{M},\tau)^* \cong L^\infty(\mathcal{M},\tau) = \mathcal{M}$ to extend the definition of Q_τ.

Specifically, if $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\text{op}}$ and

$$\ell_{u,v}(a) \coloneqq (\tau\bar{\otimes}\tau^{\text{op}})((a \otimes 1)uv^{\text{flip}}), \qquad a \in L^1(\mathcal{M},\tau),$$

then

$$|\ell_{u,v}(a)| \le \|(a\otimes 1)uv^{\text{flip}}\|_{L^1(\tau\bar{\otimes}\tau^{\text{op}})} \le \|a\otimes 1\|_{L^1(\tau\bar{\otimes}\tau^{\text{op}})}\|uv^{\text{flip}}\|_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})} = \|a\|_{L^1(\tau)}\|uv^{\text{flip}}\|_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})}$$

so that

$$\|\ell_{u,v}\|_{L^1(\mathcal{M},\tau)^*} \le \|uv^{\text{flip}}\|_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})} < \infty.$$

In particular, since $\mathcal{M}$ is dense in $L^1(\mathcal{M},\tau)$, the following definition makes sense and extends the algebraic definition of Q_τ.

Definition 7.4.7 (Extended definition of Q_τ)**.** If $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\text{op}}$, then $Q_\tau(u,v)$ is defined to be the unique element of $\mathcal{M}$ such that

$$\tau(a\,Q_\tau(u,v)) = (\tau\bar{\otimes}\tau^{\text{op}})((a \otimes 1)uv^{\text{flip}}), \qquad a \in \mathcal{M} \text{ (or } a \in L^1(\mathcal{M},\tau)).$$

It is clear from the definition that the map $Q_\tau(u,v)$ is bilinear in (u,v). Also, by the paragraph before Definition 7.4.7, if $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\text{op}}$, then

$$\|Q_\tau(u,v)\| = \|\ell_{u,v}\|_{L^1(\mathcal{M},\tau)^*} \le \|uv^{\text{flip}}\|_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})} \le \|u\|_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})}\|v\|_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})}.$$

Consequently, if $u, v \in L^2_{\text{loc}}(\mathbb{R}_+;\mathcal{M}\bar{\otimes}\mathcal{M}^{\text{op}})$, then $Q_\tau(u,v) \in L^1_{\text{loc}}(\mathbb{R}_+;\mathcal{M})$, and

$$\|Q_\tau(u,v)\|_{L^1_tL^\infty(\tau)} \le \|u\|_{L^2_tL^\infty(\tau\bar{\otimes}\tau^{\text{op}})}\|v\|_{L^2_tL^\infty(\tau\bar{\otimes}\tau^{\text{op}})}, \qquad t \ge 0, \tag{7.4.8}$$

by the Cauchy–Schwarz inequality. It is then easy to see—by starting with simple adapted biprocesses and then taking limits—that if $u, v \in \Lambda^2$, then $Q_\tau(u,v) \in L^1_{\text{loc}}(\mathbb{R}_+;\mathcal{M})$ is adapted. This is all the information we need about Q_τ, so we are now in a position to state the free Itô product rule. (However, please see Remark 7.7.18 for additional comments about Q_τ.)

Theorem 7.4.9 (Free Itô product rule)**.** *The following formulas hold.*

(i) *Suppose* $(x_1,\dots,x_n)\colon \mathbb{R}_+\to\mathcal{M}_{\mathrm{sa}}^n$ *is an* n*-dimensional semicircular Brownian motion. If, for each* $\ell\in\{1,2\}$*,* $m_\ell\colon\mathbb{R}_+\to\mathcal{M}$ *is a free Itô process satisfying*

$$\mathrm{d}m_\ell(t)=\sum_{i=1}^n u_{\ell i}(t)\#\mathrm{d}x_i(t)+k_\ell(t)\,\mathrm{d}t,$$

then

$$\mathrm{d}(m_1m_2)(t)=\mathrm{d}m_1(t)\,m_2(t)+m_1(t)\,\mathrm{d}m_2(t)+\sum_{i=1}^n Q_\tau(u_{1i}(t),u_{2i}(t))\,\mathrm{d}t.$$

Put another way,

$$\mathrm{d}m_1(t)\,\mathrm{d}m_2(t)=\sum_{i=1}^n Q_\tau(u_{1i}(t),u_{2i}(t))\,\mathrm{d}t$$

in the classical notation.

(ii) *Suppose* $(z_1,\dots,z_n)\colon\mathbb{R}_+\to\mathcal{M}^n$ *is an* n*-dimensional circular Brownian motion. If, for each* $\ell\in\{1,2\}$*,* $m_\ell\colon\mathbb{R}_+\to\mathcal{M}$ *is a free Itô process (driven by* $(z_1,\dots,z_n)$*) satisfying*

$$\mathrm{d}m_\ell(t)=\sum_{i=1}^n\big(u_{\ell i}(t)\#\mathrm{d}z_i(t)+v_{\ell i}(t)\#\mathrm{d}z_i^*(t)\big)+k_\ell(t)\,\mathrm{d}t,$$

then

$$\mathrm{d}(m_1m_2)(t)=\mathrm{d}m_1(t)\,m_2(t)+m_1(t)\,\mathrm{d}m_2(t)+\sum_{i=1}^n\big(Q_\tau(u_{1i}(t),v_{2i}(t))+Q_\tau(v_{1i}(t),u_{2i}(t))\big)\,\mathrm{d}t.$$

Put another way,

$$\mathrm{d}m_1(t)\,\mathrm{d}m_2(t)=\sum_{i=1}^n(Q_\tau(u_{1i}(t),v_{2i}(t))+Q_\tau(v_{1i}(t),u_{2i}(t)))\,\mathrm{d}t$$

in the classical notation.

By the comments following Definition 7.4.1, the second item follows from the first item with twice the dimension. Before launching into the proof of the first item, we perform a useful example calculation.

Example 7.4.10. Let $z\colon \mathbb{R}_+ \to \mathcal{M}$ be a circular Brownian motion. Written in the classical notation for quadratic covariation, Theorem 7.4.9(ii) says

$$\begin{aligned} a(t)\,\mathrm{d}z(t)\,b(t)\,\mathrm{d}z^*(t)\,c(t) &= a(t)\,\mathrm{d}z^*(t)\,b(t)\,\mathrm{d}z(t)\,c(t) \\ &= a(t)\,\tau(b(t))\,c(t)\,\mathrm{d}t \quad \text{and} \qquad (7.4.11) \\ a(t)\,\mathrm{d}z^\varepsilon(t)\,b(t)\,\mathrm{d}z^\varepsilon(t)\,c(t) &= a(t)\,\mathrm{d}z^\varepsilon(t)\,b(t)\,\mathrm{d}t\,c(t) \\ &= a(t)\,\mathrm{d}t\,b(t)\,\mathrm{d}z^\varepsilon(t)\,c(t) \\ &= a(t)\,\mathrm{d}t\,b(t)\,\mathrm{d}t\,c(t) = 0 \qquad (7.4.12) \end{aligned}$$

whenever $\varepsilon \in \{1, *\}$ and $a, b, c\colon \mathbb{R}_+ \to \mathcal{M}$ are continuous adapted processes. Now, let $n_1, n_2 \in \mathbb{N}$, and fix continuous adapted processes $a_1, b_1 \ldots, a_{n_1}, b_{n_1}, c_1, d_1 \ldots, c_{n_2}, d_{n_2}, k\colon \mathbb{R}_+ \to \mathcal{M}$. Suppose $m\colon \mathbb{R}_+ \to \mathcal{M}$ is a free Itô process satisfying

$$\mathrm{d}m(t) = \sum_{i=1}^{n_1} a_i(t)\,\mathrm{d}z(t)\,b_i(t) + \sum_{j=1}^{n_2} c_j(t)\,\mathrm{d}z^*(t)\,d_j(t) + k(t)\,\mathrm{d}t. \qquad (7.4.13)$$

Such m show up frequently "in the wild." It is often necessary, especially when m is not self-adjoint, to work with $|m|^2 = m^*m$. Then

$$\mathrm{d}|m|^2(t) = \mathrm{d}m^*(t)\,m(t) + m^*(t)\,\mathrm{d}m(t) + \mathrm{d}m^*(t)\,\mathrm{d}m(t).$$

Let us derive an expression for $\mathrm{d}m^*(t)\,\mathrm{d}m(t)$. First, we have

$$\mathrm{d}m^*(t) = \sum_{j=1}^{n_2} d_j^*(t)\,\mathrm{d}z(t)\,c_j^*(t) + \sum_{i=1}^{n_1} b_i^*(t)\,\mathrm{d}z^*(t)\,a_i^*(t) + k^*(t)\,\mathrm{d}t.$$

Therefore,

$$\mathrm{d}m^*(t)\,\mathrm{d}m(t) = \sum_{j_1,j_2=1}^{n_2} d_{j_1}^*(t)\,\tau\big(c_{j_1}^*(t)\,c_{j_2}(t)\big)\,d_{j_2}(t)\,\mathrm{d}t + \sum_{i_1,i_2=1}^{n_1} b_{i_1}^*(t)\,\tau\big(a_{i_1}^*(t)\,a_{i_2}(t)\big)\,b_{i_2}(t)\,\mathrm{d}t$$

by the free Itô product rule (in the form of Equations (7.4.11) and (7.4.12)).

Now, let $h \in \mathcal{M}_0$ be arbitrary, and suppose $g\colon \mathbb{R}_+ \to \mathcal{M}$ satisfies

$$\begin{cases} \mathrm{d}g(t) = g(t)\,\mathrm{d}z(t) \\ g(0) = h, \end{cases}$$

i.e., g is the **free multiplicative Brownian motion** starting at h. Writing

$$g_\lambda(t) := g(t) - \lambda 1 = g(t) - \lambda, \qquad \lambda \in \mathbb{C},$$

we have

$$\mathrm{d}g_\lambda(t) = g(t)\,\mathrm{d}z(t) \;\text{ and }\; \mathrm{d}g_\lambda^*(t) = \mathrm{d}z^*(t)\,g^*(t).$$

Therefore, by the formula from the previous paragraph, we have

$$\begin{aligned} \mathrm{d}|g_\lambda|^2(t) &= \mathrm{d}g_\lambda^*(t)\,g_\lambda(t) + g_\lambda^*(t)\,\mathrm{d}g_\lambda(t) + \mathrm{d}g_\lambda^*(t)\,\mathrm{d}g_\lambda(t) \\ &= \mathrm{d}g_\lambda^*(t)\,g_\lambda(t) + g_\lambda^*(t)\,\mathrm{d}g_\lambda(t) + 1\tau(g^*(t)g(t))1\,\mathrm{d}t \\ &= \mathrm{d}z^*(t)\,g^*(t)\,g_\lambda(t) + g_\lambda^*(t)\,g(t)\,\mathrm{d}z(t) + \tau\big(|g(t)|^2\big)\,\mathrm{d}t. \end{aligned}$$

We shall use this equation in Example 7.6.10.

We now turn to the proof of Theorem 7.4.9(i). Our approach is similar to that of Biane and Speicher, though we use less free probabilistic machinery by mimicking a classical approach to calculating the quadratic covariation of Itô processes: computing an L^2-limit of second-order Riemann–Stieltjes-type sums. At this time, the reader should review Notation 1.1.14.

Lemma 7.4.14. *If m_1 and m_2 are as in Theorem 7.4.9(i) and $T > 0$, then*

$$\lim_{|\Pi|\to 0} \sum_{t\in\Pi} \big(\Delta_t m_1\big)\big(\Delta_t m_2\big) = m_1(T)m_2(T) - m_1(0)m_2(0) - \int_0^T \mathrm{d}m_1(t)\,m_2(t) - \int_0^T m_1(t)\,\mathrm{d}m_2(t),$$

where the limit is in $\mathcal{M} = L^\infty(\mathcal{M},\tau)$ over partitions Π of $[0,T]$.

Proof. If Π is a partition of $[0,T]$, then

$$\begin{aligned}
\delta_T &\coloneqq m_1(T)m_2(T) - m_1(0)m_2(0) = \sum_{t\in\Pi}\big(m_1(t)m_2(t) - m_1(t_-)m_2(t_-)\big) \\
&= \sum_{t\in\Pi}\big((m_1(t_-)+\Delta_t m_1)(m_2(t_-)+\Delta_t m_2) - m_1(t_-)m_2(t_-)\big) \\
&= \sum_{t\in\Pi}\big((\Delta_t m_1)m_2(t_-) + m_1(t_-)\,\Delta_t m_2 + (\Delta_t m_1)(\Delta_t m_2)\big) \\
&= \int_0^T \mathrm{d}m_1(t)\, m_2^{\Pi}(t) + \int_0^T m_1^{\Pi}(t)\,\mathrm{d}m_2(t) + \sum_{t\in\Pi}(\Delta_t m_1)(\Delta_t m_2).
\end{aligned}$$

Now, since m_ℓ is uniformly continuous on $[0,T]$, $m_\ell^{\Pi} \to m_\ell$ uniformly on $[0,T]$ as $|\Pi| \to 0$. Therefore, by the L^∞-BDG inequality (and the dominated convergence theorem),

$$\int_0^T \mathrm{d}m_1(t)\, m_2^{\Pi}(t) \xrightarrow{|\Pi|\to 0} \int_0^T \mathrm{d}m_1(t)\, m_2(t) \;\text{ and }\; \int_0^T m_1^{\Pi}(t)\,\mathrm{d}m_2(t) \xrightarrow{|\Pi|\to 0} \int_0^T m_1(t)\,\mathrm{d}m_2(t)$$

in $\mathcal{M}$. It then follows from the calculation above that

$$\sum_{t\in\Pi}(\Delta_t m_1)(\Delta_t m_2) \xrightarrow{|\Pi|\to 0} m_1(T)m_2(T) - m_1(0)m_2(0) - \int_0^T \mathrm{d}m_1(t)\, m_2(t) - \int_0^T m_1(t)\,\mathrm{d}m_2(t)$$

in $\mathcal{M}$, as desired. □

Lemma 7.4.15. *Let $(x_1,\ldots,x_n)\colon \mathbb{R}_+ \to \mathcal{M}_{\mathrm{sa}}^n$ be an n-dimensional semicircular Brownian motion, and suppose $0 \le s < t$. Also, define $t_{k,N} \coloneqq (N-k)s/N + kt/N$ for all $N \in \mathbb{N}$ and $k \in \{0,\ldots,N\}$. If $a \in \mathcal{M}_s$, then*

$$L^2\text{-}\lim_{N\to\infty}\sum_{k=1}^{N}\big(x_i(t_{k,N}) - x_i(t_{k-1,N})\big)a\big(x_j(t_{k,N}) - x_j(t_{k-1,N})\big) = (t-s)\,\tau(a)\,\delta_{ij}$$

for all $i,j \in \{1,\ldots,n\}$.

Proof. By writing $a = (a-\tau(a)1)+\tau(a)1$, it suffices to prove the formula when a is centered and when $a = 1$. To this end, write $\Delta_{k,N}x_i \coloneqq x_i(t_{k,N}) - x_i(t_{k-1,N})$, and fix $i,j \in \{1,\ldots,n\}$. First, note that if $k \ne \ell$, then $\mathcal{M}_s$, $\Delta_{k,N}x_i$, $\Delta_{\ell,N}x_i$ are freely independent; and if, in addition, $i \ne j$, then $\mathcal{M}_s$, $\Delta_{k,N}x_i$, $\Delta_{\ell,N}x_i$, $\Delta_{k,N}x_j$, $\Delta_{\ell,N}x_j$ are freely independent. (This is because $s = t_{0,N} < t_{k,N}$

when $k \geq 1$.) Second, recall that $\|x_i(r_1) - x_i(r_2)\| = 2\sqrt{|r_1 - r_2|}$ whenever $r_1, r_2 \geq 0$. Therefore, by definition of free independence, if either 1) $i = j$ and $a \in \{b \in \mathcal{M}_s : \tau(b) = 0\}$ or 2) $i \neq j$ and $a \in \{b \in \mathcal{M}_s : \tau(b) = 0\} \cup \{1\}$, then

$$\begin{aligned}
\left\| \sum_{k=1}^N \Delta_{k,N} x_i \, a \, \Delta_{k,N} x_j \right\|_{L^2(\tau)}^2 &= \sum_{k,\ell=1}^N \tau\big(\Delta_{k,N} x_j a^* \Delta_{k,N} x_i \Delta_{\ell,N} x_i \, a \, \Delta_{\ell,N} x_j\big) \\
&= \sum_{k=1}^N \tau\big(\Delta_{k,N} x_j a^* \Delta_{k,N} x_i \Delta_{k,N} x_i \, a \, \Delta_{k,N} x_j\big) \\
&\qquad + \sum_{k \neq \ell} \tau\big(\Delta_{k,N} x_j a^* \Delta_{k,N} x_i \Delta_{\ell,N} x_i \, a \, \Delta_{\ell,N} x_j\big) \\
&= \sum_{k=1}^N \tau\big(\Delta_{k,N} x_j a^* \Delta_{k,N} x_i \Delta_{k,N} x_i \, a \, \Delta_{k,N} x_j\big) \\
&\leq \|a\|^2 \frac{16(t-s)^2}{N} \xrightarrow{N \to \infty} 0.
\end{aligned}$$

The only case that remains is when $i = j$ and $a = 1$. To take care of this case, note that if $k \neq \ell$, then the elements $(\Delta_{k,N} x_i)^2 - (t_{k,N} - t_{k-1,N})$ and $(\Delta_{\ell,N} x_i)^2 - (t_{\ell,N} - t_{\ell-1,N})$ are freely independent and centered. Thus,

$$\tau\Big(\big((\Delta_{k,N} x_i)^2 - (t_{k,N} - t_{k-1,N})\big)\big((\Delta_{\ell,N} x_i)^2 - (t_{\ell,N} - t_{\ell-1,N})\big)\Big) = 0,$$

from which it follows, as above, that

$$\begin{aligned}
\left\| \sum_{k=1}^N (\Delta_{k,N} x_i)^2 - (t-s) \right\|_{L^2(\tau)}^2 &= \left\| \sum_{k=1}^N \big((\Delta_{k,N} x_i)^2 - (t_{k,N} - t_{k-1,N})\big) \right\|_{L^2(\tau)}^2 \\
&= \sum_{k=1}^N \tau\Big(\big((\Delta_{k,N} x_i)^2 - (t_{k,N} - t_{k-1,N})\big)^2\Big) \\
&= \sum_{k=1}^N (t_{k,N} - t_{k-1,N})^2 = \frac{(t-s)^2}{N} \xrightarrow{N \to \infty} 0.
\end{aligned}$$

The third equality holds because $x \coloneqq \Delta_{k,N} x_i$ is semicircular with variance $r \coloneqq t_{k,N} - t_{k-1,N}$, so $\tau(x^{2p}) = C_p r^p$ whenever $p \in \mathbb{N}_0$, where $C_p = \binom{2p}{p}/(p+1)$ is the p^{th} Catalan number. $\square$

We are now prepared for the proof.

Proof of Theorem 7.4.9(i). By the L^∞-BDG inequality, Inequality (7.4.8), and the dominated convergence theorem, it suffices to prove the formula when $u_{\ell i} \in \mathbb{S}_\mathrm{a}$ for all $\ell \in \{1,2\}$ and $i \in \{1,\dots,n\}$. By Lemma 7.4.14, it therefore suffices to prove that if $T>0$ and $u_{\ell i} \in \mathbb{S}_\mathrm{a}$, then

$$L^2\text{-}\lim_{|\Pi|\to 0} \sum_{t\in\Pi} \big(\Delta_t m_1\big)\big(\Delta_t m_2\big) = \sum_{i=1}^n \int_0^T Q_\tau(u_{1i}(t), u_{2i}(t))\,\mathrm{d}t,$$

where the limit is over partitions Π of $[0,T]$. To this end, write

$$a_\ell := \int_0^\cdot k_\ell(t)\,\mathrm{d}t, \qquad \ell \in \{1,2\}.$$

Then

$$\begin{aligned}\sum_{t\in\Pi} \big(\Delta_t m_1\big)\big(\Delta_t m_2\big) &= \sum_{t\in\Pi} \big(\Delta_t a_1 + \Delta_t(m_1-a_1)\big)\big(\Delta_t a_2 + \Delta_t(m_2-a_2)\big) \\ &= \sum_{t\in\Pi} \Delta_t(m_1-a_1)\,\Delta_t(m_2-a_2) + \sum_{t\in\Pi}\big(\Delta_t a_1\big)\big(\Delta_t m_2\big) + \sum_{t\in\Pi}\Delta_t(m_1-a_1)\,\Delta_t a_2.\end{aligned}$$

Since $\Delta_t a_\ell = \int_{t_-}^t k_\ell(s)\,\mathrm{d}s$ whenever $t\in\Pi$,

$$\begin{aligned}\Big\|\sum_{t\in\Pi}\big(\Delta_t a_1\big)\big(\Delta_t m_2\big)\Big\| &\le \max_{s\in\Pi}\|\Delta_s m_2\|\sum_{t\in\Pi}\|\Delta_t a_1\| \\ &\le \max_{s\in\Pi}\|\Delta_s m_2\|\int_0^T \|k_1(t)\|\,\mathrm{d}t \xrightarrow{|\Pi|\to 0} 0 \ \text{ and} \\ \Big\|\sum_{t\in\Pi}\Delta_t(m_1-a_1)\,\Delta_t a_2\Big\| &\le \max_{s\in\Pi}\|\Delta_s(m_1-a_1)\|\sum_{t\in\Pi}\|\Delta_t a_2\| \\ &\le \max_{s\in\Pi}\|\Delta_s(m_1-a_1)\|\int_0^T\|k_2(t)\|\,\mathrm{d}t \xrightarrow{|\Pi|\to 0} 0\end{aligned}$$

because m_2 and m_1-a_1 are uniformly continuous on $[0,T]$. In particular, if

$$I_i[u] := \int_0^\cdot u\#\mathrm{d}x_i, \qquad u\in\Lambda^2,\ i\in\{1,\dots,n\},$$

then

$$L^2\text{-}\lim_{|\Pi|\to 0}\sum_{t\in\Pi}\big(\Delta_t m_1\big)\big(\Delta_t m_2\big) = L^2\text{-}\lim_{|\Pi|\to 0}\sum_{t\in\Pi}\sum_{i,j=1}^n \Delta_t(I_i[u_{1i}])\,\Delta_t(I_j[u_{2j}]).$$

Consequently, the proof is complete if we can show that

$$L^2\text{-}\lim_{|\Pi|\to 0}\sum_{t\in\Pi}\Delta_t(I_i[u])\,\Delta_t(I_j[v]) = \delta_{ij}\int_0^T Q_\tau(u(t),v(t))\,\mathrm{d}t, \quad u,v\in\mathbb{S}_{\mathrm{a}},\ i,j\in\{1,\dots,n\}. \tag{7.4.16}$$

Since Equation (7.4.16) is bilinear in the arguments (u,v), it suffices to prove it assuming that $u = 1_{[s_1,t_1)}a\otimes b$ and $v = 1_{[s_2,t_2)}c\otimes d$, where $[s_1,t_1),[s_2,t_2)\subseteq[0,T)$, $a,b\in\mathcal{M}_{s_1}$, $c,d\in\mathcal{M}_{s_2}$, and either $[s_1,t_1)\cap[s_2,t_2)=\emptyset$ or $[s_1,t_1)=[s_2,t_2)$. We take both cases in turn, but we first observe that if $w\in\mathbb{S}_{\mathrm{a}}$, $i\in\{1,\dots,n\}$, and $t\in\Pi$, then $\Delta_t(I_i[w]) = \int_0^\infty(1_{[t_-,t)}w)\#\mathrm{d}x_i = \int_{t_-}^t w\#\mathrm{d}x_i$. In particular, if $w\equiv 0$ on $[t_-,t)$, then $\Delta_t(I_i[w])=0$.

Case 1: $[s_1,t_1)\cap[s_2,t_2)=\emptyset$. In this case, the observation at the end of the previous paragraph gives immediately that $\sum_{t\in\Pi}\Delta_t(I_i[u])\,\Delta_t(I_j[v])=0$ when $|\Pi|$ is sufficiently small. But also $Q_\tau(u,v)\equiv 0$, so Equation (7.4.16) holds.

Case 2: $[s_1,t_1)=[s_2,t_2)=:[s,t)$. Fix $N\in\mathbb{N}$, let $\{t_{k,N}:0\le k\le N\}$ be as in Lemma 7.4.15, and suppose Π_N is a partition on $[0,T]$ such that $\{t_{k,N}:0\le k\le N\}\subseteq\Pi_N$. If $|\Pi_N|\to 0$ as $N\to\infty$, then

$$\begin{aligned}
L^2\text{-}\lim_{|\Pi|\to 0}\sum_{t\in\Pi}\Delta_t(I_i[u])\,\Delta_t(I_j[v]) &= L^2\text{-}\lim_{N\to\infty}\sum_{t\in\Pi_N}\Delta_t(I_i[u])\,\Delta_t(I_j[v])\\
&= L^2\text{-}\lim_{N\to\infty}\sum_{k=1}^N a\big(x_i(t_{k,N})-x_i(t_{k-1,N})\big)bc\big(x_j(t_{k,N})-x_j(t_{k-1,N})\big)d\\
&= (t-s)\,a\,\tau(bc)\,d\,\delta_{ij} = \delta_{ij}\int_0^T Q_\tau(u(t),v(t))\,\mathrm{d}t
\end{aligned}$$

by the observation made just before the previous paragraph, the definition of I_i, Lemma 7.4.15, and the definition of Q_τ. This completes the proof. □

Corollary 7.4.17. *If m_1 and m_2 are as in Theorem 7.4.9(i) and $T>0$, then*

$$L^\infty\text{-}\lim_{|\Pi|\to 0}\sum_{t\in\Pi}\big(\Delta_t m_1\big)\big(\Delta_t m_2\big) = \sum_{i=1}^n\int_0^T Q_\tau(u_{1i}(t),u_{2i}(t))\,\mathrm{d}t,$$

where the limit is over partitions of $[0,T]$.

Proof. Combine Lemma 7.4.14 and Theorem 7.4.9. □

7.5 Functional free Itô formula for polynomials

In this section, we prove the "functional" Itô formula for polynomials of free Itô processes (Theorem 7.5.7). We begin by defining noncommutative derivatives of polynomials. Let A be a unital $\mathbb{C}$-algebra, let $k \in \mathbb{N}$, and suppose $\tilde{a}_1, \ldots, \tilde{a}_{k+1} \in A$ are commuting elements. There exists a unique unital algebra homomorphism $\mathrm{ev}_{(\tilde{a}_1,\ldots,\tilde{a}_{k+1})} \colon \mathbb{C}[\lambda_1, \ldots, \lambda_{k+1}] \to A$ sending λ_i to $\tilde{a}_i$ whenever $i \in \{1, \ldots, k+1\}$. (This is the most basic "functional calculus.")

Definition 7.5.1 (Noncommutative derivatives of polynomials)**.** Let A be a unital $\mathbb{C}$-algebra, let $k \in \mathbb{N}$, and fix $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in A^{k+1}$. Write

$$\tilde{a}_i \coloneqq 1^{\otimes(i-1)} \otimes a_i \otimes 1^{\otimes(k+1-i)} \in A^{\otimes(k+1)}, \qquad i \in \{1, \ldots, k+1\}.$$

If $p(\lambda) = \sum_{i=0}^{n} c_i \lambda^i \in \mathbb{C}[\lambda]$, then

$$\begin{aligned} \partial^k p(\mathbf{a}) &\coloneqq k!\, \mathrm{ev}_{(\tilde{a}_1,\ldots,\tilde{a}_{k+1})}\big(p^{[k]}(\lambda_1, \ldots, \lambda_{k+1})\big) = k!\, p^{[k]}(\tilde{a}_1, \ldots, \tilde{a}_{k+1}) \\ &= k! \sum_{i=0}^{n} c_i \sum_{|\delta|=i-k} a_1^{\delta_1} \otimes \cdots \otimes a_{k+1}^{\delta_{k+1}} \in A^{\otimes(k+1)} \end{aligned} \tag{7.5.2}$$

is the $\boldsymbol{k}^{\mathbf{th}}$ **noncommutative derivative** of p evaluated at $\mathbf{a}$. We often write $\partial \coloneqq \partial^1$ and consider $\partial p(a_1, a_2)$ as an element of $A \otimes A^{\mathrm{op}}$. Finally, write

$$\partial^k p(a) \coloneqq \partial^k p\big(a_{(k+1)}\big), \qquad a \in A,$$

using Notation 1.2.5(i).

Next, we define the object appearing in the Itô correction term.

Notation 7.5.3. For $p \in \mathbb{C}[\lambda]$, $m \in \mathcal{M}$, and $u, v \in \mathcal{M} \otimes \mathcal{M}^{\mathrm{op}}$, write

$$\Delta_{u,v} p(m) \coloneqq \frac{1}{2} \mathcal{M}_\tau((1 \otimes v) \cdot \partial^2 p(m) \cdot (u \otimes 1) + (1 \otimes u) \cdot \partial^2 p(m) \cdot (v \otimes 1)), \tag{7.5.4}$$

where $\cdot$ is multiplication in $\mathcal{M} \otimes \mathcal{M}^{\mathrm{op}} \otimes \mathcal{M}$.

As was the case when we defined Q_τ, we can still make sense of the formula defining $\Delta_{u,v}p(m)$ when $u, v \in \mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}}$. And again, though the formula does not make sense as written when $u, v \in \mathcal{M} \otimes_{\min} \mathcal{M}^{\mathrm{op}}$ (let alone $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$), we can use Lemma 7.4.6 to extend $\Delta_{\cdot,\cdot}p(m) \colon (\mathcal{M} \otimes \mathcal{M}^{\mathrm{op}})^2 \to \mathcal{M}$ to a bounded bilinear map $(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})^2 \to \mathcal{M}$. At this time, we advise the reader to review Notation 7.3.2(iv), as we begin now to make heavy use of the $\#_2^{\otimes}$ operation defined therein.

Fix $p \in \mathbb{C}[\lambda]$ and $m \in \mathcal{M}$. For $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, define

$$\ell_{p,u,v}(a) \coloneqq \frac{1}{2}(\tau\bar{\otimes}\tau^{\mathrm{op}})\big((a \otimes 1)\, \partial^2 p(m \otimes 1, 1 \otimes m, m \otimes 1)\#_2^{\otimes}[uv^{\mathrm{flip}} + vu^{\mathrm{flip}}, 1 \otimes 1]\big), \qquad a \in L^1(\mathcal{M}, \tau).$$

If $u, v \in \mathcal{M} \otimes \mathcal{M}^{\mathrm{op}}$, then Lemma 7.4.6 and Equation (7.5.2) imply

$$\tau\big(a\, \Delta_{u,v}p(m)\big) = \ell_{p,u,v}(a), \qquad a \in \mathcal{M}.$$

We use this identity to extend the definition of $\Delta_{\cdot,\cdot}p(m)$. Indeed, if $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, then

$$\|\ell_{p,u,v}\|_{L^1(\mathcal{M},\tau)^*} \le \frac{1}{2}\big\|\partial^2 p(m \otimes 1, 1 \otimes m, m \otimes 1)\#_2^{\otimes}[uv^{\mathrm{flip}} + vu^{\mathrm{flip}}, 1 \otimes 1]\big\|_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})}.$$

Thus, by the duality relationship $L^1(\mathcal{M}, \tau)^* \cong \mathcal{M}$, the following definition makes sense and extends the algebraic definition of $\Delta_{u,v}p(m)$.

Definition 7.5.5 (Extended definition of $\Delta_{u,v}p(m)$)**.** If $p(\lambda) \in \mathbb{C}[\lambda]$ is a polynomial, $m \in \mathcal{M}$, and $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, then $\Delta_{u,v}p(m)$ is defined to be the unique element of $\mathcal{M}$ such that

$$\tau\big(a\Delta_{u,v}p(m)\big) = \ell_{p,u,v}(a), \qquad a \in \mathcal{M} \text{ (or } a \in L^1(\mathcal{M}, \tau)).$$

Also, we write

$$\Delta_u p(m) \coloneqq \Delta_{u,u}p(m)$$

for the $u = v$ case.

It is clear from the definition that $\Delta_{u,v}p(m)$ is trilinear in (u, v, p) and symmetric in (u, v).

Now, if $n \in \mathbb{N}_0$ and $p_n(\lambda) = \lambda^n$, then, by Equation (7.5.2) and the paragraph before Definition 7.5.5, we have

$$\begin{aligned}
&\|\Delta_{u,v} p_n(m)\| = \|\ell_{p_n,u,v}\|_{L^1(\mathcal{M},\tau)^*} \\
&\leq \frac{1}{2}\left\| 2 \sum_{|\delta|=n-2} (m \otimes 1)^{\delta_1} (uv^{\text{flip}} + vu^{\text{flip}})(1 \otimes m)^{\delta_2} (m \otimes 1)^{\delta_3} \right\|_{L^\infty(\tau \bar{\otimes} \tau^{\text{op}})} \\
&\leq 2\|u\|_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})} \|v\|_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})} \sum_{|\delta|=n-2} \|m \otimes 1\|^{\delta_1}_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})} \|1 \otimes m\|^{\delta_2}_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})} \|m \otimes 1\|^{\delta_3}_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})} \\
&= n(n-1)\|m\|^{n-2} \|u\|_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})} \|v\|_{L^\infty(\tau\bar{\otimes}\tau^{\text{op}})}.
\end{aligned}$$

Thus, if $u, v \in L^2_{\text{loc}}(\mathbb{R}_+; \mathcal{M}\bar{\otimes}\mathcal{M}^{\text{op}})$ and $m \in C(\mathbb{R}_+; \mathcal{M})$, then $\Delta_{u,v} p(m) \in L^1_{\text{loc}}(\mathbb{R}_+; \mathcal{M})$, and

$$\|\Delta_{u,v} p_n(m)\|_{L^1_t L^\infty(\tau)} \leq n(n-1)\|m\|^{n-2}_{L^\infty_t L^\infty(\tau)} \|u\|_{L^2_t L^\infty(\tau\bar{\otimes}\tau^{\text{op}})} \|v\|_{L^2_t L^\infty(\tau\bar{\otimes}\tau^{\text{op}})}, \qquad t \geq 0.$$

It is then easy to see that if $u, v \in \Lambda^2$ and $m \colon \mathbb{R}_+ \to \mathcal{M}$ is continuous and adapted, then $\Delta_{u,v} p(m) \in L^1_{\text{loc}}(\mathbb{R}_+; \mathcal{M})$ is adapted as well. The last fact we shall need about $\Delta_{u,v} p(m)$ to prove the functional free Itô formula for polynomials is the following product rule. (However, please see Remark 7.7.18 for additional comments about $\Delta_{u,v} p(m)$.)

Lemma 7.5.6 (Product rule for $\Delta_{u,v} p(m)$)**.** *If $p, q \in \mathbb{C}[\lambda]$, then*

$$\Delta_{u,v}(pq)(m) = \Delta_{u,v} p(m)\, q(m) + p(m)\, \Delta_{u,v} q(m) + Q_\tau(\partial p(m)\, u, \partial q(m)\, v) + Q_\tau(\partial p(m)\, v, \partial q(m)\, u)$$

for all $m \in \mathcal{M}$ and $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\text{op}}$.

Proof. By Proposition 1.3.3(ii) and the definition of ∂^2, if A is a unital $\mathbb{C}$-algebra and $p(\lambda), q(\lambda)$ are polynomials, then

$$\begin{aligned}
\partial^2(pq)(a_1, a_2, a_3) = {} & \partial^2 p(a_1, a_2, a_3)(1 \otimes 1 \otimes q(a_3)) \\
& + (p(a_1) \otimes 1 \otimes 1)\partial^2 q(a_1, a_2, a_3) \\
& + 2(\partial p(a_1, a_2) \otimes 1)(1 \otimes \partial q(a_2, a_3)), \qquad a_1, a_2, a_3 \in A.
\end{aligned}$$

Applying this to the algebra $A = \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ and writing $\mathbf{1} = 1 \otimes 1$ for the identity in $\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ to avoid confusion, we have

$$\begin{aligned}\partial^2(pq)(m\otimes 1, 1\otimes m, m\otimes 1) = {} & \partial^2 p(m\otimes 1, 1\otimes m, m\otimes 1)(\mathbf{1}\otimes\mathbf{1}\otimes q(m\otimes 1)) \\ & + (p(m\otimes 1)\otimes\mathbf{1}\otimes\mathbf{1})\partial^2 q(m\otimes 1, 1\otimes m, m\otimes 1) \\ & + 2(\partial p(m\otimes 1, 1\otimes m)\otimes\mathbf{1})(\mathbf{1}\otimes\partial q(1\otimes m, m\otimes 1))\end{aligned}$$

for all $m \in \mathcal{M}$. Now, notice that if $u_1, u_2 \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ and $A \in (\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})^{\otimes 3}$, then

$$((u_1\otimes\mathbf{1}\otimes\mathbf{1})A(\mathbf{1}\otimes\mathbf{1}\otimes u_2))\#_2^{\otimes}[c,d] = u_1(A\#_2^{\otimes}[c,d])u_2.$$

Since

$$p(m\otimes 1) = p(m)\otimes 1 \ \text{ and } \ q(m\otimes 1) = q(m)\otimes 1,$$

it follows from the above that if $a \in \mathcal{M}$, then

$$\begin{aligned}&\tau\big(a\,\Delta_{u,v}(pq)(m)\big) \\ &\quad= \frac{1}{2}(\tau\bar{\otimes}\tau^{\mathrm{op}})\big((a\otimes 1)\big(\partial^2 p(m\otimes 1, 1\otimes m, m\otimes 1)\#_2^{\otimes}[uv^{\mathrm{flip}} + vu^{\mathrm{flip}}, \mathbf{1}]\big)(q(m)\otimes 1)\big) \\ &\qquad+ \frac{1}{2}(\tau\bar{\otimes}\tau^{\mathrm{op}})\big((a\otimes 1)(p(m)\otimes 1)\partial^2 q(m\otimes 1, 1\otimes m, m\otimes 1)\#_2^{\otimes}[uv^{\mathrm{flip}} + vu^{\mathrm{flip}}, \mathbf{1}]\big) \\ &\qquad+ (\tau\bar{\otimes}\tau^{\mathrm{op}})\big((a\otimes 1)((\partial p(m\otimes 1, 1\otimes m)\otimes\mathbf{1})(\mathbf{1}\otimes\partial q(1\otimes m, m\otimes 1)))\#_2^{\otimes}[uv^{\mathrm{flip}} + vu^{\mathrm{flip}}, \mathbf{1}]\big) \\ &\quad= \frac{1}{2}(\tau\bar{\otimes}\tau^{\mathrm{op}})\big(((q(m)\,a)\otimes 1)\partial^2 p(m\otimes 1, 1\otimes m, m\otimes 1)\#_2^{\otimes}[uv^{\mathrm{flip}} + vu^{\mathrm{flip}}, \mathbf{1}]\big) \\ &\qquad+ \frac{1}{2}(\tau\bar{\otimes}\tau^{\mathrm{op}})\big(((a\,p(m))\otimes 1)\partial^2 q(m\otimes 1, 1\otimes m, m\otimes 1)\#_2^{\otimes}[uv^{\mathrm{flip}} + vu^{\mathrm{flip}}, \mathbf{1}]\big) + R_a \\ &\quad= \tau\big(q(m)\,a\,\Delta_{u,v}p(m)\big) + \tau\big(a\,p(m)\,\Delta_{u,v}q(m)\big) + R_a \\ &\quad= \tau(a\,\Delta_{u,v}p(m)\,q(m)) + \tau\big(a\,p(m)\,\Delta_{u,v}q(m)\big) + R_a,\end{aligned}$$

where

$$R_a = (\tau\bar{\otimes}\tau^{\mathrm{op}})\big((a\otimes 1)((\partial p(m\otimes 1, 1\otimes m)\otimes\mathbf{1})(\mathbf{1}\otimes\partial q(1\otimes m, m\otimes 1)))\#_2^{\otimes}[uv^{\mathrm{flip}} + vu^{\mathrm{flip}}, \mathbf{1}]\big).$$

Now, note that if $P_1(\lambda_1, \lambda_2) = \lambda_1^{\gamma_1}\lambda_2^{\gamma_2}$, $P_2(\lambda_1, \lambda_2) = \lambda_1^{\delta_1}\lambda_2^{\delta_2}$, $u_1 = m \otimes 1$, and $u_2 = 1 \otimes m$, then

$$\begin{aligned}
&(\tau\bar{\otimes}\tau^{\text{op}})\big((a\otimes 1)((P_1(u_1\otimes \mathbf{1}, \mathbf{1}\otimes u_2)\otimes \mathbf{1})(\mathbf{1}\otimes P_2(u_2\otimes \mathbf{1}, \mathbf{1}\otimes u_1)))\#_2^{\otimes}[uv^{\text{flip}}, \mathbf{1}]\big)\\
&\quad = (\tau\bar{\otimes}\tau^{\text{op}})\big((a\otimes 1)u_1^{\gamma_1}uv^{\text{flip}}u_2^{\gamma_2}u_2^{\delta_1}u_1^{\delta_2}\big) = (\tau\bar{\otimes}\tau^{\text{op}})\big((a\otimes 1)u_1^{\gamma_1}u_2^{\gamma_2}u(u_1^{\delta_1}u_2^{\delta_2}v)^{\text{flip}}\big)\\
&\quad = (\tau\bar{\otimes}\tau^{\text{op}})\big((a\otimes 1)\,P_1(m\otimes 1, 1\otimes m)u(P_2(m\otimes 1, 1\otimes m)v)^{\text{flip}}\big)
\end{aligned}$$

by the traciality of $\tau\bar{\otimes}\tau^{\text{op}}$, the fact that $u_2 = 1\otimes m$ commutes with both $a\otimes 1$ and $u_1 = m\otimes 1$, and the identity $u_1^{\text{flip}} = u_2$. By linearity, the above formula holds for all polynomials P_1, P_2 in two variables. Applying the formula to $P_1 = p^{[1]}$ and $P_2 = q^{[1]}$ gives

$$\begin{aligned}
R_a &= (\tau\bar{\otimes}\tau^{\text{op}})\big((a\otimes 1)\partial p(m)u(\partial q(m)v)^{\text{flip}}\big) + (\tau\bar{\otimes}\tau^{\text{op}})\big((a\otimes 1)\partial p(m)v(\partial q(m)u)^{\text{flip}}\big)\\
&= \tau(a\,Q_\tau(\partial p(m)\,u, \partial q(m)\,v)) + \tau(a\,Q_\tau(\partial p(m)\,v, \partial q(m)\,u)).
\end{aligned}$$

This completes the proof. □

We are now ready for the functional free Itô formula for polynomials.

Theorem 7.5.7 (Functional free Itô formula for polynomials)**.** *Let $p(\lambda) \in \mathbb{C}[\lambda]$ be a polynomial.*

(i) *Suppose $(x_1, \ldots, x_n)\colon \mathbb{R}_+ \to \mathcal{M}_{\text{sa}}^n$ is an n-dimensional semicircular Brownian motion. If m is a free Itô process satisfying Equation* (7.4.2)*, then*

$$\mathrm{d}p(m(t)) = \partial p(m(t))\#\mathrm{d}m(t) + \frac{1}{2}\sum_{i=1}^{n}\Delta_{u_i(t)}p(m(t))\,\mathrm{d}t.$$

(ii) *Suppose $(z_1, \ldots, z_n)\colon \mathbb{R}_+ \to \mathcal{M}^n$ is an n-dimensional circular Brownian motion. If m is a free Itô process satisfying Equation* (7.4.3)*, then*

$$\mathrm{d}p(m(t)) = \partial p(m(t))\#\mathrm{d}m(t) + \sum_{i=1}^{n}\Delta_{u_i(t),v_i(t)}p(m(t))\,\mathrm{d}t.$$

Remark 7.5.8. In either case, the map $\mathbb{R}_+ \ni t \mapsto \partial p(m(t)) \in \mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\text{op}}$ is continuous and adapted. In particular, if $\ell \in L^1_{\text{loc}}(\mathbb{R}_+; \mathcal{M})$ and $u \in \Lambda^2$, then $\partial p(m)\#\ell \in L^1_{\text{loc}}(\mathbb{R}_+; \mathcal{M})$, and, by Corollary 7.3.13, $\partial p(m)\,u \in \Lambda^2$. Thus, the integrals in Theorem 7.5.7 make sense.

Proof. Using the comments after Definition 7.4.1, it is easy to see that the second item follows from the first with twice the dimension. It therefore suffices to prove the first item. To this end, let $p(\lambda), q(\lambda(\in \mathbb{C}[\lambda]$ be polynomials, and suppose the formula in Theorem 7.5.7(i) holds for both $p(\lambda)$ and $q(\lambda)$. Then the free Itô product rule (Theorem 7.4.9), Proposition 1.3.3(ii), the definition of ∂, and Lemma 7.5.6 give

$$\begin{aligned} \mathrm{d}(pq)(m(t)) &= \mathrm{d}p(m(t))\, q(m(t)) + p(m(t))\, \mathrm{d}q(m(t)) + \mathrm{d}p(m(t))\, \mathrm{d}q(m(t)) \\ &= \big((1 \otimes q(m(t)))\partial p(m(t)) + (p(m(t)) \otimes 1)\partial q(m(t))\big)\#\mathrm{d}m(t) \\ &\qquad + \sum_{i=1}^{n} \Big(\frac{1}{2}\big(\Delta_{u_i(t)} p(m(t))\, q(m(t)) + p(m(t))\Delta_{u_i(t)} q(m(t))\big) \\ &\qquad\qquad + Q_\tau\big(\partial p(m(t))\, u_i(t), \partial q(m(t))\, u_i(t)\big)\Big)\, \mathrm{d}t \\ &= \partial(pq)(m(t))\#\mathrm{d}m(t) + \frac{1}{2}\sum_{i=1}^{n} \Delta_{u_i(t)}(pq)(m(t))\, \mathrm{d}t. \end{aligned}$$

Thus, the formula of interest holds for the polynomial pq as well.

Next, note that the formula holds trivially for $p(\lambda) = p_0(\lambda) \equiv 1$ and $p(\lambda) = p_1(\lambda) = \lambda$. Now, let $n \geq 1$, and assume the formula holds for $p(\lambda) = p_n(\lambda) = \lambda^n$. By what we just proved, this implies the formula holds for $p(\lambda) = p_n(\lambda)p_1(\lambda) = \lambda^{n+1} = p_{n+1}(\lambda)$. By induction, the formula holds for $p(\lambda) = p_n(\lambda)$ whenever $n \in \mathbb{N}_0$ is arbitrary. Since $\{p_n(\lambda) : n \in \mathbb{N}_0\}$ is a basis for $\mathbb{C}[\lambda]$, we are done. □

7.6 Traced formula

From Theorem 7.5.7 and a symmetrization argument, we obtain a highly useful "traced" functional free Itô formula. To state it, must extend the definition of noncommutative derivatives (with self-adjoint inputs). Let $\mathcal{A}$ be a unital C^*-algebra. If $k \in \mathbb{N}$ and $a_1, \ldots, a_{k+1} \in \mathcal{A}_{\mathrm{sa}}$, then

$$\mathbf{a}_\otimes := \big(1^{\otimes(i-1)} \otimes a_i \otimes 1^{\otimes(k+1-i)}\big)_{i=1}^{k+1} \in \big(\mathcal{A}^{\otimes(k+1)}\big)^{k+1} \subseteq \big(\mathcal{A}^{\otimes_{\min}(k+1)}\big)^{k+1}$$

is a $(k+1)$-tuple of commuting, self-adjoint elements in $\mathcal{A}^{\otimes_{\min}(k+1)}$ with joint spectrum equal to all of $\sigma(a_1) \times \cdots \times \sigma(a_{k+1}) \subseteq \mathbb{R}^{k+1}$; please see [CV78]. The following definition therefore makes sense using the multivariate continuous functional calculus [DL90, App., §5].

Definition 7.6.1 (Noncommutative derivatives of C^k functions)**.** Let $\mathcal{A}$ be a unital C^*-algebra. If $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in \mathcal{A}_{\mathrm{sa}}^{k+1}$ and $f \in C^k(\mathbb{R})$, then

$$\partial^k f(\mathbf{a}) \coloneqq k!\, f^{[k]}(\mathbf{a}_\otimes) \in \mathcal{A}^{\otimes_{\min}(k+1)}$$

is the $\boldsymbol{k}^{\mathbf{th}}$ **noncommutative derivative** of f evaluated at $\mathbf{a}$. As in the polynomial case, we often write $\partial \coloneqq \partial^1$ and consider $\partial f(a_1, a_2)$ as an element of $\mathcal{A} \otimes_{\min} \mathcal{A}^{\mathrm{op}}$. Also, write

$$\partial^k f(a) \coloneqq \partial^k f\big(a_{(k+1)}\big), \qquad a \in \mathcal{A}_{\mathrm{sa}},$$

using Notation 1.2.5(i).

Of course, if we view $\mathcal{A}^{\otimes(k+1)}$ as a subalgebra of $\mathcal{A}^{\otimes_{\min}(k+1)}$, then Definition 7.6.1 agrees with Definition 7.5.1 when $f(\lambda) = p(\lambda) \in \mathbb{C}[\lambda]$.

Example 7.6.2 (Wiener space functions)**.** Let $k \in \mathbb{N}$, and suppose $f = \int_{\mathbb{R}} e^{i\cdot\xi}\, \mu(\mathrm{d}\xi) \in W_k(\mathbb{R})$. If $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in \mathcal{A}_{\mathrm{sa}}^{k+1}$, it follows from Equation (1.3.16) that

$$\partial^k f(\mathbf{a}) = k! \int_{\Sigma_k} \int_{\mathbb{R}} (i\xi)^k e^{is_1\xi\, a_1} \otimes \cdots \otimes e^{is_k\xi\, a_k} \otimes e^{i(1-\sum_{j=1}^k s_j)\xi\, a_{k+1}} \mu(\mathrm{d}\xi)\, \mathrm{d}s_1 \cdots \mathrm{d}s_k$$

where the above is an iterated Bochner integral in $\mathcal{A}^{\otimes_{\min}(k+1)}$. When $k = 1$, we note for later use that actually $\partial f(a_1, a_2) = i \int_0^1 \int_{\mathbb{R}} \xi\, e^{ita_1} \otimes e^{i(1-t)a_2}\, \mu(\mathrm{d}\xi)\, \mathrm{d}t$ is an iterated Bochner integral in $\mathcal{A}\hat{\otimes}_\pi\mathcal{A}^{\mathrm{op}} \subseteq \mathcal{A} \otimes_{\min} \mathcal{A}^{\mathrm{op}}$ (with respect to $\|\cdot\|_{\mathcal{A}\hat{\otimes}_\pi\mathcal{A}^{\mathrm{op}}}$) because the map

$$[0,1] \times \mathbb{R} \ni (t, \xi) \mapsto \xi\, e^{ita_1} \otimes e^{i(1-t)a_2} \in \mathcal{A}\hat{\otimes}_\pi\mathcal{A}^{\mathrm{op}}$$

is continuous.

Remark 7.6.3. More generally, one may calculate the k^{th} noncommutative derivative of a Varopoulos C^k function by passing the result of Proposition 3.5.3(i) (with $m = k+1$ and $\varphi = f^{[k]}$) through the natural map $\mathcal{A}^{\hat{\otimes}_\pi(k+1)} \to \mathcal{A}^{\otimes_{\min}(k+1)}$.

Before stating, giving examples of, and proving our traced formula, we present a rigorous proof of a “folklore” characterization of when a free Itô process is self-adjoint.

Proposition 7.6.4. *Suppose* $(x_1,\ldots,x_n)\colon \mathbb{R}_+ \to \mathcal{M}_{\mathrm{sa}}^n$ *is an* n*-dimensional semicircular Brownian motion. For each* $\ell \in \{1,2\}$*, let* m_ℓ *be a free Itô process satisfying*

$$\mathrm{d}m_\ell(t) = \sum_{i=1}^n u_{\ell i}(t)\#\mathrm{d}x_i(t) + k_\ell(t)\,\mathrm{d}t.$$

Then $m_1 = m_2$ *if and only if* $m_1(0) = m_2(0)$*,* $k_1 = k_2$ *almost everywhere, and* $u_{1i} = u_{2i}$ *almost everywhere for all* $i \in \{1,\ldots,n\}$*.*

Proof. Let m be a free Itô process satisfying Equation (7.4.2). It suffices to show that $m \equiv 0$ if and only if $m(0) = 0$, $k \equiv 0$ almost everywhere, and $u_1 = \cdots = u_n \equiv 0$ almost everywhere. The "if" direction is obvious. For the converse, suppose $m \equiv 0$. Then

$$0 = \mathrm{d}m^*(t) = \sum_{i=1}^n u_i^\star(t)\#\mathrm{d}x_i(t) + k^*(t)\,\mathrm{d}t,$$

so that

$$0 = \mathrm{d}(mm^*)(t) = \mathrm{d}m(t)\,m^*(t) + m(t)\,\mathrm{d}m^*(t) + \sum_{i=1}^n Q_\tau(u_i(t),u_i^\star(t))\,\mathrm{d}t = \sum_{i=1}^n Q_\tau(u_i(t),u_i^\star(t))\,\mathrm{d}t$$

by the free Itô product rule. In other words,

$$\int_0^t \sum_{i=1}^n Q_\tau(u_i(s),u_i^\star(s))\,\mathrm{d}s = 0, \qquad t \geq 0.$$

Therefore, $\sum_{i=1}^n Q_\tau(u_i(t),u_i^\star(t)) = 0$ for almost every $t \geq 0$ by, for instance, the (vector-valued) Lebesgue differentiation theorem. We claim this implies $u_1 = \cdots = u_n \equiv 0$ almost everywhere. Indeed, if $u \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ is arbitrary, then, by definition of Q_τ,

$$\tau(Q_\tau(u,u^\star)) = (\tau\bar{\otimes}\tau^{\mathrm{op}})(u(u^\star)^{\mathrm{flip}}) = (\tau\bar{\otimes}\tau^{\mathrm{op}})(uu^*) = (\tau\bar{\otimes}\tau^{\mathrm{op}})(u^*u) = \|u\|_{L^2(\tau\bar{\otimes}\tau^{\mathrm{op}})}^2.$$

Our claim is then proven by an appeal to the faithfulness of $\tau\bar{\otimes}\tau^{\mathrm{op}}$. We are now left with the fact that $\int_0^t k(s)\,\mathrm{d}s = 0$ for all $t \geq 0$. Once again, it follows that $k \equiv 0$ almost everywhere. $\square$

Corollary 7.6.5. *A free Itô process m as in Equation* (7.4.2) *satisfies $m^* = m$ if and only if $m(0)^* = m(0)$, $k^* = k$ almost everywhere, and $u_i^\star = u_i$ almost everywhere for all $i \in \{1, \ldots, n\}$. Also, a free Itô process m as in Equation* (7.4.3) *satisfies $m^* = m$ if and only if $m(0)^* = m(0)$, $k^* = k$ almost everywhere, and $u_i^\star = v_i$ almost everywhere for all $i \in \{1, \ldots, n\}$.*

We now state the traced formula.

Theorem 7.6.6 (Traced Functional Free Itô Formula)**.** *The following formulas hold.*

(i) *Suppose $(x_1, \ldots, x_n) \colon \mathbb{R}_+ \to \mathcal{M}_{\mathrm{sa}}^n$ is an n-dimensional semicircular Brownian motion. If m is a free Itô process satisfying Equation* (7.4.2) *and $f \in \mathbb{C}[\lambda]$, then*

$$\tau(f(m)) = \tau(f(m(0))) + \int_0^{\cdot} \Big(\tau\big(f'(m(t))\, k(t)\big) + \frac{1}{2} \sum_{i=1}^n (\tau \bar{\otimes} \tau^{\mathrm{op}})\big(u_i^{\mathrm{flip}}(t)\, \partial f'(m(t))\, u_i(t)\big)\Big)\, \mathrm{d}t. \tag{7.6.7}$$

If $m^* = m$ (i.e., $m(0)^* = m(0)$, $k^* = k$ a.e., and $u_i^\star = u_i$ a.e. for all i), then Equation (7.6.7) *holds for any $f \colon \mathbb{R} \to \mathbb{C}$ that is C^2 in a neighborhood of the closure of $\bigcup_{t \geq 0} \sigma(m(t))$.*

(ii) *Suppose $(z_1, \ldots, z_n) \colon \mathbb{R}_+ \to \mathcal{M}^n$ is an n-dimensional circular Brownian motion. If m is a free Itô process satisfying Equation* (7.4.3) *and $f \in \mathbb{C}[\lambda]$, then*

$$\tau(f(m)) = \tau(f(m(0))) + \int_0^{\cdot} \Big(\tau\big(f'(m(t))\, k(t)\big) + \sum_{i=1}^n (\tau \bar{\otimes} \tau^{\mathrm{op}})\big(v_i^{\mathrm{flip}}(t)\, \partial f'(m(t))\, u_i(t)\big)\Big)\, \mathrm{d}t. \tag{7.6.8}$$

If $m^* = m$ (i.e., $m(0)^* = m(0)$, $k^* = k$ a.e., and $u_i^\star = v_i$ a.e. for all i), then Equation (7.6.8) *holds for any $f \colon \mathbb{R} \to \mathbb{C}$ that is C^2 in a neighborhood of the closure of $\bigcup_{t \geq 0} \sigma(m(t))$.*

Remark 7.6.9. Let m be as in Equation (7.4.2). Note that if $m^* = m$ and $f \colon \mathbb{R} \to \mathbb{C}$ is C^2 on a neighborhood of the closure of $\bigcup_{t \geq 0} \sigma(m(t))$, then

$$(\tau \bar{\otimes} \tau^{\mathrm{op}})(u_i^{\mathrm{flip}}\, \partial f'(m)\, u_i) = \langle \partial f'(m)\, u_i, u_i \rangle_{L^2(\tau \bar{\otimes} \tau^{\mathrm{op}})}$$

because $u_i^{\mathrm{flip}} = u_i^*$. By the functional-calculus-based definition of $\partial f'(m)$, we therefore may read

Equation (7.6.7) (almost everywhere) more pleasantly as

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau(f(m(t))) = \tau\big(f'(m(t))\,k(t)\big) + \frac{1}{2}\sum_{i=1}^{n}\int_{\mathbb{R}^2}\frac{f'(\lambda)-f'(\mu)}{\lambda-\mu}\,\rho_{m(t),u_i(t)}(\mathrm{d}\lambda,\mathrm{d}\mu),$$

where $\rho_{m,u_i}(\mathrm{d}\lambda,\mathrm{d}\mu) \coloneqq \langle P^{m\otimes 1, 1\otimes m}(\mathrm{d}\lambda,\mathrm{d}\mu)\,u_i,u_i\rangle_{L^2(\tau\bar{\otimes}\tau^{\mathrm{op}})}$. Here, $P^{m\otimes 1,1\otimes m}$ is the projection-valued joint spectral measure of $(m\otimes 1, 1\otimes m)$. Similar comments apply to Equation (7.6.8).

Before proving this theorem, we demonstrate its utility.

Example 7.6.10. Let $z\colon \mathbb{R}_+ \to \mathcal{M}$ be a circular Brownian motion, and let $a_i, b_i, k\colon \mathbb{R}_+ \to \mathcal{M}$ $(1 \le i \le n)$ be continuous adapted processes. Suppose $m\colon \mathbb{R}_+ \to \mathcal{M}$ satisfies

$$\mathrm{d}m(t) = \sum_{i=1}^{n}(a_i(t)\,\mathrm{d}z(t)\,b_i(t) + c_i(t)\,\mathrm{d}z^*(t)\,d_i(t)) + k(t)\,\mathrm{d}t.$$

Now, suppose in addition that $m \ge 0$ (i.e., $m^* = m$ and $\sigma(m(t)) \subseteq \mathbb{R}_+$ whenever $t \ge 0$). For example, if $\widetilde{m}$ is as in Equation (7.4.13) and $m \coloneqq |\widetilde{m}|^2 = \widetilde{m}^*\widetilde{m}$, then, as is shown in Example 7.4.10, m is a free Itô process of the form we have just described.

Now, let $\varepsilon > 0$, and define $f_\varepsilon(\lambda) \coloneqq \log(\lambda+\varepsilon)$ whenever $\lambda > -\varepsilon$ and $f_\varepsilon \equiv 0$ on $(-\infty,-\varepsilon]$. Then $f_\varepsilon \in C^\infty((-\varepsilon,\infty))$ and $\bigcup_{t\ge 0}\sigma(m(t)) \subseteq \mathbb{R}_+ \subseteq (-\varepsilon,\infty)$. Also,

$$f_\varepsilon'(\lambda) = \frac{1}{\lambda+\varepsilon} \quad\text{and}\quad (f_\varepsilon')^{[1]}(\lambda,\mu) = \frac{(\lambda+\varepsilon)^{-1}-(\mu+\varepsilon)^{-1}}{\lambda-\mu} = -\frac{1}{(\lambda+\varepsilon)(\mu+\varepsilon)}, \qquad \lambda,\mu > -\varepsilon.$$

Thus,

$$f_\varepsilon'(m) = (m+\varepsilon)^{-1} \quad\text{and}\quad \partial f_\varepsilon'(m) = (f_\varepsilon')^{[1]}(m\otimes 1, 1\otimes m) = -(m+\varepsilon)^{-1}\otimes(m+\varepsilon)^{-1}.$$

In particular, if $u = \sum_{i=1}^n a_i\otimes b_i$ and $v = \sum_{i=1}^n c_i \otimes d_i$, then

$$\begin{aligned} v^{\mathrm{flip}}\partial f_\varepsilon'(m)\,u &= -\sum_{i,j=1}^{n}(d_j\otimes c_j)((m+\varepsilon)^{-1}\otimes(m+\varepsilon)^{-1})(a_i\otimes b_i) \\ &= -\sum_{i,j=1}^{n}(d_j(m+\varepsilon)^{-1}a_i)\otimes(b_i(m+\varepsilon)^{-1}c_j). \end{aligned}$$

It follows from Theorem 7.6.6 and the fundamental theorem of calculus that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tau(f_\varepsilon(m(t))) &= \tau(f_\varepsilon'(m(t))\,k(t)) + (\tau\bar{\otimes}\tau^{\mathrm{op}})(v^{\mathrm{flip}}(t)\,\partial f_\varepsilon'(m(t))\,u(t)) \\ &= \tau((m(t)+\varepsilon)^{-1}k(t)) - \sum_{i,j=1}^{n}\tau\big(d_j(t)(m(t)+\varepsilon)^{-1}a_i(t)\big)\,\tau\big(b_i(t)(m(t)+\varepsilon)^{-1}c_j(t)\big) \quad (7.6.11)\end{aligned}$$

for all $t > 0$. Special cases of Equation (7.6.11) have shown up in the calculation of Brown measures of solutions to various free SDEs. Please see [DHK22, HZ23, DH22, HH22]. Thus far, such equations have been proven in the literature using power series arguments. Theorem 7.6.6 provides a more intuitive, natural way to do such calculations.

For concreteness, we demonstrate how Equation (7.6.11) leads to a nice re-proof of a key identity [DHK22, Lem. 5.2] that is used in the calculation of the Brown measure of the free multiplicative Brownian motion (starting at the identity). Similar calculations can be used to re-prove formulas in [HZ23, DH22, HH22].

We return to the setup of the end of Example 7.4.10, i.e.,

$$\begin{cases} \mathrm{d}g(t) = g(t)\,\mathrm{d}z(t) \\ g(0) = h, \end{cases}$$

We then take $g_\lambda \coloneqq g - \lambda$ ($\lambda \in \mathbb{C}$) and $m \coloneqq |g_\lambda|^2$. As we showed in Example 7.4.10,

$$\mathrm{d}m(t) = g_\lambda^*(t)\,g(t)\,\mathrm{d}z(t) + \mathrm{d}z^*(t)\,g^*(t)\,g_\lambda(t) + \tau(g^*(t)\,g(t))\,\mathrm{d}t.$$

By Equation (7.6.11),

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau(\log(m(t)+\varepsilon)) = \tau\big((m+\varepsilon)^{-1}\big)\tau(g^*g) - \tau\big(g^*\,g_\lambda\,(m+\varepsilon)^{-1}g_\lambda^*g\big)\tau\big((m+\varepsilon)^{-1}\big), \quad (7.6.12)$$

where the t's are suppressed on the right-hand side above for the sake of space. But now, $\tau\big(g^*g_\lambda(m+\varepsilon)^{-1}g_\lambda^*g\big) = \tau\big((m+\varepsilon)^{-1}g_\lambda^*gg^*g_\lambda\big)$, $\tau(g^*g) = \tau\big((m+\varepsilon)^{-1}(m+\varepsilon)g^*g\big)$, and

$$(m+\varepsilon)g^*g - g_\lambda^*gg^*g_\lambda = \varepsilon\,g^*g + g_\lambda^*g_\lambda g^*g - g_\lambda^*gg^*g_\lambda = \varepsilon\,g^*g$$

because $[g_\lambda, \lambda] = [g - \lambda, \lambda] = 0$. From Equation (7.6.12), we then get

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau\big(\log(|g(t)-\lambda|^2+\varepsilon)\big) = \varepsilon\,\tau\big((|g(t)-\lambda|^2+\varepsilon)^{-1}|g(t)|^2\big)\tau\big((|g(t)-\lambda|^2+\varepsilon)^{-1}\big), \qquad t > 0.$$

This is equivalent to (a generalization to arbitrary starting point of) [DHK22, Lem. 5.2].

We now begin the proof of Theorem 7.6.6, the keys to which are the following identities.

Lemma 7.6.13. *If $p(\lambda) \in \mathbb{C}[\lambda]$, $m, k \in \mathcal{M}$, and $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, then*

$$\tau(\partial p(m)\#k) = \tau(p'(m)\,k) \quad \textit{and} \quad \tau\big(\Delta_{u,v}p(m)\big) = (\tau\bar{\otimes}\tau^{\mathrm{op}})\big(v^{\mathrm{flip}}\partial p'(m)\,u\big).$$

Proof. Let $n \in \mathbb{N}_0$, and define $p_n(\lambda) \coloneqq \lambda^n$. For the first identity, note that

$$\tau(\partial p_n(m)\#k) = \sum_{\delta_1+\delta_2=n-1} \tau(m^{\delta_1}k\,m^{\delta_2}) = \sum_{\delta_1+\delta_2=n-1}\tau(m^{\delta_2}m^{\delta_1}k) = \tau(n\,m^{n-1}k) = \tau(p_n'(m)\,k).$$

By linearity, the first desired identity holds for all $p(\lambda) \in \mathbb{C}[\lambda]$. Proving the second identity is slightly more involved. We begin by making two key observations. First, fix a polynomial $P(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{C}[\lambda_1,\lambda_2,\lambda_3]$ and two elements $u_1, u_2 \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ that commute. If we define $q(\lambda_1,\lambda_2) \coloneqq P(\lambda_1,\lambda_2,\lambda_1)$ and $\mathbf{1} \coloneqq 1 \otimes 1$, then

$$(\tau\bar{\otimes}\tau^{\mathrm{op}})\big(P(u_1\otimes\mathbf{1}\otimes\mathbf{1}, \mathbf{1}\otimes u_2\otimes\mathbf{1}, \mathbf{1}\otimes\mathbf{1}\otimes u_1)\#_2^{\otimes}[u,\mathbf{1}]\big) = (\tau\bar{\otimes}\tau^{\mathrm{op}})(q(u_1,u_2)\,u),$$

as the reader may easily verify. (The computation is similar to that of R_a in the proof of Lemma 7.5.6.) Second, $(\tau\bar{\otimes}\tau^{\mathrm{op}})(u^{\mathrm{flip}}) = (\tau\bar{\otimes}\tau^{\mathrm{op}})(u)$ because $(\tau\bar{\otimes}\tau^{\mathrm{op}})(a\otimes b) = \tau(a)\,\tau(b) = (\tau\bar{\otimes}\tau^{\mathrm{op}})(b\otimes a)$ whenever $a, b \in \mathcal{M}$ and $\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}}$ is σ-weakly dense in $\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$. Now, note that

$$(q(u_1,u_2)\,u)^{\mathrm{flip}} = u^{\mathrm{flip}}q(u_1,u_2)^{\mathrm{flip}} = u^{\mathrm{flip}}q(u_1^{\mathrm{flip}},u_2^{\mathrm{flip}}), \qquad u \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}} \tag{7.6.14}$$

Combining these observations and appealing again to traciality of $\tau\bar{\otimes}\tau^{\mathrm{op}}$, we get that if, in addition, $u_1^{\mathrm{flip}} = u_2$, and if $w \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ satisfies $w^{\mathrm{flip}} = w$, then

$$(\tau\bar{\otimes}\tau^{\mathrm{op}})\big(P(u_1\otimes\mathbf{1}\otimes\mathbf{1}, \mathbf{1}\otimes u_2\otimes\mathbf{1}, \mathbf{1}\otimes\mathbf{1}\otimes u_1)\#_2^{\otimes}[w,\mathbf{1}]\big) = (\tau\bar{\otimes}\tau^{\mathrm{op}})(r(u_1,u_2)\,w), \tag{7.6.15}$$

where

$$r(\lambda_1,\lambda_2) = \frac{q(\lambda_1,\lambda_2)+q(\lambda_2,\lambda_1)}{2} = \frac{P(\lambda_1,\lambda_2,\lambda_1)+P(\lambda_2,\lambda_1,\lambda_2)}{2}.$$

Now, if $P = 2\,p^{[2]}$, then

$$r(\lambda_1,\lambda_2) = \frac{P(\lambda_1,\lambda_2,\lambda_1)+P(\lambda_2,\lambda_1,\lambda_2)}{2} = p^{[2]}(\lambda_1,\lambda_2,\lambda_1)+p^{[2]}(\lambda_2,\lambda_1,\lambda_2) = (p')^{[1]}(\lambda_1,\lambda_2),$$

as can be seen by taking $\lambda_3 \to \lambda_1$ in the definition of $p^{[2]}(\lambda_1,\lambda_2,\lambda_3)$ and using the symmetry of $p^{[1]}$. Therefore, if we apply Equation (7.6.15) with $P = 2\,p^{[2]}$, $u_1 = m \otimes 1$, $u_2 = 1 \otimes m$, and $w = (uv^{\text{flip}} + vu^{\text{flip}})/2$, then we obtain

$$\tau\big(\Delta_{u,v}p(m)\big) = \frac{1}{2}(\tau\bar{\otimes}\tau^{\text{op}})\big(\partial p'(m)\,(uv^{\text{flip}} + vu^{\text{flip}})\big) \tag{7.6.16}$$

by definition of $\Delta_{u,v}p(m)$ and noncommutative derivatives. To complete the proof, notice that if $q(\lambda_1,\lambda_2) \in \mathbb{C}[\lambda_1,\lambda_2]$ is symmetric, u_1 and u_2 satisfy $u_1^{\text{flip}} = u_2$, and $w \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\text{op}}$ is arbitrary, then, by Equation (7.6.14),

$$\begin{aligned}(\tau\bar{\otimes}\tau^{\text{op}})(q(u_1,u_2)\,w) &= (\tau\bar{\otimes}\tau^{\text{op}})\big((q(u_1,u_2)\,w)^{\text{flip}}\big) = (\tau\bar{\otimes}\tau^{\text{op}})\big(w^{\text{flip}}q(u_2,u_1)\big)\\ &= (\tau\bar{\otimes}\tau^{\text{op}})\big(w^{\text{flip}}q(u_1,u_2)\big) = (\tau\bar{\otimes}\tau^{\text{op}})\big(q(u_1,u_2)\,w^{\text{flip}}\big).\end{aligned}$$

Therefore, Equation (7.6.16) reduces to

$$\tau(\Delta_{u,v}p(m)) = (\tau\bar{\otimes}\tau^{\text{op}})(\partial p'(m)\,uv^{\text{flip}}) = (\tau\bar{\otimes}\tau^{\text{op}})(v^{\text{flip}}\partial p'(m)\,u),$$

as desired. □

Proof of Theorem 7.6.6. We prove Theorem 7.6.6(i) using Theorem 7.5.7(i). Theorem 7.6.6(ii) follows in the exact same way from Theorem 7.5.7(ii).

Fix an n-dimensional semicircular Brownian motion $(x_1,\ldots,x_n)\colon \mathbb{R}_+ \to \mathcal{M}_{\text{sa}}^n$, and suppose m is a free Itô process satisfying Equation (7.4.2). Since free stochastic integrals against x_i are noncommutative martingales that start at zero, they have trace zero. Thus, applying τ to the result of Theorem 7.5.7(i), bringing τ (which is bounded-linear) into the Bochner integrals, and appealing to Lemma 7.6.13, we have

$$\begin{aligned}\tau(p(m)) &= \tau(p(m(0))) + \int_0^{\cdot} \Big(\tau(\partial p(m(t))\#k(t)) + \frac{1}{2}\sum_{i=1}^n \tau\big(\Delta_{u_i(t)}p(m(t))\big)\Big)\,\mathrm{d}t \\ &= \tau(p(m(0))) + \int_0^{\cdot} \Big(\tau\big(p'(m(t))\,k(t)\big) + \frac{1}{2}\sum_{i=1}^n (\tau\bar{\otimes}\tau^{\mathrm{op}})\big(u_i^{\mathrm{flip}}(t)\,\partial p'(m(t))\,u_i(t)\big)\Big)\,\mathrm{d}t\end{aligned}$$

whenever $p(\lambda) \in \mathbb{C}[\lambda]$.

Suppose now that $m^* = m$ and $U \subseteq \mathbb{R}$ is an open set containing $\overline{\bigcup_{t\geq 0}\sigma(m(t))}$ such that $f \in C^2(U)$. Since m is continuous in the operator norm, m is locally bounded in the operator norm. In particular,

$$K_t \coloneqq \overline{\bigcup_{0\leq s\leq t}\sigma(m(s))} \subseteq U$$

is compact. Next, fix $t \geq 0$, and let $V_t \subseteq \mathbb{R}$ and $g_t \in C^2(\mathbb{R})$ be such that V_t is open, $K_t \subseteq V_t \subseteq U$, and $g_t = f$ on V_t. By the classical Weierstrass approximation theorem, there exists a sequence $(q_N)_{N\in\mathbb{N}}$ of polynomials such that for all $i \in \{0,1,2\}$, $q_N^{(i)} \to g_t^{(i)}$ uniformly on compact subsets of $\mathbb{R}$ as $N \to \infty$. In particular, $(q_N')^{[1]} \to (g_t')^{[1]}$ uniformly on compact subsets of $\mathbb{R}^2$ as $N \to \infty$. But now, we know from the previous paragraph that $\tau(q_N(m(t)))$ equals

$$\tau(q_N(m(0))) + \int_0^t \Big(\tau\big(q_N'(m(s))k(s)\big) + \frac{1}{2}\sum_{i=1}^n (\tau\bar{\otimes}\tau^{\mathrm{op}})\big(u_i^{\mathrm{flip}}(s)\,\partial q_N'(m(s))\,u_i(s)\big)\Big)\mathrm{d}s$$

for all $N \in \mathbb{N}$. By basic operator norm estimates on the functional calculus and the dominated convergence theorem, we can take $N \to \infty$ in this identity to conclude

$$\tau(g_t(m(t))) = \tau(g_t(m(0))) + \int_0^t \Big(\tau\big(g_t'(m(s))\,k(s)\big) + \frac{1}{2}\sum_{i=1}^n (\tau\bar{\otimes}\tau^{\mathrm{op}})\big(u_i^{\mathrm{flip}}(s)\,\partial g_t'(m(s))\,u_i(s)\big)\Big)\,\mathrm{d}s.$$

But $g_t = f$ on $V_t \supseteq K_t$ and thus $(g_t')^{[1]} = (f')^{[1]}$ on $K_t \times K_t$. We therefore have

$$g_t(m(s)) = f(m(s)) \;\text{ and }\; \partial g'(m(s)) = \partial f'(m(s)), \quad 0 \leq s \leq t.$$

Since $t \geq 0$ was arbitrary, this completes the proof. □

7.7 Functional free Itô formula for NC^2 functions

In this section, we reinterpret (and then redefine) the quantities $\partial f(m)\#k$ and $\Delta_{u,v}f(m)$ in terms of multiple operator integrals (MOIs). We shall use only the ("baby") version of the "separation of variables" approach detailed in §3.8.

We begin with a helpful observation. Fix $k \in \mathbb{N}$, and let $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in \mathcal{M}_{\mathrm{sa}}^{k+1}$. If $P(\boldsymbol{\lambda}) = \sum_{|\delta| \leq d} c_\delta \, \boldsymbol{\lambda}^\delta \in \mathbb{C}[\lambda_1, \ldots, \lambda_{k+1}]$, then

$$\big(I^{\mathbf{a}}P\big)[b] = \sum_{|\delta| \leq d} c_\delta \, a_1^{\delta_1} b_1 \cdots a_k^{\delta_k} b_k \, a_{k+1}^{\delta_{k+1}}, \qquad b = (b_1, \ldots, b_k) \in \mathcal{M}^k,$$

by definition of MOIs. In particular, by Example 1.3.8 and Equation (7.5.2), if $p(\lambda) \in \mathbb{C}[\lambda]$, then

$$\big(I^{a_1,a_2}p^{[1]}\big)[b] = \partial p(a_1, a_2)\#b \quad \text{and} \tag{7.7.1}$$

$$\big(I^{u_1,u_2,u_3}p^{[2]}\big)[v_1, v_2] = \frac{1}{2}\partial^2 p(u_1, u_2, u_3)\#_2^{\otimes}[v_1, v_2] \tag{7.7.2}$$

for all $a_1, a_2 \in \mathcal{M}_{\mathrm{sa}}$, $b \in \mathcal{M}$, $u_1, u_2, u_3 \in (\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})_{\mathrm{sa}}$, and $v_1, v_2 \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$. Recall that the operations $\#$ and $\#_2^{\otimes}$ are defined in Notation 7.3.2.

Now, for the term $\Delta_{u,v}f(m)$ in the functional free Itô formula(s) to come, we shall also need to understand MOIs of the form $\int_{\sigma(a_2)}\int_\Lambda\int_{\sigma(a_1)} \varphi(\lambda_1, \lambda_2, \lambda_3)\, P^{a_1}(\mathrm{d}\lambda_1)\, b_1\, \mu(\mathrm{d}\lambda_2)\, b_2\, P^{a_2}(\mathrm{d}\lambda_3)$, where Λ is a Polish space and μ is a Borel complex measure on Λ.

Lemma 7.7.3 (MOI with one complex measure)**.** *Let Λ be a Polish space and μ be a Borel complex measure on Λ. If $\varphi \in \ell^\infty(\sigma(a_1), \mathcal{B}_{\sigma(a_1)})\hat{\otimes}_i\ell^\infty(\Lambda, \mathcal{B}_\Lambda)\hat{\otimes}_i\ell^\infty(\sigma(a_2), \mathcal{B}_{\sigma(a_2)})$ and*

$$\varphi^\mu(\lambda_1, \lambda_3) := \int_\Lambda \varphi(\lambda_1, \lambda_2, \lambda_3)\, \mu(\mathrm{d}\lambda_2), \qquad (\lambda_1, \lambda_3) \in \sigma(a_1) \times \sigma(a_2),$$

then $\varphi^\mu \in \ell^\infty(\sigma(a_1), \mathcal{B}_{\sigma(a_1)})\hat{\otimes}_i\ell^\infty(\sigma(a_2), \mathcal{B}_{\sigma(a_2)})$. We shall write

$$\int_{\sigma(a_2)}\int_\Lambda\int_{\sigma(a_1)} \varphi(\lambda_1, \lambda_2, \lambda_3)\, P^{a_1}(\mathrm{d}\lambda_1)\, b_1\, \mu(\mathrm{d}\lambda_2)\, b_2\, P^{a_2}(\mathrm{d}\lambda_3) := (I^{a_1,a_2}\varphi^\mu)[b_1b_2] \in \mathcal{M}$$

for all $b_1, b_2 \in \mathcal{M}$.

Proof. If $(\Sigma,\rho,\varphi_1,\varphi_2,\varphi_3)$ is an ℓ^∞-IPD of φ and

$$\varphi_1^\mu(\lambda_1,\sigma) \coloneqq \varphi_1(\lambda_1,\sigma)\int_\Lambda \varphi_2(\lambda_2,\sigma)\,\mu(\mathrm{d}\lambda_2) \;\text{ and }\; \varphi_3^\mu(\lambda_3,\sigma) \coloneqq \varphi_3(\lambda_3,\sigma)$$

for all $\lambda_1 \in \sigma(a_1)$, $\lambda_3 \in \sigma(a_2)$, and $\sigma \in \Sigma$, then $(\Sigma,\rho,\varphi_1^\mu,\varphi_3^\mu)$ is an ℓ^∞-IPD of φ^μ. □

It follows from the proof above and the definition of MOIs that

$$\begin{aligned}\int_{\sigma(a_2)}\int_\Lambda\int_{\sigma(a_1)} &\varphi(\lambda_1,\lambda_2,\lambda_3)\,P^{a_1}(\mathrm{d}\lambda_1)\,b_1\,\mu(\mathrm{d}\lambda_2)\,b_2\,P^{a_2}(\mathrm{d}\lambda_3)\\ &= \int_\Sigma \mu(\varphi_2(\cdot,\sigma))\,\varphi_1(a_1,\sigma)\,b_1b_2\,\varphi_3(a_2,\sigma)\,\rho(\mathrm{d}\sigma) \qquad (7.7.4)\end{aligned}$$

whenever $(\Sigma,\rho,\varphi_1,\varphi_2,\varphi_3)$ is an ℓ^∞-IPD of φ, where $\mu(\varphi_2(\cdot,\sigma)) \coloneqq \int_\Lambda \varphi_2(\lambda,\sigma)\mu(\mathrm{d}\lambda)$.

We now identify $\partial f(m)\#k$ as an MOI.

Lemma 7.7.5. *If $f \in W_1(\mathbb{R})_{\mathrm{loc}}$ and $a_1,a_2 \in \mathcal{M}_{\mathrm{sa}}$, then*

$$\partial f(a_1,a_2) \in \mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}} \;\text{ and }\; \partial f(a_1,a_2)\#b = \big(I^{a_1,a_2}f^{[1]}\big)[b], \qquad b \in \mathcal{M}.$$

Moreover, the map $\mathcal{M}_{\mathrm{sa}}^2 \ni (a_1,a_2) \mapsto \partial f(a_1,a_2) \in \mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}}$ is continuous.

Proof. Since $W_1(\mathbb{R})_{\mathrm{loc}} \subseteq VC^1(\mathbb{R})$ and $\partial f(a_1,a_2) = f_\otimes^{[1]}(a_1,a_2)$, this is immediate from Example 3.8.18 and Proposition 3.5.12, but we provide a self-contained proof anyway.

Fix $a_1,a_2 \in \mathcal{M}_{\mathrm{sa}}$, and let $r>0$ be such that $\sigma(a_1)\cup\sigma(a_2) \subseteq [-r,r]$. By definition of $W_1(\mathbb{R})_{\mathrm{loc}}$, there exists a $g = \int_\mathbb{R} e^{i\cdot\xi}\,\mu(\mathrm{d}\xi) \in W_1(\mathbb{R})$ such that $g|_{[-r,r]} = f|_{[-r,r]}$. In particular, $g^{[1]}|_{[-r,r]^2} = f^{[1]}|_{[-r,r]^2}$, so $\partial f(a_1,a_2) = \partial g(a_1,a_2) \in \mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}}$ by Example 7.6.2. Furthermore, since $\mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}} \ni u \mapsto u\#b \in \mathcal{M}$ is a bounded linear map, the same example gives

$$\begin{aligned}\partial g(a_1,a_2)\#b &= \int_0^1\int_\mathbb{R} (i\xi)\,e^{ita_1}b\,e^{i(1-t)a_2}\,\mu(\mathrm{d}\xi)\,\mathrm{d}t\\ &= \int_{\mathbb{R}\times[0,1]} (i\xi)\,e^{ita_1}b\,e^{i(1-t)a_2}\,\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(\xi)\,|\mu|(\mathrm{d}\xi)\,\mathrm{d}t\\ &= \big(I^{a_1,a_2}g^{[1]}\big)[b] = \big(I^{a_1,a_2}f^{[1]}\big)[b]\end{aligned}$$

for all $b \in \mathcal{M}$, where the third identity holds by Equation (1.3.16) and the definition of MOIs.

For the continuity claim, note that the map

$$\mathcal{M}_{\mathrm{sa}}^2 \ni (a_1,a_2) \mapsto \int_0^1 \int_{\mathbb{R}} (i\xi)\, e^{ita_1} \otimes e^{i(1-t)a_2}\, \mu(\mathrm{d}\xi)\, \mathrm{d}t \in \mathcal{M}\hat{\otimes}_\pi \mathcal{M}^{\mathrm{op}}$$

is continuous by the dominated convergence theorem. In particular, the map $(a_1,a_2) \mapsto \partial g(a_1,a_2)$ is continuous. Since $\partial f(a_1,a_2) = \partial g(a_1,a_2)$ whenever $\sigma(a_1) \cup \sigma(a_2) \subseteq [-r,r]$, i.e., whenever $\|a_1\| \leq r$ and $\|a_2\| \leq r$, we conclude that the map $(a_1,a_2) \mapsto \partial f(a_1,a_2)$ is continuous on $\{(a_1,a_2) \in \mathcal{M}_{\mathrm{sa}}^2 : \|a_1\| \leq r,\ \|a_2\| \leq r\}$. Since $r > 0$ was arbitrary, we are done. $\square$

Since $C^2(\mathbb{R}) \subseteq W_1(\mathbb{R})_{\mathrm{loc}}$ (Proposition 3.4.6(iii)), the conclusion of Lemma 7.7.5 holds for all $f \in C^2(\mathbb{R})$. Thus, Equation (7.7.1) is a special case of Lemma 7.7.5.

Next, we make sense of $\Delta_{u,v} f(m)$ in terms of MOIs. If $f \in \mathcal{C}^{[2]}(\mathbb{R})$, $m \in \mathcal{M}_{\mathrm{sa}}$, and $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, then we define

$$\ell_{f,u,v}(a) \coloneqq (\tau\bar{\otimes}\tau^{\mathrm{op}})\big((a\otimes 1)\left(I^{m\otimes 1, 1\otimes m, m\otimes 1} f^{[2]}\right)[uv^{\mathrm{flip}} + vu^{\mathrm{flip}}, 1\otimes 1]\big), \qquad a \in L^1(\mathcal{M},\tau).$$

By Theorem 3.8.15(iii) (and Lemma 2.3.1),

$$\begin{aligned}
\|\ell_{f,u,v}\|_{L^1(\mathcal{M},\tau)^*} &\leq \big\|\left(I^{m\otimes 1, 1\otimes m, m\otimes 1} f^{[2]}\right)[uv^{\mathrm{flip}} + vu^{\mathrm{flip}}, 1\otimes 1]\big\|_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})} \\
&\leq \big\|f^{[2]}\big\|_{\ell^\infty(\sigma(m),\mathcal{B}_{\sigma(m)})^{\hat{\otimes}_i 3}} \|uv^{\mathrm{flip}} + vu^{\mathrm{flip}}\|_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})} \|1\otimes 1\|_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})} \\
&\leq 2\big\|f^{[2]}\big\|_{\ell^\infty(\sigma(m),\mathcal{B}_{\sigma(m)})^{\hat{\otimes}_i 3}} \|u\|_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})} \|v\|_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})} < \infty.
\end{aligned}$$

In particular, the following definition makes sense.

Definition 7.7.6. If $f \in \mathcal{C}^{[2]}(\mathbb{R})$, $m \in \mathcal{M}_{\mathrm{sa}}$, and $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, then we define $\Delta_{u,v} f(m)$ to be the unique element of $\mathcal{M}$ such that

$$\tau\big(a\,\Delta_{u,v} f(m)\big) = (\tau\bar{\otimes}\tau^{\mathrm{op}})\big((a\otimes 1)\left(I^{m\otimes 1, 1\otimes m, m\otimes 1} f^{[2]}\right)[uv^{\mathrm{flip}} + vu^{\mathrm{flip}}, 1\otimes 1]\big)$$

for all $a \in \mathcal{M}$ (or $a \in L^1(\mathcal{M},\tau)$). Also, write

$$\Delta_u f(m) \coloneqq \Delta_{u,u} f(m).$$

By Equation (7.7.2), Definition 7.7.6 agrees with Definition 7.5.5 when both definitions apply. Also, if $f \in \mathcal{C}^{[2]}(\mathbb{R})$, $m \in \mathcal{M}_{\mathrm{sa}}$, and $u, v \in \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, then

$$\|\Delta_{u,v}f(m)\| = \|\ell_{f,u,v}\|_{L^1(\mathcal{M},\tau)^*} \leq 2\big\|f^{[2]}\big\|_{\ell^\infty(\sigma(m),\mathcal{B}_{\sigma(m)})^{\hat{\otimes}_i 3}}\|u\|_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})}\|v\|_{L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})} \tag{7.7.7}$$

by the paragraph before Definition 7.7.6.

Lemma 7.7.8. *If $f \in NC^2(\mathbb{R})$, $m \in C(\mathbb{R}_+; \mathcal{M}_{\mathrm{sa}})$, and $u, v \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})$, then*

$$\Delta_{u,v}f(m) \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{M}) \quad \text{and} \quad \|\Delta_{u,v}f(m)\|_{L^1_t L^\infty(\tau)} \leq 2\big\|f^{[2]}\big\|_{r_t,3}\|u\|_{L^2_t L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})}\|v\|_{L^2_t L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})}$$

for all $t \geq 0$, where $r_t \coloneqq \|m\|_{L^\infty_t L^\infty(\tau)} = \sup_{0 \leq s \leq t}\|m(s)\|$.

Proof. When $f(\lambda) \in \mathbb{C}[\lambda]$, we know from §7.5 that $\Delta_{u,v}f(m) \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathcal{M})$. The claimed bound follows from applying Inequality (7.7.7) pointwise and then using the Cauchy–Schwarz inequality. If $f \in NC^2(\mathbb{R})$ is arbitrary, then there exists a sequence $(q_N)_{N\in\mathbb{N}}$ of polynomials converging in $NC^2(\mathbb{R})$ (i.e., in $\mathcal{C}^{[2]}(\mathbb{R})$) to f. What we just proved implies that the sequence $(\Delta_{u,v}q_N(m))_{N\in\mathbb{N}}$ is Cauchy in $L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{M})$, and Inequality (7.7.7) implies that $\Delta_{u,v}q_N(m) \to \Delta_{u,v}f(m)$ almost everywhere as $N \to \infty$. It follows that $\Delta_{u,v}f(m) \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{M})$ and that the claimed bound holds for $\Delta_{u,v}f(m)$ as well. □

We are finally ready for the functional free Itô formula for noncommutative C^2 functions.

Theorem 7.7.9 (Functional free Itô formula)**.** *Let $f \in NC^2(\mathbb{R})$.*

(i) *Suppose $(x_1, \ldots, x_n)\colon \mathbb{R}_+ \to \mathcal{M}^n_{\mathrm{sa}}$ is an n-dimensional semicircular Brownian motion. If m is a free Itô process satisfying Equation* (7.4.2) *and $m^* = m$, then*

$$\mathrm{d}f(m(t)) = \partial f(m(t))\#\mathrm{d}m(t) + \frac{1}{2}\sum_{i=1}^n \Delta_{u_i(t)}f(m(t))\,\mathrm{d}t. \tag{7.7.10}$$

(ii) *Suppose $(z_1, \ldots, z_n)\colon \mathbb{R}_+ \to \mathcal{M}^n$ is an n-dimensional circular Brownian motion. If m is a free Itô process satisfying Equation* (7.4.3) *and $m^* = m$, then*

$$\mathrm{d}f(m(t)) = \partial f(m(t))\#\mathrm{d}m(t) + \sum_{i=1}^n \Delta_{u_i(t),u_i^\star(t)}f(m(t))\,\mathrm{d}t.$$

Remark 7.7.11. In either case, $\mathbb{R}_+ \ni t \mapsto \partial f(m(t)) \in \mathcal{M}\hat{\otimes}_\pi\mathcal{M}^{\mathrm{op}}$ is continuous (Lemma 7.7.5) and adapted as in Example 7.3.14. In particular, if $\ell \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{M})$ and $u \in \Lambda^2$, then $\partial f(m)\#\ell \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{M})$, and, by Corollary 7.3.13, $\partial f(m)\, u \in \Lambda^2$. Thus, the integrals in the statement of Theorem 7.7.9 make sense.

Proof. As usual, the second item follows from the first with twice the dimension, so it suffices to prove the first item. To this end, let $m = m^*$ be a free Itô process satisfying Equation (7.4.2). By Theorem 7.5.7(i), Equation (7.7.10) holds when $f(\lambda) \in \mathbb{C}[\lambda]$. For general $f \in NC^2(\mathbb{R})$, let $(q_N)_{N\in\mathbb{N}}$ be a sequence of polynomials converging in $NC^2(\mathbb{R})$ to f, and fix $t \geq 0$. Since $q_N \to f$ uniformly on compact sets, $q_N(m(t)) = f(m(t))$ in $\mathcal{M}$ as $N \to \infty$. Next, let $i \in \{1,\ldots,n\}$. By Lemma 7.7.8, $\Delta_{u_i} q_N(m) \to \Delta_{u_i} f(m)$ in $L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{M})$ as $N \to \infty$. In particular,

$$L^\infty\text{-}\lim_{N\to\infty} \int_0^t \Delta_{u_i(s)} q_N(m(s))\,\mathrm{d}s = \int_0^t \Delta_{u_i(s)} f(m(s))\,\mathrm{d}s.$$

Now, if $r_t \coloneqq \sup_{0\leq s\leq t}\|m(s)\| < \infty$, then

$$\begin{aligned}\|\partial q_N(m)\,u_i - \partial f(m)\,u_i\|_{L^2_t L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})} &= \big\|(q_N - f)^{[1]}(m\otimes 1, 1\otimes m)\,u_i\big\|_{L^2_t L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})} \\ &\leq \big\|(q_N - f)^{[1]}\big\|_{\ell^\infty([-r_t,r_r]^2)}\|u_i\|_{L^2_t L^\infty(\tau\bar{\otimes}\tau^{\mathrm{op}})} \xrightarrow{N\to\infty} 0\end{aligned}$$

by basic properties of functional calculus and the fact that $\|\cdot\|_{\ell^\infty([-r_t,r_t]^2)} \leq \|\cdot\|_{r_t,2}$. Therefore,

$$L^\infty\text{-}\lim_{N\to\infty} \int_0^t (\partial q_N(m(s))\,u_i(s))\#\mathrm{d}x_i(s) = \int_0^t (\partial f(m(s))\,u_i(s))\#\mathrm{d}x_i(s)$$

by the L^∞-BDG inequality. Finally, by Lemma 7.7.5 and Theorem 3.8.15(iii),

$$\begin{aligned}\|\partial q_N(m)\#k - \partial f(m)\#k\|_{L^1_t L^\infty(\tau)} &= \big\|\big(I^{m,m}(q_N - f)^{[1]}\big)[k]\big\|_{L^1_t L^\infty(\tau)} \\ &\leq \big\|(q_N - f)^{[1]}\big\|_{r_t,2}\|k\|_{L^1_t L^\infty(\tau)} \xrightarrow{N\to\infty} 0.\end{aligned}$$

In particular,

$$L^\infty\text{-}\lim_{N\to\infty} \int_0^t \partial q_N(m(s))\#k(s)\,\mathrm{d}s = \int_0^t \partial f(m(s))\#k(s)\,\mathrm{d}s,$$

so we may deduce Equation (7.7.10) by taking $N \to \infty$ in the corresponding identity for q_N. $\square$

We end this section by deriving an explicit formula for $\Delta_{u,v}f(m)$ (with $u,v \in \mathcal{M} \otimes \mathcal{M}^{\text{op}}$) in terms of MOIs. Using this formula, we shall see directly that Theorem 7.7.9(i) generalizes [BS98, Prop. 4.3.4]. For this development, we shall view $\mathcal{M}$ as a W^*-subalgebra of $B(L^2(\mathcal{M},\tau))$ via the standard representation, i.e., as acting on $L^2(\mathcal{M},\tau)$ by left multiplication.

Proposition 7.7.12 (Explicit formula for $\Delta_{u,v}f(m)$)**.** *Let $f \in \mathcal{C}^{[2]}(\mathbb{R})$, let $m \in \mathcal{M}_{\text{sa}}$, and let $(\Sigma,\rho,\varphi_1,\varphi_2,\varphi_3)$ be an ℓ^∞-IPD of $f^{[2]}$ on $\sigma(m)^3$. If $u,v \in \mathcal{M} \otimes \mathcal{M}^{\text{op}}$, then*

$$\Delta_{u,v}f(m) = \int_\Sigma \mathcal{M}_\tau\big((1\otimes v)\cdot(\varphi_1(m,\sigma)\otimes\varphi_2(m,\sigma)\otimes\varphi_3(m,\sigma))\cdot(u\otimes 1) \\ + (1\otimes u)\cdot(\varphi_1(m,\sigma)\otimes\varphi_2(m,\sigma)\otimes\varphi_3(m,\sigma))\cdot(v\otimes 1)\big)\,\rho(\mathrm{d}\sigma),$$

where the right-hand side is a pointwise Pettis integral in $\mathcal{M} \subseteq B(L^2(\mathcal{M},\tau))$.

Proof. Write $\mathbf{1} \coloneqq 1\otimes 1$ and $\eta \coloneqq \tau\bar{\otimes}\tau^{\text{op}}$. If $a,b \in L^2(\mathcal{M},\tau)$ (so that $ab^* \in L^1(\mathcal{M},\tau)$), then

$$\begin{aligned}
\big\langle\big(\Delta_{u,v}f(m)\big)a,b\big\rangle_{L^2(\tau)} &= \tau\big(b^*\,\Delta_{u,v}f(m)a\big) = \tau\big(ab^*\,\Delta_{u,v}f(m)\big) \\
&= \eta\big((ab^*\otimes 1)\big(I^{m\otimes 1,1\otimes m,m\otimes 1}f^{[2]}\big)[uv^{\text{flip}}+vu^{\text{flip}},\mathbf{1}]\big) &\quad (7.7.13)\\
&= \eta\big((b\otimes 1)^*\big(I^{m\otimes 1,1\otimes m,m\otimes 1}f^{[2]}\big)[uv^{\text{flip}}+vu^{\text{flip}},\mathbf{1}](a\otimes 1)\big) \\
&= \big\langle\big(I^{m\otimes 1,1\otimes m,m\otimes 1}f^{[2]}\big)[uv^{\text{flip}}+vu^{\text{flip}},\mathbf{1}](a\otimes 1),b\otimes 1\big\rangle_{L^2(\eta)} \\
&= \int_\Sigma \langle\varphi_1(m\otimes 1,\sigma)(uv^{\text{flip}}+vu^{\text{flip}})\varphi_2(1\otimes m,\sigma)\,\varphi_3(m\otimes 1,\sigma)(a\otimes 1),b\otimes 1\rangle_{L^2(\eta)}\,\rho(\mathrm{d}\sigma) &\quad (7.7.14)\\
&= \int_\Sigma \langle(\varphi_1(m,\sigma)\otimes 1)(uv^{\text{flip}}+vu^{\text{flip}})(1\otimes\varphi_2(m,\sigma))(\varphi_3(m,\sigma)\otimes 1)(a\otimes 1),b\otimes 1\rangle_{L^2(\eta)}\,\rho(\mathrm{d}\sigma) \\
&= \int_\Sigma \eta\big((ab^*\otimes 1)(\varphi_1(m,\sigma)\otimes 1)(uv^{\text{flip}}+vu^{\text{flip}})(1\otimes\varphi_2(m,\sigma))(\varphi_3(m,\sigma)\otimes 1)\big)\,\rho(\mathrm{d}\sigma) \\
&= \int_\Sigma \tau\big(ab^*\,\mathcal{M}_\tau\big((1\otimes v)\cdot(\varphi_1(m,\sigma)\otimes\varphi_2(m,\sigma)\otimes\varphi_3(m,\sigma))\cdot(u\otimes 1) \\
&\qquad + (1\otimes u)\cdot(\varphi_1(m,\sigma)\otimes\varphi_2(m,\sigma)\otimes\varphi_3(m,\sigma))\cdot(v\otimes 1)\big)\big)\,\rho(\mathrm{d}\sigma) &\quad (7.7.15)\\
&= \int_\Sigma \big\langle\mathcal{M}_\tau\big((1\otimes v)\cdot(\varphi_1(m,\sigma)\otimes\varphi_2(m,\sigma)\otimes\varphi_3(m,\sigma))\cdot(u\otimes 1) \\
&\qquad + (1\otimes u)\cdot(\varphi_1(m,\sigma)\otimes\varphi_2(m,\sigma)\otimes\varphi_3(m,\sigma))\cdot(v\otimes 1)\big)a,b\big\rangle_{L^2(\tau)}\,\rho(\mathrm{d}\sigma).
\end{aligned}$$

Equation (7.7.13) holds by definition of $\Delta_{u,v}f(m)$, Equation (7.7.14) holds by definition of MOIs, and Equation (7.7.15) holds by Lemma 7.4.6 (and an elementary limiting argument). □

Corollary 7.7.16. *Retain the setting of Proposition 7.7.12. If $a, b, c, d \in \mathcal{M}$, then*

$$\Delta_{a\otimes b, c\otimes d} f(m) = \int_{\sigma(m)}\int_{\sigma(m)}\int_{\sigma(m)} f^{[2]}(\lambda_1,\lambda_2,\lambda_3)\, P^m(\mathrm{d}\lambda_1)\, a\, \tau(b\, P^m(\mathrm{d}\lambda_2)\, c)\, d\, P^m(\mathrm{d}\lambda_3)$$
$$+ \int_{\sigma(m)}\int_{\sigma(m)}\int_{\sigma(m)} f^{[2]}(\lambda_1,\lambda_2,\lambda_3)\, P^m(\mathrm{d}\lambda_1)\, c\, \tau(d\, P^m(\mathrm{d}\lambda_2)\, a)\, b\, P^m(\mathrm{d}\lambda_3).$$

Note $\mu(\mathrm{d}\lambda) = \tau(b\, P^m(\mathrm{d}\lambda)\, c)$ and $\nu(\mathrm{d}\lambda) = \tau(d\, P^m(\mathrm{d}\lambda)\, a)$ are Borel complex measures on $\sigma(m)$.

Proof. This follows from Proposition 7.7.12, the definition of $\mathcal{M}_\tau$, and Equation (7.7.4). □

Example 7.7.17 (Connection to Biane–Speicher formula)**.** Retain the setting of Proposition 7.7.12, but suppose $f \in W_2(\mathbb{R})_{\mathrm{loc}} \subseteq NC^2(\mathbb{R})$. Let $r := \|m\|$, and let $g = \int_{\mathbb{R}} e^{i\cdot\xi}\, \mu(\mathrm{d}\xi) \in W_2(\mathbb{R})$ be such that $g|_{[-r,r]} = f|_{[-r,r]}$. Since $f^{[2]}|_{[-r,r]^3} = g^{[2]}|_{[-r,r]^3}$, Equation (1.3.16) gives

$$f^{[2]}(\lambda_1,\lambda_2,\lambda_3) = \int_0^1\int_0^{1-t}\int_{\mathbb{R}} (i\xi)^2 e^{is\lambda_1\xi} e^{it\lambda_2\xi} e^{i(1-s-t)\lambda_3\xi}\, \mu(\mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t$$
$$= \int_{\mathbb{R}\times\Sigma_2} (i\xi)^2 e^{is\lambda_1\xi} e^{it\lambda_2\xi} e^{i(1-s-t)\lambda_3\xi} \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(\xi)\, |\mu|(\mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t, \qquad (\lambda_1,\lambda_2,\lambda_3) \in [-r,r]^3.$$

Consequently, by Proposition 7.7.12, if $u, v \in \mathcal{M} \otimes \mathcal{M}^{\mathrm{op}}$, then

$$\Delta_{u,v} f(m) = \int_{\mathbb{R}\times\Sigma_2} \mathcal{M}_\tau\Big((1\otimes v)\cdot\Big((i\xi)^2 e^{is\xi m} \otimes e^{it\xi m} \otimes \Big(e^{i(1-s-t)\xi m}\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(\xi)\Big)\Big)\cdot(u\otimes 1)$$
$$+ (1\otimes u)\cdot\Big((i\xi)^2 e^{is\xi m} \otimes e^{it\xi m} \otimes \Big(e^{i(1-s-t)\xi m}\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(\xi)\Big)\Big)\cdot(v\otimes 1)\Big)\, |\mu|(\mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t$$
$$= -\int_0^1\int_0^{1-t}\int_{\mathbb{R}} \xi^2 \mathcal{M}_\tau\big((1\otimes v)\cdot(e^{is\xi m}\otimes e^{it\xi m}\otimes e^{i(1-s-t)\xi m})\cdot(u\otimes 1)$$
$$+ (1\otimes u)\cdot(e^{is\xi m}\otimes e^{it\xi m}\otimes e^{i(1-s-t)\xi m})\cdot(v\otimes 1)\big)\, \mu(\mathrm{d}\xi)\, \mathrm{d}s\, \mathrm{d}t.$$

When $u = v$, this is exactly Biane and Speicher's definition of $\Delta_u f(m)$ from [BS98].[1] Moreover, since we saw in the proof of Lemma 7.7.5 that

$$\partial f(m) = i\int_0^1\int_{\mathbb{R}} \xi\, e^{itm} \otimes e^{i(1-t)m}\, \mu(\mathrm{d}\xi)\, \mathrm{d}t,$$

this demonstrates directly that Theorem 7.7.9(i) does, in fact, generalize [BS98, Prop. 4.3.4].

[1]Beware: As is noted in [BS01], the definition of $\Delta_u f(m)$ actually written in [BS98] is missing a factor of 2.

Remark 7.7.18. If X, Y, Z are topological spaces and $F\colon X \times Y \to Z$ is a function, then F is **argumentwise continuous** if the maps $F(x,\cdot)\colon Y \to Z$ and $F(\cdot,y)\colon X \to Z$ are continuous whenever $x \in X$ and $y \in Y$ are fixed. Now, fix $m \in \mathcal{M}$, $p(\lambda) \in \mathbb{C}[\lambda]$, and $f \in \mathcal{C}^{[2]}(\mathbb{R})$. Write $B\colon (\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})^2 \to \mathcal{M}$ for any one of the bilinear maps Q_τ, $\Delta_{\cdot,\cdot}p(m)$, or $\Delta_{\cdot,\cdot}f(m)$. Of course, when $B = \Delta_{\cdot,\cdot}f(m)$, we implicitly assume $m \in \mathcal{M}_{\mathrm{sa}}$. When $B \in \{Q_\tau, \Delta_{\cdot,\cdot}p(m)\}$, it is easy to see from the definition that B is argumentwise continuous with respect to the weak* topologies (i.e., σ-WOTs) on $\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$ and $\mathcal{M}$. This is also true when $B = \Delta_{\cdot,\cdot}f(m)$, but it is substantially harder to prove. The key is that MOIs are argumentwise σ-weakly continuous in their multilinear arguments; this is a special case of Corollary 5.6.10. In any case, no matter the choice of B, B is argumentwise σ-weakly continuous. Since $\mathcal{M} \otimes \mathcal{M}^{\mathrm{op}}$ is σ-weakly dense in $\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$, $B|_{(\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}})^2}$ extends uniquely to *an argumentwise σ-weakly continuous bilinear map* $(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})^2 \to \mathcal{M}$. To this extent, B is determined by its respective algebraic formula (Equations (7.4.5) or (7.5.4) or the formula in Corollary 7.7.16). However, $\mathcal{M} \otimes \mathcal{M}^{\mathrm{op}}$ is *not* necessarily norm-dense in $\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}}$. For example, if $\mathcal{M} = L^\infty([0,1])$, then $\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}} = L^\infty([0,1])\bar{\otimes}L^\infty([0,1]) = L^\infty([0,1]^2)$, and it is a standard exercise to show that if $\Delta_+ = \{(x,y) : 0 \le x \le y \le 1\}$, then $1_{\Delta_+} \in L^\infty([0,1]^2) \setminus L^\infty([0,1]) \otimes_{\min} L^\infty([0,1])$. In particular, the boundedness of $B|_{(\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}})^2}$ as a bilinear map does *not* necessarily imply that there exists a unique bounded bilinear extension of $B|_{(\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}})^2}$ to $(\mathcal{M}\bar{\otimes}\mathcal{M}^{\mathrm{op}})^2$. Such uniqueness is claimed implicitly in the paragraphs after [BS98, Def. 4.3.1 & Lem. 4.3.3]. However, this luckily does not harm Biane and Speicher's development because we *can* guarantee a unique bounded bilinear extension to $(\mathcal{M} \otimes_{\min} \mathcal{M}^{\mathrm{op}})^2$, and as we noted in Remark 7.3.8, $\Lambda^2 \subseteq L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{M} \otimes_{\min} \mathcal{M}^{\mathrm{op}})$.

7.8 Matrix stochastic calculus formulas

The purpose of this section is to motivate our main results (Theorems 7.7.9 and 7.6.6) by studying an Itô formula for C^2 scalar functions of Hermitian matrix-valued Itô processes (Theorem 7.8.13). To the author's knowledge, this formula is not written elsewhere in the literature, though related formulas are mentioned, at least for polynomials, in [Ans02]. Fix a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, P)$, with filtration satisfying the usual conditions, to which all processes to come are adapted.

Fix $n, N \in \mathbb{N}$, and, as in §7.1, let $\big(X_1^{(N)}, \ldots, X_n^{(N)}\big) = (X_1, \ldots, X_n)$ be an n-tuple of independent standard $(\mathrm{M}_N(\mathbb{C})_{\mathrm{sa}}, \langle\cdot,\cdot\rangle_N)$-valued Brownian motions. Concretely, if $\mathcal{E} \subseteq \mathrm{M}_N(\mathbb{C})_{\mathrm{sa}}$ is any orthonormal basis (ONB) for the real inner product space $(\mathrm{M}_N(\mathbb{C})_{\mathrm{sa}}, \langle\cdot,\cdot\rangle_N)$, then

$$X_i = \sum_{E \in \mathcal{E}} b_{i,E}\, E, \tag{7.8.1}$$

where $\{b_{j,E} = (b_{j,E}(t))_{t \geq 0} : j \in \{1, \ldots, n\},\, E \in \mathcal{E}\}$ is a collection of nN^2 independent standard real Brownian motions. This representation of X_i will allow us to use the following "magic formula" to identify "trace terms" in our stochastic calculus formulas. Please see [DHK13, §3.1], the paper from which the name "magic formula" originates, for a proof.

Lemma 7.8.2 (Magic formula)**.** *If $\mathcal{E} \subseteq \mathrm{M}_N(\mathbb{C})_{\mathrm{sa}}$ is a $\langle\cdot,\cdot\rangle_N$-ONB for $\mathrm{M}_N(\mathbb{C})_{\mathrm{sa}}$, then*

$$\sum_{E \in \mathcal{E}} EBE = \mathrm{tr}(B)\, I_N, \qquad B \in \mathrm{M}_N(\mathbb{C}),$$

where I_N is the $N \times N$ identity matrix.

We now make use of the $\#_k$ operation (Notation 1.5.9) on the algebra $\mathrm{M}_N(\mathbb{C})$. Importantly, we shall view the domain of $\#_1 = \#$ as $\mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C})^{\mathrm{op}}$ as opposed to $\mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C})$. Using basic linear algebra, one can show that $\#_k \colon \mathrm{M}_N(\mathbb{C})^{\otimes(k+1)} \to L_k(\mathrm{M}_N(\mathbb{C})) = B_k(\mathrm{M}_N(\mathbb{C})^k; \mathrm{M}_N(\mathbb{C}))$ is a linear isomorphism. Also, $\# \colon \mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C})^{\mathrm{op}} \to L_1(\mathrm{M}_N(\mathbb{C})) = \mathrm{End}(\mathrm{M}_N(\mathbb{C}))$ is an algebra homomorphism. In particular, we may identify $\mathrm{End}(\mathrm{M}_N(\mathbb{C}))$-valued processes $U = (U(t))_{t \geq 0}$ with $\mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C})^{\mathrm{op}}$-valued processes and write, for instance,

$$\int_0^t U(s) \# \mathrm{d}Y(s) = \int_0^t U(s)[\mathrm{d}Y(s)]$$

for the stochastic integral of U against the $\mathrm{M}_N(\mathbb{C})$-valued semimartingale Y (when this makes sense). In view of this identification and notation, we introduce $N \times N$ matrix Itô processes.

Definition 7.8.3 (Matrix Itô process)**.** An $\boldsymbol{N \times N}$ **matrix Itô process** is an adapted process M taking values in $\mathrm{M}_N(\mathbb{C})$ that satisfies

$$\mathrm{d}M(t) = \sum_{i=1}^n U_i(t) \# \mathrm{d}X_i(t) + K(t)\, \mathrm{d}t \tag{7.8.4}$$

for some predictable $\mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C})^{\mathrm{op}}$-valued processes $U_1, \ldots, U_n$ and some progressively measurable $\mathrm{M}_N(\mathbb{C})$-valued process K satisfying, almost surely,

$$\sum_{i=1}^{n} \int_0^t \|U_i(s)\|_{\otimes_N}^2 \, \mathrm{d}s + \int_0^t \|K(s)\|_N \, \mathrm{d}s < \infty, \qquad t \geq 0, \tag{7.8.5}$$

where $\|\cdot\|_{\otimes_N}$ is the norm associated to the tensor inner product $\langle \cdot, \cdot \rangle_{\otimes_N}$ on $\mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C})^{\mathrm{op}}$ induced by the usual Hilbert–Schmidt (Frobenius) inner product on $\mathrm{M}_N(\mathbb{C})$ (and $\mathrm{M}_N(\mathbb{C})^{\mathrm{op}}$).

Remark 7.8.6. The conditions in and preceding Inequality (7.8.5) guarantee that all the integrals in Equation (7.8.4) make sense and that M is a continuous $\mathrm{M}_N(\mathbb{C})$-valued semimartingale.

Now, we compute the quadratic covariation of two matrix Itô processes.

Definition 7.8.7 (Magic operator)**.** Write $\mathcal{M}_{\mathrm{tr}} \colon \mathrm{M}_N(\mathbb{C})^{\otimes 3} \to \mathrm{M}_N(\mathbb{C})$ for the linear map determined by

$$\mathcal{M}_{\mathrm{tr}}(A \otimes B \otimes C) = A \, \mathrm{tr}(B) \, C = \mathrm{tr}(B) \, AC, \qquad A, B, C \in \mathrm{M}_N(\mathbb{C}).$$

We call $\mathcal{M}_{\mathrm{tr}}$ the **magic operator**. Another way to write it is

$$\mathcal{M}_{\mathrm{tr}} = \mathfrak{m}_{\mathrm{M}_N(\mathbb{C})} \circ (\mathrm{id}_{\mathrm{M}_N(\mathbb{C})} \otimes \mathrm{tr} \otimes \mathrm{id}_{\mathrm{M}_N(\mathbb{C})}),$$

where $\mathfrak{m}_{\mathrm{M}_N(\mathbb{C})} \colon \mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C}) \to \mathrm{M}_N(\mathbb{C})$ is the linear map induced by multiplication in the algebra $\mathrm{M}_N(\mathbb{C})$.

Lemma 7.8.8. *Suppose $\mathcal{E} \subseteq \mathrm{M}_N(\mathbb{C})_{\mathrm{sa}}$ is a $\langle \cdot, \cdot \rangle_N$-orthonormal basis. If $W \in \mathrm{M}_N(\mathbb{C})^{\otimes 3}$ and $U, V \in \mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C})^{\mathrm{op}}$, then*

$$\sum_{E \in \mathcal{E}} W \#_2 [U \# E, V \# E] = \mathcal{M}_{\mathrm{tr}}((I_N \otimes V) \cdot W \cdot (U \otimes I_N)),$$

where $\cdot$ is multiplication in $\mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C})^{\mathrm{op}} \otimes \mathrm{M}_N(\mathbb{C})$; for example,

$$(A \otimes B \otimes C) \cdot (D \otimes E \otimes F) = (AD) \otimes (EB) \otimes (CF).$$

whenever $A, B, C, D, E, F \in \mathrm{M}_N(\mathbb{C})$.

Proof. It suffices to prove the formula when $U = A \otimes B$, $V = C \otimes D$, and $W = A_1 \otimes A_2 \otimes A_3$ are pure tensors. In this case,

$$\begin{aligned}\sum_{E\in\mathcal{E}} W\#_2[U\#E, V\#E] &= \sum_{E\in\mathcal{E}} W\#_2[AEB, CED] = \sum_{E\in\mathcal{E}} A_1AEBA_2CEDA_3 \\ &= A_1A\,\mathrm{tr}(BA_2C)\,DA_3 = \mathcal{M}_{\mathrm{tr}}(A_1A \otimes BA_2C \otimes DA_3) \\ &= \mathcal{M}_{\mathrm{tr}}((I_N \otimes C \otimes D) \cdot (A_1 \otimes A_2 \otimes A_3) \cdot (A \otimes B \otimes I_N)) \\ &= \mathcal{M}_{\mathrm{tr}}((I_N \otimes V) \cdot W \cdot (U \otimes I_N))\end{aligned}$$

by Lemma 7.8.2 and the definitions of $\mathcal{M}_{\mathrm{tr}}$ and the $\cdot$ operation. □

Theorem 7.8.9 (Quadratic covariation of matrix Itô processes)**.** *If, for each $\ell \in \{1, 2\}$, M_ℓ is an $N \times N$ matrix Itô process satisfying $\mathrm{d}M_\ell(t) = \sum_{i=1}^{n} U_{\ell i}(t)\#\mathrm{d}X_i(t) + K_\ell(t)\,\mathrm{d}t$ and $W = (W(t))_{t\geq 0}$ is a continuous $\mathrm{M}_N(\mathbb{C})^{\otimes 3}$-valued process, then, almost surely,*

$$\int_0^t W(s)\#_2[\mathrm{d}M_1(s), \mathrm{d}M_2(s)] = \sum_{i=1}^{n} \int_0^t \mathcal{M}_{\mathrm{tr}}((I_N \otimes U_{2i}(s)) \cdot W(s) \cdot (U_{1i}(s) \otimes I_N))\,\mathrm{d}s, \qquad t \geq 0.$$

Proof. Recall that bounded variation terms do not contribute to quadratic covariation, so we may assume $K_1 \equiv K_2 \equiv 0$. Now, using the expression (7.8.1) for X_i and the fact that

$$\mathrm{d}b_{i,E}(t)\,\mathrm{d}b_{j,F}(t) = \delta_{ij}\delta_{E,F}\,\mathrm{d}t,$$

we get

$$\begin{aligned}\int_0^t W(s)\#_2[\mathrm{d}M_1(s), \mathrm{d}M_2(s)] &= \sum_{i,j=1}^{n} \sum_{E,F\in\mathcal{E}} \int_0^t W(s)\#_2[U_{1i}(s)\#E, U_{2j}(s)\#F]\,\mathrm{d}b_{i,E}(s)\,\mathrm{d}b_{j,F}(s) \\ &= \sum_{i=1}^{n} \sum_{E\in\mathcal{E}} \int_0^t W(s)\#_2[U_{1i}(s)\#E, U_{2i}(s)\#E]\,\mathrm{d}s \\ &= \sum_{i=1}^{n} \int_0^t \left(\sum_{E\in\mathcal{E}} W(s)\#_2[U_{1i}(s)\#E, U_{2i}(s)\#E] \right) \mathrm{d}s \\ &= \sum_{i=1}^{n} \int_0^t \mathcal{M}_{\mathrm{tr}}((I_N \otimes U_{2i}(s)) \cdot W(s) \cdot (U_{1i}(s) \otimes I_N))\,\mathrm{d}s\end{aligned}$$

by Lemma 7.8.8. □

From the cases $W = A \otimes B \otimes C$, $M_1 \in \{X_i, I_N\}$, and $M_2 \in \{X_j, I_N\}$, we get Equations (7.1.6) and (7.1.7). Let us now see how Theorem 7.8.9 gives rise to a "functional" Itô formula for C^2 scalar functions of Hermitian matrix Itô processes.

Notation 7.8.10 (Noncommutative derivatives)**.** If $f \in C^k(\mathbb{R})$ and $M \in \mathrm{M}_N(\mathbb{C})_{\mathrm{sa}}$, then

$$\partial^k f(M) \coloneqq k! \sum_{\boldsymbol{\lambda} \in \sigma(M)^{k+1}} f^{[k]}(\boldsymbol{\lambda})\, P^M_{\lambda_1} \otimes \cdots \otimes P^M_{\lambda_{k+1}} \in \mathrm{M}_N(\mathbb{C})^{\otimes(k+1)}.$$

We shall view $\partial f(M) \coloneqq \partial^1 f(M)$ as an element of $\mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C})^{\mathrm{op}}$.

The key identity (7.1.8) for the derivatives of matrix functions then rewrites to

$$\partial_{B_k} \cdots \partial_{B_1} f_{\mathrm{M}_N(\mathbb{C})}(M) = \frac{1}{k!} \sum_{\pi \in S_k} \partial^k f(M) \#_k \big[B_{\pi(1)}, \ldots, B_{\pi(k)}\big], \qquad M, B_i \in \mathrm{M}_N(\mathbb{C})_{\mathrm{sa}}. \quad (7.8.11)$$

We are now ready to state and prove the (matrix) functional Itô formula that motivates our functional free Itô formula (Theorem 7.7.9).

Notation 7.8.12. If $f \in C^2(\mathbb{R})$ and $U \in \mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C})^{\mathrm{op}}$, then we define

$$\Delta_U f(M) \coloneqq \mathcal{M}_{\mathrm{tr}}((I_N \otimes U) \cdot \partial^2 f(M) \cdot (U \otimes I_N)) \in \mathrm{M}_N(\mathbb{C}),$$

where $\cdot$ is multiplication in $\mathrm{M}_N(\mathbb{C}) \otimes \mathrm{M}_N(\mathbb{C})^{\mathrm{op}} \otimes \mathrm{M}_N(\mathbb{C})$ as usual.

Theorem 7.8.13 (Functional Itô formula)**.** *Let M be an $N \times N$ matrix Itô process satisfying Equation* (7.8.4)*, and suppose $M^* = M$. If $f \in C^2(\mathbb{R})$, then*

$$\mathrm{d}f(M(t)) = \partial f(M(t)) \# \mathrm{d}M(t) + \frac{1}{2} \sum_{i=1}^{n} \Delta_{U_i(t)} f(M(t))\, \mathrm{d}t. \quad (7.8.14)$$

Proof. If $f \in C^2(\mathbb{R})$, then $f_{\mathrm{M}_N(\mathbb{C})} \in C^2(\mathrm{M}_N(\mathbb{C})_{\mathrm{sa}}; \mathrm{M}_N(\mathbb{C}))$, so we may apply Itô's formula (Equation (7.1.1)) with $F = f_{\mathrm{M}_N(\mathbb{C})}$. Doing so gives

$$\begin{aligned} \mathrm{d}f(M(t)) = \mathrm{d}f_{\mathrm{M}_N(\mathbb{C})}(M(t)) &= Df_{\mathrm{M}_N(\mathbb{C})}(M(t))[\mathrm{d}M(t)] + \frac{1}{2} D^2 f_{\mathrm{M}_N(\mathbb{C})}(M(t))[\mathrm{d}M(t), \mathrm{d}M(t)] \\ &= \partial f(M(t)) \# \mathrm{d}M(t) + \frac{1}{2} \partial^2 f(M(t)) \#_2 [\mathrm{d}M(t), \mathrm{d}M(t)] \end{aligned}$$

by Equation (7.8.11). Theorem 7.8.9 and the definition of $\Delta_U f(M)$ then yield

$$\begin{aligned}\mathrm{d}f(M(t)) &= \partial f(M(t))\#\mathrm{d}M(t) + \frac{1}{2}\sum_{i=1}^{n}\mathcal{M}_{\mathrm{tr}}((I_N\otimes U_i(t))\cdot\partial^2 f(M(t))\cdot(U_i(t)\otimes I_N))\,\mathrm{d}t\\ &= \partial f(M(t))\#\mathrm{d}M(t) + \frac{1}{2}\sum_{i=1}^{n}\Delta_{U_i(t)}f(M(t))\,\mathrm{d}t,\end{aligned}$$

as desired. □

Applying $\mathrm{tr} = \frac{1}{N}\operatorname{Tr}$ to Equation (7.8.14) and using symmetrization arguments similar to those from the proof of Lemma 7.6.13 yields the following "traced" formula that motivates Theorem 7.6.6. We leave the details to the interested reader. In the statement below, if $U = \sum_{i=1}^{k} A_i \otimes B_i \in \mathrm{M}_N(\mathbb{C})\otimes \mathrm{M}_N(\mathbb{C})^{\mathrm{op}}$, then $U^{\mathrm{flip}} := \sum_{i=1}^{k} B_i\otimes A_i \in \mathrm{M}_N(\mathbb{C})\otimes\mathrm{M}_N(\mathbb{C})^{\mathrm{op}}$. Also, we write $\mathrm{tr}^{\mathrm{op}}$ for tr considered as a function $\mathrm{M}_N(\mathbb{C})^{\mathrm{op}}\to\mathbb{C}$.

Corollary 7.8.15 (Traced functional Itô formula)**.** *Let M be an $N\times N$ matrix Itô process satisfying Equation* (7.8.4)*, and suppose $M^* = M$. If $f\in C^2(\mathbb{R})$, then*

$$\mathrm{d}\,\mathrm{tr}(f(M(t))) = \mathrm{tr}\left(f'(M(t))\,\mathrm{d}M(t)\right) + \frac{1}{2}\sum_{i=1}^{n}(\mathrm{tr}\otimes\mathrm{tr}^{\mathrm{op}})\big(U_i^{\mathrm{flip}}(t)\,\partial f'(M(t))\,U_i(t)\big)\,\mathrm{d}t,$$

where $U_i^{\mathrm{flip}}\partial f'(M)\,U_i$ is a product in the algebra $\mathrm{M}_N(\mathbb{C})\otimes\mathrm{M}_N(\mathbb{C})^{\mathrm{op}}$. Under sufficient additional boundedness conditions (e.g., U_i, K, and M are all uniformly bounded), we also have

$$\mathrm{d}\tau_N(f(M(t))) = \left(\tau_N\big(f'(M(t))\,K(t)\big) + \frac{1}{2}\sum_{i=1}^{n}(\tau_N\otimes\tau_N^{\mathrm{op}})\big(U_i^{\mathrm{flip}}(t)\,\partial f'(M(t))\,U_i(t)\big)\right)\mathrm{d}t$$

where $\tau_N = \mathbb{E}_P\circ\mathrm{tr}$ and $\tau_N^{\mathrm{op}} = \mathbb{E}_P\circ\mathrm{tr}^{\mathrm{op}}$.

7.9 Acknowledgment

Chapter 7, in part, is a reprint of the material as it appears in "Itô's formula for noncommutative C^2 functions of free Itô processes" (2022). Nikitopoulos, Evangelos A. *Documenta Mathematica*, 27, 1447–1507.

Appendix A

Auxiliary measure theory

In this appendix, we prove Pettis's measurability theorem and provide some background on vector measures and the Carathéodory–Hahn–Kluvánek extension theorem.

Standing assumptions. Fix a choice of base field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Unless otherwise specified, all vector spaces are $\mathbb{F}$-vector spaces, and all linear maps are $\mathbb{F}$-linear. Throughout, V is a Hausdorff topological vector space. In §A.2, Ω is a set.

A.1 Proof of Pettis's measurability theorem

The goal of this section is to prove Theorem 1.1.17.

Lemma A.1.1. *If α is a continuous seminorm on V and S is a separable subset of V, then there exists a countable family $\mathcal{C} \subseteq V^*$ such that $\alpha(v) = \sup\{|\ell(v)| : \ell \in \mathcal{C}\}$ whenever $v \in S$.*

Proof. Let S_0 be a countable, dense subset of S. By the Hahn–Banach theorem, if $s \in S_0$, then there exists a linear functional $\ell_s \colon V \to \mathbb{F}$ such that $\ell_s(s) = \alpha(s)$ and $|\ell_s(v)| \leq \alpha(v)$ whenever $v \in V$. Of course, $\ell_s \in V^*$ because α is continuous. We claim that $\mathcal{C} \coloneqq \{\ell_s : s \in S_0\}$ does the trick. Indeed, define $S(v) \coloneqq \sup\{|\ell(v)| : \ell \in \mathcal{C}\} = \sup\{|\ell_s(v)| : s \in S_0\}$ for all $v \in V$. By our choice of $\mathcal{C}$, $S(v) \leq \alpha(v)$ whenever $v \in V$. For the reverse inequality, observe that if $s \in S_0$ and $v \in V$, then $\ell_s(v) = \ell_s(s) + \ell_s(v - s) = \alpha(s) + \ell_s(v - s)$, and $|\ell_s(v - s)| \leq \alpha(v - s)$. Thus, $\alpha(v) \leq \alpha(s) + \alpha(v - s) \leq |\ell_s(v)| + 2\,\alpha(v - s) \leq S(v) + 2\,\alpha(v - s)$. Since S_0 is dense in S and α is continuous, if $v \in S$ and $\varepsilon > 0$, then there exists an $s \in S_0$ such that $2\,\alpha(v - s) < \varepsilon$. This yields $\alpha(v) < S(v) + \varepsilon$. Taking $\varepsilon \searrow 0$ completes the proof. □

Lemma A.1.2. *If V is locally convex and metrizable and $S \subseteq V$ is separable, then*

$$\mathcal{B}_S = \sigma(\ell|_S : \ell \in V^*).$$

Proof. If $S \subseteq V$ and $\ell \in V^*$, then $\ell|_S \colon S \to \mathbb{F}$ is continuous and therefore Borel measurable. Thus, $\sigma(\ell|_S : \ell \in V^*) \subseteq \mathcal{B}_S$. For the reverse containment, recall that, since V is locally convex and metrizable, there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ of continuous seminorms on V such that

$$d(v,w) := \sum_{k=1}^{\infty} \frac{\alpha_k(v-w)}{2^k(1+\alpha_k(v-w))}, \qquad (v,w) \in V^2, \tag{A.1.3}$$

is a metric that induces the topology of V. Now, let $\alpha \colon V \to \mathbb{R}_+$ be any continuous seminorm. If $S \subseteq V$ is separable, then $\operatorname{span} S$ is separable as well, so Lemma A.1.1 provides a countable family $\mathcal{C} \subseteq V^*$ such that

$$\alpha(v) = \sup\{|\ell(v)| : \ell \in \mathcal{C}\}, \qquad v \in \operatorname{span} S.$$

Consequently, if $w \in S$ and $\varepsilon > 0$, then

$$\{v \in S : \alpha(v-w) < \varepsilon\} = \{v \in S : \sup\{|\ell(v-w)| : \ell \in \mathcal{C}\} < \varepsilon\} \in \sigma(\ell|_S : \ell \in V^*)$$

because $\mathcal{C}$ is countable. By definition of d, it follows that

$$B_\varepsilon(w) = \{v \in S : d(v,w) < \varepsilon\} \in \sigma(\ell|_S : \ell \in V^*).$$

Since $(S, d|_{S\times S})$ is a separable metric space, the open balls $\{B_\varepsilon(w) : w \in S, \varepsilon > 0\}$ generate $\mathcal{B}_S$ as a σ-algebra, so we get $\mathcal{B}_S \subseteq \sigma(\ell|_S : \ell \in V^*)$, as desired. □

Lemma A.1.4. *If (X, d_X) is a separable pseudometric space, then there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of Borel measurable simple maps $X \to X$ converging pointwise to* id_X.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in X. If $n \in \mathbb{N}$, then we define

$$k_n(x) := \min\big\{k \in \{1,\ldots,n\} : d_X(x,x_k) = \min\big\{d_X(x,x_i) : i \in \{1,\ldots,n\}\big\}\big\}, \qquad x \in X.$$

Note that $k_n\colon X \to \{1, \ldots, n\}$ is Borel measurable. Also, define $L_n\colon \{1, \ldots, n\} \to X$ by $i \mapsto x_i$. (Of course, L_n is measurable with respect to any σ-algebra on X.) We claim that

$$s_n(x) := L_n(k_n(x)) = x_{k_n(x)}, \qquad x \in X,$$

does the job. Indeed, $s_n = L_n \circ k_n$ is Borel measurable, and $s_n(X) \subseteq \{x_1, \ldots, x_n\}$. Thus, s_n is a Borel simple map. Also, since $\{x_n : n \in \mathbb{N}\}$ is dense in X, if $\varepsilon > 0$ and $x \in X$, then there exists a $k \in \mathbb{N}$ such that $d_X(x, x_k) < \varepsilon$. By definition of $k_n(x)$, if $n \geq k$, then

$$d_X(x, s_n(x)) = d_X\big(x, x_{k_n(x)}\big) = \min\big\{d_X(x, x_i) : i \in \{1, \ldots, n\}\big\} \leq d_X(x, x_k) < \varepsilon.$$

Thus, $(s_n)_{n\in\mathbb{N}}$ converges pointwise to id_X. $\square$

This allows us to prove the first part of Theorem 1.1.17.

Proof of Theorem 1.1.17(i). Suppose $F\colon \Omega \to V$ is strongly measurable. We observed after Definition 1.1.8 that F is Baire measurable. Since V is metrizable, $\mathcal{B}_V^a = \mathcal{B}_V$. Thus, F is Borel measurable. Now, if $(F_n)_{n\in\mathbb{N}}$ is a sequence of simple maps $\Omega \to V$ converging pointwise to F, then $F(\Omega) \subseteq \overline{\bigcup_{n\in\mathbb{N}} F_n(\Omega)} =: S$. Since $F_n(\Omega)$ is finite for all $n \in \mathbb{N}$, S is separable. Since S is separable and metrizable, any subset of S is separable. Thus, $F(\Omega)$ is separable.

Next, choose a metric d on V that induces the topology of V. If $F\colon \Omega \to V$ is weakly measurable and $S := F(\Omega) \subseteq V$ is separable, then, by Lemma A.1.2, the map $F\colon \Omega \to S$ is $(\mathscr{F}, \mathcal{B}_S)$-measurable. Now, apply Lemma A.1.4 to the separable metric space $(X, d_X) = (S, d|_{S\times S})$ to get a sequence $(s_n)_{n\in\mathbb{N}}$ of Borel simple maps $S \to S$ converging pointwise to id_S. Then $(F_n)_{n\in\mathbb{N}} := (\iota_S \circ s_n \circ F)_{n\in\mathbb{N}}$ is a sequence of simple maps $\Omega \to V$ converging pointwise to F. (Here, $\iota_S\colon S \hookrightarrow V$ is the inclusion.) Thus, F is strongly measurable. $\square$

For the second part, we need to refine Lemma A.1.4 when X is a vector space and d_X is given by a seminorm on X.

Lemma A.1.5. *If $S \subseteq V$ is a separable linear subspace and α is a continuous seminorm on V, then there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of Borel simple maps $S \to S$ such that $\sup_{n\in\mathbb{N}} \alpha(s_n(v)) \leq \alpha(v)$ and $\lim_{n\to\infty} \alpha(s_n(v) - v) = 0$ whenever $v \in S$.*

Proof. Let S_{00} be a countable, dense subset of S, and write S_0 for the $\mathbb{F} \cap (\mathbb{Q} + i\mathbb{Q})$-span of S_0. Then S_0 is also a countable, dense subset of S. Let $(v_n)_{n\in\mathbb{N}}$ be an enumeration of S_0 with $v_1 = 0$. If $n \in \mathbb{N}$ and $v \in S$, then we define

$$K_n(v) \coloneqq \{k \in \{1, \ldots, n\} : \alpha(v_k) \leq \alpha(v)\} \text{ and}$$
$$k_n(v) \coloneqq \min\{k \in K_n(v) : \alpha(v - v_k) = \min\{\alpha(v - v_i) : i \in K_n(v)\}\}.$$

(Note that $1 \in K_n(v)$ always.) Since α is continuous, $k_n \colon S \to \{1, \ldots, n\}$ is Borel measurable. Also, define $L_n \colon \{1, \ldots, n\} \to S$ by $i \mapsto v_i$. We claim that

$$s_n(v) \coloneqq L_n(k_n(v)) = v_{k_n(v)}, \quad v \in S,$$

does the job. Indeed, $s_n = L_n \circ k_n$ is Borel measurable, and $s_n(S) \subseteq \{v_1, \ldots, v_n\}$. Thus, s_n is a Borel simple map. Also, if $v \in S$, then, by definition of $K_n(v)$, $\alpha(s_n(v)) = \alpha\big(v_{k_n(v)}\big) \leq \alpha(v)$.

It remains to show that $(s_n)_{n\in\mathbb{N}}$ converges pointwise to id_S. To this end, fix $v \in S$ and $\varepsilon > 0$. If $\alpha(v) = 0$, then $k_n(v) = 1$, and $\alpha(s_n(v) - v) = 0$ for all $n \in \mathbb{N}$. Now, assume that $\alpha(v) > 0$, and define

$$\delta \coloneqq \frac{\alpha(v)}{\alpha(v) + 2}\,\varepsilon = \frac{\varepsilon}{1 + \frac{2}{\alpha(v)}} > 0.$$

Since S_0 is dense in S, there exists a $w \in S_0$ such that $\alpha(v - w) \leq \delta$. Now, let $r \in \mathbb{Q}$ be such that

$$1 + \frac{\delta}{\alpha(v)} \leq r < 1 + \frac{2\delta}{\alpha(v)}. \tag{A.1.6}$$

By definition of S_0, $r^{-1}w \in S_0$, so there exists an $m \in \mathbb{N}$ such that $r^{-1}w = v_m$. Note that

$$\alpha(v_m) = \frac{\alpha(w)}{r} \leq \frac{\alpha(v) + \delta}{r} \leq \alpha(v) \text{ and} \tag{A.1.7}$$
$$\alpha(v - v_m) \leq \alpha(v - w) + \frac{r-1}{r}\,\alpha(w) \leq \delta + \frac{2\delta}{\alpha(v) + \delta} < \varepsilon. \tag{A.1.8}$$

by Inequality (A.1.6) and our choices of r and δ. Crucially, Inequality (A.1.7) says that if $n \geq m$, then $m \in K_n(v)$. Consequently, if $n \geq m$, then $\alpha(s_n(v) - v) = \alpha\big(v_{k_n(v)} - v\big) \leq \alpha(v_m - v) < \varepsilon$ by definition of k_n and Inequality (A.1.8). This completes the proof. $\square$

Proof of Theorem 1.1.17(ii). We observed after Definition 1.1.8 that if $F\colon \Omega \to V$ is strongly integrable and α is a continuous seminorm on V, then $\int_\Omega \alpha(F)\,\mathrm{d}\mu < \infty$.

Let $F\colon \Omega \to V$ be a strongly measurable map such that $\int_\Omega \alpha(F)\,\mathrm{d}\mu < \infty$ whenever α is a continuous seminorm on V. To show F is strongly integrable, we first reduce to the σ-finite case. To this end, let $(\alpha_k)_{k\in\mathbb{N}}$ be as in the proof of Lemma A.1.2; without loss of generality, we assume also that $\alpha_k \le \alpha_{k+1}$ whenever $k \in \mathbb{N}$. Since $\int_\Omega \alpha_k(F)\,\mathrm{d}\mu < \infty$, the set $\{\omega \in \Omega : \alpha_k(F(\omega)) > 0\}$ is σ-finite for μ. Therefore, the set $\Omega_0 \coloneqq \{\omega \in \Omega : F(\omega) \neq 0\} = \bigcup_{k\in\mathbb{N}}\{\omega \in \Omega : \alpha_k(F(\omega)) > 0\}$ is also σ-finite for μ. Now, suppose $\big(F_n^0\big)_{n\in\mathbb{N}}$ is a sequence of $\mu|_{\Omega_0}$-integrable simple maps $\Omega_0 \to V$ such that $F_n^0 \to F|_{\Omega_0}$ pointwise as $n \to \infty$ and $\int_{\Omega_0} \alpha\big(F_n^0 - F\big)\,\mathrm{d}\mu \to 0$ as $n \to \infty$ whenever α is a continuous seminorm on V. For $n \in \mathbb{N}$, define $F_n\colon \Omega \to V$ by $F_n|_{\Omega\setminus\Omega_0} \equiv 0 \equiv F|_{\Omega\setminus\Omega_0}$ and $F_n|_{\Omega_0} \coloneqq F_n^0$. Then $F_n\colon \Omega \to V$ is a μ-integrable simple map, $F_n \to F$ pointwise as $n \to \infty$, and $\int_\Omega \alpha(F_n - F)\,\mathrm{d}\mu = \int_{\Omega_0} \alpha\big(F_n^0 - F\big)\,\mathrm{d}\mu \to 0$ as $n \to \infty$ whenever α is a continuous seminorm on V. Therefore, we may and do assume μ is σ-finite, though we shall not use this assumption until the last paragraph of the proof.

Next, since $S \coloneqq \operatorname{span} F(\Omega)$ is separable, Lemma A.1.5 says that if $k \in \mathbb{N}$, then there exists a sequence $\big(s_n^k\big)_{n\in\mathbb{N}}$ of Borel simple maps $S \to S$ such that

$$\sup_{n\in\mathbb{N}} \alpha_k\big(s_n^k(v)\big) \le \alpha_k(v) \quad \text{and} \quad \lim_{n\to\infty} \alpha_k\big(s_n^k(v) - v\big) = 0, \qquad v \in S. \tag{A.1.9}$$

By Lemma A.1.2 and the weak measurability of F, $G_n^k \coloneqq s_n^k \circ F\colon \Omega \to S$ is $(\mathscr{F}, \mathcal{B}_S)$-measurable and simple. By Relation (A.1.9) and the dominated convergence theorem,

$$\lim_{n\to\infty} \int_\Omega \alpha_k\big(G_n^k - F\big)\,\mathrm{d}\mu = 0.$$

Consequently, if $n \in \mathbb{N}$, then there exists an $N_n \in \mathbb{N}$ such that

$$\int_\Omega \alpha_n\big(G_{N_n}^n - F\big)\,\mathrm{d}\mu \le \frac{1}{n}.$$

Since $(\alpha_k)_{k\in\mathbb{N}}$ is increasing, this gives

$$\lim_{n\to\infty} \int_\Omega \alpha_k\big(G_{N_n}^n - F\big)\,\mathrm{d}\mu = 0, \qquad k \in \mathbb{N}.$$

Since L^1-convergence implies convergence in measure, we get that if $k \in \mathbb{N}$, then $\alpha_k\big(G^n_{N_n} - F\big) \to 0$ in measure as $n \to \infty$. Consequently, if d is as in Equation (A.1.3), then $d\big(G^n_{N_n}, F\big) \to 0$ in measure as $n \to \infty$. Therefore, there is a subsequence

$$(G_k)_{k\in\mathbb{N}} := \left(G^{n_k}_{N_{n_k}}\right)_{k\in\mathbb{N}}$$

such that $d(G_k, F) \to 0$ almost everywhere, i.e., $G_k \to F$ almost everywhere, as $k \to \infty$. Let $N \in \mathscr{F}$ be such that $\mu(N) = 0$ and $G_k(\omega) \to F(\omega)$ as $k \to \infty$ whenever $\omega \in \Omega_0 := \Omega \setminus N$.

Since the map $F|_N \colon N \to V$ is still strongly measurable, there exists a sequence $\big(F^{00}_n\big)_{n\in\mathbb{N}}$ of simple maps $N \to V$ converging pointwise to $F|_N$. If, for $n \in \mathbb{N}$, we define $F^0_n \colon \Omega \to V$ by $F^0_n|_{\Omega_0} := G_n|_{\Omega_0}$ and $F^0_n|_N := F^{00}_n$, then $F^0_n \colon \Omega \to V$ is a simple map, and $F^0_n \to F$ pointwise (*everywhere*) as $n \to \infty$. Since $\mu(N) = 0$, we still have $\int_\Omega \alpha_k\big(F^0_n - F\big)\,\mathrm{d}\mu = \int_\Omega \alpha_k(G_n - F)\,\mathrm{d}\mu \to 0$ as $n \to \infty$ whenever $k \in \mathbb{N}$.

Finally, let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$ such that $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$ and $\mu(\Omega_n) < \infty$ whenever $n \in \mathbb{N}$. Then $(F_n) := \big(1_{\Omega_n} F^0_n\big)_{n\in\mathbb{N}}$ is a sequence of μ-integrable simple maps $\Omega \to V$ such that $F_n \to F$ pointwise as $n \to \infty$ and, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_\Omega \alpha_k(F_n - F)\, d\mu = 0, \qquad k \in \mathbb{N}.$$

Since $\{\alpha_k : k \in \mathbb{N}\}$ generates the topology of V, this completes the proof. □

A slight variation of the proof of Lemma A.1.5 yields another interesting characterization of strong (measurability and) integrability.

Lemma A.1.10. *If $S \subseteq V$ is a separable linear subspace and α is a continuous seminorm on V, then there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of Borel σ-simple maps $S \to S$ such that*

$$\sup_{n\in\mathbb{N}} \alpha(s_n) \le \alpha|_S \quad \textit{and} \quad \lim_{n\to\infty} \sup_{v\in S} \alpha(s_n(v) - v) = 0.$$

Sketch of proof. Let S_{00} be a countable, dense subset of S, and write S_0 for the $\mathbb{F} \cap (\mathbb{Q} + i\mathbb{Q})$-span of S_0. Then S_0 is also a countable, dense subset of S. Let $(v_n)_{n\in\mathbb{N}}$ be an enumeration of S_0 with $v_1 = 0$. Also, for $\varepsilon > 0$ and $n \in \mathbb{N}$, define $B^\varepsilon_n := \{v \in S : \alpha(v - v_n) \le \varepsilon,\ \alpha(v_n) \le \alpha(v)\} \in \mathcal{B}_S$.

If $A_n^\varepsilon \coloneqq B_n^\varepsilon \setminus \bigcup_{k=1}^{n-1} B_k^\varepsilon \in \mathcal{B}_S$, then $A_n^\varepsilon \cap A_m^\varepsilon = \emptyset$ whenever $n \neq m$, and $\bigcup_{n\in\mathbb{N}} A_n^\varepsilon = S$. Consequently, if $\varepsilon_n \coloneqq 1/n$ and

$$s_n(v) \coloneqq \sum_{k=1}^{\infty} 1_{A_k^{\varepsilon_n}}(v)\, v_k, \qquad v \in S,$$

then the sequence $(s_n)_{n\in\mathbb{N}}$ has the desired properties. □

Theorem A.1.11. *Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and suppose V is locally convex and metrizable. A map $F\colon \Omega \to V$ is strongly integrable if and only if there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of σ-simple maps $\Omega \to V$ such that for all continuous seminorms α on V, $\int_\Omega \alpha(F_n)\,\mathrm{d}\mu < \infty$ whenever n is large enough, and*

$$\lim_{n\to\infty} \sup_{\omega\in\Omega} \alpha(F_n(\omega) - F(\omega)) = \lim_{n\to\infty} \int_\Omega \alpha(F_n - F)\,\mathrm{d}\mu = 0.$$

Sketch of proof. The "if" direction follows from Theorem 1.1.17. For the "only if" direction, let $(\alpha_k)_{k\in\mathbb{N}}$ be as in the proof of Lemma A.1.2 with $(\alpha_k)_{k\in\mathbb{N}}$ increasing. If $F\colon \Omega \to V$ is strongly integrable, then $S \coloneqq \operatorname{span} F(\Omega)$ is separable. By Lemma A.1.10, if $k \in \mathbb{N}$, then there exists a sequence $\left(s_n^k\right)_{n\in\mathbb{N}}$ of Borel σ-simple maps $S \to S$ such that

$$\sup_{n\in\mathbb{N}} \alpha_k\left(s_n^k\right) \leq \alpha_k|_S \quad \text{and} \quad \lim_{n\to\infty} \sup_{v\in S} \alpha_k\left(s_n^k(v) - v\right) = 0. \tag{A.1.12}$$

The map $G_n^k \coloneqq \iota_S \circ s_n^k \circ F\colon \Omega \to V$ is σ-simple. By Relation (A.1.12) and the dominated convergence theorem, $\int_\Omega \alpha_k\left(G_n^k\right)\mathrm{d}\mu < \infty$, and

$$\lim_{n\to\infty} \sup_{\omega\in\Omega} \alpha_k\left(G_n^k(\omega) - F(\omega)\right) = 0 = \lim_{n\to\infty} \int_\Omega \alpha_k\left(G_n^k - F\right)\mathrm{d}\mu.$$

Consequently, if $n \in \mathbb{N}$, then there exists an $N_n \in \mathbb{N}$ such that

$$\max\left\{\sup_{\omega\in\Omega} \alpha_n\left(G_{N_n}^n(\omega) - F(\omega)\right),\ \int_\Omega \alpha_n\left(G_{N_n}^n(\omega) - F(\omega)\right)\mu(\mathrm{d}\omega)\right\} \leq \frac{1}{n}.$$

The sequence $(F_n)_{n\in\mathbb{N}} \coloneqq \left(G_{N_n}^n\right)_{n\in\mathbb{N}}$ has the desired properties. □

Note that when $\mu \equiv 0$, strong μ-integrability is precisely strong measurability.

A.2 Vector measures and their extension

In this section, we review some terminology and notation from vector measure theory and state some results on the extension of vector measures that are useful in §5.10.

Definition A.2.1 (Vector-valued set functions)**.** Suppose $\mathscr{E} \subseteq 2^\Omega$ satisfies $\emptyset, \Omega \in \mathscr{E}$, and let $(G_n)_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathscr{E}$. A function $\mu\colon \mathscr{E} \to V$ is

(i) **finitely additive** if $\mu\big(\bigcup_{i=1}^n G_i\big) = \sum_{i=1}^n \mu(G_i)$ whenever $\bigcup_{i=1}^n G_i \in \mathscr{E}$,

(ii) (**weakly**) **countably additive** if $\mu\big(\bigcup_{n\in\mathbb{N}} G_n\big) = \sum_{n=1}^\infty \mu(G_n)$ in the (weak) topology of V whenever $\bigcup_{n\in\mathbb{N}} G_n \in \mathscr{E}$,

(iii) **strongly additive** if $\sum_{n=1}^\infty \mu(G_n)$ always exists in the topology of V,

(iv) a **finitely additive vector measure** if $\mathscr{E}$ is an algebra and μ is finitely additive, and

(v) a **vector measure** if $\mathscr{E}$ is a σ-algebra and μ is countably additive.

Of course, if $\mu\colon \mathscr{E} \to V$ is a (finitely additive) vector measure, then $\ell \circ \mu$ is a (finitely additive) complex measure for all $\ell \in V^*$.

Definition A.2.2 (Semivariation)**.** If $\mathscr{A} \subseteq 2^\Omega$ is an algebra, V is a normed vector space, and $\mu\colon \mathscr{A} \to V$ is a finitely additive vector measure, then

$$\|\mu\|(G) \coloneqq \sup\{|\ell \circ \mu|(G) : \ell \in V^*,\ \|\ell\|_{V^*} \le 1\} \in [0,\infty], \qquad G \in \mathscr{A},$$

and $\|\mu\|_{\mathrm{svar}} \coloneqq \|\mu\|(\Omega)$. The function $\|\mu\|$ is the **semivariation** of μ. If $\|\mu\|_{\mathrm{svar}} < \infty$, then μ has **bounded semivariation**. If $\mathscr{F}$ is a σ-algebra, then $M(\Omega,\mathscr{F};V)$ is the set of V-valued vector measures on $(\Omega,\mathscr{F})$ of bounded semivariation.

Here now are some results about extending vector measures.

Lemma A.2.3 (Extending to an algebra)**.** *Let $\mathscr{E} \subseteq 2^\Omega$ be an elementary family (in the sense of [Fol99, §1.2]). If $\mu_{00}\colon \mathscr{E} \to V$ is finitely (respectively, (weakly) countably) additive, then μ_{00} extends uniquely to a finitely (respectively, (weakly) countably) additive function $\mu_0\colon \mathrm{alg}(\mathscr{E}) \to V$, where $\mathrm{alg}(\mathscr{E}) \subseteq 2^\Omega$ is the algebra generated by $\mathscr{E}$.*

Sketch of proof. Since $\mathscr{E}$ is an elementary family, $\operatorname{alg}(\mathscr{E})$ is the set of finite disjoint unions of elements of $\mathscr{E}$. Consequently, if $G_1, \ldots, G_n \in \mathscr{E}$ are disjoint and $G := \bigcup_{i=1}^n G_i \in \operatorname{alg}(\mathscr{E})$, then we must take $\mu_0(G) := \sum_{i=1}^n \mu_{00}(G_i)$. It then follows from standard arguments that μ_0 is well defined and finitely (respectively, (weakly) countably) additive on $\operatorname{alg}(\mathscr{E})$ because μ_{00} is finitely (respectively, (weakly) countably) additive on $\mathscr{E}$. □

Theorem A.2.4 (Carathéodory–Hahn–Kluvánek extension theorem [DU77, Thm. I.5.2])**.** *Let $\mathscr{A} \subseteq 2^{\Omega}$ be an algebra, let V be a Banach space, and let $\mu_0 \colon \mathscr{A} \to V$ be a weakly countably additive function of bounded semivariation. If μ_0 is strongly additive, then μ_0 extends uniquely to a vector measure $\mu \colon \sigma(\mathscr{A}) \to V$, and $\|\mu\|_{\mathrm{svar}} = \|\mu_0\|_{\mathrm{svar}}$.*

It can be difficult to verify that a given finitely additive vector measure of bounded semivariation is strongly additive. Luckily, there is a full characterization of the situation in which one *never* has to do so.

Theorem A.2.5 (Diestel–Faires theorem [DU77, Thm. I.4.2])**.** *Define*

$$c_0 := \Big\{(a_n)_{n\in\mathbb{N}} \in \ell^{\infty}(\mathbb{N};\mathbb{F}) : \lim_{n\to\infty} a_n = 0\Big\},$$

and let V be a Banach space. The following are equivalent:

(i) *V contains a copy of c_0 (i.e., there is a linear map $T \colon c_0 \to V$ and constants $\varepsilon, C > 0$ such that $\varepsilon\|a\|_{c_0} \le \|Ta\|_V \le C\|a\|_{c_0}$ for all $a \in c_0$); and*

(ii) *there exists a set S, an algebra $\mathscr{A} \subseteq 2^S$, and a finitely additive vector measure $\mu_0 \colon \mathscr{A} \to V$ of bounded semivariation that is not strongly additive.*

Corollary A.2.6. *If V is a weakly sequentially complete Banach space, then every finitely additive V-valued vector measure of bounded semivariation is strongly additive.*

Sketch of proof. By the Diestel–Faires theorem, it suffices to show that V cannot contain a copy of c_0. By the Hahn–Banach theorem and Mazur's theorem, any closed linear subspace of V is weakly sequentially complete, so it suffices to show that c_0 is not weakly sequentially complete. To this end, let $e_n \in c_0$ be the n^{th} standard basis vector, and define $s_n := \sum_{i=1}^n e_i \in c_0$. It is an easy exercise to show that $(s_n)_{n\in\mathbb{N}}$ is weakly Cauchy but not weakly convergent. □

By combining Theorem A.2.4 and Corollary A.2.6, we obtain the following result, which is the form of the Carathéodory–Hahn–Kluvánek theorem used in §5.10.

Theorem A.2.7. *Let $\mathscr{A} \subseteq 2^\Omega$ be an algebra, let V be a Banach space, and let $\mu_0 \colon \mathscr{A} \to V$ be a weakly countably additive function of bounded semivariation. If V is weakly sequentially complete, then μ_0 extends uniquely to a vector measure $\mu \colon \sigma(\mathscr{A}) \to V$, and $\|\mu\|_{\mathrm{svar}} = \|\mu_0\|_{\mathrm{svar}}$.*

In particular, this result holds for reflexive Banach spaces.

Proposition A.2.8. *Reflexive Banach spaces are weakly sequentially complete.*

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a weakly Cauchy sequence in V, and write $\mathrm{ev} \colon V \to V^{**}$ for the natural embedding. Since $\mathbb{F}$ is complete, if $\ell \in V^*$, then there exists $\eta(\ell) \in \mathbb{F}$ such that $\ell(v_n) \to \eta(\ell)$ as $n \to \infty$. Clearly, $\eta \colon V^* \to \mathbb{F}$ is linear. By the principle of uniform boundedness, η is bounded. Since V is reflexive, $\eta = \mathrm{ev}(v)$ for some $v \in V$, and by definition, $v_n \to v$ weakly as $n \to \infty$. □

Appendix B

Proof of Peller's theorem

In this appendix, we provide a full proof of Theorem 6.6.9. We shall use basic facts about tempered distributions and their Fourier transforms freely; please see [Hör83, Rud91] for the relevant material. In particular, we recall that, as a consequence of the Paley–Wiener theorem, if $f \in \mathscr{S}'(\mathbb{R}^m)$ is such that $\operatorname{supp} \widehat{f}$ is compact, then f is a smooth function.

B.1 The key decomposition

First, we set some notation that we shall use to write an expression (Theorem B.1.3 below) that is key to the endeavor of proving Theorem 6.6.9.

Notation B.1.1. We define two families $(r_u)_{u\in\mathbb{R}_+}$ and $(\mu_u)_{u\in\mathbb{R}_+}$ of tempered distributions on $\mathbb{R}$ by requiring that $r_0 \coloneqq \delta_0$, $\mu_0 \coloneqq 0$, and, for $u > 0$ and $\xi \in \mathbb{R}$,

$$\begin{aligned} \widehat{r_u}(\xi) &\coloneqq 1_{[0,u]}(|\xi|) + \frac{u}{|\xi|}1_{(u,\infty)}(|\xi|), \\ \widehat{\mu_u}(\xi) &\coloneqq \frac{|\xi| - u}{|\xi|}1_{(u,\infty)}(|\xi|) = 1 - \widehat{r_u}(\xi). \end{aligned}$$

In other words, $\mu_u = \check{1} - r_u = \delta_0 - r_u$ for all $u \geq 0$.

Here are important properties of the families $(r_u)_{u\in\mathbb{R}_+}$ and $(\mu_u)_{u\in\mathbb{R}_+}$.

Proposition B.1.2. *Let $f \colon \mathbb{R} \to \mathbb{C}$ be a Borel measurable function. Write $f * \mu \colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ for the map $(u, x) \mapsto (f * \mu_u)(x) = f(x) - (f * r_u)(x)$ when it makes sense.*

(i) *If $u > 0$, then $r_u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Specifically, $r_u = u\, r_1(u\cdot)$, $\|r_1\|_{L^2} = \sqrt{2\pi^{-1}}$, and $\|r_1\|_{L^1} < 2 < \infty$, so that $\|r_u\|_{L^2} = \sqrt{2(\pi u)^{-1}}$ and $\|r_u\|_{L^1} < 2$.*

(ii) *If f is bounded, then $f * \mu$ is bounded and Borel measurable with*

$$\|f * \mu\|_{\ell^\infty(\mathbb{R}_+\times\mathbb{R})} \le \|f\|_{\ell^\infty(\mathbb{R})}(1 + \|r_1\|_{L^1}) \le 3\|f\|_{\ell^\infty(\mathbb{R})}.$$

(And we can replace the ℓ's with L's.) If, in addition, $f \in C(\mathbb{R})$, then $f*\mu \in C((0,\infty)\times\mathbb{R})$.

(iii) *If $f \in L^1(\mathbb{R})$, then $\|f * \mu_u\|_{L^1} \le \|f\|_{L^1}(1 + \|r_1\|_{L^1}) \le 3\|f\|_{L^1}$ for all $u \ge 0$ as well.*

(iv) *Let $\sigma > 0$. If f is bounded and $\operatorname{supp}\widehat{f} \subseteq [0,\sigma]$, then $f * \mu_u \equiv 0$ whenever $u > \sigma$. In particular, $(f * \mu_.)(x) \in C_c(\mathbb{R}_+)$ for all $x \in \mathbb{R}$.*

Proof. We take each item in turn but postpone the proof of (iv) until just after Lemma B.1.7.

(i) First, note that $\|\widehat{r}_1\|_{L^2} = 2$ by an easy calculation. Therefore, by Plancherel's theorem, $\|r_1\|_{L^2} = (2\pi)^{-1/2}\|\widehat{r}_1\|_{L^2} = \sqrt{2\pi^{-1}}$. Next, fix $u > 0$ and $\xi \in \mathbb{R}$. Notice that $\widehat{r}_u(\xi) = \widehat{r}_1(\xi/u)$, from which it follows, by Fourier inversion on L^2, that $r_u = \mathcal{F}^{-1}(\widehat{r}_1(\cdot/u)) = u\, r_1(u\,\cdot)$. In particular, $\|r_u\|_{L^2} = u^{-1/2}\|r_1\|_{L^2}$, and $\|r_u\|_{L^1} = \|r_1\|_{L^1}$, as claimed.

It now suffices to prove $\|r_1\|_{L^1} < 2$. To this end, note $\int_{-1}^1 |r_1(x)|\,\mathrm{d}x \le \sqrt{2}\|r_1\|_{L^2} = 2\pi^{-1/2}$ by the previous paragraph. Now, for almost every $x \in \mathbb{R}$,

$$\begin{aligned} r_1(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{r}_1(\xi)\, e^{ix\xi}\,\mathrm{d}\xi = -\frac{1}{2\pi x^2}\int_{-\infty}^{\infty} \widehat{r}_1(\xi)\frac{d^2}{d\xi^2}e^{ix\xi}\,\mathrm{d}\xi \\ &= -\frac{1}{2\pi x^2}\int_{-\infty}^{\infty} \frac{d^2}{d\xi^2}\widehat{r}_1(\xi)\, e^{ix\xi}\,\mathrm{d}\xi = -\frac{1}{2\pi x^2}\int_{|\xi|>1} \frac{2}{|\xi|^3}e^{ix\xi}\,\mathrm{d}\xi, \end{aligned}$$

using integration by parts, where all of the above are improper Riemann integrals. (Since $r_1 = L^2\text{-}\lim_{R\to\infty}\frac{1}{2\pi}\int_{|\xi|\le R}\widehat{r}_1(\xi)e^{i\cdot\xi}\,\mathrm{d}\xi$, we should really take a particular sequence $R_k \to \infty$ as $k \to \infty$ for the first couple of integrals.) Now, notice

$$\left|\int_{|\xi|>1} \frac{2}{|\xi|^3}e^{ix\xi}\,\mathrm{d}\xi\right| \le 4\int_1^{\infty}\frac{1}{\xi^3}\,\mathrm{d}\xi = 2.$$

It follows that

$$\int_{|x|>1} |r_1(x)|\,\mathrm{d}x \le \frac{2}{\pi}\int_1^{\infty}\frac{1}{x^2}\,\mathrm{d}x = \frac{2}{\pi}.$$

We finally conclude that $\|r_1\|_{L^1} \le 2\pi^{-1/2} + 2\pi^{-1} < 2$, as desired.

(ii) Fix $u > 0$ and $x \in \mathbb{R}$. Then, recalling $r_u = u\, r_1(u\cdot)$,

$$f * r_u(x) = \int_{\mathbb{R}} f(x-y)\, r_u(y)\, \mathrm{d}y = \int_{\mathbb{R}} f\big(x - u^{-1}t\big)\, r_1(t)\, \mathrm{d}t.$$

The measurability of $f * \mu$ follows from this identity and the fact that $f * \mu(0, \cdot) = 0$. The bounds are also immediate from this identity (because $f * \mu = f - f * r_\cdot$) and the first part. Finally, the joint continuity of $(0,\infty) \times \mathbb{R} \ni (u,x) \mapsto f * r_u(x) \in \mathbb{R}$ follows from the continuity of f and the dominated convergence theorem (which applies because f is bounded and $r_1 \in L^1(\mathbb{R})$).

(iii) This is immediate from Young's convolution inequality (when $u > 0$), the fact that $f * \mu_0 = 0$ (when $u = 0$), and the first part. □

In order to bound integral projective tensor norms, one must exhibit expressions for the functions in question as integrals that "separate variables" in a particular way. Here is one such expression, which we take the rest of the section to prove.

Theorem B.1.3. *Fix $\sigma > 0$. If $f \in \ell^\infty(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ satisfies* $\operatorname{supp} \widehat{f} \subseteq [0,\sigma]$*, then*

$$f^{[k]}(\boldsymbol{\lambda}) = i^k \sum_{j=1}^{k+1} \int_{\mathbb{R}_+^k} \left(\prod_{m=1}^{j-1} e^{i\lambda_m u_m}\right) (f * \mu_{|\vec{u}|})(\lambda_j)\, e^{-i\lambda_j|\vec{u}|} \left(\prod_{m=j+1}^{k+1} e^{i\lambda_m u_{m-1}}\right) \mathrm{d}\vec{u} \qquad \text{(B.1.4)}$$

for all $k \in \mathbb{N}$ and $\boldsymbol{\lambda} \in \mathbb{R}^{k+1}$, where $|\vec{u}| \coloneqq \sum_{m=1}^k u_m$ and empty products are defined to be 1*.*

Remark B.1.5. Equation (B.1.4) was written in [Pel85] and [Pel06] in the $k = 1$ and $k = 2$ cases, respectively, in a slightly different form. The use of μ_u was inspired by [Pel16], in which Equation (B.1.4) is written exactly as stated in the cases $k \in \{1, 2\}$.

For example,

$$f^{[1]}(\lambda_1,\lambda_2) = i \int_{\mathbb{R}_+} \big((f * \mu_u)(\lambda_1)\, e^{-i\lambda_1 u} e^{i\lambda_2 u} + e^{i\lambda_1 u} (f * \mu_u)(\lambda_2)\, e^{-i\lambda_2 u}\big)\, \mathrm{d}u \quad \text{and} \qquad \text{(B.1.6)}$$

$$f^{[2]}(\lambda_1,\lambda_2,\lambda_3) = -\int_{\mathbb{R}_+^2} \big((f * \mu_{u+v})(\lambda_1)\, e^{-i\lambda_1(u+v)} e^{i\lambda_2 u} e^{i\lambda_3 v} + e^{i\lambda_1 u} (f * \mu_{u+v})(\lambda_2)\, e^{-i\lambda_2(u+v)} e^{i\lambda_3 v}$$
$$+ e^{i\lambda_1 u} e^{i\lambda_2 v} (f * \mu_{u+v})(\lambda_3)\, e^{-i\lambda_3(u+v)}\big)\, \mathrm{d}u\, \mathrm{d}v$$

for all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

Notice that Proposition B.1.2 allows us to make sense of Equation (B.1.4) in the first place. By Proposition B.1.2(iv), the integrand in Equation (B.1.4) is bounded, continuous, and vanishes whenever $|\vec{u}| > \sigma$. Therefore, the integral above is really over $\{\vec{u} \in \mathbb{R}_+^k : |\vec{u}| \leq \sigma\}$, which has finite measure. This, together with the continuity part of Proposition B.1.2(ii) and the dominated convergence theorem, also implies the right-hand side of (B.1.4) is continuous in $\boldsymbol{\lambda}$.

Equation (B.1.4) is proven, inspired by the sketch in [Pel06], in the following steps.

Step 1. Use an approximation procedure (Lemma B.1.7) to reduce to when $f, \widehat{f} \in L^1(\mathbb{R})$.

Step 2. Use an inductive argument to reduce to the $k = 1$ case, i.e., Equation (B.1.6).

Step 3. Prove Equation (B.1.6) assuming $f, \widehat{f} \in L^1(\mathbb{R})$.

The approximation procedure in Step 1 will also help us to prove Proposition B.1.2(iv).

Lemma B.1.7. *For the remainder of this section, fix $\sigma > 0$ and a function $f \in \ell^\infty(\mathbb{R}, \mathcal{B}_\mathbb{R})$ with* $\operatorname{supp} \widehat{f} \subseteq [0, \sigma]$. *Let $0 \leq \omega \in C_c^\infty(\mathbb{R})$ be such that* $\operatorname{supp} \omega \subseteq [0, 1]$ *and $\int_\mathbb{R} \omega(\xi)\, \mathrm{d}\xi = 2\pi$. Define*

$$\omega_n \coloneqq n\, \omega(n\cdot) \ \ \textit{and} \ \ f_n \coloneqq \check{\omega}_n f, \qquad n \in \mathbb{N}.$$

Then

(i) $\|f_n\|_{\ell^\infty(\mathbb{R})} \leq \|f\|_{\ell^\infty(\mathbb{R})}$ *and $f_n \to f$ pointwise as $n \to \infty$,*

(ii) $f_n \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

(iii) $\widehat{f_n} \in \mathscr{S}(\mathbb{R}) \subseteq L^1(\mathbb{R})$ *and* $\operatorname{supp} \widehat{f_n} \subseteq [0, \sigma + 1/n] \subseteq [0, \sigma + 1]$, *and*

(iv) $f_n * \mu \to f * \mu$ *boundedly on $\mathbb{R}_+ \times \mathbb{R}$ as $n \to \infty$.*

Proof. We take each item in turn.

(i) Notice that if $x \in \mathbb{R}$, then

$$\check{\omega}_n(x) = \widetilde{n\, \omega(n\cdot)}(x) = \check{\omega}(n^{-1}x) \xrightarrow{n\to\infty} \check{\omega}(0) = \frac{1}{2\pi} \int_\mathbb{R} \omega(\xi)\, \mathrm{d}\xi = 1.$$

Also, since $\omega \geq 0$, $\left|\check{\omega}_n(x)\right| = \left|\frac{1}{2\pi} \int_\mathbb{R} \omega(\xi)\, e^{in^{-1}x\xi}\, \mathrm{d}\xi\right| \leq \frac{1}{2\pi} \int_\mathbb{R} \omega(\xi)\, \mathrm{d}\xi = 1$, i.e., $\|\check{\omega}_n\|_{\ell^\infty(\mathbb{R})} \leq 1$. This takes care of the first part.

(ii) Of course, $\check{\omega}_n \in \mathscr{S}(\mathbb{R}) \subseteq L^1(\mathbb{R})$, so that $\|f_n\|_{L^1} \leq \|f\|_{L^\infty}\|\check{\omega}_n\|_{L^1} < \infty$.

(iii) By the basic properties of the Fourier transform on tempered distributions, we have $\widehat{f_n} = \mathcal{F}(\check{\omega}_n f) = \omega_n * \widehat{f}$. Since $\widehat{f}$ has compact support, $\omega_n * \widehat{f} \in \mathscr{S}(\mathbb{R})$, and

$$\operatorname{supp}\widehat{f_n} = \operatorname{supp}\big(\omega_n * \widehat{f}\big) \subseteq \overline{\operatorname{supp}\omega_n + \operatorname{supp}\widehat{f}} \subseteq \big[0, n^{-1}\big] + [0,\sigma] = \big[0, \sigma + n^{-1}\big],$$

as claimed.

(iv) By Proposition B.1.2 and the first part, $\|f_n * \mu\|_{\ell^\infty(\mathbb{R}_+\times\mathbb{R})} \leq 3\|f_n\|_{\ell^\infty(\mathbb{R})} \leq 3\|f\|_{\ell^\infty(\mathbb{R})}$ for all $n \in \mathbb{N}$. Now, fix $u > 0$ and $x \in \mathbb{R}$. (The case $u = 0$ is obvious.) By the proof of Proposition B.1.2(ii) and the dominated convergence theorem,

$$(f_n * r_u)(x) = \int_{\mathbb{R}} f_n\big(x - u^{-1}y\big)\, r_1(y)\,\mathrm{d}y \xrightarrow{n\to\infty} \int_{\mathbb{R}} f\big(x - u^{-1}y\big)\, r_1(y)\,\mathrm{d}y = (f * r_u)(x).$$

Therefore, $(f_n * \mu_u)(x) = f_n(x) - (f_n * r_u)(x) \to f(x) - (f * r_u)(x) = (f * \mu_u)(x)$ as $n \to \infty$. □

Proof of Proposition B.1.2(iv). Suppose first that $f, \widehat{f} \in L^1(\mathbb{R})$. Recall from Proposition B.1.2(iii) that $\|f * \mu_u\|_{L^1} \leq 3\|f\|_{L^1}$, so that $f * \mu_u \in L^1(\mathbb{R})$. Also,

$$\mathcal{F}(f * \mu_u) = \widehat{f}\ \widehat{\mu_u} \in L^1(\mathbb{R}).$$

But $\operatorname{supp}\widehat{\mu_u} = (-\infty, -u] \cup [u, \infty)$ whenever $u > 0$, and $\operatorname{supp}\widehat{f} \subseteq [0,\sigma]$. Therefore, if $u > \sigma$, then $\mathcal{F}(f * \mu_u) \equiv 0$. Therefore, by the Fourier inversion theorem,

$$f * \mu_u = \mathcal{F}^{-1}(\mathcal{F}(f * \mu_u)) \equiv 0$$

as well. (Recall $f * \mu_u \in C(\mathbb{R})$, so this equality is *everywhere*.)

Now, for general f as in Proposition B.1.2(iv), let $(f_n)_{n\in\mathbb{N}}$ be as in Lemma B.1.7. Since $f_n, \widehat{f_n} \in L^1(\mathbb{R})$ and $\operatorname{supp}\widehat{f_n} \subseteq [0, \sigma + 1/n]$, we know from the previous paragraph that $f_n * \mu_u \equiv 0$ whenever $u > \sigma + 1/n$. Now, suppose $u > \sigma$. Then, choosing $n_1 \in \mathbb{N}$ such that $u > \sigma + 1/n$ for all $n \geq n_1$, we know that $f_n * \mu_u \equiv 0$ whenever $n \geq n_1$. Since $f_n * \mu \to f * \mu$ pointwise as $n \to \infty$, we conclude that $f * \mu_u \equiv 0$ as well. □

We now begin the proof of Theorem B.1.3 in earnest.

Notation B.1.8. If $j \in \{1, \ldots, k+1\}$, then

$$\varepsilon_{k,j}^{f}(\boldsymbol{\lambda}, \vec{u}) \coloneqq \left(\prod_{m=1}^{j-1} e^{i\lambda_m u_m}\right)(f * \mu_{|\vec{u}|})(\lambda_j)\, e^{-i\lambda_j |\vec{u}|}\left(\prod_{m=j+1}^{k+1} e^{i\lambda_m u_{m-1}}\right), \qquad \boldsymbol{\lambda} \in \mathbb{R}^{k+1},\ \vec{u} \in \mathbb{R}_+^k.$$

Step 1. Suppose Equation (B.1.4) holds when we also assume $f, \widehat{f} \in L^1(\mathbb{R})$. For arbitrary f, let $(f_n)_{n\in\mathbb{N}}$ be as in Lemma B.1.7. Since $f_n, \widehat{f_n} \in L^1(\mathbb{R})$, we know that Equation (B.1.4) holds for f_n in place of f. We must take $n \to \infty$ to obtain Equation (B.1.4) for f. To this end, first let $\lambda_1, \ldots, \lambda_{k+1} \in \mathbb{R}$ be distinct, and write $\boldsymbol{\lambda} \coloneqq (\lambda_1, \ldots, \lambda_{k+1}) \in \mathbb{R}^{k+1}$ as usual. Then, by the recursive definition of the k^{th} divided difference, $f_n^{[k]}(\boldsymbol{\lambda}) \to f^{[k]}(\boldsymbol{\lambda})$ as $n \to \infty$ because $f_n \to f$ pointwise as $n \to \infty$. Second, $\varepsilon_{k,j}^{f_n} \to \varepsilon_{k,j}^{f}$ boundedly on $\mathbb{R}^{k+1} \times \mathbb{R}_+^k$ as $n \to \infty$ by Lemma B.1.7(iv). Third, by Proposition B.1.2(iv), the integral $\int_{\mathbb{R}_+^k} \varepsilon_{k,j}^{f_n}(\boldsymbol{\lambda}, \vec{u})\, \mathrm{d}\vec{u}$ is really only over $\{\vec{u} \in \mathbb{R}_+^k : |\vec{u}| \le \sigma + 1\}$ for all $n \in \mathbb{N}$ and $j \in \{1, \ldots, k+1\}$. Therefore, by the assumption and the dominated convergence theorem,

$$f_n^{[k]}(\boldsymbol{\lambda}) = i^k \sum_{j=1}^{k+1} \int_{\mathbb{R}_+^k} \varepsilon_{k,j}^{f_n}(\boldsymbol{\lambda}, \vec{u})\, \mathrm{d}\vec{u} \xrightarrow{n\to\infty} i^k \sum_{j=1}^{k+1} \int_{\mathbb{R}_+^k} \varepsilon_{k,j}^{f}(\boldsymbol{\lambda}, \vec{u})\, \mathrm{d}\vec{u}, \qquad \boldsymbol{\lambda} \in \mathbb{R}^{k+1}.$$

We conclude that

$$f^{[k]}(\boldsymbol{\lambda}) = i^k \sum_{j=1}^{k+1} \int_{\mathbb{R}_+^k} \varepsilon_{k,j}^{f}(\boldsymbol{\lambda}, \vec{u})\, \mathrm{d}\vec{u}$$

whenever $\lambda_1, \ldots, \lambda_{k+1} \in \mathbb{R}$ are distinct. Since $\{\boldsymbol{\lambda} \in \mathbb{R}^{k+1} : \lambda_1, \ldots, \lambda_{k+1} \in \mathbb{R} \text{ are distinct}\}$ is dense in $\mathbb{R}^{k+1}$ and both sides of the above are continuous in $\boldsymbol{\lambda}$, we are done. □

Next comes Step 2, which is a bit painful and may be skipped on a first read. We warm up with two easy lemmas.

Lemma B.1.9. *If $u \ge 0$ and $h(\lambda) \coloneqq e^{i\lambda u}$, then $h^{[1]}(\lambda_1, \lambda_2) = i \int_0^u e^{i\lambda_1 v} e^{i\lambda_2 (u-v)}\, \mathrm{d}v$.*

Proof. The result is obvious if $u = 0$, so we assume $u > 0$. By Proposition 1.3.3(iii),

$$h^{[1]}(\lambda_1, \lambda_2) = \int_0^1 h'(t\lambda_1 + (1-t)\lambda_2)\, \mathrm{d}t = i \int_0^1 u e^{i(t\lambda_1 + (1-t)\lambda_2)u}\, \mathrm{d}t = i \int_0^u e^{i\lambda_1 v} e^{i\lambda_2 (u-v)}\, \mathrm{d}v,$$

where we substituted $v \coloneqq tu$. □

Lemma B.1.10. *For the remainder of this section, assume that $f, \widehat{f} \in L^1(\mathbb{R})$ as well. If $u > 0$ and $g(\lambda) \coloneqq (f * \mu_u)(\lambda)\, e^{-i\lambda u}$, then*

$$(g * \mu_v)(\lambda) = (f * \mu_{u+v})(\lambda)\, e^{-i\lambda u}, \qquad v > 0,\ \lambda \in \mathbb{R}.$$

Proof. Note that $\widehat{g}(\xi) = \mathcal{F}(f * \mu_u)(\xi + u) = \widehat{f}(\xi + u)\, \widehat{\mu_u}(\xi + u)$, so that

$$\mathcal{F}(g * \mu_v)(\xi) = \widehat{g}(\xi)\, \widehat{\mu_v}(\xi) = \widehat{f}(\xi + u)\, \widehat{\mu_u}(\xi + u)\, \widehat{\mu_v}(\xi).$$

But

$$\begin{aligned} \widehat{\mu_u}(\xi + u)\, \widehat{\mu_v}(\xi) &= \frac{\xi + u - u}{\xi + u} 1_{(u,\infty)}(\xi + u)\, \frac{\xi - v}{\xi} 1_{(v,\infty)}(\xi) \\ &= \frac{\xi - v}{\xi + u} 1_{(u+v,\infty)}(\xi + u) = \widehat{\mu_{u+v}}(\xi + u), \end{aligned}$$

so that

$$\mathcal{F}(g * \mu_u)(\xi) = \widehat{f}(\xi + u)\, \widehat{\mu_{u+v}}(\xi + u) = \mathcal{F}((f * \mu_{u+v})\, e^{-i \cdot u})(\xi).$$

The result follows from the Fourier inversion theorem. □

We are now ready for Step 2.

Step 2. Assume for some $\ell \in \mathbb{N}$ that Equation (B.1.4) holds whenever $k \in \{1, \ldots, \ell\}$ (and all relevant f). Suppose $k \in \{1, \ldots, \ell\}$, and fix distinct $\lambda_1, \ldots, \lambda_{k+2} \in \mathbb{R}$. Then

$$\begin{aligned} f^{[k+1]}(\lambda_1, \ldots, \lambda_{k+2}) &= \frac{f^{[k]}(\lambda_1, \ldots, \lambda_{k+1}) - f^{[k]}(\lambda_1, \ldots, \lambda_k, \lambda_{k+2})}{\lambda_{k+1} - \lambda_{k+2}} \\ &= i^k \sum_{j=1}^{k+1} \int_{\mathbb{R}_+^k} \frac{\varepsilon_{k,j}^f(\lambda_1, \ldots, \lambda_{k+1}, \vec{u}) - \varepsilon_{k,j}^f(\lambda_1, \ldots, \lambda_k, \lambda_{k+2}, \vec{u})}{\lambda_{k+1} - \lambda_{k+2}}\, \mathrm{d}\vec{u}. \end{aligned}$$

We now examine each term in the above sum. Define

$$\delta_j(\vec{u}) \coloneqq \frac{\varepsilon_{k,j}^f(\lambda_1, \ldots, \lambda_{k+1}, \vec{u}) - \varepsilon_{k,j}^f(\lambda_1, \ldots, \lambda_k, \lambda_{k+2}, \vec{u})}{\lambda_{k+1} - \lambda_{k+2}}$$

for ease of notation.

First, suppose $1 \le j < k+1$. Then, by definition of the $\varepsilon_{k,j}$'s and Lemma B.1.9,

$$\begin{aligned}\delta_j(\vec{u}) &= \prod_{m=1}^{j-1} e^{i\lambda_m u_m} (f * \mu_{|\vec{u}|})(\lambda_j)\, e^{-i\lambda_j|\vec{u}|} \prod_{m=j+1}^{k} e^{i\lambda_m u_{m-1}} \frac{e^{i\lambda_{k+1}u_k} - e^{i\lambda_{k+2}u_k}}{\lambda_{k+1} - \lambda_{k+2}} \\ &= i\int_0^{u_k} \prod_{m=1}^{j-1} e^{i\lambda_m u_m} (f * \mu_{|\vec{u}|})(\lambda_j)\, e^{-i\lambda_j|\vec{u}|} \prod_{m=j+1}^{k} e^{i\lambda_m u_{m-1}}\, e^{i\lambda_{k+1}v} e^{i\lambda_{k+2}(u_k - v)}\, \mathrm{d}v.\end{aligned}$$

Now, this allows us to write

$$\int_{\mathbb{R}_+^k} \delta_j(\vec{u})\, \mathrm{d}\vec{u} = i\int_{\mathbb{R}_+^k}\int_0^{u_k} \prod_{m=1}^{j-1} e^{i\lambda_m u_m} (f * \mu_{|\vec{u}|})(\lambda_j)\, e^{-i\lambda_j|\vec{u}|} \prod_{m=j+1}^{k} e^{i\lambda_m u_{m-1}}\, e^{i\lambda_{k+1}v} e^{i\lambda_{k+2}(u_k - v)}\, \mathrm{d}v\, \mathrm{d}\vec{u}.$$

We now manipulate this integral expression. Changing the order of integration yields

$$i\int_{\mathbb{R}_+^k}\int_0^{u_k} \cdot\, \mathrm{d}v\, \mathrm{d}\vec{u} = i\int_{\mathbb{R}_+}\int_{\mathbb{R}_+^k} 1_{\{u_k \ge v\}}(\vec{u}) \cdot\, \mathrm{d}\vec{u}\, \mathrm{d}v.$$

Changing variables by $(u_1, \ldots, u_k, v) \mapsto (u_1, \ldots, u_{k-1}, v, u_k - v) =: (v_1, \ldots, v_{k+1}) = \mathbf{v}$ yields

$$i\int_{\mathbb{R}_+}\int_{\mathbb{R}_+^k} 1_{\{u_k \ge v\}}(\vec{u})\, g(\vec{u}, v)\, \mathrm{d}\vec{u}\, \mathrm{d}v = i\int_{\mathbb{R}_+^{k+1}} g(v_1, \ldots, v_{k-1}, v_k + v_{k+1}, v_k)\, \mathrm{d}\mathbf{v}$$

whenever $g \colon \mathbb{R}_+^{k+1} \to \mathbb{C}$ is a nice enough function. (In particular, this change of variables converts $|\vec{u}|$ to $|\mathbf{v}|$.) This yields

$$\begin{aligned}\int_{\mathbb{R}_+^k} \delta_j(\vec{u})\, \mathrm{d}\vec{u} &= i\int_{\mathbb{R}_+^{k+1}} \prod_{m=1}^{j-1} e^{i\lambda_m v_m} (f * \mu_{|\mathbf{v}|})(\lambda_j)\, e^{-i\lambda_j|\mathbf{v}|} \prod_{m=j+1}^{k} e^{i\lambda_m v_{m-1}}\, e^{i\lambda_{k+1}v_k} e^{i\lambda_{k+2}v_{k+1}}\, \mathrm{d}\mathbf{v} \\ &= i\int_{\mathbb{R}_+^{k+1}} \prod_{m=1}^{j-1} e^{i\lambda_m v_m} (f * \mu_{|\mathbf{v}|})(\lambda_j)\, e^{-i\lambda_j|\mathbf{v}|} \prod_{m=j+1}^{k+2} e^{i\lambda_m v_{m-1}}\, \mathrm{d}\mathbf{v} \\ &= i\int_{\mathbb{R}_+^{k+1}} \varepsilon_{k+1,j}^f(\lambda_1, \ldots, \lambda_{k+2}, \mathbf{v})\, \mathrm{d}\mathbf{v},\end{aligned}$$

which is one of the terms we wanted to see.

Second, for the $j = k+1$ term, notice that

$$\delta_{k+1}(\vec{u}) = \prod_{m=1}^{k} e^{i\lambda_m u_m} \big(f * \mu_{|\vec{u}|}\, e^{-i\cdot|\vec{u}|}\big)^{[1]}(\lambda_{k+1}, \lambda_{k+2}).$$

But the function

$$g(\lambda) := (f * \mu_{|\vec{u}|})(\lambda)\, e^{-i\lambda|\vec{u}|}, \quad ; \ \lambda \in \mathbb{R},$$

satisfies $g, \widehat{g} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\operatorname{supp} \widehat{g} \subseteq [0,\sigma]$. Consequently, by assumption and Lemma B.1.10, if $|\vec{u}| > 0$, then

$$\begin{aligned} g^{[1]}(\lambda_{k+1}, \lambda_{k+2}) &= i \int_{\mathbb{R}_+} \Big((g * \mu_v)(\lambda_{k+1})\, e^{-i\lambda_{k+1}v} e^{i\lambda_{k+2}v} + (g * \mu_v)(\lambda_{k+2})\, e^{-i\lambda_{k+2}v} e^{i\lambda_{k+1}v} \Big)\, \mathrm{d}v \\ &= i \int_{\mathbb{R}_+} \Big((f * \mu_{|\vec{u}|+v})(\lambda_{k+1})\, e^{-i\lambda_{k+1}(|\vec{u}|+v)} e^{i\lambda_{k+2}v} \\ &\qquad\qquad + (f * \mu_{|\vec{u}|+v})(\lambda_{k+2})\, e^{-i\lambda_{k+2}(|\vec{u}|+v)} e^{i\lambda_{k+1}v} \Big)\, \mathrm{d}v. \end{aligned}$$

Therefore, renaming $(u_1, \ldots, u_k, v)$ to $(v_1, \ldots, v_{k+1}) = \mathbf{v}$,

$$\begin{aligned} \int_{\mathbb{R}^k_+} \delta_{k+1}(\vec{u})\, \mathrm{d}\vec{u} &= i \int_{\mathbb{R}^k_+} \int_{\mathbb{R}_+} \Bigg(\prod_{m=1}^{k} e^{i\lambda_m u_m} (f * \mu_{|\vec{u}|+v})(\lambda_{k+1})\, e^{-i\lambda_{k+1}(|\vec{u}|+v)} e^{i\lambda_{k+2}v} \\ &\qquad\qquad + \prod_{m=1}^{k} e^{i\lambda_m u_m} (f * \mu_{|\vec{u}|+v})(\lambda_{k+2})\, e^{-i\lambda_{k+2}(|\vec{u}|+v)} e^{i\lambda_{k+1}v} \Bigg)\, \mathrm{d}v\, \mathrm{d}\vec{u} \\ &= i \int_{\mathbb{R}^{k+1}_+} \Bigg(\prod_{m=1}^{k} e^{i\lambda_m v_m} (f * \mu_{|\mathbf{v}|})(\lambda_{k+1})\, e^{-i\lambda_{k+1}|\mathbf{v}|} e^{i\lambda_{k+2}v_{k+1}} \\ &\qquad\qquad + \prod_{m=1}^{k+1} e^{i\lambda_m v_m} (f * \mu_{|\mathbf{v}|})(\lambda_{k+2})\, e^{-i\lambda_{k+2}|\mathbf{v}|} \Bigg)\, \mathrm{d}\mathbf{v} \\ &= i \int_{\mathbb{R}^{k+1}_+} \Big(\varepsilon^f_{k+1,k+1}(\lambda_1, \ldots, \lambda_{k+2}, \mathbf{v}) + \varepsilon^f_{k+1,k+2}(\lambda_1, \ldots, \lambda_{k+2}, \mathbf{v}) \Big)\, \mathrm{d}\mathbf{v}, \end{aligned}$$

which are the remaining terms we needed.

Finally, putting it all together, we have

$$f^{[k+1]}(\lambda_1, \ldots, \lambda_{k+2}) = i^k \sum_{j=1}^{k+1} \int_{\mathbb{R}^k_+} \delta_j(\vec{u})\, \mathrm{d}\vec{u} = i^{k+1} \sum_{j=1}^{k+2} \int_{\mathbb{R}^{k+1}_+} \varepsilon^f_{k+1,j}(\lambda_1, \ldots, \lambda_{k+2}, \mathbf{v})\, \mathrm{d}\mathbf{v}$$

whenever $\lambda_1, \ldots, \lambda_{k+2} \in \mathbb{R}$ are distinct. Since both sides of the equation above are continuous in $(\lambda_1, \ldots, \lambda_{k+2})$, this completes the proof. □

We are left with Step 3 (the easiest step), i.e., the base case of the induction in Step 2.

Step 3. First, we claim that

$$f^{[1]}(\lambda_1,\lambda_2) = \frac{i}{2\pi}\int_{\mathbb{R}_+^2} \widehat{f}(u+v)\, e^{i\lambda_1 u} e^{i\lambda_2 v}\,\mathrm{d}u\,\mathrm{d}v, \qquad \lambda_1,\lambda_2\in\mathbb{R}.$$

(The integral above makes sense because $\widehat{f}$ is compactly supported and belongs to $L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$.) Indeed, by the Fourier inversion theorem, the continuity of f, and the fact that $\operatorname{supp}\widehat{f}\subseteq\mathbb{R}_+$,

$$f(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}_+} \widehat{f}(\xi)\, e^{i\lambda\xi}\,\mathrm{d}\xi, \qquad \lambda\in\mathbb{R}.$$

Consequently, if $\lambda_1,\lambda_2\in\mathbb{R}$ are distinct, then

$$\begin{aligned} f^{[1]}(\lambda_1,\lambda_2) &= \frac{1}{2\pi}\int_{\mathbb{R}_+} \widehat{f}(\xi)\frac{e^{i\lambda_1\xi}-e^{i\lambda_2\xi}}{\lambda_1-\lambda_2}\,\mathrm{d}\xi = \frac{i}{2\pi}\int_{\mathbb{R}_+}\int_0^\xi \widehat{f}(\xi)\, e^{i\lambda_1 v}e^{i\lambda_2(\xi-v)}\,\mathrm{d}v\,\mathrm{d}\xi \\ &= \frac{i}{2\pi}\int_{\mathbb{R}_+}\int_v^\infty \widehat{f}(\xi)\, e^{i\lambda_1 v}e^{i\lambda_2(\xi-v)}\,\mathrm{d}\xi\,\mathrm{d}v = \frac{i}{2\pi}\int_{\mathbb{R}_+^2} \widehat{f}(u+v)\, e^{i\lambda_1 v}e^{i\lambda_2 u}\,\mathrm{d}u\,\mathrm{d}v, \end{aligned}$$

by Lemma B.1.9 and the change of variable $u \coloneqq \xi - v$. Swapping the roles of u and v in the above integral gives the desired expression. As usual, the continuity of both sides in (λ_1,λ_2) allows us to pass from distinct λ_1,λ_2 to arbitrary λ_1,λ_2.

Therefore, our goal is to show that

$$i\int_{\mathbb{R}_+}\big((f*\mu_u)(\lambda_1)\, e^{-i\lambda_1 u}e^{i\lambda_2 u} + (f*\mu_u)(\lambda_2)\, e^{-i\lambda_2 u}e^{i\lambda_1 u}\big)\,\mathrm{d}u = \frac{i}{2\pi}\int_{\mathbb{R}_+^2}\widehat{f}(u+v)\, e^{i\lambda_1 u}e^{i\lambda_2 v}\,\mathrm{d}u\,\mathrm{d}v$$

for all $\lambda_1,\lambda_2\in\mathbb{R}$. To this end, notice that for all $u\geq 0$, the function $g(\lambda)\coloneqq(f*\mu_u)(\lambda)\, e^{-i\lambda u}$ satisfies $g,\widehat{g}\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ and $g\in C(\mathbb{R})$. Also, if $u>0$, then

$$\widehat{g}(\xi) = \widehat{f}(\xi+u)\,\widehat{\mu_u}(\xi+u) = \widehat{f}(\xi+u)\frac{\xi}{\xi+u}1_{(0,\infty)}(\xi), \qquad \xi\in\mathbb{R}.$$

Consequently, by the Fourier inversion theorem and the continuity of g,

$$g(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{g}(\xi)\, e^{i\lambda\xi}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{\mathbb{R}_+}\widehat{f}(\xi+u)\frac{\xi}{\xi+u}e^{i\lambda\xi}\,\mathrm{d}\xi, \qquad \lambda\in\mathbb{R},$$

whenever $u > 0$. Therefore,

$$\begin{aligned}
i\int_{\mathbb{R}_+}(f*\mu_u)(\lambda_1)\,e^{-i\lambda_1 u}e^{i\lambda_2 u}\,\mathrm{d}u &= \frac{i}{2\pi}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\widehat{f}(\xi+u)\frac{\xi}{\xi+u}e^{i\lambda_1\xi}e^{i\lambda_2 u}\,\mathrm{d}\xi\,\mathrm{d}u \\
&= \frac{i}{2\pi}\int_{\mathbb{R}_+^2}\widehat{f}(u+v)\frac{u}{u+v}e^{i\lambda_1 u}e^{i\lambda_2 v}\,\mathrm{d}u\,\mathrm{d}v \quad \text{and} \\
i\int_{\mathbb{R}_+}(f*\mu_u)(\lambda_2)\,e^{-i\lambda_2 u}e^{i\lambda_1 u}\,\mathrm{d}u &= \frac{i}{2\pi}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\widehat{f}(\xi+u)\frac{\xi}{\xi+u}e^{i\lambda_2\xi}e^{i\lambda_1 u}\,\mathrm{d}\xi\,\mathrm{d}u \\
&= \frac{i}{2\pi}\int_{\mathbb{R}_+^2}\widehat{f}(u+v)\frac{v}{u+v}e^{i\lambda_1 u}e^{i\lambda_2 v}\,\mathrm{d}v\,\mathrm{d}u.
\end{aligned}$$

Adding these together yields

$$\frac{i}{2\pi}\int_{\mathbb{R}_+^2}\widehat{f}(u+v)\,e^{i\lambda_1 u}e^{i\lambda_2 v}\,\mathrm{d}u\,\mathrm{d}v,$$

as desired. This completes the proof. □

B.2 The estimate

We now use Theorem B.1.3 to prove Theorem 6.6.9.

Proposition B.2.1. *If $v,\sigma > 0$ and $f\colon \mathbb{R}\to\mathbb{C}$ is as in Lemma B.1.7, then $f*\mu_v \in BOC(\mathbb{R})$. Please see Definition 6.6.1 for the definition of $BOC(\mathbb{R})$.*

Proof. First, suppose $f,\widehat{f}\in L^1(\mathbb{R})$ as well, and let $g \coloneqq f*\mu_v$. Then $\widehat{g}=\widehat{f}\,\widehat{\mu}_v$ is compactly supported in $\mathbb{R}_+$. By Theorem B.1.3 (really, only Step 3 of its proof), we have

$$g^{[1]}(\lambda_1,\lambda_2) = i\int_{\mathbb{R}_+}\big((g*\mu_u)(\lambda_1)\,e^{-i\lambda_1 u}e^{i\lambda_2 u} + (g*\mu_u)(\lambda_2)\,e^{-i\lambda_2 u}e^{i\lambda_1 u}\big)\,\mathrm{d}u.$$

For arbitrary f as in the statement and f_n as in Lemma B.1.7, we have that

$$(f_n*\mu_v)^{[1]}(\lambda_1,\lambda_2) = i\int_{\mathbb{R}_+}\big(((f_n*\mu_v)*\mu_u)(\lambda_1)\,e^{-i\lambda_1 u}e^{i\lambda_2 u} + ((f_n*\mu_v)*\mu_u)(\lambda_2)\,e^{-i\lambda_2 u}e^{i\lambda_1 u}\big)\,\mathrm{d}u.$$

As in Step 1 of the proof of Theorem B.1.3, we may take $n\to\infty$ when $\lambda_1\neq\lambda_2$ to conclude

$$(f*\mu_v)^{[1]}(\lambda_1,\lambda_2) = i\int_{\mathbb{R}_+}\big(((f*\mu_v)*\mu_u)(\lambda_1)\,e^{-i\lambda_1 u}e^{i\lambda_2 u} + ((f*\mu_v)*\mu_u)(\lambda_2)\,e^{-i\lambda_2 u}e^{i\lambda_1 u}\big)\,\mathrm{d}u.$$

Since the right-hand side is continuous in (λ_1, λ_2), we conclude that this identity holds when $\lambda_1 = \lambda_2$ as well. This is a ℓ^∞-integral projective decomposition of $(f * \mu_v)^{[1]}$. By Corollary 6.5.12, if H is a complex Hilbert space, $a \in C(H)_{\mathrm{sa}}$, and $c \in B(H)_{\mathrm{sa}}$, then

$$\|(f * \mu_v)(a + c) - (f * \mu_v)(a)\| \leq \big\|(f * \mu_v)^{[1]}\big\|_{\ell^\infty(\mathbb{R},\mathcal{B}_\mathbb{R})\hat{\otimes}_i \ell^\infty(\mathbb{R},\mathcal{B}_\mathbb{R})}\|c\|,$$

so that $f * \mu_v \in OC(\mathbb{R})$. Since $f * \mu_v$ is bounded by Proposition B.1.2(ii), we are done. □

Proposition B.2.2. *If $k \in \mathbb{N}$, then there is a constant $a_k < \infty$ such that whenever $\sigma > 0$ and $f \in \ell^\infty(\mathbb{R}, \mathcal{B}_\mathbb{R})$ satisfies $\operatorname{supp} \widehat{f} \subseteq \big[-\sigma, -\frac{\sigma}{4}\big] \cup \big[\frac{\sigma}{4}, \sigma\big]$, we have that*

$$\big\|f^{[k]}\big\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}} \leq a_k\sigma^k\|f\|_{L^\infty}.$$

Proof. Let f be as in the statement of the proposition, and fix a bump function $\psi_1 \in C_c^\infty(\mathbb{R})$ such that $\psi_1 \equiv 1$ on $[1/4, 1]$ and $\operatorname{supp}\psi_1 \subseteq [1/8, 2]$. Write $\psi_1^\sigma(\xi) := \psi_1(\sigma\xi)$, $\psi_2^\sigma(\xi) := \psi_1^\sigma(-\xi)$, $\chi_1^\sigma := \mathcal{F}^{-1}(\psi_1^\sigma)$ and $\chi_2^\sigma := \mathcal{F}^{-1}(\psi_2^\sigma)$. Then $f = \chi_1^\sigma * f + \chi_2^\sigma * f$ because $\widehat{f} = \psi_1^\sigma\widehat{f} + \psi_2^\sigma\widehat{f}$. But $f_1 := \chi_1^\sigma * f$ satisfies the hypotheses of Theorem B.1.3. By Lemma 6.6.5, Proposition B.2.1, the fact that $BOC(\mathbb{R})$ is an algebra, and the comments about when the integrand in Equation (B.1.4) vanishes, Theorem B.1.3 yields a BOCIPD of $f_1^{[k]}$ that implies

$$\begin{aligned}\big\|f_1^{[k]}\big\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}} &\leq \sum_{j=1}^{k+1}\int_{\{\vec{u}\in\mathbb{R}_+^k : |\vec{u}|\leq\sigma\}}\|f_1 * \mu_{|\vec{u}|}\|_{\ell^\infty(\mathbb{R})}\,\mathrm{d}\vec{u} \leq 3(k+1)\frac{\sigma^k}{k!}\|f_1\|_{L^\infty} \\ &\leq 3\|\chi_1^\sigma\|_{L^1}(k+1)\frac{\sigma^k}{k!}\|f\|_{L^\infty} = 3\big\|\chi_1^1\big\|_{L^1}(k+1)\frac{\sigma^k}{k!}\|f\|_{L^\infty}\end{aligned}$$

by the bounds from Proposition B.1.2 and Young's convolution inequality. Next, $x \mapsto f_2(-x)$ also satisfies the hypotheses of Theorem B.1.3. This allows us to conclude

$$\big\|f_2^{[k]}\big\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}} \leq 3\big\|\chi_2^1\big\|_{L^1}(k+1)\frac{\sigma^k}{k!}\|f\|_{L^\infty}$$

as well. Combining these two estimates completes the proof and shows that we may take $a_k = 3(k+1)\big(\big\|\chi_1^1\big\|_{L^1} + \big\|\chi_2^1\big\|_{L^1}\big)/k!$ in the statement of proposition. □

We now transfer this result into the desired statement about Besov spaces. Recall from Definition 3.6.1 that we fixed $\eta \in C_c^\infty(\mathbb{R}^m)$ such that $0 \leq \eta \leq 1$, $\operatorname{supp} \eta \subseteq \{\xi \in \mathbb{R}^m : |\xi|_2 \leq 2\}$, and $\eta \equiv 1$ on $\{\xi \in \mathbb{R}^m : |\xi|_2 \leq 1\}$. We also defined $\eta_i(\xi) := \eta\big(2^{-i}\xi\big) - \eta\big(2^{-i+1}\xi\big)$ for all $i \in \mathbb{Z}$ and $\xi \in \mathbb{R}^m$. It is easy to see that $0 \leq \eta_i \leq 1$, $\operatorname{supp} \eta_i \subseteq \big\{\xi \in \mathbb{R}^m : 2^{i-1} \leq |\xi|_2 \leq 2^{i+1}\big\}$, $\eta + \sum_{i=1}^n \eta_i = \eta(2^{-n}\cdot)$ for all $n \in \mathbb{N}$ (so that $\eta + \sum_{i=1}^\infty \eta_i \equiv 1$ everywhere), and $\sum_{i=-\infty}^\infty \eta_i = 1_{\mathbb{R}^m \setminus \{0\}}$. From these bump functions, we get the Littlewood–Paley sequences or decompositions of a tempered distribution. Indeed, if $f \in \mathscr{S}'(\mathbb{R}^m)$ and $n \in \mathbb{Z}$, then

$$f = \widecheck{\eta(2^{-n+1}\boldsymbol{\cdot})} * f + \sum_{i=n}^\infty \check{\eta}_i * f \tag{B.2.3}$$

in the weak* topology of $\mathscr{S}'(\mathbb{R}^m)$. Therefore, the series $\sum_{i=-\infty}^\infty \check{\eta}_i * f$ converges in the weak* topology if and only if $(\widecheck{\eta(2^n\cdot)} * f)_{n \in \mathbb{N}}$ converges in the weak* topology, if and only if $(\eta(2^n\cdot)\widehat{f})_{n\in\mathbb{N}}$ converges in the weak* topology. In particular, the identity

$$f = \sum_{i=-\infty}^\infty \check{\eta}_i * f \tag{B.2.4}$$

holds if and only if $\mathrm{w}^*\text{-}\lim_{n\to\infty} \widecheck{\eta(2^n\boldsymbol{\cdot})} * f = 0$, if and only if $\mathrm{w}^*\text{-}\lim_{n\to\infty} \eta(2^n\boldsymbol{\cdot})\widehat{f} = 0$. Equation (B.2.3) with $n = 1$ is the **inhomogeneous Littlewood–Paley decomposition** of f. Equation (B.2.4) (or at the least the formal series therein) is the **homogeneous Littlewood–Paley decomposition** of f. Sometimes they are also called the **Calderón reproducing formulas**. The proofs boil down to the weak* continuity of the Fourier transform and the fact that if $\psi \in \mathscr{S}(\mathbb{R}^m)$, then $\eta\big(R^{-1}\cdot\big)\psi(\cdot) \to \psi$ in $\mathscr{S}(\mathbb{R}^m)$ as $R \to \infty$, which is a nice exercise to prove.

Note that if $\sum_{i=-\infty}^\infty \check{\eta}_i * f$ converges in the weak* topology, then it is easy to see that $P := f - \sum_{i=-\infty}^\infty \check{\eta}_i * f \in \mathscr{S}'(\mathbb{R}^m)$ satisfies $\operatorname{supp} \widehat{P} \subseteq \{0\}$. As a result, $P \in \mathbb{C}[\lambda_1, \ldots, \lambda_m]$ is a polynomial, and

$$f = \sum_{i=-\infty}^\infty \check{\eta}_i * f + P.$$

This observation will come in handy later. The most important fact about Besov spaces for us is that the inhomogeneous Littlewood–Paley series of the k^{th} derivative of a function belonging to $\dot{B}_1^{k,\infty}(\mathbb{R})$ converges uniformly. To prove this, we use Bernstein's inequality.

Lemma B.2.5 (Bernstein's inequality)**.** *Suppose* $\alpha \in \mathbb{N}_0^m$ *and* $1 \le r \le p \le \infty$*. There is a constant* $b_{\alpha,r,p} < \infty$ *such that for all* $R > 0$ *and* $u \in \mathscr{S}'(\mathbb{R}^m)$ *with* $\operatorname{supp} \widehat{u} \subseteq \{\xi \in \mathbb{R}^m : |\xi|_2 \le R\}$*,*

$$\left\|\partial^\alpha u\right\|_{L^p} \le b_{\alpha,r,p} R^{|\alpha|+m(\frac{1}{r}-\frac{1}{p})} \|u\|_{L^r}.$$

Proof. Defining $u_R := R^{-m} u\big(R^{-1}\,\cdot\big)$, we see that $\operatorname{supp} \widehat{u}_R \subseteq \{\xi \in \mathbb{R}^m : |\xi|_2 \le 1\}$. Supposing we know the desired inequality when $R = 1$, we have $\|\partial^\alpha u_R\|_{L^p} \lesssim \|u_R\|_{L^r}$. Since

$$\partial^\alpha u_R = R^{-|\alpha|} R^{-m} (\partial^\alpha u)\big(R^{-1}\,\cdot\big) \text{ and } \|u_R\|_{L^q} = R^{m(\frac{1}{q}-1)} \|u\|_{L^q}, \qquad q \in [1, \infty],$$

we conclude that

$$R^{-|\alpha|+m(\frac{1}{p}-1)} \left\|\partial^\alpha u\right\|_{L^p} = \left\|\partial^\alpha u_R\right\|_{L^p} \lesssim \|u_R\|_{L^r} = R^{m(\frac{1}{r}-1)} \|u\|_{L^r},$$

whence the desired inequality follows. Therefore, we may and do assume $R = 1$.

Next, we notice there are really two inequalities in the one we would like to prove: $\|u\|_{L^p} \lesssim \|u\|_{L^r}$ and $\left\|\partial^\alpha u\right\|_{L^p} \lesssim \|u\|_{L^p}$. To prove these, it is key to notice that, by taking Fourier transforms of both sides and recalling that $\eta \equiv 1$ on $\{\xi \in \mathbb{R}^m : |\xi|_2 \le 1\}$, we have $u = \check{\eta} * u$ and $\partial^\alpha u = \check{\eta} * \partial^\alpha u = (\partial^\alpha \check{\eta}) * u$. Consequently, by Young's convolution inequality,

$$\|u\|_{L^p} = \|\check{\eta} * u\|_{L^p} \le \|\check{\eta}\|_{L^q} \|u\|_{L^r},$$

where $1/q = 1 + 1/p - 1/r \in [0, 1]$ (using that $1 \le r \le p \le \infty$). By the same inequality,

$$\left\|\partial^\alpha u\right\|_{L^p} = \left\|(\partial^\alpha \check{\eta}) * u\right\|_{L^p} \le \left\|\partial^\alpha \check{\eta}\right\|_{L^1} \|u\|_{L^p}.$$

This completes the proof. □

We actually learned from the proof that we can take

$$b_{\alpha,p,p} \le \left\|\partial^\alpha \check{\eta}\right\|_{L^1} \text{ and } b_{\vec{0},r,p} \le \|\check{\eta}\|_{L^q}, \tag{B.2.6}$$

where $1/q = 1 - (1/r - 1/p)$. In particular, we can take $b_{\alpha,r,p} \le \|\partial^\alpha \check{\eta}\|_{L^1} \|\check{\eta}\|_{L^q}$.

Proposition B.2.7. *Fix $s \in \mathbb{R}$, $p \in [1, \infty]$, and $f \in \dot{B}_1^{s,p}(\mathbb{R}^m)$. If $\alpha \in \mathbb{N}_0^m$ and $|\alpha| = s - \frac{m}{p}$, then $\sum_{i=-\infty}^{\infty} \check{\eta}_i * \partial^\alpha f = \sum_{i=-\infty}^{\infty} \partial^\alpha(\check{\eta}_i * f)$ is absolutely uniformly convergent.*

Proof. Since the Fourier transform of $\check{\eta}_i * \partial^\alpha f$ is supported in $\{\xi \in \mathbb{R}^m : |\xi|_2 \leq 2^{i+1}\}$,

$$\sum_{i=-\infty}^{\infty} \|\check{\eta}_i * \partial^\alpha f\|_{L^\infty} \leq b_{\alpha,p,\infty} \sum_{i=-\infty}^{\infty} (2^{i+1})^{|\alpha|+\frac{m}{p}} \|\check{\eta}_i * f\|_{L^p} = 2^s b_{\alpha,p,\infty} \sum_{i=-\infty}^{\infty} 2^{is} \|\check{\eta}_i * f\|_{L^p}$$

by Bernstein's inequality. Since $\sum_{i=-\infty}^{\infty} 2^{is}\|\check{\eta}_i * f\|_{L^p} = \|f\|_{\dot{B}_1^{s,p}} < \infty$, we are done. $\square$

Let us record a special bound we learned in the proof about our case of interest. If $m = 1$ and $(s, p, q) = (k, \infty, 1)$ for some $k \in \mathbb{N}_0$, then Relation (B.2.6) gives

$$\sum_{i=-\infty}^{\infty} \|(\check{\eta}_i * f)^{(k)}\|_{L^\infty} = \sum_{i=-\infty}^{\infty} \|\check{\eta}_i * f^{(k)}\|_{L^\infty} \leq \underbrace{2^k \|\check{\eta}^{(k)}\|_{L^1}}_{:=b_k} \|f\|_{\dot{B}_1^{k,\infty}}, \qquad f \in \dot{B}_1^{k,\infty}(\mathbb{R}). \tag{B.2.8}$$

In particular, if $f \in \dot{B}_1^{k,\infty}(\mathbb{R})$, then there exists a polynomial $P_k \in \mathbb{C}[\lambda]$ such that

$$f^{(k)} = \sum_{i=-\infty}^{\infty} (\check{\eta}_i * f)^{(k)} + P_k \in BC(\mathbb{R}) + \mathbb{C}[\lambda] \tag{B.2.9}$$

as tempered distributions. Consequently, $f \in C^k(\mathbb{R})$. In other words, $\dot{B}_1^{k,\infty}(\mathbb{R}) \subseteq C^k(\mathbb{R})$.

Proof of Theorem 6.6.9. For the first part, let $f \in \dot{B}_1^{k,\infty}(\mathbb{R})$. By Relations (B.2.8) and (B.2.9), $\sum_{i\in\mathbb{Z}} \|(\check{\eta}_i * f)^{(k)}\|_{L^\infty} < \infty$, and $f^{(k)}$ differs from the bounded continuous function $\sum_{i\in\mathbb{Z}} (\check{\eta}_i * f)^{(k)}$ by a polynomial P_k. If $f \in PB^k(\mathbb{R})$, then $f^{(k)}$ is itself bounded, so P_k must also be bounded and therefore constant. Write $C \in \mathbb{C}$ for this constant.

Now, let $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_{k+1}) \in \mathbb{R}^{k+1}$. By Proposition 1.3.3(iii) (twice), the uniform convergence of the series, and the fact that $\rho_k(\Delta_k) = 1/k!$,

$$\begin{aligned} f^{[k]}(\boldsymbol{\lambda}) &= \int_{\Delta_k} f^{(k)}(\mathbf{t} \cdot \boldsymbol{\lambda})\, \rho_k(d\mathbf{t}) = \int_{\Delta_k} \Big(C + \sum_{i\in\mathbb{Z}} (\check{\eta}_i * f)^{(k)}(\mathbf{t} \cdot \boldsymbol{\lambda}) \Big)\, \rho_k(d\mathbf{t}) \\ &= \frac{C}{k!} + \sum_{i\in\mathbb{Z}} \int_{\Delta_k} \big(\check{\eta}_i * f\big)^{(k)}(\mathbf{t} \cdot \boldsymbol{\lambda})\, \rho_k(d\mathbf{t}) = \frac{C}{k!} + \sum_{i\in\mathbb{Z}} (\check{\eta}_i * f)^{[k]}(\boldsymbol{\lambda}). \end{aligned}$$

Next, for all $i \in \mathbb{Z}$, $\check{\eta}_i * f$ satisfies the hypotheses of Proposition B.2.2 with $\sigma = 2^{i+1}$. The completeness of $BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}$, Proposition B.2.2, and the definition of $\|\cdot\|_{\dot{B}_1^{k,\infty}}$ then give

$$\begin{aligned}\big\|f^{[k]}\big\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}} &\leq \frac{|C|}{k!}+\sum_{i\in\mathbb{Z}}\big\|(\check{\eta}_i * f)^{[k]}\big\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}} \\ &\leq \frac{|C|}{k!}+\sum_{i\in\mathbb{Z}} a_k(2^{i+1})^k\|\check{\eta}_i * f\|_{L^\infty} = \frac{|C|}{k!}+2^k a_k\|f\|_{\dot{B}_1^{k,\infty}}.\end{aligned}$$

Finally, recalling the definition of C and using Inequality (B.2.8) again, we get

$$|C| \leq \inf_{t\in\mathbb{R}}\big|f^{(k)}(t)\big| + \sum_{i\in\mathbb{Z}}\big\|(\check{\eta}_i * f)^{(k)}\big\|_{L^\infty} \leq \inf_{t\in\mathbb{R}}\big|f^{(k)}(t)\big| + b_k\|f\|_{\dot{B}_1^{k,\infty}}.$$

It follows that we may take $c_k = b_k/k! + 2^k a_k$ in the first part of the theorem.

For the second part, let $f \in PB^1(\mathbb{R})$. By the above, there is some $C \in \mathbb{C}$ such that

$$f' = C + \sum_{i\in\mathbb{Z}}\check{\eta}_i * f' = C + \sum_{i\in\mathbb{Z}}(\check{\eta}_i * f)'. \tag{B.2.10}$$

Since it is easy to see that $\dot{B}_1^{1,\infty}(\mathbb{R}) \cap \dot{B}_1^{k,\infty}(\mathbb{R}) = \bigcap_{i=1}^k \dot{B}_1^{i,\infty}(\mathbb{R})$, if $f \in PB^1(\mathbb{R}) \cap \dot{B}_1^{k,\infty}(\mathbb{R})$, then $\sum_{i\in\mathbb{Z}}\|(\check{\eta}_i * f)^{(\ell)}\|_{L^\infty} < \infty$ for all $\ell \in \{1,\ldots,k\}$. This ensures that we can differentiate the series in Equation (B.2.10) to conclude that $f^{(\ell)} = \sum_{i\in\mathbb{Z}}\big(\check{\eta}_i * f\big)^{(\ell)}$ for all $\ell \in \{2,\ldots,k\}$, so that $f \in PB^\ell(\mathbb{R})$. This proves $PB^1(\mathbb{R}) \cap \dot{B}^{k,\infty}(\mathbb{R}) = PB^1(\mathbb{R}) \cap \cdots \cap PB^k(\mathbb{R})$. In addition, if $k \geq 2$, then the previous paragraph's analysis gives $\big\|f^{[k]}\big\|_{BOC(\mathbb{R})^{\hat{\otimes}_i(k+1)}} \leq 2^k a_k\|f\|_{\dot{B}_1^{k,\infty}}$. This completes the proof. □

Remark B.2.11. If we require $f \in \dot{B}_1^{1,\infty}(\mathbb{R})$ and $f' = \sum_{i\in\mathbb{Z}}\check{\eta}_i * f'$ instead of only $f \in PB^1(\mathbb{R})$, then the proof above yields $\|f^{[1]}\|_{BOC(\mathbb{R})\hat{\otimes}_i BOC(\mathbb{R})} \leq 2a_1\|f\|_{\dot{B}_1^{1,\infty}}$, i.e., we can get rid of the $\inf_{x\in\mathbb{R}}|f'(x)|$ term.

B.3 Acknowledgment

Appendix B, in part, is a reprint of the material as it appears in "Higher derivatives of operator functions in ideals of von Neumann algebras" (2023). Nikitopoulos, Evangelos A. *Journal of Mathematical Analysis and Applications*, 519, 126705.

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