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Hybrid Stabilization of Linear Systems with Reverse Polytopic Input Constraints

Pedro Casau, Ricardo G. Sanfelice, Carlos Silvestre

***Abstract*— This paper addresses the problem of globally uniformly exponentially stabilizing a linear system to the origin by output feedback while avoiding a prescribed set of input values. We consider that the set of input values to avoid is given by the union of a finite number of closed polytopes that do not contain the origin and we refer to this restriction as a reverse polytopic input constraint. We show that the synthesis of the hybrid controller can be performed by solving a set of linear matrix inequalities for the full state feedback case, and by solving a set of bilinear matrix inequalities for the output feedback case. The resulting closed-loop hybrid system is shown to satisfy key conditions for well-posedness and robustness to small measurement noise. Furthermore, we apply the proposed hybrid controller to the stabilization of a single-input linear system subject to reverse polytopic constraints on the norm of the input. The behavior of the corresponding closed-loop hybrid system is validated by simulations.**

I. Introduction

A. Background & Motivation

The development of controllers for systems subject to input constraints has been on the spotlight over the last few decades. In [1], it was shown that the global asymptotic stabilization of linear systems subject to saturation constraints by static state feedback is possible if and only if the system is small-input asymptotically null-controllable, that is, if there exists a globally stabilizing control law for arbitrarily small bounds on the input. Moreover, to find such controller the plant must not have eigenvalues with positive real part, as shown in [2]. Alternatively, it is possible to semi-globally exponentially stabilize input-constrained linear plants with no eigenvalues with positive real part using linear feedback, as shown in [3], [4], that is, for each compact set of initial conditions, there exists a linear feedback that exponentially stabilizes all solutions starting from that set, without violating the constraints. More recently, the problem of semi-global stabilization with rate and magnitude constraints on the input has been addressed in [5]. Finally, let us remark that a more computationally expensive method for the stabilization of linear systems subject to input constraints consists on the real-time computation of a finite input sequence that minimizes a given cost function and that belongs to a given (bounded) set [6], [7]. The set of input constraints is usually considered to be a polytope, because these sets can be described by a finite number of linear inequalities, rendering the optimization problem easier to solve.

The aforementioned control strategies can be used for asymptotic or semi-global stabilization of underactuated mechanical systems, for example. Unfortunately, solutions to these problems often overlook the situations where continuous control laws are ineffective (see e.g. [8, Eq. (25)]). The limitations of continuous feedback in the stabilization of dynamic systems were first recognized in [9] and [10], providing the motivation for the research of nonsmooth control techniques which is still a topic of active research. In this paper, we acknowledge the limitations of continuous feedback, and make use of novel hybrid control techniques to address the problem of stabilizing a linear system subject to reverse polytopic input constraints. This constraint can be used to enclose a set of input values that should be avoided by the controller. Particular applications where such constraints may be useful include: 1) the stabilization of a quadrotor vehicle, where one wishes to avoid the free-fall condition (c.f. [11]); 2) tracking for an autonomous surface vehicle, where the condition of zero thrust should be avoided (c.f. [12]); 3) spacecraft stabilization by means of two control moment gyros, where the gimbal-locking condition is to be avoided (c.f. [13]). We make extensive use of the hybrid systems framework in [14] which, in particular, provides conditions for the stability of a set for a hybrid system under the influence of small perturbations.

B. Contributions & Organization of the Paper

In Section III, we address the problem of designing a controller that globally uniformly exponentially stabilizes the origin of a linear plant subject to polyhedral input constraints. The proposed hybrid controller relies on appropriate switching between multiple linear controllers that must satisfy the following conditions: 1) each linear controller must be a stabilizing controller for the linear plant in the absence of input constraints, and; 2) there must be a large enough number of linear controllers to guarantee that, whenever the input approaches the unwanted polyhedra, another controller that does not violate the input constraints is available in the collection. Under the previous conditions, we show that the maximal solutions to the closed-loop system are complete, the origin of the linear plant is globally uniformly exponentially stable, and the stability is robust to small perturbations.

In Section IV, we show that if the region of input values to avoid is a polytope that does not contain the origin, then

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it is possible to stabilize the linear system and avoid the given input values by switching between two linear controllers. Interestingly, these controllers can be designed using controller synthesis tools from the literature of robust control. In Section IV-A, we demonstrate the applicability of the proposed strategy to the stabilization of a single-input system subject to constraints on the norm of the input and we compare the hybrid feedback that we propose with saturated feedback for the double integrator, under the influence of exogenous disturbances. In Section V, we provide some concluding remarks and, in the next section, we introduce some notation as well as fundamental results that are used throughout the paper. The paper [15] is a preliminary version of this work that deals with the problem of stabilizing a linear system subject to singular constraints of the input, rather than reverse polytopic constraints.

II. Preliminaries

$\mathbb{N}$ denotes the set of natural numbers and 0. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, with the inner product $\langle u,v\rangle = u^\top v$ defined for each $u,v\in\mathbb{R}^n$, and the induced norm $|v| = \sqrt{\langle v,v\rangle}$ for each $v\in\mathbb{R}^n$. For each $i\in\{1,2,\ldots,n\}$, $e_i\in\mathbb{R}^n$ is equal to zero, except the i-th entry, which is equal to 1 and $\mathbf{1}_n\in\mathbb{R}^n$ denotes a vector where each entry is equal to 1. Given $v,u\in\mathbb{R}^n$, we say that $v\preceq u$ ($v\succeq u$) if $e_i^\top v\leq e_i^\top u$ ($e_i^\top v\geq e_i^\top u$) for each $i\in\{1,2,\ldots,n\}$. Similarly, we say that $v\prec u$ ($v\succ u$) if $e_i^\top v< e_i^\top u$ ($e_i^\top v> e_i^\top u$) for each $i\in\{1,2,\ldots,n\}$. The set of $n\times m$ matrices with real entries is denoted by $\mathbb{R}^{n\times m}$. The set of $n\times n$ real symmetric matrices whose eigenvalues are positive (non-negative) is denoted by $\mathbb{S}^n_{>0}$ ($\mathbb{S}^n_{\geq 0}$). Similarly, the set of $n\times n$ real symmetric matrices whose eigenvalues are all negative (non-positive) is denoted by $\mathbb{S}^n_{<0}$ ($\mathbb{S}^n_{\leq 0}$). We use (v,u) in place of $\begin{bmatrix}v^\top & u^\top\end{bmatrix}^\top$ at times as each element (v,u) of a product space $\mathbb{R}^n\times\mathbb{R}^m$ can be identified with a column vector as follows: $(v,u)=\begin{bmatrix}v^\top & u^\top\end{bmatrix}^\top$.

The interior of a set $S\subset\mathbb{R}^n$ is denoted by $\operatorname{int} S$, its closure is denoted by $\operatorname{cl} S$, and its boundary is denoted by $\operatorname{bd} S$. The *polar set* of $S\subset\mathbb{R}^n$ is

$$S^\circ := \bigcap_{x\in S}\{y\in\mathbb{R}^n : \langle x,y\rangle\leq 1\}.$$

The convex hull of a set $S\subset\mathbb{R}^n$ is the set of all possible convex combinations of points within S and it is denoted by $\operatorname{conv}(S)$. The domain of a set-valued map $\mathcal{M}:\mathbb{R}^m\rightrightarrows\mathbb{R}^n$ is given by $\operatorname{dom}\mathcal{M} := \{x\in\mathbb{R}^m : \mathcal{M}(x)\neq\emptyset\}$. A polyhedra $\mathcal{P}_{(M,m)}\subset\mathbb{R}^p$ is given by

$$\mathcal{P}_{(M,m)} := \{u\in\mathbb{R}^p : Mu\preceq m\} \tag{1}$$

for some $(M,m)\in\mathbb{R}^{s\times p}\times\mathbb{R}^s$ and for some $s\in\mathbb{N}\backslash\{0\}$, see [16] for more information.

Lemma 1. *For each $(M,m)\in\mathbb{R}^{s\times p}\times\mathbb{R}^s$ and for each $\delta>0$, we have that $\mathcal{P}_{(M,m)}\subset\mathcal{P}_{(M,m+\mathbf{1}_s\delta)}$.*

Lemma 2. *For each $(M,m)\in\mathbb{R}^{s\times p}\times\mathbb{R}^s$ and for each $\delta>0$, if $\mathcal{P}_{(M,m)}\notin\{\emptyset,\mathbb{R}^p\}$ then*

$$\mathcal{P}_{(M,m+\mathbf{1}_s\delta)}\notin\{\emptyset,\mathbb{R}^p\}, \tag{2a}$$
$$\mathcal{P}_{(M,m)}\subset\operatorname{int}(\mathcal{P}_{(M,m+\mathbf{1}_s\delta)}). \tag{2b}$$

Proof. If $\mathcal{P}_{(M,m+\mathbf{1}_s\delta)}=\emptyset$, then it follows from Lemma 1 that $\mathcal{P}_{(M,m)}\subset\mathcal{P}_{(M,m+\mathbf{1}_s\delta)}=\emptyset$, which is a contradiction, since we have assumed that $\mathcal{P}_{(M,m)}\neq\emptyset$. On the other hand, if $\mathcal{P}_{(M,m+\mathbf{1}_s\delta)}=\mathbb{R}^p$, then $M=0$ and $m+\mathbf{1}_s\delta\succeq 0$ (any other combination of values for M and $m+\mathbf{1}_s\delta$ would generate either the empty set or a finite intersection of halfspaces (c.f. [16, Section 1.1]). Since $M=0$ in this case, it follows that $\mathcal{P}_{(M,m)}$ is equal to either $\mathbb{R}^p$ or $\emptyset$, which is a contradiction. This proves (2a). The boundary of $\mathcal{P}_{(M,m+\mathbf{1}_s\delta)}$ is given by

$$\operatorname{bd}\left(\mathcal{P}_{(M,m+\mathbf{1}_s\delta)}\right)=\bigcup_{i=1}^{s}\{u\in\mathbb{R}^p : e_i^\top Mu=e_i^\top(m+\mathbf{1}_s\delta)\}\cap\mathcal{P}_{(M,m+\mathbf{1}_s\delta)},$$

(c.f. [16, Theorem 8.2]). Since $\mathcal{P}_{(M,m+\mathbf{1}_s\delta)}$ is not equal to $\emptyset$ nor $\mathbb{R}^p$, then, from the observation that no points in the boundary of $\mathcal{P}_{(M,m+\mathbf{1}_s\delta)}$ belong to $\mathcal{P}_{(M,m)}$, it follows that $\mathcal{P}_{(M,m)}\subset\operatorname{int}(\mathcal{P}_{(M,m+\mathbf{1}_s\delta)})$. This concludes the proof of (2b). □

Lemma 3. *Given $(M,m)\in\mathbb{R}^{s\times p}\times\mathbb{R}^s$, the following holds:*

$$\mathcal{P}_{(M,m)}\cap\operatorname{cl}(\mathbb{R}^p\backslash\mathcal{P}_{(M,m+\mathbf{1}_s\delta)})=\emptyset \tag{3}$$

for each $\delta>0$.

Proof. Given $(M,m)\in\mathbb{R}^{s\times p}\times\mathbb{R}^s$, we have that

$$\operatorname{cl}(\mathbb{R}^p\backslash\mathcal{P}_{(M,m+\mathbf{1}_s\delta)})=\bigcup_{i=1}^{s}\{u\in\mathbb{R}^p : e_i^\top Mu\geq e_i^\top(m+\mathbf{1}_s\delta)\}$$

Therefore, for each $u\in\operatorname{cl}(\mathbb{R}^p\backslash\mathcal{P}_{(M,m+\mathbf{1}_s\delta)})$, it follows that $e_i^\top Mu\geq e_i^\top(m+\mathbf{1}_s\delta)$, for some $i\in\{1,2,\ldots,s\}$. However, if $u\in\mathcal{P}_{(M,m)}$ then it satisfies $e_i^\top Mu\leq e_i^\top m$, for each $i\in\{1,2,\ldots,s\}$; hence, since $\delta>0$, $u\in\mathbb{R}^p$ does not belong simultaneously to $\operatorname{cl}(\mathbb{R}^p\backslash\mathcal{P}_{(M,m+\mathbf{1}_s\delta)})$ and $\mathcal{P}_{(M,m)}$, which proves (3). □

Theorem 1. *Given $(M,m)\in\mathbb{R}^{s\times p}\times\mathbb{R}^s$ for some $s\in\mathbb{N}\backslash\{0\}$, if $\mathcal{P}_{(M,m)}$ is bounded, then there exists $a\in\mathbb{R}^s$ such that $a\succ 0$, $M^\top a=0$, and $\mathbf{1}_s^\top a=1$.*

Proof. Let $m_i=e_i^\top m$ for each $i\in\{1,2,\ldots,s\}$ and let $\widetilde{M}\in\mathbb{R}^{s\times p}$ be such that $e_i^\top\widetilde{M}=e_i^\top M/m_i$ for each $i\in\{1,2,\ldots,s\}$. Since $e_i^\top Mx\leq m_i \iff \frac{1}{m_i}e_i^\top Mx\leq 1$ for each $x\in\mathbb{R}^p$, then

$$\mathcal{P}_{(M,m)}=\mathcal{P}_{(\widetilde{M},\mathbf{1}_s)}. \tag{4}$$

From [16, Theorem 9.1] and (4), it follows that

$$\mathcal{P}^\circ_{(M,m)}=\operatorname{conv}\left(\{0,\widetilde{M}^\top e_1,\widetilde{M}^\top e_2,\ldots,\widetilde{M}^\top e_s\}\right).$$

It follows from [16, Theorem 6.1] and the assumption that $\mathcal{P}_{(M,m)}$ is bounded that $0\in\operatorname{int}(\mathcal{P}^\circ_{(M,m)})$. Thus, using the definition of convex hull, we see that $\mathcal{P}^\circ_{(M,m)}$ can be rewritten as follows

$$\mathcal{P}^\circ_{(M,m)}=\{x\in\mathbb{R}^p : x=\widetilde{M}^\top b \text{ for some } b\succeq 0,\ \mathbf{1}_s^\top b=1\}. \tag{5}$$

In particular, since $0\in\operatorname{int}(\mathcal{P}^\circ_{(M,m)})$, it follows that there exists $\epsilon>0$ such that $\epsilon\mathbb{B}:=\{x\in\mathbb{R}^p : |x|<\epsilon\}\subset\operatorname{int}(\mathcal{P}^\circ_{(M,m)})$. Hence, $\epsilon\frac{\widetilde{M}^\top e_i}{|\widetilde{M}^\top e_i|}\in\mathcal{P}^\circ_{(M,m)}$, which, from (5), implies the existence of $b_i\in\mathbb{R}^s$ satisfying $\widetilde{M}^\top b_i=-\epsilon\frac{\widetilde{M}^\top e_i}{|\widetilde{M}^\top e_i|}$, $b_i\succeq 0$ and $\mathbf{1}^\top b_i=1$ for each $i\in\{1,2,\ldots,s\}$. Let

$$\widetilde{a}_i := \frac{\left|\widetilde{M}^\top e_i\right|}{\epsilon\left(1+\frac{|\widetilde{M}^\top e_i|}{\epsilon}\right)}b_i+\frac{1}{\left(1+\frac{|\widetilde{M}^\top e_i|}{\epsilon}\right)}e_i$$

for each $i \in \{1,2,\dots,s\}$. It is straightforward to verify that $\widetilde{M}^\top \widetilde{a}_i = 0$, $\mathbf{1}_s^\top \widetilde{a}_i = 1$ and $\widetilde{a}_i \succeq 0$ for each $i \in \{1,2,\dots,s\}$. In particular, since $e_i^\top b_i \geq 0$ and $\epsilon > 0$, it follows that

$$e_i^\top \widetilde{a}_i \geq \frac{\epsilon}{\epsilon + \left|\widetilde{M}^\top e_i\right|} > 0,$$

for each $i \in \{1,2,\dots,s\}$. Then, defining $a := \frac{1}{s}\sum_{i=1}^s \widetilde{a}_i$, it follows that $\widetilde{M}^\top a = 0$, $\mathbf{1}_s^\top a = 1$ and $a \succ 0$, which concludes the proof. □

A hybrid system $\mathcal{H}$ with state space $\mathbb{R}^n$ is defined as follows:

$$\begin{aligned} \dot{\xi} &\in \mathcal{F}(\xi) \quad \xi \in \mathcal{C} \\ \xi^+ &\in \mathcal{G}(\xi) \quad \xi \in \mathcal{D} \end{aligned} \tag{6}$$

where $\mathcal{C} \subset \mathbb{R}^n$ is the flow set, $\mathcal{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map, $\mathcal{D} \subset \mathbb{R}^n$ denotes the jump set, and $\mathcal{G} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the jump map. A solution ξ to $\mathcal{H}$ is parametrized by (t,j), where t denotes ordinary time and j denotes the jump time, and its domain $\operatorname{dom}\xi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain: for each $(T,J) \in \operatorname{dom}\xi$, $\operatorname{dom}\xi \cap ([0,T] \times \{0,1,\dots J\})$ can be written in the form $\cup_{j=0}^{J-1}([t_j,t_{j+1}],j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$, where $I_j := [t_j, t_{j+1}]$ and the t_j's define the jump times. A solution ξ to a hybrid system is said to be *maximal* if it cannot be extended by flowing nor jumping and *complete* if its domain is unbounded. The projection of solutions onto the t direction is given by $\xi{\downarrow_t}(t) := \xi(t, \mathcal{J}(t))$ where $\mathcal{J}(t) := \max\{j : (t,j) \in \operatorname{dom}\xi\}$. If a hybrid system satisfies the so-called hybrid basic conditions, then its set of solutions has good structural properties, which, in particular, enabled the development of a robust stability theory for hybrid systems [14, Assumption 6.5]. Let $\mathcal{A} \subset \mathbb{R}^n$ denote a closed set and $|\xi|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |\xi - y|$. The set $\mathcal{A}$ is globally uniformly exponentially stable in the t-direction for the hybrid system $\mathcal{H}$ if each maximal solution ξ is complete with $\operatorname{dom}\xi$ unbounded in the t-direction and if there exist strictly positive real numbers k, λ such that each solution ξ satisfies

$$|\xi(t,j)|_{\mathcal{A}} \leq k \exp(-\lambda t)\, |\xi(0,0)|_{\mathcal{A}} \tag{7}$$

for each $(t,j) \in \operatorname{dom}\xi$. The gradient of a function $V : \mathbb{R}^n \to \mathbb{R}$ at ξ is denoted by $\nabla V(\xi)$ and the generalized directional derivative of V evaluated at ξ with direction v is denoted by $V^\bullet(\xi, v)$ (see [17] for more information).

Theorem 2. *If the hybrid system* (6) *satisfies the hybrid basic conditions [14, Assumption 6.5], then a closed set $\mathcal{A} \subset \mathbb{R}^n$ is globally uniformly exponentially stable in the t-direction for* (6) *if each maximal solution ξ to the hybrid system is complete and there exist positive real numbers $\underline{\alpha}, \overline{\alpha}, \eta, p$ and a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ that is locally Lipschitz on an open neighborhood of* $\operatorname{cl}(\mathcal{C})$ *satisfying*

$$\underline{\alpha}\, |\xi|_{\mathcal{A}}^p \leq V(\xi) \leq \overline{\alpha}\, |\xi|_{\mathcal{A}}^p \quad \forall \xi \in \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}(\mathcal{D}) \tag{8a}$$

$$V^\bullet(\xi, f) \leq -\eta V(\xi) \quad \forall \xi \in \mathcal{C},\ f \in \mathcal{F}(\xi) \tag{8b}$$

$$V(g) \leq V(\xi) \quad \forall \xi \in \mathcal{D},\ g \in \mathcal{G}(\xi) \tag{8c}$$

$$\mathcal{G}(\mathcal{D}) \cap \mathcal{D} = \emptyset. \tag{8d}$$

Proof. Let ξ denote a solution to (6) defined for each $(t,j) \in \operatorname{dom}\xi$ and, with a slight abuse of notation, let $V(t,j) := V(\xi(t,j))$. From (8a), we have that $V(t,j)$ is non-negative for each $(t,j) \in \operatorname{dom}\xi$. From (8b), it follows that V strictly decreases during flows and, from (8c), it follows that $V(t,j+1) \leq V(t,j)$ for each $(t,j), (t,j+1) \in \operatorname{dom}\xi$. From arguments similar to those in [18, Lemma C.1], it follows that $V(t,j) \leq V(0,0)\exp(-\eta t)$ for each $(t,j) \in \operatorname{dom}\xi$. From the set of inequalities (8a), we have that

$$|\xi(t,j)|_{\mathcal{A}} \leq \left(\frac{\overline{\alpha}}{\underline{\alpha}}\right)^{\frac{1}{p}} |\xi(0,0)|_{\mathcal{A}} \exp\left(-\frac{\eta}{p} t\right)$$

for each $(t,j) \in \operatorname{dom}\xi$. This implies that every solution to (6) is bounded and, since they are complete by assumption, we conclude that they are precompact, i.e., complete and bounded.

Since the hybrid system (6) satisfies the hybrid basic conditions and its solutions are precompact, it follows from [19, Lemma 2.7] that there exists $\varsigma > 0$ such that $t_{j+1} - t_j \geq \varsigma$ for each $(t_{j+1}, j), (t_j, j) \in \operatorname{dom}\xi$ with $j \geq 1$, thus $t \geq (j-1)\varsigma$ for each $(t,j) \in \operatorname{dom}\xi$. Consequently, we have that $t \geq (t+j)\frac{\varsigma}{1+\varsigma} - \varsigma$, thus $t \to \infty$ as $t + j \to \infty$, and $\operatorname{dom}\xi$ is unbounded in the t direction. □

Remark 1. The notion of uniform exponential stability discussed in this paper is borrowed from [20] and it reflects the fact that the bound (7) is satisfied uniformly over all possible jumps of the solutions to the hybrid system.

III. Controller Design

In this section, we address the problem of exponentially stabilizing

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{9}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $y \in \mathbb{R}^m$ denotes the output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{m \times n}$ denote the data of the system for some $m, n, p \in \mathbb{N}\backslash\{0\}$, and $u \in \mathbb{R}^p$ denotes the input, which is subject to reverse polytopic input constraints, defined next.

Problem 1. *Given a finite subset $\mathcal{M}$ of $\mathbb{N}$, a linear system* (9)*, and a family of closed convex polytopes $\{\mathcal{P}_{(M_i,m_i)}\}_{i \in \mathcal{M}}$ in $\mathbb{R}^p$ such that $0 \notin \mathcal{P}_{(M_i,m_i)}$ for each $i \in \mathcal{M}$, design a hybrid controller $\mathcal{H}_c = (\mathcal{C}_c, \mathcal{F}_c, \mathcal{D}_c, \mathcal{G}_c)$ with state variable $x_c \in \mathbb{R}^\ell$ satisfying*

$$\begin{aligned} \dot{x}_c &\in \mathcal{F}_c(x_c, y) \quad (x_c, y) \in \mathcal{C}_c \subset \mathbb{R}^\ell \times \mathbb{R}^m, \\ x_c &\in \mathcal{G}_c(x_c, y) \quad (x_c, y) \in \mathcal{D}_c \subset \mathbb{R}^\ell \times \mathbb{R}^m, \end{aligned} \tag{10}$$

and output $\kappa(x_c, y) \in \mathbb{R}^p$ for each $(x_c, y) \in \mathbb{R}^\ell \times \mathbb{R}^m$, such that

$$\widetilde{\mathcal{A}} := \{(x, x_c) \in \mathbb{R}^n \times \mathbb{R}^\ell : x = 0\} \tag{11}$$

is globally uniformly exponentially stable in the t-direction for the closed-loop system resulting from the interconnection between (9) *and* (10) *while satisfying the constraint*

$$\kappa(x_c, y) \notin \bigcup_{i \in \mathcal{M}} \mathcal{P}_{(M_i,m_i)} \tag{12}$$

for each $(x_c, y) \in \mathcal{C}_c$. □

To illustrate the applicability of a controller that is able to tackle reverse polytopic input constraints (12), we provide the following examples.

Example 1. Consider the full-state feedback problem for (9) with data

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and assume that $|u|$ should not take values in the interval $[0.5, 1.5]$. This constraint can be cast in the form of (12) with $\mathcal{M} = \{1, 2\}$ and

$$M_1 = -M_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^\top, \quad m_2 = m_1 = \begin{bmatrix} 1.5 & -0.5 \end{bmatrix}^\top.$$

□

Example 2. Consider the following system

$$\dot{p} = v, \quad \dot{v} = r\widetilde{u} + g$$

where $p \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ are the states of the system, $r \in \mathbb{R}^3$ satisfying $|r| = 1$ and $\widetilde{u} \in \mathbb{R}$ are the inputs and $g \in \mathbb{R}^3$ is a constant different than 0. This system models the evolution of the position and velocity of a quadrotor vehicle [8] and, by setting $r = (u - g)/|u - g|$ and $\widetilde{u} = |u - g|$ with $u \in \mathbb{R}^3$, it takes the form of (9) with

$$A = \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_3 \end{bmatrix},$$

as long as $u \neq g$. This constraint can be encompassed by (12) by setting $\mathcal{M} = \{1\}$, $M_1 = \begin{bmatrix} I_3 & -I_3 \end{bmatrix}^\top$ and $m_1 = \begin{bmatrix} g^\top & -g^\top \end{bmatrix}^\top$. □

A. Outline of Proposed Solution

To solve Problem 1, we propose a collection of N linear controllers of the form

$$\dot{z} = A_\sigma z + B_\sigma y \tag{13a}$$

$$u := \kappa_\sigma(z, y) := K_\sigma \begin{bmatrix} z \\ y \end{bmatrix}, \tag{13b}$$

where $z \in \mathbb{R}^k$ represents the controller state, $u \in \mathbb{R}^p$ is the input which is assigned to the control law $(z, y) \mapsto \kappa_\sigma(z, y)$, $y \in \mathbb{R}^m$ is the input of the controller, and $A_\sigma \in \mathbb{R}^{k\times k}$, $B_\sigma \in \mathbb{R}^{k\times m}$, $K_\sigma \in \mathbb{R}^{p\times(k+m)}$ for each $\sigma \in \mathcal{N} := \{1, 2, \ldots, N\}$. The design of an appropriate controller switching law guarantees that (12) is satisfied. Furthermore, we provide conditions on the parameters N and $(A_\sigma, B_\sigma, K_\sigma)$ for each $\sigma \in \mathcal{N}$, that guarantee global uniform exponential stability of (11) in the t-direction.

Given $(M_i, m_i) \in \mathbb{R}^{s_i\times p} \times \mathbb{R}^{s_i}$ with $s_i \in \mathbb{N}$ for each $i \in \mathcal{M}$, a family of N controllers as in (13) and $\delta^+ > 0$, we define the set-valued map

$$\rho(z, y) := \left\{\sigma \in \mathcal{N} : \kappa_\sigma(z, y) \in \mathcal{C}^+\right\} \tag{14}$$

for each $(z, y) \in \mathbb{R}^k \times \mathbb{R}^m$, where

$$\mathcal{C}^+ := \mathrm{cl}\left(\bigcap_{i\in\mathcal{M}} \left(\mathbb{R}^p \backslash \mathcal{P}_{(M_i, m_i+\mathbf{1}_{s_i}\delta^+)}\right)\right).$$

The map (14) identifies the index $\sigma \in \mathcal{N}$ of those controllers whose output $\kappa_\sigma(z, y)$ does not violate the given input constraints.

Defining $x_c := (z, \sigma) \in \mathbb{R}^k \times \mathcal{N}$ and $\Upsilon := \mathbb{R}^k \times \mathcal{N} \times \mathbb{R}^m$, we construct the hybrid controller

$$\mathcal{F}_c(x_c, y) := \begin{pmatrix} A_\sigma z + B_\sigma y \\ 0 \end{pmatrix} \quad \forall (x_c, y) \in \mathcal{C}_c := \left\{(x_c, y) \in \Upsilon : \kappa_\sigma(z, y) \in \mathcal{C}^-\right\}, \tag{15a}$$

$$\mathcal{G}_c(x_c, y) := (z, \rho(z, y)) \quad \forall (x_c, y) \in \mathcal{D}_c := \{(x_c, y) \in \Upsilon : \kappa_\sigma(z, y) \in \mathcal{D}^-\}, \tag{15b}$$

where

$$\mathcal{C}^- := \mathrm{cl}\left(\bigcap_{i\in\mathcal{M}} \left(\mathbb{R}^p \backslash \mathcal{P}_{(M_i, m_i+\mathbf{1}_{s_i}\delta^-)}\right)\right), \tag{16a}$$

$$\mathcal{D}^- := \bigcup_{i\in\mathcal{M}} \mathcal{P}_{(M_i, m_i+\mathbf{1}_{s_i}\delta^-)}. \tag{16b}$$

The interconnection between (9) and (15) results in the hybrid system $\mathcal{H} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ with state space $\mathcal{E} := \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{N}$ and dynamics

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \\ \dot{\sigma} \end{pmatrix} &= \mathcal{F}(x, x_c) := \begin{pmatrix} Ax + B\kappa_\sigma(z, Cx) \\ A_\sigma z + B_\sigma Cx \\ 0 \end{pmatrix} & (x, x_c) \in \mathcal{C} \\ \begin{pmatrix} x^+ \\ z^+ \\ \sigma^+ \end{pmatrix} &\in \mathcal{G}(x, x_c) := \begin{pmatrix} x \\ z \\ \rho(z, Cx) \end{pmatrix} & (x, x_c) \in \mathcal{D} \end{aligned} \tag{17}$$

with

$$\mathcal{C} := \{(x, x_c) \in \mathcal{E} : (x_c, Cx) \in \mathcal{C}_c\}, \tag{18a}$$

$$\mathcal{D} := \{(x, x_c) \in \mathcal{E} : (x_c, Cx) \in \mathcal{D}_c\}. \tag{18b}$$

Note that the reverse polytopic input constraint (12) is satisfied for the hybrid system (17), as shown next.

Lemma 4. *Let $\delta^- > 0$, condition (12) holds for each $(x_c, y) \in \mathcal{C}_c$ and $(z, x) \notin \bigcup_{i\in\mathcal{M}} \mathcal{P}_{(M_i^\sigma, m_i)}$ for each $(x, x_c) \in \mathcal{C}$, where, for each $i \in \mathcal{M}$,*

$$M_i^\sigma := M_i K_\sigma \begin{bmatrix} I_k & 0 \\ 0 & C \end{bmatrix}. \tag{19}$$

Proof. It follows from Lemma 2 that $\mathcal{P}_{(M,m)} \subset \mathrm{int}\left(\mathcal{P}_{(M,m+\mathbf{1}_s\delta)}\right)$ for each $(M, m) \in \mathbb{R}^{s\times p} \times \mathbb{R}^s$, $s \in \mathbb{N}\backslash\{0\}$ and $\delta > 0$. Thus

$$\bigcup_{i\in\mathcal{M}} \mathcal{P}_{(M_i, m_i)} \subset \mathrm{int}\left(\bigcup_{i\in\mathcal{M}} \mathcal{P}_{(M_i, m_i+\mathbf{1}_{s_i}\delta^-)}\right).$$

Since

$$\mathrm{cl}\left(\bigcap_{i\in\mathcal{M}} \left(\mathbb{R}^p \backslash \mathcal{P}_{(M_i, m_i+\mathbf{1}_{s_i}\delta^-)}\right)\right) = \mathbb{R}^p \backslash \mathrm{int}\left(\bigcup_{i\in\mathcal{M}} \mathcal{P}_{(M_i, m_i+\mathbf{1}_{s_i}\delta^-)}\right)$$

it follows that

$$\kappa_\sigma(z, y) \notin \mathrm{int}\left(\bigcup_{i\in\mathcal{M}} \mathcal{P}_{(M_i, m_i+\mathbf{1}_{s_i}\delta^-)}\right)$$

for each $(x_c, y) \in \mathcal{C}_c$, yielding the first part of the lemma.

From the definition of $\mathcal{P}_{(M,m)}$ in (1), the condition $\kappa_\sigma(z, Cx) \in \mathcal{P}_{(M,m)}$ is satisfied if and only if $MK_\sigma \begin{bmatrix} z \\ Cx \end{bmatrix} \preceq m$, thus (12) is equivalent to $(z, x) \notin \mathcal{P}_{(M_i^\sigma, m)}$ where M_i^σ is given by (19). □

B. Properties of Solutions to $\mathcal{H}$

To guarantee that maximal solutions to (17) are complete, one needs to guarantee that $\mathcal{D} \subset \mathrm{dom}\,\mathcal{G}$, which is to say: $\rho(z, y) \neq \emptyset$ for all $(x_c, y) \in \mathcal{D}_c$. Sufficient conditions that guarantee this property are provided next.

Lemma 5. *Given finite sets $\mathcal{N}, \mathcal{M} \subset \mathbb{N}$, $(M_i, m_i) \in \mathbb{R}^{s_i\times p} \times \mathbb{R}^{s_i}$ for each $i \in \mathcal{M}$, $K_\sigma \in \mathbb{R}^{p\times(k+m)}$ for each $\sigma \in \mathcal{N}$, and $\delta^+ > \delta^- > 0$, if, for each $(i, \sigma) \in \mathcal{M} \times \mathcal{N}$, there exists*

$\sigma' \in \mathcal{N}$ such that, for each $j \in \mathcal{M}$, there is a solution $\mu \succeq 0$ to

$$\mu^\top \begin{bmatrix} M_i^\sigma \\ M_j^{\sigma'} \end{bmatrix} = 0, \quad \mu^\top \begin{bmatrix} m_i + \mathbf{1}_{s_i}\delta^- \\ m_j + \mathbf{1}_{s_j}\delta^+ \end{bmatrix} < 0 \tag{20}$$

where M_i^σ is given by (19)*, then $\mathcal{D} \subset \operatorname{dom}\mathcal{G}$.* □

Proof. If, for each $(i,\sigma) \in \mathcal{M}\times\mathcal{N}$, there exists $\sigma' \in \mathcal{N}$ such that, for each $j \in \mathcal{M}$, the following set of inequalities is not feasible

$$M_i^\sigma \begin{bmatrix} z \\ x \end{bmatrix} \preceq m_i + \mathbf{1}_{s_i}\delta^-, \quad M_j^{\sigma'} \begin{bmatrix} z \\ x \end{bmatrix} \preceq m_j + \mathbf{1}_{s_j}\delta^+, \tag{21}$$

then, for each $(z,x) \in \mathcal{P}_{(M_i^\sigma, m_i + \mathbf{1}_{s_i}\delta^-)}$, there exists $\sigma' \in \mathcal{N}$ such that $(z,x) \notin \mathcal{P}_{(M_j^{\sigma'}, m_j + \mathbf{1}_{s_j}\delta^+)}$ for all $j \in \mathcal{M}$. The feasibility of the conditions in (21) may be cast as follows:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & M_i^\sigma w \preceq m_i + \mathbf{1}_{s_i}\delta^- \\ & M_j^{\sigma'} w \preceq m_j + \mathbf{1}_{s_j}\delta^+ \end{array} \tag{22}$$

where we have used $w := (z,x)$ for the sake of compactness. From [21, Section 5.2.2], it follows that (22) is not feasible if the dual problem is unbounded. The dual problem to (22) is:

$$\begin{array}{ll} \text{maximize} & -\mu^\top \begin{bmatrix} m_i + \mathbf{1}_{s_i}\delta^- \\ m_j + \mathbf{1}_{s_j}\delta^+ \end{bmatrix} \\ \text{subject to} & \mu^\top \begin{bmatrix} M_i^\sigma \\ M_j^{\sigma'} \end{bmatrix} = 0, \ \mu \succeq 0. \end{array} \tag{23}$$

From (20), it follows that there exists a feasible point μ to (23) such that

$$-\mu^\top \begin{bmatrix} m_i + \mathbf{1}_s\delta^- \\ m_j + \mathbf{1}_s\delta^+ \end{bmatrix} > 0.$$

Noticing that, for each $a > 0$, $a\mu$ is still a solution to (23), we conclude that (23) is unbounded provided that (20) hold. □

The conditions (20) ensure that $\mathcal{D} \subset \operatorname{dom}\mathcal{G}$, which is a key point in proving that $\mathcal{H}$ in (17) satisfies the hybrid basic conditions given next.

Lemma 6. *Given finite sets $\mathcal{N}, \mathcal{M} \subset \mathbb{N}$, $(M_i, m_i) \in \mathbb{R}^{s_i\times p} \times \mathbb{R}^{s_i}$ for each $i \in \mathcal{M}$, $K_\sigma \in \mathbb{R}^{p\times(k+m)}$ for each $\sigma \in \mathcal{N}$, and $\delta^+ > \delta^- > 0$, if* (20) *holds, then the hybrid system* (17) *satisfies the hybrid basic conditions [14, Assumption 6.5].*

Proof. The set $\mathcal{C}$ in (18a) is closed because $\mathcal{C}^-$ in (16a) is closed. The set $\mathcal{D}$ in (18b) is closed because $\mathcal{D}^-$ (16b) is the union of closed sets. The function $\mathcal{F}$ in (17) is single-valued, continuous, and defined for each $(x, x_c) \in \mathcal{C}$. The property $\mathcal{D} \subset \operatorname{dom}\mathcal{G}$ follows from Lemma 5 and local boundedness relative to $\mathcal{D}$ follows from the fact that ρ takes values on a compact set. To prove outer-semicontinuity, let $\{(x_j, z_j, \sigma_j)\}_{j\in\mathbb{N}}$ denote a sequence in $\mathcal{D}$ that converges to (x,z,σ), and $\{\sigma'_j\}_{j\in\mathbb{N}}$ denote a sequence satisfying $\sigma'_j \in \rho(z_j, Cx_j)$ for each $j \in \mathbb{N}$ converging to σ'. Suppose that $\sigma' \notin \rho(x,z,\sigma)$. Then, there exists $\ell \in \mathcal{M}$ such that $M_\ell^{\sigma'} \begin{bmatrix} z & Cx \end{bmatrix}^\top < m_\ell + \mathbf{1}_{s_\ell}\delta^+$, and, by continuity, $M_\ell^{\sigma'_j} \begin{bmatrix} z_j & Cx_j \end{bmatrix}^\top < m_\ell + \mathbf{1}_{s_\ell}\delta^+$, for sufficiently large j, where $M_\ell^{\sigma'_j}$ is given by (19). However, this contradicts that $\sigma'_j \in \rho(z_j, Cx_j)$ for each $j \in \mathbb{N}$, thus proving the outer semicontinuity of $\mathcal{G}$ relative to $\mathcal{D}$. □

Central to the proof of uniform exponential stability is the completeness of maximal solutions, because, in particular, if there exists $(x,z,\sigma) \in \mathcal{D}$ such that $\mathcal{G}(x,z,\sigma) \cap (\mathcal{C} \cup \mathcal{D}) = \emptyset$, then, since C is closed, the solution to the hybrid system cannot be extended, which is something we want to avoid.

Lemma 7. *Given finite sets $\mathcal{N}, \mathcal{M} \subset \mathbb{N}$, $(M_i, m_i) \in \mathbb{R}^{s_i\times p} \times \mathbb{R}^{s_i}$ for each $i \in \mathcal{M}$, $K_\sigma \in \mathbb{R}^{p\times(k+m)}$ for each $\sigma \in \mathcal{N}$, and $\delta^+ > \delta^- > 0$, if* (20) *holds, then every maximal solution to* (17) *is complete.*

Proof. This proof follows from an application of [14, Proposition 2.10]. □

C. Stability Properties of $\mathcal{H}$

Given $\{K_\sigma\}_{\sigma\in\mathcal{N}} \subset \mathbb{R}^{p\times(k+n)}$, it follows from (17) that the dynamics of the variables (x,z) during flows are described by

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \widetilde{A}_\sigma \begin{bmatrix} x \\ z \end{bmatrix} \tag{24}$$

where

$$\widetilde{A}_\sigma := \begin{bmatrix} A + BK_\sigma^y C & BK_\sigma^z \\ B_\sigma C & A_\sigma \end{bmatrix},$$

for each $\sigma \in \mathcal{N}$ and for some $\{K_\sigma^z\}_{\sigma\in\mathcal{N}} \subset \mathbb{R}^{p\times k}$, $\{K_\sigma^y\}_{\sigma\in\mathcal{N}} \subset \mathbb{R}^{p\times m}$ satisfying $K_\sigma := \begin{bmatrix} K_\sigma^z & K_\sigma^y \end{bmatrix}$. Next, we show that, using the controller (15), if the controller gain K_σ is selected so that the origin of the system (24) is exponentially stable, then the set

$$\mathcal{A} := \{(x, x_c) \in \mathcal{E} : x = z = 0\} \subset \widetilde{\mathcal{A}} \tag{25}$$

where $\widetilde{\mathcal{A}}$ is given by (11), is globally uniformly exponentially stable in the t-direction for the hybrid system (17). Moreover, the input constraint (12) is satisfied, provided that the conditions (20) hold.

Theorem 3. *Given finite sets $\mathcal{N}, \mathcal{M} \subset \mathbb{N}$ and $(M_i, m_i) \in \mathbb{R}^{s_i\times p} \times \mathbb{R}^{s_i}$ for each $i \in \mathcal{M}$, if there exist $P \in \mathbb{S}^{n+k}_{>0}$, $K_\sigma \in \mathbb{R}^{p\times(k+m)}$, $A_\sigma \in \mathbb{R}^{k\times k}$, $B_\sigma \in \mathbb{R}^{k\times m}$ and $\delta^-, \delta^+ \in \mathbb{R}$ satisfying $0 < \delta^- < \delta^+$ such that* (20) *holds and*

$$\widetilde{A}_\sigma^\top P + P\widetilde{A}_\sigma \in \mathbb{S}^{n+k}_{<0} \quad \forall \sigma \in \mathcal{N} \tag{26}$$

then the set $\mathcal{A}$ in (25) *is globally uniformly exponentially stable in the t-direction for system* (17) *and the constraint* (12) *holds.*

Proof. Since $P \in \mathbb{S}^{n+k}_{>0}$, the function

$$V(x,z,\sigma) := \begin{bmatrix} x^\top & z^\top \end{bmatrix} P \begin{bmatrix} x \\ z \end{bmatrix}$$

is positive definite relative to $\mathcal{A}$, and satisfies (8a) with $\underline{\alpha} = \lambda_{\min}(P)$ and $\overline{\alpha} = \lambda_{\max}(P)$. Moreover, since (26) holds, the derivative of V is negative definite relative to (25) and (8b) holds for some $\eta > 0$. Since during jumps only the variable σ changes, we have that $V(g) = V(x,z,\sigma)$ for each $g \in \mathcal{G}(x,z,\sigma)$ and for each $(x,z,\sigma) \in \mathcal{D}$. Thus, V is non-increasing on each jump and (8c) is satisfied.

Noting that jumps of the hybrid system map the output $\kappa_\sigma(z,y)$ from $\mathcal{D}^-$ to $\mathcal{C}^+$, it follows from $0 < \delta^- < \delta^+$ and Lemma 3 that $\mathcal{D}^- \cap \mathcal{C}^+ = \emptyset$ and, consequently, we have $\mathcal{G}(\mathcal{D}) \cap \mathcal{D} = \emptyset$; thus, (8d) holds, meaning that there is a lower bound on the time between jumps. The desired result follows from Theorem 2 by noticing that V is strictly decreasing during flows and the conditions of Lemmas 6 and 7 are satisfied by assumption. It follows from Lemma 4 that (12) holds along any solution to the closed-loop system. □

D. Robustness to "Outer Perturbations"

Let us consider the perturbation of $\mathcal{H} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ given by the hybrid system $\mathcal{H}_\Delta = (\mathcal{C}_\Delta, \mathcal{F}_\Delta, D_\Delta, G_\Delta)$ with data

$$\begin{aligned}\mathcal{F}_\Delta(x, x_c) \coloneqq \operatorname{conv}(&\mathcal{F}(((x, x_c) + \Delta\nu((x, x_c))\mathbb{B}) \cap \mathcal{C})) \\ &+ \Delta\nu((x, x_c))\mathbb{B} \quad \forall (x, x_c) \in \mathcal{C}_\Delta\end{aligned} \tag{27a}$$

$$\begin{aligned}\mathcal{G}_\Delta((x, x_c)) \coloneqq \{&(\bar{x}, \bar{x}_c) \in \mathbb{R}^n \times \mathbb{R}^\ell : (\bar{x}, \bar{x}_c) \in g + \Delta\nu(g)\mathbb{B}, \\ &g \in \mathcal{G}(((x, x_c) + \Delta\nu((x, x_c))\mathbb{B}) \cap \mathcal{D})\} \quad \forall (x, x_c) \in \mathcal{D}_\Delta\end{aligned} \tag{27b}$$

where

$$\begin{aligned}\mathcal{C}_\Delta &\coloneqq \{(x, x_c) \in \mathcal{E} : ((x, x_c) + \Delta\nu(x, x_c)\mathbb{B}) \cap \mathcal{C} \neq \emptyset\}, \\ \mathcal{D}_\Delta &\coloneqq \{(x, x_c) \in \mathcal{E} : ((x, x_c) + \Delta\nu(x, x_c)\mathbb{B}) \cap \mathcal{D} \neq \emptyset\},\end{aligned}$$

with $\Delta \in (0, 1)$ and $\nu : \mathbb{R}^{n+\ell} \to \mathbb{R}_{\geq 0}$ is a continuous function which outer perturbs the dynamics of the nominal hybrid system $\mathcal{H}$, capturing perturbations such as measurement noise, unmodeled dynamics and exogenous disturbances that are bounded by the perturbation function ν. If these perturbations are small enough, then the properties of the original system carry over to the perturbed hybrid system, as proved next.

Theorem 4. *Given finite sets $\mathcal{N}, \mathcal{M} \subset \mathbb{N}$ and $(M_i, m_i) \in \mathbb{R}^{s_i \times p} \times \mathbb{R}^{s_i}$ for each $i \in \mathcal{M}$, if there exist $P \in \mathbb{S}^{n+k}_{>0}$, $K_\sigma \in \mathbb{R}^{p \times (k+m)}$, $A_\sigma \in \mathbb{R}^{k \times k}$, $B_\sigma \in \mathbb{R}^{k \times m}$ and $\delta^-, \delta^+ \in \mathbb{R}$ satisfying $0 < \delta^- < \delta^+$ such that* (20) *and* (26) *hold, then there exist $k, \lambda > 0$ such that each solution ξ to $\mathcal{H}$ in* (17) *satisfies*

$$|\xi(t, j)|_\mathcal{A} \leq k \exp(-\lambda t)\, |\xi(0, 0)|_\mathcal{A} \tag{29}$$

for all $(t, j) \in \operatorname{dom}\xi$. Moreover, for each compact set $E \subset \mathcal{E}$ and $\epsilon > 0$, there exists $\Delta^\star > 0$ such that, for each $\Delta \in (0, \Delta^\star]$, the maximal solutions ξ_Δ to $\mathcal{H}_\Delta$ in (27) *from E satisfy*

$$|\xi_\Delta(t, j)|_\mathcal{A} \leq k \exp(-\lambda t)\, |\xi_\Delta(0, 0)|_\mathcal{A} + \epsilon$$

for all $(t, j) \in \operatorname{dom}\xi_\Delta$.

Proof. It follows from Lemma 6 that the hybrid system (17) satisfies the hybrid basic conditions. Thus, (27) also does (c.f. [22, Theorem 5.4]). Then, the theorem is a direct consequence [22, Theorem 6.6] and Theorem 3. □

In other words, the set $\mathcal{A}$ is said to be semi-globally practically asymptotically stable for the perturbed system (27), in the sense that, for each compact set of initial conditions and ultimate bound on the tracking error, $\epsilon > 0$, there exists a small enough Δ, such that the norm of each solution to (27) is upper bounded by the sum of ϵ and the exponential upper bound of the original system (29).

IV. Avoiding Polytopic Input Constraints with Two Controllers

We are able to solve Problem 1 using (15) with $\mathcal{N} = \{1, 2\}$ and controller gains K_1, K_2 satisfying $K_2 = \gamma K_1$ for some $\gamma \in \mathbb{R}$, provided that the following assumption holds.

Assumption 1. *Given $(M, m) \in \mathbb{R}^{s \times p} \times \mathbb{R}^s$, $\mathcal{P}_{(M,m)}$ is a compact set such that $0 \notin \mathcal{P}_{(M,m)}$.* □

In this direction, notice that for any bounded set $\mathcal{P}_{(M,m)} \subset \mathbb{R}^p$, it follows from Theorem 1 that there exists a vector $\mathrm{v} \in \mathbb{R}^s$ such that $M^\top \mathrm{v} = 0$ and $\mathrm{v} \succ 0$. Let $\underline{\mathrm{v}} : \mathbb{R}^s \times \mathbb{R}_{\geq 0} \to \mathbb{R}^s$ and $\overline{\mathrm{v}} : \mathbb{R}^s \times \mathbb{R}_{\geq 0} \to \mathbb{R}^s$ be functions that satisfy

$$e_j^\top \underline{\mathrm{v}}(\mathrm{v}, \delta^+) = \begin{cases} e_j^\top \mathrm{v} & \text{if } e_j^\top(m + \mathbf{1}_s \delta^+) < 0 \\ 0 & \text{otherwise}\end{cases} \tag{30a}$$

$$e_j^\top \overline{\mathrm{v}}(\mathrm{v}, \delta^+) = \begin{cases} e_j^\top \mathrm{v} & \text{if } e_j^\top(m + \mathbf{1}_s \delta^+) > 0 \\ 0 & \text{otherwise}\end{cases} \tag{30b}$$

for each $j \in \{1, 2, \ldots, s\}$. These quantities are used in the next corollary to compute the range of values of $\gamma \in \mathbb{R}$ that allow Problem 1 to be solved.

Corollary 1. *Given $(M, m) \in \mathbb{R}^{s \times p} \times \mathbb{R}^s$, let $\mathcal{N} \coloneqq \{1, 2\}$ and suppose that Assumption 1 holds, so that there exists $\mathrm{v} \in \mathbb{R}^s$ satisfying $M^\top \mathrm{v} = 0$ and $\mathrm{v} \succ 0$. Then, for each $\delta^-, \delta^+ \in \mathbb{R}$ satisfying*

$$0 < \delta^- < \delta^+ < -\frac{\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top m}{\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top \mathbf{1}_s}, \tag{31}$$

where $\underline{\mathrm{v}} : \mathbb{R}^s \times \mathbb{R}_{\geq 0} \to \mathbb{R}^s$ satisfies (30a)*, there exist $\overline{\gamma}, \underline{\gamma} \in \mathbb{R}$ satisfying $\underline{\gamma} < \overline{\gamma}$ such that, for each $\gamma \in (0, \underline{\gamma}) \cup (\overline{\gamma}, +\infty)$, if there exists $P \in \mathbb{S}^{n+k}_{>0}$ such that* (26) *holds with $K_2 = \gamma K_1$, then the set* (25) *is globally uniformly exponentially stable in the t-direction for* (17) *and the constraint* (12) *holds.*

Proof. The desired result follows from Theorem 3 by showing that the conditions (20) are satisfied for each $\sigma \in \mathcal{N} \coloneqq \{1, 2\}$. Using the fact that $K_2 = \gamma K_1$ and $\mu \coloneqq \begin{bmatrix}\mu_1^\top & \mu_2^\top\end{bmatrix}^\top$, we rewrite (20) for $\sigma = 1$ as follows:

$$\begin{cases} \begin{bmatrix} I_k & 0 \\ 0 & C^\top \end{bmatrix} K_1^\top M^\top (\mu_1 + \gamma\mu_2) = 0 \\ \mu_1, \mu_2 \succeq 0 \\ -\mu_1^\top (m + \mathbf{1}_s \delta^-) - \mu_2^\top (m + \mathbf{1}_s \delta^+) > 0 \end{cases} \tag{33}$$

Choosing $\mu_1 + \gamma\mu_2 = \mathrm{v}$, where $\mathrm{v} \in \mathbb{R}^s$ is such that $M^\top \mathrm{v} = 0$ and $\mathrm{v} \succ 0$, we have that the conditions (33) are satisfied if

$$-\gamma\mu_1^\top (m + \mathbf{1}_s \delta^-) - (\mathrm{v} - \mu_1)^\top (m + \mathbf{1}_s \delta^+) > 0, \tag{34}$$

since $\gamma > 0$ by assumption. Notice that, for each $\xi \in \mathcal{P}_{(M,m)}$, we have $M\xi \preceq m$ which, together with the fact that $\mathrm{v} \succ 0$, implies that

$$\mathrm{v}^\top m \geq 0. \tag{35}$$

For functions $\underline{\mathrm{v}}, \overline{\mathrm{v}}$ satisfying (30) it follows that $\mathrm{v} = \overline{\mathrm{v}}(\mathrm{v}, \delta^+) + \underline{\mathrm{v}}(\mathrm{v}, \delta^+)$ and from (35) we have that $\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top m \geq -\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top m$. Since $\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top m \leq \underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^+) \leq 0$ and $\delta^- > 0$, it follows that $\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-) > 0$. Then, condition (34) is equivalent to

$$\gamma < \frac{(\overline{\mathrm{v}}(\mathrm{v}, \delta^+) - \mathrm{v})^\top (m + \mathbf{1}_s \delta^+)}{\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-)} = -\frac{\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^+)}{\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-)},$$

when $\mu_1 = \overline{\mathrm{v}}(\mathrm{v}, \delta^+)$. Since $\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-) < \underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^+) < 0$, it follows that

$$\frac{\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^+)}{\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-)} < 0.$$

On the other hand, if $\mu_1 = \underline{\mathrm{v}}(\mathrm{v}, \delta^+)$, then conditions (33) are satisfied if

$$\gamma > -\frac{\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^+)}{\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-)}.$$

Similarly, conditions (20) with $\sigma = 2$ may also be verified with $\mu_1 = \underline{\mathrm{v}}(\mathrm{v}, \delta^+)$ or $\mu_1 = \overline{\mathrm{v}}(\mathrm{v}, \delta^+)$ provided that

$$\gamma < -\frac{\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-)}{\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^+)} \quad \text{or } \gamma > -\frac{\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-)}{\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^+)},$$

respectively. We conclude that the conditions (20) are satisfied for all $\gamma \in (0, \underline{\gamma}) \cup (\overline{\gamma}, +\infty)$ with

$$\underline{\gamma} := \min\left\{ -\frac{\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-)}{\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^+)}, -\frac{\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^+)}{\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-)} \right\} \tag{36a}$$

$$\overline{\gamma} := \max\left\{ -\frac{\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-)}{\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^+)}, -\frac{\overline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^+)}{\underline{\mathrm{v}}(\mathrm{v}, \delta^+)^\top (m + \mathbf{1}_s \delta^-)} \right\} \tag{36b}$$

Since condition (26) is satisfied, the desired result follows from Theorem 3. $\square$

It is important to note that, for any $\delta^-, \delta^+ \in \mathbb{R}$ satisfying $0 < \delta^- < \delta^+$, the jump set $\mathcal{D}$ and its image through the jump map $\mathcal{G}(\mathcal{D})$ do not intersect. This is a pivotal property in the proof of Corollary 1, because it guarantees that there are no discrete solutions to the hybrid system. In addition, by maximizing the difference $\delta^+ - \delta^-$, one increases the distance between the $\mathcal{P}_{(M, m+\mathbf{1}_s\delta^-)}$ and $\mathcal{P}_{(M, m+\mathbf{1}_s\delta^+)}$, making the closed-loop system less prone to chattering.

The parameters $\underline{\gamma}$ and $\overline{\gamma}$ determine how far γ must be from $\gamma = 1$ in order to satisfy the conditions (20). In the statement of Corollary 1, we do not provide $\underline{\gamma}$ nor $\overline{\gamma}$ explicitly, but (36) provides some conservative estimates for these bounds. More stricter values for these variables can be obtained numerically for each particular application.

Since $\underline{\gamma}, \overline{\gamma}$ in (36) do not depend on the controller gains, one may select γ prior to the design of the controller gains. Then, for a fixed γ, one computes the controller gains that satisfy (26) as follows: find $B_c \in \mathbb{R}^{k\times m}$, $A_c \in \mathbb{R}^{k\times k}$, $K^y \in \mathbb{R}^{p\times m}$, $K^z \in \mathbb{R}^{p\times k}$ and $P \in \mathbb{S}^{n+k}_{>0}$ such that (32) holds and it can be cast as an LMI optimization problem for the full-state feedback case [21]. For the output feedback case, the problem can be cast as BMI optimization problem [23]. The following example illustrates the previous remarks for the double integrator system.

Example 3. Using $M = \begin{bmatrix}1 & -1\end{bmatrix}^\top$ and $m = \begin{bmatrix}1.5 & -0.5\end{bmatrix}^\top$, we see that $\mathrm{v} = \mathbf{1}_2^\top$ satisfies $M^\top \mathrm{v} = 0$. From (30) and with this choice of v, we have that $\overline{\mathrm{v}}(\mathrm{v}, \delta^+) = \begin{bmatrix}1 & 0\end{bmatrix}^\top$ and $\underline{\mathrm{v}}(\mathrm{v}, \delta^+) = \begin{bmatrix}0 & 1\end{bmatrix}^\top$ for any $\delta^+ \in [0, 0.5)$. If $\delta^+ = 0.5$, then $\underline{\mathrm{v}}(\mathrm{v}, \delta^+) = 0$ and $\overline{\mathrm{v}}(\mathrm{v}, \delta^+) = \begin{bmatrix}1 & 0\end{bmatrix}^\top$. If $\delta^+ > 0.5$, then $\overline{\mathrm{v}}(\mathrm{v}, \delta^+) = \mathbf{1}_2$ and $\underline{\mathrm{v}}(\mathrm{v}, \delta^+) = 0$.

Choosing $\delta^+ = 0.1$, $\delta^- = 0.01$, it is straightforward to check that $\underline{\gamma} \approx 0.265$ and $\overline{\gamma} \approx 3.77$. Selecting $\gamma = 3.8$, we find a controller gain $K_1 \in \mathbb{R}^{1\times 2}$ that satisfies (32) using the strategy outlined in [21, Section 7.3.1] for LTI systems (with $B_w = B$), yielding $K_1 = \begin{bmatrix}-0.4834 & -0.9429\end{bmatrix}$. $\square$

A. Satisfying an Input Norm Constraint with Two Controllers

In this section, we consider the stabilization of the single-input system of the form (9) that is subject to norm constraints on the input: $|u| \notin \mathcal{P}_{(M,m)}$ for some closed segment satisfying Assumption 1.

Corollary 2. *Given $M_1 = -M_2 := \begin{bmatrix}1 & -1\end{bmatrix}^\top$ and $m_1 = m_2 = \begin{bmatrix}m^+ & m^-\end{bmatrix}^\top$, suppose that Assumption 1 holds and let $\mathcal{N} := \{1, 2\}$. Then, for each $\delta^-, \delta^+ \in \mathbb{R}$ satisfying*

$$0 < \delta^- < \delta^+ < -m^-, \tag{37}$$

there exist $\overline{\gamma}, \underline{\gamma} \in \mathbb{R}$ satisfying $\underline{\gamma} < \overline{\gamma}$ such that, for each $\gamma \in (0, \underline{\gamma}) \cup (\overline{\gamma}, +\infty)$, if (26) *holds, then the set* (25) *is globally uniformly exponentially stable in the t-direction for the closed-loop system resulting from the interconnection of* (9) *(with $p = 1$) and* (15) *with $K_2 = \gamma K_1$ and the constraint* (12) *holds.*

Proof. The desired result follows from Theorem 3 by showing that the conditions (20) are satisfied for each $\sigma \in \mathcal{N} := \{1, 2\}$ and K_1, K_2 as defined in the statement. It turns out that the conditions (20) are the same as (33), thus the desired results holds by noting that (37) equates to (31) when $\mathrm{v} = \mathbf{1}_2$. $\square$

As we have mentioned in Section I, it is possible to asymptotically stabilize the origin of double integrator in Example 1 without violating the input constraints $|u| \notin [0.5, 1.5]$, using the saturated feedback $u = -\lambda\, \mathrm{sat}(K_1 x/\lambda) + \mathrm{w}$, where $\mathrm{w} \in \mathbb{R}$ represents an exogenous disturbance, $\mathrm{sat}(x) := \max\{\min\{x, 1\}, -1\}$ and $\lambda \in (0, 0.5)$. In the sequel, we compare the behavior of the double integrator subject to the constraint $|u| \notin [0.5, 1.5]$ using the saturated control law and the hybrid controller developed in this section, by means of simulation results. We recall that the aforementioned input constraint can be cast as a reverse polytopic input constraint (12) with $\mathcal{M} := \{1, 2\}$, $M_1 = -M_2 = \begin{bmatrix}1 & -1\end{bmatrix}^\top$ and $m_1 = m_2 = \begin{bmatrix}1.5 & 0.5\end{bmatrix}^\top$, as shown in Example 1. Moreover, we make use of the controller data K_1, K_2, δ^+ and δ^- that was selected in Example 3. Fig. 1 depicts the behavior of the double integrator for the hybrid controller on the left and for the saturated controller on the right, using a saturation level $\lambda = 0.25$ and initial condition $(x_1(0,0), x_2(0,0), q(0,0)) = (-3.6205, 2.1210, 2)$. It is possible to observe that both controllers are able to drive the state towards the equilibrium point, but the hybrid controller provides a faster convergence rate, as expected. It is also possible to observe that there exists a positive lower bound for the time between switches of the controller, which is a consequence of $\mathcal{G}(\mathcal{D}) \cap \mathcal{D} = \emptyset$ and [19, Lemma 2.7], thus preventing Zeno solutions.

Fig. 2 depicts the behaviour of both closed-loop systems under the influence of a disturbance w for the same initial conditions as in the Fig 1. The disturbance w that was constructed following the ideas in [24] and its magnitude is given in Fig. 3. In this situation, both the hybrid controller and

$$\begin{aligned}
&\begin{bmatrix} A^\top + C^\top K^{y\top} B^\top & C^\top B_c^\top \\ K^{z\top} B^\top & A_c^\top \end{bmatrix} P + P \begin{bmatrix} A + BK^y C & BK^z \\ B_c C & A_c \end{bmatrix} \in \mathbb{S}^{n+k}_{<0} \\
&\begin{bmatrix} A^\top + \gamma C^\top K^{y\top} B^\top & C^\top B_c^\top \\ \gamma K^{z\top} B^\top & A_c^\top \end{bmatrix} P + P \begin{bmatrix} A + \gamma BK^y C & \gamma BK^z \\ B_c C & A_c \end{bmatrix} \in \mathbb{S}^{n+k}_{<0}.
\end{aligned} \tag{32}$$

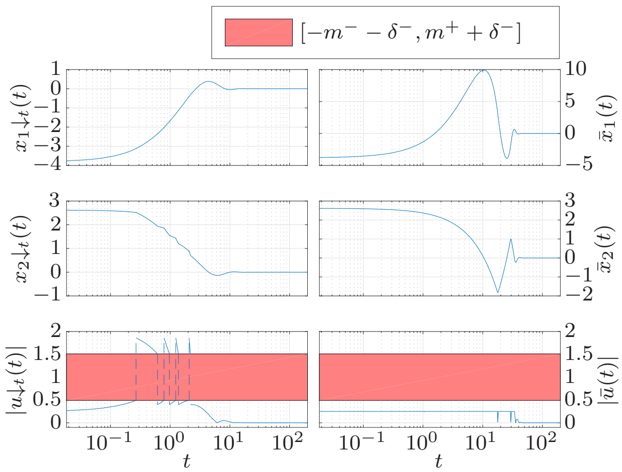

Fig. 1. Comparison between the hybrid controller on the left and the saturated feedback law on the right without exogenous disturbances. The dashed vertical lines represent jumps on the hybrid controller.

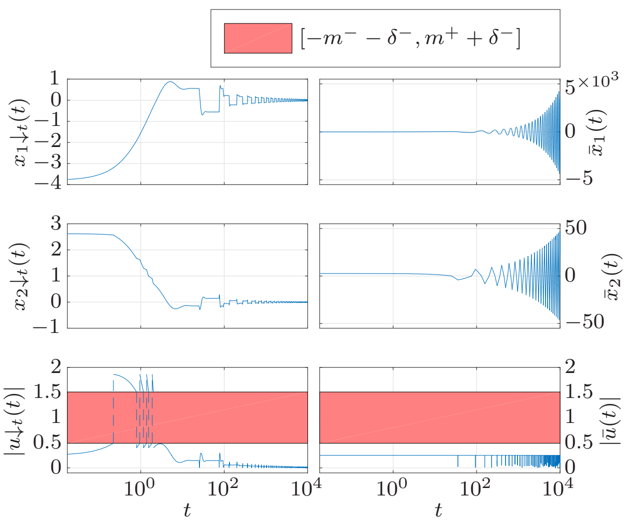

Fig. 2. Comparison between the hybrid controller on the left and the saturated feedback law on the right with exogenous disturbances. The dashed vertical lines represent jumps on the hybrid controller.

the saturated feedback law satisfy the input constraints, but, in the case of the saturated feedback law, the state is diverging.

V. Conclusion

In this paper, we presented a hybrid controller that is able to globally uniformly exponentially stabilize in the t-direction a linear system subject to reverse polytopic input constraints.

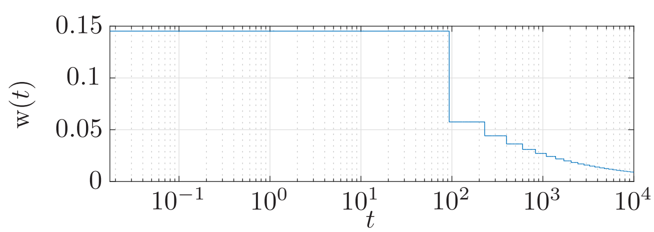

Fig. 3. Magnitude of the disturbance used in the simulation associated with Fig. 2.

We made use of a numerical example to illustrate that: 1) the controller design for the full state feedback case can be solved by means of linear matrix inequalities; 2) the controller parameters can be tuned to reduce chattering and increase robustness to measurement noise/disturbances; 3) the proposed hybrid controller can be applied to the stabilization of single-input systems subject to norm constraints on the input.

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