Title

The overflow in the Katona Theorem

Journal

Combinatorial Theory, 6(2)

ISSN

2766-1334

Authors

Frankl, Peter

Wang, Jian

Publication Date

2026-08-20

THE OVERFLOW IN THE KATONA THEOREM

Peter Frankl[1] and Jian Wang*[2]

[1] *Rényi Institute, Budapest, Hungary*
frankl.peter@renyi.hu
[2] *School of Mathematics, Sichuan University, Chengdu, China*
wangjianmath01@scu.edu.cn

Submitted: Jun 2, 2023; Accepted: Oct 8, 2025; Published: Aug 20, 2026

Abstract. Let $n > 2r > 0$ be integers. We consider families $\mathcal{F}$ of subsets of an n-element set, in which the union of any two members has size at most $2r$. One of our results states that for $n \geqslant 8r$ the number of members of size exceeding r in $\mathcal{F}$ is at most $\binom{n-2}{r-1}$. Another result shows that for $n > 3.5r$ the number of sets of size at least r is at most $\binom{n}{r}$. Both bounds are best possible and the latter sharpens the classical Katona Theorem. Similar results are proved for the odd case of the Katona Theorem as well.

Keywords. The Katona theorem, overflow, shifting, the random walk method

Mathematics Subject Classifications. 05D05

1. Introduction

Let $[n] = \{1, 2, \dots, n\}$ be the standard n-element set and $2^{[n]}$ its power set. For $0 \leqslant k \leqslant n$ let $\binom{[n]}{k} \subset 2^{[n]}$ be the collection of all k-subsets. Subsets of $2^{[n]}$ are called *families*. If $\mathcal{F} \subset \binom{[n]}{k}$ it is called k-uniform.

For a positive integer t, the family $\mathcal{F}$ is called t-*intersecting* if $|F \cap F'| \geqslant t$ for all $F, F' \in \mathcal{F}$. Also for a positive integer u, $\mathcal{F}$ is called u-*union* if $|F \cup F'| \leqslant u$ for all $F, F' \in \mathcal{F}$.

Define $\mathcal{F}^c = \{[n] \setminus F \colon F \in \mathcal{F}\}$, the *family of complements*. It should be obvious that $\mathcal{F} \subset 2^{[n]}$ is t-intersecting if and only if $\mathcal{F}^c$ is $(n-t)$-union.

Let us state the two most important results concerning intersecting families, the Erdős–Ko–Rado Theorem and the Katona Theorem.

Theorem 1.1 ([EKR61]). *Let $n > k \geqslant t > 0$ be integers and $\mathcal{F} \subset \binom{[n]}{k}$ be t-intersecting. Then for $n > n_0(k, t)$,*

$$|\mathcal{F}| \leqslant \binom{n-t}{k-t}. \tag{1.1}$$

*Supported by National Natural Science Foundation of China Grant no. 12471316 and Natural Science Foundation of Shanxi Province, China Grant no. RD2500002993.

Let us mention that the exact value $n_0(k,t) = (k-t+1)(t+1)$ was determined by Erdős, Ko and Rado for $t = 1$, by Frankl [Fra78] for $t \geqslant 15$ and finally Wilson [Wil84] closed the gap $2 \leqslant t \leqslant 14$ with a completely different proof valid for all t. Let us note also that the *full t-star*

$$\mathcal{S}(n,k,t) = \left\{ S \in \binom{[n]}{k} : [t] \subset S \right\}$$

shows that (1.1) is best possible.

The corresponding problem for *non-uniform* families was solved by Katona. Let us state it for u-union families.

Theorem 1.2 ([Kat64]). *Let $n > u > 0$ and suppose that $\mathcal{F} \subset 2^{[n]}$ is u-union. Then (i) or (ii) holds.*

(i) $u = 2d$ *and*

$$|\mathcal{F}| \leqslant \sum_{0 \leqslant i \leqslant d} \binom{n}{i}. \tag{1.2}$$

(ii) $u = 2d+1$ *and*

$$|\mathcal{F}| \leqslant 2 \sum_{0 \leqslant i \leqslant d} \binom{n-1}{i}. \tag{1.3}$$

Let us define the *Katona Families* $\mathcal{K}(n,u)$:

$$\mathcal{K}(n,2d) = \{K \subset [n] \colon |K| \leqslant d\},\ \mathcal{K}(n,2d+1) = \{K \subset [n] \colon |K \setminus \{1\}| \leqslant d\}.$$

These families show that both (1.2) and (1.3) are best possible. As Katona [Kat64] showed, for $u \leqslant n-2$, $u = 2d$ the family $\mathcal{K}(n,2d)$ is the unique family attaining equality in (1.2). Similarly, for $u \leqslant n-2$, $u = 2d+1$, $\mathcal{K}(n,2d+1)$ is unique up to isomorphism.

Let us mention that in the case of $u = n-1$ both bounds (1.2) and (1.3) reduce to $|\mathcal{F}| \leqslant 2^{n-1}$ which was already known to Erdős, Ko and Rado. In this case there are many non-isomorphic extremal families (cf. [EKR61]).

An intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ is called *non-trivial* if $\cap\mathcal{F} = \varnothing$. Define the *Hilton–Milner Family* $\mathcal{H}(n,k)$ and the *Triangle Family* $\mathcal{T}(n,k)$ as:

$$\mathcal{H}(n,k) = \left\{ H \in \binom{[n]}{k} : 1 \in H,\ H \cap [2,k+1] \neq \varnothing \right\} \cup \{[2,k+1]\}$$

and

$$\mathcal{T}(n,k) = \left\{ F \in \binom{[n]}{k} : |F \cap [3]| \geqslant 2 \right\}.$$

It is easy to see that they are both non-trivial intersecting.

For the $t = 1$, $n > 2k$ case of the Erdős–Ko–Rado Theorem, Hilton and Milner proved a very important stability result.

Theorem 1.3. *Let $n > 2k$ and let $\mathcal{F} \subset \binom{[n]}{k}$ be non-trivial intersecting, then*

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1. \tag{1.4}$$

The construction $\mathcal{H}(n,k)$ shows that (1.4) is best possible.

The corresponding statement for the Katona Theorem was proved by Frankl [Fra17b]. Define $\mathcal{K}^*(n,u)$ as:

$$\mathcal{K}^*(n,2d) = \mathcal{K}(n,2d) \setminus \left\{ H \in \binom{[n]}{d} : H \cap [d+1] = \varnothing \right\} \cup \{[d+1]\}$$

and

$$\mathcal{K}^*(n,2d+1) = \mathcal{K}(n,2d+1) \cup \left\{ H \in \binom{[n]}{d+1} : 1 \in H, H \cap [2,d+2] = \varnothing \right\} \cup \{[2,d+2]\}.$$

Theorem 1.4 ([Fra17b])**.** *Suppose that $\mathcal{F} \subset 2^{[n]}$ is u-union, $2 \leqslant u < n$ and $\mathcal{F} \not\subset \mathcal{K}(n,u)$ holds. Then*

$$|\mathcal{F}| \leqslant |\mathcal{K}^*(n,u)|.$$

Let us note that $|\mathcal{H}(n,k) \setminus \mathcal{S}(n,k,1)| = 1$ and $|\mathcal{K}^*(n,u) \setminus \mathcal{K}(n,u)| = 1$, i.e., both families are in a sense extremely close to the original optimal families. For $x \in [n]$, define

$$\mathcal{F}(\bar{x}) = \{F \in \mathcal{F} : x \notin \mathcal{F}\}.$$

Let us define the *diversity* $\gamma(n,k)$ for $n > 2k$ by

$$\gamma(n,k) = \max_{\mathcal{F}} \min_{x \in [n]} |\mathcal{F}(\bar{x})|,$$

where the maximum is taken over all 1-intersecting families $\mathcal{F} \subset \binom{[n]}{k}$. The problem of determining $\gamma(n,k)$ was raised by Katona (cf. [LP08]). Improving earlier results ([LP08], [Fra17a], [Kup18]), it is proved that

Theorem 1.5 ([Fra20b], [FW24])**.** *For $n > 36k$,*

$$\gamma(n,k) = \binom{n-3}{k-2}. \tag{1.5}$$

In case of the Katona Theorem, for $u = 2d$, there is no special element in $\mathcal{K}(n,u)$. Therefore we prefer a new name for the maximum of $|\mathcal{F} \setminus \mathcal{K}(n,u)|$.

Definition 1.6. For $n > 2d+1$ define the *overflow* $\sigma_{2d}(\mathcal{F})$ of $\mathcal{F}$ as $|\mathcal{F} \setminus \mathcal{K}(n,2d)|$ and set

$$\sigma(n,2d) = \max \left\{ \sigma_{2d}(\mathcal{F}) : \mathcal{F} \subset 2^{[n]} \text{ is } 2d\text{-union} \right\}.$$

A natural family with relatively large overflow is

$$\mathcal{B}(n,2d)=\{B\subset[n]\colon |B\setminus[2]|\leqslant d-1\}.$$

Obviously,

$$\mathcal{B}(n,2d)\setminus\mathcal{K}(n,2d)=\left\{B\in\binom{[n]}{d+1}\colon [2]\subset B\right\}.$$

Our main result shows that $\sigma(n,2d)=\binom{n-2}{d-1}$ for $n\geqslant 8d$.

Theorem 1.7. *Suppose that $n\geqslant 8d$, $\mathcal{F}\subset 2^{[n]}$ is $2d$-union. Then*

$$|\mathcal{F}\setminus\mathcal{K}(n,2d)|\leqslant\binom{n-2}{d-1}. \tag{1.6}$$

Recall the standard notations

$$\mathcal{F}^{(\ell)}=\{F\in\mathcal{F}\colon |F|=\ell\}\text{ and }\mathcal{F}^{(\geqslant\ell)}=\{F\in\mathcal{F}\colon |F|\geqslant\ell\}.$$

With these notations $\mathcal{F}\setminus\mathcal{K}(n,2d)=\mathcal{F}^{(\geqslant d+1)}=\mathcal{F}^{(d+1)}\cup\mathcal{F}^{(d+2)}\cup\ldots\cup\mathcal{F}^{(n)}$. Moreover, if $\mathcal{F}$ is u-union then $\mathcal{F}^{(\ell)}=\varnothing$ for $\ell>u$.

Let us define one more $2d$-union family:

$$\mathcal{D}=\{D\subset[n]\colon |D\setminus[4]|\leqslant d-2\}.$$

Its overflow is

$$|\mathcal{D}\setminus\mathcal{K}(n,2d)|=|\mathcal{D}^{(d+1)}|+|\mathcal{D}^{(d+2)}|=5\binom{n-4}{d-2}+\binom{n-4}{d-3}.$$

For $n\leqslant 4d-2$,

$$\begin{aligned}5\binom{n-4}{d-2}+\binom{n-4}{d-3}-\binom{n-2}{d-1}&=4\binom{n-4}{d-2}+\binom{n-3}{d-2}-\binom{n-2}{d-1}\\&=4\binom{n-4}{d-2}-\binom{n-3}{d-1}\\&=\frac{4d-1-n}{d-1}\binom{n-4}{d-2}>0.\end{aligned}$$

This shows that (1.6) does not hold for $n<4d-1$.

Let

$$\mathcal{K}_x(n,2d+1)=\{K\subset[n]\colon |K\setminus\{x\}|\leqslant d\}.$$

Define the *overflow* $\sigma_{2d+1}(\mathcal{F})$ of $\mathcal{F}$ as $\min_{x\in[n]}|\mathcal{F}\setminus\mathcal{K}_x(n,2d+1)|$.

For the odd case, we obtain the following result.

Theorem 1.8. *Suppose that* $n > 36(d+1)$, $\mathcal{F} \subset 2^{[n]}$ *is* $(2d+1)$*-union. Then*

$$\sigma_{2d+1}(\mathcal{F}) \leqslant 2\binom{n-3}{d-1}. \tag{1.7}$$

Define

$$\mathcal{G} = \mathcal{G}(n, 2d+1) = \{G \subset [n] \colon |G \cap [4,n]| \leqslant d-1\}.$$

Then $\mathcal{G}^{(d+2)}$ is the star of $[3]$ and $\mathcal{G}^{(d+1)}$ is the triangle family $\mathcal{T}(n, d+1)$. Hence for $x \in [3]$,

$$|\mathcal{G} \setminus \mathcal{K}_x(n, 2d+1)| = 2\binom{n-3}{d-1}.$$

Thus (1.7) is best possible.

Our next result is a sharpening of the Katona Theorem in the case $u = 2r$.

Theorem 1.9. *Let* $n \geqslant 3.5r + 1$ *and let* $\mathcal{F} \subset 2^{[n]}$ *be* $2r$*-union. Then*

$$|\mathcal{F}^{(\geqslant r)}| \leqslant \binom{n}{r} \tag{1.8}$$

with equality if and only if $\mathcal{F}^{(\geqslant r)} = \binom{[n]}{r}$.

To see that Theorem 1.9 implies the Katona Theorem just note that $\mathcal{F}^{(<r)} \leqslant \sum_{0 \leqslant i < r} \binom{n}{i}$ is evident for any $\mathcal{F} \subset 2^{[n]}$.

The coefficient 3.5 in the above theorem could most likely be replaced by a smaller number. However the example and computation below show that the statement is no longer true for $n < 2.2r$.

Recall the $2r$-union family:

$$\mathcal{D} = \{D \subset [n] \colon |D \setminus [6]| \leqslant r-3\}.$$

It is easy to see that $\mathcal{D}^{(\geqslant r+4)} = \varnothing$ and

$$\begin{aligned}
|\mathcal{D}^{(r)}| &= 20\binom{n-6}{r-3} + 15\binom{n-6}{r-4} + 6\binom{n-6}{r-5} + \binom{n-6}{r-6},\\
|\mathcal{D}^{(r+1)}| &= 15\binom{n-6}{r-3} + 6\binom{n-6}{r-4} + \binom{n-6}{r-5},\\
|\mathcal{D}^{(r+2)}| &= 6\binom{n-6}{r-3} + \binom{n-6}{r-4},\\
|\mathcal{D}^{(r+3)}| &= \binom{n-6}{r-3}.
\end{aligned}$$

Thus,

$$|\mathcal{D}^{(\geqslant r)}| = 42\binom{n-6}{r-3} + 22\binom{n-6}{r-4} + 7\binom{n-6}{r-5} + \binom{n-6}{r-6}.$$

Since

$$\binom{n}{r}=\binom{n-6}{r}+6\binom{n-6}{r-1}+15\binom{n-6}{r-2}+20\binom{n-6}{r-3}+15\binom{n-6}{r-4}+6\binom{n-6}{r-5}+\binom{n-6}{r-6},$$

it follows that

$$\begin{aligned}|\mathcal{D}^{(\geqslant r)}|-\binom{n}{r}&=22\binom{n-6}{r-3}+7\binom{n-6}{r-4}+\binom{n-6}{r-5}\\&\quad-\binom{n-6}{r}-6\binom{n-6}{r-1}-15\binom{n-6}{r-2}.\end{aligned}\tag{1.9}$$

Note that for any $0\leqslant i\leqslant 5$ and $n=(1+c)r$,

$$\frac{\binom{n-6}{r-i}}{\binom{n-6}{r-i-1}}=\frac{n-r+i-5}{r-i}=\frac{cr+i-5}{r-i}\to c$$

as r tends to infinity. Thus (1.9)>0 is essentially equivalent to

$$1+7c+22c^2-15c^3-6c^4-c^5>0.$$

This holds for $1<c<1.2$. Thus to establish (1.8) one needs at least $n\geqslant 2.2r$.

For the odd case we have the following result.

Theorem 1.10. *Let $n>6r$ and let $\mathcal{F}\subset 2^{[n]}$ be $(2r+1)$-union. Then*

$$|\mathcal{F}^{(\geqslant r+1)}|\leqslant\binom{n-1}{r}\tag{1.10}$$

with equality if and only if $\mathcal{F}^{(\geqslant r+1)}=\mathcal{S}(n,r+1,1)$ up to isomorphism.

Computation similar to the even case, using the example $\{F\in[n]:|F\setminus[5]|\leqslant r-2\}$ shows that $6r$ cannot be replaced by $2.88r$.

Two families $\mathcal{A}\subset\binom{[n]}{a}$ and $\mathcal{B}\subset\binom{[n]}{b}$ are called *cross t-intersecting* if $|A\cap B|\geqslant t$ for any $A\in\mathcal{A}$ and $B\in\mathcal{B}$. We need the following simple fact.

Fact 1.11. *Suppose that $\mathcal{F}\subset 2^{[n]}$ is $(2d+h)$-union then (i), (ii) hold.*

(i) $\mathcal{F}^{(d+i)}$ is $(2i-h)$-intersecting.

(ii) $\mathcal{F}^{(d+i)}$ and $\mathcal{F}^{(d+j)}$ are cross $(i+j-h)$-intersecting.

Proof. Since (i) is the special case $i=j$ of (ii), we only prove (ii). Choose $F\in\mathcal{F}^{(d+i)}$, $G\in\mathcal{F}^{(d+j)}$. Then

$$|F\cap G|=|F|+|G|-|F\cup G|\geqslant(d+i)+(d+j)-(2d+h)=i+j-h.\qquad\square$$

Let us recall the following common notations: for $P\subset Q\subset[n]$, let

$$\mathcal{F}(P,Q)=\{F\setminus Q\colon F\in\mathcal{F},\ F\cap Q=P\}.$$

When $P=Q$, we simply write $\mathcal{F}(P)$.

2. Shifting and random walks

In this section, we recall the shifting method invented by Erdős, Ko and Rado [EKR61] and the random walk method developed by the first author [Fra78, Fra87].

For a family $\mathcal{F} \subset 2^{[n]}$ and two integers $1 \leqslant i < j \leqslant n$, one define the $i \leftarrow j$ shift S_{ij} by

$$S_{ij}(\mathcal{F}) = \{S_{ij}(F) \colon F \in \mathcal{F}\},$$

where

$$S_{ij}(F) = \begin{cases} F' := (F \setminus \{j\}) \cup \{i\}, & j \in F, i \notin F \text{ and } F' \notin \mathcal{F}; \\ F, & \text{otherwise.} \end{cases}$$

Obviously, $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$, $|S_{ij}(F)| = |F|$ and it is known (cf. e.g. [Fra87]) that the $i \leftarrow j$ shift maintains the t-intersecting, u-union and cross t-intersecting properties.

Repeated application of the $i \leftarrow j$ shift leads to a family $\tilde{\mathcal{F}}$ satisfying $S_{ij}(\tilde{\mathcal{F}}) = \tilde{\mathcal{F}}$ for all $1 \leqslant i < j \leqslant n$. Such families are called *initial*.

Let $(x_1, \ldots, x_k)$ denote the k-set $\{x_1, \ldots, x_k\}$ where we know that $x_1 < \ldots < x_k$. If $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_k) \in \binom{[n]}{k}$ then we say that $(x_1, \ldots, x_k)$ precedes $(y_1, \ldots, y_k)$ and denote it by $(x_1, \ldots, x_k) \prec (y_1, \ldots, y_k)$ if and only if $x_i \leqslant y_i$ for all $1 \leqslant i \leqslant k$.

The initial family has the following key property.

Proposition 2.1 ([Fra87]). *Let $\mathcal{F} \subset 2^{[n]}$. Then $\mathcal{F}$ is initial if and only if for all $F, G \subset [n]$ with $|F| = |G|$, $G \in \mathcal{F}$ and $F \prec G$ imply $F \in \mathcal{F}$.*

For t-intersecting initial families and cross t-intersecting initial families, the first author proved the following two results.

Proposition 2.2 ([Fra87]). *Suppose that $F \in \mathcal{F} \subset 2^{[n]}$ and $\mathcal{F}$ is initial and t-intersecting. Then*

$$|F \cap [t + 2i]| \geqslant t + i \text{ for some } i \geqslant 0. \tag{2.1}$$

Proposition 2.3 ([Fra87]). *Suppose that $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are cross t-intersecting and both initial, $F \in \mathcal{F}$, $G \in \mathcal{G}$. Then there exists some $s \geqslant t$ so that*

$$|F \cap [s]| + |G \cap [s]| \geqslant s + t. \tag{2.2}$$

One can deduce (2.1) from (2.2). On the other hand (2.1) has a geometric interpretation. Let us associate with every subset $R \subset [n]$ a walk $w(R)$ of n steps in the plane starting from the origin. If after the ith step we are in point (x, y) then we move to $(x, y + 1)$ or $(x + 1, y)$ according to whether $i + 1 \in R$ or $i + 1 \notin R$. If $|R| = k$ then $w(R)$ is a walk from $(0, 0)$ to $(n - k, k)$ with each step goes up a unit or goes right a unit.

Choose i in (2.1) to be maximal implies equality and thereby $w(F)$ is in the point $(i, t + i)$ after $t + 2i$ steps. That is, it is on the line $y = x + t$.

Corollary 2.4 ([Fra78]). *If $\mathcal{F} \subset 2^{[n]}$ is t-intersecting and initial, then $w(F)$ hits the line $y = x + t$ for each $F \in \mathcal{F}$.*

The reflection of $(0,0)$ on the line $y=x+t$ is $(-t,t)$. The *reflection principle* says that the number of walks from $(-t,t)$ to any fixed point V under the line is the same as the number of walks from $(0,0)$ to V hitting the line. This method was first used in [Fra78] to derive a general upper bound on the size of a uniform t-intersecting family.

Proposition 2.5 ([Fra78])**.** *Suppose that $\mathcal{F}\subset 2^{[n]}$ is t-intersecting. Then*

$$|\mathcal{F}^{(t+\ell)}|\leqslant \binom{n}{\ell} \text{ for all } \ell\geqslant 0. \tag{2.3}$$

Similarly, we have the following more general statement.

Proposition 2.6. *Let (a,b) and $(n-k,k)$ be two points below the line $y=x+t$ and assume $a+t\leqslant k$. Then the number of walks from (a,b) to $(n-k,k)$ hitting the line $y=x+t$ is $\binom{n-a-b}{k-a-t}$.*

Proof. Consider a walk P from (a,b) to $(n-k,k)$ hitting the line $y=x+t$ with each step goes up a unit or goes right a unit. Let $(c,c+t)$ be the first point on the line $y=x+t$ touched by P. Let P_1 be the subwalk of P from (a,b) to $(c,c+t)$ and let P_2 be the subwalk of P from $(c,c+t)$ to $(n-k,k)$. We reflect P_1 about the line $y=x+t$ and obtain a walk Q_1 from $(b-t,a+t)$ to $(c,c+t)$. Following Q_1 with P_2, we get a walk Q from $(b-t,a+t)$ to $(n-k,k)$. It is easy to check that this established a one-to-one correspondence between the walks from (a,b) to $(n-k,k)$ hitting the line $y=x+t$ and the walks from $(b-t,a+t)$ to $(n-k,k)$. Since the number of walks from $(b-t,a+t)$ to $(n-k,k)$ is exactly $\binom{n-a-b}{k-a-t}$, the proposition follows. $\square$

It should be mentioned that the universal bound (2.3) for t-intersecting families was improved by Frankl [Fra20a].

Theorem 2.7 ([Fra20a])**.** *Suppose that $\mathcal{F}\subset 2^{[n]}$ is t-intersecting. Then*

$$|\mathcal{F}^{(t+\ell)}|\leqslant \binom{n-1}{\ell} \text{ for } \ell\leqslant \frac{n-t-1}{2}. \tag{2.4}$$

The following proposition was proved in [Fra78] and we include its proof for completeness.

Proposition 2.8 ([Fra78])**.** *Let $\mathcal{F}\subset\binom{[n]}{k}$ be an initial family. If $(1,2,\ldots,p,p+2,p+4,\ldots,2k-p)\notin\mathcal{F}$ for some $0\leqslant p\leqslant k-1$, then*

$$|\mathcal{F}|\leqslant \binom{n}{k-p-1}. \tag{2.5}$$

If $(p,p+2,p+4,\ldots,p+2k-2)\notin\mathcal{F}$ for some $2\leqslant p\leqslant n-2k+2$, then

$$|\mathcal{F}|\leqslant \binom{n}{k}-\binom{n-p+2}{k}+\binom{n-p+2}{k-1}. \tag{2.6}$$

Proof. Assume $A := (1, 2, \ldots, p, p+2, p+4, \ldots, 2k-p) \notin \mathcal{F}$. Then by Proposition 2.1, $A \prec F$ cannot hold for all $F \in \mathcal{F}$. If $w(F)$ does not hit the line $y = x + p + 1$ then $w(F)$ would go below $w(A)$, implying $A \prec F$, a contradiction. Thus $w(F)$ hits the line $y = x + p + 1$ for all $F \in \mathcal{F}$. By Proposition 2.6,

$$|\mathcal{F}| \leqslant \binom{n}{k-p-1}.$$

Note that

$$|\{F \in \mathcal{F} \colon F \cap [p-2] \neq \varnothing\}| \leqslant \binom{n}{k} - \binom{n-p+2}{k}. \tag{2.7}$$

Suppose that $B := (p, p+2, p+4, \ldots, p+2k-2) \notin \mathcal{F}$. Let $F \in \mathcal{F}$ with $F \cap [p-2] = \varnothing$. If $w(F)$ does not hit the line $y = x - p + 3$, then $w(F)$ goes below $w(B)$, implying $B \prec F$, contradicting Proposition 2.1. Then $w(F)$ hits the line $y = x - p + 3$ for all $F \in \mathcal{F}$ with $F \cap [p-2] = \varnothing$. By Proposition 2.6,

$$\begin{aligned} &|\{F \in \mathcal{F} \colon F \cap [p-2] = \varnothing\}| \\ &\leqslant \text{the number of walks from } (p-2, 0) \text{ to } (n-k, k) \text{ hitting the line } y = x - p + 3 \\ &= \binom{n-p+2}{k-1}. \end{aligned} \tag{2.8}$$

Adding (2.7) and (2.8), (2.6) follows. □

We need the following inequality in the proofs.

Proposition 2.9. *Let n, r, a, b be positive integers, $n \geqslant r + a$, $n > r > b$. Then*

$$\frac{\binom{n-a}{r}}{\binom{n}{r-b}} \geqslant \left(\frac{n-r+b-a+1}{n-a+1}\right)^a \left(\frac{n-r+b-a}{r}\right)^b. \tag{2.9}$$

Proof. The inequality follows directly from

$$\begin{aligned} \frac{\binom{n-a}{r}}{\binom{n}{r-b}} &= \frac{(n-r+b)(n-r+b-1)\ldots(n-r+b-a+1)}{n(n-1)\ldots(n-a+1)} \\ &\quad \times \frac{(n-r+b-a)(n-r+b-a-1)\ldots(n-r-a+1)}{r(r-1)\ldots(r-b+1)} \\ &\geqslant \left(\frac{n-r+b-a+1}{n-a+1}\right)^a \left(\frac{n-r+b-a}{r}\right)^b. \end{aligned}$$

□

3. Proof of Theorem 1.7

For $E = (a_1, a_2, \dots, a_k)$, let $E - p$ denote the set $(a_1 - p, a_2 - p, \dots, a_k - p)$. Let $\mathcal{F} \subset \binom{[p+1,n]}{k}$ be an initial family. Define the *left-translate* $L_p(\mathcal{F})$ of $\mathcal{F}$ as

$$L_p(\mathcal{F}) := \{F - p \colon F \in \mathcal{F}\}.$$

Clearly, $L_p(\mathcal{F}) \subset \binom{[n-p]}{k}$ is also an initial family.

Lemma 3.1. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an initial t-intersecting family. Then for any $R \subsetneq [t]$, $\mathcal{F}(R, [t])$ is $2(t - |R|)$-intersecting.*

Proof. Suppose that there exist $E_1, E_2 \in \mathcal{F}(R, [t])$ such that $|E_1 \cap E_2| < 2(t - |R|)$. Since $\mathcal{F}$ is t-intersecting, we infer that $|E_1 \cap E_2| \geqslant t - |R|$. Choose $S \subset E_1 \cap E_2$ with $|S| = t - |R|$. Then by initiality and $(E_1 \setminus S) \cup [t] \prec R \cup E_1$ we have $(E_1 \setminus S) \cup [t] \in \mathcal{F}$. However,

$$\left|\Big((E_1 \setminus S) \cup [t]\Big) \cap (E_2 \cup R)\right| = |E_1 \cap E_2| - |S| + |R| < 2(t - |R|) - (t - |R|) + |R| = t,$$

a contradiction. Thus the lemma holds. □

Proof of Theorem 1.7. Noting that shifting does not change the overflow $|\mathcal{F} \setminus \mathcal{K}(n, 2d)|$, we may assume that $\mathcal{F}$ is initial. Since adding a subset $F_0 \subset F \in \mathcal{F}$ to $\mathcal{F}$ will not affect the u-union property, we may assume that $\mathcal{F}$ is a *complex*, that is, $F_0 \subset F \in \mathcal{F}$ implies $F_0 \in \mathcal{F}$.

If $\mathcal{F}^{(\ell)} = \varnothing$ for all $\ell \geqslant d + 2$ then $\mathcal{F} \setminus \mathcal{K}(n, 2d) = \mathcal{F}^{(d+1)}$. By Fact 1.11, $\mathcal{F}^{(d+1)}$ is 2-intersecting. Thus by (1.1), $|\mathcal{F}^{(d+1)}| \leqslant \binom{n-2}{d-1}$ for $n > 3d$ and the theorem holds.

From now on we assume $\mathcal{F}^{(\ell)} \neq \varnothing$ for some $\ell \geqslant d + 2$. Since $\mathcal{F}$ is a complex, $\mathcal{F}^{(d+2)} \neq \varnothing$. By initiality $[d+2] \in \mathcal{F}$ follows. By Fact 1.11, $\mathcal{F}^{(d+1)}, \mathcal{F}^{(d+2)}$ are cross 3-intersecting. Note that $|[d+2] \cap (1, 2, d+3, d+5, \dots, 3d-1)| = 2$. It follows that $(1, 2, d+3, d+5, \dots, 3d-1) \notin \mathcal{F}^{(d+1)}$.

Let $\mathcal{F}_0^{(d+1)} = \{F \in \mathcal{F}^{(d+1)} \colon \{1, 2\} \subset F\}$ and let $\mathcal{F}_1^{(d+1)} = \mathcal{F}^{(d+1)} \setminus \mathcal{F}_0^{(d+1)}$. We distinguish two cases.

Case 1. $(1, 2, 4, 6, 8, \dots, 2d) \notin \mathcal{F}^{(d+1)}$.

Since $(1, 2, 4, 6, 8, \dots, 2d) \notin \mathcal{F}^{(d+1)}$, $(2, 4, 6, \dots, 2d - 2) \notin L_2(\mathcal{F}^{(d+1)}([2]))$. By (2.5) we have

$$|\mathcal{F}_0^{(d+1)}| = |L_2(\mathcal{F}^{(d+1)}([2]))| \leqslant \binom{n-2}{(d-1)-1} = \binom{n-2}{d-2}. \tag{3.1}$$

Since $\mathcal{F}^{(d+1)}$ is 2-intersecting, by Lemma 3.1, $\mathcal{F}(\{\ell\}, [2])$ is 2-intersecting for $\ell = 1, 2$ and $\mathcal{F}(\varnothing, [2])$ is 4-intersecting. By (2.4) we have $|\mathcal{F}(\{\ell\}, [2])| \leqslant \binom{n-3}{d-2}$. Using (2.4) we have $|\mathcal{F}(\varnothing, [2])| \leqslant \binom{n-3}{d-3}$. Thus,

$$\begin{aligned}|\mathcal{F}_1^{(d+1)}| &= |\mathcal{F}(\{1\}, [2])| + |\mathcal{F}(\{2\}, [2])| + |\mathcal{F}(\varnothing, [2])| \\ &\leqslant 2\binom{n-3}{d-2} + \binom{n-3}{d-3} = \binom{n-3}{d-2} + \binom{n-2}{d-2}.\end{aligned} \tag{3.2}$$

By Fact 1.11 and Proposition 2.5,

$$|\mathcal{F}^{(d+i)}| \leqslant \binom{n}{d+i-2i} = \binom{n}{d-i}.$$

Note that $n \geqslant 6d$ implies that

$$\frac{\binom{n}{d-i}}{\binom{n}{d-i-1}} = \frac{n-d+i+1}{d-i} > 5.$$

It follows that

$$\sum_{2\leqslant i\leqslant d} |\mathcal{F}^{(d+i)}| \leqslant \sum_{2\leqslant i\leqslant d} \binom{n}{d-i} \leqslant \binom{n}{d-2} + \frac{5}{4}\binom{n}{d-3} \leqslant \binom{n}{d-2} + \frac{5}{4}\frac{n^2}{(n-d)^2}\binom{n-2}{d-3}$$
$$\leqslant \binom{n}{d-2} + \frac{9}{5}\binom{n-2}{d-3}. \tag{3.3}$$

Adding (3.1), (3.2) and (3.3), we get

$$\sum_{1\leqslant i\leqslant d} |\mathcal{F}^{(d+i)}| \leqslant \binom{n-3}{d-2} + 2\binom{n-2}{d-2} + \binom{n}{d-2} + \frac{9}{5}\binom{n-2}{d-3}$$
$$= \binom{n-3}{d-2} + \frac{1}{5}\binom{n-2}{d-2} + \frac{9}{5}\binom{n-1}{d-2} + \binom{n}{d-2}.$$

By $n \geqslant 6d$,

$$\sum_{1\leqslant i\leqslant d} |\mathcal{F}^{(d+i)}| \leqslant \left(\frac{d-1}{n-2} + \frac{1}{5}\frac{d}{n-d} + \frac{9}{5}\frac{dn}{(n-d)^2} + \frac{dn^2}{(n-d)^3}\right)\binom{n-2}{d-1}$$
$$\leqslant \left(\frac{1}{6} + \frac{1}{5}\times\frac{1}{5} + \frac{9}{5}\times\frac{6}{5^2} + \frac{6^2}{5^3}\right)\binom{n-2}{d-1}$$
$$< \binom{n-2}{d-1}.$$

Case 2. $(1, 2, 4, 6, 8, \ldots, 2d) \in \mathcal{F}^{(d+1)}$.

Choose the minimal j such that $(1, 2, j, j+2, j+4, \ldots, j+2d-4) \notin \mathcal{F}_0^{(d+1)}$. Clearly we have $5 \leqslant j \leqslant d+3$. By minimality of j, $(1, 2, j-1, j+1, j+3, \ldots, j+2d-5) \in \mathcal{F}_0^{(d+1)}$. Then the 2-intersecting property implies $(3, \ldots, j-2, j, j+2, \ldots, j+2(d-j+3)) \notin \mathcal{F}^{(d+1)}(\{\ell\}, [2])$ for $\ell = 1, 2$. It follows that

$$(1, 2, \ldots, j-4, j-2, j, \ldots, j+2(d-j+2)) \notin L_2(\mathcal{F}^{(d+1)}(\{\ell\}, [2])).$$

By (2.5), we infer that

$$|\mathcal{F}^{(d+1)}(\{\ell\}, [2])| = |L_2(\mathcal{F}^{(d+1)}(\{\ell\}, [2]))| \leqslant \binom{n-2}{d-(j-4)-1} = \binom{n-2}{d+3-j}.$$

Similarly, $(1,2,\dots,j-4,j-3,j-2,j,\dots,j+2(d-j+2)) \notin L_2(\mathcal{F}^{(d+1)}(\varnothing,[2]))$ and

$$|\mathcal{F}^{(d+1)}(\varnothing,[2])| = |L_2(\mathcal{F}^{(d+1)}(\varnothing,[2]))| \leqslant \binom{n-2}{(d+1)-(j-2)-1} = \binom{n-2}{d+2-j}.$$

Thus,

$$\begin{aligned}|\mathcal{F}_1^{(d+1)}| &\leqslant |\mathcal{F}^{(d+1)}(\{1\},[2])| + |\mathcal{F}^{(d+1)}(\{2\},[2])| + |\mathcal{F}^{(d+1)}(\varnothing,[2])|\\ &= 2\binom{n-2}{d+3-j} + \binom{n-2}{d+2-j} = \binom{n-2}{d+3-j} + \binom{n-1}{d+3-j}. \end{aligned} \tag{3.4}$$

Claim 3.2.

$$|\mathcal{F}_0^{(d+1)}| \leqslant \binom{n-2}{d-1} - \binom{n-j+2}{d-1} + \binom{n-j+2}{d-2}. \tag{3.5}$$

Proof. Since $(1,2,j,j+2,j+4,\dots,j+2d-4) \notin \mathcal{F}_0^{(d+1)}$,

$$(j-2,j,j+2,\dots,j+2d-6) \notin L_2(\mathcal{F}_0^{(d+1)}([2])).$$

By (2.6) it follows that

$$|\mathcal{F}_0^{(d+1)}| = |L_2(\mathcal{F}^{(d+1)}([2]))| \leqslant \binom{n-2}{d-1} - \binom{n-j+2}{d-1} + \binom{n-j+2}{d-2}. \qquad \square$$

By Fact 1.11, we know that $\mathcal{F}^{(d+1)}, \mathcal{F}^{(d+i)}$ are cross $(i+1)$-intersecting. Since $(1,2,j-1,j+1,j+3,\dots,j+2d-5) \in \mathcal{F}_0^{(d+1)}$,

$$(1,2,3,\dots,j+2i-8,j+2i-7,j+2i-6,j+2i-4,\dots,2d-j+6) \notin \mathcal{F}^{(d+i)}.$$

By (2.5),

$$|\mathcal{F}^{(d+i)}| \leqslant \binom{n}{d+i-(j+2i-6)-1} = \binom{n}{d+5-i-j}.$$

Since $n \geqslant 6d$ implies

$$\frac{\binom{n}{d+5-i-j}}{\binom{n}{d+4-i-j}} = \frac{n-d-4+i+j}{d+5-i-j} \geqslant 5,$$

we have

$$\begin{aligned}\sum_{2\leqslant i\leqslant d} |\mathcal{F}^{(d+i)}| &< \binom{n}{d+3-j} + \frac{5}{4}\binom{n}{d+2-j}\\ &< \binom{n}{d+3-j} + \frac{5}{4}\frac{n}{n-d}\binom{n-1}{d+2-j}\\ &< \binom{n}{d+3-j} + 2\binom{n-1}{d+2-j}. \end{aligned} \tag{3.6}$$

Adding (3.4), (3.5) and (3.6),

$$|\mathcal{F} \setminus \mathcal{K}(n, 2d)| = \sum_{1 \leqslant i \leqslant d} |\mathcal{F}^{(d+i)}| \leqslant 3\binom{n}{d+3-j} + \binom{n-2}{d-1} - \binom{n-j+2}{d-1} + \binom{n-j+2}{d-2}.$$

We are left to show that

$$\binom{n-j+2}{d-1} - \binom{n-j+2}{d-2} - 3\binom{n}{d+3-j} > 0. \tag{3.7}$$

Since $n \geqslant 6d$ and $j \leqslant d+3$ imply

$$\frac{\binom{n-j+2}{d-1}}{\binom{n-j+2}{d-2}} = \frac{n-j-d+4}{d-1} > 4,$$

it follows that $\binom{n-j+2}{d-1} - \binom{n-j+2}{d-2} > \frac{3}{4}\binom{n-j+2}{d-1}$. By (2.9), $5 \leqslant j \leqslant d+3$ and $n \geqslant 6d$,

$$\frac{\binom{n-j+2}{d-1}}{\binom{n}{d+3-j}} \geqslant \left(\frac{n-d}{n-j+3}\right)^{j-2} \left(\frac{n-d-1}{d-1}\right)^{j-4} \geqslant 5^{2j-6}/6^{j-2}.$$

For $j \geqslant 6$,

$$\begin{aligned}
\frac{3}{4}\binom{n-j+2}{d-1} - 3\binom{n}{d+3-j} &\geqslant \frac{3}{4}\binom{n}{d+3-j}\left(5^{2j-6}/6^{j-2} - 4\right) \\
&\geqslant \frac{3}{4}\binom{n}{d+3-j}\left(5^{6}/6^{4} - 4\right) > 0.
\end{aligned}$$

Thus (3.7) holds.

For $j = 5$, by $n \geqslant 8d$ we have

$$\begin{aligned}
&\binom{n-3}{d-1} - \binom{n-3}{d-2} - 3\binom{n}{d-2} \\
&= \binom{n}{d-2}\left(\frac{(n-d+2)(n-d+1)(n-d)(n-d-1)}{(d-1)n(n-1)(n-2)} - 4\right) \\
&\geqslant \binom{n}{d-2}\left(\frac{(n-d)^4}{dn^3} - 4\right) \\
&\geqslant \binom{n}{d-2}\left(\frac{7^4}{8^3} - 4\right) > 0.
\end{aligned}$$

Thus (3.7) holds and the theorem is proven. □

4. Proof of Theorem 1.8

We are going to derive Theorem 1.8 from the following proposition.

Proposition 4.1. *Let* $\mathcal{F}_i \subset \binom{[n]}{d+1+i}$, $i = 1, 2, \ldots, d$. *Suppose that* $\mathcal{F}_i$ *is* $(2i+1)$*-intersecting and* $\mathcal{F}_i, \mathcal{F}_j$ *are cross* $(i+j+1)$*-intersecting. Then for* $n \geqslant 9d$,

$$\sum_{1\leqslant i\leqslant d} |\mathcal{F}_i| \leqslant \binom{n-3}{d-1}.$$

Proof. Assume that $\mathcal{F}_i$ is initial for all $i = 1, 2, \ldots, d$. Let $\mathcal{F}_{1,0} = \{F \in \mathcal{F}_1 \colon \{1, 2, 3\} \subset F\}$ and let $\mathcal{F}_{1,1} = \mathcal{F}_1 \setminus \mathcal{F}_{1,0}$. We distinguish three cases.

Case 1. $(1, 2, 3, d+4, d+6, \ldots, 3d) \in \mathcal{F}_1$.

The 3-intersecting property implies $(1, 2, 4, 5, \ldots, d+3) \notin \mathcal{F}_1$. It follows that every $F \in \mathcal{F}_1$ contains $\{1, 2, 3\}$. Also, for $i \geqslant 2$,

$$|\{1, 2, 3, d+4, d+6, \ldots, 3d\} \cap [d+1+i]| < i+2.$$

Since $\mathcal{F}_1, \mathcal{F}_i$ are cross $(i+2)$-intersecting, $[d+1+i] \notin \mathcal{F}_i$. By the initiality $\mathcal{F}_i = \varnothing$ for all $i = 2, 3, \ldots, d$. Since $\mathcal{F}_1$ is 3-intersecting and $n \geqslant 4(k-2)$, by (1.1) we have $|\mathcal{F}_1| \leqslant \binom{n-3}{d-1}$. Thus,

$$\sum_{1\leqslant i\leqslant d} |\mathcal{F}_i| = |\mathcal{F}_1| \leqslant \binom{n-3}{d-1}.$$

Case 2. $(1, 2, 3, 5, 7, 9, \ldots, 2d+1) \notin \mathcal{F}_1$.

Then $(2, 4, 6, \ldots, 2d-2) \notin L_3(\mathcal{F}_1([3]))$. By (2.5) we have

$$|\mathcal{F}_{1,0}| = |L_3(\mathcal{F}_1([3]))| \leqslant \binom{n-3}{d-2}. \tag{4.1}$$

Since $\mathcal{F}_1$ is 3-intersecting, by Lemma 3.1, $\mathcal{F}_1([3] \setminus \{\ell\}, [3])$ is 2-intersecting, $\mathcal{F}_1(\{\ell\}, [3])$ is 4-intersecting for $\ell = 1, 2, 3$ and $\mathcal{F}_1(\varnothing, [3])$ is 6-intersecting. By (2.4) we obtain that

$$\begin{aligned}
|\mathcal{F}_{1,1}| &= \sum_{1\leqslant \ell\leqslant 3} |\mathcal{F}_1([3] \setminus \{\ell\}, [3])| + \sum_{1\leqslant \ell\leqslant 3} |\mathcal{F}_1(\{\ell\}, [3])| + |\mathcal{F}_1(\varnothing, [3])| \\
&\leqslant 3\binom{n-4}{d-2} + 3\binom{n-4}{(d+1)-4} + \binom{n-4}{(d+2)-6} \\
&= 3\binom{n-4}{d-2} + 3\binom{n-4}{d-3} + \binom{n-4}{d-4} \\
&= \binom{n-4}{d-2} + \binom{n-3}{d-2} + \binom{n-2}{d-2}.
\end{aligned} \tag{4.2}$$

Since $\mathcal{F}_i$ is $(2i+1)$-intersecting, by Proposition 2.5

$$|\mathcal{F}_i| \leqslant \binom{n}{d+1+i-2i-1} = \binom{n}{d-i}.$$

Note that $n \geqslant 7d$ implies that

$$\frac{\binom{n}{d-i}}{\binom{n}{d-i-1}} = \frac{n-d+i+1}{d-i} > 6.$$

Since $\binom{n}{d-3} = \frac{n}{n-d+3}\binom{n-1}{d-3} < \frac{7}{6}\binom{n-1}{d-3}$, it follows that

$$\sum_{2\leqslant i\leqslant d} |\mathcal{F}_i| \leqslant \sum_{2\leqslant i\leqslant d} \binom{n}{d-i} \leqslant \binom{n}{d-2} + \frac{6}{5}\binom{n}{d-3} < \binom{n}{d-2} + 2\binom{n-1}{d-3}. \tag{4.3}$$

Adding (4.1), (4.2) and (4.3),

$$\sum_{1\leqslant i\leqslant d} |\mathcal{F}_i| \leqslant \binom{n-4}{d-2} + 2\binom{n-3}{d-2} + \binom{n-2}{d-2} + \binom{n}{d-2} + 2\binom{n-1}{d-3} \leqslant 5\binom{n}{d-2}.$$

By $n \geqslant 9d$,

$$\begin{aligned}\sum_{1\leqslant i\leqslant d} |\mathcal{F}_i| \leqslant 5\binom{n}{d-2} &= \frac{5n(n-1)(n-2)(d-1)}{(n-d+2)(n-d+1)(n-d)(n-d-1)}\binom{n-3}{d-1} \\ &\leqslant \frac{5n^3 d}{(n-d)^4}\binom{n-3}{d-1} \\ &< \binom{n-3}{d-1}.\end{aligned}$$

Case 3. $(1,2,3,d+4,d+6,\ldots,3d) \notin \mathcal{F}_1$ and $(1,2,3,5,7,9,\ldots,2d+1) \in \mathcal{F}_1$.

Choose the minimal j such that $(1,2,3,j,j+2,j+4,\ldots,j+2d-4) \notin \mathcal{F}_{1,0}$. Then $6 \leqslant j \leqslant d+4$.

By minimality of j, $(1,2,3,j-1,j+1,j+3,\ldots,j+2d-5) \in \mathcal{F}_{1,0}$. Then the 3-intersecting property implies $(1,2,4,5,\ldots,j-2,j,j+2,\ldots,j+2(d-j+3),j+2(d-j+4)) \notin \mathcal{F}_{1,1}$. It follows that for all $\ell \in [3]$,

$$(1,2,\ldots,j-5,j-3,j-1,\ldots,j+2(d-j+3)-3,j+2(d-j+4)-3) \notin L_3(\mathcal{F}_1([3]\setminus\{\ell\},[3])).$$

By (2.5),

$$|\mathcal{F}_1([3]\setminus\{\ell\},[3])| = |L_3(\mathcal{F}_1([3]\setminus\{\ell\},[3]))| \leqslant \binom{n-3}{d-(j-5)-1} = \binom{n-3}{d+4-j}.$$

Similarly,

$$(1,2,\ldots,j-5,j-4,j-3,j-1,\ldots,j+2(d-j+3)-3,j+2(d-j+4)-3) \notin L_3(\mathcal{F}_1(\{\ell\},[3]))$$

and

$$(1,2,\ldots,j-2,j-1,j+1\ldots,j+2(d-j+3)-3,j+2(d-j+4)-3)\notin L_3(\mathcal{F}_1(\varnothing,[3])).$$

By (2.5),

$$|\mathcal{F}_1(\{\ell\},[3])|\leqslant\binom{n-3}{d+3-j}\text{ and }|\mathcal{F}_1(\varnothing,[3])|\leqslant\binom{n-3}{d+2-j}.$$

Thus,

$$\begin{aligned}|\mathcal{F}_{1,1}|&=\sum_{1\leqslant\ell\leqslant3}|\mathcal{F}_1([3]\setminus\{\ell\},[3])|+\sum_{1\leqslant\ell\leqslant3}|\mathcal{F}_1(\{\ell\},[3])|+|\mathcal{F}_1(\varnothing,[3])|\\&\leqslant3\binom{n-3}{d+4-j}+3\binom{n-3}{d+3-j}+\binom{n-3}{d+2-j}\\&<3\binom{n-1}{d+4-j}.\end{aligned}\tag{4.4}$$

Since $(1,2,3,j,j+2,j+4,\ldots,j+2d-4)\notin\mathcal{F}_1$,

$$(j-3,j-1,j+1,\ldots,j+2d-7)\notin L_3(\mathcal{F}_1([3])).$$

By (2.6),

$$|\mathcal{F}_{1,0}|=|L_3(\mathcal{F}_1([3]))|\leqslant\binom{n-3}{d-1}-\binom{n-j+2}{d-1}+\binom{n-j+2}{d-2}.\tag{4.5}$$

Since $\mathcal{F}_1,\mathcal{F}_i$ are cross $(i+2)$-intersecting, by $(1,2,3,j-1,j+1,j+3,\ldots,j+2d-5)\in\mathcal{F}_{1,0}$ we have

$$(1,2,3,\ldots,j+2i-8,j+2i-7,j+2i-6,j+2i-4,\ldots,2d-j+8)\notin\mathcal{F}_i.$$

By (2.5) for $2\leqslant i\leqslant d$,

$$|\mathcal{F}_i|\leqslant\binom{n}{(d+1+i)-(j+2i-6)-1}=\binom{n}{d+6-i-j}.$$

Since $n\geqslant 9d$ implies

$$\frac{\binom{n}{d+6-i-j}}{\binom{n}{d+5-i-j}}=\frac{n-d-5+i+j}{d+6-i-j}\geqslant8,$$

we infer that

$$\sum_{2\leqslant i\leqslant d}|\mathcal{F}_i|\leqslant\binom{n}{d+4-j}+\frac{8}{7}\binom{n}{d+3-j}<\binom{n}{d+4-j}+2\binom{n-1}{d+3-j}.\tag{4.6}$$

Adding (4.4), (4.5) and (4.6),

$$\sum_{1\leqslant i\leqslant d}|\mathcal{F}_i|\leqslant4\binom{n}{d+4-j}+\binom{n-3}{d-1}-\binom{n-j+2}{d-1}+\binom{n-j+2}{d-2}.$$

We are left to show that

$$\binom{n-j+2}{d-1} - \binom{n-j+2}{d-2} - 4\binom{n}{d+4-j} > 0. \tag{4.7}$$

Since $n \geqslant 9d$ implies that

$$\frac{\binom{n-j+2}{d-1}}{\binom{n-j+2}{d-2}} = \frac{n-j-d+4}{d-1} > 7,$$

it follows that $\binom{n-j+2}{d-1} - \binom{n-j+2}{d-2} \geqslant \frac{6}{7}\binom{n-j+2}{d-1}$. As $j \geqslant 6$ and $n \geqslant 9d$, by (2.9)

$$\begin{aligned}\frac{\binom{n-j+2}{d-1}}{\binom{n}{d+4-j}} \geqslant \left(\frac{n-d-1}{n-j+3}\right)^{j-2}\left(\frac{n-d-2}{d-1}\right)^{j-5} &\geqslant \left(\frac{n-d-1}{n-3}\right)^{j-2}\left(\frac{n-d}{d}\right)^{j-5}\\ &\geqslant \left(\frac{8}{9}\right)^{j-2} 8^{j-5} = 8^{2j-7}/9^{j-2}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{6}{7}\binom{n-j+2}{d-1} - 4\binom{n}{d+4-j} &\geqslant \frac{6}{7}\binom{n}{d+4-j}\left(8^{2j-7}/9^{j-2} - \frac{14}{3}\right)\\ &\geqslant \frac{5}{6}\binom{n}{d+4-j}\left(8^{5}/9^{4} - \frac{14}{3}\right) > 0\end{aligned}$$

and the proposition is proven. □

Proof of Theorem 1.8. Let $\mathcal{F} \subset 2^{[n]}$ be $(2d+1)$-union. Let $\mathcal{F}_i = \mathcal{F}^{(d+1+i)}$, $i = 1, 2, \ldots, d$. By Fact 1.11, $\mathcal{F}_i$ is $(2i+1)$-intersecting and $\mathcal{F}_i, \mathcal{F}_j$ are cross $(i+j+1)$-intersecting. By Proposition 4.1,

$$\sum_{1\leqslant i\leqslant d} |\mathcal{F}^{(d+1+i)}| = \sum_{1\leqslant i\leqslant d} |\mathcal{F}_i| \leqslant \binom{n-3}{d-1}.$$

Since the $(2d+1)$-union property implies that $\mathcal{F}^{(d+1)}$ is intersecting, by Theorem 1.5 for $n > 36(d+1)$ there exists $x \in [n]$ such that

$$|\mathcal{F}^{(d+1)}(\bar{x})| \leqslant \binom{n-3}{d-1}.$$

Thus,

$$|\mathcal{F} \setminus \mathcal{K}_x(n, 2d+1)| \leqslant \sum_{1\leqslant i\leqslant d} |\mathcal{F}^{(d+1+i)}| + |\mathcal{F}^{(d+1)}(\bar{x})| \leqslant 2\binom{n-3}{d-1}.$$

□

5. Proof of Theorem 1.9

We need Hilton's Lemma [Hil76], which is a useful reformulation of the Kruskal–Katona Theorem [Kru63, Kat64].

To state it let us recall the definition of the lexicographic order on $\binom{[n]}{k}$. For two distinct sets $F, G \in \binom{[n]}{k}$ we say that F *precedes* G if

$$\min\{i \colon i \in F \setminus G\} < \min\{i \colon i \in G \setminus F\}.$$

E.g., $(1,7)$ precedes $(2,3)$. For positive integers n, k, m, let $\mathcal{L}(n,k,m)$ denote the family of first m members of $\binom{[n]}{k}$.

Lemma 5.1 ([Hil76]). *Let n, a, b be positive integers, $n > a + b$. Suppose that $\mathcal{A} \subset \binom{[n]}{a}$ and'$\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting. Then $\mathcal{L}(n, a, |\mathcal{A}|)$ and $\mathcal{L}(n, b, |\mathcal{B}|)$ are cross-intersecting as well.*

For $\mathcal{F} \subset \binom{[n]}{k}$ and $0 \leqslant \ell < k$, define the *ℓth shadow* $\partial^{(\ell)}\mathcal{F}$ as

$$\partial^{(\ell)}\mathcal{F} = \left\{ E \in \binom{[n]}{\ell} \colon \text{ there exists } F \in \mathcal{F} \text{ such that } E \subset F \right\}.$$

Proposition 5.2 ([Spe28]). *For $\mathcal{F} \subset \binom{[n]}{k}$ and $0 \leqslant \ell < k$,*

$$\frac{|\partial^{(\ell)}\mathcal{F}|}{\binom{n}{\ell}} \geqslant \frac{|\mathcal{F}|}{\binom{n}{k}}. \tag{5.1}$$

We need the following consequence of Sperner's bound (Proposition 5.2).

Corollary 5.3. *Let $\mathcal{A} \subset \binom{[n]}{a}$, $\mathcal{B} \subset \binom{[n]}{b}$ be cross-intersecting families. Then*

$$\frac{|\mathcal{A}|}{\binom{n}{a}} + \frac{|\mathcal{B}|}{\binom{n}{b}} \leqslant 1. \tag{5.2}$$

Proof. Let $\mathcal{A}^c = \{[n] \setminus A \colon A \in \mathcal{A}\}$. Since $\mathcal{A}, \mathcal{B}$ are cross-intersecting, $\partial^{(b)}\mathcal{A}^c \cap \mathcal{B} = \varnothing$. It follows that

$$|\partial^{(b)}\mathcal{A}^c| + |\mathcal{B}| \leqslant \binom{n}{b}.$$

By Sperner's bound (Proposition 5.2), we have

$$\frac{|\partial^{(b)}\mathcal{A}^c|}{\binom{n}{b}} \geqslant \frac{|\mathcal{A}^c|}{\binom{n}{n-a}} = \frac{|\mathcal{A}|}{\binom{n}{a}}.$$

Thus (5.2) follows. □

Proof of Theorem 1.9. Let $\mathcal{F} \subset 2^{[n]}$ be an initial $2r$-union family. Since adding a subset $F_0 \subset F \in \mathcal{F}$ to $\mathcal{F}$ will not affect the $2r$-union property, we may assume that $\mathcal{F}$ is a complex.

Claim 5.4. *If* $|\mathcal{F}^{(r)}| > \binom{n-1}{r-1} + \binom{n-2}{r-1}$, *then*

$$|\mathcal{F}^{(r)}| + 2.5|\mathcal{F}^{(r+1)}| \leqslant \binom{n}{r}. \tag{5.3}$$

Proof. By Fact 1.11, $\mathcal{F}^{(r)}$ and $\mathcal{F}^{(r+1)}$ are cross-intersecting. Let $\mathcal{A} = \mathcal{L}(n, r, |\mathcal{F}^{(r)}|)$ and $\mathcal{B} = \mathcal{L}(n, r+1, |\mathcal{F}^{(r+1)}|)$. By Hilton's Lemma (Lemma 5.1) $\mathcal{A}, \mathcal{B}$ are cross-intersecting as well. Since $|\mathcal{A}| = |\mathcal{F}^{(r)}| > \binom{n-1}{r-1} + \binom{n-2}{r-1}$, we see that

$$\left\{A \in \binom{[n]}{r} : A \cap \{1,2\} \neq \varnothing\right\} \subset \mathcal{A} \text{ and thereby } |\mathcal{B}| = |\mathcal{B}(1,2)|.$$

Then $\mathcal{A}(\bar{1}, \bar{2}), \mathcal{B}(1,2)$ are cross-intersecting families on the ground set $[3, n]$. By (5.2),

$$\frac{|\mathcal{A}(\bar{1}, \bar{2})|}{\binom{n-2}{r}} + \frac{|\mathcal{B}(1,2)|}{\binom{n-2}{r-1}} \leqslant 1.$$

Hence,

$$|\mathcal{A}(\bar{1}, \bar{2})| + 2.5|\mathcal{B}(1,2)| \leqslant |\mathcal{A}(\bar{1}, \bar{2})| + \frac{n-r-1}{r}|\mathcal{B}(1,2)| \leqslant \binom{n-2}{r}.$$

By $|\mathcal{A}(\bar{1}, \bar{2})| = |\mathcal{A}| - \binom{n-1}{r-1} - \binom{n-2}{r-1}$ and $|\mathcal{B}| = |\mathcal{B}(1,2)|$, (5.3) follows. □

Let us distinguish two cases.

Case 1. $(2, 4, 6, \ldots, 2r) \notin \mathcal{F}^{(r)}$.
By (2.5),

$$|\mathcal{F}^{(r)}| \leqslant \binom{n}{r-1}. \tag{5.4}$$

Using that $\mathcal{F}$ is a complex, for $i = 1, 2, \ldots, r$ we have

$$(1, 2, \ldots, 2i, 2i+2, \ldots, 2r) \notin \mathcal{F}^{(r+i)}.$$

Then by (2.5),

$$|\mathcal{F}^{(r+i)}| \leqslant \binom{n}{r-i-1}. \tag{5.5}$$

Since $n > 3r$, $\binom{n}{r-i}/\binom{n}{r-i-1} = \frac{n-r+i+1}{r-i} > 2$. Thus (5.4) and (5.5) imply

$$\sum_{i \geqslant 0} |\mathcal{F}^{(r+i)}| < 2\binom{n}{r-1} = 2\frac{r}{n-r+1}\binom{n}{r} < \binom{n}{r}. \tag{5.6}$$

Case 2. $(2,4,6,\dots,2r)\in\mathcal{F}^{(r)}$.

By Fact 1.11, $\mathcal{F}^{(r+i)}$ is $2i$-intersecting for $i=1,2,\dots,r$. By (1.1) and $n>3r$ we have

$$|\mathcal{F}^{(r+1)}|\leqslant\binom{n-2}{r-1}. \tag{5.7}$$

Using $n>2r$ and (2.4),

$$|\mathcal{F}^{(r+i)}|\leqslant\binom{n-1}{r-i} \text{ for } i=2,3,\dots,r.$$

For $n>3.5r$, $\binom{n-1}{r-i}/\binom{n-1}{r-i-1}=\frac{n-r+i}{r-i}>2.5$, implying

$$\sum_{i\geqslant 2}|\mathcal{F}^{(r+i)}|\leqslant\sum_{i\geqslant 2}\binom{n-1}{r-i}<\frac{5}{3}\binom{n-1}{r-2}. \tag{5.8}$$

We distinguish three subcases according to the size of $\mathcal{F}^{(r)}$.

Subcase 2.1. $|\mathcal{F}^{(r)}|\leqslant\binom{n-1}{r-1}+\binom{n-2}{r-1}$.

By (5.7) and (5.8),

$$\begin{aligned}
|\mathcal{F}^{(\geqslant r)}|&=|\mathcal{F}^{(r)}|+|\mathcal{F}^{(r+1)}|+\sum_{i\geqslant 2}|\mathcal{F}^{(r+i)}|\\
&\leqslant\binom{n-1}{r-1}+2\binom{n-2}{r-1}+\frac{5}{3}\binom{n-1}{r-2}\\
&\leqslant\binom{n-1}{r-1}+2\binom{n-2}{r-1}+2\binom{n-1}{r-2}\\
&=\binom{n-1}{r-1}\left(1+\frac{2(n-r)}{n-1}+\frac{2(r-1)}{n-r+1}\right).
\end{aligned}$$

As $n>3.5r$, using $\frac{n-1}{r-1}>3.5$ and $n-r=(n-1)-(r-1)$, we infer

$$\begin{aligned}
1+\frac{2(n-r)}{n-1}+\frac{2(r-1)}{n-r+1}&=3+2(r-1)\left(\frac{1}{n-r+1}-\frac{1}{n-1}\right)\\
&=3+\frac{2(r-1)(r-2)}{(n-1)(n-r+1)}\\
&<3+\frac{2}{3.5\times 2.5}<3.5.
\end{aligned}$$

Thus $|\mathcal{F}^{(\geqslant r)}|<3.5\binom{n-1}{r-1}\leqslant\binom{n}{r}$.

Subcase 2.2. $\binom{n-1}{r-1}+\binom{n-2}{r-1}<|\mathcal{F}^{(r)}|\leqslant 2\binom{n-1}{r-1}$.

If $\sum_{i\geqslant 2}|\mathcal{F}^{(r+i)}|\leqslant 1.5|\mathcal{F}^{(r+1)}|$ then by Claim 5.4,

$$|\mathcal{F}^{(\geqslant r)}|=|\mathcal{F}^{(r)}|+\sum_{i\geqslant 1}|\mathcal{F}^{(r+i)}|\leqslant|\mathcal{F}^{(r)}|+2.5|\mathcal{F}^{(r+1)}|\leqslant\binom{n}{r}$$

and we are done. Thus we may assume that

$$|\mathcal{F}^{(r+1)}|<\frac{2}{3}\sum_{i\geqslant 2}|\mathcal{F}^{(r+i)}|. \tag{5.9}$$

By (5.8),

$$\sum_{i\geqslant 1}|\mathcal{F}^{(r+i)}|\leqslant\frac{25}{9}\binom{n-1}{r-2}.$$

Thus,

$$\begin{aligned}|\mathcal{F}^{(\geqslant r)}|\leqslant 2\binom{n-1}{r-1}+\frac{25}{9}\binom{n-1}{r-2}&=\binom{n-1}{r-1}\left(2+\frac{25}{9}\frac{r-1}{n-r+1}\right)\\&<\binom{n-1}{r-1}\left(2+\frac{25}{9}\times\frac{2}{5}\right)\\&<3.2\binom{n-1}{r-1}<\binom{n}{r}.\end{aligned}$$

Subcase 2.3. $|\mathcal{F}^{(r)}|>2\binom{n-1}{r-1}$.

By Claim 5.4 we may assume that

$$|\mathcal{F}^{(r+1)}|\leqslant\frac{2}{3}\sum_{i\geqslant 2}|\mathcal{F}^{(r+i)}|. \tag{5.10}$$

Since $|\mathcal{F}^{(r)}|>2\binom{n-1}{r-1}=\binom{n}{r}-\binom{n-3+2}{r}+\binom{n-3+2}{r-1}$, by (2.6) we infer that $(3,5,7,\dots,2r+1)\in\mathcal{F}^{(r)}$.

Choose the maximal j such that $(j,j+2,\dots,j+2r-2)\in\mathcal{F}^{(r)}$. Then $(j+1,j+3,\dots,j+2r-1)\notin\mathcal{F}^{(r)}$ and $j\geqslant 3$. Recall that $\mathcal{F}$ is an initial complex. If $\mathcal{F}^{(r+1)}=\varnothing$ then $|\mathcal{F}^{(\geqslant r)}|=|\mathcal{F}^{(r)}|\leqslant\binom{n}{r}$. Thus we may assume that $\mathcal{F}^{(r+1)}\neq\varnothing$. By (5.10), $\mathcal{F}^{(r+2)}\neq\varnothing$. Then the initiality implies $[r+2]\in\mathcal{F}^{(r+2)}$. Since $\mathcal{F}^{(r)},\mathcal{F}^{(r+2)}$ are cross 2-intersecting, $j\leqslant r$. Hence $3\leqslant j\leqslant r$.

Since $(j+1,j+3,\dots,j+2r-1)\notin\mathcal{F}^{(r)}$, by (2.6)

$$|\mathcal{F}^{(r)}|\leqslant\binom{n}{r}-\binom{n-j+1}{r}+\binom{n-j+1}{r-1}. \tag{5.11}$$

Recall that $\mathcal{F}^{(r)},\mathcal{F}^{(r+i)}$ are cross i-intersecting. By $(j,j+2,\dots,j+2r-2)\in\mathcal{F}^{(r)}$, we infer that

$$(1,2,\dots,j+2i-5,j+2i-4,j+2i-3,j+2i-1,\dots,2r-j+1)\notin\mathcal{F}^{(r+i)}.$$

By (2.5) we have

$$|\mathcal{F}^{(r+i)}| \leqslant \binom{n}{(r+i)-(j+2i-3)-1} = \binom{n}{r-i-j+2}.$$

Since $n > 3.5r$ implies $\binom{n}{r-i-j+3}/\binom{n}{r-i-j+2} = \frac{n-r+i+j-2}{r-i-j+3} > 2.5 = \frac{5}{2}$,

$$\sum_{2\leqslant i\leqslant r} |\mathcal{F}^{(r+i)}| \leqslant \sum_{2\leqslant i\leqslant r} \binom{n}{r-i-j+2} \leqslant \frac{5}{3}\binom{n}{r-j}.$$

By (5.10),

$$\sum_{1\leqslant i\leqslant r} |\mathcal{F}^{(r+i)}| \leqslant \left(1+\frac{2}{3}\right)\times\frac{5}{3}\binom{n}{r-j} = \frac{25}{9}\binom{n}{r-j}.$$

Thus,

$$|\mathcal{F}^{(\geqslant r)}| \leqslant \binom{n}{r} - \binom{n-j+1}{r} + \binom{n-j+1}{r-1} + \frac{25}{9}\binom{n}{r-j}.$$

We are left to show that

$$\binom{n-j+1}{r} - \binom{n-j+1}{r-1} - \frac{25}{9}\binom{n}{r-j} > 0. \tag{5.12}$$

Since $n > 3.5r$ and $j \leqslant r$ imply $\binom{n-j+1}{r}/\binom{n-j+1}{r-1} = \frac{n-r-j+2}{r} \geqslant 1.5$,

$$\binom{n-j+1}{r} - \binom{n-j+1}{r-1} \geqslant \frac{1}{3}\binom{n-j+1}{r}.$$

As $3 \leqslant j \leqslant r$ and $n > 3.5r$, by (2.9)

$$\frac{\binom{n-j+1}{r}}{\binom{n}{r-j}} \geqslant \left(\frac{n-r+2}{n-j+2}\right)^{j-1}\left(\frac{n-r+1}{r}\right)^{j} \geqslant \left(\frac{n-r+2}{n-1}\right)^{j-1}\left(\frac{n-r}{r}\right)^{j} \geqslant \frac{2.5^{2j-1}}{3.5^{j-1}}.$$

Since $2.5^7/3.5^3 > 14$, for $j \geqslant 4$ we have

$$\begin{aligned}\frac{1}{3}\binom{n-j+1}{r} - \frac{25}{9}\binom{n}{r-j} &\geqslant \frac{1}{3}\binom{n}{r-j}\left(\frac{2.5^7}{3.5^3}-\frac{25}{9}\right)\\ &> \frac{1}{3}\binom{n}{r-j}\left(14-\frac{25}{3}\right) > 0\end{aligned}$$

and (5.12) follows.

For $j = 3$, using $n \geqslant 3.5r+1$ we have $\binom{n-2}{r}/\binom{n-2}{r-1} = \frac{n-r-1}{r} \geqslant 2.5$. It follows that

$$\binom{n-2}{r} - \binom{n-2}{r-1} \geqslant \frac{3}{5}\binom{n-2}{r}.$$

Thus,

$$\frac{3}{5}\binom{n-2}{r} - \frac{25}{9}\binom{n}{r-3} > \binom{n}{r-3}\left(\frac{3}{5}\times\frac{2.5^5}{3.5^2}-\frac{25}{9}\right) > 0.$$

Thus (5.12) holds and the theorem is proven. □

6. Proof of Theorem 1.10

Proof of Theorem 1.10. Assume that $\mathcal{F} \subset 2^{[n]}$ is an initial $(2r+1)$-union complex. We distinguish two cases.

Case 1. $(1, 3, 5, \ldots, 2r+1) \notin \mathcal{F}^{(r+1)}$.

By (2.5),

$$|\mathcal{F}^{(r+1)}| \leqslant \binom{n}{r-1}.$$

Using the fact that $\mathcal{F}$ is a complex, we infer $(1, 2, 3, 4, 5, \ldots, 2i-2, 2i-1, 2i+1, \ldots, 2r+1) \notin \mathcal{F}^{(r+i)}$ for $i = 2, 3, \ldots, r+1$. By (2.5),

$$|\mathcal{F}^{(r+i)}| \leqslant \binom{n}{(r+i)-(2i-1)-1} = \binom{n}{r-i}.$$

Since $n \geqslant 4r$ implies $\binom{n}{r-i}/\binom{n}{r-i-1} = \frac{n-r+i+1}{r-i} > 3$,

$$\sum_{i \geqslant 1} |\mathcal{F}^{(r+i)}| \leqslant \sum_{i \geqslant 1} \binom{n}{r-i} \leqslant \frac{3}{2}\binom{n}{r-1} \leqslant \frac{3rn}{2(n-r+1)(n-r)}\binom{n-1}{r} < \binom{n-1}{r}.$$

Case 2. $(1, 3, 5, \ldots, 2r+1) \in \mathcal{F}^{(r+1)}$.

Choose maximal j such that $(1, j, j+2, \ldots, j+2r-2) \in \mathcal{F}^{(r+1)}$. Then the maximality implies $(1, j+1, j+3, \ldots, j+2r-1) \notin \mathcal{F}^{(r+1)}$. Clearly $j \geqslant 3$. Recall the definition of the left-translate

$$L_1(\mathcal{F}) = \{\{x_1-1, x_2-1, \ldots, x_k-1\} \colon \{x_1, x_2, \ldots, x_k\} \in \mathcal{F}\}.$$

It follows that $(j, j+2, \ldots, j+2r-2) \notin L_1(\mathcal{F}^{(r+1)}(1))$. Thus by (2.6)

$$|\mathcal{F}^{(r+1)}(1)| = |L_1(\mathcal{F}^{(r+1)}(1))| \leqslant \binom{n-1}{r} - \binom{n-j+1}{r} + \binom{n-j+1}{r-1}. \tag{6.1}$$

By Fact 1.11, $\mathcal{F}^{(r+1)}$ is intersecting. If $\mathcal{F}^{(\geqslant r+2)} = \varnothing$, then by (1.1) for $n > 2(r+1)$,

$$|\mathcal{F}^{(\geqslant r+1)}| = |\mathcal{F}^{(r+1)}| \leqslant \binom{n-1}{r}$$

with equality holding if and only if $\mathcal{F}^{(r+1)} = \mathcal{S}(n, r+1, 1)$, the full star up to isomorphism. Thus we may assume that $\mathcal{F}^{(\geqslant r+2)} \neq \varnothing$. As $\mathcal{F}$ is a complex, $[r+2] \in \mathcal{F}$. By Fact 1.11 we know that $\mathcal{F}^{(r+1)}, \mathcal{F}^{(r+2)}$ are cross 2-intersecting, implying that $j \leqslant r+2$.

As $\mathcal{F}^{(r+1)}$ is intersecting, $(1, j, j+2, \ldots, j+2r) \in \mathcal{F}^{(r+1)}$ implies that

$$(2, 3, \ldots, j-1, j+1, \ldots, 2r-j+5) \notin \mathcal{F}^{(r+1)}(\bar{1}).$$

That is, $(1, 2, \ldots, j-2, j, \ldots, 2r-j+4) \notin L_1(\mathcal{F}^{(r+1)}(\bar{1}))$. Using (2.5),

$$|\mathcal{F}^{(r+1)}(\bar{1})| = |L_1(\mathcal{F}^{(r+1)}(\bar{1}))| \leqslant \binom{n-1}{r+1-(j-2)-1} = \binom{n-1}{r-j+2}. \tag{6.2}$$

Note also that $\mathcal{F}^{(r+1)}, \mathcal{F}^{(r+i)}$ are cross i-intersecting. Then $(1, j, j+2, \ldots, j+2r) \in \mathcal{F}^{(r+1)}$ implies

$$(1, 2, \ldots, j+2i-6, j+2i-5, j+2i-3, \ldots, 2r-j+5) \notin \mathcal{F}^{(r+i)}.$$

Hence, by (2.5)

$$|\mathcal{F}^{(r+i)}| \leqslant \binom{n}{r+i-(j+2i-5)-1} = \binom{n}{r-i-j+4}.$$

Since $\binom{n}{r-i-j+4}/\binom{n}{r-i-j+3} = \frac{n-r+i+j-3}{r-i-j+4} > 3$,

$$\sum_{i\geqslant 2} |\mathcal{F}^{(r+i)}| \leqslant \sum_{i\geqslant 2} \binom{n}{r-i-j+4} \leqslant \frac{3}{2}\binom{n}{r-j+2}. \tag{6.3}$$

Adding (6.1), (6.2) and (6.3), we obtain that

$$|\mathcal{F}^{(\geqslant r+1)}| \leqslant \binom{n-1}{r} - \binom{n-j+1}{r} + \binom{n-j+1}{r-1} + \frac{5}{2}\binom{n}{r-j+2}.$$

We are left to show that

$$\binom{n-j+1}{r} - \binom{n-j+1}{r-1} - \frac{5}{2}\binom{n}{r-j+2} > 0. \tag{6.4}$$

Note that $\binom{n-j+1}{r}/\binom{n-j+1}{r-1} = \frac{n-j-r+2}{r} \geqslant 2$. It implies that $\binom{n-j+1}{r} - \binom{n-j+1}{r-1} \geqslant \frac{1}{2}\binom{n-j+1}{r}$. By (2.9), $3 \leqslant j \leqslant r+2$ and $n > 4r$,

$$\frac{\binom{n-j+1}{r}}{\binom{n}{r-j+2}} \geqslant \left(\frac{n-r}{n-j+2}\right)^{j-1}\left(\frac{n-r-1}{r}\right)^{j-2} \geqslant 3^{2j-3}/4^{j-1}.$$

For $j \geqslant 5$,

$$\begin{aligned}
\binom{n-j+1}{r} - \binom{n-j+1}{r-1} - \frac{5}{2}\binom{n}{r-j+2} &> \frac{1}{2}\binom{n-j+1}{r} - \frac{5}{2}\binom{n}{r-j+2}\\
&\geqslant \frac{1}{2}\binom{n}{r-j+2}\left(3^{2j-3}/4^{j-1} - 5\right)\\
&\geqslant \frac{1}{2}\binom{n}{r-j+2}\left(3^{7}/4^{4} - 5\right) > 0.
\end{aligned}$$

For $j = 4$, using $n > 4r$ we have

$$\begin{aligned}
&\binom{n-3}{r} - \binom{n-3}{r-1} - \frac{5}{2}\binom{n}{r-2} \\
&= \binom{n-3}{r}\left(1 - \frac{r}{n-r-2} - \frac{5}{2}\frac{r(r-1)n(n-1)(n-2)}{(n-r+2)(n-r+1)(n-r)(n-r-1)(n-r-2)}\right) \\
&\geqslant \binom{n-3}{r}\left(1 - \frac{r}{n-r-2} - \frac{5}{2}\frac{r(r-1)n(n-1)(n-2)}{(n-r+1)(n-r)(n-r-1)^3}\right) \\
&\geqslant \binom{n-3}{r}\left(1 - \frac{1}{3} - \frac{5}{2} \times \frac{4^3}{3^5}\right) > 0.
\end{aligned}$$

For $j = 3$ and $n > 6r$,

$$\begin{aligned}
&\binom{n-2}{r} - \binom{n-2}{r-1} - \frac{5}{2}\binom{n}{r-1} \\
&= \binom{n-2}{r}\left(1 - \frac{r}{n-r-1} - \frac{5}{2}\frac{rn(n-1)}{(n-r+1)(n-r)(n-r-1)}\right) \\
&\geqslant \binom{n-2}{r}\left(1 - \frac{1}{5} - \frac{5}{2} \times \frac{6^2}{5^3}\right) > 0.
\end{aligned}$$

Thus (6.4) holds and the theorem is proven. □

7. Concluding remarks

In the present paper we investigated the maximal overflow in the Katona Theorem. We established best possible bounds under linear constraints. However there is still a gap that remains.

Open Problem. Determine the maximal overflow

$$\sigma(n,u) = \max\left\{\sigma_u(\mathcal{F})\colon \mathcal{F} \subset 2^{[n]},\ \mathcal{F} \text{ initial and } u\text{-union}\right\}, n \geqslant u+2.$$

It is not hard to show that in the case $n = u + 1$, u is even the maximum is attained for $\mathcal{F} = 2^{[u]}$.

To introduce a closely related problem, let us first recall the notation $A \triangle B = (A \setminus B) \cup (B \setminus A)$ for *symmetric difference*.

Let us mention that $2^{[n]}$ endowed with the distance $d(A, B) = |A \triangle B|$ is a *metric space*. The *diameter* of $\mathcal{A} \subset 2^{[n]}$ is defined as $\delta(\mathcal{A}) = \max\{|A \triangle B|\colon A, B \in \mathcal{A}\}$.

Since $|A \triangle B| \leqslant |A \cup B|$, $\delta(\mathcal{A}) \leqslant u$ for a u-union family $\mathcal{F}$, the following result strengthens the Katona Theorem.

Kleitman's Diameter Theorem ([Kle66]). Let $n > u > 0$. Suppose that $\mathcal{F} \subset 2^{[n]}$ satisfies $\delta(\mathcal{F}) \leqslant u$. Then

$$|\mathcal{F}| \leqslant |\mathcal{K}(n,u)|. \tag{7.1}$$

Unlike in the Katona Theorem, there are many families attaining equality. In fact for every $A \subset 2^{[n]}$, $\mathcal{K}_A(n,u) = \{K \triangle A \colon K \in \mathcal{K}(n,u)\}$ satisfies $\delta(\mathcal{K}_A(n,u)) = u$. For $n > u+1$ Bezrukov [Bez87] proved that these are the only optimal families.

For $u = 2d$ the family $\mathcal{K}_A(n,u) = \{K \subset [n] \colon |K \triangle A| \leqslant d\}$ and it is called the *ball* of *radius* d with *center* A. By analogy for $u = 2d+1$ we call $\mathcal{K}_A(n,u)$ the *double ball*.

For his proof Kleitman [Kle66] introduced a very useful operation, the *down-shift* D_i:

$$D_i(\mathcal{F}) = \{D_i(F) \colon F \in \mathcal{F}\},$$

where

$$D_i(F) = \begin{cases} F \setminus \{i\}, & \text{if } i \in F \in \mathcal{F} \text{ and } F \setminus \{i\} \notin \mathcal{F}; \\ F, & \text{otherwise.} \end{cases}$$

As $\delta(D_i(\mathcal{F})) \leqslant \delta(\mathcal{F})$, applying the down-shift D_i, $1 \leqslant i \leqslant n$ repeatedly will eventually produce a complex $\widetilde{\mathcal{F}}$ with $|\widetilde{\mathcal{F}}| = |\mathcal{F}|$ and $\delta(\widetilde{\mathcal{F}}) \leqslant \delta(\mathcal{F})$. For a complex $\widetilde{\mathcal{F}}$, $\delta(\widetilde{\mathcal{F}}) = \max\{|A \cup B| \colon A, B \in \widetilde{\mathcal{F}}\}$. Consequently, $|\mathcal{F}| = |\widetilde{\mathcal{F}}| \leqslant |\mathcal{K}(n, \delta(\mathcal{F}))|$.

For a fixed pair (n,u), let us define the *(diametral) overflow* $\kappa_u(\mathcal{F})$ as

$$\kappa_u(\mathcal{F}) = \min\{|\mathcal{F} \setminus \mathcal{K}_A(n,u)| \colon A \subset [n]\}.$$

The following two facts are easy to prove.

Fact 7.1. *Let $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$, $i \in [n]$. Then*

$$|D_i(\mathcal{F}) \cap D_i(\mathcal{G})| \geqslant |\mathcal{F} \cap \mathcal{G}|.$$

Fact 7.2. $D_i(\mathcal{K}_A(n,u)) = \mathcal{K}_{A \setminus \{i\}}(n,u)$.

Corollary 7.3. *Let $\mathcal{F} \subset 2^{[n]}$ be a complex. Then*

$$\sigma_u(\mathcal{F}) = \kappa_u(\mathcal{F}).$$

Recall the definition of the families:

$$\mathcal{B}(n,2d) = \{B \subset [n] \colon |B \setminus [2]| \leqslant d-1\}$$

and

$$\mathcal{G}(n,2d+1) = \{G \subset [n] \colon |G \setminus [3]| \leqslant d-1\}.$$

Then Corollary 7.3 implies

$$\kappa_{2d}(\mathcal{B}(n,2d)) = \binom{n-2}{d-1}, \ \kappa_{2d+1}(\mathcal{G}(n,2d+1)) = 2\binom{n-3}{d-1}.$$

Based on these examples let us state:

Conjecture 7.4. Let $\mathcal{F} \subset 2^{[n]}$ with $\delta(\mathcal{F}) \leqslant u$. There exists an absolute constant c such that for $n > cu$,

$$\kappa_u(\mathcal{F}) \leqslant \binom{n-2}{d-1} \text{ for } u = 2d, \ \kappa_u(\mathcal{F}) \leqslant 2\binom{n-3}{d-1} \text{ for } u = 2d+1.$$

Acknowledgements

We thank the anonymous referees for their helpful comments.

References

[Bez87] Sergei Bezrukov. Specification of the maximal sized subsets of the unit cube with respect to given diameter. *Problems of Information Transmission*, 1:106–109, 1987.

[EKR61] P. Erdős, Chao Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12(1):313–320, 1961. doi:10.1093/qmath/12.1.313.

[Fra78] Peter Frankl. The Erdős-Ko-Rado theorem is true for $n = ckt$. In *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I*, volume 18 of *Colloq. Math. Soc. János Bolyai*, pages 365–375. North-Holland, Amsterdam-New York, 1978.

[Fra87] Peter Frankl. The shifting technique in extremal set theory. *Surveys in combinatorics*, 123:81–110, 1987.

[Fra17a] Peter Frankl. Antichains of fixed diameter. *Mosc. J. Comb. Number Theory*, 7(3):3–33, 2017.

[Fra17b] Peter Frankl. A stability result for the Katona theorem. *Journal of Combinatorial Theory, Series B*, 122:869–876, 2017. doi:10.1016/j.jctb.2016.10.003.

[Fra20a] Peter Frankl. An improved universal bound for t-intersecting families. *European J. Combin.*, 87:103134, 4, 2020. doi:10.1016/j.ejc.2020.103134.

[Fra20b] Peter Frankl. Maximum degree and diversity in intersecting hypergraphs. *Journal of Combinatorial Theory, Series B*, 144:81–94, 2020. doi:10.1016/j.jctb.2020.01.001.

[FW24] Peter Frankl and Jian Wang. Improved bounds on the maximum diversity of intersecting families. *European Journal of Combinatorics*, 118:103885, 2024. doi:10.1016/j.ejc.2023.103885.

[Hil76] AJW Hilton. The Erdős–Ko–Rado theorem with valency conditions. Unpublished manuscript, 1976.

[Kat64] Gyula Katona. Intersection theorems for systems of finite sets. *Acta Math. Acad. Sci. Hungar.*, 15:329–337, 1964. doi:10.1007/BF01897141.

[Kle66] Daniel J Kleitman. On a combinatorial conjecture of Erdős. *Journal of Combinatorial Theory*, 1(2):209–214, 1966. doi:10.1016/S0021-9800(66)80027-3.

[Kru63] Joseph B Kruskal. *The number of simplices in a complex*, volume 10, pages 251–278. University of California Press, 1963. doi:10.1525/9780520319875-014.

[Kup18] Andrey Kupavskii. Diversity of uniform intersecting families. *European Journal of Combinatorics*, 74:39–47, 2018. doi:10.1016/j.ejc.2018.07.005.

[LP08] Nathan Lemons and Cory Palmer. The unbalance of set systems. *Graphs and Combinatorics*, 24(4):361–365, 2008. doi:10.1007/s00373-008-0793-9.

[Spe28] Emanuel Sperner. Ein satz über untermengen einer endlichen menge. *Mathematische Zeitschrift*, 27(1):544–548, 1928. doi:10.1007/bf01171114.

[Wil84] Richard M. Wilson. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica*, 4(2-3):247–257, 1984. doi:10.1007/BF02579226.