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BY THE FADDEEV EQUATIONS

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ABSTRACT

We have applied the Faddeev equations to the calculation of resonances in a state of three pions with the quantum numbers of the ω^0 particle. Only the kinematics is made relativistic and the pion-pion scattering amplitude which appears in the kernel of the equations is approximated by the ρ-contribution alone. Two resonances are found, one of which has a mass and a width reasonably close to those of the ω^0. The second resonance has an approximate mass of 1600 MeV.

APPENDIX

In this Appendix, we indicate how to separate parity in the Faddeev equations.

Let us consider a wave function of three momenta $\chi(\vec{p}_1, \vec{p}_2, \vec{p}_3)$. Its projection on a state of well defined angular momentum J is given by

$$\chi^J_{Mm}(p_1, p_2, p_3) = (8\pi^2)^{-1} \int \mathcal{D}^J_{mM}(\alpha, \beta, \gamma)\, \chi(\vec{p}_1, \vec{p}_2, \vec{p}_3) dR \qquad (1)$$

where m is the projection of the total angular momentum on a space-fixed axis while M is its projection on an axis Oz normal to the plane of $\vec{p}_1, \vec{p}_2, \vec{p}_3 (\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0)$. R is the rotation, with Euler angles (α, β, γ) which brings the space-fixed system of axis upon a system linked to the momenta. $dR = \sin\beta\, d\alpha\, d\beta\, d\gamma$.

The action of parity upon $\chi(\vec{p}_1, \vec{p}_2, \vec{p}_3)$ is given by

$$P\, \chi(\vec{p}_1, \vec{p}_2, \vec{p}_3) = \epsilon R_0\, \chi(\vec{p}_1, \vec{p}_2, \vec{p}_3) \qquad (2)$$

where R_0 is the rotation of an angle π around the Oz axis and ϵ is the quotient of the intrinsic parity of the three-particle system by the product of the component particles intrinsic parities. Using

$$\mathcal{D}^J_{mM}(\alpha, \beta, \gamma)\, \mathcal{D}^J(R_0) = (-1)^M\, \mathcal{D}^J_{mM}(\alpha, \beta, \gamma) \qquad (3)$$

we get

$$P\, \chi^J_{Mm}(p_1, p_2, p_3) = (-1)^M \epsilon\, \chi^J_{Mm}(p_1, p_2, p_3)\,. \qquad (4)$$

Equation (4) can be used to separate parity in any equation of the type

$$\chi^J_{Mm} = \sum_{M'} K^J_{MM'}\, \chi^J_{M'm} \qquad (5)$$

where the kernel K commutes with parity. In fact, it is sufficient to keep M and M' even when ϵ is positive and odd when ϵ is negative.

We want to thank Dr. Maurice Jacob for a useful conversation on the subject of this Appendix.

1. INTRODUCTION

One of the most important problems in strong interaction physics is to find a satisfactory dynamical treatment of three-body systems. Two methods seem promising, which we shall discuss for the special case of the ω^0-particle.

The first method consists in taking full advantage of our understanding of the two-body system to reduce the three-body problem to a two-body problem. The three-pion scattering amplitude has a pole when the mass of a two-pion system is equal to the ρ mass. The residue of the three-body scattering amplitude at such a pole is essentially a π-ρ scattering amplitude to which one tries to apply the N/D technique.[1] This method is well adapted to a treatment of the exchange of particles, i.e., of the poles in a crossed channel. (From the point of view of a three-particle system, such exchanges are three-body forces.) However, the three-body nature of the problem reappears in two ways:

(a) There appear very important anomalous thresholds owing to the fact that one of the scattered particles is unstable.[2]

(b) It is difficult to take into account the three-body cut in the unitarity relation.

The second method is in many instances complementary to the first one. It consists in using the Faddeev equations for the three-body scattering amplitudes.[3] It is an off-the-energy-shell method, which means that one must introduce form factors, not well-known nor well defined, which act as cutoffs. As a compensation, there is no anomalous threshold. This method takes an exact account of the three-body unitarity.[4] Although it does not take crossing into account, it is possible, in principle, to introduce the effect of exchanged particles as a three-body force.

One obvious drawback of the Faddeev equations is that they are true only in a nonrelativistic approximation. However, it is easy to make the kinematics relativistic and to consider the equations as a convenient device for satisfying three-body unitarity.[5]

According to their complementary characters, the two methods will presumably not apply with the same success to any given problem. For instance, the three-particle channel of a baryon resonance, the overlapping of two resonances in a three-body final state can seemingly be treated by the Faddeev equation method.

The present paper reports a preliminary attempt at solving the Faddeev equations. Our intention for doing that work was twofold:

(1). The Faddeev equations, as originally written by Faddeev in the momentum representation, are very cumbersome. After separation of the total angular momentum, they contain an integration upon two energies while the inhomogeneous term contains delta functions.[6] In this form the kernel of the equation is not completely continuous although its square is. Unfortunately, when iterated once, the separated Faddeev equations contain a summation upon three energies which is practically beyond the possibilities of a computer. We tried, therefore, to solve the noniterated equations.

(2). We wanted to know if the effect of three-particle unitarity, as embodied in the Faddeev equations, was enough to generate a resonance in elementary-particle interactions. Clearly, the simplest problem of that type is provided by a three-pion system with the quantum numbers of the ω^0. Owing to the peculiar symmetry properties of this system, the number of amplitudes to be introduced is small and each amplitude has symmetry properties with respect to the partial energies which reduce the domain of integration.

We have therefore tried to solve the equations with the following approximations:

(a) The pion-pion scattering amplitude is approximated by the contribution of the ρ-resonance, which is represented by a Breit-Wigner formula.[7]

(b) Owing to limitations by the computer, the integrations are replaced by summations over a finite number of energies.

(c) The range of integration being automatically limited by approximation (b), the off-the-energy-shell dependence of the pion-pion amplitude was neglected. This means that our choice of energies where the summations are made also defines a cutoff and, in this first crude attempt, we have not investigated the effect of this cutoff. Qualitatively, it can be said that our range of energies is so small that we certainly underestimate the attraction due to pion-pion integration and we expect to underestimate the binding energy of the resonances.

With these approximations, we have solved the homogeneous Faddeev equations and found two possible resonances, one being very similar to the physical ω^0, and another having a higher mass.

In Section 2, we give the equations which have been solved and in Section 3 the numerical results. In the Appendix, we indicate how to separate the homogeneous Faddeev equations according to different values of parity.

2. DERIVATION OF THE INTEGRAL EQUATIONS

Owing to the relativistic kinematics, the equations derived in Reference 1 will be slightly modified. It is convenient to take the pion mass as unity and define the total energies of the particles in the center of mass by

$$\omega_i = (p_i^2 + 1)^{1/2} \quad , \tag{1}$$

where, as in Reference 6, p is the momentum of the *i*th particle in the total center of mass. We also have

$$\vec{q}_{23} = (\vec{p}_2 - \vec{p}_3)/2 \quad ,$$

$$4q_{23}^2 = 2(\omega_2^2 + \omega_3^2) - \omega_1^2 - 3 \quad , \tag{2}$$

where $\vec{q}_{23}$ is the momentum of particle 2 relative to the center of mass of particles 2 and 3 . The total energy of particles 2 and 3 in this center of mass is

$$E_{23} = (4q_{23}^2 + 4)^{1/2} = \left[2(\omega_2^2 + \omega_3^2) - \omega_1^2 + 1 \right]^{1/2} . \tag{3}$$

As in Reference 6 we have, for the angle between the momenta $\vec{p}_2$ and $\vec{p}_3$,

$$\cos \theta_{23} = \frac{p_2^2 + p_3^2 - p_1^2}{2p_2p_3} . \tag{4}$$

We also need γ_1 , the angle between $\vec{p}_1$ and $\vec{q}_{23}$. We have

$$\cos \gamma_1 = \frac{\vec{p}_1 \cdot \vec{q}_{23}}{p_1\, q_{23}} = \frac{\omega_3^{\,2} - \omega_2^{\,2}}{(\omega_1^{\,2} - 1)^{1/2} \left[2(\omega_2^{\,2} + \omega_3^{\,3}) - \omega_1^{\,2} - 3\right]^{1/2}} . \tag{5}$$

Furthermore, the normalization coefficient A of Eq. (24) of Reference 6 will be changed to

$$A = \left[\frac{2J + 1}{8\pi^2\, \omega_1\, \omega_2\, \omega_3}\right]^{1/2} . \tag{6}$$

Finally, the inhomogeneous term of the Faddeev equation will be written as

$$\langle \omega'\ J'\ M_1' \,|\mathcal{T}_{23}(z)|\ \omega\ J\ M_1 \rangle = (\omega_1\, \omega_2\, \omega_3\, \omega_1'\, \omega_2'\, \omega_3')^{1/2}\, \delta_{M_1' M_1}$$
$$\times F_{23}(\omega,\ \omega',\ u,\ z - E_1)\, e^{i' M_1' u}\, \frac{\delta(\omega_1' - \omega_1)}{\omega_1\, p_1}\, du , \tag{7}$$

with the z axis chosen along $\vec{p}_1$.

For our purposes here, it is more convenient to choose the z axis perpendicular to the plane of the triangle defined by the three momenta $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$. This can be done by a rotation of $\pi/2$ around the y axis, which has so far been chosen to be in the plane of the triangle. We have[8]

$$|JM_f\rangle = D^J_{M_i M_f}(0, \tfrac{-\pi}{2}, 0)|JM_i\rangle = d^J_{M_i M_f}(\tfrac{-\pi}{2})|JM_i\rangle = \Delta^J_{M_f M_i}|JM_i\rangle . \tag{8}$$

This rotation brings the x axis along $\vec{p}_1$ and the z axis perpendicular to the plane of $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$. In this new coordinate system we have, instead of Eq. (7) ,

$$\langle \omega_1' \; J' \; M_1' | \mathscr{C}_{23}(z) | \omega \; J \; M_1 \rangle = (\omega_1 \, \omega_2 \, \omega_3 \, \omega_1' \, \omega_2' \, \omega_3')^{1/2}$$

$$\times \int \sum_{\mu} F_{23}(\omega, \omega', u, z - E_1) \, e^{i\mu u} \, du \frac{\delta(\omega_1' - \omega_1)}{\omega_1 \, p_1} \, \Delta^{J*}_{M_1' \mu} \, \Delta^{J}_{M_1 \mu} \quad . \tag{9}$$

Now, writing the partial wave expansion of $F_{23}(\omega, \omega', u, z - E_1)$, and keeping one term only, we have[6]

$$F_{23}(\omega, \omega', u, z - E_1) \doteq f_{23} \, P_\ell(\cos \chi_1, \cos \gamma_1', \sin \chi_1, \sin \gamma_1', \cos u)(2\ell + 1) \; , \tag{10}$$

where f_{23} is given by the Breit-Wigner formula. We have[9]

$$f_{23} = -4\pi \, a_{23} \quad , \tag{11}$$

where in terms of the phase shifts we have

$$a_{23} = \frac{1}{\nu^{1/2}} \, e^{i \delta} \sin \delta \quad . \tag{12}$$

Correspondingly we have

$$a_{23} = \frac{-1}{\nu^{1/2}} \frac{\Gamma/2}{E - E_0 + i\Gamma/2} \tag{13}$$

and

$$f_{23} = \frac{4\pi}{\nu^{1/2}} \frac{\Gamma/2}{E - E_0 + i\Gamma/2} . \tag{14}$$

We identify ν with $q_{23}^{\,2}$, E with E_{23} , E_0 and Γ with the mass and width of the resonance in the two-body amplitude. We then have

$$f_{23} = \frac{4\pi}{\left[2(\omega_2^{\,2} + \omega_3^{\,2}) - \omega_1^{\,2} - 3\right]^{1/2}} \frac{\Gamma}{E_{23} - E_0 + i\Gamma/2} . \tag{15}$$

Next we write E_{23} in terms of z , the total energy of the system in the three-particle center of mass. We have

$$z = \omega_1 + (p_1^{\,2} + 4q_{23}^{\,2} + 4)^{1/2}$$

or

$$E_{23} = (4q_{23}^{\,2} + 4)^{1/2} = \left[(z - \omega_1)^2 - p_1^{\,2}\right]^{1/2} = (z^2 - 2z\omega_1 + 1)^{1/2} . \tag{16}$$

Therefore we have

$$f_{23} = \frac{4\pi}{\left[2(\omega_2^{\,2} + \omega_3^{\,2}) - \omega_1 - 3\right]^{1/2}} \frac{\Gamma}{(z^2 - 2z\omega_1 + 1)^{1/2} - E_0 + i\Gamma/2} \tag{17}$$

Now, using Equations (9) and (10), we have

$$\langle \omega' \, J' \, M_1' | \mathcal{T}_{23}(z) | \omega \, J \, M_1 \rangle = (\omega_1 \omega_2 \omega_3 \omega_1' \omega_2' \omega_3')^{1/2} (2\ell + 1) f_{23} \sum_{\mu}$$

$$\times \int e^{i\mu\mu} \, du \frac{\delta(\omega_1' - \omega_1)}{\omega_1 \, p_1} \Delta^{J*}_{M_1'\mu} \Delta^{J}_{M_1\mu} P_\ell(\cos\gamma_1 \cos\gamma_1' + \sin\gamma_1 \sin\gamma_1' \cos u)$$

$$= \sum_{\mu} 8\pi^2 (\omega_1 \omega_2 \omega_3 \omega_1' \omega_2' \omega_3')^{1/2} f_{23} \Delta^{J*}_{M_1'\mu} \Delta^{J}_{M_1\mu} \frac{\delta(\omega_1 - \omega_1')}{\omega_1 \, p_1} Y^*_{\ell,\mu}(\gamma_1, 0) \, Y_{\ell,\mu}(\gamma_1', 0) \quad . \tag{18}$$

with f_{23} given by (17) .

In our case, since the two-body amplitude is approximated by the ρ , we have $\ell = 1$ and, since the three-body amplitude is chosen to have the quantum numbers of the ω , we have $J = 1$. In Equation (18) let us consider the factor

$$X_{MM'}(\gamma_1, \gamma_1') \equiv \sum_{\mu} \Delta^{J*}_{M'\mu} \Delta^{J}_{M\mu} Y^*_{\ell,\mu}(\gamma_1, 0) \, Y_{\ell,\mu}(\gamma_1', 0)$$

with $J = 1$, $\ell = 1$. (19)

Using explicit values of $Y_{1,\mu}$ and $\Delta^{1}_{M\mu}$ and summing over μ , we obtain

$$X_{MM'}(\gamma_1, \gamma_1') = \frac{3}{16\pi} \Big[\sin\gamma_1 \sin\gamma_1' \, (3M^2 M'^2 - 2M^2 - 2M'^2 + 2)$$

$$+ 2MM' \sin\gamma_1 \sin\gamma_1' \Big] = \frac{3}{16\pi} Z_{MM'}$$

(20)

Finally, using Eqs. (17) , (18) , and (20) we obtain

$$\langle \omega' \; J' \; M_1' | \mathcal{T}_{23}(z) | \; \omega \; J \; M_1 \rangle \; = $$

$$\frac{6\pi^2 (\omega_1\omega_2\omega_3\omega_1'\omega_2'\omega_3')^{1/2} \; \delta(\omega_1' - \omega_1) \; \Gamma Z_{M_1 M_1'}(\gamma_1, \gamma_1')}{\omega_1 \; p_1 \left[2(\omega_2^{\,2} + \omega_3^{\,2}) - \omega_3^{\,2} - 3\right]^{1/2} \left[(z^2 - 2\omega_1 \; z + 1)^{1/2} - E_p + i\Gamma/2\right]} \; , \tag{21}$$

where $E_p = 5.4$ and $\Gamma = 0.7$ in pion mass units.

Now the Faddeev equations will be (leaving out the inhomogeneous term)

$$\langle \omega' \; J \; M' | \mathcal{T}^1 | \; \omega \; J \; M \rangle \; = \; -6\pi^2 \; \Gamma \sum_{M''}$$

$$\times \int \frac{d\omega_1'' \; d\omega_2'' \; d\omega_3'' (\omega_1' \; \omega_2' \; \omega_3' \; \omega_1'' \; \omega_2'' \; \omega_3'')^{1/2} \; \delta(\omega_1' - \omega_1'') \; Z_{M'M''}(\gamma_1', \gamma_1'')}{\omega_1' \; p_1' \; (\sum \omega'' - z) \left[2(\omega_2'^{\,2} + \omega_3'^{\,2}) - \omega_1'^{\,2} - 3\right]^{1/2} \left[(z^2 - 2\omega_1' z + 1)^{1/2} - E_p + i\Gamma/2\right]}$$

$$\times \left[\langle \omega'' \; J \; M'' | \mathcal{T}^2 | \; \omega \; J \; M \rangle + \langle \omega'' \; J \; M'' | \mathcal{T}^3 | \; \omega \; J \; M \rangle\right]$$

$$= \sum_{M''} \int K_{M'M''}(\omega', \omega'') \; d\omega'' \left[\langle \omega'' \; J \; M'' | \mathcal{T}^2 | \; \omega \; J \; M \rangle + \langle \omega'' \; J \; M'' | \mathcal{T}^3 | \; \omega \; J \; M \rangle\right] \; , \tag{22}$$

where $K_{M'M''}(\omega', \omega'')$ is given by the right-hand side of Eq. (21) . We need not write the rest of the equations since, as we shall see, owing to the symmetry of the problem, the matrix elements of $\mathcal{T}^2$ and $\mathcal{T}^3$ will be rewritten in terms of the matrix elements of $\mathcal{T}^1$, and Eq. (22) reduces to an equation involving $\mathcal{T}^1$ alone.

From the symmetry of the problem we have

$$\langle 1\omega'' \; J \; M''|\mathcal{C}^1|\;\omega \; J \; M \; 1\rangle \;=\; \langle 2\omega^{3}{}'' \; J \; M''|\mathcal{C}^2|\omega^3 \; J \; M2\rangle \quad , \tag{23}$$

where $\omega^3 \equiv (\omega_2, \omega_1, \omega_3)$ and the state $|\;\omega \; J \; M1\rangle$ means the x axis is chosen along $\vec{p}_1$. We go from $|\;\omega \; J \; M1\rangle$ to $|\;\omega \; J \; M'2\rangle$ by a rotation θ_{12} around the z axis. Therefore we have

$$|J \; M2\rangle \;=\; D^J_{M'M}(\theta_{12})|\;JM'1\rangle \;=\; e^{-iM\theta_{12}}\,\delta_{M'M}|\;J \; M'1\rangle \;=\; e^{-iM\theta_{12}}|\;J \; M1\rangle \;, \tag{24}$$

and so

$$\Big[\langle 1\omega'' \; J \; M''|\mathcal{C}^2|\omega' \; J \; M'1\rangle + \langle 1\omega'' \; J \; M''|\mathcal{C}^3|\;\omega' \; J \; M'1\rangle\Big]$$

$$= \Big[e^{iM'\theta'_{12}}\langle \overset{3}{\omega}{}'' \; J \; M''|\mathcal{C}^1|\;\overset{3}{\omega}{}' \; J \; M'\rangle + e^{iM'\theta'_{13}}\,e^{-iM''\theta''_{13}}$$

$$\times \quad \langle \overset{2}{\omega}{}'' \; J \; M''|\mathcal{C}^1|\overset{2}{\omega}{}' \; J \; M'\rangle\Big] \quad . \tag{25}$$

The state ω is antisymmetric in the isospin space and therefore, owing to the Bose statistic, it should also be antisymmetric in ordinary space. This means that the states should be antisymmetrized according to

$$|A \; \omega'_1 \; \omega'_2 \; \omega'_3 \; J \; M'\rangle \;=\; |\omega'_1 \; \omega'_2 \; \omega'_3\rangle - |\omega'_1 \; \omega'_3 \; \omega'_2\rangle + e^{iM'\theta'_{12}}(|\omega'_3 \; \omega'_1 \; \omega'_2\rangle$$

$$- |\omega'_2 \; \omega'_1 \; \omega'_3\rangle) + e^{iM'\theta'_{13}}(|\omega'_2 \; \omega'_3 \; \omega'_1\rangle - |\omega'_3 \; \omega'_2 \; \omega'_1\rangle) \quad , \tag{26}$$

and so

$$P|A\,\omega_1'\,\omega_2'\,\omega_3'\rangle = (-1)^P e^{-iM'\theta_{1ip}'} |A\,\omega_1'\,\omega_2'\,\omega_3'\,J\,M'\rangle \quad . \tag{27}$$

where P is a permutation operator and the factor $e^{-iM'\theta_{1ip}'}$ means that whenever the permutation involves particle 1 and particle i the coordinate axes are rotated by θ_{1ip} around the O_z axis. Therefore we have

$$e^{-iM''\theta_{12}''} e^{iM'\theta_{12}'} \langle A\overset{3}{\omega}''|\mathcal{G}^1|A\overset{3}{\omega}'\rangle = \langle\omega''JM''|\mathcal{G}^1|\omega'JM'\rangle \quad . \tag{28}$$

Combining (28) with (25) and (22), we have

$$\langle A\omega\,J\,M|\mathcal{G}^1|A\omega''\,J\,M''\rangle = -12\pi^2\Gamma$$

$$\int\sum_{M'} \frac{(\omega_1'\omega_2'\omega_3'\omega_1\omega_2\omega_3)^{1/2}}{\omega_1 p_1(\sum\omega' - Z)} \frac{\delta(\omega_1 - \omega_1')\, Z_{MM'}(\gamma_1 - \gamma_1')\, d\omega_1'\, d\omega_2'\, d\omega_3'}{\left[2(\omega_2^2 + \omega_3^2) - \omega_1^2 - 3\right]^{1/2}\left[(Z^2 - 2\omega_1 Z + 1)^{1/2} - E_p + i\Gamma/2\right]}$$

$$\times\langle A\omega'\,J\,M'|\mathcal{G}^1|\,A\omega''\,J\,M''\rangle \quad . \tag{29}$$

The kernel is given by

$$K_{MM'}(\omega,\omega') = \frac{-12\pi^2\Gamma(\omega_1\omega_2\omega_3\omega_1'\omega_2'\omega_3')^{1/2}\delta(\omega_1 - \omega_1')Z_{MM'}(\gamma_1,\gamma_1')}{\omega_1 p_1(\sum\omega' - Z)\left[2(\omega_2^2 + \omega_3^2) - \omega_1^2 - 3\right]^{1/2}\left[(Z^2 - 2\omega_1 Z + 1)^{1/2} - E_p + i\Gamma/2\right]} \quad . \tag{30}$$

This kernel should be symmetrized to ensure the symmetry of the solution due to the Bose statistics which has already been imposed. Equation (30) will then become

$$K_{MM'}(\omega, \omega') = -12\pi^2\Gamma \sum_P (-1)^P \frac{(\omega_1\ \omega_2\ \omega_3\ \omega_1'\ \omega_2'\ \omega_3')^{1/2}}{\omega_1 p_1(\sum\omega' - z)}$$

$$\times \quad \frac{\delta(\omega_1 - R\omega_1')\ PZ_{MM'}(\gamma_1,\gamma_1')\ e^{-iM'\theta_{1ip}'}}{\left[2(\omega_2^2 + \omega_3^2) - \omega_1^2 - 3\right]^{1/2}\left[(z^2 - 2\omega_1 z + 1)^{1/2} - E_p + i\Gamma/2\right]}, \tag{31}$$

where it is now understood that the integration in Eq. (29) is restricted to the condition $p_1' < p_2' < p_3'$. We should also remember that the momenta p_1', p_2', p_3' are further restricted to form a triangle. In Eq. (31) the permutation operator on $Z_{MM'}(\gamma_1, \gamma_1')$ is defined to exchange the momenta defining γ_1'. From Eqs. (31) and (29) we have an equation of the form

$$T' = K(z)\ T' \quad . \tag{32}$$

The solutions z which make the above equation satisfied are discussed in the next section.

III. NUMERICAL CALCULATIONS

In Eq. (29) we choose a finite mesh size and change the integration into a summation. In Eq. (32) K is then a matrix which is the direct product of the matrix in MM' indices and a matrix in ω , ω' indices. For a general z , Eq. (32) is taken to be of the form

$$K(z)\,T' = \lambda(z)\,T' \quad . \tag{33}$$

Our numerical solution consists of finding all possible values of (complex) z for which $\lambda(z) = 1$. We shall then interpret the real and imaginary parts of z as the mass and width of the three-body resonance. Owing to the practical difficulties we are forced to use not too large a matrix (of the order of 100 by 100). This automatically leads to a cutoff and a fairly large step size of integration. The mesh size of integration over each ω_i' has been chosen to be 1/3 and a cutoff at 8/3 . Once all the matrix elements of K(z) are known for a given z , all the eigenvalues $\lambda(z)$ of Eq. (33) are calculated. For a small (real) z , say $z = 1$, the eigenvalue with smallest nonzero magnitude has a real part which is positive, but it is considerably smaller than unity. This eigenvalue has a fairly small imaginary part. When z is increased to $z = 6$ this eigenvalue moves close to unity but with a nonzero imaginary part. Next z is allowed to become complex and the value of z for which $\lambda(z)$ is close to unity is sought. We obtain the solution

$$Z_r = 6.25m_\pi \ , \qquad Z_i = -0.15m_\pi \quad .$$

This solution corresponds to a calculated mass of ω particle about 870 MeV and a width of about 20 MeV . Of all the other eigenvalues for Z_r in the range of 1 to 6 , none is close enough to unity. As the value of Z_r is further increased another eigenvalue, which so far had a large imaginary part, moves closer to unity and we obtain another solution, $Z_r = 9.5$ and $Z_i = -0.25$. This solution would correspond to a resonance at about 1400 MeV and a width of about 20 MeV . As Z_r is further increased this eigenvalue moves away from unity. We have not increased the z value beyond $Z_r = 12$.

Let us now make a few remarks about the sensitivity of the solution to the imput data:

(a) Increasing E_ρ increases Z_r , the mass of the three-body resonance. This result is expected from the energy denominators of the kernel in Eq. (31) .

(b) Increasing the width of the ρ makes Z_r decrease. This effect can be interpreted as an increase in the force and therby an increase in the binding energy of the three-body system.

(c) When a larger mesh size (1/2 instead of 1/3) , and therby a larger cutoff (7/2 instead of 8/3) , is chosen, the solution for Z_r is also increased. We obtain $Z_r = 7.25$ instead of $Z_r = 6.25$. This result is in contrast with (b) and we believe that it is due to the crudeness of approximation (too large a mesh size) rather than the sensitivity to the cutoff.

Although the method used in this paper is entirely crude, we believe that the present results are very encouraging for the use of Faddeev equation to solve analogous problems. The next step will be to use better approximation techniques, and particularly variational techniques, and to explore other channels of the $\pi - \pi - \pi$, $K - K - \pi$, and $K - \bar{K} - \pi$ systems. Also a better understanding of the properties of the eigenvalues of the Faddeev kernel will have to be gained.

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APPENDIX

In this Appendix we reduce the Faddeev equations according to the two parity states. We shall call $\mathcal{E}$ the product of the parity of a bound state by the intrinsic parities of the three component particles. Let $X^{(1)}(\vec{p}_1,\vec{p}_2,\vec{p}_3)$ be the bound state or resonance wave functions. They satisfy the homogeneous Faddeev equations, the first of which is .

$$(H_0-z)\, X^{(1)}(\vec{p}_1,\vec{p}_2,\vec{p}_3) = \int F_{23}(\vec{p}_1,\vec{p}_2,\vec{p}_3;\vec{p}_1',\vec{p}_2',\vec{p}_3'\; z - E_1)\, \delta(\vec{p}_1' - \vec{p}_1) \times \left[X^{(2)}(\vec{p}_1',\vec{p}_2',\vec{p}_3') + X^{(3)}(\vec{p}_1',\vec{p}_2',\vec{p}_3')\right] d^3\vec{p}_1'\, d^3\vec{p}_2' \quad . \tag{A.1}$$

The reduction of angular momentum is made through an eigenfunction expansion,

$$X(\vec{p}_1,\vec{p}_2,\vec{p}_3) = \sum \frac{(2J+1)}{8\pi^2}\, X^J_{M\mu}(p_1,p_2,p_3)\, \mathcal{D}^J_{M\mu}(\alpha,\beta,\gamma) \quad . \tag{A.2}$$

Here (α, β, γ) are the Euler angles of the rotation which brings a space-fixed system of axis to the body-fixed system. The index μ , which is the projection of the total angular momentum upon a fixed axis, is a dummy index and we shall put it equal to 0 . In abstract form Eq. (A.2) reads

$$\langle \vec{p}_1,\vec{p}_2,\vec{p}_3|X\rangle = \langle p_1 p_2 p_3|\, \langle J\, M\, 0|X\rangle \quad . \tag{A.3}$$

Introducing Eq. (A.2) into (A.1), one gets the reduced Faddeev equation

$$(\sum_{i=1}^{3} \omega - z)\, X_m^{(1)J}(p) = \int A_{23}\, F_{23}(p, p', \mu, z - E_1)$$

$$\times \left[X_{m'}^{(2)J}(p') + X_{m'}^{(3)J}(p') \right] \mathfrak{D}^J_{m'm}(V)\, d\omega_2'\, d\omega_3'\, d\mu \quad , \tag{A.4}$$

where V is the rotation of Eq. (9) and A_{23} a kinematical factor.

In order to use parity conservation, let us write

$$P|J\ M\ 0\rangle = (-1)^{J+M}\, |J,\ -M,\ 0\rangle^* \quad , \tag{A.5}$$

where P is the parity operator.

When z is real F_{23} is a real function and so is X, therefore

$$\langle J\ M\ 0|PX\rangle = \mathcal{E}\langle J\ M\ 0|X\rangle = (-1)^{J+M}\,\langle J,\ -M\ 0|{*X}\rangle$$

$$= (-1)^{J+M}\,\langle J, -M\ 0|X\rangle^* \quad . \tag{A.6}$$

For complex values of z, Eq. (A.6) reads

$$X_m^J(z) = (-1)^m\, \mathcal{E}(-1)^J\, X_{-m}^{*J}(z^*) \quad , \tag{A.7}$$

and it is indeed easy to verify that the right-hand member of Eq. (A.7) satisfies Eq. (A.4) .

Equation (A.6) can be further reduced by introducing the new function

$$\Phi_M^{(1)J} = X_M^{1J} + \mathcal{E}(-1)^J\, X_M^{*1J}(z^*) \quad , \tag{A.8}$$

where $M \geqslant 0$.

In fact, using the relation

$$\mathcal{D}^{J}_{-m'-m}(V) = (-1)^{m-m'}\mathcal{D}^{J}_{m'm}(V) \quad , \tag{A.9}$$

one gets

$$(\omega_1 + \omega_2 + \omega_3 - z)\, \phi_M^{(1)J}(p) = A_{23}\int F_{23}(p,p',\mu,z - E_1)\, du\, d\omega_2\, d\omega_3$$

$$\times \sum_{M'=0}^{J} \left[\phi_{M'}^{(2)J}(p') + \phi_{M'}^{(3)J}(p')\right] \alpha_{M'}\left[\mathcal{D}^{J}_{M'M}(V) + (-1)^{M'}\mathcal{E}(-1)^{J}\mathcal{D}^{J}_{-M'M}(V)\right] \quad , \tag{A.10}$$

where $M \geqslant 0$, $\alpha_0 = 1/2$, and $\alpha_M = 1$ for $M \neq 0$.

Equation (A.10) can be further reduced when some of the particles are identical as indicated in Section 2.

FOOTNOTES AND REFERENCES

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† Present address: University of California at San Diego.

1. G. F. Chew and S. Mandelstam, Phys. Rev. 119, 667 (1960).

2. R. Karplus, C. M. Sommerfield, and E. H. Wichmann, Phys. Rev. 111, 1187 (1958).

3. L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. 39, 1659 (1960) [English transl. Soviet Physics JETP 12, 1016 (1961)].

4. C. Lovelace, Lectures at Edinburgh Summer School, 1963 (unpublished).

5. This approach has been analyzed recently by C. Lovelace, "A Practical Theory of Three-Particle States I. Nonrelativistic," Imperial College (London) preprint.

6. R. Omnes, Phys. Rev. 136, 1358B (1964).

7. Our model of the ω^0 is therefore identical with the one proposed by G. F. Chew, Phys. Rev. Letters 6, 142 (1960).

8. See A. R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton University Press, Princeton, New Jersey, 1959).

9. See, for example, A. Messiah, Quantum Mechanics, Chapter XIX, Dunod, Paris, 1959.