Title
QUANTUM SOLITONS AND THE VARIATIONAL METHOD: A SAMPLE SINE-GORDON CALCULATION

Author
Ingermanson, R.

Publication Date
1986

LBL-20844
Preprint

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

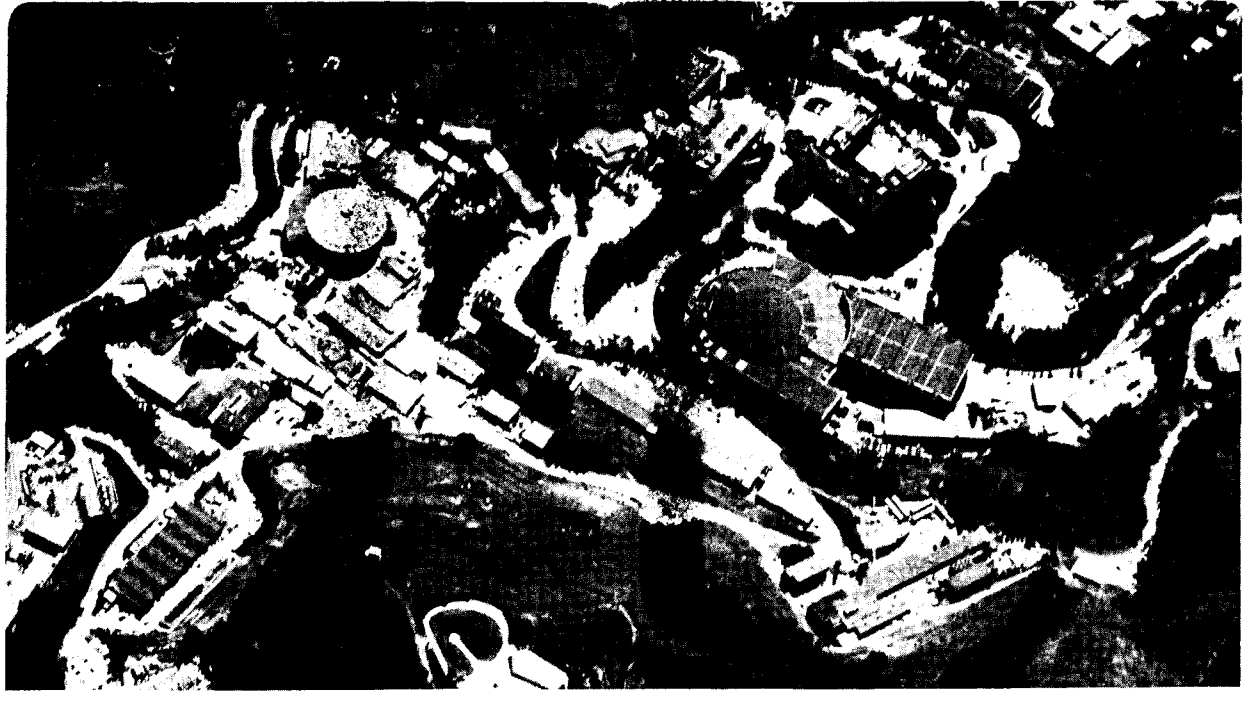
Physics Division

Submitted for publication

QUANTUM SOLITONS AND THE VARIATIONAL METHOD: A SAMPLE SINE-GORDON CALCULATION

R. Ingermanson

January 1986



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

DISCLAIMER

January 17, 1986 LBL-20844

QUANTUM SOLITONS AND THE VARIATIONAL METHOD: A SAMPLE SINE-GORDON CALCULATION *

Randall Ingermanson

Department of Physics
and
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720, U.S.A.

ABSTRACT

We outline a general, practical method of computing quantum corrections to the energy of solitons. The method is illustrated by working out the details for the sine-Gordon model. The results differ by a small amount from the two-loop answer, which is much better than the one-loop result and close to the exact answer for weak and intermediate values of the coupling constant, $\beta^2 \lesssim 2\pi$.

*This work was supported in part by the Director, Office of Energy Research, Office of High Energy Physics and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098, and in part by the National Science Foundation, under Research Grant No. PHY-81-18547.

1 Introduction

During the past few years, variational methods have enjoyed a modest resurgence in interest among field theorists. For a variety of approaches and applications, see [1-4]. The practicality of the variational method depends on our ability to find simple trial states which accuratcly reflcct the essential physics. For this reason, most authors have restricted attention to the vacuum state. Translational invariance then simplifies the class of trial states, so that calculations become manageable.

The question naturally arises whether variational techniques might be useful also in soliton calculations. Several years ago, Coleman [5] applied the method to a broad class of $1+1$ dimensional models, but the results were rather disappointing. The simplest reasonable trial state yields the classical energy. Thus, the classical soliton mass is an upper bound to the mass of the quantum soliton. For an attempt to improve on this type of calculation, see [6].

The variational method, as applied to solitons, has been generally eclipsed by techniques such as the WKB approximation [7], S-matrix technology [8], and the quantum inverse scattering method [9]. While these methods have yielded exact results for the sine-Gordon model, their application requires great mathematical ingenuity. Consequently, they seem to be limited to two-dimensional models.

The central claim of this paper is that the variational method 1) does not require extraordinary cleverness, and yet 2) provides a good approximation for a reasonably simple class of trial states. This two-fold claim will be explicitly demonstrated for the sine-Gordon model.

We can hope that a suitable extension of the variational method to higher dimensions would be useful in more interesting models, such as the Skyrme model or magnetic monopole models. This possibility will be discussed in the concluding section.

2 The Vacuum

The system we would like to study is the sine-Gordon model in two dimensions, given by the Lagrangian

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{\beta^2}[\cos(\beta\phi) - 1] \\ &= \tfrac{1}{2}\dot{\phi}_x^2 - V(\phi) \\ V(\phi) &\equiv \tfrac{1}{2}(\partial_x\phi)^2 + \frac{m^2}{\beta^2}[1 - \cos(\beta\phi)].\end{aligned}$$

(We use the notation $\phi(x) = \phi_x$.)

Then the canonical momentum is

$$\Pi_x = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_x} = \dot{\phi}_x,$$

and the Hamiltonian is

$$H = \int_x \left[\tfrac{1}{2}\Pi_x^2 + V(\phi_x)\right].$$

Quantization requires us to implement the commutation relation

$$[\phi_x, \Pi_y] = i\delta_{xy},$$

which is trivial to do via a functional derivative:

$$\Pi_x = -i\frac{\delta}{\delta\phi_x}.$$

What is not trivial is to find the energy eigenstates, solutions of the functional Schrödinger equation

$$H\Psi = E\Psi.$$

Our approach will be to approximate Ψ variationally.

An extremely simple variational ansatz is the general Gaussian wavefunctional

$$\Psi = N_f \exp\left[i\int_x P_x\phi_x - \tfrac{1}{2}\int_{x,y}(\phi - \Phi)_x f_{xy}(\phi - \Phi)_y\right], \qquad (2.1)$$

where N_f is a normalization constant. P_x and Φ_x have simple physical interpretations:

$$\begin{aligned}\Phi_x &= \langle\Psi|\,\phi_x\,|\Psi\rangle \\ P_x &= \langle\Psi|\,\Pi_x\,|\Psi\rangle.\end{aligned}$$

The meaning of f_{xy} is less obvious. (With no loss of generality, we can require $f_{xy} = f_{yx}$, since the anti-symmetric part of f_{xy} does not contribute to (2.1).) In the limit $\beta \to 0$, we would have a free-field theory, and one can show [3] that the exact vacuum state would then be of the form (2.1), with $P_x = 0$ and

$$f_{xy} = \int \frac{dp}{2\pi}\cos[p(x-y)]\sqrt{p^2 + m^2}. \qquad (2.2)$$

Thus, f_{xy} is linked to the energy of one-particle excitations.

Φ_x is the classical field. In this section, we restrict ourselves to the static case, $\dot{\Phi}_x = 0$. Using standard path integral methods, we have

$$\begin{aligned}&E[\Phi, P, f] \equiv \langle\Psi|\,H\,|\Psi\rangle \\ &= \int_x \left[\tfrac{1}{2}P_x^2 + \tfrac{1}{2}(\partial_x\Phi_x)^2 - \frac{m^2}{\beta^2}\left[Z_x\cos(\beta\Phi_x) - 1\right] + \tfrac{1}{4}\left[f_{xx} - \int_y \delta_{xy}\nabla_x^2 f_{xy}^{-1}\right]\right],\end{aligned}$$

where

$$Z_x[f] \equiv \exp\left[-\tfrac{1}{4}\beta^2 f_{xx}^{-1}\right].$$

In order to minimize the energy, we should clearly set $P_x = 0$.

Note that, if f_{xy} is anything like the free-field case (2.2), then E contains some divergences. We should therefore make renormalization our first priority. For this purpose, it is sufficient to let Φ be a constant function, $\partial_x\Phi_x = 0$, a condition which will hold throughout the rest of this section.

Now, f_{xy} is to be determined variationally. It is the solution of the equation

$$\frac{\delta E}{\delta f_{xy}} = 0.$$

It is quite easy to show that this has the solution

$$\overline{f}_{xy} = \int \frac{dp}{2\pi}\cos[p(x-y)]\sqrt{p^2 + \mu^2(\Phi)}, \qquad (2.3)$$

where $\mu^2(\Phi)$ is defined by

$$\mu^2(\Phi) = m^2 Z_x\left[\overline{f}\right]\cos(\beta\Phi). \qquad (2.4)$$

Since Z_x depends on $(\overline{f}_{xx})^{-1}$, which depends on $\mu^2(\Phi)$, it is far from trivial to show that (2.3) and (2.4) uniquely specify $\overline{f}_{xy}$ and $\mu^2(\Phi)$. The technical details are worked out elsewhere [10]. A unique solution exists if $\cos(\beta\Phi) \geq 0$ and $\beta^2 < 8\pi$. Furthermore, $\overline{f}_{xy}$ is at a *minimum* of the energy functional (rather than at a maximum) only if $\beta^2 < 8\pi$. (This latter fact was first noted by Coleman [11].)

Defining

$$I_n(\mu^2) \equiv \int \frac{dp}{2\pi} (p^2+\mu^2)^{\frac{1}{2}-n}$$
$$Z_m(\mu^2) \equiv \exp\left[-\tfrac{1}{4} I_1(\mu^2)\right],$$

we find that the unrenormalized effective potential is

$$\begin{aligned}\mathcal{V}(\Phi) &\equiv \frac{E[\Phi,0,\overline{f}]}{\text{volume}} \\ &= \tfrac{1}{2} I_0(\mu^2) - \tfrac{1}{4}\mu^2 I_1(\mu^2) - \frac{m^2}{\beta^2}\left[Z_m(\mu^2)\cos(\beta\Phi) - 1\right].\end{aligned}$$

In view of (2.4), we can rewrite our results as

$$\mathcal{V}(\Phi) = \tfrac{1}{2} I_0(\mu^2) - \tfrac{1}{4}\mu^2 I_1(\mu^2) + \frac{m^2-\mu^2}{\beta^2}$$
$$\mu^2(\Phi) = m^2 Z_m(\mu^2)\cos(\beta\Phi).$$

Note that I_0 and I_1 are divergent integrals. It is not necessary to explicitly regulate them; any reasonable regulator will preserve the formal differentiability properties

$$\frac{\partial I_0}{\partial(\mu^2)} = \tfrac{1}{2} I_1(\mu^2)$$
$$\frac{\partial I_1}{\partial(\mu^2)} = -\tfrac{1}{2} I_2(\mu^2).$$

From these, one can then compute

$$\frac{\partial \mathcal{V}}{\partial(\mu^2)} = -\frac{1}{\beta^2}\left(1 - \tfrac{1}{8}\mu^2\beta^2 I_2(\mu^2)\right)$$
$$\frac{\partial \mu^2}{\partial \Phi} = -\beta\mu^2\tan(\beta\Phi)\left(1 - \tfrac{1}{8}\mu^2\beta^2 I_2(\mu^2)\right)^{-1}.$$

Therefore,

$$\frac{\partial \mathcal{V}}{\partial \Phi} = \frac{\mu^2}{\beta}\tan(\beta\Phi).$$

Clearly, the minimum of the effective potential is at $\Phi = 0$ and we define the renormalized mass

$$m_R^2 \equiv \left.\frac{\partial^2 \mathcal{V}}{\partial \Phi^2}\right|_{\Phi=0} = \mu^2(0) = m^2 Z_m\big(\mu^2(0)\big) = m^2 \exp\left[-\tfrac{1}{4}\beta^2 I_1(m_R^2)\right]. \quad (2.5)$$

This determines the bare mass m in terms of β, m_R and the divergent integral $I_1(m_R^2)$. We then obtain the vacuum energy density

$$\mathcal{V}_{vac} = \tfrac{1}{2} I_0(m_R^2) + \frac{m^2}{\beta^2} - \frac{m_R^2}{\beta^2}\left[1 + \tfrac{1}{4}\beta^2 I_1(m_R^2)\right]. \quad (2.6)$$

(2.5) and (2.6) are enough to renormalize the model. For the purposes of this paper, this is all that we need to know about the vacuum.

3 The Soliton

In this section, we turn to the problem of computing the soliton mass. A classical static soliton centered at the origin has the field configuration

$$\hat{\Phi}_x \equiv \frac{4}{\beta}\tan^{-1}(\exp[mx]) \quad (3.1)$$

and the mass

$$\int_x V(\hat{\Phi}_x) = \frac{8m}{\beta^2}.$$

In discussing the quantum case, we will consider fluctuations about a mean field $\hat{\Phi}$, where $\hat{\Phi}$ is some arbitrary static function which asymptotically approaches $2\pi n_\pm/\beta$ as $x \to \pm\infty$.

It is tempting to try a wave-functional for the quantized soliton of the form

$$\Psi_{soliton} = N_f \exp\left[i\int_x P_x(\phi-\hat{\Phi})_x - \tfrac{1}{2}\int_{x,y}(\phi-\hat{\Phi})_x f_{xy}(\phi-\hat{\Phi})_y\right], \quad (3.2)$$

where f_{xy} has the same general form

$$f_{xy} = \int \frac{dp}{2\pi}\cos[p(x-y)]\sqrt{p^2+\mathcal{M}^2} \quad (3.3)$$

as it did in the vacuum case. $\mathcal{M}$ is some parameter to be determined variationally. One can show that (3.2) and (3.3) describe a coherent state, an eigenstate of the destruction operators $a(p)$. Several years ago, Coleman performed a variational calculation using coherent states [5]. His results, though expressed in a different notation, are easy to translate into ours. The functions P, f and $\hat{\Phi}$ which minimize the energy are

$$\begin{aligned} P_x &= 0 \\ f_{xy} &= \int \frac{dp}{2\pi}\cos[p(x-y)]\sqrt{p^2+m_R^2} \\ \hat{\Phi}_x &= \frac{4}{\beta}\tan^{-1}(\exp[m_R x]), \end{aligned} \quad (3.4)$$

and the minimum of the energy functional is

$$E_{trial} = E_{vac} + \frac{8m_R}{\beta^2}.$$

Using the formulae of section 2, it is straightforward to verify these results in our functional notation.

We observe that the only effect of quantum mechanics (at least, in the coherent-state approximation) is to renormalize the mass and shift the energy. This being a variational calculation, the classical energy is an upper bound to the true quantum energy. To improve on the classical results, we need to improve our trial state Ψ.

One possible improvement in Ψ would be to let f_{xy} be completely general, rather than assuming that it takes the form (3.3). One could then try to solve the general variational calculation

$$\frac{\delta E}{\delta f_{xy}} = 0.$$

This turns out to require the solution of a differential equation. In the sine-Gordon problem, this is tractable, though in higher dimensions, it becomes increasingly more difficult. A more serious flaw in this scheme is that it ignores a major facet of the physics of solitons— the zero modes—associated with the symmetries of the system. Note that a Gaussian wave-functional can *never* adequately describe these zero modes.

However, there is a well-known way to quantize zero modes, by introducing collective coordinates. In our case, for example, we would assume that the mean-field has the form

$$\Phi(x,t) = \hat{\Phi}\big(x - a(t)\big).$$

The collective coordinate here is $a(t)$, which is interpreted as the position of the soliton. If $a(t)$ were the only quantum coordinate of the problem, then the Hamiltonian would describe a free particle and the energy eigenfunctions would be the momentum eigenstates

$$\psi_q(a) \propto e^{iqa}$$

of momentum q. But there *are* other coordinates describing the soliton, the field coordinates $\phi_x(t)$. A natural ansatz for the variational wave-functional is therefore

$$\Psi = N e^{iqa} \exp\left[i \int_x P_x(\phi - \Phi)_x - \tfrac{1}{2} \int_{x,y} (\phi - \Phi)_x f_{xy} (\phi - \Phi)_y \right], \qquad (3.5)$$

where

$$\Phi(x;a) = \hat{\Phi}(x-a)$$
$$P(x;a) = \hat{P}(x-a).$$

This ansatz is simple in appearance, but it has a hidden difficulty: fluctuations in ϕ must be constrained so as not to vibrate in the direction of the zero mode, which is already described by the collective coordinate. This problem is best tackled by Dirac's method of constrained quantization [12]. An excellent review of this method, in the context of solitons, is given by Tomboulis and Woo [13], which we will follow very closely.

We begin at the classical level, later making the transition to quantum mechanics by replacing Poisson brackets with commutators. The classical Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\dot{\phi}^2 - V(\phi).$$

We shift the field, defining $\xi(x,t)$ by

$$\phi(x,t) = \Phi\big(x; a(t)\big) + \xi(x,t).$$

Then

$$\dot{\phi}_x = -\dot{a}(\partial_x \Phi_x) + \dot{\xi}_x$$

and the Lagrangian can be rewritten as

$$\begin{aligned} L(t) &\equiv \int_x \mathcal{L} \\ &= \tfrac{1}{2} M \dot{a}^2 - \dot{a} \int_x (\partial_x \Phi_x)\dot{\xi}_x + \int_x \left[\tfrac{1}{2}\dot{\xi}_x^2 - V \right], \\ M[\Phi] &\equiv \int_x (\partial_x \Phi_x)^2 \quad . \end{aligned}$$

The conjugate momenta are

$$\begin{aligned} P &\equiv \frac{\partial L}{\partial \dot{a}} = M\dot{a} - \int_x (\partial_x \Phi_x)\dot{\xi}_x \\ \Pi_x &\equiv \frac{\delta L}{\delta \dot{\xi}_x} = -\dot{a}(\partial_x \Phi_x) + \dot{\xi}_x \quad , \end{aligned}$$

which satisfy a constraint

$$\psi \equiv P + \int_x \Pi_x (\partial_x \Phi_x) = 0. \qquad (3.6)$$

It is convenient to shift Π_x also. We define

$$\overline{\Pi}_x \equiv \Pi_x + \frac{(\partial_x \Phi_x) P}{M}.$$

The constraint (3.6) can then be rewritten in the "linearized" form

$$\overline{\psi} \equiv \int_x \overline{\Pi}_x (\partial_x \Phi_x) = 0. \qquad (3.7)$$

The classical Hamiltonian works out to be

$$H \equiv P\dot{a} + \int_x \Pi_x \dot{\xi}_x - L$$
$$= \frac{P^2}{2M} + \int_x \left[\tfrac{1}{2}\overline{\Pi}_x^2 + V\right].$$

But, because of the constraint, there are ambiguities in the Hamiltonian, as discussed more fully by Tomboulis and Woo [13]. It is necessary to impose some subsidiary condition on the coordinates to remove this ambiguity. A condition which is convenient, due to its close resemblance to (3.7), is

$$\chi \equiv \int_x (\partial_x \Phi_x)\xi_x = 0. \tag{3.8}$$

Since (3.8) is time-independent, $\dot{\chi}$ vanishes. This implies the relation

$$\int_x \dot{\xi}_x(\partial_x \Phi_x) = \dot{a}\Gamma \tag{3.9}$$

where

$$\Gamma \equiv \int_x (\partial_x^2 \Phi)\xi_x.$$

(3.9) is useful, because it enables us to prove the identity

$$P\dot{a} + \int_x \Pi_x \dot{\xi}_x = P\left(1 - \frac{\Gamma}{M}\right)\dot{a} + \int_x \overline{\Pi}_x \dot{\xi}_x. \tag{3.10}$$

Defining

$$\overline{P} \equiv P\left(1 - \frac{\Gamma}{M}\right),$$

(3.10) tells us that the set $\{P, a, \Pi_x, \xi_x\}$ is related by a canonical transformation to $\{\overline{P}, a, \overline{\Pi}_x, \xi_x\}$. One set is as good as the other, though the second is more convenient. In terms of the latter set, the Hamiltonian becomes

$$H = \frac{1}{2M}\overline{P}\left[1 - \frac{\Gamma}{M}\right]^{-2}\overline{P} + \int_x \left[\tfrac{1}{2}\overline{\Pi}_x^2 + V\right].$$

We must now discuss the constraints more fully. Suppose $f(\overline{P}, a, \overline{\Pi}, \xi)$ is some arbitrary function of the canonical variables. If we use the usual definition of the Poisson brackets

$$\{a, \overline{P}\} = 1$$
$$\{\xi_x, \overline{\Pi}_y\} = \delta_{xy},$$

then, in general, it will turn out that

$$\{f, \overline{\psi}\} \neq 0 \neq \{f, \chi\}.$$

This conflicts with our demand that $\overline{\psi}$ and χ vanish. Dirac's solution was to modify the brackets:

$$\{f,g\}^* = \{f,g\} - \frac{1}{\{\overline{\psi},\chi\}}\left[\{f,\chi\}\{\overline{\psi},g\} - \{f,\overline{\psi}\}\{\chi,g\}\right],$$

where f and g are arbitrary functions of the canonical variables. It is easily verified that

$$\{f,\overline{\psi}\}^* = 0 = \{f,\chi\}^*.$$

Having defined brackets that respect the constraints, we can now quantize the system by

$$\{f,g\}^* \to i[f,g].$$

It is a tedious, but straightforward task to work out the commutators:

$$[\overline{P}, a] = -i$$
$$[f(a,\xi,\overline{\Pi}), a] = 0$$
$$\left[\overline{P}, \xi_x\right] = -i\frac{\Gamma}{M}(\partial_x \Phi_x)$$
$$\left[\overline{\Pi}_x, \xi_y\right] = -i\left[\delta_{xy} - \frac{1}{M}(\partial_x \Phi_x)(\partial_y \Phi_y)\right].$$

These are the only commutators that we need in order to compute

$$E_{soliton} \equiv \frac{\langle\Psi|H|\Psi\rangle'}{\langle\Psi|\Psi\rangle'}.$$

The primes indicate that the path integrals over ξ_x should be carried out only on the subspace on which $\chi[\xi]$ vanishes. Equivalently, we can insert a $\delta\left[\chi[\xi]\right]$ in the path integral, and integrate over *all* ξ_x's . This can be done with the help of the integral representation of the delta-function

$$\delta(\chi) = \int_{-\infty}^{\infty} \frac{d\rho}{2\pi}\exp(i\rho\chi).$$

Note that the path integral remains Gaussian, because $\chi[\xi]$ is a *linear* functional of ξ.

As a warmup exercise, one can use the above formalism to show that

$$\frac{\langle\Psi|\overline{P}|\Psi\rangle'}{\langle\Psi|\Psi\rangle'} = q,$$

where q is the momentum of the soliton that we expect by naively examining (3.5).

4 Results

Before describing the final results, we split the Hamiltonian into a translational and a vibrational piece:

$$H = H_{trans} + H_{vib}$$

where

$$H_{trans} = \frac{1}{2M}\overline{P}\left[1 - \frac{\Gamma}{M}\right]^{-2}\overline{P}$$
$$H_{vib} = \int_x \left[\tfrac{1}{2}\overline{\Pi}_x^2 + V\right].$$

After a lengthy calculation, we find

$$E_{vib} \equiv \frac{\langle\Psi|\, H_{vib}|\Psi\rangle'}{\langle\Psi\,|\Psi\rangle'}$$
$$= E_{vac} + M - \frac{J_1[\hat{\Phi}]}{4M} + \frac{m_R^2 J_{-2}[\hat{\Phi}] - M}{4J_{-1}[\hat{\Phi}]}$$
$$- \frac{m_R^2}{\beta^2}\int_x \cos(\beta\hat{\Phi}_x)\left[\exp\left(\frac{\beta^2\left[\int_y f_{xy}^{-1}(\partial_y\hat{\Phi}_y)\right]^2}{4J_{-1}[\hat{\Phi}]}\right) - 1\right],$$

where

$$J_n[\hat{\Phi}] \equiv \int_{x,y}(\partial_x\hat{\Phi}_x) f_{xy}^n (\partial_y\hat{\Phi}_y). \tag{4.1}$$

It is somewhat easier to compute the translational piece. We need to use the trick

$$\frac{1}{\alpha^2} = -\lim_{\epsilon\to 0+}\int_0^\infty ds\, s\cos(s\alpha)\exp(-\epsilon s),$$

in order to handle the factor

$$\left(1 - \frac{\Gamma[\xi]}{M}\right)^{-2}$$

in the path integral. This trick retains the Gaussian nature of the path integral. Then we compute

$$E_{trans} \equiv \frac{\langle\Psi|\, H_{trans}|\Psi\rangle'}{\langle\Psi\,|\Psi\rangle'}$$
$$= -\lim_{\epsilon\to 0+}\int_0^\infty ds\, s\cos(s)\exp\left(-\tfrac{1}{4}s^2 K[\hat{\Phi}]\right)\left[\frac{q^2}{2M} + \frac{K[\hat{\Phi}]}{8M}\left(J_1[\hat{\Phi}] - \frac{M^2}{J_{-1}[\hat{\Phi}]}\right)\right],$$

where

$$K[\hat{\Phi}] \equiv \frac{1}{M^2}\int_{x,y}(\partial_x^2\hat{\Phi}_x) f_{xy}^{-1}(\partial_y^2\hat{\Phi}_y). \tag{4.2}$$

We are interested in the rest energy, so we can set $q = 0$. If we substitute the classical solution (3.4) into (4.1) and (4.2), then we find numerically that

$$J_1[\hat{\Phi}] = 9.09502\frac{m_R^2}{\beta^2}$$
$$J_{-1}[\hat{\Phi}] = 7.19580\frac{1}{\beta^2}$$
$$J_{-2}[\hat{\Phi}] = 6.57974\frac{1}{m_R\beta^2}$$
$$K[\hat{\Phi}] = .029675\,\beta^2 \quad .$$

Numerical evaluation of (4.1) and (4.2) shows that E_{trans} is wholly irrelevant. It is never larger than $2\times 10^{-3} m_R$, which is much smaller than the vibrational correction to the classical energy:

$$\Delta E_{vib} \equiv E_{vib} - E_{vac} - M.$$

This is plotted in Fig. 1, along with the one-loop [7], two-loop [14, 15] and exact [7] corrections:

$$\Delta E_{1-loop} = \frac{-m_R}{\pi}$$
$$\Delta E_{2-loop} = m_R\left[-\frac{1}{\pi} + \frac{\beta^2}{192}\right]$$
$$\Delta E_{exact} = m_R\left[\frac{1}{2\sin\left[\frac{1}{16}\beta'^2\right]} - \frac{8}{\beta^2}\right],$$

where

$$\beta'^2 \equiv \frac{\beta^2}{1 - \frac{\beta^2}{8\pi}}.$$

5 Conclusion

Our results are offset by a small amount from the two-loop answer. By inserting a better mean-field function $\hat{\Phi}_x$, we could presumably lower our variational estimate of the energy slightly. However, we are already so close that this is unnecessary.

Compared to the two-loop computation [14,15], we have lost slightly in accuracy, but gained substantially in simplicity. This gain would be even more pronounced in more realistic soliton models in four dimensions. The reason is that the perturbative calculation requires the solution of a nonlinear differential equation. For the sine-Gordon model, this equation can be

solved analytically. In other models, it would be necessary to find numerical solutions, thereby greatly complicating the calculation. Our method avoids this problem by using only plane-waves. (This is a useful approximation, which could be improved, if necessary. As we have seen, for the sine-Gordon model, the plane-wave approximation is perfectly adequate.) A bonus implicit in our method is that it sums a certain class of diagrams to *all* orders in perturbation theory [3].

There is one puzzling aspect of our results. We have done a variational calculation, so one might expect that our estimate for the soliton energy should be higher than the exact energy. Such an inequality holds for weak coupling, but not for strong. Why should this be?

The answer is that our results are expressed as a ratio $\Delta E/m_R$. Both ΔE and m_R were computed variationally, in terms of β, the bare mass m and a divergent integral. Thus, both ΔE and m_R are upper bounds, so their ratio is *not* an upper bound.

Finally, we should comment on possible applications of our methods to more interesting models.

It is difficult to compute quantum corrections in the Skyrme model [16], primarily because it is non-renormalizable, but also because of the higher-derivative terms. These problems would arise in *any* computational scheme. If they can be overcome, then the methods outlined in this paper would be a useful tool in this model. In two dimensions, quantum solitons are lighter than their classical counterparts [5]. It is plausible that this holds in higher dimensions. If so, then quantum effects would lower the energy from the semi-classical estimates of Adkins, Nappi and Witten [17], which are several hundred MeV too large, when the experimental value of F_π is used.

Another interesting application of our methods would be to magnetic monopoles. Lacking experimental data, this topic is less pressing than is the Skyrme model. However, it would undoubtedly be an easier calculation.

Acknowledgements

I would like to thank Adlai Smith for suggesting the trick used in Section 4, S. Mahajan and I. Hinchliffe for instruction in the use of the plotting software, and O. Alvarez for reading and discussing this paper. This work was supported in part by the Director, Office of Energy Research, Office of High Energy Physics and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098, and in part by the National Science Foundation, under Research Grant No. PHY-81-18547.

References

[1] W. A. Bardeen and M. Moshe, Phys. Rev. **D28** (1983), 1372.

[2] J. Kuti (unpublished), described in J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. **D10** (1974), 2428.

[3] T. Barnes and G. I. Ghandour, Phys. Rev. **D28** (1980), 924.

[4] P. M. Stevenson, Phys. Rev. **D30** (1984), 1712; Phys. Rev. **D32** (1985), 1389.

[5] S. Coleman in "New Phenomena in Subnuclear Physics" (A. Zichichi, ed.) part A, p. 297, Plenum Press, New York, 1977.

[6] K. Cahill, Phys. Lett. **53B** (1974), 174.

[7] R. F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. **D11** (1975), 3424.

[8] A. B. Zamolodchikov, Comm. Math. Phys. **55** (1977), 183.

[9] E. K. Sklyanin, L. A. Takhtadzhyan and L. D. Faddeev, Theor. Math. Phys. **40** (1980), 688.

[10] R. Ingermanson, preprint LBL-20453, (October, 1985), to be published in Nucl. Phys. **B**.

[11] S. Coleman, Phys. Rev. **D11** (1975), 2088.

[12] P. A. M. Dirac, Proc. Roy. Soc. London **A246** (1958), 326.

[13] E. Tomboulis and G. Woo, Annals of Physics **98** (1976), 1.

[14] H. J. De Vega, Nucl. Phys. **B115** (1976), 411.

[15] J. Verwaest, Nucl. Phys. **B123** (1977), 100.

[16] T. H. R. Skyrme, Proc. Roy. Soc. **A260** (1961), 127.

[17] G. S. Adkins, C. R. Nappi and E. Witten, Nucl. Phys. **B228** (1983), 552.

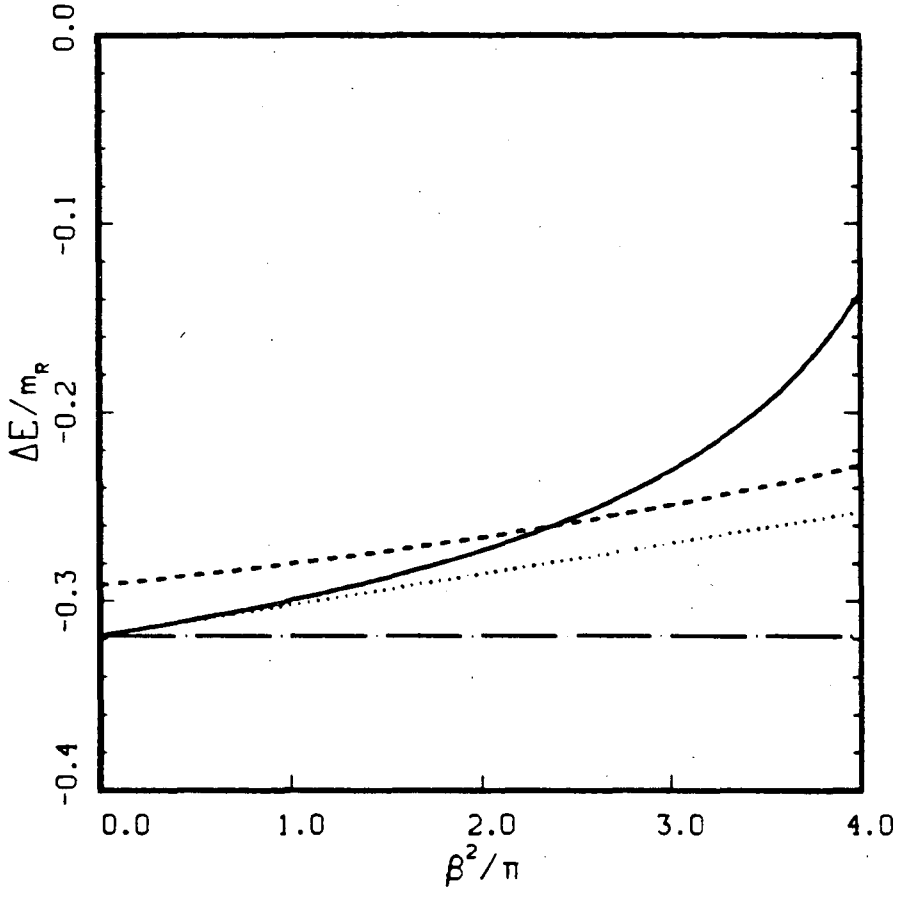


Figure 1: Plot of $\Delta E/m_R$ *vs* β^2/π. Solid line is exact, long dashes are one-loop, dots are two-loop and short dashes are our variational results.

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.

LAWRENCE BERKELEY LABORATORY
TECHNICAL INFORMATION DEPARTMENT
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720