Title

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Journal

Combinatorial Theory, 3(3)

ISSN

2766-1334

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Publication Date

2023

DOI

10.5070/C63362800

Harmonic differential forms for pseudo-reflection groups II. Bi-degree bounds

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Submitted: Aug 24, 2022; Accepted: Nov 13, 2023; Published: Dec 22, 2023

Abstract. This paper studies three results that describe the structure of the super-coinvariant algebra of pseudo-reflection groups over a field of characteristic 0. Our most general result determines the top component in total degree, which we prove for all Shephard–Todd groups $G(m, p, n)$ with $m \neq p$ or $m = 1$. Our strongest result gives tight bi-degree bounds and is proven for all $G(m, 1, n)$, which includes the Weyl groups of types A and B/C. For symmetric groups (i.e. type A), this provides new evidence for a recent conjecture of Zabrocki related to the Delta Conjecture of Haglund–Remmel–Wilson. Finally, we examine analogues of a classic theorem of Steinberg and the Operator Theorem of Haiman.

Our arguments build on the type-independent classification of semi-invariant harmonic differential forms carried out in the first paper in this sequence. In this paper we use concrete constructions including Gröbner and Artin bases for the classical coinvariant algebras of the pseudo-reflection groups $G(m, p, n)$, which we describe in detail. We also prove that exterior differentiation is exact on the super-coinvariant algebra of a general pseudo-reflection group. Finally, we discuss related conjectures and enumerative consequences.

Keywords. Coinvariant algebras, pseudo-reflection groups, Gröbner basis, Artin basis, differential forms, exterior derivatives

Mathematics Subject Classifications. 05E16 (Primary), 20F55, 05A15 (Secondary)

1. Introduction

1.1. Overview of results

A polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n]$ is $\mathfrak{S}_n$-*harmonic* if it is annihilated by $\frac{\partial^k}{\partial x_1^k} + \cdots + \frac{\partial^k}{\partial x_n^k}$ for all $k \geqslant 1$. Here $\mathfrak{S}_n$ is the symmetric group on n elements. A classic result of Steinberg [Ste64] describes the harmonic polynomials as precisely those of the form $\partial_g \Delta_n$ where $\Delta_n \coloneqq \prod_{1 \leqslant i < j \leqslant n}(x_j - x_i)$ is the Vandermonde determinant and ∂_g is a polynomial in the partial derivatives $\frac{\partial}{\partial x_i}$. In particular, the top-degree harmonic polynomial is Δ_n, which transforms by the sgn representation of $\mathfrak{S}_n$.

Steinberg's result extends uniformly to an arbitrary *pseudo-reflection group* G. We more generally consider the problem of determining the *harmonic differential forms* of a pseudo-reflection group G. We have been motivated by Steinberg's result, Haiman's Operator Theorem [Hai02, Hai94], and a recent series of conjectures of Zabrocki [Zab19] and Haglund–Remmel–Wilson [HRW18] related to higher coinvariant algebras described in more detail below.

The paper [SW21] provides a starting point for such a description by giving a complete, type-independent construction of the $\det$-isotypic harmonic differential forms. In analogy with Steinberg's result, one may expect the $\det$-isotypic forms to be the "top" harmonics in some precise sense. Our main results are as follows:

1. The top total-degree forms are $\det$-isotypic for "almost all" irreducible pseudo-reflection groups $G = G(m, p, n)$, namely those with $p \neq m$ or $m = 1$ (Theorem 1.15).
2. The top bi-degree forms occur at the bi-degrees of the $\det$-isotypic elements for the groups $G(m, 1, n)$ (Theorem 1.12), which includes $\mathfrak{S}_n$ and the Coxeter groups of type B.
3. All harmonic forms are obtained by applying partial derivatives to $\det$-isotypic forms when the rank is $\leqslant 2$ and either $G = G(m, 1, n)$ or G is real (Theorem 1.6) and also for multiples of the volume form when $G = G(m, 1, n)$ (Theorem 1.7).
4. The $t = 0, z = -q$ specialization of the Hilbert series of the super-diagonal harmonics of the symmetric group agrees with Zabrocki's conjecture (see Section 1.10).

We also show that Theorem 1.6, Theorem 1.7, and Theorem 1.12 fail for certain G; see Remark 1.8 and Remark 1.13. We furthermore provide conjectures describing when they hold more broadly; see Conjecture 1.9 and Conjecture 1.16.

Our arguments rely on the following tools which may be of independent interest.

5. The exterior derivative cochain complex on super-harmonic differential forms (or super coinvariants) is exact for all pseudo-reflection groups G (Theorem 1.10).
6. The Artin and Gröbner bases for the coinvariant ideal for $G(m, p, n)$ (Section 5.1, Theorem 5.5, Theorem 5.6).

In the following subsections, we summarize classical properties of coinvariant algebras and harmonics for pseudo-reflection groups, introduce super analogues of these constructions, and state our main results and conjectures. See Section 2 and [SW21, §2] for additional detailed background.

1.2. Coinvariant algebras and harmonics in type A

Let $\mathfrak{S}_n$ be the symmetric group on n elements. The classical *coinvariant algebra* of $\mathfrak{S}_n$ is the quotient

$$\mathcal{R}_n := \mathbb{Q}[x_1, \ldots, x_n]/\mathcal{I}_n,$$

where $\mathcal{I}_n := \langle \mathbb{Q}[x_1, \ldots, x_n]_+^{\mathfrak{S}_n} \rangle$ is the *coinvariant ideal* generated by all homogeneous symmetric polynomials of positive degree. A great deal is known about the structure of $\mathcal{R}_n$ as a graded $\mathfrak{S}_n$-module [Sta79]. The top-degree component of $\mathcal{R}_n$ is the full subspace of elements which transform by $\sigma \cdot x = \operatorname{sgn}(\sigma)x$ and is spanned by the image of the Vandermonde determinant

$$\Delta_n := \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i).$$

The *harmonics* of $\mathcal{I}_n$ are

$$\mathcal{H}_n := \{f \in \mathbb{Q}[x_1, \ldots, x_n] : \partial_g f = 0 \text{ for all } g \in \mathcal{I}_n\}.$$

Here ∂_g is the partial differential operator defined by extending $x_i \mapsto \partial_{x_i} = \frac{\partial}{\partial x_i}$ multiplicatively and $\mathbb{Q}$-linearly. The harmonics $\mathcal{H}_n$ are the orthogonal complement of $\mathcal{I}_n$ under the natural positive-definite Hermitian form

$$(f, g) := \text{degree zero component of } \partial_g f, \tag{1.1}$$

so the natural projection $\mathcal{H}_n \to \mathcal{R}_n$ is a graded $\mathfrak{S}_n$-module isomorphism.

1.3. Coinvariant algebras and harmonics of pseudo-reflection groups

More generally, suppose $K \subset \mathbb{C}$ is a subfield closed under complex conjugation, $V = K^n$, and $G \subset \mathrm{GL}(V)$ is a *pseudo-reflection group*. That is, G is a finite group of unitary matrices generated by *pseudo-reflections*, which are non-identity transformations of finite order which fix a hyperplane pointwise.

The pseudo-reflection groups were famously classified by Shephard–Todd [ST54] into an infinite family $G(m, p, n)$ where $m, p, n \in \mathbb{Z}_{\geqslant 1}$ and $p \mid m$ together with 34 exceptional groups. The group $G(1, 1, n) = \mathfrak{S}_n$ consists of $n \times n$ permutation matrices. The group $G(m, 1, n)$ consists of $n \times n$ *pseudo-permutation matrices*, namely matrices such that each row and column has one non-zero entry which is an mth complex root of unity. In particular, $G(2, 1, n) = \mathfrak{B}_n$ is the Weyl group of signed permutations. Finally, the group $G(m, p, n)$ is the index-p normal subgroup of $G(m, 1, n)$ consisting of pseudo-permutation matrices where the product of the non-zero elements taken to the $\frac{m}{p}$th power is 1. In particular, $G(2, 2, n) = \mathfrak{D}_n$ is the Weyl group of signed permutations with an even number of signs.

The constructions in Section 1.2 generalize from $\mathfrak{S}_n$ to G for any pseudo-reflection group G (see Section 2). The *coinvariant algebra of* G and the G-*harmonics* are given as follows:

$$\begin{aligned}
\mathcal{R}_G &:= K[x_1, \ldots, x_n]/\mathcal{I}_G \qquad \text{where} \\
\mathcal{I}_G &:= \langle K[x_1, \ldots, x_n]_+^G \rangle \qquad \text{and} \\
\mathcal{H}_G &:= \{f \in K[x_1, \ldots, x_n] : \partial_g f = 0 \text{ for all } g \in \mathcal{I}_G\},
\end{aligned}$$

where ∂_g is defined by extending $x \mapsto \partial_{x_i} = \frac{\partial}{\partial x_i}$ multiplicatively and *conjugate*-linearly in K. For $G = G(m,p,n)$, the G-action is given by taking x_i to be the ith coordinate function on K^n. As before, $\mathcal{H}_G \to \mathcal{R}_G$ is an isomorphism of graded G-modules.

Chevalley [Che55] showed that $\mathcal{R}_G$, and hence $\mathcal{H}_G$, carries the regular representation. Furthermore, the top-degree component of the harmonics $\mathcal{H}_G$ is spanned by an element

$$\Delta_G \in K[x_1, \ldots, x_n]$$

which is unique up to non-zero scalar multiples and which transforms according to $g \cdot \Delta_G = \det(g)\Delta_G$. We call Δ_G the *Vandermondian* of G, since $\Delta_{\mathfrak{S}_n} = \Delta_n$.

It is clear from the definition that if f is harmonic, then so is $\partial_g f$. Steinberg showed that every G-harmonic can be obtained "from the top down" starting with Δ_G as follows.

Theorem 1.1 (Steinberg, [Ste64, Thm. 1.3(c)])**.** *For any pseudo-reflection group G,*

$$\mathcal{H}_G = K[\partial_{x_1}, \ldots, \partial_{x_n}]\Delta_G = \{\partial_g \Delta_G : g \in K[x_1, \ldots, x_n]\}.$$

1.4. Haiman's Operator Theorem

A famous extension of the classical coinvariant algebra for $G = \mathfrak{S}_n$ was introduced by Garsia and Haiman [GH93]. It involves two sets of commuting variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ with diagonal $\mathfrak{S}_n$-action given by $\sigma(x_i) \coloneqq x_{\sigma(i)}, \sigma(y_i) \coloneqq y_{\sigma(i)}$. The *diagonal coinvariants* and *diagonal harmonics* are $\mathfrak{S}_n$-modules bi-graded by x- and y-degree and are defined by

$$\begin{aligned}
\mathcal{DR}_n &\coloneqq \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]/\mathcal{K}_n \qquad &&\text{where} \\
\mathcal{K}_n &\coloneqq \langle \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]_+^{\mathfrak{S}_n} \rangle &&\text{and} \\
\mathcal{DH}_n &\coloneqq \{f \in \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n] : \partial_g f = 0 \text{ for all } g \in \mathcal{K}_n\}.
\end{aligned}$$

As usual, $\mathcal{DH}_n \cong \mathcal{DR}_n$ as bi-graded $\mathfrak{S}_n$-modules. Haiman conjectured and later proved the following description of the diagonal harmonics. Let $E_p \coloneqq \sum_{i=1}^n y_i \partial_{x_i}^p$.

Theorem 1.2 ([Hai02, Thm. 4.2])**.** *We have*

$$\mathcal{DH}_n = \mathbb{Q}[\partial_{x_1}, \ldots, \partial_{x_n}, E_1, \ldots, E_{n-1}]\Delta_n. \tag{1.2}$$

Haiman's proof of this theorem, which was originally conjectured in [Hai94, Conj. 5.1.1], involves the deep use of algebraic geometry. An elementary proof of the y-degree 1 component of Theorem 1.2 was given by Alfano [Alf98].

1.5. Super coinvariant algebras and harmonic differential forms

A recent conjecture of Zabrocki [Zab19] introduced the *super-diagonal coinvariant algebra* as a potential representation-theoretic model for the *Delta Conjecture* of Haglund–Remmel–Wilson [HRW18][1]. The $t = 0$ case of Zabrocki's conjecture sparked significant interest in the

[1]The "rise" version has now been independently proven by D'Adderio–Mellit [DM22] and Blasiak–Haiman–Morse–Pun–Seelinger [BHM+23].

following extension of the classical coinvariant algebras, which is our main object of interest. See [SW21, §1.2] for an overview of this and related work and further references. This paper studies the extension of the polynomial ring $K[x_1, \ldots, x_n]$ by adjoining *anti-commuting* variables $\theta_1, \ldots, \theta_n$ where $\theta_i\theta_j = -\theta_j\theta_i$ and $x_i\theta_j = \theta_j x_i$. Here θ_i is conceptually the differential 1-form $\mathrm{d}x_i$, so the products $\theta_{i_1} \cdots \theta_{i_k}$ represent differential k-forms $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$. We use the θ variables for consistency with existing literature. Since $\theta_i^2 = 0$, we often take $i_1 < \cdots < i_k$. Let $K[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]$ be the ring of differential forms with polynomial coefficients. The G-action on the θ_i is the same as on the x_i and is extended multiplicatively and K-linearly to $K[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]$, which is a G-module bi-graded by x-degree and θ-degree. Abstractly, this is the ring $\mathrm{S}(V^*) \otimes \wedge V^*$; see [SW21, §2] for details.

Definition 1.3. The *super coinvariant algebra* of a pseudo-reflection group G is the quotient

$$\mathcal{SR}_G \coloneqq K[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]/\mathcal{J}_G$$

where $\mathcal{J}_G \coloneqq \langle K[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]_+^G \rangle$ is the *super coinvariant ideal* generated by all bi-homogeneous G-invariant differential forms of positive total degree.

We write $\mathcal{SR}_G^k$ for the component of $\mathcal{SR}_G$ consisting of images of k-forms, i.e. the component of θ-degree k. Note that $\mathcal{SR}_G^0 = \mathcal{R}_G$, so the super coinvariant algebra of G contains the classical coinvariant algebra of G.

To define the harmonics in this context requires the extension of the partial differential operators ∂_g to differential forms. Let ∂_{θ_i} be the usual *interior product* defined by

$$\partial_{\theta_i}\theta_{i_1} \cdots \theta_{i_k} = \begin{cases} (-1)^{\ell-1}\theta_{i_1} \cdots \widehat{\theta_{i_\ell}} \cdots \theta_{i_k} & \text{if } i = i_\ell \\ 0 & \text{otherwise.} \end{cases}$$

Here ∂_{x_i} is θ-linear and ∂_{θ_i} is x-linear. Note that ∂_{x_i} and ∂_{θ_j} satisfy the same (anti-)commutation relations as the x_i and θ_j. For $\omega \in K[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]$, let ∂_ω be obtained by replacing each x_i with ∂_{x_i}, replacing each θ_j with ∂_{θ_j}, reversing the order of the θ's, and taking the conjugate of the coefficients. These twists ensure the extension of (1.1) remains positive-definite.

Definition 1.4. The *super harmonics* of a pseudo-reflection group G are

$$\mathcal{SH}_G \coloneqq \{\eta \in K[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n] : \partial_\omega \eta = 0 \text{ for all } \omega \in \mathcal{J}_G\}.$$

The natural projection $\mathcal{SH}_G \to \mathcal{SR}_G$ is again an isomorphism of bi-graded G-modules, so $\mathcal{SR}_G$ and $\mathcal{SH}_G$ are frequently interchangeable.

1.6. The $\det$-isotypic harmonic differential forms

From Steinberg's Theorem 1.1, the $\det$-isotypic component of the harmonics,

$$\mathcal{H}_G^{\det} \coloneqq \{f \in \mathcal{H}_G : \sigma \cdot f = \det(\sigma) f\} = \mathrm{Span}_K\{\Delta_G\},$$

plays a special role. In [SW21], the authors gave a "top-down" construction of $\mathcal{SH}_G^{\det}$ in the spirit of Steinberg's Theorem 1.1 and Haiman's Theorem 1.2 using certain differential operators $\mathrm{d}_1, \ldots, \mathrm{d}_r \in \mathrm{End}_K(\mathcal{SH}_G)$. Here $r \coloneqq \dim(V/V^G)$. We call these operators *generalized exterior derivatives* since in general we may take

$$\mathrm{d}_1 = \mathrm{d} \coloneqq \sum_{i=1}^n \frac{\partial}{\partial x_i}\theta_i \tag{1.3}$$

to be the *exterior derivative*, where θ_i here denotes left multiplication by θ_i. If $e_1^*, \ldots, e_r^*$ are the positive *co-exponents* of G, then d_i lowers x-degree by e_i^* and raises θ-degree by 1. See Section 2.3 for details and Table 7.1 for explicit descriptions of $\mathrm{d}_1, \ldots, \mathrm{d}_r$ for $G(m, p, n)$. Table 7.1 lists the real cases $G = \mathfrak{S}_n, \mathfrak{B}_n, \mathfrak{D}_n, \mathfrak{Dih}_{2n}$ explicitly.

The main result of [SW21] gives a basis of 2^r elements for the $\det$-isotypic component of the super harmonics.

Theorem 1.5 ([SW21, Thm. 5.7])**.** *Let G be a pseudo-reflection group. Then*

$$\mathcal{SH}_G^{\det} = \mathrm{Span}_K\{\mathrm{d}_{i_1} \cdots \mathrm{d}_{i_k}\Delta_G : 1 \leqslant i_1 < \cdots < i_k \leqslant r\}. \tag{1.4}$$

1.7. Differential operator results

The partial derivatives ∂_g for $g \in K[x_1, \ldots, x_n]$ clearly preserve $\mathcal{SH}_G$, so $\partial_g \mathcal{SH}_G^{\det} \subset \mathcal{SH}_G$. Motivated by Steinberg's Theorem 1.1 and Haiman's Theorem 1.2, we show that the following reverse containments hold.

Theorem 1.6. *Let $G \subset \mathrm{GL}(V)$ be a pseudo-reflection group with rank $r = \dim(V/V^G)$. Then if $r \leqslant 2$ and either $G = G(m, 1, n)$ or G is real,*

$$\begin{aligned} \mathcal{SH}_G &= K[\partial_{x_1}, \ldots, \partial_{x_n}]\mathcal{SH}_G^{\det} \\ &= \{\partial_g \mathrm{d}_{i_1} \cdots \mathrm{d}_{i_k}\Delta_G : g \in K[x_1, \ldots, x_n], i_j \in [r]\}. \end{aligned} \tag{1.5}$$

Theorem 1.7. *Let $G = G(m, 1, n)$ or let G be real. Then the θ-degree r component of (1.5) holds.*

Remark 1.8. In Lemma 3.8, we show that in fact the $\mathcal{SH}_G^r$ component of (1.5) holds for $G = G(m, p, n)$ if and only if $r \leqslant 2$, $G = G(m, 1, n)$, or $G = G(2, 2, n)$. Hence (1.5) cannot possibly hold for any groups in the infinite family $G(m, p, n)$ beyond these. However, computational data shows that (1.5) *fails* for $G = \mathfrak{D}_4$ and $\mathfrak{D}_5$. It appears likely to fail for $\mathfrak{D}_n$ with $n \geqslant 4$. On the other hand, we have verified that (1.5) holds for $\mathfrak{S}_n$ with $n \leqslant 6$, $\mathfrak{B}_n$ with $n \leqslant 4$, $G(3, 1, 4)$, and $G(5, 1, 3)$, among others. See Table 7.2 for additional data. We also note that the $\mathcal{SH}_{\mathfrak{S}_n}^1$ case of (1.5) is equivalent to the y-degree 1 case of Haiman's Theorem 1.2. Consequently, we are lead to the following conjecture.

Conjecture 1.9 (Differential Operator Conjecture[2])**.** If $G = G(m, 1, n)$, then (1.5) holds.

[2]The type A case of this conjecture has now been proven by Rhoades–Wilson [RW23]. See Remark 1.19 for further discussion.

Theorem 1.6 includes the dihedral groups $G(m, m, 2) = \mathfrak{Dih}_{2m}$ $(m \geqslant 1)$ and 6 exceptional groups. We do not have a complete determination or conjecture for the exceptional groups for which (1.5) holds. It does hold for H_3, though perhaps surprisingly it fails for F_4.

Our proof of Theorem 1.6 is mostly uniform and relies on the following result which may be of independent interest. See Section 1.10 for additional consequences.

Theorem 1.10. *For any pseudo-reflection group $G \subset \mathrm{GL}(V)$ with $r = \dim(V/V^G)$, the exterior derivative cochain complex*

$$0 \to K \to \mathcal{SH}_G^0 \xrightarrow{\mathrm{d}} \mathcal{SH}_G^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{SH}_G^r \xrightarrow{\mathrm{d}} 0 \tag{1.6}$$

is exact.

The complex in Theorem 1.10 is a finite-dimensional, algebraic analogue of the de Rham complex of a smooth manifold. Exactness is proved with an analogue of Hodge theory using total Laplacians. See Section 4 for the proofs of Theorem 1.6 and Theorem 1.10.

1.8. Bi-degree bound results

When (1.5) holds, the harmonics

$$\mathrm{d}_{i_1} \cdots \mathrm{d}_{i_k} \Delta_G$$

are the "top-most" elements of $\mathcal{SH}_G$. In particular, it implies that for each k, the top x-degree elements of $\mathcal{SH}_G$ belong to $\mathcal{SH}_G^{\det}$ and fully describes the non-zero bi-degree components of $\mathcal{SH}_G$ as follows. Write $\mathcal{SH}_G^{i,k}$ for the x-degree i, θ-degree k component of $\mathcal{SH}_G$.

Lemma 1.11. *Let $G \subset \mathrm{GL}(V)$ have rank $r = \dim(V/V^G)$ and co-exponents $1 \leqslant e_1^* \leqslant \cdots \leqslant e_r^*$. Suppose (1.5) holds. Then*

$$\mathcal{SH}_G^{i,k} \neq 0 \quad \Leftrightarrow \quad \begin{array}{l} 0 \leqslant k \leqslant r \text{ and} \\ 0 \leqslant i \leqslant \deg \Delta_G - e_1^* - \cdots - e_k^*. \end{array} \tag{1.7}$$

Moreover, if $i = \deg \Delta_G - e_i^* - \cdots - e_k^*$, then $\mathcal{SH}_G^{i,k} \subset \mathcal{SH}_G^{\det}$.

Our strongest result is to show that the following consequence of Conjecture 1.9 is true unconditionally. In particular, it verifies the predicted bi-degree support of Zabrocki's super coinvariant algebra conjecture when $t = 0$, providing additional evidence for that conjecture.

Theorem 1.12. *Let $G = G(m, 1, n)$. Then $\mathcal{SH}_{G(m,1,n)}^{i,k} \neq 0$ if and only if $i, k \geqslant 0$ and*

$$i + k + m\binom{k}{2} \leqslant m\binom{n}{2} + (m-1)n. \tag{1.8}$$

Remark 1.13. Note that Lemma 1.11 may hold even when (1.5) fails. By Table 7.2, this indeed occurs for $\mathfrak{D}_4, \mathfrak{D}_5, F_4$. In Lemma 3.8, we show that the $\mathcal{SH}_G^r$ component of (1.7) holds for $G(m, p, n)$ if and only if $r \leqslant 2$, $G = G(m, 1, n)$, or $G = G(2, 2, n) = \mathfrak{D}_n$. In contrast to Conjecture 1.9, our data do not rule out the possibility that (1.7) holds for $\mathfrak{D}_n$.

In [SS22], the exact Hilbert series for $\mathcal{SH}_{G(m,1,n)}^{n-k}$ is conjectured in terms of generalized ordered q-Stirling numbers, which is consistent with Theorem 1.12.

Our proof of Theorem 1.12 uses Gröbner and Artin bases of $\mathcal{R}_{G(m,1,n)}$ developed in Section 5. See Section 6 for the proof of Theorem 1.12.

1.9. Total degree bound results

The bi-degree support and top components from Lemma 1.11 imply the following total degree support and top components of $\mathcal{SH}_G$, or equivalently $\mathcal{SR}_G$.

Lemma 1.14. *Let G be a pseudo-reflection group. Suppose Lemma 1.11 holds. Then*

$$\bigoplus_{i+k=d} \mathcal{SH}_G^{i,k} \neq 0 \quad \Leftrightarrow \quad 0 \leqslant d \leqslant \deg \Delta_G. \tag{1.9}$$

Moreover,

$$\bigoplus_{i+k=\deg \Delta_V} \mathcal{SH}_G^{i,k} \subset \mathcal{SH}_G^{\det}. \tag{1.10}$$

We show that this weaker description is true even in many cases where Lemma 1.11 fails to hold. Our most general result is the following.

Theorem 1.15. *Let $G = G(m,p,n)$ with $p \neq m$ or $p = 1$. Then*

$$\bigoplus_{i+k=\ell} \mathcal{SH}_{G(m,p,n)}^{i,k} \neq 0 \quad \Leftrightarrow \quad 0 \leqslant \ell \leqslant m\binom{n}{2} + n\left(\frac{m}{p} - 1\right).$$

Moreover, if $\ell = m\binom{n}{2} + n\left(\frac{m}{p}-1\right)$, then

$$\bigoplus_{i+k=\ell} \mathcal{SH}_{G(m,p,n)}^{i,k} = (K + K\mathrm{d}) \prod_{1\leqslant i<j\leqslant n} (x_j^m - x_i^m) \cdot (x_1 \cdots x_n)^{m/p-1}.$$

Our proof of Theorem 1.15 again uses Gröbner and Artin bases of $\mathcal{R}_{G(m,1,n)}$ developed in Section 5. See Section 7 for the proof of Theorem 1.15.

Our results and computations have uncovered no cases in which Theorem 1.15 fails to hold. Indeed, our argument shows that it must hold for all $G = G(m,p,n)$ except possibly when $m = p > 1$ and $k \in \{n-1, n-2\}$. Consequently, we conjecture the following.

Conjecture 1.16 (Total Degree Bounds Conjecture)**.** Let G be a pseudo-reflection group. Then

$$\bigoplus_{i+k=d} \mathcal{SH}_G^{i,k} \neq 0 \quad \Leftrightarrow \quad 0 \leqslant d \leqslant \deg \Delta_G.$$

Moreover, $\bigoplus_{i+k=\deg \Delta_V} \mathcal{SH}_G^{i,k} \subset \mathcal{SH}_G^{\det}$.

1.10. Hilbert series considerations

We finish this introduction with some additional enumerative consequences of the preceding results which support some further conjectures.

The *Hilbert series* of $\mathcal{SR}_G^k$ and $\mathcal{SR}_G$ are

$$\mathrm{Hilb}(\mathcal{SR}_G^k; q) := \sum_{i \geqslant 0} q^i \dim_K(\mathcal{SR}_G^{i,k})$$

$$\mathrm{Hilb}(\mathcal{SR}_G; q, z) := \sum_{k \geqslant 0} z^k \,\mathrm{Hilb}(\mathcal{SR}_G^k; q),$$

where q tracks x-degree and z tracks θ-degree. An immediate consequence of Theorem 1.10 is the following enumerative corollary.

Corollary 1.17. *For any pseudo-reflection group G,*

$$\mathrm{Hilb}(\mathcal{SR}_G; q, -q) = \sum_{k \geqslant 0} (-q)^k \,\mathrm{Hilb}(\mathcal{SR}_G^k; q) = 1.$$

Our overarching goal has been to provide evidence for Zabrocki's conjecture [Zab19] for the tri-graded Frobenius series of the type A super-diagonal coinvariant algebra. In particular, we may check Zabrocki's conjecture against Corollary 1.17.

To do so, let $\mathrm{Stir}_q(n, k)$ be a *q-Stirling number of the second kind* [Car48, §3], defined recursively by $\mathrm{Stir}_q(n, k) = \mathrm{Stir}_q(n-1, k-1) + [k]_q \,\mathrm{Stir}_q(n-1, k)$ and $\mathrm{Stir}_q(0, k) = \delta_{k=0}$. Here $[k]_q := 1 + q + \cdots + q^{k-1}$ is a *q-integer* and $\delta_P = 1$ if P is true and 0 otherwise. Zabrocki's conjecture specializes as follows.

Conjecture 1.18 (Zabrocki, [Zab19])**.** For $0 \leqslant k \leqslant n$,

$$\mathrm{Hilb}(\mathcal{SR}_{\mathfrak{S}_n}^k; q) = [n-k]_q! \,\mathrm{Stir}_q(n, n-k).$$

Remark 1.19. Since this paper was written, Rhoades–Wilson [RW23] have proven Conjecture 1.18. Somewhat surprisingly, the argument simultaneously proves the type A case of Conjecture 1.9. A key insight in [RW23] involves the construction of certain families of regular sequences, which refines the upper bound argument in Theorem 1.12.

Based on computational evidence including Table 7.2, the first author has introduced a type B analogue of Conjecture 1.18. Let $\mathrm{Stir}_q^B(n, k)$ be a *type B q-Stirling number of the second kind*, defined recursively by $\mathrm{Stir}_q^B(n, k) = \mathrm{Stir}_q^B(n-1, k-1) + [2k+1]_q \,\mathrm{Stir}_q^B(n-1, k)$ and $\mathrm{Stir}_q^B(0, k) = \delta_{k=0}$. The type B analogue of Conjecture 1.18 is as follows.

Conjecture 1.20. For $0 \leqslant k \leqslant n$,

$$\mathrm{Hilb}(\mathcal{SR}_{\mathfrak{B}_n}^k; q) = [n-k]_{q^2}! [2]_q^{n-k} \,\mathrm{Stir}_q^B(n, n-k).$$

Consequently, Corollary 1.17 leads us to the following conjecture. It has been proven in [SS23] both algebraically and with a sign-reversing involution.

Conjecture 1.21. We have

$$\sum_{k=0}^{n} (-q)^k [n-k]_q! \,\mathrm{Stir}_q(n, n-k) = 1 = \sum_{k=0}^{n} (-q)^k [n-k]_{q^2}! [2]_q^{n-k} \,\mathrm{Stir}_q^B(n, n-k).$$

Conjectured monomial bases for $\mathcal{SR}_{\mathfrak{S}_n}$ and $\mathcal{SR}_{\mathfrak{B}_n}$ are also given in [SS23].

One may obtain a different complex from (1.6) by replacing the differentials d with d_i for $1 \leqslant i \leqslant r$, though the result is typically not exact. The graded Euler characteristic of the complex is

$$\chi(H^*(\mathcal{SH}_G, \mathrm{d}_i); q) = \mathrm{Hilb}(\mathcal{SR}_G; q, -q^{e_i^*}) - 1.$$

A potential approach to Conjecture 1.18 and Conjecture 1.20 is to find homotopic complexes with the correct Euler characteristic. More concretely, Conjecture 1.18 is equivalent to the following variation on Corollary 1.17.

Conjecture 1.22. For $1 \leqslant j \leqslant n-1$,

$$\mathrm{Hilb}(\mathcal{SR}_{\mathfrak{S}_n}; q, -q^j) = \sum_{k=0}^{n} (-q^j)^k [n-k]_q! \,\mathrm{Stir}_q(n, n-k).$$

1.11. Paper organization

The second author previously presented the heart of the argument for Theorem 1.12 in [Wal19]. The present work extends, generalizes, and supercedes [Wal19].

Section 2 gives background on polynomial differential forms, generalized exterior derivatives, and the invariant theory of pseudo-reflection groups. Section 3 analyzes the structure of the top θ-degree component of $\mathcal{SH}_G$ and proves Theorem 1.7. Section 4 set up our algebraic Hodge theory argument proving exactness of exterior differentiation on the harmonics, Theorem 1.10, as well as Theorem 1.6. In Section 5 we describe the Artin and Gröbner bases for $G(m,p,n)$; see Theorem 5.5 and Theorem 5.6. Section 6 proves the bi-degree bounds in Theorem 1.12. Section 7 considers the top total degree and proves Theorem 1.15.

2. Background

2.1. Polynomial differentials

We first briefly introduce the standard G-module structures and differential operators underlying our results. See [SW21, §2] for a general, abstract version. The following concrete version is included in the spirit of much of the combinatorics literature and is intended to make these developments more accessible.

Let $K \subset \mathbb{C}$ be a subfield closed under complex conjugation. Let $V = K^n$. Suppose $G \subset U(n, K)$ is a subgroup of unitary matrices, so $(\sigma^{-1})_{ij} = \overline{\sigma}_{ji}$ with σ_{ij} defined by $\sigma \cdot f_j = \sum_{i=1}^n \sigma_{ij} f_i$ where $f_1, \ldots, f_n$ is the standard orthonormal basis. Let $x_1, \ldots, x_n \in V^* := \mathrm{Hom}_K(V, K)$ be the dual basis $x_i(f_j) = \delta_{i=j}$.

The ring $K[x_1, \ldots, x_n]$ consists of the polynomial functions $f \colon V \to K$. The group G acts naturally on polynomial functions via the *contragredient* action $(\sigma \cdot f)(v) := f(\sigma^{-1}(v))$. Concretely, $\sigma \cdot x_j = \sum_{i=1}^n \overline{\sigma}_{ij} x_i$ and $\sigma \cdot \sum c_\alpha x^\alpha = \sum c_\alpha \sigma(x_1)^{\alpha_1} \cdots \sigma(x_n)^{\alpha_n}$. Define a *conjugate-*

linear bijection

$$\tau\colon V \to V^* \qquad \text{where} \qquad \sum_{i=1}^{n} c_i f_i \mapsto \sum_{i=1}^{n} \overline{c}_i x_i$$

Then τ is G-equivariant, i.e. $\tau(\sigma \cdot v) = \sigma \cdot \tau(v)$ for all $\sigma \in G$.

The derivative of $f \in K[x_1, \ldots, x_n]$ in the direction $v \in V$ is the polynomial function $\partial_v f$ defined by $(\partial_v f)(w) \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f(w + tv)$. To simplify the exposition, we transfer these derivatives to V^* by defining operators $\partial_x f \coloneqq \partial_{\tau^{-1}(x)} f$. Concretely, $\partial_{x_i} = \frac{\partial}{\partial x_i}$ is the usual partial derivative. More generally, we extend these partial derivatives to polynomials in $K[x_1, \ldots, x_n]$ multiplicatively and *conjugate*-linearly.

Definition 2.1. If $g = \sum_\alpha c_\alpha x^\alpha \in K[x_1, \ldots, x_n]$ then set

$$\partial_g \coloneqq \sum_\alpha \overline{c}_\alpha \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}.$$

We use two fundamental properties of ∂_g. First, $\sigma \cdot \partial_g f = \partial_{\sigma \cdot g}(\sigma \cdot f)$. In particular, if g is G-invariant, then $\sigma \cdot \partial_g f = \partial_g(\sigma \cdot f)$ and ∂_g is G-equivariant. Second, if g is homogeneous and non-zero, then $\partial_g g = \sum_\alpha |c_\alpha|^2 \alpha! > 0$.

We may define a G-invariant positive-definite Hermitian form on $K[x_1, \ldots, x_n]$ by

$$(f, g) \coloneqq (\partial_g f)(0, \ldots, 0).$$

This form is linear in f and conjugate-linear in g. Under this form, the coinvariant ideals and harmonics from Section 1.2 are orthogonal complements, $\mathcal{I}_G^\perp = \mathcal{H}_G$ and $\mathcal{H}_G^\perp = \mathcal{I}_G$. Furthermore, the adjoint of ∂_{x_i} with respect to this form is multiplication by x_i, i.e. $\partial_{x_i}^\dagger f = x_i f$. More generally, $\partial_g^\dagger f = gf$. See for example [SW21, Lemmas 2.7, 2.11, 4.2].

2.2. Differential forms

In place of polynomial functions $f(v)$ on V, we may consider alternating multilinear k-forms $\eta\colon V^k \to K$ on V. We again have the natural contragredient action $(\sigma \cdot \eta)(v_1, \ldots, v_k) \coloneqq \eta(\sigma^{-1}(v_1), \ldots, \sigma^{-1}(v_k))$. Concretely, the alternating k-forms on V have $\binom{n}{k}$ basis elements indexed by $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and determined by

$$(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k})(f_{j_1}, \ldots, f_{j_k}) = \begin{cases} 1 & \text{if } j_1 = i_1, \ldots, j_k = i_k \\ 0 & \text{if } \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\}. \end{cases}$$

Following a standard convention in this area of algebraic combinatorics, we abbreviate $\theta_i \coloneqq \mathrm{d}x_i$ and typically suppress the $\wedge$ symbol. The alternating multilinear forms on V under the wedge product together form the K-algebra $K[\theta_1, \ldots, \theta_n]$ generated by $\theta_1, \ldots, \theta_n$ subject to the relations $\theta_i \theta_j = -\theta_j \theta_i$, and in particular $\theta_i^2 = 0$. The G-action is given by $\sigma \cdot \theta_j = \sum_{i=1}^n \overline{\sigma}_{ij} \theta_i$ and $\sigma \cdot \sum c_{i_1,\ldots,i_k} \theta_{i_1} \cdots \theta_{i_k} = \sum c_{i_1,\ldots,i_k} \sigma(\theta_{i_1}) \cdots \sigma(\theta_{i_k})$.

The analogue of the directional derivative ∂_v for alternating forms is the *interior product*. If $\eta\colon V^k \to K$ is alternating and $v \in V$, then $(\partial_v \eta)(v_1, \ldots, v_{k-1}) \coloneqq \eta(v, v_1, \ldots, v_{k-1})$.

In contrast to partial derivatives, which commute, interior products anti-commute according to $\partial_v \partial_w = -\partial_w \partial_v$. We again have $\sigma \cdot \partial_v \eta = \partial_{\sigma(v)}(\sigma \cdot \eta)$. By a slight abuse of notation, now let $\tau \colon V \to V^*$ be the conjugate-linear bijection given by $\sum_{i=1}^n c_i f_i \mapsto \sum_{i=1}^n \overline{c}_i \theta_i$, which is again G-equivariant. Let $\partial_\theta := \partial_{\tau^{-1}(\theta)}$. Concretely, if $i_1 < \cdots < i_k$,

$$\partial_{\theta_i} \theta_{i_1} \cdots \theta_{i_k} = \begin{cases} (-1)^{\ell-1} \theta_{i_1} \cdots \widehat{\theta}_{i_\ell} \cdots \theta_{i_k} & \text{if } i = i_\ell \\ 0 & \text{otherwise,} \end{cases}$$

where $\widehat{\theta}_{i_\ell}$ means θ_{i_ℓ} is omitted.

Definition 2.2. Let $K[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]$ be the ring of differential forms on V with coefficients which are polynomial functions on V. This is the K-algebra generated by indeterminates $x_1, \ldots, x_n, \theta_1, \ldots, \theta_n$ subject to the relations $x_i x_j = x_j x_i$, $\theta_i \theta_j = -\theta_j \theta_i$, and $x_i \theta_j = \theta_j x_i$.

As before, we extend the interior product operators multiplicatively and conjugate-linearly to $K[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]$. Here we write I in place of $1 \leqslant i_1 < \cdots < i_k \leqslant n$. Note the reversal of the order of the ∂_θ operators.

Definition 2.3. If

$$\omega = \sum_{\alpha, I} c_{\alpha,I} x^\alpha \theta_{i_1} \cdots \theta_{i_k} \in K[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]$$

then set

$$\partial_\omega := \sum_{\alpha, I} \overline{c}_{\alpha,I} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \partial_{\theta_{i_k}} \cdots \partial_{\theta_{i_i}}. \tag{2.1}$$

Note that $\sigma \cdot \partial_\omega f = \partial_{\sigma \cdot \omega}(\sigma \cdot f)$. Moreover, one may check that if ω is bi-homogeneous and non-zero, then $\partial_\omega \omega = \sum_{\alpha,I} |c_{\alpha,I}|^2 \alpha! > 0$. Hence we may again define a G-invariant positive-definite Hermitian form on $K[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]$ by

$$(f, \omega) := \text{constant coefficient of } \partial_\omega f.$$

This form is linear in f and conjugate-linear in ω.

Under this form, the super coinvariant ideals and super harmonics from Section 1.5 are orthogonal complements, $\mathcal{J}_G^\perp = \mathcal{SH}_G$ and $\mathcal{SH}_G^\perp = \mathcal{J}_G$. Furthermore, the adjoint of ∂_{θ_i} with respect to this form is left multiplication by θ_i, i.e. $\partial_{\theta_i}^\dagger f = \theta_i f$. More generally, $\partial_\omega^\dagger = \omega f$. See [SW21, Lemmas 2.7, 2.11, 5.4] for more information.

2.3. Generalized exterior derivatives

Recall the exterior derivative (1.3), which is defined by $\mathrm{d} = \sum_{i=1}^n \partial_{x_i} \theta_i$. Here and elsewhere, θ_i refers to left multiplication by θ_i. Equation (1.3) is consistent with our usage of $\mathrm{d}x_i$ in Section 2.2. We now describe the operators d_i generalizing (1.3) and underlying our construction of $\mathcal{SH}_G^{\det}$ in [SW21].

We have $\sigma \cdot \mathrm{d}f = \mathrm{d}(\sigma \cdot f)$ for all $f \in K[x_1, \dots, x_n, \theta_1, \dots, \theta_n]$. The G-equivariance of the exterior derivative conceptually arises from the G-invariance of the form $\overline{x}_1\theta_1 + \cdots + \overline{x}_n\theta_n$, or equivalently the G-invariance of the norm-squared function $\overline{x}_1 x_1 + \cdots + \overline{x}_n x_n$, which is G-invariant since $G \subset U(n, K)$ consists of unitary matrices. Here $\overline{x}_i(v) := \overline{x_i(v)}$.

More generally, every G-invariant element of $K[\overline{x}_1, \dots, \overline{x}_n, \theta_1, \dots, \theta_n]$ gives rise to a corresponding G-equivariant operator in

$$K[\partial_{x_1}, \dots, \partial_{x_n}, \theta_1, \dots, \theta_n]$$

obtained by replacing $\overline{x}_i$ with ∂_{x_i}. We now summarize the structure of these operators and define the operators d_i used in Section 1.6.

When G is a pseudo-reflection group, Shephard–Todd [ST54] and later Chevalley [Che55] showed that $K[x_1, \dots, x_n]^G = K[f_1, \dots, f_n]$ for algebraically independent homogeneous G-invariants $f_1, \dots, f_n$ called *basic invariants* of G. The basic invariants are not unique, though the multiset $\{d_1, \dots, d_n\}$ of their degrees is uniquely determined and is called the multiset of *degrees* of G. The *exponents* of G are the multiset $\{d_1 - 1, \dots, d_n - 1\}$, which is the multiset of the x-degrees of $\mathrm{d}f_i$.

Solomon [Sol63] described the G-invariants $K[x_1, \dots, x_n, \theta_1, \dots, \theta_n]^G$. He showed that they have a K-basis given by

$$\{f_1^{\alpha_1} \cdots f_n^{\alpha_n} \mathrm{d}f_{i_1} \cdots \mathrm{d}f_{i_k} : \alpha_i \geqslant 0, 1 \leqslant i_1 < \cdots < i_k \leqslant n, k \in [n]\}. \tag{2.2}$$

Consequently,

$$\mathcal{J}_G = \langle K[x_1, \dots, x_n, \theta_1, \dots, \theta_n]_+^G \rangle = \langle f_1, \dots, f_n, \mathrm{d}f_1, \dots, \mathrm{d}f_n \rangle. \tag{2.3}$$

Orlik–Solomon [OS80] generalized Solomon's exterior algebra construction of the invariants to certain Galois conjugates. Translated to the present language of differential operators, we have the following.

Theorem 2.4 ([OS80]; see [SW21, §3.1]). *There are bi-homogeneous, G-equivariant operators $\mathrm{d}_1, \dots, \mathrm{d}_n$ which raise θ-degree by 1 such that a K-basis for the G-equivariant operators in $K[\partial_{x_1}, \dots, \partial_{x_n}, \theta_1, \dots, \theta_n]^G$ is given by*

$$\{\partial_{f_1}^{\alpha_1} \cdots \partial_{f_n}^{\alpha_n} \mathrm{d}_{i_1} \cdots \mathrm{d}_{i_k} : \alpha_i \geqslant 0, 1 \leqslant i_1 < \cdots < i_k \leqslant n, k \in [n]\}. \tag{2.4}$$

Definition 2.5. The d_i satisfying equation (2.4) will be called *generalized exterior derivatives* on $K[x_1, \dots, x_n, \theta_1, \dots, \theta_n]$.

Like "the" fundamental invariants f_i, the generalized exterior derivatives d_i are not unique, and (2.4) has many solutions. See Remark 2.8 for an explicitly computable criterion to determine if a proposed set of generalized exterior derivatives satisfies (2.4). See Table 7.1 for an explicit description of convenient sets of generalized exterior derivatives for $G(m, p, n)$. For example, when $G = \mathfrak{S}_n$, one may use

$$\mathrm{d}_i = \sum_{j=1}^{n} \partial_{x_j}^i \theta_j$$

where $0 \leqslant i \leqslant n-1$; see Section 2.6 for further details. When $K \subset \mathbb{R}$, so G is a (real) reflection group, then by (2.2) we may take $\mathrm{d}_i = \sum_{j=1}^n \partial_{f_{ij}} \theta_j$ where $f_{ij} = \partial_{x_j} f_i$.

Since the generalized exterior derivatives raise θ-degree by 1, they satisfy $\mathrm{d}_i \mathrm{d}_j = -\mathrm{d}_j \mathrm{d}_i$, so in particular $\mathrm{d}_i^2 = 0$. While the d_i are not unique, the multiset of degrees $\{e_1^*, \ldots, e_n^*\}$ by which they lower x-degree is uniquely determined. This multiset by definition consists of the *co-exponents* of G.

2.4. Removing invariant vectors

Our description of the semi-invariant differential forms in Theorem 1.5 from Section 1.6 involves r operators $\mathrm{d}_1, \ldots, \mathrm{d}_r$ where $r = \dim(V/V^G) \leqslant n$, rather than n operators. This arises from a slight mismatch between Theorem 1.5 and the result [SW21, Thm. 5.7] underlying it. That result involves semi-invariant harmonic forms in $\mathrm{S}(V^*) \otimes \wedge M^*$, where M is a finite-dimensional G-module with $M^G = 0$. The generalization of the basis (1.4) from [SW21] involves $\dim M$ differential operators in general. When $G = \mathfrak{S}_n$ acts on $\mathbb{Q}^n$ by permutation matrices, $V^G = \mathrm{Span}_{\mathbb{Q}}\{(1, \ldots, 1)\} \neq 0$, so the result does not directly apply. However, we may use $M = V/V^G$ in [SW21, Thm. 5.7], which in the case of $G = \mathfrak{S}_n$ is the *standard representation*.

The relationship between the coinvariants and harmonics of $\mathrm{S}(V^*) \otimes \wedge(V/V^G)^*$ and $\mathrm{S}(V^*) \otimes \wedge V^*$ is straightforward. Write $\mathcal{J}'_G$, $\mathcal{SH}'_G$, and $\mathcal{SR}'_G$ for the super coinvariant ideal, super harmonics, and super coinvariant algebra of $\mathcal{S}(V^*) \otimes \wedge(V/V^G)^*$.

Lemma 2.6. *The super harmonics $\mathcal{SH}_G$ and $\mathcal{SH}'_G$ are naturally isomorphic.*

Proof. (Sketch.) We have a natural map $\Phi\colon \mathrm{S}(V^*) \otimes \wedge V^* \to \mathrm{S}(V^*) \otimes \wedge(V/V^G)^*$. By replacing $\theta_1, \ldots, \theta_n$ with a basis $\psi_1, \ldots, \psi_r, \psi_{r+1}, \ldots, \psi_n$ for which $\psi_{r+1}, \ldots, \psi_n$ is a basis for $\wedge^1(V^G)^*$, we may think of $\mathrm{S}(V^*) \otimes \wedge(V/V^G)^*$ as being obtained from $\mathrm{S}(V^*) \otimes \wedge V^*$ by setting $\psi_1, \ldots, \psi_r = 0$. Moreover, we may suppose $\mathrm{Span}_K\{\psi_1, \ldots, \psi_r\}$ is G-stable. We then find that $\Phi(\mathcal{J}_G) = \mathcal{J}'_G$, Φ descends to an isomorphism $\mathcal{SR}_G \xrightarrow{\sim} \mathcal{SR}'_G$, and Φ restricts to an isomorphism $\mathcal{SH}_G \xrightarrow{\sim} \mathcal{SH}'_G$. □

In particular, the det-isotypic component $\mathcal{SH}_G^{\det}$ has dimension 2^r rather than 2^n. Similarly, the "volume form" in $\mathcal{SH}_G$ is an r-form rather than an n-form, namely $\psi_1 \cdots \psi_r$ in the notation of the proof, and $\mathcal{SH}_G^r$ is the top non-zero component of $\mathcal{SH}_G$.

We may hence choose generalized exterior derivatives $\mathrm{d}_{r+1}, \ldots, \mathrm{d}_n$ which each fix the x-degree and act as 0 on $\mathcal{SH}_G$. The corresponding co-exponents $e_{r+1}^*, \ldots, e_n^*$ are all zero and will be ignored. The remaining generalized exterior derivatives $\mathrm{d}_1, \ldots, \mathrm{d}_r$ strictly decrease x-degree and their co-exponents are positive. The description of $\mathcal{SH}_G^{\det}$ in (1.4) now follows from [SW21, Thm. 5.7].

2.5. Vandermondians and Jacobians

We now briefly give an explicit construction of the key element Δ_G from Section 1.2, which we call the *Vandermondian* of G, along with a related element Δ_G^*, which we call the *co-Vandermondian* of G. See [SW21, §3] for a more complete summary and references to the literature. We continue the notation from Section 2.3, so G is a pseudo-reflection group and $f_1, \ldots, f_n$ are basic invariants of G.

By Steinberg's Theorem 1.1, the top-degree component of $\mathcal{H}_G \subset K[x_1, \ldots, x_n]$ is spanned by an element $\Delta_G \in \mathcal{H}_G$, uniquely defined up to non-zero scalar multiples, which transforms according to $g \cdot \Delta_G = \det(g)\Delta_G$. Similarly, there is an element $\Delta_G^* \in \mathcal{H}_G$, uniquely defined up to non-zero scalar multiples, which transforms according to $g \cdot \Delta_G^* = \overline{\det(g)}\Delta_G$. By Steinberg's result, $\deg \Delta_G \geqslant \deg \Delta_G^*$. In fact, $\Delta_G^* \mid \Delta_G$ and equality holds if and only if G is generated by reflections (that is, order 2 pseudo-reflections). These facts may be read off from a formula of Gutkin [Gut73], which expresses Δ_G and Δ_G^* explicitly in terms of the reflecting hyperplanes of G as follows.

Let $\mathcal{A}(G)$ be the set of reflecting hyperplanes of G, i.e. the fixed spaces of pseudo-reflections of G. For each $H \in \mathcal{A}(G)$, fix some $\alpha_H \in V^*$ with $\ker \alpha_H = H$. Let G_H denote the subgroup of G fixing H pointwise. The Vandermondian is defined uniquely up to a non-zero scalar by

$$\Delta_G = \prod_{H \in \mathcal{A}(G)} \alpha_H^{|G_H|-1}.$$

The co-Vandermondian is defined uniquely up to a non-zero scalar by

$$\Delta_G^* = \prod_{H \in \mathcal{A}(G)} \alpha_H.$$

Furthermore, $\deg \Delta_G = \sum_{i=1}^n (\deg(f_i) - 1)$ is the sum of the exponents of G, which is the x-degree of $\mathrm{d}f_1 \cdots \mathrm{d}f_n$. Similarly, the sum of the co-exponents is the number of reflecting hyperplanes, $e_1^* + \cdots + e_n^* = |\mathcal{A}(G)|$, which is the amount the x-degree is lowered by $\mathrm{d}_1 \cdots \mathrm{d}_r$.

Remark 2.7. Given a set of homogeneous G-invariants $f_1, \ldots, f_n$, one may verify that they are indeed basic invariants using Saito's criterion [OT92, Thm. 4.19] or an appropriate generalization [OS80, Thm. 3.1], which says that it suffices to check that the Jacobian determinant of $f_1, \ldots, f_n$ agrees with the Vandermondian of G. That is, we require $J_G := \det(\partial_{x_j} f_i)_{1 \leqslant i,j \leqslant n} = \Delta_G$, up to a non-zero constant. Note that in this case $\mathrm{d}f_1 \cdots \mathrm{d}f_n$ is a non-zero multiple of $\Delta_G\, \theta_1 \cdots \theta_n$.

Remark 2.8. Likewise, for the generalized exterior derivatives $\mathrm{d}_i = \sum_{j=1}^n \partial_{f_{ij}} \theta_j$, the corresponding Jacobian determinant $J_G^* := \det(f_{ij})_{1 \leqslant i,j \leqslant n}$ transforms according to $\sigma \cdot J_G^* = \overline{\det(\sigma)} J_G^*$. Given a proposed set of bi-homogeneous G-equivariant operators, one may verify that they are indeed generalized exterior derivatives by checking that J_G^* agrees with the co-Vandermondian of G, $\det(f_{ij})_{1 \leqslant i,j \leqslant n} = \Delta_G^*$, up to a non-zero constant. Note that $\mathrm{d}_1 \cdots \mathrm{d}_n = \partial_{\Delta_G^*}\, \theta_1 \cdots \theta_n$.

2.6. Explicit formulas

We now give explicit descriptions for the basic invariants, generalized exterior derivatives, Vandermondians, co-Vandermondians, and co-exponents of the Shephard–Todd pseudo-reflection groups $G = G(m, p, n)$ described in Section 1.3. See Table 7.1 at the end of the paper for a quick summary. The special cases of real reflection groups $G(1, 1, n) = \mathfrak{S}_n$, $G(2, 1, n) = \mathfrak{B}_n$, $G(2, 2, n) = \mathfrak{D}_n$, and $G(m, m, 2) = \mathfrak{Dih}_m$ are also written out in Table 7.1.

When $G = G(1,1,n) = \mathfrak{S}_n$ is the symmetric group, we may use power-sums for the basic invariants, namely $f_i = \sum_{j=1}^n x_j^i$ for $1 \leqslant i \leqslant n$. The reflections are the $\binom{n}{2}$ transpositions with reflecting hyperplanes $x_j - x_i = 0$, so $\Delta_{\mathfrak{S}_n} = \prod_{1\leqslant i<j\leqslant n}(x_j - x_i)$ is the classical Vandermonde determinant of degree $\binom{n}{2}$. Since G is a real reflection group, the co-Vandermondian is equal to the Vandermondian, and we may use the exterior derivatives of the f_i to construct the generalized exterior derivatives d_i. It is most convenient to use $\mathrm{d}_i = \sum_{j=1}^n \partial_{x_j^i}\theta_j$ for $0 \leqslant i \leqslant n-1$. In particular, $\mathrm{d}_1 = \mathrm{d}$ is the exterior derivative and $\mathrm{d}_0 = \theta_1 + \cdots + \theta_n$ reflects the fact that $x_1 + \cdots + x_n$ is $\mathfrak{S}_n$-invariant. Note that d_0 acts as 0 on $\mathcal{SH}_{\mathfrak{S}_n}$, so only $\mathrm{d}_1, \ldots, \mathrm{d}_{n-1}$ are of interest, and $r = n-1$. The co-exponents are $e_i^* = i$ and their sum is $\binom{n}{2}$.

When $G = G(m,1,n)$ is the group of pseudo-permutation matrices whose non-zero entries belong to the group μ_m of mth complex roots of unity, we may use basic invariants $f_i = \sum_{j=1}^n x_j^{mi}$ for $1 \leqslant i \leqslant n$. The pseudo-reflections come in two types. First, the $m\binom{n}{2}$ generalized transpositions indexed by $1 \leqslant i < j \leqslant n$ and $a \in \mu_m$ where $\sigma(f_i) = af_j, \sigma(f_j) = a^{-1}f_i, \sigma(f_k) = f_k$ for $k \notin \{i,j\}$, which have reflecting hyperplanes $ax_i - x_j$ with $|G_H| = 2$. Second, the $n(m-1)$ "rotations" indexed by $1 \leqslant i \leqslant n$ and $a \in \mu_m - \{1\}$ where $\sigma(f_i) = af_i$, $\sigma(f_k) = f_k$ for $k \neq i$, which have reflecting hyperplanes $x_i = 0$ with $|G_H| = m$. Now $\Delta_{G(m,1,n)} = (x_1 \cdots x_n)^{m-1}\prod_{1\leqslant i<j\leqslant n}(x_j^m - x_i^m)$, which has degree $m\binom{n}{2} + n(m-1)$. When $m > 1$, the co-Vandermondian is $\Delta^*_{G(m,1,n)} = (x_1 \cdots x_n)\prod_{1\leqslant i<j\leqslant n}(x_j^m - x_i^m)$, which has degree $m\binom{n}{2} + n$. In this case, the generalized exterior derivatives are $\mathrm{d}_i = \sum_{j=1}^n \partial_{x_j^{(i-1)m+1}}\theta_j$ for $1 \leqslant i \leqslant n$, and the co-exponents are $e_i^* = (i-1)m + 1$.

When $G = G(m,p,n)$ is the subgroup of $G(m,1,n)$ where the product of the non-zero entries raised to the (m/p)th power is 1, we may use basic invariants $f_i = \sum_{j=1}^n x_j^{mi}$ for $1 \leqslant i \leqslant n-1$ and $f_n = (x_1 \cdots x_n)^{m/p}$. The pseudo-reflections come in the same types as for $G(m,1,n)$, except that there are $n(m/p - 1)$ "rotations" which additionally require $a \in \mu_{m/p} - \{1\}$, and $|G_H| = m/p$. Now $\Delta_{G(m,p,n)} = (x_1 \cdots x_n)^{m/p-1}\prod_{1\leqslant i<j\leqslant n}(x_j^m - x_i^m)$, which has degree $m\binom{n}{2} + n(m/p - 1)$. When $p \neq m$, the co-Vandermondian is $x_1 \cdots x_n \prod_{1\leqslant i<j\leqslant n}(x_j^m - x_i^m)$, which has degree $m\binom{n}{2} + n$. In this case, the generalized exterior derivatives are $\mathrm{d}_i = \sum_{j=1}^n \partial_{x_j^{(i-1)m+1}}\theta_j$, and the co-exponents are $e_i^* = (i-1)m + 1$. When $p = m$, the only pseudo-reflections are the generalized transpositions and the co-Vandermondian is $\prod_{1\leqslant i<j\leqslant m}(x_j^m - x_i^m)$, which has degree $m\binom{n}{2}$. In this case, the generalized exterior derivatives are $\mathrm{d}_i = \sum_{j=1}^n \partial_{x_j^{(i-1)m+1}}\theta_j$ for $1 \leqslant i \leqslant n-1$ and $\mathrm{d}_n = \sum_{j=1}^n \partial_{(x_1\cdots\widehat{x}_j\cdots x_n)^{m-1}}\theta_j$, with co-exponents $e_i^* = (i-1)m + 1$ for $1 \leqslant i \leqslant n-1$ and $e_n^* = (n-1)(m-1)$.

3. The structure of $\mathcal{SH}_G^r$

We begin by considering the top θ-degree component of the key differential operator equation (1.5). Let G be a pseudo-reflection group with basic invariants $f_1, \ldots, f_n$. By Lemma 2.6, we may streamline our exposition by supposing without loss of generality throughout this subsection that $V^G = 0$. Hence $r = n$ and the basic invariants $f_1, \ldots, f_n$ all have degree at least 2.

Recall that $\Delta_G, \Delta_G^* \in \mathcal{H}_G \subset K[x_1, \ldots, x_n]$ are the unique elements up to non-zero scalar multiples in $\mathcal{H}_G^{\det}$ and $\mathcal{H}_G^{\overline{\det}}$, respectively. Since $\mathcal{H}_G$ is isomorphic to the regular representation of G, there is also a non-zero element

$$\Gamma_G \in \mathcal{H}_G^{\det^2}$$

unique up to non-zero scalar multiples. We may describe Γ_G in terms of Δ_G and Δ_G^* as follows.

Lemma 3.1. *We have $\Gamma_G = \partial_{\Delta_G^*}\Delta_G$ up to non-zero scalar multiples.*

Proof. Since $\Delta_G \in \mathcal{H}_G$, we have $\partial_{\Delta_G^*}\Delta_G \in \mathcal{H}_G$. We have

$$\begin{aligned}\sigma \cdot \partial_{\Delta_G^*}\Delta_G &= \partial_{\sigma(\Delta_G^*)}\sigma(\Delta_G)\\ &= \partial_{\overline{\det(\sigma)}\Delta_G^*}\det(\sigma)\Delta_G\\ &= \det(\sigma)^2\partial_{\Delta_G^*}\Delta_G,\end{aligned}$$

so $\partial_{\Delta_G^*}\Delta_G \in \mathcal{H}_G^{\det^2}$. By Gutkin's formula in Section 2.5, $\Delta_G^* \mid \Delta_G$, so

$$0 \neq \partial_{\Delta_G}\Delta_G = \partial_{\Delta_G/\Delta_G^*}\partial_{\Delta_G^*}\Delta_G.$$

Hence $\partial_{\Delta_G^*}\Delta_G \neq 0$. □

Consider the following analogues of the classical coinvariant ideal $\mathcal{I}_G$ and harmonics $\mathcal{H}_G$.

Definition 3.2. Let

$$\mathcal{I}'_G := \langle \partial_{x_j} f_i : i, j \in [n]\rangle$$

and

$$\mathcal{H}'_G := \{f \in K[x_1, \ldots, x_n] : \partial_g f = 0 \text{ for all } g \in \mathcal{I}'_G\}.$$

The ideal $\mathcal{I}'_G$ is the $(n-1)$st Fitting ideal of the Jacobian of the basic invariants $f_1, \ldots, f_n$. Since $\deg f_i \geqslant 2$, $\mathcal{I}'_G$ is proper. It is independent of the basis $x_1, \ldots, x_n$.

Lemma 3.3. *We have $\mathcal{I}'_G \supset \mathcal{I}_G$, $\mathcal{H}'_G \subset \mathcal{H}_G$, and $K[x_1, \ldots, x_n] = \mathcal{H}'_G \oplus \mathcal{I}'_G$.*

Proof. Recall Euler's formula, $\sum_{j=1}^n x_j\partial_{x_j} f = \deg(f)f$ for homogeneous f. Thus $f_i \in \mathcal{I}'_G$ for all i, so $\mathcal{I}'_G \supset \mathcal{I}_G$. We now see directly that $\mathcal{H}'_G \subset \mathcal{H}_G$. Finally, $(\mathcal{I}'_G)^\perp = \mathcal{H}'_G$. □

Lemma 3.4. *Suppose $V^G = 0$. Then*

$$\mathcal{SH}_G^n = \mathcal{H}'_G\theta_1 \cdots \theta_n.$$

Proof. The n-forms $\omega \in \mathcal{SH}_G^n$ are of the form $g\,\theta_1 \cdots \theta_n$ for some $g \in K[x_1, \ldots, x_n]$. We have $\omega \in \mathcal{SH}_G^n$ if and only if $\partial_{f_i}\omega = 0$ and $\partial_{\mathrm{d}f_i}\omega = 0$ for all $1 \leqslant i \leqslant n$. The first condition occurs if and only if $g \in \mathcal{H}_G$. Noting that $\mathrm{d}f_i = \sum_{j=1}^n \partial_{x_j} f_i\theta_j$, the second condition implies that

$$\partial_{\mathrm{d}f_i} g\,\theta_1 \cdots \theta_n = \sum_{j=1}^n \pm\partial_{\partial_{x_j} f_i} g\,\theta_1 \cdots \widehat{\theta_j} \cdots \theta_n.$$

Hence $\partial_{\mathrm{d}f_i}\omega = 0$ if and only if $\partial_{\partial_{x_j} f_i} g = 0$ for all j, so the second condition is equivalent to $g \in \mathcal{H}'_G$. The result follows since $\mathcal{H}'_G \subset \mathcal{H}_G$. □

Definition 3.5. For $F \in K[x_1, \ldots, x_n]$, write

$$\operatorname{Ann} F \coloneqq \{g \in K[x_1, \ldots, x_n] : \partial_g F = 0\}.$$

Steinberg proved Theorem 1.1 by first showing [Ste64, Thm. 1.3(b)]

$$\operatorname{Ann} \Delta_G = \mathcal{I}_G. \tag{3.1}$$

We now consider $\operatorname{Ann} \Gamma_G$.

Lemma 3.6. *Suppose $V^G = 0$. Then $\Gamma_G \in \mathcal{H}'_G$. Equivalently,*

$$\operatorname{Ann} \Gamma_G \supset \mathcal{I}'_G.$$

Proof. As noted in Section 2.5, $\mathrm{d}_1 \cdots \mathrm{d}_n = \partial_{\Delta_G^*} \theta_1 \cdots \theta_n$, so $\Gamma_G \theta_1 \cdots \theta_n = \partial_{\Delta_G^*} \Delta_G \theta_1 \cdots \theta_n = \mathrm{d}_1 \cdots \mathrm{d}_n \Delta_G$. By (1.4) and Lemma 3.4, $\mathrm{d}_1 \cdots \mathrm{d}_n \Delta_G \in \mathcal{SH}_G^n = \mathcal{H}'_G \theta_1 \cdots \theta_n$, so $\Gamma_G \in \mathcal{H}'_G$. Hence $\partial_g \Gamma_G = 0$ for all $g \in \mathcal{I}'_G$, so $\mathcal{I}'_G \subset \operatorname{Ann} \Gamma_G$. □

We will now show that $\operatorname{Ann} \Gamma_G = \mathcal{I}'_G$ is equivalent to the case of (1.5) with n generalized exterior derivatives when $V^G = 0$. We then restate and prove Theorem 1.7.

Proposition 3.7. *Suppose that G is a pseudo-reflection group and $V^G = 0$. Then*

$$\mathcal{SH}_G^n = K[\partial_{x_1}, \ldots, \partial_{x_n}](\mathcal{SH}_G^n)^{\det} \tag{3.2}$$

if and only if

$$\operatorname{Ann} \Gamma_G = \mathcal{I}'_G. \tag{3.3}$$

Proof. By (1.4) and Lemma 3.1,

$$\begin{aligned}(\mathcal{SH}_G^n)^{\det} &= K\, \mathrm{d}_1 \cdots \mathrm{d}_n \Delta_G \\ &= K\, \partial_{\Delta_G^*} \Delta_G \theta_1 \cdots \theta_n \\ &= K\, \Gamma_G \theta_1 \cdots \theta_n.\end{aligned}$$

Hence

$$K[\partial_{x_1}, \ldots, \partial_{x_n}](\mathcal{SH}_G^n)^{\det} = K[\partial_{x_1}, \ldots, \partial_{x_n}] \Gamma_G \theta_1 \cdots \theta_n.$$

By Lemma 3.4, equation (3.2) is hence equivalent to

$$\mathcal{H}'_G = \{\partial_g \Gamma_G : g \in K[x_1, \ldots, x_n]\}. \tag{3.4}$$

Consider the conjugate-linear map $\psi\colon K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$ given by $\psi(g) \coloneqq \partial_g \Gamma_G$, so $\ker \psi = \operatorname{Ann} \Gamma_G$. Equation (3.4) is equivalent to $\operatorname{im} \psi = \mathcal{H}'_G$. By Lemma 3.6, we have $\psi\colon K[x_1, \ldots, x_n] \to \mathcal{H}'_G$. Again by Lemma 3.6, $\ker \psi \supset \mathcal{I}'_G$. Thus by Lemma 3.3, $\operatorname{im} \psi = \operatorname{im} \psi|_{\mathcal{H}'_G}$ and $\ker \psi = \ker \psi|_{\mathcal{H}'_G} \oplus \mathcal{I}'_G$. Since $\mathcal{H}'_G$ is finite-dimensional, $\psi|_{\mathcal{H}'_G}$ is surjective if and only if it is injective, which occurs if and only if $\ker \psi = \mathcal{I}'_G$. This is a restatement of (3.3). □

Theorem 1.7. *Let $G = G(m, 1, n)$ or let G be real. Then the θ-degree r component of (1.5) holds.*

Proof. First suppose G is real. Then $\det^2 = 1$, so $\Gamma_G = 1$ and $\operatorname{Ann}\Gamma_G = \langle x_1, \ldots, x_n\rangle$. Furthermore, $x_1^2 + \cdots + x_n^2$ is G-invariant, so $\mathcal{I}'_G = \langle x_1, \ldots, x_n\rangle$ as well. The result follows from Proposition 3.7.

If $G = G(m, 1, n)$ for $m > 1$, then $r = n$. Table 7.1 gives $\mathcal{I}'_G = \langle x_1^{m-1}, \ldots, x_n^{m-1}\rangle$. Furthermore, we claim $\Gamma_G = (x_1 \cdots x_n)^{m-2}$. First, this is annihilated by ∂_{f_i} by degree considerations, so it is harmonic. Also, it transforms under G by $\det^2$. Indeed, if $\sigma(f_i) = \lambda_i f_{\tau(i)}$, $\det(\sigma) = \lambda_1 \cdots \lambda_n \det(\tau)$, so $\det(\sigma)^2 = (\lambda_1 \cdots \lambda_n)^2$ and

$$\sigma(x_1 \cdots x_n)^{m-2} = (\overline{\lambda_1} \cdots \overline{\lambda_n} x_1 \cdots x_n)^{m-2} = (\lambda_1 \cdots \lambda_n)^2 (x_1 \cdots x_n)^{m-2}.$$

Finally, the annihilator of $(x_1 \cdots x_n)^{m-2}$ is precisely $\langle x_1^{m-1}, \ldots, x_n^{m-1}\rangle$. □

The assertion is false for $G(m, p, n)$ not covered by the above theorem. Here we consider the cyclic groups $G(m, p, 1) = G(m/p, 1, 1)$ as part of the family $G(m, 1, n)$. The simplest example with strict containment is $G(4, 2, 2)$, which is generated by reflections but is not real.

Lemma 3.8. *If $G = G(m, p, n)$ is not of the form $G(1, 1, n)$, $G(2, 1, n)$, $G(2, 2, n)$, $G(m, p, 1) = G(m/p, 1, 1)$, $G(m, m, 2)$, or $G(m, 1, n)$, then $r = n$ and $\operatorname{Ann}\Gamma_G \neq \mathcal{I}'_G$. Hence (3.3) does not hold, so the r-form component of (1.5) does not hold.*

Moreover, in these cases, the top x-degree component of $\mathcal{SH}^r_G$ is strictly higher than the top x-degree component of $(\mathcal{SH}^r_G)^{\det}$, so the r-form component of (1.7) does not hold.

Proof. By Table 7.1, we may use basic invariants $f_i = \sum_{j=1}^n x_j^{mi}$ for $1 \leqslant i \leqslant n-1$ and $f_n = (x_1 \cdots x_n)^{m/p}$. Hence

$$\mathcal{I}'_G = \langle x_1^{m-1}, \ldots, x_n^{m-1}, x_1^{-1}(x_1 \cdots x_n)^{m/p}, \ldots, x_n^{-1}(x_1 \cdots x_n)^{m/p}\rangle.$$

If $p = m$, $\Delta_G = \Delta_G^*$ (even though $G(m, m, n)$ is typically not real), so $\Gamma_G = 1$ and $\operatorname{Ann}\Gamma_G = \langle x_1, \ldots, x_n\rangle$. The top x-degree component of $(\mathcal{SH}^n_G)^{\det}$ is hence degree 0. By assumption, $m \geqslant 3$, so the generators x_i^{m-1} of $\mathcal{I}'_G$ are at least quadratic. Also by assumption, $n \geqslant 3$, so the generators $x_i^{-1}(x_1 \cdots x_n)$ are also at least quadratic, so $\operatorname{Ann}\Gamma_G \supsetneq \mathcal{I}'_G$. Moreover, $\mathcal{I}'_G$ does not contain the linear polynomials, so $\mathcal{H}'_G$ contains all linear polynomials, and the top x-degree component of $\mathcal{SH}^n_G$ is at least degree 1.

If $p \neq m$, one finds $\Gamma_G = (x_1 \cdots x_n)^{m/p-2}$ as in the proof of Theorem 1.7, so $\operatorname{Ann}\Gamma_G = \langle x_1^{m/p-1}, \ldots, x_n^{m/p-1}\rangle$. The generators of $\mathcal{I}'_G$ have strictly larger degree since $n \geqslant 2$, so the containment is strict. Indeed, it is easy to see that $x_1(x_1 \cdots x_n)^{m/p-2} \notin \mathcal{I}'_G$, so the top x-degree component of $\mathcal{SH}^n_G$ is at least 1 higher than that of $(\mathcal{SH}^n_G)^{\det}$. □

Example 3.9. When $G = G(4, 2, 2)$, we have $\mathcal{I}'_G = \langle x_1^3, x_2^3, x_1^2 x_2, x_1 x_2^2\rangle$ while $\Gamma_G = 1$ has annihilator $\langle x_1, x_2\rangle$. Hence $\mathcal{H}'_G$ consists of all g of degree at most 2 and $\mathcal{SH}^2_G = \{g\,\theta_1\theta_2 : \deg g \leqslant 2\}$. On the other hand, $(\mathcal{SH}^2_G)^{\det} = K\,\theta_1\theta_2$. Hence (1.5) and (1.7) are false in this case.

We may also use the explicit description of $\mathcal{H}'_G$ to determine the highest degree of multiples of the volume form in $\mathcal{SH}_G$. This will be used below in Section 7.

Lemma 3.10. *Let $G = G(m,p,n)$. Then the top x-degree component of $\mathcal{SH}_G^r$ has degree*

$$\max\{i : \mathcal{SH}_G^{i,r} \neq 0\} = \begin{cases} 0 & \text{if } n=1, m/p \leqslant 2 \\ m/p-2 & \text{if } n=1, m/p \geqslant 3 \\ 0 & \text{if } n \geqslant 2, m \leqslant 2 \\ m/p-2+(n-1)(m-2) & \text{if } n \geqslant 2, m \geqslant 3. \end{cases}$$

Proof. If $n = 1 = m/p$, then $\mathcal{SH}_G^0 = K$ has degree 0. If $n = 1$ and $m/p \geqslant 2$, then $f_1 = x_1^{m/p}$ with $\mathrm{d}f_1 = m/p \cdot x_1^{m/p-1}\theta_1$. Hence $x_1^s \in \mathcal{SH}_G^1$ if and only if $\partial_{x_1}^{m/p-1} x^s = 0$, or if and only if $s \leqslant m/p-2$.

Now take $n \geqslant 2$. As above, we have

$$\mathcal{I}'_G = \langle x_1^{m-1}, \ldots, x_n^{m-1}, x_1^{-1}(x_1\cdots x_n)^{m/p}, \ldots, x_n^{-1}(x_1\cdots x_n)^{m/p}\rangle$$

for $(m,p) \neq (1,1)$. If $m = p = 1$, then G is real and $\mathcal{SH}_G^r = K$ has top degree 0 by Theorem 1.7. If $p = 1$ and $m \geqslant 2$, we have $\mathcal{I}'_G = \langle x_1^{m-1}, \ldots, x_n^{m-1}\rangle$, so the top-degree monomial not in $\mathcal{I}'_G$ is $(x_1\cdots x_n)^{m-2}$ which has degree $n(m-2)$. If $p = m = 2$, then $\mathcal{I}'_G = \langle x_1, \ldots, x_n\rangle$ and again $\mathcal{SH}_G^r = K$.

Now take $n \geqslant 2$, $m \geqslant 3$, and $p \geqslant 2$. Here $m/p \leqslant m/2 < m-1$. Suppose $x^\alpha \notin \mathcal{I}'_G$ for $|\alpha|$ maximal. By symmetry, we may suppose $\alpha_1 \leqslant \cdots \leqslant \alpha_n$. If $\alpha_1 \geqslant m/p$, then $(x_1\cdots x_n)^{m/p} \mid x^\alpha$ and $x^\alpha = 0$, so $\alpha_1 \leqslant m/p-1$. If $\alpha_1 = m/p-1$, then $\alpha_2 = m/p-1$, and we find $x^\alpha = x_1^{m/p-1}x_2^{m/p-1}(x_3\cdots x_n)^{m-2}$. If $\alpha_1 = m/p-2$, then we find $x^\alpha = x_1^{m/p-2}(x_2\cdots x_n)^{m-2}$. The degree of the latter minus the degree of the former is $m-2-m/p \geqslant 0$, so the latter is the top-degree element. □

Corollary 3.11. *Let $G = G(m,p,n)$. The top total degree component of $\mathcal{SH}_G^r$ has degree strictly below $\deg \Delta_G$, except when $n = 1$ or $(m,p,n) \in \{(1,1,2),(2,2,2)\}$ when equality holds.*

Proof. In this case,

$$\deg \Delta_G = m\binom{n}{2} + n\left(\frac{m}{p}-1\right).$$

First take $n = 1$. If $m = p = 1$, then $r = 0$ and both degrees are 0. If $m/p \geqslant 2$, then $r = 1$, $\deg \Delta_G = m/p-1$, and the top *total* degree of $\mathcal{SH}_G^r$ is $m/p-2+1 = m/p-1$ by Lemma 3.10, so equality holds.

Now consider $n \geqslant 2$. If $m = 1$, then $\binom{n}{2} \geqslant r = n-1$, and the inequality is strict for $n \geqslant 3$. If $m = 2$, then $m\binom{n}{2}+n(m/p-1) \geqslant n$, and the inequality is strict except when $m = p = n = 2$. If $m \geqslant 3$, we have

$$\begin{aligned}\left(m\binom{n}{2} + n\left(\frac{m}{p}-1\right)\right) - \left(\frac{m}{p}-2+(n-1)(m-2)+n\right) \\ = \frac{m(n-1)(p(n-2)+2)}{2p} > 0.\end{aligned}$$

□

4. Exactness of exterior differentiation on $\mathcal{SH}_G$

The exactness of the polynomial de Rham complex $(K[x_1,\ldots,x_n,\theta_1,\ldots,\theta_n],\mathrm{d})$ is well-known and is usually attributed to Koszul. In this section we introduce a generalization of the complex (1.6) for the harmonics of a pseudo-reflection group G. We then summarize an algebraic analogue of well-known results in Hodge theory. Finally we prove Theorem 1.10 and deduce Theorem 1.6. Our argument follows an approach to proving exactness of the Koszul complex; see for example [GW98, Prop. 7.2.11].

4.1. Super harmonic cochain complexes

We begin by showing that two "dual" types of operators preserve the super harmonics. The first of these appeared in [SW21].

Lemma 4.1. *Let $d=\sum_{j=1}^n \partial_{g_j}\theta_j$ with $g_j\in K[x_1,\ldots,x_n]$ be a G-equivariant operator which strictly lowers x-degree. Then $d\colon \mathcal{SH}_G\to\mathcal{SH}_G$ preserves the super harmonics of G. In particular, we have a cochain complex*

$$0\to K\to \mathcal{SH}_G^0 \xrightarrow{d} \mathcal{SH}_G^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{SH}_G^n\to 0. \tag{4.1}$$

Proof. See [SW21, Cor. 5.6]. The argument in the next proof is very similar. □

In particular, the generalized exterior derivatives $\mathrm{d}_1,\ldots,\mathrm{d}_r$ preserve the super harmonics $\mathcal{SH}_G$. Moreover, we have the following "dual" result.

Lemma 4.2. *Let $\delta=\sum_J g_J\partial_{\theta_J}$ be a G-equivariant operator which strictly lowers θ-degree (i.e. $g_\varnothing=0$), where each $g_J\in K[x_1,\ldots,x_n]$ is linear. Then $\delta\colon\mathcal{SH}_G\to\mathcal{SH}_G$ preserves the super harmonics of G.*

Proof. Suppose $\eta\in\mathcal{SH}_G$, so $\partial_\omega\eta=0$ for all $\omega\in\mathcal{J}_G$. We must show $\partial_\omega\delta\eta=0$. Let $\omega=\sum_K h_K\theta_K$. We may suppose δ and ω are bi-homogeneous, so $|J|$ and $|K|$ are constant in the expansions, and that ω is G-invariant. Since $\partial_{\theta_J}\partial_{\theta_K}=(-1)^{|J||K|}\partial_{\theta_K}\partial_{\theta_J}$, we have

$$\begin{aligned}\partial_\omega\delta-(-1)^{|J||K|}\delta\partial_\omega &= \sum_{J,K}\partial_{h_K}\partial_{\theta_K}g_J\partial_{\theta_J}-(-1)^{|J||K|}g_J\partial_{\theta_J}\partial_{h_K}\partial_{\theta_K}\\ &=\sum_{J,K}(\partial_{h_K}g_J-g_J\partial_{h_K})\partial_{\theta_K}\partial_{\theta_J}.\end{aligned}$$

It is easily seen that $\partial_h g-g\partial_h\in K[\partial_{x_1},\ldots,\partial_{x_n}]$ when $h,g\in K[x_1,\ldots,x_n]$ and g is linear. Indeed, it may be reduced to the identity $\partial_{x_i}^a x_i-x_i\partial_{x_i}^a=a\partial_{x_i}^{a-1}$. Thus $\partial_\omega\delta-(-1)^{|J||K|}\delta\partial_\omega=\partial_\lambda$ for some $\lambda\in K[x_1,\ldots,x_n,\theta_1,\ldots,\theta_n]$. Since ω is G-invariant and δ is G-equivariant, λ is G-invariant. Since δ strictly lowers θ-degree, λ has positive θ-degree or is 0, so $\lambda\in\mathcal{J}_G$, and we find $\partial_\omega\delta\eta=0$. □

In particular, $\mathrm{d}^\dagger = \sum_{j=1}^n x_j \partial_{\theta_j}$ satisfies the hypotheses of Lemma 4.2 and hence preserves the harmonics $\mathcal{SH}_G$. However, the adjoints $\mathrm{d}_i^\dagger$ of the generalized exterior derivatives do not in general preserve the harmonics.

4.2. Hodge theory and Laplacians

A classic technique due to Hodge [Hod41, Ch. III] for analyzing cohomology on Riemannian manifolds involves replacing the cohomology groups with kernels of *Laplacians*. An algebraic version of this decomposition for cell complexes was introduced by Eckmann [Eck44] (see also [Eck00] and [DR02, §3] for further references). We state the version of this decomposition appropriate to our context and include a standard proof sketch using elementary linear algebra for the benefit of the reader.

Lemma 4.3. *Suppose*

$$\begin{tikzcd} A \arrow[r, bend left, "\delta^\dagger"] & B \arrow[l, bend left, "\delta"] \arrow[r, bend left, "d"] & C \arrow[l, bend left, "d^\dagger"] \end{tikzcd}$$

is a sequence of linear maps between finite-dimensional K-vector spaces with non-degenerate Hermitian forms, the adjoints are taken with respect to these forms, and $d\delta^\dagger = 0$. Then

$$B = \operatorname{im} \delta^\dagger \oplus \ker L \oplus \operatorname{im} d^\dagger \qquad \textit{and} \tag{4.2}$$

$$\ker d = \operatorname{im} \delta^\dagger \oplus \ker L \tag{4.3}$$

where

$$L := d^\dagger d + \delta^\dagger \delta$$

is the "total Laplacian".

Proof. (Sketch.) Since $\ker d \perp \operatorname{im} d^\dagger$ and B is finite-dimensional, $B = \ker d \oplus \operatorname{im} d^\dagger$, so we must show (4.3). Write $a = d^\dagger d$, $\alpha = \delta^\dagger \delta$. Now (4.3) becomes $\ker a = \operatorname{im} \alpha \oplus \ker(a + \alpha)$. Clearly $\ker a \cap \ker \alpha \subset \ker(a + \alpha)$. Since $a\alpha = \alpha a = 0$ and $a = a^\dagger, \alpha = \alpha^\dagger$, one finds $\ker(a + \alpha) = \ker a \cap \ker \alpha$. Hence we must show $\ker a = \operatorname{im} \alpha \oplus (\ker a \cap \ker \alpha)$.

Since $\alpha = \alpha^\dagger$, we have $B = \operatorname{im} \alpha \oplus (\operatorname{im} \alpha)^\perp = \operatorname{im} \alpha \oplus \ker \alpha$, so $\ker a = (\operatorname{im} \alpha \oplus \ker \alpha) \cap \ker a$. Now $\operatorname{im} d^\dagger \subset \ker d$ implies $\operatorname{im} \alpha \subset \ker a$, so $(\operatorname{im} \alpha \oplus \ker \alpha) \cap \ker a = \operatorname{im} \alpha \oplus (\ker \alpha \cap \ker a)$. □

Corollary 4.4. *The homology of the sequence $A \xrightarrow{\delta^\dagger} B \xrightarrow{d} C$ from Lemma 4.3 at B is*

$$H := \frac{\ker d}{\operatorname{im} \delta^\dagger} \cong \ker L$$

where $L = d^\dagger d + \delta^\dagger \delta$. In particular, the sequence is exact at B if and only if L is invertible.

When $d = \mathrm{d}$ is the exterior derivative, the adjoint operator is $\mathrm{d}^\dagger = \sum_{j=1}^n x_j \partial_{\theta_j}$, and

$$L = \mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger = \sum_{i=1}^n \left(\theta_i \partial_{\theta_i} + x_i \partial_{x_i}\right).$$

Thus L acts by multiplying by the total degree, so it is invertible except on constants. Indeed, many of the operators d_i in Table 7.1 are of the form $d = \sum_{j=1}^n \partial_{x_j^N} \theta_j$ for some $N \in \mathbb{Z}_{\geqslant 0}$, which act diagonally on the monomial basis with kernel given by the span of all x^α where $\alpha_j < N$ for all $j = 1, \ldots, n$. Since we do not use this fact, we leave the details for the interested reader.

4.3. Proving exactness and the rank 2 case

We may now restate and prove Theorem 1.10 from the introduction.

Theorem 1.10. *For any pseudo-reflection group $G \subset \mathrm{GL}(V)$ with $r = \dim(V/V^G)$, the exterior derivative cochain complex*

$$0 \to K \to \mathcal{SH}_G^0 \xrightarrow{\mathrm{d}} \mathcal{SH}_G^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{SH}_G^r \xrightarrow{\mathrm{d}} 0$$

is exact.

Proof. By Lemma 4.1, the exterior derivative d preserves $\mathcal{SH}_G$ and the complex is well-defined. By Lemma 4.2, the adjoint $\mathrm{d}^\dagger$ also preserves $\mathcal{SH}_G$. Thus by Corollary 4.4, exactness at $\mathcal{SH}_G^k$ for $k > 0$ is equivalent to invertibility of the total Laplacian $L = \mathrm{d}^\dagger\mathrm{d} + \mathrm{d}\mathrm{d}^\dagger$ acting on $\mathcal{SH}_G^k$. Since L is multiplication by the total degree, the result follows. □

Corollary 4.5. *For $i + k \geqslant 1$,*

$$\mathcal{SH}_G^{i,k} = \mathrm{d}\mathcal{SH}_G^{i+1,k-1} \oplus \mathrm{d}^\dagger \mathcal{SH}_G^{i-1,k+1}. \tag{4.4}$$

Proof. Apply Theorem 1.10 and Lemma 4.3. □

Remark 4.6. For $i > 1$, the adjoints $\mathrm{d}_i^\dagger$ do not typically preserve $\mathcal{SH}_G$. While the corresponding total Laplacian frequently acts diagonally on monomials, it cannot directly be used to analyze the homology of the complex $(\mathcal{SH}_G^\bullet, \mathrm{d}_i)$. A possible approach to Conjecture 1.22 would be to quantify this failure.

Before restating and proving Theorem 1.6, we give a criterion for the second component in (4.4) to satisfy (1.5). Here we use the notation

$$\mathcal{D}_G := K[\partial_{x_1}, \ldots, \partial_{x_n}]\mathcal{SH}_G^{\det}.$$

Lemma 4.7. *Suppose $\mathcal{SH}_G^k \subset \mathcal{D}_G$. Then*

$$\mathrm{d}^\dagger \mathcal{SH}_G^k \subset \mathcal{D}_G$$

if and only if

$$\partial_{\theta_i} (\mathcal{SH}_G^k)^{\det} \subset \mathcal{D}_G$$

for all $i = 1, \ldots, n$.

Proof. First suppose that $\mathrm{d}^\dagger\mathcal{SH}_G^k \subset \mathcal{D}_G$. A straightforward computation yields $\partial_{x_i}\mathrm{d}^\dagger - \mathrm{d}^\dagger\partial_{x_i} = \partial_{\theta_i}$. Thus

$$\begin{aligned}\partial_{\theta_i}(\mathcal{SH}_G^k)^{\det} &= (\partial_{x_i}\mathrm{d}^\dagger - \mathrm{d}^\dagger\partial_{x_i})(\mathcal{SH}_G^k)^{\det}\\ &\in \partial_{x_i}\mathcal{D}_G + \mathrm{d}^\dagger\mathcal{SH}_G^k \subset \mathcal{D}_G.\end{aligned}$$

Now suppose $\partial_{\theta_i}(\mathcal{SH}_G^k)^{\det} \subset \mathcal{D}_G$ for all $i = 1, \ldots, n$. Let $\omega \in \mathcal{SH}_G^k$. Since $\mathcal{SH}_G^k \subset \mathcal{D}_G$, it suffices to take $\omega = \partial_g\eta$ for some homogeneous $g \in K[x_1, \ldots, x_n]$ and $\eta \in (\mathcal{SH}_G^k)^{\det}$. We show that $\mathrm{d}^\dagger\omega \in \mathcal{D}_G$ by induction on $\deg g$. In the base case when g is constant, $\omega \in (\mathcal{SH}_G^k)^{\det}$, so $\mathrm{d}^\dagger\omega \in (\mathcal{SH}_G^{k-1})^{\det} \subset \mathcal{D}_G$. If $\deg g > 0$, write $g = \sum_{j=1}^n x_j g_j$. Then

$$\begin{aligned}d^\dagger\omega = d^\dagger\partial_g\eta &= \sum_{j=1}^n \mathrm{d}^\dagger\partial_{x_j}\partial_{g_j}\eta = \sum_{j=1}^n(\partial_{x_j}\mathrm{d}^\dagger - \partial_{\theta_j})\partial_{g_j}\eta\\ &= \sum_{j=1}^n \partial_{x_j}\mathrm{d}^\dagger\partial_{g_j}\eta - \partial_{g_j}\partial_{\theta_j}\eta \in \sum_{j=1}^n \partial_{x_j}\mathcal{D}_G + \partial_{g_j}\mathcal{D}_G \subset \mathcal{D}_G.\end{aligned}$$

□

Theorem 1.6. *Let $G \subset \mathrm{GL}(V)$ be a pseudo-reflection group with rank $r = \dim(V/V^G)$. Then if $r \leqslant 2$ and either $G = G(m,1,n)$ or G is real,*

$$\begin{aligned}\mathcal{SH}_G &= K[\partial_{x_1}, \ldots, \partial_{x_n}]\mathcal{SH}_G^{\det}\\ &= \{\partial_g\mathrm{d}_{i_1}\cdots\mathrm{d}_{i_k}\Delta_G : g \in K[x_1, \ldots, x_n], i_j \in [r]\}.\end{aligned}$$

Proof. The $\mathcal{SH}_G^0$ component of $\mathcal{SH}_G$ satisfies (1.5) by Steinberg's Theorem 1.1 in the sense that $\mathcal{SH}_G^0 \subset K[\partial_{x_1}, \ldots, \partial_{x_n}]\mathcal{SH}_G^{\det} =: \mathcal{D}_G$. Hence the $\mathrm{d}\mathcal{SH}_G^0$ summand of $\mathcal{SH}_G^1$ from Corollary 4.5 satisfies (1.5) as well. The result follows for $r \leqslant 1$, so take $r = 2$.

First suppose G is real. By Theorem 1.7, $\mathcal{SH}_G^2 = \mathrm{Span}_K\{\mathrm{vol}\}$ where vol is the volume form on V/V^G, which transforms by $\det$. Since $\mathrm{d}^\dagger$ is G-equivariant, $\mathrm{d}^\dagger\mathcal{SH}_G^2 \subset (\mathcal{SH}_G^1)^{\det} \subset \mathcal{D}_G$. Hence $\mathcal{SH}_G^1 \subset \mathcal{D}_G$ by Corollary 4.5, and the result follows.

Now suppose $G = G(m,1,n)$ has rank $r = 2$, so $n = 2$ and $m \geqslant 3$. By Theorem 1.7, $\mathcal{SH}_G^2 \subset \mathcal{D}_G$. By Lemma 4.7 and Theorem 1.5, we may check that $\mathrm{d}^\dagger\mathcal{SH}_G^2 \subset \mathcal{D}_G$ by instead checking that $\partial_{\theta_i}\mathrm{d}_1\mathrm{d}_2\Delta_G \in \mathcal{D}_G$. We verify this condition directly for $G = G(m,1,2)$.

From Table 7.1, we have

$$\begin{aligned}\Delta_G &= x_1^{m-1}x_2^{2m-1} - x_1^{2m-1}x_2^{m-1}\\ \mathrm{d}_1 &= \partial_{x_1}\theta_1 + \partial_{x_2}\theta_2\\ \mathrm{d}_2 &= \partial_{x_1^{m+1}}\theta_1 + \partial_{x_2^{m+1}}\theta_2.\end{aligned}$$

Hence $(\mathcal{SH}_G^2)^{\det} = K\,x_1^{m-2}x_2^{m-2}\theta_1\theta_2$. Thus we must show $x_1^{m-2}x_2^{m-2}\theta_i \in \mathcal{D}_G$. We directly compute

$$\partial_{x_1}(\mathrm{d}_2 + \partial_{x_2^m}\mathrm{d}_1)\Delta_G = 2(2m-1)\underline{m+1}(m-1)x_1^{m-2}x_2^{m-2}\theta_2.$$

We may obtain $x_1^{m-2}x_2^{m-2}\theta_1$ symmetrically, which completes the proof. □

5. Gröbner and Artin bases for $G(m, p, n)$

Our proofs of Theorem 1.12 and Theorem 1.15 will use explicit Gröbner and monomial bases of the coinvariant algebras $\mathcal{R}_{G(m,p,n)}$ with respect to the lexicographic order on monomials. Artin [Art98, p.41] implicitly gave the first monomial basis in type A, which corresponds in a natural way to the *inversion* statistic on permutations. Garsia[3] [Gar80] gave a separate *descent basis* in type A which corresponds to the *major index* statistic. The descent basis was subsequently generalized to Weyl groups and $G(m, p, n)$ by a variety of authors; see [BB07, p.324] for details and further references.

The Artin bases are very well-known for $\mathfrak{S}_n$, they appear to be folklore for $G(m, 1, n)$, and we have been unable to locate a description of them for $G(m, p, n)$ with $p > 1$. We give Artin bases $\mathcal{A}(m, p, n)$ for general $G(m, p, n)$ in Section 5.1; see Definition 5.3. In Section 5.2, we give some combinatorial properties of $\mathcal{A}(m, p, n)$. In Section 5.3, we use Gröbner bases to prove the main results of this section, Theorem 5.5 and Theorem 5.6. Our arguments are elementary and self-contained, except for an appeal to Chevalley's result that $\dim_K \mathcal{R}_G = |G|$ when G is a (pseudo-)reflection group.

5.1. Artin and Gröbner bases of $\mathcal{R}_{G(m,p,n)}$

The classical Artin basis for $\mathcal{R}_n = \mathbb{C}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n\rangle$ comes from [Art98, p.41] and consists of the $n!$ monomials

$$\mathcal{A}(1, 1, n) := \{x_1^{a_1} \cdots x_n^{a_n} : 0 \leqslant a_i < i, \forall i \in [n]\}. \tag{5.1}$$

These monomials may be visualized as "sub-staircase diagrams"; see Figure 5.1.

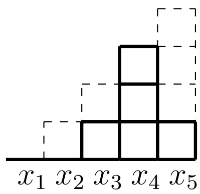


Figure 5.1: A vertically oriented sub-staircase diagram visualizing the monomial $x_1^0 x_2^0 x_3^1 x_4^3 x_5^1$ in the Artin basis for $\mathfrak{S}_5$.

The corresponding Artin basis for $G(m, 1, n)$ consists of the image of the $m^n n!$ monomials

$$\mathcal{A}(m, 1, n) := \{x_1^{a_1} \cdots x_n^{a_n} : 0 \leqslant a_i < im, \forall i \in [n]\}. \tag{5.2}$$

These too may be visualized as sub-staircase diagrams; see Figure 5.2. We typically draw these sub-staircase diagrams horizontally rather than vertically for convenience.

Definition 5.1. A *sub-staircase diagram of type* $G(m, 1, n)$ is a left-justified arrangement of n rows consisting of $a_1, \ldots, a_n$ square cells from top to bottom satisfying $0 \leqslant a_i < im$ for all $i \in [n]$.

[3] An anonymous referee notes, "In fact, this basis was discovered earlier by Steinberg [Ste75] in the context of general Weyl groups W."

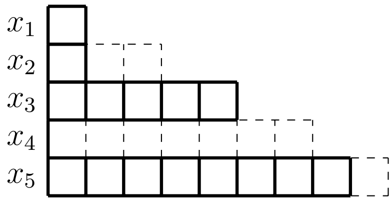
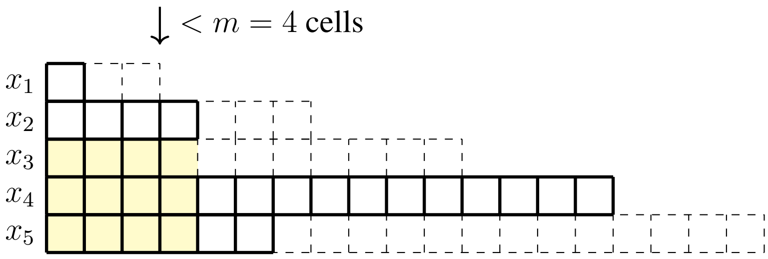


Figure 5.2: A horizontally oriented sub-staircase diagram which visualizes the monomial $x_1^{a_1} \cdots x_5^{a_5} = x_1 x_2 x_3^5 x_5^8$ in the Artin basis for $\mathfrak{B}_5 = G(2,1,5)$.

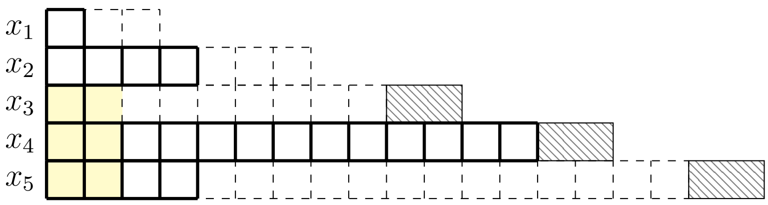
The Artin basis for general $G(m,p,n)$ is an index p subset of $\mathcal{A}(m,1,n)$ which we may describe using the following operation on sub-staircase diagrams.

Definition 5.2. Suppose $m, p, n \in \mathbb{Z}_{\geqslant 1}$ with $p \mid m$. Let A be a sub-staircase diagram for $G(m,1,n)$. Define the *p-contraction* of A as follows. Let i be the largest index such that the ith row of A has fewer than m cells. Take the lower-left rectangle of width m using rows $i, i+1, \ldots, n$ and shrink this rectangle horizontally by a factor of p. If this creates a partial cell, delete it. The result is the p-contraction of A.

Note that such an i exists since the first row has length $< m$. See Figure 5.3 below for an example.



(a) A sub-staircase diagram for $G(4,1,5)$ representing the monomial $x_1 x_2^4 x_3^3 x_4^{15} x_5^6$. The bottom-most row with $< m$ cells is row $i = 3$. The lower-left rectangle involved in p-contraction is highlighted.



(b) The result of 2-contracting the previous sub-staircase diagram. The half-cell arising from contracting row 3's three cells by a factor of two has been removed to create a sub-staircase diagram of type $G(4,2,5)$ representing the monomial $x_1 x_2^4 x_3 x_4^{13} x_5^4$.

Figure 5.3: An example of p-contraction.

Definition 5.3. A *sub-staircase diagram* of type $G(m,p,n)$ is defined to be the result of applying p-contraction to any sub-staircase diagram of type $G(m,1,n)$. The *Artin basis* of $G(m,p,n)$ is the corresponding set of monomials:

$$\begin{aligned}\mathcal{A}(m,p,n) := \{x_1^{a_1}\cdots x_n^{a_n} : \exists j \in [n] \text{ s.t. } & 0 \leqslant a_i < im, \forall i < j, \text{ and} \\ & 0 \leqslant a_j < \frac{m}{p}, \text{ and} \\ & 0 \leqslant a_i - \frac{m}{p} < (i-1)m, \forall i > j\}.\end{aligned} \tag{5.3}$$

Example 5.4. The dihedral group of order $2m$ is $G(m,m,2)$. The $2m$ sub-staircase diagrams of type $G(m,m,2) = \mathfrak{Dih}_{2m}$ are

$$\{(0,0),\dots,(m-1,0),(0,1),\dots,(0,m)\}$$

with corresponding Artin basis

$$\mathcal{A}(m,m,2) = \{1,x,\dots,x^{m-1},y,\dots,y^m\}$$

where $x := x_1$ and $y := x_2$.

In Section 5.3 we will show that $\mathcal{A}(m,p,n)$ is indeed a monomial basis:

Theorem 5.5. *The Artin basis $\mathcal{A}(m,p,n)$ descends to a basis for the classical coinvariant algebra $\mathcal{R}_{G(m,p,n)} = K[x_1,\dots,x_n]/\mathcal{I}_{G(m,p,n)}$.*

The Gröbner basis for the type A coinvariant ideal relative to lexicographic order (actually relative to any linear order with $x_1 > \cdots > x_n$) is well-known and can be extracted from Artin's original argument in [Art98] with some effort. Let $h_j(x_1,\dots,x_n)$ denote the *complete homogeneous symmetric polynomial* of degree j in n variables. The general Gröbner basis is as follows, which is proved in Section 5.3.

Theorem 5.6. *The reduced Gröbner basis of $\mathcal{I}_{G(m,p,n)}$ with respect to the lexicographic term order with $x_1 > \cdots > x_n$ is:*

- *If $p = 1$: $\{h_j(x_j^m,\dots,x_n^m) : j \in [n]\}$.*
- *If $p > 1$:*

$$\begin{aligned}&\{h_j(x_j^m,\dots,x_n^m) : j \in [n-1]\} \\ &\qquad \sqcup \{h_{j-1}(x_j^m,\dots,x_n^m)(x_j\cdots x_n)^{m/p} : j \in [n]\}.\end{aligned}$$

Remark 5.7. Recall that a Gröbner basis $\mathcal{G} \subset F[x_1,\dots,x_n]$ is *reduced* if for all $g \in \mathcal{G}$, the leading coefficient of g is 1 and for all monomials m in all $h \in \mathcal{G} - \{g\}$, the leading monomial of g does not divide m. See [CLO15, §2.4, 2.7] for details.

5.2. Combinatorial properties of the Artin bases for $G(m,p,n)$

We now give some combinatorial properties of the Artin bases $\mathcal{A}(m,p,n)$.

Lemma 5.8. *There are $m^n n!/p$ sub-staircase diagrams of type $G(m,p,n)$. Furthermore, the Hilbert series of the Artin basis of type $G(m,p,n)$ is*

$$\mathrm{Hilb}(\mathcal{A}(m,p,n);q) = [m]_q[2m]_q \cdots [(n-1)m]_q[nm/p]_q. \tag{5.4}$$

Proof. Note that p-contraction results in some fraction k/p of a cell, $k \in \{0,\ldots,p-1\}$, which is then discarded. It follows that the fibers of the p-contraction map are all of cardinality p. Hence there are $m^n n!/p$ sub-staircase diagrams of type $G(m,p,n)$.

For the Hilbert series, condition on which row j is the lowest with length $< m$. The resulting Hilbert series is easily seen to be

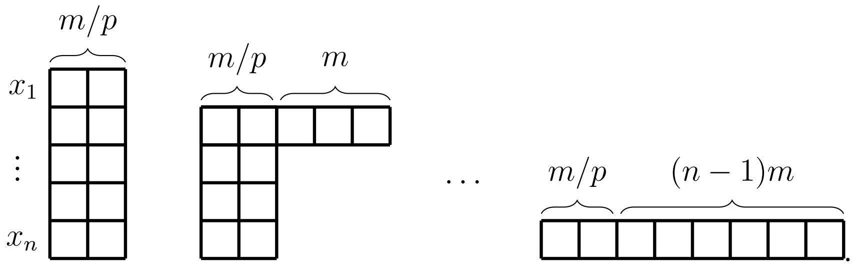
$$\begin{aligned}
&\sum_{j=1}^{m}[m]_q[2m]_q\cdots[(j-1)m]_q[m/p]_q q^{m/p}[jm]_q\cdots q^{m/p}[(n-1)m]_q \\
&\quad= \sum_{j=1}^{n}\left(\prod_{i=1}^{n-1}[im]_q\right) q^{(n-j)m/p}[m/p]_q \\
&\quad= \prod_{i=1}^{n-1}[im]_q \cdot [m/p]_q \cdot \sum_{j=1}^{n} q^{(n-j)m/p}.
\end{aligned}$$

The result follows by observing

$$\begin{aligned}
[m/p]_q \cdot \sum_{j=1}^{n} q^{(n-j)m/p} &= \frac{1-q^{m/p}}{1-q}\cdot\frac{1-q^{nm/p}}{1-q^{m/p}} \\
&= [nm/p]_q.
\end{aligned}$$

$\square$

Lemma 5.9. *A sub-staircase diagram of type $G(m,1,n)$ is a sub-staircase diagram of type $G(m,p,n)$ if and only if it does not contain any of the "hook" diagrams*



Proof. First suppose to the contrary that there exists a sub-staircase diagram A of type $G(m,p,n)$ containing the jth hook diagram. We may reverse the p-contraction process by horizontally expanding the width m/p rectangle in A involving rows $i, i+1, \ldots, n$ where i is the lowest row of length $< m/p$. The result, call it B, necessarily expands the rectangle of width m/p involving rows $j, j+1, \ldots, n$ in the hook diagram. After expansion, row j of B has length $\geqslant m + (j-1)m = jm$, contradicting the fact that B must be a sub-staircase diagram of type $G(m,1,n)$. Thus sub-staircase diagrams of type $G(m,p,n)$ do not contain the above hooks.

Conversely, suppose we have a sub-staircase diagram $A = (a_1, \ldots, a_n)$ of type $G(m,1,n)$ which does not contain the above hooks. Since A does not contain the first hook, an m/p by n rectangle, there is some row in A of length $< m/p$. Thus we may apply the reverse p-contraction process described above to A starting at some row j. We must only show the result, call it $B = (b_1, \ldots, b_n)$, remains a sub-staircase diagram of type $G(m,1,n)$, i.e. $b_i < im$ for all $i \in [n]$. We have $b_1 = a_1, \ldots, b_{j-1} = a_{j-1}$, so the necessary constraint is satisfied for $i < j$. When $i = j$, we have $b_j = a_j p < (m/p)p = m \leqslant jm$ as required. Finally, since rows $i > j$ of A have length $\geqslant m/p$ and yet the ith hook is not contained in the diagram, we must have $a_i < (m/p) + (i-1)m$ for $i > j$. Thus

$$b_i = a_i + (m - m/p) < (i-1)m + m = im,$$

as required. □

We also have the following recursive description of the Artin bases. The Hilbert series formula (5.4) also follows easily from this description and induction.

Lemma 5.10. *The Artin bases of types $G(m,p,n)$ are defined recursively by*

$$\mathcal{A}(m,p,n) = \{x_1^{a_1} \cdots x_n^{a_n} : \forall i \in [n-1], 0 \leqslant a_i < im \text{ and } 0 \leqslant a_n < m/p\} \sqcup \bigsqcup_{j=m/p}^{(n-1)m+m/p-1} x_n^j \mathcal{A}(m,p,n-1)$$

for $n \geqslant 2$, with base cases

$$\mathcal{A}(m,p,1) = \{x_1^{a_1} : 0 \leqslant a_1 < m/p\}.$$

Proof. The base cases are immediate. For the recursive formula, the first term consists of elements where $j = n$ when performing p-contraction. When $j < n$, we may keep or remove the final row without affecting the procedure materially, and that row has length satisfying $0 \leqslant a_n - m/p < (n-1)m$. □

5.3. Gröbner bases for $G(m,p,n)$

We now prove Theorem 5.5 and Theorem 5.6.

Lemma 5.11. *The polynomials in the proposed Gröbner bases in Theorem 5.6 belong to $\mathcal{I}_{m,p,n}$.*

Proof. We first prove by induction on j that $h_k(x_j^m, \ldots, x_n^m) \in \mathcal{I}_{m,p,n}$ whenever $k \geqslant j$. In the base case $j = 1$, $h_k(x_1^m, \ldots, x_n^m)$ is $G(m,p,n)$-invariant, so $h_k(x_1^m, \ldots, x_n^m) \in \mathcal{I}_{m,p,n}$ for all $k \geqslant 1$.

For the inductive step, first recall

$$h_{k+1}(x_j, \ldots, x_n) = h_{k+1}(x_{j+1}, \ldots, x_n) + x_j h_k(x_j, \ldots, x_n). \tag{5.5}$$

Suppose $h_k(x_j, \ldots, x_n) \in \mathcal{I}_{m,p,n}$ for all $k \geqslant j$. Since $h_{k+1}(x_j, \ldots, x_n)$ and $h_k(x_j, \ldots, x_n)$ belong to $\mathcal{I}_{m,p,n}$, (5.5) gives $h_{k+1}(x_{j+1}, \ldots, x_n) \in \mathcal{I}_{m,p,n}$, completing the induction. If $p = 1$, we are now done, so take $p > 1$.

We next prove by induction on j that $h_{j-1}(x_j^m, \ldots, x_n^m)(x_j \cdots x_n)^{m/p} \in \mathcal{I}_{m,p,n}$. In the base case $j = 1$, it is easy to see directly that $(x_1 \cdots x_n)^{m/p}$ is $G(m,n,p)$-invariant. For the inductive step, suppose

$$h_{j-1}(x_j^m, \ldots, x_n^m)(x_j \cdots x_n)^{m/p} \in \mathcal{I}_{m,p,n}.$$

Letting $k = j-1$ and $x_i \mapsto x_i^m$ in (5.5), multiplying by $(x_{j+1} \cdots x_n)^{m/p}$, and rearranging gives

$$\begin{aligned} h_j(x_{j+1}^m, \ldots, x_n^m)(x_{j+1} \cdots x_n)^{m/p} &= (x_{j+1} \cdots x_n)^{m/p} h_j(x_j^m, \ldots, x_n^m) \\ &\quad - x_j^{m-m/p}(x_j \cdots x_n)^{m/p} h_{j-1}(x_j^m, \ldots, x_n^m) \in \mathcal{I}_{m,n,p}. \end{aligned}$$

□

Lemma 5.12. *The leading monomials of the polynomials in the proposed Gröbner basis in Theorem 5.6 with respect to the lexicographic term order with $x_1 > \cdots > x_n$ are as follows.*

- *If $p = 1$: $\{x_j^{jm} : j \in [n]\}$.*
- *If $p > 1$: $\{x_j^{jm} : j \in [n-1]\} \sqcup \{x_j^{(j-1)m}(x_j \cdots x_n)^{m/p} : j \in [n]\}$.*

Proof. This may be read off directly from the proposed Gröbner basis. □

We may now complete the proof of Theorem 5.5 and Theorem 5.6.

Proof of Theorem 5.5. Let $\operatorname{LT}(S)$ denote the ideal generated by leading terms of a subset $S \subset K[x_1, \ldots, x_n]$. Consider the monomial ideal $\operatorname{LT}(\mathcal{G})$ generated by the leading terms of the proposed Gröbner basis $\mathcal{G}$ for $G(m,p,n)$. By Lemma 5.11, $\operatorname{LT}(\mathcal{G}) \subset \operatorname{LT}(\mathcal{I}_{m,p,n})$, so

$$K[x_1, \ldots, x_n]/\operatorname{LT}(\mathcal{G}) \supset K[x_1, \ldots, x_n]/\operatorname{LT}(\mathcal{I}_{m,p,n}). \tag{5.6}$$

The standard monomial basis of $K[x_1, \ldots, x_n]/\operatorname{LT}(\mathcal{G})$ consists of all monomials not divisible by any of the leading terms of $\mathcal{G}$. Encoding monomials in diagrams as in Section 5.1, Lemma 5.9 and Lemma 5.12 together show that the Artin basis $\mathcal{A}(m,p,n)$ is precisely the standard monomial basis for $K[x_1, \ldots, x_n]/\operatorname{LT}(\mathcal{G})$. By Lemma 5.8, this basis has size $m^n n!/p$.

As for the right-hand side of (5.6), by Gröbner theory, the vector space dimensions of $K[x_1, \ldots, x_n]/\operatorname{LT}(\mathcal{I}_{m,p,n})$ and $K[x_1, \ldots, x_n]/\mathcal{I}_{m,p,n}$ agree [CLO15, Prop. 5.3.4]. Chevalley [Che55] proved

$$\dim K[x_1, \ldots, x_n]/\mathcal{I}_{m,p,n} = |G(m,p,n)| = m^n n!/p.$$

Thus equality must hold in (5.6), so $\operatorname{LT}(\mathcal{G}) = \operatorname{LT}(\mathcal{I}_{m,p,n})$ and the result follows. □

Proof of Theorem 5.6. By the preceding argument, we have $\mathrm{LT}(\mathcal{G}) = \mathrm{LT}(\mathcal{I}_{m,p,n})$, so the proposed Gröbner basis is in fact a Gröbner basis. All that is left is to check that $\mathcal{G}$ is reduced. The leading coefficients are all 1, so we must only verify the divisibility condition.

We begin with the $p = 1$ case. Consider a general term

$$x_j^{a_j m} \cdots x_n^{a_n m} \text{ in } h_j(x_j^m, \ldots, x_n^m)$$

where $j \in [n]$. Suppose to the contrary $x_k^{km} \mid x_j^{a_j m} \cdots x_n^{a_n m}$ for some $k \in [n]$ with $k \neq j$. Then $a_k m \geqslant km$, so $a_k \geqslant 1$, and hence $j \leqslant k$. Now $h_j(x_j^m, \ldots, x_n^m)$ has degree jm while x_k^{km} has degree km, so $km \leqslant jm$, forcing $k = j$, a contradiction.

We now turn to the case $p > 1$. Again consider a general term $x_j^{a_j m} \cdots x_n^{a_n m}$ in the polynomial $h_j(x_j^m, \ldots, x_n^m)$ where $j \in [n-1]$. By the above argument, $x_k^{km} \nmid a_j^{a_j m} \cdots x_n^{a_n m}$ for all $k \in [n-1]$. Now suppose to the contrary that $x_k^{(k-1)m}(x_k \cdots x_n)^{m/p} \mid x_j^{a_j m} \cdots x_n^{a_n m}$ for some $k \in [n]$. Since $a_j m + \cdots + a_n m = jm$, we have $a_k \leqslant j$. As before, $j \leqslant k$. On the other hand, $(k-1)m + m/p \leqslant a_k m$, so $a_k > k - 1$, hence $a_k \geqslant k \geqslant j$. Thus we have $a_k = k = j$ and $x_j^{a_j m} \cdots x_n^{a_n m} = x_k^{km}$. However, $k < n$, so $x_k^{(k-1)m}(x_k \cdots x_n)^{m/p} \nmid x_k^{km}$, as required.

Finally, consider a general term

$$x_j^{a_j m} \cdots x_n^{a_n m}(x_j \cdots x_n)^{m/p} \text{ in } h_{j-1}(x_j^m, \ldots, x_n^m)(x_j \cdots x_n)^{m/p}$$

for $j \in [n]$. Note that $a_j m + \cdots + a_n m = (j-1)m$, so $a_k \leqslant j - 1$. First, suppose to the contrary that $x_k^{km} \mid x_j^{a_j m} \cdots x_n^{a_n m}(x_j \cdots x_n)^{m/p}$ for $k \in [n-1]$. Then $km \leqslant a_k m + m/p$, so $k \leqslant a_k$ since $p > 1$. Since $a_k \geqslant 1$, we have $j \leqslant k$. Now $j \leqslant k \leqslant a_k \leqslant j - 1$, a contradiction. Finally, suppose to the contrary that $x_k^{(k-1)m}(x_k \cdots x_n)^{m/p} \mid x_j^{a_j m} \cdots x_n^{a_n m}(x_j \cdots x_n)^{m/p}$ for some $k \in [n]$ with $k \neq j$. Clearly $j \leqslant k$. Also, $(k-1)m + m/p \leqslant a_k m + m/p$, so $k - 1 \leqslant a_k \leqslant j - 1$, meaning $k \leqslant j$. Hence $j = k$, a final contradiction. □

6. Bi-degree bounds for $G(m, 1, n)$

We now prove one of our main results, Theorem 1.12. The argument uses the Gröbner bases in Theorem 5.6. We first record two recursive relations involving the elements $\partial_{x_i} h_j(x_1^m, \ldots, x_n^m)$.

Lemma 6.1. *Let $j, m \in \mathbb{Z}_{\geqslant 1}$. Then*

$$\begin{aligned}\partial_{x_i} h_j(x_1^m, \ldots, x_n^m) = m x_i^{m-1} h_{j-1}(x_1^m, \ldots, x_n^m)\\ + x_i^m \partial_{x_i} h_{j-1}(x_1^m, \ldots, x_n^m).\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}&\sum_{i=1}^{n} (x_1 \cdots \widehat{x}_i \cdots x_n)^{m-1} \partial_{x_i} h_j(x_1^m, \ldots, x_n^m)\\ &\quad = (x_1 \cdots x_n)^{m-1} m(n + j - 1) h_{j-1}(x_1^m, \ldots, x_n^m).\end{aligned} \tag{6.2}$$

Proof. To prove (6.1), start with

$$h_j(x_1, \ldots, x_n) = h_j(x_1, \ldots, \widehat{x}_i, \ldots, x_n) + x_i h_{j-1}(x_1, \ldots, x_n), \tag{6.3}$$

apply $x_\ell \mapsto x_\ell^m$ for all $1 \leqslant \ell \leqslant n$, and differentiate with respect to x_i. Equation (6.2) is implied by (6.1) and Euler's identity $\sum_{i=1}^n x_i \partial_{x_i} h = \deg(h) h$ for h homogeneous. □

The argument in the following key lemma is based on one given by François Brunault [Bru12] in his proof that the h_j are typically irreducible.

Lemma 6.2. *For each $j, m \in \mathbb{Z}_{\geqslant 1}$, the n polynomials*

$$\{\partial_{x_i} h_j(x_1^m, \ldots, x_n^m) : i \in [n]\}$$

have no common zero in $\mathbb{C}^n - \{0\}$.

Proof. We argue first by induction on n, and for each n by induction on j. The result is clear in the base cases $n = 1$ or $j = 1$, so take $n, j \geqslant 2$. Suppose $z = (z_1, \ldots, z_n)$ is a common zero. If $z_\ell = 0$ for some $\ell \in [n]$, then $(z_1, \ldots, \widehat{z}_\ell, \ldots, z_n)$ is a common zero of the set $\{\partial_{x_i} h_j(x_1^m, \ldots, \widehat{x}_\ell, \ldots, x_n^m) : i \in [n] - \{\ell\}\}$ by (6.3), so $z = 0$ by induction on n. Hence we suppose $z_\ell \neq 0$ for all $\ell \in [n]$.

By (6.2) and the fact that $z_\ell \neq 0$, $h_{j-1}(z_1^m, \ldots, z_n^m) = 0$. Equation (6.1) implies that $(z_1, \ldots, z_n)$ is a zero of $\partial_{x_i} h_{j-1}(x_1^m, \ldots, x_n^m)$ for all $i \in [n]$. Thus $z = 0$ by induction on j, completing the proof. □

Remark 6.3. Lemma 6.2 and the Nullstellensatz combine to show that the radical of the ideal $\langle \partial_{x_i} h_j(x_1^m, \ldots, x_n^m) : i \in [n] \rangle$ is $\langle x_1, \ldots, x_n \rangle$, i.e.

$$\{\partial_{x_i} h_j(x_1^m, \ldots, x_n^m)\}$$

forms a *homogeneous system of parameters* for $\mathbb{C}[x_1, \ldots, x_n]$ (for $j \geqslant 2$). Since $\mathbb{C}[x_1, \ldots, x_n]$ is *Cohen–Macaulay*, they form a *regular sequence*. Consequently,

$$\operatorname{Hilb}\left(\frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle \partial_{x_i} h_j(x_1^m, \ldots, x_n^m) : i \in [n] \rangle}; q\right) = \left(\frac{1 - q^{mj-1}}{1 - q}\right)^n = [mj - 1]_q^n, \tag{6.4}$$

where the exponents in the numerator arise from the degrees of the elements in the regular sequence. See [Sta83, I.5, p.39-42] for details and [CKW09] for results similar to Lemma 6.2.

Corollary 6.4. *Let $K \subset \mathbb{C}$. For each $j, m \in \mathbb{Z}_{\geqslant 1}$ and homogeneous polynomial $p(x_j, \ldots, x_n) \in K[x_j, \ldots, x_n]$, we have*

$$\deg p > (mj - 2)(n - j + 1) \quad \Rightarrow \quad p \in \langle \partial_{x_i} h_j(x_j^m, \ldots, x_n^m) : i = j, \ldots, n \rangle.$$

Proof. By (6.4) and Lemma 6.2,

$$\begin{aligned}\operatorname{Hilb}\left(\frac{K[x_j, \ldots, x_n]}{\langle \partial_{x_i} h_j(x_j^m, \ldots, x_n^m) : i = j, \ldots, n \rangle}; q\right) &= \left(\frac{1 - q^{mj-1}}{1 - q}\right)^{n-j+1} \\ &= [mj - 1]_q^{n-j+1}.\end{aligned}$$

The right-hand side has degree $(mj - 2)(n - j + 1)$. □

The final ingredient needed for Theorem 1.12 is the following observation. Afterwards we restate and prove Theorem 1.12.

Lemma 6.5. *If $j \in [n]$ and $i \in [n]$, then*

$$\partial_{x_i} h_j(x_j^m, \dots, x_n^m)\, \theta_j \cdots \theta_n \in \mathcal{J}_{G(m,p,n)}.$$

Proof. The result is trivial if $i < j$, so take $i \geqslant j$. Since the exterior derivative $\mathrm{d} = \sum_{\ell=1}^n \partial_{x_\ell} \theta_\ell$ is G-equivariant and an anti-derivation, it preserves $\mathcal{J}_{G(m,p,n)}$. By Theorem 5.6,

$$h_j(x_j^m, \dots, x_n^m) \in \mathcal{I}_{G(m,p,n)} \subset \mathcal{J}_{G(m,p,n)}.$$

Thus

$$\begin{aligned} &\mathrm{d}h_j(x_j^m, \dots, x_n^m)\, \theta_j \cdots \widehat{\theta_i} \cdots \theta_n \\ &\qquad = \pm \partial_{x_i} h_j(x_j^m, \dots, x_n^m)\, \theta_j \cdots \theta_n \in \mathcal{J}_{G(m,p,n)}. \end{aligned}$$

□

Theorem 1.12. *Let $G = G(m,1,n)$. Then $\mathcal{SH}^{i,k}_{G(m,1,n)} \neq 0$ if and only if $i,k \geqslant 0$ and*

$$i + k + m\binom{k}{2} \leqslant m\binom{n}{2} + (m-1)n.$$

Proof. First suppose $i + k + m\binom{k}{2} > m\binom{n}{2} + (m-1)n$. Let $f \in \mathcal{SR}^{i,k}_{G(m,1,n)}$. We must show $f = 0$. It suffices to suppose $f = x^\alpha \theta_I$ is a monomial with $|\alpha| = i$ and $|I| = k$ and show $x^\alpha \theta_I \in \mathcal{J}_{G(m,1,n)}$. Since $\mathcal{SR}_{G(m,1,n)}$ is $\mathfrak{S}_n$-invariant, we may take $\theta_I = \theta_j \cdots \theta_n$ where $j = n - k + 1$. Since $\mathcal{J}_{G(m,1,n)}$ contains $\mathcal{I}_{G(m,1,n)} \cdot K[\theta_1, \dots, \theta_n]$, we may further suppose x^α belongs to the Artin basis $\mathcal{A}(m,1,n)$ for $\mathcal{R}(m,1,n)$, so $\alpha_\ell \leqslant \ell m - 1$ for all $\ell \in [n]$ by Theorem 5.5.

The degree of $x_j^{\alpha_j} \cdots x_n^{\alpha_n}$ is

$$\begin{aligned} \alpha_j + \cdots + \alpha_n &= i - \alpha_1 - \cdots - \alpha_{j-1} \\ &\geqslant i - \sum_{\ell=1}^{j-1} (\ell m - 1) = i - m\binom{j}{2} + (j-1) \\ &> m\binom{n}{2} + (m-1)n - k - m\binom{k}{2} - m\binom{j}{2} + (j-1) \\ &= (mj-2)(n-j+1). \end{aligned}$$

By Corollary 6.4, $x_j^{\alpha_j} \cdots x_n^{\alpha_n}$ belongs to the ideal generated by

$$\partial_{x_\ell} h_r(x_j^m, \dots, h_n^m).$$

Hence by Lemma 6.5,

$$x_j^{\alpha_j} \cdots x_n^{\alpha_n}\, \theta_j \cdots \theta_n \in \mathcal{J}_{G(m,1,n)}.$$

Thus $x^\alpha\,\theta_I \in \mathcal{J}_{G(m,1,n)}$, which proves necessity.

We now prove sufficiency. We first show that if $i + k + m\binom{k}{2} = m\binom{n}{2} + (m-1)n$, then $\mathcal{SH}^{i,k}_{G(m,1,n)} \neq 0$. By Table 7.1, $\deg \Delta_V = m\binom{n}{2} + (m-1)n$ and the operators d_j lower x-degree by $m(j-1)+1$ while raising θ-degree by 1. By Theorem 1.5, the element $\mathrm{d}_1 \cdots \mathrm{d}_k \Delta_V \in \mathcal{SH}_G$ is a non-zero harmonic k-form of x-degree

$$
\begin{aligned}
m\binom{n}{2} &+ (m-1)n - \sum_{j=1}^{k} (m(j-1)+1) \\
&= m\binom{n}{2} + (m-1)n - m\binom{k}{2} - k \\
&= i.
\end{aligned}
$$

Thus $\mathcal{SH}^{i,k}_{G(m,1,n)} \neq 0$.

Finally, since $\mathcal{SH}_G$ is closed under partial differentiation, $\mathcal{SH}^{i,k}_G \neq 0$ implies $\mathcal{SH}^{i',k'}_G \neq 0$ for all $i' \leqslant i$ and $k' \leqslant k$. Indeed, one may pick a monomial $cx^\alpha\theta_I$ in a non-zero element $\omega \in \mathcal{SH}^{i,k}_G$. Now $\partial_{x^\alpha\theta_I}\omega = \langle cx^\alpha\theta_I, x^\alpha\theta_I\rangle$ is a non-zero constant, so $0 \neq \partial_{x^\beta\theta_J}\omega \in \mathcal{SH}^{i',k'}_G$ for $\beta \leqslant \alpha$, $J \subset K$ with $|\beta| = i'$, $|J| = k'$. □

7. Total degree bounds for $G(m,p,n)$

We next prove Theorem 1.15. We begin with the case of 1-forms.

Lemma 7.1. *Let $G = G(m,p,n)$. The top-degree component of $\mathcal{SH}^1_G$ is $K\,\mathrm{d}\Delta_G$, except when $G = G(2,2,2)$ when it is*

$$\mathrm{Span}_K\{\mathrm{d}\Delta_G, \mathrm{d}_2\Delta_G\} = \mathrm{Span}_K\{x_1\theta_1 + x_2\theta_2, x_2\theta_1 + x_1\theta_2\}.$$

Proof. Suppose $\omega \in \mathcal{SH}^1_G$ is homogeneous with $\omega = \sum_{j=1}^n \omega_j\theta_j$. Since $\omega_j \in \mathcal{H}_G$, we have $\deg \omega_j \leqslant \deg \Delta_G$.

If $\deg \omega_j = \deg \Delta_G$, then $\omega_j = c_j \Delta_G$ for some $c_j \in K$. Then

$$\mathrm{d}^\dagger\omega = \sum_{j=1}^n x_j \partial_{\theta_j}\omega = \sum_{j=1}^n c_j x_j \Delta_G \in \mathcal{SH}^0_G = \mathcal{H}_G.$$

Hence $\sum_{j=1}^n c_j x_j = 0$, so $c_j = 0$ and $\omega = 0$.

If $\deg \omega_j = \deg \Delta_G - 1$, then $\omega_j = \partial_{\ell_j}\Delta_G$ for some ℓ_j of degree 1. Hence

$$0 = \partial_{\mathrm{d}f_i}\omega = \sum_{j=1}^n \partial_{\partial_{x_j} f_i}\omega_j = \sum_{j=1}^n \partial_{\ell_j \partial_{x_j} f_i}\Delta_G.$$

By (3.1), $\sum_{j=1}^n \ell_j \partial_{x_j} f_i \in \mathcal{I}_G$. From Table 7.1, we may use $f_i = \sum_{j=1}^n x_j^{mi}$ for $1 \leqslant i \leqslant n-1$ and $f_n = (x_1 \cdots x_n)^{m/p}$. The $i = n$ condition becomes

$$\sum_{j=1}^n \ell_j x_j^{-1}(x_1\cdots x_n)^{m/p} \in \langle (x_1\cdots x_n)^{m/p}, \sum_{j=1}^n x_j^{mi} \text{ for } 1 \leqslant i \leqslant n-1\rangle. \quad (7.1)$$

If $n = 1$, we have $\ell_1 = cx_1$ for $c \in K$, so $\omega = c \cdot \mathrm{d}\Delta_G$. We may thus assume $n \geqslant 2$.

First suppose $m/p + 1 < m$. In this case, the x_k-degree of monomials on the left-hand side of (7.1) is at most $m/p + 1$, while the x_k-degree of the generators $\sum_{j=1}^{n} x_j^{im}$ are at least m, so they cannot contribute. Hence $\sum_{j=1}^{n} \ell_j x_j^{-1}$ is constant. We leave it to the reader to check that this implies $\ell_j = c_j x_j$ for $c_j \in K$. The condition for $i = 1$ is therefore

$$\sum_{j=1}^{n} c_j x_j^m \in \langle (x_1 \cdots x_n)^{m/p}, \sum_{j=1}^{n} x_j^{mi} \text{ for } 1 \leqslant i \leqslant n-1 \rangle \tag{7.2}$$

Hence we have $c \in K$ and $f \in K[x_1, \ldots, x_n]$ such that

$$\sum_{j=1}^{n} (c_j - c) x_j^m = f \cdot (x_1 \cdots x_n)^{m/p}.$$

Since in the monomial expansion of the right-hand side of this equation every monomial is of the form x^β for $\beta_i > 0$ for all $1 \leqslant i \leqslant n$ and $n \geqslant 2$, both sides of this equation must be zero. Thus $f = 0$ and $c_j = c$ for all j. Hence $\omega = c \cdot \mathrm{d}\Delta_G$.

Finally, suppose $m/p + 1 \geqslant m$. If $p = 1$, the result follows from Theorem 1.12, so take $p > 1$. If $m > 2$, then $m/p + 1 \leqslant m/2 + 1 < m/2 + m/2 = m$, contrary to our assumption, so $m = 2 = p$ and $m/p + 1 = m$. The higher generators $\sum_{j=1}^{n} x_j^{mi}$ for $i \geqslant 2$ here cannot contribute to (7.1), so we have $c \in K$ and $g \in K[x_1, \ldots, x_n]$ such that

$$\sum_{j=1}^{n} \ell_j x_j^{-1} x_1 \cdots x_n = c x_1 \cdots x_n + g(x_1^2 + \cdots x_n^2).$$

The x_k-degree of the left-hand side is at most 2, forcing the x_k-degree of g to be 0 for all k. Hence g is constant. If $n \geqslant 3$, we see $g = 0$, and the previous arguments apply to give $\omega = c \cdot \mathrm{d}\Delta_G$.

We are left with the case $m = p = n = 2$. Here we have

$$\mathcal{J}_{G(2,2,2)} = \langle x_1^2 + x_2^2, x_1 x_2, x_1\theta_1 + x_2\theta_2, x_2\theta_1 + x_1\theta_2 \rangle.$$

It is straightforward to verify directly that the claimed elements are the only top-degree elements of $\mathcal{SH}_G^1$. □

Example 7.2. For $G = G(2,2,2)$, the elements of maximal total degree are spanned by $\Delta_G = x_1^2 - x_2^2$, $\mathrm{d}\Delta_G/2 = x_1\theta_1 - x_2\theta_2$, $-\mathrm{d}_2\Delta_G/2 = x_2\theta_1 - x_1\theta_2$, and $\mathrm{d}\mathrm{d}_2\Delta_G/4 = \theta_1\theta_2$.

We now give an analogue of the key argument we used in Section 6.

Lemma 7.3. *Let $G = G(m,p,n)$ where $p > 1$. If $j \geqslant 1$ and $f \in K[x_j, \ldots, x_n]$, then*

$$\begin{aligned} &\deg f > (n-j+1)(m(j-1)-1) \\ &\qquad \Rightarrow \quad f \cdot (x_j \cdots x_n)^{m/p} \theta_j \cdots \theta_n \in \mathcal{J}_{G(m,p,n)}. \end{aligned}$$

Proof. If $j = 1$, then $(x_1 \cdots x_n)^{m/p} \in \mathcal{I}_G \subset \mathcal{J}_G$ and the result follows. Now take $1 < j \leqslant n$.

Suppose we have a common zero of the set of polynomials

$$P = \{x_i \partial_{x_i} h_{j-1}(x_j^m, \dots, x_n^m) : i = j, \dots, n\}.$$

We have

$$\left(\sum_{i=j}^{n} x_i \partial_{x_i}\right) h_{j-1}(x_j^m, \dots, x_n^m) = m(j-1) h_{j-1}(x_j^m, \dots, x_n^m) = 0,$$

so $h_{j-1}(x_j^m, \dots, x_n^m) = 0$. Hence

$$\begin{aligned}
\partial_{x_i} h_j(x_j^m, \dots, x_n^m) &= \partial_{x_i}(h_j(x_j^m, \dots, \widehat{x}_i, \dots, x_n^m) + x_i^m h_{j-1}(x_j^m, \dots, x_n^m)) \\
&= m x_i^{m-1} h_{j-1}(x_j^m, \dots, x_n^m) + x_i^m \partial_{x_i} h_{j-1}(x_j^m, \dots, x_n^m) \\
&= 0.
\end{aligned}$$

Thus the only common zero of P is 0 by Lemma 6.2, so P forms a regular sequence in the ring $K[x_j, \dots, x_n]$. The Hilbert series of the quotient is hence $[m(j-1)]_q^{n-j+1}$, which has degree $(n-j+1)(m(j-1)-1)$. Thus $f \in \langle P \rangle$.

By Theorem 5.6, $h_{j-1}(x_j^m, \dots, x_n^m)(x_j \cdots x_n)^{m/p} \in \mathcal{I}_G$. Also,

$$\begin{aligned}
&x_i \partial_{x_i} h_{j-1}(x_j^m, \dots, x_n^m)(x_j \cdots x_n)^{m/p} \\
&\qquad = x_i \left(\partial_{x_i} h_{j-1}(x_j^m, \dots, x_n^m)\right)(x_j \cdots x_n)^{m/p} \\
&\qquad + \frac{m}{p} h_{j-1}(x_j^m, \dots, x_n^m)(x_j \cdots x_n)^{m/p}.
\end{aligned}$$

Hence

$$\begin{aligned}
&x_i \mathrm{d} h_{j-1}(x_j^m, \dots, x_n^m)(x_j \cdots x_n)^{m/p} \theta_j \cdots \widehat{\theta}_i \cdots \theta_n \\
&\qquad = \pm \left(x_i \partial_{x_i} h_{j-1}(x_j^m, \dots, x_n^m)\right)(x_j \cdots x_n)^{m/p} \theta_j \cdots \theta_n \\
&\qquad \pm \frac{m}{p} h_{j-1}(x_j^m, \dots, x_n^m)(x_j \cdots x_n)^{m/p} \theta_j \cdots \theta_n \\
&\qquad \in \mathcal{J}_G.
\end{aligned}$$

Thus $\langle P \rangle (x_j \cdots x_n)^{m/p} \theta_j \cdots \theta_n \subset \mathcal{J}_G$, and the result follows. □

We may now restate and prove Theorem 1.15.

Theorem 1.15. *Let $G = G(m, p, n)$ with $p \neq m$ or $p = 1$. Then*

$$\bigoplus_{i+k=\ell} \mathcal{SH}^{i,k}_{G(m,p,n)} \neq 0 \quad \Leftrightarrow \quad 0 \leqslant \ell \leqslant m\binom{n}{2} + n\left(\frac{m}{p} - 1\right).$$

Moreover, if $\ell = m\binom{n}{2} + n\left(\frac{m}{p} - 1\right)$, then

$$\bigoplus_{i+k=\ell} \mathcal{SH}^{i,k}_{G(m,p,n)} = (K + K\mathrm{d}) \prod_{1 \leqslant i < j \leqslant n} (x_j^m - x_i^m) \cdot (x_1 \cdots x_n)^{m/p-1}.$$

Proof. If $p = 1$, the result follows from Theorem 1.12 and Lemma 7.1, so take $p > 1$.

Consider a monomial $x^\alpha \theta_I \notin \mathcal{J}_G$ with $|\alpha| = i$ and $|I| = k$. Applying an element of $\mathfrak{S}_n$, we may take $\theta_I = \theta_{n-k+1} \cdots \theta_n$. Since $\mathcal{I}_G \subset \mathcal{J}_G$, we may further take $x^\alpha \in \mathcal{A}(m, p, n)$.

Suppose x^α is obtained by p-contracting rows $j - 1, j, \ldots, n$. Then $\alpha_{j-1} \leqslant \frac{m}{p} - 1$ and $\alpha_j, \ldots, \alpha_n \geqslant \frac{m}{p}$, so $(x_j \cdots x_n)^{m/p} \mid x^\alpha$.

We first consider the case when $j \leqslant n - k + 1$. We may write

$$x^\alpha = x_1^{\alpha_1} \cdots x_{n-k}^{\alpha_{n-k}} f(x_{n-k+1}, \ldots, x_n)(x_{n-k+1} \cdots x_n)^{m/p}.$$

If $\deg f > k(m(n-k) - 1)$ then $x^\alpha \theta_I \in \mathcal{J}_G$ by Lemma 7.3, so $\deg f \leqslant k(m(n-k) - 1)$. Since p-contraction starts at $j - 1 \leqslant n - k$, $\alpha_1 + \cdots + \alpha_{n-k}$ is maximized by p-contracting the maximal-degree element of $\mathcal{A}(m, 1, n)$, which results in x^β with

$$\beta_1 = \frac{m}{p} - 1, \beta_2 = \frac{m}{p} - 1 + m, \ldots, \beta_{n-k} = \frac{m}{p} - 1 + (n-k-1)m.$$

Hence

$$\alpha_1 + \cdots + \alpha_{n-k} \leqslant \sum_{\ell=1}^{n-k} \left(\left(\frac{m}{p} - 1 \right) + (\ell - 1)m \right).$$

Combining these bounds, we find that

$$\begin{aligned}
\deg \Delta_G - \deg x^\alpha \theta_I &\geqslant m\binom{n}{2} + n\left(\frac{m}{p} - 1\right) \\
&\quad - \sum_{\ell=1}^{n-k} \left(\left(\frac{m}{p} - 1 \right) + (\ell - 1)m \right) \\
&\quad - k(m(n-k) - 1) - k\frac{m}{p} - k \\
&= m\binom{k}{2} - k.
\end{aligned}$$

This difference is positive if $k \geqslant 2$ since $m > p > 1$, in which case $\deg \Delta_G > i + k$ as required. If $k = 0$, the result follows from Theorem 1.1, and if $k = 1$, the result follows from Lemma 7.1.

Now consider the case when $j > n - k + 1$. Since $(x_j \cdots x_n)^{m/p} \mid x^\alpha$, we may write

$$x^\alpha = x_1^{\alpha_1} \cdots x_{j-1}^{\alpha_{j-1}} f(x_j, \ldots, x_n)(x_j \cdots x_n)^{m/p}.$$

If $\deg f > (n - j + 1)(m(j-1) - 1)$ then Lemma 7.3 implies that

$$f \cdot (x_j \cdots x_n)^{m/p} \theta_j \cdots \theta_n \in \mathcal{J}_G,$$

so $x^\alpha \theta_I \in \mathcal{J}_G$. Hence $\deg f \leqslant (n - j + 1)(m(j-1) - 1)$. Since p-contraction begins at row $j - 1$, we have

$$\alpha_1 + \cdots + \alpha_{j-1} \leqslant \sum_{\ell=1}^{j-2} ((m-1) + (\ell - 1)m) + \frac{m}{p} - 1.$$

In all, we find

$$\begin{aligned}
\deg \Delta_G - \deg x^\alpha \theta_I \geqslant m\binom{n}{2} + n\left(\frac{m}{p} - 1\right) \\
&- \sum_{\ell=1}^{j-2} ((m-1) + (\ell-1)m) - \frac{m}{p} + 1 \\
&- (n-j+1)(m(j-1)-1) - (n-j+1)\frac{m}{p} - k \\
&= \frac{m}{p}(j-2) + m\binom{n-j+1}{2} - k. \qquad (7.3)
\end{aligned}$$

Suppose $n - j \geqslant 2$. Then since $p > 1$, the bound in (7.3) becomes

$$\begin{aligned}
&\frac{m}{p}(j-2) + \frac{m}{2}(n-j+1)(n-j) - k \\
&\qquad \geqslant \frac{m}{p}\left((j-2) + (n-j+1)(n-j)\right) - k \\
&\qquad \geqslant \frac{m}{p}\left((j-2) + 2(n-j+1)\right) - k \\
&\qquad > \frac{m}{p}\left(j-2+n-j+2\right) - n \\
&\qquad = n\left(\frac{m}{p} - 1\right) \geqslant 0.
\end{aligned}$$

Thus, as before we have $\deg \Delta_G > i + k$.

We are left with the cases $j \in \{n-1, n, n+1\}$. When $n = 1$ or $n = 2$, the result follows from Lemma 7.1 and Corollary 4.5, so take $n \geqslant 3$. If $j = n+1$, then since $m > p$, the bound in (7.3) is

$$\frac{m}{p}(n-1) - k \geqslant 2(n-1) - k \geqslant 2(n-1) - n = n - 2 \geqslant 1.$$

If $j = n - 1$, the bound becomes

$$\frac{m}{p}(n-3) + m - k \geqslant 2(n-3) + 4 - n = n - 2 \geqslant 1.$$

Finally, if $j = n$, we have

$$\frac{m}{p}(n-2) - k \geqslant 2(n-2) - k \geqslant 2(n-2) - n = n - 4,$$

so the result follows when $n \geqslant 5$. We are left with the cases $n \in \{3, 4\}$. If $n = 3$, we have $2(n-2) - k = 2 - k$, so the result holds at $k = 0, 1$. It also holds at $k = 3$ by Corollary 3.11, so it holds at $k = 2$ by Corollary 4.5. If $n = 4$, we have $2(n-2) - k = 4 - k$, so the result holds at $k = 0, 1, 2, 3$. It also holds at $k = 4$ by Corollary 3.11. This completes the proof. $\square$

G	f_i	Δ_G	$\deg \Delta_G$	d_i	e_i^*	Δ_G^*	$\deg \Delta_G^*$
$G(1,1,n)$ $=\mathfrak{S}_n$	$\prod_{j=1}^n x_j^i$	$\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)$	$\binom{n}{2}$	$\sum_{j=1}^n \partial_{x_j^i}\theta_j$	i	$\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)$	$\binom{n}{2}$
$G(2,1,n)$ $=\mathfrak{B}_n$	$\sum_{j=1}^n x_j^{2i}$	$\prod_{1\leqslant i<j\leqslant n}(x_j^2-x_i^2)$ $\cdot x_1\cdots x_n$	n^2	$\sum_{j=1}^n \partial_{x_j^{2i-1}}\theta_j$	$2i-1$	$\prod_{1\leqslant i<j\leqslant n}(x_j^2-x_i^2)$ $\cdot x_1\cdots x_n$	n^2
$G(2,2,n)$ $=\mathfrak{D}_n$	$\sum_{j=1}^n x_j^{2i}$ (for $1\leqslant i\leqslant n-1$) $x_1\cdots x_n$ (for $i=n$)	$\prod_{1\leqslant i<j\leqslant n}(x_j^2-x_i^2)$	$n(n-1)$	$\sum_{j=1}^n \partial_{x_j^{2i-1}}\theta_j$ (for $1\leqslant i\leqslant n-1$) $\sum_{j=1}^n \partial_{x_1\cdots\widehat{x}_j\cdots x_n}\theta_j$ (for $i=n$)	$2i-1$ (for $1\leqslant i\leqslant n-1$) $n-1$ (for $i=n$)	$\prod_{1\leqslant i<j\leqslant n}(x_j^2-x_i^2)$	$n(n-1)$
$G(m,m,2)$ $=\mathfrak{Dih}_{2m}$ for $m\geqslant 2$	$x_1^m+x_2^m$, x_1x_2	$x_2^m-x_1^m$	m	$\partial_{x_1}\theta_1+\partial_{x_2}\theta_2$, $\partial_{x_2^{m-1}}\theta_1+\partial_{x_1^{m-1}}\theta_2$	1, $m-1$	$x_2^m-x_1^m$	m
$G(m,1,n)$ for $m>1$	$\sum_{j=1}^n x_j^{mi}$	$\prod_{1\leqslant i<j\leqslant n}(x_j^m-x_i^m)$ $\cdot(x_1\cdots x_n)^{m-1}$	$m\binom{n}{2}+n(m-1)$	$\sum_{j=1}^n \partial_{x_j^{(i-1)m+1}}\theta_j$	$(i-1)m+1$	$\prod_{1\leqslant i<j\leqslant n}(x_j^m-x_i^m)$ $\cdot x_1\cdots x_n$	$m\binom{n}{2}+n$
$G(m,p,n)$ for $p\neq m$	$\sum_{j=1}^n x_j^{mi}$ (for $1\leqslant i\leqslant n-1$) $(x_1\cdots x_n)^{m/p}$ (for $i=n$)	$\prod_{1\leqslant i<j\leqslant n}(x_j^m-x_i^m)$ $\cdot(x_1\cdots x_n)^{m/p-1}$	$m\binom{n}{2}+n\left(\frac{m}{p}-1\right)$	$\sum_{j=1}^n \partial_{x_j^{(i-1)m+1}}\theta_j$	$(i-1)m+1$	$\prod_{1\leqslant i<j\leqslant n}(x_j^m-x_i^m)$ $\cdot x_1\cdots x_n$	$m\binom{n}{2}+n$
$G(m,m,n)$	$\sum_{j=1}^n x_j^{mi}$ (for $1\leqslant i\leqslant n-1$) $x_1\cdots x_n$ (for $i=n$)	$\prod_{1\leqslant i<j\leqslant n}(x_j^m-x_i^m)$	$m\binom{n}{2}$	$\sum_{j=1}^n \partial_{x_j^{(i-1)m+1}}\theta_j$ (for $1\leqslant i\leqslant n-1$) $\sum_{j=1}^n \partial_{(x_1\cdots\widehat{x}_j\cdots x_n)^{m-1}}\theta_j$ (for $i=n$)	$(i-1)m+1$ (for $1\leqslant i\leqslant n-1$) $(n-1)(m-1)$ (for $i=n$)	$\prod_{1\leqslant i<j\leqslant n}(x_j^m-x_i^m)$	$m\binom{n}{2}$

Table 7.1: Summary of explicit descriptions of the basic invariants f_i, the Vandermondians Δ_G and their degrees, the generalized exterior derivatives d_i, the co-exponents e_i^*, and the co-Vandermondians Δ_G^* and their degrees for all non-exceptional pseudo-reflection groups G. Here f_i ranges over $1\leqslant i\leqslant n$, and d_i and e_i^* range over $1\leqslant i\leqslant r$ where $r=n$ except $r=n-1$ when $G=\mathfrak{S}_n$.

G	$\mathrm{Hilb}(\mathcal{SH}_G; 1, z)$	$\mathrm{Hilb}(K[\partial_{x_1}, \ldots, \partial_{x_n}]\mathcal{SH}_G^{\mathrm{det}}; 1, z)$
$\mathfrak{S}_3$	$z^2 + 6z + 6$	(same)
$\mathfrak{S}_4$	$z^3 + 14z^2 + 36z + 24$	(same)
$\mathfrak{S}_5$	$z^4 + 30z^3 + 150z^2 + 240z + 120$	(same)
$\mathfrak{S}_6$	$z^5 + 62z^4 + 540z^3 + 1560z^2 + 1800z + 720$	(same)
$\mathfrak{B}_4$	$z^4 + 80z^3 + 464z^2 + 768z + 384$	(same)
$\mathfrak{B}_5$	$z^5 + 242z^4 + 2640z^3 + 8160z^2 + 9600z + 3840$	(same)
$\mathfrak{D}_2$	$z^2 + 4z + 4$	(same)
$\mathfrak{D}_3$	$z^3 + 14z^2 + 36z + 24$	(same)
$\mathfrak{D}_4$	$z^4 + 48z^3 + 240z^2 + 384z + 192$	$z^4 + 46z^3 + 238z^2 + 384z + 192$
$\mathfrak{D}_5$	$z^5 + 162z^4 + 1440z^3 + 4160z^2 + 4800z + 1920$	$z^5 + 147z^4 + 1405z^3 + 4140z^2 + 4800z + 1920$
H_3	$z^3 + 62z^2 + 180z + 120$	(same)
F_4	$z^4 + 244z^3 + 1396z^2 + 2304z + 1152$	$z^4 + 220z^3 + 1372z^2 + 2304z + 1152$
$G(3,1,2)$	$4z^2 + 21z + 18$	(same)
$G(4,1,2)$	$9z^2 + 40z + 32$	(same)
$G(5,1,2)$	$16z^2 + 65z + 50$	(same)
$G(5,1,3)$	$64z^3 + 665z^2 + 1350z + 750$	(same)
$G(3,1,4)$	$16z^4 + 609z^3 + 2862z^2 + 4212z + 1944$	(same)
$G(4,1,4)$	$81z^4 + 2320z^3 + 9920z^2 + 13824z + 6144$	(same)
$G(4,2,4)$	$76z^4 + 1451z^3 + 5408z^2 + 7104z + 3072$	$z^4 + 544z^3 + 3616z^2 + 6144z + 3072$
$G(4,4,4)$	$33z^4 + 416z^3 + 1920z^2 + 3072z + 1536$	$z^4 + 286z^3 + 1822z^2 + 3072z + 1536$

Table 7.2: Computer calculations of Hilbert series of the terms in (1.5). The $q = 1$ specialization has been taken to make the output manageable. (1.5) holds for a given G if and only if the two columns agree. In each case, (1.7) and (1.9) indeed hold, except (1.7) fails for $G(4, 2, 4)$ and $G(4, 4, 4)$. Calculations were done using SageMath [The21]. The exceptional group calculations were done with [Meh88].

Remark 7.4. The preceding arguments handle the vast majority of the groups with $m = p$ as well. One finds that the result holds except possibly when $k \in \{n-2, n-1\}$ and when p-contraction starts at the $(n-1)$st row. Note that the "extra" elements $\mathrm{d}_2\Delta_{G(2,2,2)}$ and $\mathrm{dd}_2\Delta_{G(2,2,2)}$ from Example 7.2 are of this form, though they are det-isotypic.

Acknowledgements

The first author thanks Brendon Rhoades, Bruce Sagan, and Mike Zabrocki for insightful discussions, as well as the anonymous referees for their careful reading and useful suggestions.

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