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Stable perfect isometries between blocks of finite groups

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STABLE PERFECT ISOMETRIES BETWEEN BLOCKS OF FINITE GROUPS

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by

Çisil Karagüzel

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Abstract

Stable perfect isometries between blocks of finite groups

by

Çisil Karagüzel

Let $(\mathbb{K}, \mathcal{O}, F)$ be a large enough p-modular system for finite groups G and H. Let A be a p-block of the group algebra $\mathcal{O}G$ and B be a p-block of the group algebra $\mathcal{O}H$. Broué introduced the definition of a perfect isometry between the p-blocks A and B which is a generalized $\mathbb{K}$-valued character leading to a special bijection between the sets of irreducible $\mathbb{K}$-characters of A and B. We introduce and investigate the notion of a stable perfect isometry, a way to consider perfect isometries up to generalized projective characters of the corresponding p-blocks. The main interest lies in understanding for which blocks all stable perfect self-isometries can be lifted to perfect self-isometries. We verify this for algebras of abelian p-groups and certain cases of blocks with cyclic defect group as well as blocks with Klein four defect group. We also introduce the notion of a stable p-permutation equivalence. Given block A, we show that if all stable perfect self-isometries of A lift to perfect self-isometries, then all stable p-permutation self-equivalences of A lift to p-permutation self-equivalences.

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Introduction

Let $(\mathbb{K}, \mathcal{O}, F)$ be a large enough p-modular system for a finite group G. The group algebra $\mathcal{O}G$ decomposes into a direct product of indecomposable algebras which are called block algebras. To understand the module category of $\mathcal{O}G$, it is sufficient to understand the module categories of its block algebras.

The number of irreducible characters, irreducible Brauer characters and the module category $\mathrm{mod}(B)$ are some of the characteristics of a block algebra B of a finite group G. These are called global invariants of a block algebra B. On the other hand, a block algebra B of a finite group G has local invariants such as defect groups and fusion systems.

In the representation theory, there are conjectures that concern the local-global features of block algebras. Broué's abelian defect group conjecture suggests an explanatory mechanism for some of these conjectures.

Conjecture 1. *(Broué's abelian defect group conjecture): Let G be a finite group, b be a block of $\mathcal{O}G$, and D be a defect group of b. Moreover, let c denote the Brauer correspondent of b. If D is abelian, then there is an equivalence between the bounded derived categories of $\mathcal{O}Gb$ and $\mathcal{O}N_G(D)c$.*

There are different versions of the abelian defect group conjecture depending on the chosen equivalence. A splendid Rickard equivalence gives the strongest version.

The abelian defect group conjecture has only been proven in certain cases such as for blocks of symmetric groups by Chuang and Rouquier [7] and for the cyclic defect group case over F by Rickard [20] and over $\mathcal{O}$ by Linckelmann [11] as well as by Rouquier [21] with a description of a splendid Rickard complex for a block with cyclic defect group.

Being one of the strongest block equivalences, a splendid Rickard equivalence implies many other block equivalences. On the Grothendieck group level, a version of such a block equivalence is a p-permutation equivalence introduced by Boltje and Xu in [2] in a restricted case, and extended and studied in detail by Boltje and Perepelitsky in [1]. It is known that such an equivalence preserves various local invariants of blocks such as defect groups, block fusion systems and Külshammer-Puig classes. Moreover, every p-permutation equivalence gives us a perfect isometry, which is a weaker form of a block equivalence. Perfect isometries were introduced by Broué in [5].

If there is a perfect isometry between two block algebras A and B, then they have the same number of irreducible characters and isomorphic centers as $\mathcal{O}$-algebras, see [5] for further details. The set of perfect self-isometries of a block B has a group structure that is denoted by $PI(B)$. Linckelmann [11] proved that given a block B of $\mathcal{O}G$ with a cyclic defect group D and with the inertial quotient E, B and $\mathcal{O}(D \rtimes E)$ are derived equivalent. This result implies that $PI(B) \cong PI(\mathcal{O}(D \rtimes E))$. The precise group structure of $PI(\mathcal{O}(D \rtimes E))$ can be found in Ruengrot [22] and Sambale [24].

Examples of equivalences of blocks on the categorical level are Morita equivalences, derived equivalences and splendid Rickard equivalences. A Morita equivalence and a splendid Rickard equivalence imply a derived equivalence which implies a perfect isometry.

The stable versions of most of these block equivalences on the categorical level have been defined and studied. For example, stable equivalences of Morita type are defined and studied by Broué [6]. Linckelmann [13] proves that there exists a stable equivalence of Morita type between two algebras $\mathcal{O}P$ and $\mathcal{O}Q$ of non-trivial p-groups if and only if there is a Morita equivalence between $\mathcal{O}P$ and $\mathcal{O}Q$.

Although it may be a straightforward question to ask if there are stable versions of the block equivalences on the Grothendieck group level, to the best of our knowledge, they have not been considered before. In this thesis, we will introduce and study the notions of stable perfect isometries as well as stable p-permutation equivalences.

Chapters 1, 2, and 3, we review basics of the block theory and representation theory, including the definitions and properties of perfect isometries and p-permutation equivalences.

In Chapter 4, we introduce the notion of a stable perfect isometry between blocks of finite groups. We let $\mathrm{SPI}(A, B)$ denote the set of stable perfect isometries between the blocks A and B. We introduce the set of cosets $\overline{\mathrm{SPI}}(A, B)$ and the group structure of $\overline{\mathrm{SPI}}(B)$. We define the group homomorphism $\Phi : \mathrm{PI}(B) \to \overline{\mathrm{SPI}}(B)$. We discuss the finiteness of the group $\overline{\mathrm{SPI}}(B)$. Mimicking the result of Broué in [6] for the stable equivalences of Morita type regarding the stable centers, we show that stable perfect isometry also induces an $\mathcal{O}$-module isomorphism between stable centers of the corresponding blocks.

In Chapter 5, we verify the surjectivity of the map $\Phi : \mathrm{PI}(\mathcal{O}P) \to \overline{\mathrm{SPI}}(\mathcal{O}P)$ for an abelian p-group P. We prove that for two abelian p-groups P and Q, there is a perfect isometry between $\mathcal{O}P$ and $\mathcal{O}Q$ if and only if there is a stable perfect isometry between $\mathcal{O}P$ and $\mathcal{O}Q$.

In Chapter 6, we study the stable perfect isometries of blocks with cyclic defect groups. We will have a partial answer to the surjectivity of the map $\Phi : \mathrm{PI}(B) \to \overline{\mathrm{SPI}}(B)$. The results of this chapter are based on the result of Linckelmann [11] which is that there is a derived equivalence between the block algebra B and $\mathcal{O}(D \rtimes E)$ where B has a cyclic defect group D and the inertial quotient E.

In Chapter 7, we verify the surjectivity of the map $\Phi : \mathrm{PI}(B) \to \overline{\mathrm{SPI}}(B)$ for a block algebra B with a Klein four defect group. The result of this chapter is based on the fact that such a block algebra B is perfectly isometric to either $\mathcal{O}V_4$ or $\mathcal{O}A_4$, which follows from [12].

In Chapter 8, we introduce the notion of a stable p-permutation equivalence. We show that a stable p-permutation equivalence induces a stable perfect isometry. We show that to verify the surjectivity of the map $\Psi : T^{\Delta}_{\circ}(B, B) \to \overline{\mathrm{stab}_{\circ} T^{\Delta}(B)}$, it suffices to verify the surjectivity of $\Phi : \mathrm{PI}(B) \to \overline{\mathrm{SPI}}(B)$.

Chapter 1

Preliminaries

In this chapter, we review the background information and notation that will be used throughout this thesis. In Section 1.1, we review the notion of p-modular systems, and in Section 1.2 introduce definitions and notations in block theory. In Sections 1.3 and 1.4 we review p-permutation modules and Grothendieck groups, respectively. Finally, in Sections 1.5 and 1.6 we will review some known block equivalences on the categorical level, which motivate the main topic of this thesis.

1.1 p-modular systems

The main reference of this section is [17].

A **valuation** of a field $\mathbb{K}$ is a function $\nu : \mathbb{K}^\times \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}^\times$,

$$\nu(ab) = \nu(a) + \nu(b) \text{ and } \nu(a+b) \geq \min(\nu(a), \nu(b)).$$

Such a valuation is called **discrete** if the image is isomorphic to $\mathbb{Z}$. One can extend ν via $\nu(0) = \infty$ and define $\nu : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$. We define **the valuation ring**

$$\mathcal{O} := \{a \in \mathbb{K} \mid \nu(a) \geq 0\}$$

which is a subring of $\mathbb{K}$ and which has a unique maximal ideal called, **the valuation ideal**,

$$\wp := \{a \in \mathbb{K} \mid \nu(a) > 0\}.$$

Hence, one can define the field $F := \mathcal{O}/\wp$ called the residue field of $(\mathbb{K}, \nu)$.

By using the notion of "equivalence" on the discrete valuations, one can assume that ν is **normalized**, i.e. $\nu(\mathbb{K}^\times) = \mathbb{Z}$. For such a normalized valuation ν, we let $\pi \in \mathbb{K}$ be an element such that $\nu(\pi) = 1$. In this case, one has $\mathcal{O}^\times = \{a \in \mathbb{K} \mid \nu(a) = 0\}$ and every element $a \in \mathbb{K}^\times$ can be written as $a = u\pi^n$ with unique elements $u \in \mathcal{O}^\times$ and $n \in \mathbb{Z}$. Furthermore, in this case, $\mathbb{K}$ is the field of fractions of $\mathcal{O}$ and the ideals of $\mathcal{O}$ are as follows:

$$0 = \bigcap_{n \in \mathbb{N}} \pi^n \mathcal{O} \subseteq \cdots \subseteq \pi^2 \mathcal{O} \subseteq \pi \mathcal{O} \subseteq \mathcal{O}.$$

Next, for a given pair $(\mathbb{K}, \nu)$ as above, one can define a metric on $\mathbb{K}$ as follows: choose $\gamma \in \mathbb{R}$ with $0 < \gamma < 1$. Then, $d(a, b) := \gamma^{\nu(a-b)}$ is a metric on $\mathbb{K}$. $\mathbb{K}$ is a topological space with respect to this metric. It should be noted that such a topology is independent of "equivalence classes" of ν and γ. Now we are ready to define a p-modular system:

Definition 2. *Let p be a prime. A p-modular system is a triple $(\mathbb{K}, \mathcal{O}, F)$ where $\mathbb{K}$ is a field of characteristic 0 which has a valuation ν such that $\mathcal{O}$ is its complete discrete valuation ring, F is its residue field and has characteristic p.*

We say a p-modular system $(\mathbb{K}, \mathcal{O}, F)$ is *large enough* for a given finite group G if $\mathbb{K}$ contains a root of unity of order $\exp(G)$. In this case, $\mathbb{K}$ and F are splitting fields for G and all of its subgroups. Such a p-modular system always exists, e.g., consider $(\mathbb{Q}_p(\zeta), \mathbb{Z}_p[\zeta], \mathbb{F}_q)$ where q is the smallest power of p such that the p'-part of $\exp(G)$ divides $q-1$ and ζ is a root of unity of order $\exp(G)$. Throughout, we will always assume that a given p-modular system is large enough for groups that are considered.

1.2 Block theory

In this section, we review the basics of block theory. Further details can be found in [15] and [17].

Let G be a finite group and let $(\mathbb{K}, \mathcal{O}, F)$ be a large enough p-modular system for G. One can write a decomposition of $1 \in Z(\mathcal{O}G)$ such that

$$1 = e_1 + e_2 + \cdots + e_t$$

where $e_1, e_2, \cdots, e_t$ are primitive idempotents of $Z(\mathcal{O}G)$. Letting $B_i := \mathcal{O}Ge_i$, we get the decomposition of the group algebra $\mathcal{O}G$:

$$\mathcal{O}G = B_1 \times B_2 \times \cdots \times B_t.$$

In this case, each e_i is called **a block idempotent** of $\mathcal{O}G$ and each B_i is called **a block algebra** of $\mathcal{O}G$. The set of block idempotents of $\mathcal{O}G$ is denoted by $\mathrm{Bl}(\mathcal{O}G)$.

We set $\mathbb{K}B := \mathbb{K} \otimes_{\mathcal{O}} B$ for a block B of $\mathcal{O}G$. We let $\mathrm{Irr}_{\mathbb{K}}(G)$ denote the set of irreducible $\mathbb{K}$-characters of G and we let

$$\mathrm{Irr}_{\mathbb{K}}(B_i) := \{\chi \in \mathrm{Irr}_{\mathbb{K}}(G) : \text{corresponding irreducible } \mathbb{K}G\text{-module } V_\chi \text{ belongs to } e_i\}.$$

Similarly, we let $\mathrm{IBr}_F(G)$ denote the set of irreducible Brauer characters of G, and we let

$$\mathrm{IBr}_F(B_i) := \{\varphi \in \mathrm{IBr}(G) : \text{corresponding simple } FG\text{-module belongs to } \bar{e}_i\}$$

where $\bar{e}_i$ is the image of e_i under the map $- : \mathcal{O}G \twoheadrightarrow FG \cong \mathcal{O}G/\pi\mathcal{O}G$.

We have $\mathrm{Irr}_{\mathbb{K}}(G) = \bigsqcup_{i=1}^{t} \mathrm{Irr}_{\mathbb{K}}(B_i)$, and similarly $\mathrm{IBr}_F(G) = \bigsqcup_{i=1}^{t} \mathrm{IBr}_F(B_i)$. Let χ be in $\mathrm{Irr}_{\mathbb{K}}(G)$, one sets

$$e_\chi := \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(x^{-1})x.$$

The primitive idempotents of $Z(\mathbb{K}G)$ are $\{e_\chi \mid \chi \in \mathrm{Irr}_{\mathbb{K}}(G)\}$. For $B \in \mathrm{Bl}(\mathcal{O}G)$, one has $e_B = \sum_{\chi \in \mathrm{Irr}_{\mathbb{K}}(B)} e_\chi$.

Remark 3. *([16, Theorem 2.13]) Let $h \in G$ and χ_i, χ_j be irreducible ordinary characters of G. Then the following holds:*

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(gh)\chi_j(g^{-1}) = \delta_{i,j} \frac{\chi_i(h)}{\chi_i(1)}.$$

Definition 4. *Let g be an element of a finite group G.*

- *g is called p-element if $\mathrm{o}(g) = p^b$ for some $b \in \mathbb{N}_\circ$.*
- *g is called p-regular (or p'-element) if $p \nmid \mathrm{o}(g)$.*
- *g is called p-singular if $p \mid \mathrm{o}(g)$.*

Proposition 5. *Let g be an element of a finite group G. There exists a unique pair $(g_p, g_{p'})$ of elements in G such that*

- *$g = g_p \cdot g_{p'}$.*
- *g_p is a p-element and $g_{p'}$ is a p'-element.*
- *$g_p \cdot g_{p'} = g_{p'} \cdot g_p$.*

Notation 6. *Let G be a finite group. We let $G_{p'}$ denote the set of p-regular elements, and let G_p denote the set of p-singular elements.*

Notation 7. *Let G be a finite group, and e a block of $\mathcal{O}G$ and $B := \mathcal{O}Ge$ be the corresponding block algebra.*

- *$k(G) := |\mathrm{Irr}_{\mathbb{K}}(G)|$.*
- *$l(G) := |\mathrm{IBr}_F(G)|$.*
- *$k(B) := |\mathrm{Irr}_{\mathbb{K}}(B)|$.*
- *$l(B) := |\mathrm{IBr}_F(B)|$.*

Remark 8. *Let $\{\chi_1, \chi_2, \cdots, \chi_k\}$ denote the set of irreducible $\mathbb{K}$-characters of G, and $\{\varphi_1, \cdots, \varphi_l\}$ denote the set of irreducible Brauer characters of G. Let $\{\eta_1, \eta_2, \cdots, \eta_l\}$ denote the projective indecomposable characters of G. Then, one has*

- $D := (d_{i,j}) \in \mathrm{Mat}_{k\times l}(\mathbb{N}_\circ)$ *is the decomposition matrix, where* $\mathrm{Res}^G_{G_{p'}}(\chi) = \sum_{j=1}^{l} d_{ij}\varphi_j$*.*
- $C := (c_{i,j}) \in \mathrm{Mat}_{l\times l}(\mathbb{N}_\circ)$ *is the Cartan matrix such that* $C = D^t D$*.*
- *For each* $i \in \{1, 2, \cdots, l\}$*, we have,* $\eta_i = \sum_{i=1}^{k} d_{ji}\chi_j$*.*
- *Let* $e \in \mathrm{Bl}(\mathcal{O}G)$*. If* χ_i *and* η_j *belong to* e *and* φ_r *does not belong to* e*, then* $d_{ir} = 0$ *and* $c_{jr} = 0$*. If we reorder* $\chi_1, \cdots, \chi_k$*,* $\varphi_1, \cdots, \varphi_l$ *and* $\eta_1, \cdots, \eta_l$ *according to the block algebras* $B_1 := \mathcal{O}Ge_1, \cdots, B_t := \mathcal{O}Ge_t$ *that they belong, one has the following:*

$$D = \begin{pmatrix} D_{B_1} & 0 & \cdots & 0 \\ 0 & D_{B_2} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D_{B_t} \end{pmatrix}$$

so that $D^t_{B_i} D_{B_i} = C_{B_i}$ *where* $D_{B_i} \in \mathrm{Mat}_{l(B_i)\times k(B_i)}$ *with* $k(B_i) := |\mathrm{Irr}_{\mathbb{K}}(B_i)|$ *and also* $l(B_i) := |\mathrm{IBr}_F(B_i)|$*. We will refer the matrix* D_{B_i} *as the decomposition matrix of the block* B_i*, and* C_{B_i} *as the Cartan matrix of the block* B_i*.*

We note that the following concepts and results can be found in [14] and [15].

Notation 9. *Let* $\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(G)$ *be the free* $\mathbb{Z}$*-span of the irreducible* $\mathbb{K}$*-characters of* G*. The elements of* $\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(G)$ *will be referred as generalized characters or virtual characters. For a block algebra* B *of* $\mathcal{O}G$*, we let* $\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(B)$ *denote the free* $\mathbb{Z}$*-span of the irreducible* $\mathbb{K}$*-characters of the block algebra* B *of* G*.*

Notation 10. *Let* $\mathbb{Z}\mathrm{IBr}_F(G)$ *be the free* $\mathbb{Z}$*-span of the irreducible Brauer characters of* G*. For a block algebra* B *of* $\mathcal{O}G$*, we let* $\mathbb{Z}\mathrm{IBr}_F(B)$ *denote the free* $\mathbb{Z}$*-span of the irreducible Brauer characters of the block algebra* B *of* G*.*

Note that since $\mathrm{Irr}_{\mathbb{K}}(G) = \bigsqcup_{i=1}^{t} \mathrm{Irr}_{\mathbb{K}}(B_i)$, one has $\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(G) = \bigoplus_{i=1}^{t} \mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(B)$. Furthermore, since $\mathrm{IBr}_F(G) = \bigsqcup_{i=1}^{t} \mathrm{IBr}_F(B_i)$, one has $\mathbb{Z}\mathrm{IBr}_F(G) = \bigoplus_{i=1}^{t} \mathbb{Z}\mathrm{IBr}_F(B)$. For the details, we refer the reader to [15], Chapter 6, Theorem 6.5.3.

Notation 11. *Let* $\mathrm{Pr}(\mathcal{O}G)$ *denote the subgroup of* $\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(G)$ *which is generated by the characters of finitely generated (left) projective* $\mathcal{O}G$*-modules. Similarly, for a block algebra* $B := \mathcal{O}Ge$ *of*

$\mathcal{O}G$, we let $\mathrm{Pr}(\mathcal{O}Ge)$ denote the subgroup of $\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)$ which is generated by the characters of finitely generated (left) projective $\mathcal{O}Ge$-modules. Similarly, we define $\mathrm{Pr}(FG)$ and $\mathrm{Pr}(FG\bar{e})$ as subgroups of $\mathbb{Z}\mathrm{IBr}_F(G)$ and $\mathbb{Z}\mathrm{IBr}_F(\mathcal{O}Ge)$, respectively.

Recall that we always assume that $(\mathbb{K}, \mathcal{O}, F)$ is a p-modular system for G. In fact, the next result is true when $(\mathbb{K}, \mathcal{O}, F)$ is large enough for a given finite group G.

Remark 12. *(*[14, Theorem 5.14.1]*) The decomposition map*

$$\begin{aligned} d_G : \mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(G) &\to \mathbb{Z}\mathrm{IBr}_F(G) \\ \chi &\mapsto \mathrm{Res}^G_{G_{p'}}(\chi) \end{aligned}$$

is a surjective group homomorphism.

Let $e \in \mathrm{Bl}(\mathcal{O}G)$ and $B = \mathcal{O}Ge$ be the corresponding block algebra. The decomposition map d_G in Remark 12 induces a group homomorphism $d_{G,B} : \mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(B) \to \mathbb{Z}\mathrm{IBr}_{\mathbb{F}}(B)$.

Notation 13. *(*[15, Chapter 6, Section 5]*) $L^0(B)$ denotes the kernel of $d_{G,B}$. Hence $L^0(B)$ consists of all generalized characters in $\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(B)$ that vanish on all p'-elements of G.*

Remark 14. *(*[15, Theorem 6.5.11]*) Let G be a finite group and e be a block of $\mathcal{O}G$ and $B = \mathcal{O}Ge$ be the corresponding block algebra. Suppose that $\mathbb{K}$ is a splitting field for $\mathbb{K}Ge$. The following hold:*

(i) The decomposition map $d_{G,B} : \mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(B) \to \mathbb{Z}\mathrm{IBr}_F(B)$ is surjective, and it induces an isomorphism $\mathrm{Pr}(\mathcal{O}Ge) \cong \mathrm{Pr}(FG\bar{e})$.

(ii) The matrix C_B is positive definite, and $\det(C_B) > 0$.

(iii) One has $L^0(\mathcal{O}Ge)^\perp = \mathrm{Pr}(\mathcal{O}Ge)$; equivalently, $\mathrm{Pr}(\mathcal{O}Ge)$ consists of all generalized characters in $\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(B)$ that vanish on all p-singular elements of G.

Remark 15. *(*[15, Chapter 6]*) Note that by Remark 14, we have an isomorphism of abelian groups*

$$\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)/(L^0(\mathcal{O}Ge) \oplus \mathrm{Pr}(\mathcal{O}Ge)) \cong \mathbb{Z}\mathrm{IBr}_F(FG\bar{e})/\mathrm{Pr}(FG\bar{e}).$$

The order of this group is $|\det(C_B)|$ *where* $B = \mathcal{O}Ge$*, and the orders of the cyclic direct factors of this group are the elementary divisors of* C_B*.*

Definition 16. *Let* G *be a finite group, and* $H \leq G$*. We consider the* H*-fixed points of* $\mathcal{O}G$ *under conjugation,*

$$(\mathcal{O}G)^H := \{x \in \mathcal{O}G \mid {}^h x = x \text{ for all } h \in H\}. \tag{1.1}$$

If $K \leq H \leq G$*, one has* $(\mathcal{O}G)^H \subseteq (\mathcal{O}G)^K$*. On the other side, we have the relative trace map*

$$\mathrm{Tr}_K^H : (\mathcal{O}G)^K \to (\mathcal{O}G)^H, \tag{1.2}$$

given by $x \mapsto \sum_{h \in [H/K]} {}^h x$*. We let* $(\mathcal{O}G)_K^H := \mathrm{Im}(\mathrm{Tr}_K^H)$*.*

Definition 17. *(*[14, Theorem 5.4.1]*) Let* G *be a finite group and* P *be a* p*-subgroup of* G*. The canonical* F*-linear projection* $FG \to FC_G(P)$ *induces a split surjective homomorphism of* $N_G(P)$*-algebras over* F*,*

$$\mathrm{Br}_P : (FG)^P \to FC_G(P) \tag{1.3}$$

where $\ker(\mathrm{Br}_P) = \sum_{Q<P} (FG)_Q^P$*. One can also precompose this morphism with* $(\mathcal{O}G)^P \to (FG)^P$ *and we again will denote this map with* $\mathrm{Br}_P : (\mathcal{O}G)^P \to FC_G(P)$*, which is an* $\mathcal{O}$*-algebra homomorphism which is not necessarily split any more.*

Definition 18. *Let* $e \in \mathrm{Bl}(\mathcal{O}G)$*. A defect group of* e *is a subgroup* P *of* G *minimal with respect to the property that* $e \in (\mathcal{O}G)_P^G$*. Equivalently, it is a subgroup* P *of* G *maximal with respect to the property that* $\mathrm{Br}_P(e) \neq 0$*.*

Theorem 19. *Let* P *be a defect group of* $e \in \mathrm{Bl}(\mathcal{O}G)$*. Then, the following hold:*

(i) P *is a* p*-subgroup of* G*.*

(ii) The defect groups of a block e *form a* G*-conjugacy class of* p*-subgroups of* G*.*

Definition 20. *The defect of a block* $e \in \mathrm{Bl}(\mathcal{O}G)$ *is the unique integer* $d_p(e) := d$ *such that the order of defect groups of* e *is* p^d*.*

Definition 21. *([15, Definition 6.7.7]) Let $b \in \mathrm{Bl}(\mathcal{O}G)$ and let P be a defect group of b. There is a unique block c of $\mathcal{O}N_G(P)$ with P as a defect group with the property that $\mathrm{Br}_P(b) = \mathrm{Br}_P(c)$. The block c is called the Brauer correspondent of b. If $e \in \mathrm{Bl}(\mathcal{O}C_G(P))$ such that $ec = e$, then the group $E = N_G(P, e)/PC_G(P)$ is called the inertial quotient of b.*

1.3 p-permutation modules

The further details of this section can be found in Chapter 5 of [14].

Definition 22. *Let M be a finitely generated indecomposable $\mathcal{O}G$-module. A subgroup Q of G is called a vertex of M if Q is a minimal with the property that M is relatively Q-projective. If Q is a vertex of M, an $\mathcal{O}Q$-source of M is an indecomposable $\mathcal{O}Q$-module V such that M is isomorphic to a direct summand of $\mathrm{Ind}_Q^G(V)$.*

Remark 23. *A finitely generated indecomposable $\mathcal{O}G$-module U is projective if and only if U has the trivial group as a vertex and $\mathcal{O}$ as a source.*

Definition 24. *An indecomposable $\mathcal{O}G$-module M is called a trivial source module if for some vertex Q of M the trivial $\mathcal{O}Q$-module $\mathcal{O}$ is a source of M.*

Remark 25. *An indecomposable $\mathcal{O}G$-module M is a trivial source module if and only if M is a direct summand of a permutation module.*

Definition 26. *An $\mathcal{O}G$-module M is called a p-permutation module if for any p-subgroup P of G, $\mathrm{Res}_P^G(M)$ is a permutation $\mathcal{O}P$-module.*

Remark 27. *The following hold:*

(i) A finitely generated $\mathcal{O}G$-module M is a p-permutation module if and only if M is a direct sum of trivial source $\mathcal{O}G$-modules.

(ii) Any direct summand of a p-permutation module is again a p-permutation module.

Note that projective modules are p-permutation modules.

Notation 28. *We let ${}_{\mathcal{O}G}triv$ denote the category of p-permutation $\mathcal{O}G$-modules.*

1.4 Grothendieck groups

Given a finite-dimensional algebra A over a field k, the **Grothendieck group**, $R(A)$, with respect to short exact sequences, is the free abelian group with $\mathbb{Z}$-basis given by isomorphism classes $[S]$ of simple (left) A-modules. We let $A = \mathbb{K}G$ be the group algebra over the field $\mathbb{K}$. We let $R(\mathbb{K}G)$ to denote the Grothendieck group of finitely generated (left) $\mathbb{K}G$-modules. We can identify $R(\mathbb{K}G)$ with $\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(G)$. Similarly, one can define $R(FG)$ and identify it with $\mathbb{Z}\mathrm{IBr}_F(G)$.

Let e be a block of $\mathcal{O}G$ and let $R(\mathbb{K}Ge)$ denote the Grothendieck group of (left) $\mathbb{K}Ge$-modules. We can identify $R(\mathbb{K}Ge)$ with $\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)$. We let $R(FG\bar{e})$ denote the Grothendieck group of $FG\bar{e}$-modules and identify it with $\mathbb{Z}\mathrm{IBr}_F(FG\bar{e})$.

We consider the $\mathbb{Z}$-span of the isomorphism classes of indecomposable projective (left) $\mathcal{O}G$-modules and we use the same notation $\mathrm{Pr}(\mathcal{O}G)$. Similarly, we use the notations $\mathrm{Pr}(\mathcal{O}Ge)$, $\mathrm{Pr}(FG)$ and $\mathrm{Pr}(FG\bar{e})$ with the obvious meanings.

We let $T(\mathcal{O}G)$ denote the Grothendieck group of the category ${}_{\mathcal{O}Ge}triv$ with respect to direct sums. For a block e of $\mathcal{O}G$, one can define $T(\mathcal{O}Ge)$ and $\mathrm{Pr}(\mathcal{O}Ge)$. Note that clearly $\mathrm{Pr}(\mathcal{O}G) \subseteq T(\mathcal{O}G)$ and $\mathrm{Pr}(\mathcal{O}Ge) \subseteq T(\mathcal{O}Ge)$. Similarly, one can define $T(FG)$ and $T(FG\bar{e})$. We have $\mathrm{Pr}(FG) \subseteq T(FG)$ and $\mathrm{Pr}(FG\bar{e}) \subseteq T(FG\bar{e})$.

One has

$$Pr(\mathcal{O}Ge) \subseteq T(\mathcal{O}Ge) \xrightarrow{\kappa_G} R(\mathbb{K}Ge) \tag{1.4}$$

where the map κ_G is induced by the scalar extension $\mathbb{K} \otimes_{\mathcal{O}} -$. It is also important to note since κ_G is injective on $\mathrm{Pr}(\mathcal{O}Ge)$, we also use the same notation for their characters in $R(\mathbb{K}Ge)$. Recall from Remark 14 that an element $\chi \in R(\mathbb{K}Ge)$ belongs to $\mathrm{Pr}(\mathcal{O}Ge)$ if and only if $\chi(g) = 0$ for all p-singular elements $g \in G$.

1.5 Morita equivalences

Although the notion of Morita equivalence can be defined in a more general setting, e.g., for symmetric algebras, in this thesis, we are interested in Morita equivalence for block algebras. The further details of this section can be found in [6] and [14, Chapter 2].

We let A and B be block algebras of $\mathcal{O}G$ and $\mathcal{O}H$, respectively for finite groups G and H. We let mod(A) denote the category of finitely generated (left) A-modules and similarly mod(B) denotes the category of finitely generated (left) B-modules.

The block algebras A and B are called **Morita equivalent** if the categories mod(A) and mod(B) are equivalent. Equivalently, if there exists an (A, B)-bimodule M and a (B, A)-bimodule N satisfying the following:

(j) $M \otimes_B N \cong A$ as an (A, A)-bimodule,

(ii) $N \otimes_A M \cong B$ as a (B, B)-bimodule,

(iii) M and N are finitely generated projective as left and right modules.

In this case, one can define the functors

$$M \otimes_B - : \mathrm{mod}(B) \to \mathrm{mod}(A)$$
$$N \otimes_A - : \mathrm{mod}(A) \to \mathrm{mod}(B)$$

which sends simple modules to simple modules.

Furthermore, the following invariants coincide for the Morita equivalent block algebras A and B:

- $Z(A) \cong Z(B)$.
- $k(A) = k(B)$ and $l(A) = l(B)$.
- $D_A = D_B$ and $C_A = C_B$.

Related to our interest, one can also define a stable version of a Morita equivalence. Later on, we will use a similar idea for perfect isometries to define stable perfect isometries and obtain similar traits.

1.6 Stable equivalences of Morita type

Although stable equivalences of Morita type are defined in a more general setting, namely for symmetric algebras, we again limit our attention to block algebras of finite groups.

For further details we refer to [6].

Let A be a block algebra of $\mathcal{O}G$ and B be a block algebra of $\mathcal{O}H$ for finite groups G and H. A and B are called *stably equivalent of Morita type* if there exists an (A, B)-bimodule M and a (B, A)-bimodule N satisfying:

(j) $M \otimes_B N \cong A \oplus V$ as an (A, A)-bimodule, for some finitely generated projective (A, A)-bimodule V,

(ii) $N \otimes_A M \cong B \oplus W$ as a (B, B)-bimodule, for some finitely generated projective (B, B)-bimodule W,

(iii) M and N are finitely generated projective as left and right modules.

In this case, the stable module categories of A and B are equivalent.

Definition 29. *([6], Section 5) Let $e \in \mathrm{Bl}(\mathcal{O}G)$ and $A = \mathcal{O}Ge$. We set $Z^{st}(A) := Z(A)/Z^{pr}(A)$, called the stable center of A. Here $Z^{pr}(A) := \{\sum_{g\in G} {}^g a \mid a \in A\}$, called the projective center of A.*

Proposition 30. *([6], 5.4 Proposition) Let A and B be block algebras of $\mathcal{O}G$ and $\mathcal{O}H$, respectively. A stable equivalence of Morita type between A and B induces an $\mathcal{O}$-algebra isomorphism between $Z^{st}(A)$ and $Z^{st}(B)$.*

We refer the reader to [6] for further details on stable equivalences of Morita type.

A similar statement to Proposition 30 will be observed for stable perfect isometries later in Proposition 66. In our case, we will only have an $\mathcal{O}$-module isomorphism instead of an $\mathcal{O}$-algebra isomorphism.

Further invariants of stable equivalences of Morita type are listed in [9]. More details can also be found in [15]. We list some of them here:

In the following set up, assume $A = \mathcal{O}Ge$ and $B = \mathcal{O}Hf$.

- $R(\mathbb{K}Ge)/\mathrm{Pr}(\mathcal{O}Ge) \cong R(\mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf)$
- $R(FG\bar{e})/\mathrm{Pr}(FG\bar{e}) \cong R(FH\bar{f})/\mathrm{Pr}(FH\bar{f})$.
- $L^0(A) \cong L^0(B)$.

1.6.1 Stable equivalences of Morita type for p-groups

Let P and Q be non-trivial p-groups. Linckelmann [13, Theorem 3.1] shows that if there is a stable equivalence of Morita type between $\mathcal{O}P$ and $\mathcal{O}Q$, then $\mathcal{O}P$ and $\mathcal{O}Q$ are Morita equivalent. The next theorem follows from [13, Theorem 3.1], [13, Corollary 3.2] and [13, Corollary 3.3]:

Theorem 31. *(*[13, Section 3]*) Let P and Q be non-trivial p-groups. Then, the following are equivalent:*

(a) $\mathcal{O}P$ and $\mathcal{O}Q$ are Morita equivalent.

(b) There is a stable equivalence of Morita type between $\mathcal{O}P$ and $\mathcal{O}Q$.

(c) $P \cong Q$.

In Chapter 5, we will try to mimic this theorem in the context of perfect isometries and stable perfect isometries.

Chapter 2

Perfect Isometries

Throughout this chapter, G, H and K denote finite groups. We assume that the p-modular system $(\mathbb{K}, \mathcal{O}, F)$ is large enough for G, H and K. In this section, we recall and introduce notations, concepts and well-known results related to perfect isometries. For more details on perfect isometries we refer the reader to [1], [5], and [15]. For the details on the perfect isometries of blocks with cyclic defect group, we refer to [22] and [24].

Recall that we identify $R(\mathbb{K}G)$ with $\mathbb{Z}\mathrm{Irr}_{\mathbb{K}}(G)$. We set $R(\mathbb{K}G, \mathbb{K}H) := R(\mathbb{K}G \otimes (\mathbb{K}H)^{\circ})$ where $(\mathbb{K}H)^{\circ}$ is the opposite algebra of $\mathbb{K}H$ and we identify $(\mathbb{K}H)^{\circ}$ with $\mathbb{K}H$ via the map $h^{\circ} \mapsto h^{-1}$. As a result, we identify $\mathbb{K}G \otimes (\mathbb{K}H)^{\circ}$ with $\mathbb{K}[G \times H]$ via $(g, h^{\circ}) \mapsto (g, h^{-1})$.

Throughout we let $e \in \mathrm{Bl}(\mathcal{O}G)$ and $f \in \mathrm{Bl}(\mathcal{O}H)$, and $B := \mathcal{O}Ge$ and $A := \mathcal{O}Hf$. We set $R(\mathbb{K}Ge, \mathbb{K}Hf) := R(\mathbb{K}[G \times H](e \otimes f^{\star}))$ where f^{*} is defined through the convention of opposite algebras above. Furthermore, if $\mu \in R(\mathbb{K}Ge, \mathbb{K}Hf)$, μ° denotes the $\mathbb{K}$-dual of μ in $R(\mathbb{K}Hf, \mathbb{K}Ge)$.

Given $\mu \in R(\mathbb{K}G, \mathbb{K}H)$ and $\nu \in R(\mathbb{K}G, \mathbb{K}K)$, the character $\mu \cdot_H \nu \in R(\mathbb{K}G, \mathbb{K}K)$ is defined as follows: for any $(g, k) \in G \times K$,

$$(\mu \cdot_H \nu)(g, k) = \frac{1}{|H|} \sum_{h \in H} \mu(g, h)\nu(h, k)$$

and in this way we have the following bilinear map

$$R(\mathbb{K}G, \mathbb{K}H) \times R(\mathbb{K}H, \mathbb{K}K) \to R(\mathbb{K}G, \mathbb{K}K)$$

sending (μ, ν) to $\mu \cdot_H \nu$. This map is induced by taking the tensor product $- \otimes_{\mathbb{K}H} -$ of bimodules.

As a special case, if we take $K = 1$, then one can see that every generalized character $\mu \in R(\mathbb{K}G, \mathbb{K}H)$ induces a group homomorphism

$$I_\mu : R(\mathbb{K}H) \to R(\mathbb{K}G) \tag{2.1}$$

given by $\psi \mapsto \mu \cdot_H \psi$.

2.1 Definition

Note that if $\mu \in R(\mathbb{K}Ge, \mathbb{K}Hf)$, then $I_\mu : R(\mathbb{K}Hf) \to R(\mathbb{K}Ge)$ for any blocks e, f of $\mathcal{O}G$ and $\mathcal{O}H$, respectively. A generalized character $\mu \in R(\mathbb{K}Ge, \mathbb{K}Hf)$ is called an **isometry** between $\mathcal{O}Ge$ and $\mathcal{O}Hf$ if the map $I_\mu : R(\mathbb{K}Hf) \to R(\mathbb{K}Ge)$ is bijective, and it preserves inner products, namely

$$\langle \psi, \psi' \rangle_H = \langle I_\mu(\psi), I_\mu(\psi') \rangle_G \tag{2.2}$$

for all $\psi, \psi' \in R(\mathbb{K}Hf)$.

The next remark provides equivalent definitions of an isometry between two blocks $\mathcal{O}Ge$ and $\mathcal{O}Hf$.

Remark 32. *(Remark 8.4(a),* [1]*) The following are equivalent:*

(i) μ *is an isometry between* $\mathcal{O}Ge$ *and* $\mathcal{O}Hf$.

(ii) $\mu \cdot_H \mu^\circ = \sum_{\chi \in \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)} \chi \times \chi^\circ$ *in* $R(\mathbb{K}Ge, \mathbb{K}Ge)$ *and* $\mu^\circ \cdot_G \mu = \sum_{\psi \in \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Hf)} \psi \times \psi^\circ$ *in* $R(\mathbb{K}Hf, \mathbb{K}Hf)$.

(iii) There exists a bijection $\mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Hf) \xrightarrow{\sim} \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)$, $\psi \mapsto \chi_\psi$ *and some* $\epsilon_\psi = \pm 1$ *such that one has* $\mu = \sum_{\psi \in \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Hf)} \epsilon_\psi \cdot \chi_\psi \times \psi^\circ$.

Notation 33. *We will use the notation* $[\mathbb{K}Ge]$ *for the character* $\sum_{\chi\in\mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)} \chi\times\chi^{\circ}$ *as it is the character of* $\mathbb{K}Ge$ *as a* $(\mathbb{K}G,\mathbb{K}G)$*-bimodule.*

A generalized character $\mu \in R(\mathbb{K}Ge, \mathbb{K}Hf)$ is called **perfect** (see [5]) if the following two conditions are satisfied:

(i) For all $(g,h)\in G\times H$, $\mu(g,h)$ is divisible by $|C_G(g)|$ and $|C_H(h)|$ in $\mathcal{O}$. This condition is called **the integrality condition**.

(ii) If $(g,h)\in G\times H$ is such that $\mu(g,h)\neq 0$, then g is a p'-element if and only if h is a p'-element. This condition is called **the separability condition**.

If only condition (ii) holds, we call μ **quasi-perfect**. We let $\mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)$ denote the set of quasi-perfect generalized characters in $R(\mathbb{K}Ge, \mathbb{K}Hf)$. If an isometry $\mu\in R(\mathbb{K}Ge, \mathbb{K}Hf)$ is perfect, μ is called **a perfect isometry** between $\mathcal{O}Ge$ and $\mathcal{O}Hf$. Perfect isometries are first defined by Broué in [5].

The next theorem provides a refined condition to check when a quasi-perfect character is perfect. The result is due to Kiyota [10], Kiyota's theorem is cited and explained in [22] and [24]. We note that the original result of Kiyota is only for an isometry; however, the theorem is also true for quasi-perfect generalized characters. The proof for the quasi-perfect generalized characters is almost the same as the original proof in [10].

We will also benefit from Kiyota's theorem later in this thesis.

Theorem 34. *(*[10, Theorem 2.2]*) Let* $\mu\in R(\mathbb{K}Ge, \mathbb{K}Hf)$ *be a generalized quasi-perfect character. Then* μ *is perfect if for all* p*-singular elements* $g\in G$*, and* $h\in H$*,* $\mu(g,h)$ *is divisible by* $|C_G(g)|$ *and* $|C_H(h)|$ *in* $\mathcal{O}$*.*

Lemma 35. *(*[18]*, Lemma 2.21) Let* $x\in G$ *and* $\varphi\in\mathrm{IBr}_F(G)$ *where* η_φ *is the corresponding indecomposable projective character of* G*. Then, one has*

$$\frac{\eta_\varphi(x)}{|C_G(x)|_p}\in\mathcal{O}. \tag{2.3}$$

As an application of Theorem 34, we review the following lemma which is well-known. We will use this lemma later in this thesis.

Lemma 36. *Let $\mu \in R(\mathbb{K}Ge, \mathbb{K}Hf)$ and $\mu_{proj} \in Pr(\mathcal{O}Ge, \mathcal{O}Hf)$. Then $\mu + \mu_{proj}$ is perfect if and only if μ is perfect.*

Proof. Note that if at least one of g or h is p-singular, then $\mu(g,h) + \mu_{proj}(g,h) = \mu(g,h)$, by the definition of a projective character. This implies that μ satisfies the separation axiom if and only if $\mu + \mu_{proj}$ satisfies the separation axiom. Additionally, the integrality condition follows from Theorem 34. $\square$

The next result will be referred to later. The ideas in the proof can be found in Sambale [24] in the context of perfect isometries. We prove the result for the convenience of the reader.

Lemma 37. *([24]) Let $e \in \mathrm{Bl}(\mathcal{O}G)$, $f \in \mathrm{Bl}(\mathcal{O}H)$ and $e_K \in \mathrm{Bl}(\mathcal{O}K)$. Let $\mu \in R(\mathbb{K}Ge, \mathbb{K}Hf)$ and $\tau \in R(\mathbb{K}Hf, \mathbb{K}Ke_K)$. Suppose that μ and τ satisfies the integrality condition. Then, $\mu \cdot_H \tau$ also satisfies the integrality condition.*

Proof. Let $(g,k) \in G \times K$. Letting $\mathcal{R}$ denote a set of representatives of the conjugacy classes of H, we obtain that

$$\begin{aligned}\frac{(\mu \cdot_H \tau)(g,k)}{|C_G(g)|} &= \frac{1}{|H||C_G(g)|}\sum_{h\in H}\mu(g,h)\tau(h,k)\\ &= \frac{1}{|H||C_G(g)|}\sum_{h\in\mathcal{R}}|H:C_H(h)|\mu(g,h)\tau(h,k)\\ &= \sum_{h\in\mathcal{R}}\frac{\mu(g,h)}{|C_G(g)|}\frac{\tau(h,k)}{|C_H(h)|} \in \mathcal{O}\end{aligned}$$

because both μ and τ satisfies the integrality condition. Similarly, one can show $|C_K(k)|$ divides $(\mu \cdot_H \tau)(g,k)$ in $\mathcal{O}$. $\square$

The next argument can also be found in [24]. We state and prove it in our context.

Lemma 38. *([24]) Let $e \in \mathrm{Bl}(\mathcal{O}G)$, $f \in \mathrm{Bl}(\mathcal{O}H)$ and $e_K \in \mathrm{Bl}(\mathcal{O}K)$. Suppose that μ is an element of $\mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)$ and ψ is an element of $\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Ke_K)$. Then, $\mu \cdot_H \psi$ is in $\mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Ke_K)$.*

Proof. Let $(g,k) \in G \times K$ such that $(\mu \cdot_H \psi)(g,k) \neq 0$. That is we have

$$0 \neq (\mu \cdot_H \psi)(g,k) \tag{2.4}$$

$$= \frac{1}{|H|} \sum_{h \in H} \mu(g,h)\psi(h,k). \tag{2.5}$$

Thus there is at least one element $h \in H$ such that $\mu(g,h)\psi(h,k) \neq 0$. In particular, $\mu(g,h) \neq 0$ and $\psi(h,k) \neq 0$. Since μ is quasi-perfect, g is a p'-element if and only if h is a p'-element. Furthermore, since ψ is quasi-perfect, h is a p'-element if and only if k is a p'-element. Hence the proof follows. $\square$

Suppose that there is a perfect isometry $\mu \in \mathrm{R}(\mathbb{K}Ge, \mathbb{K}Hf)$ between the blocks $\mathcal{O}Ge$ and $\mathcal{O}Hf$ with defect groups P, Q, respectively. Broué in [5] and [6] showed that the following properties hold:

(i) One has $|P| = |Q|$.

(ii) The group isomorphism $I_\mu : \mathrm{R}(\mathbb{K}Hf) \to \mathrm{R}(\mathbb{K}Ge)$ maps $\mathrm{Pr}(\mathcal{O}Hf)$ to $\mathrm{Pr}(\mathcal{O}Ge)$.

(iii) The Cartan matrices of e and f have the same determinant and elementary divisors with the same multiplicities.

(iv) $Z(\mathcal{O}Ge) \cong Z(\mathcal{O}Hf)$ as $\mathcal{O}$-algebras and this isomorphism restricts to projective centers, $Z^{pr}(\mathcal{O}Ge) \cong Z^{pr}(\mathcal{O}Hf)$.

The next result relates stable equivalence of Morita type and perfect isometries.

Proposition 39. *(*[9, Proposition 3.3]*) Given finite groups G, H, $e \in \mathrm{Bl}(\mathcal{O}G)$ and $f \in \mathrm{Bl}(\mathcal{O}H)$, let $A = \mathcal{O}Ge$ and $B = \mathcal{O}Hf$. Suppose that $(\mathbb{K}, \mathcal{O}, F)$ is large enough and suppose that there is a stable equivalence of Morita type between A and B achieved by a (B,A)-bimodule M which is finitely generated projective as a left and right module. Furthermore, assume that the isometry $L^0(A) \cong L^0(B)$ induced by Φ_M extends to an isometry $\Phi : R(\mathbb{K}A) \cong R(\mathbb{K}B)$. Then, $\chi_\Phi - \chi_M \in \mathrm{Pr}(A,B)$. In particular, Φ is a perfect isometry.*

2.2 Perfect self-isometry group of a block

Let $e \in \mathrm{Bl}(\mathcal{O}G)$. We use the notation of Ruengrot in [22], namely $\mathrm{PI}(\mathcal{O}Ge)$ for the group of perfect self-isometries of the block $\mathcal{O}Ge$. The group operation of $\mathrm{PI}(\mathcal{O}Ge)$ is $- \cdot_G -$ and its identity is $[\mathbb{K}Ge] = \sum_{\chi \in \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)} \chi \times \chi^\circ$ in $\mathrm{R}(\mathbb{K}Ge, \mathbb{K}Ge)$, and given $\mu \in \mathrm{PI}(\mathcal{O}Ge)$, its inverse is $\mu^\circ \in \mathrm{PI}(\mathcal{O}Ge)$.

The next theorem describes the group structure of perfect self-isometries of the algebra $\mathcal{O}P$ for an abelian p-group P. A special case of the following theorem, when $P = C_p$, can also be found in [23].

Theorem 40. *(*[22, Theorem 5.1.1]*) Let P be an abelian p-group. Then every perfect isometry has a homogeneous sign and* $\mathrm{PI}(\mathcal{O}P) \cong (P \rtimes \mathrm{Aut}(P)) \times \langle -id \rangle$.

The next result is regarding the group structure of perfect self-isometries of blocks with cyclic defect group. The theorem uses Linckelmann's result in [11] which says that the derived category of a block with cyclic defect group is equivalent to the derived category of the semidirect product of its defect group and its inertial quotient. Since the derived equivalence implies perfect isometry, to understand the perfect self-isometries of a block with cyclic defect group, it is sufficient to understand $\mathrm{PI}(\mathcal{O}(D \rtimes E))$ where D is a defect group of the block and E is the inertial quotient. Ruengrot provides a detailed study of the group $D \rtimes E$ in [22, Chapter 6]. To summarize, E acts Frobeniusly on D, so it also acts on the irreducible $\mathbb{K}$-characters of D. The size of the orbit of any non-trivial $\mathbb{K}$-character of D is $|E| = e$, hence $|D| - 1$ many non-trivial characters of D break up into orbits of equal size $|E| = e$. We let $t = (|D| - 1)/e$, and each of these orbits induce an irreducible character of $D \rtimes E$, which are denoted by $\Phi_1, \Phi_2, \cdots, \Phi_t$. These characters are referred as *the exceptional characters* of $D \rtimes E$. Additionally, $D \rtimes E$ also has irreducible $\mathbb{K}$-characters inflated from E, denoted by $\{\chi_1, \chi_2, \cdots, \chi_e\}$, and these are called *the non-exceptional characters* of $D \rtimes E$. In fact, one has $\mathrm{Irr}_{\mathbb{K}}(D \rtimes E) = \{\chi_1, \cdots, \chi_e\} \sqcup \{\Phi_1, \cdots, \Phi_t\}$. The group of perfect self-isometries of a block B with a cyclic defect group is computed by Ruengrot [22] and Sambale [24].

Theorem 41. *(*[22] *Theorem 6.0.5,* [24]*, Theorem 3.5) Let G be a finite group and $B \in \mathrm{Bl}(\mathcal{O}G)$ with a cyclic defect group D of order p^n and inertial quotient E with $e := |E|$. Let $t = (p^n - 1)/e$. Then,* $\mathrm{Irr}(B) = \{\chi_1, \cdots, \chi_e\} \sqcup \{\Phi_1, \cdots, \Phi_t\}$.

(i) If $e = 1$*, then* $\mathrm{PI}(B) \cong \mathrm{PI}(\mathcal{O}D) \cong (D \rtimes \mathrm{Aut}(D)) \times \langle -id \rangle$.

(ii) If $e \geq 1$ *and* $t = 1$ *then* $|\mathrm{Irr}(B)| = p$*. Every permutation on* $\mathrm{Irr}(B)$ *gives a perfect isometry (with a choice of sign), and* $\mathrm{PI}(B) \cong S_p \times \langle -id \rangle$.

(iii) If $1 < e < |D| - 1$ *and* $t > 1$*, then* $\mathrm{PI}(B) \cong \langle -id \rangle \times S_e \times C_{\varphi(|D|)/e}$ *where* S_e *permutes the non-exceptional characters, namely* $\{\chi_1, \cdots, \chi_e\}$*, and* $C_{\Phi(|D|)/e}$ *permutes the exceptional characters of* B*, namely* $\{\Phi_1, \cdots, \Phi_t\}$.

Note that $D \rtimes E = C_p \rtimes C_{p-1}$ in the case (ii). It is important to note that the results of Ruengrot and Sambale indicate that perfect isometries behave mindfully with respect to non-exceptional and exceptional characters with the exception of case (ii) above.

Chapter 3

p-permutation equivalences

In this section, we will review the definition and some properties of p-permutation equivalences. The further details of this section can be found in [1] and [19]. The p-permutation equivalences were first defined by Boltje–Xu in [2] in certain restricted situations, and then extended to a greater generality and studied in detail by Boltje–Perepeletsky in [1].

Throughout this section, we again let $(\mathbb{K}, \mathcal{O}, F)$ be large enough for finite groups G and H. For this thesis, we also always assume that $e \in \mathrm{Bl}(\mathcal{O}G)$ and $f \in \mathrm{Bl}(\mathcal{O}H)$. Recall that $T(\mathcal{O}Ge)$ denotes the Grothendieck group of the category ${}_{\mathcal{O}Ge}triv$ with respect to direct sums. Similarly as before, we let $T(\mathcal{O}Ge, \mathcal{O}Hf) := T(\mathcal{O}[G \times H](e \otimes f^*))$. The tensor product of bimodules induces maps on the Grothendieck group level and we use the same notation $\cdot_H$.

We will begin by reviewing some basic notions which can be found in [3].

Definition 42. *Let G and H be finite groups and $X \leq G \times H$. Let $p_1 : G \times H \to G$ and $p_2 : G \times H \to H$ be the canonical projections. Let*

$$k_1(X) := \{g \in G \mid (g, 1) \in X\} \quad \textit{and} \quad k_2(X) := \{h \in H \mid (1, h) \in X\}.$$

Note that $k_1(X) \trianglelefteq p_1(X)$ and $k_2(X) \trianglelefteq p_2(X)$.

Definition 43. *Given $P \leq G$, $Q \leq H$ and a group isomorphism $\phi : Q \to P$, one defines* *a* ***twisted diagonal subgroup*** *of* $G \times H$ *as follows:*

$$\Delta(P, \phi, Q) := \{(\phi(y), y) \mid y \in Q\} \leq G \times H. \tag{3.1}$$

Remark 44. *Broué in* [5] *showed that given any* $e \in \mathrm{Bl}(\mathcal{O}G)$ *and* $f \in \mathrm{Bl}(\mathcal{O}H)$*, the character of an indecomposable module* $M \in {}_{\mathcal{O}Ge}\mathrm{triv}_{\mathcal{O}Hf}$ *with twisted diagonal vertex is perfect.*

Definition 45. *Let* $X \leq G \times H$ *and* $Y \leq H \times K$*. Then,*

$$X \star Y := \{(g, k) \in G \times K \mid \exists h \in H : (g, h) \in X, (h, k) \in Y\}. \tag{3.2}$$

Note that $X \star Y \leq G \times K$*.*

Next, we will review the extended tensor product. This construction was first described by Bouc in [4]. For further details on extended tensor products, see [1, Chapter 6].

Definition 46. *Let* $X \leq G \times H$ *and* $Y \leq H \times K$*. Let* $M \in {}_{\mathcal{O}X}\mathrm{mod}$ *and* $N \in {}_{\mathcal{O}Y}\mathrm{mod}$*. Consider the* $(\mathcal{O}[k_1(X)], \mathcal{O}[k_2(Y)])$*-bimodule,* $M \otimes_{[\mathcal{O}k_2(X) \cap k_1(Y)]} N$*. Note that* $k_1(X) \times k_2(Y) \leq X \star Y$*. One can extend this module structure to an* $\mathcal{O}[X \star Y]$*-module structure such that given* $(g, k) \in X \star Y$*, and* $m \in M$*,* $n \in N$*,* $(g, k) \cdot (m \otimes n) = (g, h)m \otimes (h, k)n$ *where* $h \in H$ *is chosen such that* $(g, h) \in X$ *and* $(h, k) \in Y$*. We will denote this extended tensor product by* $M \overset{X,Y}{\underset{\mathcal{O}H}{\otimes}} N$ *and obtain the functor:*

$$- \overset{X,Y}{\underset{\mathcal{O}H}{\bigotimes}} - : {}_{\mathcal{O}X}\mathrm{mod} \times {}_{\mathcal{O}Y}\mathrm{mod} \to {}_{\mathcal{O}[X \star Y]}\mathrm{mod}.$$

Theorem 47. *(*[4, Corollary 3.4]*) Let* $X \leq G \times H$*,* $Y \leq H \times K$*,* $M \in {}_{\mathcal{O}X}$*mod and* $N \in {}_{\mathcal{O}Y}$*mod. Then, there is an isomorphism*

$$\mathrm{Ind}_X^{G \times H}(M) \otimes_{\mathcal{O}H} \mathrm{Ind}_Y^{H \times K}(N) \cong \bigoplus_{t \in [p_2(X) \backslash H / p_1(Y)]} \mathrm{Ind}_{X \star {}^{(t,1)}Y}^{G \times K}(M \overset{X, {}^{(t,1)}Y}{\underset{\mathcal{O}H}{\bigotimes}} {}^{(t,1)}N) \tag{3.3}$$

of $(\mathcal{O}G, \mathcal{O}H)$*-bimodules.*

Lemma 48. *(*[1, Lemma 7.2]*) Let* $X \leq G \times H$*,* $Y \leq H \times K$*.*

(a) If $M \in {}_{\mathcal{O}X}triv$ and $N \in {}_{\mathcal{O}Y}triv$, then $M \overset{X,Y}{\underset{\mathcal{O}H}{\bigotimes}} N \in {}_{\mathcal{O}(X\star Y)}triv$.

(b) If $M \in {}_{\mathcal{O}X}mod$ and $N \in {}_{\mathcal{O}Y}mod$ are indecomposable with twisted diagonal vertices then every indecomposable direct summand of the $\mathcal{O}[X\star Y]$-module $M \overset{X,Y}{\underset{\mathcal{O}H}{\bigotimes}} N$ has twisted diagonal vertices.

Given a subgroup $X \leq G \times H$, and $d \in Z(\mathcal{O}[G \times H])$, one denotes by $T^{\Delta}(\mathcal{O}Xd)$ the subgroup of $T(\mathcal{O}Xd)$ which is spanned by the isomorphism classes of indecomposable p-permutation $\mathcal{O}Xd$-module with twisted diagonal vertices. Given blocks e, f of $\mathcal{O}G$ and $\mathcal{O}H$, respectively, one sets $T^{\Delta}(\mathcal{O}Ge, \mathcal{O}Hf) := T^{\Delta}(\mathcal{O}[G \times H](e \otimes f^*))$.

Definition 49. *Let $e \in \mathrm{Bl}(\mathcal{O}G)$ and $f \in \mathrm{Bl}(\mathcal{O}H)$. A p-permutation equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$ is an element $\gamma \in T^{\Delta}(\mathcal{O}Ge, \mathcal{O}Hf)$ satisfying*

$$\gamma \cdot_H \gamma^{\circ} = [\mathcal{O}Ge] \in T^{\Delta}(\mathcal{O}Ge, \mathcal{O}Ge) \text{ and } \gamma^{\circ} \cdot_G \gamma = [\mathcal{O}Hf] \in T^{\Delta}(\mathcal{O}Hf, \mathcal{O}Hf) \tag{3.4}$$

Notation 50. *The set of p-permutation equivalences between $\mathcal{O}Ge$ and $\mathcal{O}Hf$ will be denoted by $T^{\Delta}_{\circ}(\mathcal{O}Ge, \mathcal{O}Hf)$.*

The next proposition uses Remark 44 which says that a p-permutation equivalence induces a perfect isometry between two blocks. We will mimic this result in the stable set up.

Proposition 51. *([1, Proposition 9.9]) Let $e \in \mathrm{Bl}(\mathcal{O}G)$ and $f \in \mathrm{Bl}(\mathcal{O}H)$. Let $\gamma \in T^{\Delta}_{\circ}(\mathcal{O}Ge, \mathcal{O}Hf)$. Then, $\mu := \kappa_{G\times H}(\gamma) \in R(\mathbb{K}Ge, \mathbb{K}Hf)$ is a perfect isometry between blocks $\mathcal{O}Ge$ and $\mathcal{O}Hf$.*

Chapter 4

Stable perfect isometries between blocks of finite groups

In this section, we introduce and investigate the notion of a stable perfect isometry, a way to consider a perfect isometry up to projective modules. We will prove some properties regarding the group of the set of cosets of stable perfect isometries.

Throughout this chapter, we always assume that e, f are blocks of $\mathcal{O}G, \mathcal{O}H$, respectively. Recall that as before, we identify the element $\sum_{\chi \in \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)} \chi \times \chi^{\circ}$ in $\mathrm{R}(\mathbb{K}Ge, \mathbb{K}Ge)$ with $[\mathbb{K}Ge]$.

4.1 Definition

Definition 52. *A stable isometry between the algebras $\mathcal{O}Ge$ and $\mathcal{O}Hf$ is a generalized character μ in $R(\mathbb{K}Ge, \mathbb{K}Hf)$ satisfying the following two conditions:*

(l) $\mu \cdot_H \mu^{\circ} = [\mathbb{K}Ge] + \pi_G$ *in* $\mathrm{R}(\mathbb{K}Ge, \mathbb{K}Ge)$ *for some* $\pi_G \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$.

(r) $\mu^{\circ} \cdot_G \mu = [\mathbb{K}Hf] + \pi_H$ *in* $\mathrm{R}(\mathbb{K}Hf, \mathbb{K}Hf)$ *for some* $\pi_H \in \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$.

Definition 53. *Let $e \in \mathrm{Bl}(\mathcal{O}G)$ and $f \in \mathrm{Bl}(\mathcal{O}H)$. A stable perfect isometry between the block algebras $\mathcal{O}Ge$ and $\mathcal{O}Hf$ (respectively stable quasi-perfect isometry) is a stable isometry μ in $R(\mathbb{K}Ge, \mathbb{K}Hf)$ which is also perfect (respectively quasi-perfect).*

Notation 54. *Let* $\mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Hf)$ *denote the set of stable perfect isometries from* $\mathcal{O}Hf$ *to* $\mathcal{O}Ge$*. For convenience, we let* $\mathrm{SPI}(\mathcal{O}Ge) := \mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Ge)$*.*

Next, we make the following observation which is well-known to the experts.

Lemma 55. *Let* e, f, e_K *and* f_L *be in of* $\mathrm{Bl}(\mathcal{O}G), \mathrm{Bl}(\mathcal{O}H), \mathrm{Bl}(\mathcal{O}K)$ *and* $\mathrm{Bl}(\mathcal{O}L)$*, respectively. Let* μ *and* η *be quasi-perfect generalized characters in* $\mathrm{R}(\mathbb{K}Ge, \mathbb{K}Hf)$ *and* $\mathrm{R}(\mathbb{K}Ke_K, \mathbb{K}Lf_L)$ *and let* $\pi \in \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Ke_K)$*. Then,* $\mu \cdot_H \pi \cdot_K \eta$ *is in* $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Lf_L)$*.*

Proof. Let $(g, l) \in G \times L$ be a p-singular element. Without loss of generality, we can assume that l is p-singular, then

$$\begin{aligned}
(\mu \cdot_H \pi \cdot_K \eta)(g, l) &= \frac{1}{|K|} \sum_{x \in K} (\mu \cdot_H \pi)(g, x)\eta(x, l) \\
&= \frac{1}{|K|} \sum_{x \in K_p} (\mu \cdot_H \pi)(g, x)\eta(x, l) \\
&= \frac{1}{|H||K|} \sum_{x \in K_p} \sum_{y \in H} \mu(g, y)\pi(y, x)\eta(x, l) \\
&= 0
\end{aligned}$$

by using the quasi-perfectness of η and the last equality holds since π is in $\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Ke_K)$ so it vanishes on the p-singular elements (y, x) by Remark 14. Now by the characterization of elements in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Lf_L)$, we see that $\mu \cdot_H \pi \cdot_K \eta$ is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Lf_L)$. □

Lemma 56. *Let* $e \in \mathrm{Bl}(\mathcal{O}G)$*,* $f \in \mathrm{Bl}(\mathcal{O}H)$*. Let* $\mu \in \mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Hf)$ *and* $\pi \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$*. Then,* $\mu + \pi$ *is also in* $\mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Hf)$*. Therefore, for any* $\mu \in \mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Hf)$ *the set* $\mu + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf) := \{\mu + \pi \mid \pi \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)\}$ *is a subset of* $\mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Hf)$*.*

Proof. Firstly, by Lemma 36 if μ is a perfect generalized character, then so is $\mu + \pi$ where π is a generalized projective character. This means that we only need to check the conditions (i)-(ii) in the Definition 52 for the generalized character $\mu + \pi$.

Note that

$$
\begin{aligned}
(\mu+\pi)\cdot_H(\mu+\pi)^\circ &= (\mu+\pi)\cdot_H(\mu^\circ+\pi^\circ)\\
&= \mu\cdot_H(\mu^\circ+\pi^\circ)+\pi\cdot_H(\mu^\circ+\pi^\circ)\\
&= \mu\cdot_H\mu^\circ+\mu\cdot_H\pi^\circ+\pi\cdot_H\mu^\circ+\pi\cdot_H\pi^\circ\\
&= [\mathbb{K}Ge]+\pi^{'}+\mu\cdot_H\pi^\circ+\pi\cdot_H\mu^\circ+\pi\cdot_H\pi^\circ
\end{aligned}
$$

where $\mu\cdot_H\mu^\circ=[\mathbb{K}Ge]+\pi^{'}$ for some $\pi^{'}\in\mathrm{Pr}(\mathcal{O}Ge,\mathcal{O}Ge)$. Note that by Lemma 55 $\mu\cdot_H\pi^\circ$ and $\pi\cdot_H\mu^\circ$ are in $\mathrm{Pr}(\mathcal{O}Ge,\mathcal{O}Ge)$, and therefore $\pi^{'}+\mu\cdot_H\pi^\circ+\pi\cdot_H\mu^\circ+\pi\cdot_H\pi^\circ$ is in $\mathrm{Pr}(\mathcal{O}Ge,\mathcal{O}Ge)$. This shows that $\mu+\pi$ satisfies the condition (i) in Definition 52. A similar proof will follow to show condition (ii) where we use $\mu^\circ\cdot\mu=[\mathbb{K}Hf]+\pi^{''}$ for some $\pi^{''}\in\mathrm{Pr}(\mathcal{O}Hf,\mathcal{O}Hf)$.

The final part $\mu+\mathrm{Pr}(\mathcal{O}Ge,\mathcal{O}Hf)\subseteq\mathrm{SPI}(\mathcal{O}Ge,\mathcal{O}Hf)$ is now straightforward. □

Notation 57. *Given $e\in\mathrm{Bl}(\mathcal{O}G)$ and $f\in\mathrm{Bl}(\mathcal{O}H)$, and $\mu\in\mathrm{SPI}(\mathcal{O}Ge,\mathcal{O}Hf)$, we denote the set of the cosets $\mu+\mathrm{Pr}(\mathcal{O}Ge,\mathcal{O}Hf)$ by $\overline{\mathrm{SPI}}(\mathcal{O}Ge,\mathcal{O}Hf)$. We also have the following map*

$$\Phi_{G,H}:\mathrm{SPI}(\mathcal{O}Ge,\mathcal{O}Hf)\to\overline{\mathrm{SPI}}(\mathcal{O}Ge,\mathcal{O}Hf)$$

given by $\mu\mapsto\mu+\mathrm{Pr}(\mathcal{O}Ge,\mathcal{O}Hf)$. We set $\overline{\mathrm{SPI}}(\mathcal{O}Ge):=\overline{\mathrm{SPI}}(\mathcal{O}Ge,\mathcal{O}Ge)$.

Lemma 58. *Let e,f,e_K be in of $\mathrm{Bl}(\mathcal{O}G),\mathrm{Bl}(\mathcal{O}H),\mathrm{Bl}(\mathcal{O}K)$, respectively. Then, $-\cdot_H-$ induces the following bilinear maps*

$$\mathrm{SPI}(\mathcal{O}Ge,\mathcal{O}Hf)\times\mathrm{SPI}(\mathcal{O}Hf,\mathcal{O}Ke_K)\to\mathrm{SPI}(\mathcal{O}Ge,\mathcal{O}Ke_K)\tag{4.1}$$

by $(\mu,\nu)\mapsto\mu\cdot_H\nu$, and

$$\overline{\mathrm{SPI}}(\mathcal{O}Ge,\mathcal{O}Hf)\times\overline{\mathrm{SPI}}(\mathcal{O}Hf,\mathcal{O}Ke_K)\to\overline{\mathrm{SPI}}(\mathcal{O}Ge,\mathcal{O}Ke_K)\tag{4.2}$$

by $(\mu+\mathrm{Pr}(\mathcal{O}Ge,\mathcal{O}Hf),\nu+\mathrm{Pr}(\mathcal{O}Hf,\mathcal{O}Ke_K))\mapsto(\mu\cdot_H\nu)+\mathrm{Pr}(\mathcal{O}Ge,\mathcal{O}Ke_K)$.

In particular, one has the following commutative diagram

$$\begin{array}{ccc}
\mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Hf) \times \mathrm{SPI}(\mathcal{O}Hf, \mathcal{O}Ke_K) & \longrightarrow & \mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Ke_K) \\
\downarrow & & \downarrow \\
\overline{\mathrm{SPI}}(\mathcal{O}Ge, \mathcal{O}Hf) \times \overline{\mathrm{SPI}}(\mathcal{O}Hf, \mathcal{O}Ke_K) & \longrightarrow & \overline{\mathrm{SPI}}(\mathcal{O}Ge, \mathcal{O}Ke_K).
\end{array}$$

Proof. Firstly, we show that given $\mu \in \mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Hf)$ and $\nu \in \mathrm{SPI}(\mathcal{O}Hf, \mathcal{O}Ke_K)$, one has $\mu \cdot_H \nu \in \mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Ke_K)$. For this, note that

$$\begin{aligned}
(\mu \cdot_H \nu) \cdot_K (\mu \cdot_H \nu)^\circ &= \mu \cdot_H \nu \cdot_K \nu^\circ \cdot_H \mu^\circ \\
&= \mu \cdot_H ([\mathbb{K}Hf] + \pi) \cdot_H \mu^\circ \\
&= \mu \cdot_H [\mathbb{K}Hf] \cdot_H \mu^\circ + \mu \cdot_H \pi \cdot_H \mu^\circ \\
&= \mu \cdot_H \mu^\circ + \mu \cdot_H \pi \cdot_H \mu^\circ \\
&= [\mathbb{K}Ge] + \pi^{'} + \mu \cdot_H \pi \cdot_H \mu^\circ
\end{aligned}$$

where $\nu \cdot_K \nu^\circ = [\mathbb{K}Hf] + \pi$ and $\mu \cdot_H \mu^\circ = [\mathbb{K}Hf] + \pi^{'}$ for some $\pi, \pi^{'} \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$. By Lemma 55 we know that $\pi^{'} + \mu \cdot_H \pi \cdot_H \mu^\circ$ is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$. Similar argument follows for $(\mu \cdot_H \nu)^\circ \cdot_G (\mu \cdot_H \nu) = [\mathbb{K}Ke_K] + \pi^{''}$ for some $\pi^{''} \in \mathrm{Pr}(\mathcal{O}Ke_K, \mathcal{O}Ke_K)$. Furthermore, as both μ and ν are perfect, so is $\mu \cdot_H \nu$ by Lemma 37 and Lemma 38. Hence we showed that $\mu \cdot_H \nu$ is in $\mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Ke_K)$.

Note that the second map is also well-defined as for any $\pi \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$ and $\pi^{'} \in \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Ke_K)$, one has

$$[(\mu + \pi) \cdot_H (\nu + \pi^{'})] = \mu \cdot_H \nu + \pi \cdot_H \nu + \mu \cdot_H \pi^{'} + \pi \cdot_H \pi^{'}$$

and by Lemma 55 $\pi \cdot_H \nu + \mu \cdot_H \pi^{'} + \pi \cdot_H \pi^{'}$ is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ke_K)$ which implies that

$$(\mu \cdot_H \nu) + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ke_K) = [(\mu + \pi) \cdot_H (\nu + \pi^{'})] + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ke_K).$$

$\square$

4.2 The group $\overline{\mathrm{SPI}}(\mathcal{O}Ge)$ and $\Phi : \mathrm{PI}(\mathcal{O}Ge) \to \overline{\mathrm{SPI}}(\mathcal{O}Ge)$

We note that the set of stable perfect isometries $\mathrm{SPI}(\mathcal{O}Ge)$ is in general not a group.

Theorem 59. *The set* $\mathrm{SPI}(\mathcal{O}Ge)$ *has a monoid structure with respect to* $\cdot_G$ *and with the identity element* $[\mathbb{K}Ge]$*. For any* $\mu \in \mathrm{SPI}(\mathcal{O}Ge)$*, the coset* $\mu + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$ *is also in* $\mathrm{SPI}(\mathcal{O}Ge)$ *and we denote the set of all such cosets by* $\overline{\mathrm{SPI}}(\mathcal{O}Ge)$*. Then,* $\overline{\mathrm{SPI}}(\mathcal{O}Ge)$ *has a group structure induced by* $\cdot_G$*. Furthermore, there is a group homomorphism* $\Phi : \mathrm{PI}(\mathcal{O}Ge) \to \overline{\mathrm{SPI}}(\mathcal{O}Ge)$ *given by* $\mu \mapsto \mu + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$*.*

Proof. By Lemma 58, we see that $\mathrm{SPI}(\mathcal{O}Ge)$ has a monoid structure. As before, we consider the binary operation $\star : \overline{\mathrm{SPI}}(\mathcal{O}Ge) \times \overline{\mathrm{SPI}}(\mathcal{O}Ge) \to \overline{\mathrm{SPI}}(\mathcal{O}Ge)$ defined by

$$(\mu + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge), \eta + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)) \mapsto (\mu \cdot_G \eta) + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge).$$

Since $[\mathbb{K}Ge]$, the character of the block itself, is a perfect isometry, in particular, a stable perfect isometry, the element $[\mathbb{K}Ge] + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$ in $\overline{\mathrm{SPI}}(\mathcal{O}Ge)$ is the identity with respect to $\star$. For any element $\mu + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$ in $\overline{\mathrm{SPI}}(\mathcal{O}Ge)$, we have that $\mu^\circ + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$ is the inverse in $\overline{\mathrm{SPI}}(\mathcal{O}Ge)$ as μ° is also a stable perfect isometry. Finally, associativity of $\star$ follows from that of $\cdot_G$. Now, we conclude that $\overline{\mathrm{SPI}}(\mathcal{O}Ge)$ has a group structure with respect to the operation $\star$ which is induced by $\cdot_G$.

Finally, we have for any $\mu, \eta \in \mathrm{PI}(\mathcal{O}Ge) \subseteq \mathrm{SPI}(\mathcal{O}Ge)$, one has the following:

$$\begin{aligned}\Phi(\mu \cdot_G \eta) &= \mu \cdot_G \eta + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)\\ &= (\mu + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)) \star (\eta + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge))\\ &= \Phi(\mu) \star \Phi(\eta).\end{aligned}$$

□

Now, we denote the subgroup of the quasi-perfect generalized characters in $\mathrm{R}(\mathbb{K}Ge, \mathbb{K}Hf)$ by $\mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)$. Since every projective character is perfect, $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf) \subseteq \mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)$. We would like to prove that the group $\overline{\mathrm{SPI}}(\mathcal{O}Ge)$ is finite. More generally, we will show that there are only finite many cosets in $\overline{\mathrm{SPI}}(\mathcal{O}Ge, \mathcal{O}Hf)$.

4.3 Finiteness of the group $\overline{\mathrm{SPI}}(\mathcal{O}Ge)$

Remark 60. *Let $e \in \mathrm{Bl}(\mathcal{O}G)$ and $f \in \mathrm{Bl}(\mathcal{O}H)$. Assume that ψ is in $L^0(\mathbb{K}Ge, \mathbb{K}Hf)$ and π is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$. Then, $\langle \psi, \pi \rangle_{G\times H} = 0$.*

Proof. The proof follows from Remark 14. □

Theorem 61. *There are only finitely many distinct cosets of stable quasi-perfect isometries between two blocks $\mathcal{O}Ge$ and $\mathcal{O}Hf$ with respect to $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$ in $\mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)$. In particular, $\overline{\mathrm{SPI}}(\mathcal{O}Ge, \mathcal{O}Hf)$ is a finite set, and $\overline{\mathrm{SPI}}(\mathcal{O}Ge)$ is a finite group.*

Proof. For a given stable quasi-perfect isometry μ in $\mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)$, we define the following map

$$\tilde{I}_\mu : \mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf) \to \mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$$

by $\psi + \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf) \mapsto \mu \cdot_H \psi + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$ for any $\psi \in \mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)$. Note that this map is well-defined by Lemma 38, and since if $\psi \in \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$, then $\mu \cdot_H \psi$ is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$ by Lemma 55. Moreover, we observe that for any $\pi \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$, one has $\tilde{I}_\mu = \tilde{I}_{\mu+\pi}$, which once again follows from the fact that $\pi \cdot_H \psi$ is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$ for any $\psi \in \mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)$. Now, we consider the subgroup

$$\begin{aligned} \mathrm{S}(\mathbb{K}Ge, \mathbb{K}Hf) &:= \mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf) \cap [L^0(\mathbb{K}Ge, \mathbb{K}Hf) \oplus \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)] \\ &= [\mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf) \cap L^0(\mathbb{K}Ge, \mathbb{K}Hf)] \oplus \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf) \end{aligned}$$

of $R(\mathbb{K}Ge, \mathbb{K}Hf)$. Next, we will show that the restriction of $\tilde{I}_\mu$ to $\mathrm{S}(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$ is a map into $\mathrm{S}(\mathbb{K}Ge, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$. It suffices to show that given an element ψ in $\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf) \cap L^0(\mathbb{K}Hf, \mathbb{K}Hf)$, one has $\mu \cdot_H \psi \in \mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf) \cap L^0(\mathbb{K}Ge, \mathbb{K}Hf)$. Let (x, y) be a p'-element in $G \times H$, then

$$\begin{aligned} (\mu \cdot_H \psi)(x, y) &= \frac{1}{|H|} \sum_{h\in H} \mu(x, h)\psi(h, y) \\ &= \frac{1}{|H|} \sum_{h\in H_{p'}} \mu(x, h)\psi(h, y) \\ &= 0 \end{aligned}$$

in which the second equality follows from the fact that if h is not in $H_{p'}$, then we have $\mu(x,h)=0$ since x is a p'-element and μ is quasi-perfect. The final equality follows from the fact that $\psi \in L^0(\mathbb{K}Hf, \mathbb{K}Ge)$ so $\psi(h,y)=0$. This shows that $\mu \cdot_H \psi$ is in $L^0(\mathbb{K}Ge, \mathbb{K}Hf)$ and it is also in $QR(\mathbb{K}Ge, \mathbb{K}Hf)$ by Lemma 38. Hence $\mu \cdot_H \psi$ is in $\mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf) \cap L^0(\mathbb{K}Ge, \mathbb{K}Hf)$.

Now, we let $\psi, \psi^{'} \in \mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf) \cap L^0(\mathbb{K}Hf, \mathbb{K}Hf)$. One has

$$\begin{aligned}
\langle I_\mu(\psi), I_\mu(\psi^{'})\rangle_{G\times H} &= \langle \psi, \mu^\circ \cdot_G \mu \cdot_H \psi^{'}\rangle_{H\times H} \\
&= \langle \psi, [\mathbb{K}Hf] \cdot_H \psi^{'}\rangle_{H\times H} + \langle \psi, \pi_H \cdot_H \psi^{'}\rangle_{H\times H} \\
&= \langle \psi, \psi^{'}\rangle_{H\times H}
\end{aligned}$$

where $\mu^\circ \cdot_G \mu = [\mathbb{K}Hf] + \pi_H$ for some $\pi_H \in \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$. Note $\langle \psi, \pi_H \cdot_H \psi^{'}\rangle_{H\times H} = 0$ in the equation above, since $\pi_H \cdot_H \psi^{'} \in \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$ and $\psi \in L^0(\mathbb{K}Hf, \mathbb{K}Hf)$.

Now, let $\psi_1, \ldots, \psi_n$ be a $\mathbb{Z}$-basis for $\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf) \cap L^0(\mathbb{K}Hf, \mathbb{K}Hf)$. Suppose that $\langle \psi_i, \psi_i\rangle_{H\times H} = n_i$. Then, by the above observation, we know that $\langle I_\mu(\psi_i), I_\mu(\psi_i)\rangle_{G\times H} = n_i$. For any group, and natural number n_i, there are only finitely many generalized characters with norm n_i. Therefore, there are only finitely many possibilities for $\tilde{I}_\mu \mid_{S(\mathbb{K}Hf,\mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf,\mathcal{O}Hf)}$, the restriction of $\tilde{I}_\mu$ to $S(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$.

Next, we consider the restriction map:

$$\begin{array}{c}
\mathrm{Hom}(\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf), \mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)) \\
\big\downarrow{\scriptstyle res} \\
\mathrm{Hom}(S(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf), \mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)),
\end{array}$$

sending ϕ to $\phi \mid_{S(\mathbb{K}Hf,\mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf,\mathcal{O}Hf)}$. We claim that this map is injective. Let ϕ be an element in $\mathrm{Hom}(\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf), \mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf))$ such that its restriction, $\phi \mid_{S(\mathbb{K}Hf,\mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf,\mathcal{O}Hf)}$, is the zero map.

Since $R(\mathbb{K}Hf, \mathbb{K}Hf)$ is $\mathbb{Z}$-free, we have $\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)$ is $\mathbb{Z}$-free. Next, we show that $\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$ is torsion-free. Let $0 \neq n \in \mathbb{Z}$ and $\psi \in \mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)$ be such that $n \cdot \psi \in \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$. By the characterization of elements of $\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$ in Remark 14, $n \cdot \psi$ vanishes on p-singular elements of $H \times H$. This implies that ψ vanishes on p-singular elements of $H \times H$ and hence again by Remark 14, we have $\psi \in \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$.

Furthermore, recall that by Remark 15, $L^0(\mathbb{K}Hf, \mathbb{K}Hf) \oplus \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$ has a finite index in $R(\mathbb{K}Hf, \mathbb{K}Hf)$ and we have

$$\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/S(\mathbb{K}Hf, \mathbb{K}Hf) =$$
$$\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/(QR(\mathbb{K}Hf, \mathbb{K}Hf) \cap (L^0(\mathbb{K}Hf, \mathbb{K}Hf) \oplus \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf))) \cong$$
$$\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf) + (L^0(\mathbb{K}Hf, \mathbb{K}Hf) \oplus \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf))/(L^0(\mathbb{K}Hf, \mathbb{K}Hf) \oplus \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)),$$

which is in $R(\mathbb{K}Hf, \mathbb{K}Hf)/(L^0(\mathbb{K}Hf, \mathbb{K}Hf) \oplus \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf))$. Therefore, $S(\mathbb{K}Hf, \mathbb{K}Hf)$ has a finite index in $\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)$.

Next, since $\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$ is a finitely generated torsion-free module over a PID, $\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$ is free. Since $S(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$ is a submodule of $\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$, $\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$ has a basis, $e_1, \ldots, e_n$ and there exists (all non-zero) invariant factors $d_1 \mid d_2 \mid \cdots \mid d_n$ in $\mathbb{Z}$ such that $d_1 e_1, \ldots, d_n e_n$ is a basis of the submodule $S(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$.

By our assumption, we must have $\phi(d_i e_i) = 0$ for each $i \in \{1, 2, \ldots, n\}$. Then, we have $d_i \phi(e_i) = 0$ for all $i \in \{1, 2, \ldots, n\}$. Since $\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$ is torsion-free and also $d_i \neq 0$, we have $\phi(e_i) = 0$ for each $i \in \{1, 2, \ldots, n\}$. Hence ϕ is the zero map to start with.

Now, combining injectivity of the restriction map and the fact that there are only finitely many possibilities for $\tilde{I}_\mu \mid_{S(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)}$, we conclude that there are only finitely many possibilities for $\tilde{I}_\mu$.

For the final part, note that if $\sigma, \sigma^{'} \in \mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)$ such that $\tilde{I}_\sigma = \tilde{I}_{\sigma'}$, one has $(\sigma - \sigma^{'}) \cdot_H \psi \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$ for all $\psi \in \mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)$. Taking $\psi = [\mathbb{K}Hf]$, we conclude that $\sigma - \sigma^{'} \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$. Therefore, the map

$$\begin{array}{c} \mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf) \\ \big\downarrow \Lambda \\ \mathrm{Hom}(\mathrm{QR}(\mathbb{K}Hf, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf), \mathrm{QR}(\mathbb{K}Ge, \mathbb{K}Hf)/\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)) \end{array}$$

given by $\sigma + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf) \mapsto \tilde{I}_\sigma$, is injective.

Now combining all of our results with the injective map Λ, we conclude that there are only finitely many cosets $\mu + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$ where $\mu \in \mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Hf)$. $\square$

4.4 Stable centers

Throughout we let $e \in \mathrm{Bl}(\mathcal{O}G)$ and $f \in \mathrm{Bl}(\mathcal{O}H)$. We aim to show that a stable perfect isometry between $\mathcal{O}Ge$ and $\mathcal{O}Hf$ gives an $\mathcal{O}$-module isomorphism between the stable centers $Z^{st}(\mathcal{O}Ge)$ and $Z^{st}(\mathcal{O}Hf)$. For this section, we start with reviewing some results that are already known and used in different contexts.

Broué in [5] proved that a perfect isometry induces an algebra isomorphism between the centers of corresponding block algebras. In the next results, we adapt Broué's proofs to our situation where we have a stable perfect isometry. Therefore, we state the following results in our context and prove it for the convenience to the reader.

Lemma 62. *([5]) Let $\mu \in R(\mathbb{K}Ge, \mathbb{K}Hf)$ be a generalized character such that it satisfies the integrality condition. Then, the $\mathbb{K}$-linear map,*

$$\rho_\mu : \mathbb{K}H \to \mathbb{K}G$$

given by $h \mapsto \sum_{g \in G} \frac{1}{|H|}\mu(g^{-1}, h^{-1})g$ maps into $Z(\mathbb{K}Ge)$. Moreover, it also restricts an $\mathcal{O}$-linear map $\mathcal{O}H \to Z(\mathcal{O}Ge)$.

Proof. For any $h \in H$,

$$\begin{aligned}
\rho_\mu(h) &= \sum_{g \in G} \frac{1}{|H|}\mu(g^{-1}, h^{-1})g \\
&= \sum_{x \in \mathcal{R}} \sum_{t \in G/C_G(x)} \frac{1}{|H|}\mu(tx^{-1}t^{-1}, h^{-1})txt^{-1} \\
&= \sum_{x \in \mathcal{R}} \frac{\mu(x^{-1}, h^{-1})}{|H|}\Big(\sum_{t \in G/C_G(x)} txt^{-1}\Big) \in Z(\mathbb{K}G),
\end{aligned}$$

where $\mathcal{R}$ denotes a set of representatives of conjugacy classes of G.

Now suppose that $\chi \notin \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)$. Then, next we will show that $\rho_\mu(h)e_\chi = 0$. Let $\mu = \sum_{\psi \in \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)} \sum_{\lambda \in \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Hf)} a_{\lambda,\psi} \cdot \lambda \times \psi^\circ$, then

$$
\begin{aligned}
\rho_\mu(h)e_\chi &= \sum_{g\in G}\frac{1}{|H|}\sum_{x\in G}\mu(g^{-1},h^{-1})\frac{\chi(1)}{|G|}\chi(x^{-1})gx \quad \text{(letting } y=gx\text{)}\\
&= \sum_{y\in G}\frac{1}{|H|}\sum_{g\in G}\mu(g^{-1},h^{-1})\frac{\chi(1)}{|G|}\chi(y^{-1}g)y\\
&= \sum_{y\in G}\frac{\chi(1)}{|H||G|}\sum_{g\in G}\sum_{\psi\in\mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)}\sum_{\lambda\in\mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Hf)}a_{\lambda,\psi}\lambda(h)[\sum_{g\in G}\psi(g^{-1})\chi(y^{-1}g)]y\\
&= 0
\end{aligned}
$$

by Remark 3 and the fact that $\psi \in \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)$ and $\chi \notin \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Ge)$. This implies that ρ_μ maps into $Z(\mathbb{K}Ge)$.

Finally, suppose that $\sum\limits_{h\in H}\alpha_h h \in \mathcal{O}H$, then we have

$$
\begin{aligned}
\rho_\mu(\sum_{h\in H}\alpha_h h) &= \sum_{g\in G}(\frac{1}{|H|}\sum_{h\in H}\mu(g^{-1},h^{-1})\alpha_h)g\\
&= \sum_{g\in G}(\sum_{h\in\mathcal{R}}\frac{\mu(g^{-1},h^{-1})}{|C_H(h)|}\alpha_h)g \in Z(\mathcal{O}G)
\end{aligned}
$$

since $\frac{\mu(g^{-1},h^{-1})}{|C_H(h)|} \in \mathcal{O}$ by the integrality condition. In particular ρ_μ maps into $Z(\mathcal{O}Ge)$. □

Lemma 63. *Let $\mu \in R(\mathbb{K}G, \mathbb{K}H)$ and $\lambda \in R(\mathbb{K}H, \mathbb{K}I)$. Then, $\rho_\mu \circ \rho_\lambda = \rho_{\mu\cdot_H\lambda}$.*

Proof. For $t \in I$, we have

$$
\begin{aligned}
\rho_\mu\circ\rho_\lambda(t) &= \frac{1}{|I|}\sum_{h\in H}\lambda(h^{-1},t^{-1})\rho_\mu(h)\\
&= \frac{1}{|I|}\sum_{g\in G}(\frac{1}{|H|}\sum_{h\in H}\mu(g^{-1},h^{-1})\lambda(h^{-1},t^{-1}))g\\
&= \frac{1}{|I|}\sum_{g\in G}(\mu\cdot_H\lambda)(g^{-1},t^{-1})g\\
&= \rho_{\mu\cdot_H\lambda}(t).
\end{aligned}
$$

□

Remark 64. *Let $\mu_1, \mu_2 \in R(\mathbb{K}G, \mathbb{K}H)$. Then, $\rho_{\mu_1+\mu_2} = \rho_{\mu_1} + \rho_{\mu_2}$.*

Proof. The proof is straightforward. □

Lemma 65. *$\rho_{[KH]}(a) = a$ for all $a \in Z(\mathbb{K}H)$.*

Proof. It is sufficient to show that $\rho_{[\mathbb{K}H]}(\sum_{h\in H/C_H(y)} hyh^{-1}) = \sum_{h\in H/C_H(y)} hyh^{-1}$ for all $y \in H$.

$$\begin{aligned}
\rho_{[\mathbb{K}H]}(\sum_{h\in H/C_H(y)} hyh^{-1}) &= \sum_{h\in H/C_H(y)} \rho_{[\mathbb{K}H]}(hyh^{-1}) \\
&= \sum_{h\in H/C_H(y)} \sum_{x\in H} \frac{1}{|H|}[\mathbb{K}H](x^{-1}, hy^{-1}h^{-1})x \\
&= \sum_{h\in H/C_H(y)} \frac{1}{|H|} \sum_{x\in H} (\sum_{\chi\in \mathrm{Irr}_{\mathbb{K}}(\mathcal{O}Hf)} \chi(x^{-1})\chi(y))x \\
&= \sum_{h\in H/C_H(y)} \frac{1}{|H|} \sum_{x=_H y} |C_H(y)|x \\
&= \sum_{x=_H y} x \\
&= \sum_{h\in H/C_H(y)} hyh^{-1}.
\end{aligned}$$

□

Now, we are ready to study stable centers under a stable perfect isometry. We will observe that as in the case of a stable equivalence of Morita type, Section 1.6, we have an isomorphism between stable centers. However, in our case, we only could prove the isomorphism as $\mathcal{O}$-modules, not necessarily as algebras.

Proposition 66. *Let $\mu \in R(\mathbb{K}Ge, \mathbb{K}Hf)$ be a stable perfect isometry. Then, ρ_μ induces an $\mathcal{O}$-module isomorphism between $Z^{st}(\mathcal{O}Ge)$ and $Z^{st}(\mathcal{O}Hf)$.*

Proof. Firstly, we show $\rho_\mu(Tr_1^H(y)) \subseteq Tr_1^G(\mathcal{O}G)$ for all $y \in H$. Let $\mathcal{R}$ denote a set of representatives of conjugacy classes of G. We consider

$$\begin{aligned}
\rho_\mu(\sum_{h\in H} hyh^{-1}) &= \sum_{h\in H} \rho_\mu(hyh^{-1}) \\
&= \sum_{h\in H} \frac{1}{|H|} \sum_{g\in G} \mu(g^{-1}, y^{-1})g
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{g\in G} \mu(g^{-1}, y^{-1}) g \\
&= \sum_{g\in\mathcal{R}} \sum_{x\in G/C_G(g)} \mu(xg^{-1}x^{-1}, y^{-1}) xgx^{-1} \\
&= \sum_{g\in\mathcal{R}} \sum_{x\in G} \frac{\mu(g^{-1}, y^{-1})}{|C_G(g)|} xgx^{-1} \\
&= Tr_1^G(\sum_{g\in\mathcal{R}} \lambda g) \in Tr_1^G(\mathcal{O}G)
\end{aligned}
$$

where $\lambda = \frac{\mu(g^{-1},y^{-1})}{|C_G(g)|} \in \mathcal{O}$ by the integrality condition. Hence

$$\rho_\mu(Tr_1^H(\mathcal{O}H)) \subseteq Tr_1^G(\mathcal{O}G) \cap \mathbb{K}Ge = Tr_1^G(\mathcal{O}G)e = Tr_1^G(\mathcal{O}Ge).$$

Thus, ρ_μ induces an $\mathcal{O}$-module homomorphism $\bar{\rho}_\mu : Z^{st}(\mathcal{O}Hf) \to Z^{st}(\mathcal{O}Ge)$.

Let $a \in Z(\mathbb{K}H)$. Note that by Remark 64, one has

$$\rho_{[\mathbb{K}H]}(a) = \rho_{[\mathbb{K}Hf]}(a) + \rho_{[\mathbb{K}H(1-f)]}(a)$$

and since $\rho_{[\mathbb{K}Hf]}(a) \in Z(\mathbb{K}Hf)$, we have $\rho_{[\mathbb{K}Hf]}(a)f = \rho_{[\mathbb{K}Hf]}(a)$. Therefore

$$\rho_{[\mathbb{K}H]}(a) = \rho_{[\mathbb{K}Hf]}(a)f + \rho_{[\mathbb{K}H(1-f)]}(a)(1-f).$$

and by Lemma 65, one has

$$af = \rho_{[\mathbb{K}H]}(a)f = \rho_{[\mathbb{K}Hf]}(a)f = \rho_{[\mathbb{K}Hf]}(a).$$

Now since μ is a stable perfect isometry we have $\mu^o \cdot_G \mu = [\mathbb{K}Hf] + \gamma$ where γ is in $\Pr(\mathcal{O}Hf, \mathcal{O}Hf)$. Next, note that given $a = \sum_{h\in H/C_H(y)} hyh^{-1}$ for $y \in H$, we have

$$
\begin{aligned}
(\rho_{\mu^o} \circ \rho_\mu)(a) &= \rho_{(\mu^0 \cdot_G \mu)}(a) \\
&= \rho_{[KHf]}(a) + \rho_\gamma(a) \\
&= af + \rho_\gamma(a).
\end{aligned}
$$

Note that

$$\begin{aligned}
\rho_\gamma(a) &= \rho_\gamma(\sum_{h\in H/C_H(y)} hyh^{-1}) \\
&= \sum_{h\in H/C_H(y)} \rho_\gamma(hyh^{-1}) \\
&= \sum_{h\in H/C_H(y)} \rho_\gamma(y) \\
&= \frac{|H|}{|C_H(y)|}\rho_\gamma(y) \\
&= \frac{1}{|C_H(y)|}\sum_{h\in H}\gamma(h^{-1},y^{-1})h \\
&= \frac{1}{|C_H(y)|}\sum_{h\in \mathcal{R}}\sum_{x\in H/C_H(h)}\gamma(xh^{-1}x^{-1},y^{-1})xhx^{-1} \\
&= \frac{1}{|C_H(y)|}\sum_{h\in \mathcal{R}}\sum_{x\in H/C_H(h)}\gamma(h^{-1},y^{-1})xhx^{-1} \\
&= \sum_{h\in \mathcal{R}}\sum_{x\in H}\frac{\gamma(h^{-1},y^{-1})}{|C_H(h)||C_H(y)|}xhx^{-1} \\
&= Tr_1^H(\sum_{h\in\mathcal{R}}\frac{\gamma(h^{-1},y^{-1})}{|C_{H\times H}(h,y)|}h) \\
&= Tr_1^H(u)
\end{aligned}$$

with $u = \sum\limits_{h\in\mathcal{R}} \frac{\gamma(h^{-1},y^{-1})}{|C_{H\times H}(h,y)|}h$ in $\mathcal{O}H$ since $\frac{\gamma(h^{-1},y^{-1})}{|C_{H\times H}(h,y)|}$ is in $\mathcal{O}$ by Remark 35. Hence

$$(\rho_{\mu^o}\circ\rho_\mu)(a) = af + Tr_1^H(u)$$

which implies

$$(\rho_{\mu^o}\circ\rho_\mu)(a)f = af + Tr_1^H(u)f$$
$$(\rho_{\mu^o}\circ\rho_\mu)(a)f = af + Tr_1^H(uf)$$

where $uf \in \mathcal{O}Hf$. Now, note that there is a well-defined $\mathcal{O}$-module homomorphism

$$\bar{\rho}_\mu : Z^{st}(\mathcal{O}Hf) \to Z^{st}(\mathcal{O}Ge)$$

induced by ρ_μ and similarly $\bar{\rho}_{\mu^o} : Z^{st}(\mathcal{O}Ge) \to Z^{st}(\mathcal{O}Hf)$.

By above observation, $\bar{\rho}_{\mu^o} \circ \bar{\rho}_\mu = Id_{Z^{st}(\mathcal{O}Hf)}$ and similarly, $\bar{\rho}_\mu \circ \bar{\rho}_{\mu^o} = Id_{Z^{st}(\mathcal{O}Ge)}$ proving the proposition. □

4.5 Further notes

Throughout we assume that $e \in \mathrm{Bl}(\mathcal{O}G)$ and $f \in \mathrm{Bl}(\mathcal{O}H)$. In this section, we will observe that whenever two blocks $\mathcal{O}Ge$, $\mathcal{O}Hf$ have a perfect isometry between them, if the map $\Phi_G : \mathrm{PI}(\mathcal{O}Ge) \to \overline{\mathrm{SPI}}(\mathcal{O}Ge)$ is surjective then $\Phi_H : \mathrm{PI}(\mathcal{O}Hf) \to \overline{\mathrm{SPI}}(\mathcal{O}Hf)$ is surjective. Similarly, if the map $\Phi_G : \mathrm{PI}(\mathcal{O}Ge) \to \overline{\mathrm{SPI}}(\mathcal{O}Ge)$ is surjective then so is the map

$$\Phi_{G,H} : \mathrm{PI}(\mathcal{O}Ge, \mathcal{O}Hf) \to \overline{\mathrm{SPI}}(\mathcal{O}Ge, \mathcal{O}Hf).$$

We start with the following result which can be found in [22].

Remark 67. *([22, Proposition 4.0.7]) Let $\mu \in R(\mathbb{K}Ge, \mathbb{K}Hf)$ be a perfect isometry between $\mathcal{O}Ge$ and $\mathcal{O}Hf$. Then, there is an isomorphism*

$$\theta : \mathrm{PI}(\mathcal{O}Hf) \to \mathrm{PI}(\mathcal{O}Ge)$$

defined by $\alpha \mapsto \mu \cdot_H \alpha \cdot_H \mu^\circ$ with the inverse

$$\theta^{-1} : \mathrm{PI}(\mathcal{O}Ge) \to \mathrm{PI}(\mathcal{O}Hf)$$

is given by $\beta \mapsto \mu^\circ \cdot_G \beta \cdot_G \mu$.

Now, we are ready to introduce the main result of this section which motivates us to work with stable perfect self-isometries and the surjectivity question in that case.

Lemma 68. *Let $e \in \mathrm{Bl}(\mathcal{O}G)$ and $f \in \mathrm{Bl}(\mathcal{O}H)$. Let $\mathrm{PI}(\mathcal{O}Ge, \mathcal{O}Hf)$ denote the set of perfect isometries between $\mathcal{O}Ge$ and $\mathcal{O}Hf$. Assume that $\mathrm{PI}(\mathcal{O}Ge, \mathcal{O}Hf) \neq \emptyset$. If the map*

$$\Phi_G : \mathrm{PI}(\mathcal{O}Ge) \to \overline{\mathrm{SPI}}(\mathcal{O}Ge)$$

is surjective, then

(i) $\Phi_H : \mathrm{PI}(\mathcal{O}Hf) \to \overline{\mathrm{SPI}}(\mathcal{O}Hf)$ *is surjective.*

(ii) $\Phi_{G,H} : \mathrm{PI}(\mathcal{O}Ge, \mathcal{O}Hf) \to \overline{\mathrm{SPI}}(\mathcal{O}Ge, \mathcal{O}Hf)$ *is surjective.*

Proof. By our assumption, $\mathrm{PI}(\mathcal{O}Ge, \mathcal{O}Hf) \neq \emptyset$. Thus, let $\mu \in \mathrm{PI}(\mathcal{O}Ge, \mathcal{O}Hf)$.

(i) Let $\tau + \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf) \in \overline{\mathrm{SPI}}(\mathcal{O}Hf)$. By the definition of $\overline{\mathrm{SPI}}(\mathcal{O}Hf)$, τ is in $\mathrm{SPI}(\mathcal{O}Hf)$.

Define $\tilde{\theta} : \overline{\mathrm{SPI}}(\mathcal{O}Hf) \to \overline{\mathrm{SPI}}(\mathcal{O}Ge)$ by $\tau + \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf) \mapsto \mu \cdot_H \tau \cdot_H \mu^\circ + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$. It is clear from Lemma 58 that $\tilde{\theta}$ is a well-defined group homomorphism.

By our assumption, $\Phi_G : \mathrm{PI}(\mathcal{O}Ge) \to \overline{\mathrm{SPI}}(\mathcal{O}Ge)$ is surjective. So there exists $\alpha \in \mathrm{PI}(\mathcal{O}Ge)$ such that $\alpha + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge) = \mu \cdot_H \tau \cdot_H \mu^\circ + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$, i.e., $\alpha - \mu \cdot_H \tau \cdot_H \mu^\circ$ is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$. By Lemma 55, $\mu^\circ \cdot_G (\alpha - \mu \cdot_H \tau \cdot_H \mu^\circ) \cdot_G \mu$ is in $\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$. Since $\mu \in \mathrm{PI}(\mathcal{O}Ge, \mathcal{O}Hf)$, we have

$$\mu^\circ \cdot_G (\alpha - \mu \cdot_H \tau \cdot_H \mu^\circ) \cdot_G \mu = (\mu^\circ \cdot_G \alpha \cdot_G \mu) - \tau.$$

Therefore, we have $(\mu^\circ \cdot_G \alpha \cdot_G \mu) - \tau$ is in $\mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$, i.e.

$$\tau + \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf) = (\mu^\circ \cdot_G \alpha \cdot_G \mu) + \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf).$$

Note that $\mu^\circ \cdot_G \alpha \cdot_G \mu$ is in $\mathrm{PI}(\mathcal{O}Hf)$ by Remark 67. Thus, the map

$$\Phi_H : \mathrm{PI}(\mathcal{O}Hf) \to \overline{\mathrm{SPI}}(\mathcal{O}Hf)$$

is surjective.

(ii) Let $\sigma + \overline{\mathrm{SPI}}(\mathcal{O}Ge, \mathcal{O}Hf)$ so $\sigma \in \mathrm{SPI}(\mathcal{O}Ge, \mathcal{O}Hf)$. Note that $\sigma \cdot_H \mu^\circ + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$ is in $\overline{\mathrm{SPI}}(\mathcal{O}Ge)$ by using Lemma 58. Now using the surjectivity of Φ_G, there exists τ in $\mathrm{PI}(\mathcal{O}Ge)$ such that $\Phi_G(\tau) = \sigma \cdot_H \mu^\circ + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$, i.e., $\tau - (\sigma \cdot_H \mu^\circ)$ is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$.

Then, clearly, $\tau \cdot_G \mu$ is again perfect, and furthermore,

$$\begin{aligned}
(\tau \cdot_G \mu) \cdot_H (\tau \cdot_G \mu)^\circ &= \tau \cdot_G \mu \cdot_H \mu^\circ \cdot_H \tau^\circ \\
&= \tau \cdot_G [\mathbb{K}Ge] \cdot_G \tau^\circ \\
&= \tau \cdot_G \tau^\circ \\
&= [\mathbb{K}Ge].
\end{aligned}$$

Similarly, one has $(\tau \cdot_G \mu)^\circ \cdot (\tau \cdot_G \mu) = [\mathbb{K}Hf]$. By using Remark 32, $\tau \cdot_G \mu$ is an isometry. Now, combining the last two results, $\tau \cdot_G \mu$ is in $\mathrm{PI}(\mathcal{O}Ge, \mathcal{O}Hf)$. For the final part, note that since $\tau - (\sigma \cdot_H \mu^\circ)$ is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$, by using Lemma 55, we have $(\tau \cdot_G \mu) - (\sigma \cdot_H \mu^\circ \cdot_G \mu)$ is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$. Note that

$$(\tau \cdot_G \mu) - (\sigma \cdot_H \mu^\circ \cdot_G \mu) = \tau \cdot_G \mu - \sigma$$

is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$ since $\mu^\circ \cdot_G \mu = [\mathbb{K}Hf]$. Hence, we have

$$\tau \cdot_G \mu + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf) = \sigma + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$$

with $\tau \cdot_G \mu \in \mathrm{PI}(\mathcal{O}Ge, \mathcal{O}Hf)$ and so $\Phi_{G,H}$ is surjective.

$\square$

4.6 Stable isometry in terms of matrices

Remark 69. *Let π be an element in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$. Then, $\pi = \sum_{i=1}^{k} \sum_{j=1}^{k} v_{i,j} \cdot \kappa(P_i \otimes_{\mathcal{O}} P_j^*)$ for some integers $v_{i,j}$ where P_i, P_j runs through the indecomposable projective $\mathcal{O}Ge$-modules. Let $V = (v_{i,j})$ and note that $\kappa(P_i) = \sum_{m=1}^{k} d_{mi}\chi_m$ and $\kappa(P_j^*) = \sum_{n=1}^{k} d_{nj}\chi_n^\circ$ where $D_B = (d_{ij})$.*

Therefore, we have

$$
\begin{aligned}
\pi &= \sum_{i=1}^{k}\sum_{j=1}^{k} v_{i,j}\cdot \kappa(P_i \otimes_{\mathcal{O}} P_j^*)\\
&= \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{m=1}^{k}\sum_{n=1}^{k} v_{i,j} d_{mi} d_{nj}\cdot(\chi_m \times \chi_n^\circ)\\
&= \sum_{m=1}^{k}\sum_{n=1}^{k}\sum_{i=1}^{k} d_{mi}\sum_{j=1}^{k} v_{ij} d_{nj}(\chi_m \times \chi_n^\circ)\\
&= \sum_{m=1}^{k}\sum_{n=1}^{k}\sum_{i=1}^{k} d_{mi}\sum_{j=1}^{k} V_{ij} D_{jn}^T(\chi_m \times \chi_n^\circ)\\
&= \sum_{m=1}^{k}\sum_{n=1}^{k}\sum_{i=1}^{k} D_{mi}(VD^T)_{in}(\chi_m \times \chi_n^\circ)\\
&= \sum_{m=1}^{k}\sum_{n=1}^{k}(DVD^T)_{mn}(\chi_m \times \chi_n^\circ).
\end{aligned}
$$

Remark 70. *Let* $\mu = \sum_{i,j=1}^{k} a_{ij}\cdot \chi_i \times \chi_j^\circ$ *an element in* $R(\mathbb{K}Ge, \mathbb{K}Ge)$ *and let* $A = (a_{ij})$ *be the corresponding integral matrix. Note that* $\mu^\circ = \sum_{i',j'=1}^{k} a_{i'j'}\cdot \chi_{j'} \times \chi_{i'}^\circ$. *Recall that one has*

$$
(\chi_i \times \chi_j^\circ)\cdot_G (\chi_{j'} \times \chi_{i'}^\circ) = \begin{cases} \chi_i \times \chi_{i'}^\circ & \text{if } j = j', \\ 0 & \text{otherwise} \end{cases}
$$

and this implies that

$$
\begin{aligned}
\mu \cdot_G \mu^\circ &= (\sum_{i,j=1}^{k} a_{ij}\cdot \chi_i \times \chi_j^\circ)\cdot_G (\sum_{i',j'=1}^{k} a_{i'j'}\cdot \chi_{j'} \times \chi_{i'}^\circ)\\
&= \sum_{i,i'=1}^{k}(\sum_{j=1}^{k} a_{ij}a_{i'j})\cdot \chi_i \times \chi_{i'}^\circ\\
&= \sum_{i,i'=1}^{k}(\sum_{j=1}^{k} A_{i,j}A_{j,i'}^t)\cdot \chi_i \times \chi_{i'}^\circ\\
&= \sum_{i,i'=1}^{k}(AA^t)_{i,i'}\cdot \chi_i \times \chi_{i'}^\circ.
\end{aligned}
$$

Remark 71. *Let* $\mu = \sum_{i,j=1}^{k} a_{ij} \cdot \chi_i \times \chi_j^\circ$ *be an element in* $\mathrm{SPI}(\mathcal{O}Ge)$*. Let* $A = (a_{ij})$*. Then,*

(i) $AA^t = I + DVD^t$ *for some* $V \in \mathrm{Mat}_{k\times k}(\mathbb{Z})$*,*

(ii) $A^tA = I + DTD^t$ *for some* $T \in \mathrm{Mat}_{k\times k}(\mathbb{Z})$*,*

(iii) μ *is perfect.*

Chapter 5

Stable perfect isometries of abelian p-groups

In this section, we would like to provide a full description of $\overline{\mathrm{SPI}}(\mathcal{O}P)$ and prove the surjectivity of the map $\Phi : \mathrm{PI}(\mathcal{O}P) \to \overline{\mathrm{SPI}}(\mathcal{O}P)$ where P is an abelian p-group. Throughout this section E is the $k \times k$-matrix whose entries are all 1 and $k = |P|$.

The only indecomposable projective $(\mathcal{O}P, \mathcal{O}P)$-bimodule is $\mathcal{O}(P \times P)$, hence $\mathrm{Pr}(\mathcal{O}P, \mathcal{O}P)$ is generated by the character $\kappa(\mathcal{O}(P \times P)) = \sum\limits_{i,j=1}^{k} (\psi_i \times \psi_j^\circ)$ where $\mathrm{Irr}_{\mathbb{K}}(G) = \{\psi_1, \cdots, \psi_k\}$. Now let $\mu = \sum_{i,j=1}^{k} a_{i,j}(\psi_i \times \psi_j^\circ)$ be a stable self-isometry of $\mathcal{O}P$ so we have

$$\mu \cdot_P \mu^\circ = \sum_{i=1}^{k} \psi_i \times \psi_i^\circ + r \sum_{i,j=1}^{k} (\psi_i \times \psi_j^\circ)$$

and

$$\mu^\circ \cdot_P \mu = \sum_{i=1}^{k} \psi_i \times \psi_i^\circ + s \sum_{i,j=1}^{k} (\psi_i \times \psi_j^\circ)$$

for some integers r and s. Now this implies that letting $A = (a_{i,j})$, we have

$$AA^T = I_{k\times k} + rE \text{ and } A^T A = I_{k\times k} + sE \text{ for integers r and s as given above.} \tag{5.1}$$

Therefore, to understand the stable self-isometries of $\mathcal{O}P$, one has to understand $k \times k$-matrices with integer entries that satisfy the equation (5.1). The next proposition is crucial for the results in this chapter.

5.1 The key proposition

Notation 72. *Let E denote the $k \times k$-matrix whose entries are all 1.*

The following remark will help us to prove our main proposition below by providing a decent reduction to cases to be examined.

Remark 73. *Let $k \geq 2$, and $A \in \mathrm{Mat}_{k\times k}(\mathbb{Z})$ such that $A^T A = I + rE = A^T A$ for some $r \in \mathbb{Z}$. For any $\sigma, \beta \in Sym(k)$, we let P_σ and P_β denote the corresponding permutation matrices P_σ and P_β. Then, one has $(P_\sigma A P_\beta)^T (P_\sigma A P_\beta) = I + rE$.*

Proof. It is a straightforward proof once we note that E commutes with permutation matrices and permutation matrices are orthogonal. □

Proposition 74. *Let $k \geq 2$, and $A \in \mathrm{Mat}_{k\times k}(\mathbb{Z})$. If $AA^t = I + rE$ and $A^t A = I + sE$ for some $r, s \in \mathbb{Z}$ then $r = s \geq 0$ and precisely one of the following occurs:*

(i) $r = 0$ and A is a signed permutation matrix.

(ii) $r \neq 0$ and $A = \epsilon P + aE$ for some $\epsilon \in \{-1, 1\}$, and some permutation matrix P, and some $a \neq 0$ in $\mathbb{Z}$.

Proof. Note that $(AA^t)_{i,i} = \sum_{l=1}^{k} {a_{i,l}}^2 = 1 + r \geq 0$. If $r = -1$, then A is the zero matrix, and it does not satisfy $AA^t = I + rE$, so we must have $r \geq 0$. Similarly, $(A^t A)_{i,i} = \sum_{l=1}^{k} {a_{l,i}}^2 = 1 + s \geq 0$. Similarly, if $s = -1$, then A is the zero matrix. Hence, $s \geq 0$.

Next, note that

$$\sum_{i=1}^{k} A_{i^{th} row} \cdot A_{i^{th} row} = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij}^2 = k(1 + r),$$

and similarly,

$$\sum_{i=1}^{k} A^{i^{th} col.} \cdot A^{i^{th} col.} = \sum_{j=1}^{k} \sum_{i=1}^{k} a_{ij}^2 = k(1 + s).$$

These two sums are the same so $r = s$.

Firstly, assume that $r = 0$. Then, A is an orthogonal matrix. Hence, for all $i \in \{1, \cdots, k\}$ we have $A_{i^{th}row} \cdot A_{i^{th}row} = 1$, that is, $a_{i1}^2 + ... + a_{ik}^2 = 1$. Since A has integer entries, there is only one j such that $a_{i,j} \in \{-1, 1\}$ and the rest is 0. Now we consider $A^{j^{th}col} \cdot A^{j^{th}col} = 1$, that is, $a_{1j}^2 + ... + a_{kj}^2 = 1$. We know that $a_{i,j} \neq 0$ so $a_{xj} = 0$ for all $x \in \{1, ..., k\} - \{i\}$. Hence there is only one non-zero entry $a_{i,j} \in \{-1, 1\}$ in the i-th row and j-th column, which is true for every i and j. Thus A must be a signed permutation matrix.

From now on assume $r \neq 0$. It suffices to show the statement in (ii). Recall that $(AA^t)_{i,j} = A_{i^{th}row} \cdot A_{j^{th}row} = \delta_{i,j} + r$ and $(A^tA)_{i,j} = A^{i^{th}col} \cdot A^{j^{th}col} = \delta_{i,j} + r$. If $i \neq j$, then $(A_{i^{th}row} - A_{j^{th}row}) \cdot (A_{i^{th}row} - A_{j^{th}row}) = 2$ which implies $\sum_{l=1}^{k}(a_{il} - a_{jl})^2 = 2$ and we will call this **the row condition**. Similarly $(A^{i^{th}col} - A^{j^{th}col}) \cdot (A^{i^{th}col} - A^{j^{th}col}) = 2$ which implies $\sum_{l=1}^{k}(a_{li} - a_{lj})^2 = 2$ and we will call this **the column condition**. By using the fact that A is a matrix with integer entries, the row (respectively column) condition implies that comparing entries of the i-th row (respectively column) and the j-th row (column), they differ at precisely two positions and there they differ by ± 1. We will refer these properties as the results of row and column properties. By Remark 73 we can assume that these differences occur in the first and second entries of the first two rows and we can assume $a_{11} - a_{21} = 1$ and $a_{12} - a_{22} \in \{-1, 1\}$, therefore we have $a_{1i} = a_{2i}$ for all $i \in \{3, 4, ..., k\}$. Hence we have two cases:

Case I: Assume that $a_{12} - a_{22} = 1$.

In this case, we have

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1k} \\ a_{11} - 1 & a_{12} - 1 & a_{13} & ... & a_{1k} \\ .... & .... & .... & ... & .... \end{pmatrix}$$

Next we compare the first and second row to obtain a relation between a_{11} and a_{12}. We have $A_{1^{st}row} \cdot A_{1^{st}row} = 1 + r = A_{2^{nd}row} \cdot A_{2^{nd}row}$ which implies $a_{11}^2 + a_{12}^2 = (a_{11} - 1)^2 + (a_{12} - 1)^2$

an so $a_{12} = 1 - a_{11}$. Hence the picture is:

$$A = \begin{pmatrix} a_{11} & 1 - a_{11} & a_{13} & ... & a_{1k} \\ a_{11} - 1 & -a_{11} & a_{13} & ... & a_{1k} \\ .... & .... & .... & ... & .... \end{pmatrix}$$

Since $a_{11} \in \mathbb{Z}$, one has $a_{11} \neq 1 - a_{11}$. Then using **the column condition** we have

$$(a_{11} - (1 - a_{11}))^2 = 1,$$

i.e., $(2a_{11} - 1)^2 = 1$, so either $2a_{11} - 1 = 1$ implying $a_{11} = 1$ or $2a_{11} - 1 = -1$ implying $a_{11} = 0$.

SUBCASES:

(a) Assume $a_{11} = 1$. By using **the column condition** we obtain that $a_{l1} = a_{l2}$ for all $l \in \{3, 4, ..., k\}$ so we have

$$A = \begin{pmatrix} 1 & 0 & a_{13} & ... & a_{1k} \\ 0 & -1 & a_{13} & ... & a_{1k} \\ a_{31} & a_{31} & ... & ... & ... \\ a_{41} & a_{41} & ... & ... & ... \\ ... & ... & ... & ... & ... \\ a_{k1} & a_{k1} & ... & ... & ... \end{pmatrix}$$

By **the row condition** one can see that $a_{l1} = 0$ for all $l \in \{3, 4, ..., k\}$. However, this would imply that the norm of the first column is 1, which implies that $r = 0$, a contradiction.

(b) Assume $a_{11} = 0$. Then $a_{12} = 1$, and hence

$$A = \begin{pmatrix} 0 & 1 & a_{13} & ... & a_{1k} \\ -1 & 0 & a_{13} & ... & a_{1k} \\ a_{31} & a_{31} & ... & ... & ... \\ a_{41} & a_{41} & ... & ... & ... \\ ... & ... & ... & ... & ... \\ a_{k1} & a_{k1} & ... & ... & ... \end{pmatrix}$$

By Remark 73, multiplying with a permutation matrix that changes first and second column, we are back to case (a) and we already know that there is no solution for our case.

This finishes the proof of **Case I**.

Case II: Assume that $a_{12} - a_{22} = -1$. In this case we have

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1k} \\ a_{11} - 1 & a_{12} + 1 & a_{13} & ... & a_{1k} \\ .... & .... & .... & ... & .... \end{pmatrix}$$

Since the norms of first and second rows are equal, we have

$$a_{11}^2 + a_{12}^2 = (a_{11} - 1)^2 + (a_{12} + 1)^2$$

implying $a_{12} = a_{11} - 1$. If $k = 2$, then

$$A = \begin{pmatrix} a_{11} & a_{11} - 1 \\ a_{11} - 1 & a_{11} \end{pmatrix} = -P_{(12)} + a_{11}E$$

and we are done. Thus we assume $k \geq 3$. Since $a_{12} = a_{11} - 1$, we have

$$A = \begin{pmatrix} a_{11} & a_{11}-1 & a_{13} & ... & a_{1k} \\ a_{11}-1 & a_{11} & a_{13} & ... & a_{1k} \\ a_{31} & a_{31} & ... & ... & ... \\ a_{41} & a_{41} & ... & ... & ... \\ \vdots & \vdots & ... & ... & ... \\ a_{k1} & a_{k1} & ... & ... & ... \end{pmatrix}$$

By **the result of column condition** we have $a_{1l} \in \{a_{11} - 1, a_{11}\}$ for all $l \in \{3, 4, ..., k\}$. Similarly by **the result of row condition** we have $a_{l1} \in \{a_{11} - 1, a_{11}\}$ for all $l \in \{3, 4, ..., k\}$. Now, we examine possibilities for a_{13} and a_{31}. In total, we got 4 subcases:

SUBCASES:

Subcase 1: $a_{13} = a_{11} - 1$ **and** $a_{31} = a_{11} - 1$

In this case we have

$$A = \begin{pmatrix} a_{11} & a_{11}-1 & a_{11}-1 & a_{14} & \cdots & a_{1k} \\ a_{11}-1 & a_{11} & a_{11}-1 & a_{14} & \cdots & a_{1k} \\ a_{11}-1 & a_{11}-1 & a_{33} & a_{34} & \cdots & a_{3k} \\ a_{41} & a_{41} & a_{43} & a_{44} & \cdots & a_{4k} \\ \vdots & \vdots & \vdots & ... & ... & \\ a_{k1} & a_{k1} & a_{k3} & ... & ... & \end{pmatrix}$$

By **the result of column** or **row** condition one has $a_{33} \in \{a_{11} - 2, a_{11} - 1, a_{11}\}$. We will separately study each of these cases.

Subsubcase (a): Firstly, suppose $a_{33} = a_{11}$. Hence

$$A = \begin{pmatrix} a_{11} & a_{11}-1 & a_{11}-1 & a_{14} & \cdots & a_{1k} \\ a_{11}-1 & a_{11} & a_{11}-1 & a_{14} & \cdots & a_{1k} \\ a_{11}-1 & a_{11}-1 & a_{11} & a_{14} & \cdots & a_{1k} \\ a_{41} & a_{41} & a_{41} & a_{44} & \cdots & a_{4k} \\ \vdots & \vdots & \vdots & ... & ... & \\ a_{k1} & a_{k1} & a_{k1} & ... & ... & \end{pmatrix}.$$

Note that if $k = 3$, it follows that $A = I + (a_{11} - 1)E$. Hence we can assume $k \geq 4$. In this case we note that $a_{l1} = a_{1l} = a_{11} - 1$ for all $l \in \{4, ..., k\}$. (Explanation: we have $a_{41} \in \{a_{11}, a_{11} - 1\}$ and if $a_{41} = a_{11}$ then use row condition and compare the norm of first and fourth row and see $(a_{11} - 1)^2 = a_{11}^2$ so no integer solutions. Similar argument follows for a_{l1} and a_{1l}). Hence we get

$$A = \begin{pmatrix} a_{11} & a_{11}-1 & a_{11}-1 & a_{11}-1 & \cdots & a_{11}-1 \\ a_{11}-1 & a_{11} & a_{11}-1 & a_{11}-1 & \cdots & a_{11}-1 \\ a_{11}-1 & a_{11}-1 & a_{11} & a_{11}-1 & \cdots & a_{11}-1 \\ a_{11}-1 & a_{11}-1 & a_{11}-1 & a_{44} & \cdots & a_{4k} \\ \vdots & \vdots & \vdots & ... & ... & \\ a_{11}-1 & a_{11}-1 & a_{11}-1 & ... & ... & \end{pmatrix}$$

By Remark 73 we can assume $a_{44} \neq a_{11} - 1$ implying by either **the result of row** or **column condition** $a_{44} \in \{a_{11} - 2, a_{11}\}$. Next we compare $A^{3^{rd}col} \cdot A^{3^{rd}col} = A^{4^{th}col} \cdot A^{4^{th}col}$ implying $a_{11}^2 + (a_{11} - 2)^2 = (a_{11} - 1)^2 + a_{44}^2$, that is $a_{44}^2 = a_{11}^2$. Hence either $a_{44} = a_{11}$ or $a_{44} = -a_{11}$.

If $a_{44} = -a_{11}$, then since $a_{44} \in \{a_{11} - 2, a_{11}\}$, we have either $-a_{11} = a_{11} - 2$ implying $a_{11} = 1$ or $-a_{11} = a_{11}$ implying $a_{11} = 0$. If $a_{11} = 1$ then the norm of the first row is equal to 1 which implies $r = 0$, a contradiction. Hence $a_{11} = 0$ so $a_{44} = 0$. By Remark 73 we can take $a_{55} \neq -1$. Hence $a_{55} \in \{-2, 0\}$. But comparing the norm of 4-th and 5-th columns, we find that $a_{55} = 0$. Continuing this way, we find $A = I + (-1)E$ is a solution.

Next we assume $a_{44} = a_{11}$. Hence

$$A = \begin{pmatrix} a_{11} & a_{11}-1 & a_{11}-1 & a_{11}-1 & \cdots & a_{11}-1 \\ a_{11}-1 & a_{11} & a_{11}-1 & a_{11}-1 & \cdots & a_{11}-1 \\ a_{11}-1 & a_{11}-1 & a_{11} & a_{11}-1 & \cdots & a_{11}-1 \\ a_{11}-1 & a_{11}-1 & a_{11}-1 & a_{11} & \cdots & a_{4k} \\ a_{11}-1 & a_{11}-1 & a_{11}-1 & a_{11}-1 & \cdots & a_{4k} \\ \vdots & \vdots & \vdots & \vdots & ... & \\ a_{11}-1 & a_{11}-1 & a_{11}-1 & a_{11}-1 & ... & \end{pmatrix}$$

Next, by Remark 73, can assume $a_{55} \neq a_{11} - 1$, implying $a_{55} \in \{a_{11} - 2, a_{11}\}$. Comparing norms of third and fifth columns, we get $a_{11}^2 = a_{55}^2$ so either $a_{55} = a_{11}$ or $a_{55} = -a_{11}$. Similarly as before, if $a_{55} = -a_{11}$ then either $-a_{11} = a_{11} - 2$ implying $a_{11} = 1$ or $-a_{11} = a_{11}$ implying $a_{11} = 0$. In the case $a_{11} = 1$, it is easy to see that A would be an orthogonal matrix so $r = 0$ so not a solution. If $a_{11} = 0$, then $A = I + (-1)E$ is a solution with $k \geq 3$ for our case. Now, assume $a_{55} = a_{11}$. Note with the same logic, we can take $a_{66} = ... = a_{kk} = a_{11}$ (otherwise repeat the same argument as above). Hence, in this case $A = I + (a_{11} - 1)E$ is a solution for $a_{11} \neq 1$. This finishes the first subsubcase in which we assumed $a_{33} = a_{11}$.

Subsubcase (b): We assume $a_{33} = a_{11} - 2$. Hence we have

$$A = \begin{pmatrix} a_{11} & a_{11}-1 & a_{11}-1 & a_{14} & \cdots & a_{1k} \\ a_{11}-1 & a_{11} & a_{11}-1 & a_{14} & \cdots & a_{1k} \\ a_{11}-1 & a_{11}-1 & a_{11}-2 & a_{34} & \cdots & a_{3k} \\ a_{41} & a_{41} & a_{43} & a_{44} & \cdots & a_{4k} \\ \vdots & \vdots & \vdots & ... & ... & \\ a_{k1} & a_{k1} & a_{k3} & ... & ... & \end{pmatrix}$$

Note by **the result of column condition**, $a_{l1} = a_{l3}$ for all $l \in \{4, 5, ..., k\}$. Also note by **the result of row condition** $a_{3l} = a_{1l}$ for all $l \in \{4, 5, ..., k\}$. Now the norm of the second column and third column is the same which implies $a_{11} = 1$. These three observations imply that $a_{1l} = a_{l1} = 0$ for all $l \in \{4, 5, ..., k\}$. But then by taking the norm of the first row, it implies

$r = 0$, a contradiction.

Subsubcase (c): We assume $a_{33} = a_{11} - 1$. Then we have

$$A = \begin{pmatrix} a_{11} & a_{11}-1 & a_{11}-1 & a_{14} & \cdots & a_{1k} \\ a_{11}-1 & a_{11} & a_{11}-1 & a_{14} & \cdots & a_{1k} \\ a_{11}-1 & a_{11}-1 & a_{11}-1 & a_{34} & \cdots & a_{3k} \\ a_{41} & a_{41} & a_{43} & a_{44} & \cdots & a_{4k} \\ \vdots & \vdots & \vdots & ... & ... & \\ a_{k1} & a_{k1} & a_{k3} & ... & ... & \end{pmatrix}$$

which implies $k \geq 4$. In this case by **the result of row and column conditions** we note $a_{l1}, a_{1l} \in \{a_{11} - 1, a_{11}\}$. By Remark 73 one can take $a_{34} \neq a_{11} - 1$ so $a_{34} \in \{a_{11} - 2, a_{11}\}$. If $a_{34} = a_{11} - 2$, then by exchanging the third and fourth columns to obtain the contradiction in the Subsubcase (b). If $a_{34} = a_{11}$, then $a_{14} = a_{11} - 1$, then by using a permutation, note that we go back to the Subsubcase (a) where $a_{33} = a_{11}$.

Subcase 2: $a_{13} = a_{11} - 1$ **and** $a_{31} = a_{11}$ We have

$$A = \begin{pmatrix} a_{11} & a_{11}-1 & a_{11}-1 & a_{14} & \cdots & a_{1k} \\ a_{11}-1 & a_{11} & a_{11}-1 & a_{14} & \cdots & a_{1k} \\ a_{11} & a_{11} & a_{33} & a_{34} & \cdots & a_{3k} \\ a_{41} & a_{41} & a_{43} & a_{44} & \cdots & a_{4k} \\ \vdots & \vdots & \vdots & ... & ... & \\ a_{k1} & a_{k1} & a_{k3} & ... & ... & \end{pmatrix}$$

Note that $a_{33} \in \{a_{11}-1, a_{11}\}$. If $a_{33} = a_{11}$, then note $a_{1l} = a_{3l}$ for all $l \in \{4, 5, ..., k\}$. comparing norm of second and third row, we get a contradiction so no solution. If $a_{33} = a_{11} - 1$, then comparing norm of second and third column, we again get a contradiction so no solution.

Subcase 3: $a_{13} = a_{11}$ **and** $a_{31} = a_{11} - 1$. Note that this is the transpose of A in Subcase 2, so no solution.

Subcase 4: $a_{13} = a_{11}$ **and** $a_{31} = a_{11}$ Hence we have

$$A = \begin{pmatrix} a_{11} & a_{11}-1 & a_{11} & a_{14} & \cdots & a_{1k} \\ a_{11}-1 & a_{11} & a_{11} & a_{14} & \cdots & a_{1k} \\ a_{11} & a_{11} & a_{33} & a_{34} & \cdots & a_{3k} \\ a_{41} & a_{41} & a_{43} & a_{44} & \cdots & a_{4k} \\ \vdots & \vdots & \vdots & \ldots & \ldots & \\ a_{k1} & a_{k1} & a_{k3} & \ldots & \ldots & \end{pmatrix}.$$

Note that if A has the property of $AA^t = I + rE = A^tA$, then so does $-A$. Note that changing the first and second rows of $-A$ and we are again in Subcase 1 with $-a_{11}+1$ playing the role of a_{11}. □

5.2 Surjectivity of the map $\Phi : \mathrm{PI}(\mathcal{O}P) \to \overline{\mathrm{SPI}}(\mathcal{O}P)$

Proposition 74 provides us with an understanding of integral $k \times k$-matrices A with the property that $AA^T = I + rE$ and $A^TA = I + sE$, for integers r, s. We are now ready to describe the stable perfect self-isometries of the p-block $\mathcal{O}P$ for an abelian p-group P and prove the surjectivity of the map $\Phi : \mathrm{PI}(\mathcal{O}P) \to \overline{\mathrm{SPI}}(\mathcal{O}P)$.

Corollary 75. *Suppose that P is an abelian p-group. Then, the map $\Phi : \mathrm{PI}(\mathcal{O}P) \twoheadrightarrow \overline{\mathrm{SPI}}(\mathcal{O}P)$ is surjective.*

Proof. Let $\mu \in \mathrm{SPI}(\mathcal{O}P)$ and $A = (a_{i,j})$ be the integral $k \times k$-matrix where $\mu = \sum_{i,j=1}^{k} a_{i,j}(\psi_i \times \psi_j^\circ)$. Since μ is a stable isometry, we know that A satisfies the Equation 5.1, i.e., $AA^T = I_{k\times k} + rE$ and $AA^T = I_{k\times k} + sE$ for some integers r and s. Now, we can apply Proposition 74 for A. Then, one of the following occurs:

If $r = 0$ and A is a signed permutation matrix, then by Remark 32, μ is an isometry. Note that μ is perfect as it is assumed to be a stable perfect isometry to begin with. Hence, μ is in $\mathrm{PI}(\mathcal{O}P)$.

If $r \neq 0$ and $A = \epsilon P + aE$ for $\epsilon = \pm 1$, and some permutation matrix P and some non-zero integer a. We can interpret this result as follows:

$$\mu = \epsilon \cdot \mu_P + a \cdot \kappa[\mathcal{O}(P \times P)] \tag{5.2}$$

where μ_P is the generalized character associated with the permutation matrix P. Next, recall that μ is perfect, and $a \cdot \kappa[\mathcal{O}(P \times P)]$ is a generalized projective character, in particular also perfect. It follows from the Equation 5.2 that $\epsilon \cdot \mu_P$ must also be perfect. On the other hand, note that $(\epsilon P) \cdot (\epsilon P^T) = I_{k\times k}$ implies that $(\epsilon\mu_P) \cdot_P (\epsilon\mu_P)^\circ = [\mathbb{K}P]$ and similarly $(\epsilon P^T) \cdot (\epsilon P) = I_{k\times k}$ implies that $(\epsilon\mu_P)^\circ \cdot_P (\epsilon\mu_P) = [\mathbb{K}P]$. By Remark 32, this means that $\epsilon\mu_P$ is also an isometry, so $\epsilon\mu_P$ is in $\mathrm{PI}(\mathcal{O}P)$. Hence, μ can be lifted to the perfect isometry $\epsilon\mu_P$, proving the surjectivity of the map Φ. $\square$

5.3 Main result

Now, we will generalize this result and show that any stable perfect isometry between p-blocks $\mathcal{O}P$ and $\mathcal{O}Q$ can be lifted to a perfect isometry for abelian p-groups P and Q. We will need the following theorem for our main result.

Theorem 76 ([8])**.** *Let G and H be finite p-groups and $\mathbb{F}$ be a field of characteristic p. Suppose that G is abelian. If $\mathbb{F}G \cong \mathbb{F}H$ then $G \cong H$.*

Recall that Linckelmann in [11] proved Theorem 31 in the context of stable Morita equivalences. Our next theorem will mimic that in the stable perfect isometry case.

Theorem 77. *Assume that P and Q are abelian p-groups. The following are equivalent:*

(a) $P \cong Q$.

(b) There exists a perfect isometry between $\mathcal{O}P$ and $\mathcal{O}Q$.

(c) There exists a stable perfect isometry between $\mathcal{O}P$ and $\mathcal{O}Q$.

Proof. The statements (a) implies (b) and (b) implies (c) are trivial.

Suppose that (b) holds, i.e., there exists a perfect isometry between the blocks $\mathcal{O}P$ and $\mathcal{O}Q$. Since every perfect isometry induces an algebra isomorphism between the centers of the corresponding algebras, and since $Z(\mathcal{O}P) = \mathcal{O}P$ and $Z(\mathcal{O}Q) = \mathcal{O}Q$ for abelian groups P and Q, we have $\mathcal{O}P \cong \mathcal{O}Q$ as $\mathcal{O}$-algebras. This implies that $FP \cong FQ$ as F-algebras. By using Theorem 76 we conclude that $P \cong Q$, so (a) holds.

Next, suppose that (c) holds, i.e. there exists a stable perfect isometry μ from $\mathcal{O}P$ to $\mathcal{O}Q$. We have $Z^{st}(\mathcal{O}P) \cong (\mathcal{O}/|P|\mathcal{O})P$ and $Z^{st}(\mathcal{O}Q) \cong (\mathcal{O}/|Q|\mathcal{O})Q$. We proved in Proposition 66, a stable perfect isometry induces an $\mathcal{O}$-module isomorphism between the corresponding stable centers. Therefore we have $(\mathcal{O}/|P|\mathcal{O})P \cong (\mathcal{O}/|Q|\mathcal{O})Q$ as $\mathcal{O}$-modules which implies $|P| = |Q|$. Therefore we have $|\mathrm{Irr}(\mathbb{K}P)| = |\mathrm{Irr}(\mathbb{K}Q)| = |P| = |Q| = k$. Recall that $\mathcal{O}(P \times P)$ is the only projective indecomposable $(\mathcal{O}P, \mathcal{O}P)$-bimodule and $\mathcal{O}(Q \times Q)$ is the only projective indecomposable $(\mathcal{O}Q, \mathcal{O}Q)$-bimodule. Since μ is a stable perfect isometry, for some $r, z \in \mathbb{Z}$, it satisfies:

$$\mu \cdot_Q \mu^\circ = [\mathbb{K}P] + r \cdot \kappa(\mathcal{O}(P \times P)) \text{ and } \mu^\circ \cdot_P \mu = [\mathbb{K}Q] + s \cdot \kappa(\mathcal{O}(Q \times Q)).$$

We have $\kappa(\mathcal{O}(P \times P)) = \sum_{m=1}^{k} \sum_{n=1}^{k} \psi_m \times \psi_n^\circ$ where ψ_m, ψ_n runs through the irreducible $\mathbb{K}$-characters of P. Similarly, we have $\kappa(\mathcal{O}(Q \times Q)) = \sum_{m=1}^{k} \sum_{n=1}^{k} \chi_m \times \chi_n^\circ$ where χ_m, χ_n runs through the irreducible $\mathbb{K}$-characters of Q. Therefore if we let $A = (a_{i,j})$ be the $k \times k$-matrix associated with μ, we have

$$AA^T = I_{k \times k} + rE \text{ and } A^T A = I_{k \times k} + sE.$$

Now, by Proposition 74, either $r = s = 0$ and A is a signed permutation matrix, or $r = s \neq 0$ and $A = \epsilon P' + aE$ for some $\epsilon = \pm 1$ and some permutation matrix P' and $a \in \mathbb{Z}$.

If $r = s = 0$ and A is a signed permutation matrix, Remark 32 implies μ is isometry, and μ is also perfect by assumption.

If $r = s \neq 0$ and $A = \epsilon P' + aE$ for some $\epsilon = \pm 1$ and some permutation matrix P' and $a \in \mathbb{Z}$, let $\mu_{P'}$ be the generalized character in $R(\mathbb{K}P, \mathbb{K}Q)$ associated with the $k \times k$-matrix permutation matrix P'. Then, we have

$$\mu_A = \epsilon \cdot \mu_{P'} + a \cdot \kappa(\mathcal{O}(P \times Q)) = \mu_{P'} + a \sum_{m=1}^{k} \sum_{n=1}^{k} (\psi_m \times \chi_n^\circ).$$

Since μ_A and $\kappa(\mathcal{O}(P \times Q))$ are both perfect, we have $\epsilon \cdot \mu_{P'}$ is also perfect. Since $P^{'}$ is a permutation matrix, it is clear that $\epsilon \cdot \mu_{P'}$ is isometry. Combining these, we have proved that $\epsilon \cdot \mu_{P'}$ is a perfect isometry, hence (b) holds. $\square$

Chapter 6

Stable perfect isometries of the blocks with cyclic defect groups

Let B be a block algebra of $\mathcal{O}G$ with a cyclic defect group D and inertial quotient E. Then, by Linckelmann [11], $\mathcal{O}(D \rtimes E)$ and B are derived equivalent, in particular, perfectly isometric. Then, by Lemma 68 to verify the surjectivity of the map $\Phi : \mathrm{PI}(B) \to \overline{\mathrm{SPI}}(B)$, it is sufficient to understand the surjectivity of the map $\Phi : \mathrm{PI}(\mathcal{O}(D \rtimes E)) \to \overline{\mathrm{SPI}}(\mathcal{O}(D \rtimes E))$.

Following the notation of Ruengrot in [22], we let $e := |E|$ and $t = \frac{|D|-1}{e}$. Then, there are three possible cases:

(i) The case $e = 1$, i.e., $\mathcal{O}(D \rtimes E) = \mathcal{O}D$.

(ii) The case $e > 1$ and $t = 1$ which is the case $\mathcal{O}(D \rtimes E) = \mathcal{O}(C_p \rtimes C_{p-1})$ as shown in [22].

(iii) The case $e > 1$ and $t > 1$.

In this chapter, the question on the surjectivity of the map $\Phi : \mathrm{PI}(B) \to \overline{\mathrm{SPI}}(B)$ will be answered in the case (i) and (ii) above; however, we do not have an answer for the surjectivity of the case (iii).

We prove the following results:

Theorem 78. *Let B be a block algebra of a finite group G with a cyclic defect group D. Assume* *that the inertial quotient E is trivial, i.e., $e = 1$. Then, the map*

$$\Phi : \mathrm{PI}(\mathcal{O}D) \to \overline{\mathrm{SPI}}(\mathcal{O}D)$$

is surjective.

Theorem 79. *If $\mu \in \mathrm{SPI}(\mathcal{O}(C_p \rtimes C_{p-1}))$ with the coefficient matrix A then $A = \epsilon I + DWD^t$ for $\epsilon = \pm 1$ and some $W \in \mathrm{Mat}_{l\times l}(\mathbb{Z})$. In particular, the map*

$$\Phi : \mathrm{PI}(\mathcal{O}(C_p \rtimes C_{p-1})) \to \overline{\mathrm{SPI}}(\mathcal{O}(C_p \rtimes C_{p-1}))$$

is surjective.

6.1 The case: $e = 1$

Theorem 80. *Let B be a block algebra of a finite group G with a cyclic defect group D. Assume that the inertial quotient E is trivial, i.e., $e = 1$. Then, the map*

$$\Phi : \mathrm{PI}(\mathcal{O}D) \to \overline{\mathrm{SPI}}(\mathcal{O}D)$$

is surjective.

Proof. This is the case where $E = 1$ and $D = C_{p^n}$. In particular, D is an abelian p-group and by Corollary 75, we know that $\Phi : \mathrm{PI}(\mathcal{O}D) \to \overline{\mathrm{SPI}}(\mathcal{O}D)$ is surjective. □

Recall that by Linckelmann [11], $\mathcal{O}(D \rtimes E)$ and B are derived equivalent for blocks with cyclic defect group D, in particular, perfectly isometric. Then, Lemma 68 and Theorem 80 imply that $\Phi : \mathrm{PI}(B) \to \overline{\mathrm{SPI}}(B)$ is surjective in the case $e = 1$.

Recall that we also know $\mathrm{PI}(B) \cong (D \rtimes \mathrm{Aut}(D)) \times \langle -id\rangle$ proven by Ruengrot in [22].

6.2 The case: $e > 1$ and $t = 1$

Let B be a block algebra of $\mathcal{O}G$ with a cyclic defect group D and the inertial quotient E where $e > 1$ and $t = 1$. The following is shown by Ruengrot in [22]: if $e > 1$ and $t = 1$, then we have $D = C_p$ and $E = C_{p-1}$. Hence $\mathcal{O}(D \rtimes E) = \mathcal{O}(C_p \rtimes C_{p-1})$.

We let $D = \langle a \rangle$ and $E = \langle b \rangle$. Using Ruengrot's notation in [22], the conjugacy classes of $G = (C_p \rtimes C_{p-1})$ are $\{1\}$, $\{b^i\}^G$ for each $i \in \{1, \cdots, p-2\}$ and $\{a\}^G$. Note that the only p-singular conjugacy class is $\{a\}^G$. Moreover, the non-exceptional characters are $\{\chi_1, \cdots, \chi_e\}$, and there is only one exceptional character which is $\{\chi_p = \Phi_p\}$.

The character table of $C_p \rtimes C_{p-1}$ is as follows:

$C_p \rtimes C_{p-1}$	1	b	$\cdots$	b^i	$\cdots$	b^{p-2}	a
χ_1	1	1	$\cdots$	1	$\cdots$	1	1
χ_2	1	ξ	$\cdots$	ξ^i	$\cdots$	ξ^{p-2}	1
$\vdots$	1	$\vdots$	$\cdots$	$\vdots$		$\vdots$	1
χ_{j+1}	1	ξ^j	$\cdots$	ξ^{ji}	$\cdots$	$\xi^{(p-2)j}$	1
$\vdots$	1	$\vdots$		$\vdots$		$\vdots$	1
χ_{p-1}	1	ξ^{e-1}	$\cdots$	$\cdots$	$\cdots$	ξ	1
$\chi_p = \Phi_p$	$p-1$	0	$\cdots$	0	$\cdots$	0	-1

where ξ denotes a primitive $(p-1)^{st}$ root of unity.

We denote the $p \times p$-matrix coming from the character table of $C_p \rtimes C_{p-1}$ by X with respect to the conjugacy class arrangement as above. We let Y denote the $(p-1)\times(p-1)$-matrix formed from X by removing the p^{th} column and p^{th} row. Note that Y is the character table of C_{p-1} as the non-exceptional characters of $C_p \rtimes C_{p-1}$ are the ones inflated from the irreducible characters of C_{p-1}.

6.2.1 Supplementary observations

Our aim is to understand separability and integrality conditions of a given stable perfect isometry μ by using its coefficient matrix A and the character table of $X = C_p \rtimes C_{p-1}$. For this we will need the following observations:

Lemma 81. *Let X be the matrix of the character table of $C_p \rtimes C_{p-1}$ and $Y = (y_{i,j})$ be the matrix of the character table of C_{p-1} as above. Then,*

(i) $Y^{-1} = \frac{1}{p-1}\overline{Y}$ *where* $\overline{Y} = (\bar{y}_{i,j})$.

(ii) Furthermore, one has

$$X^{-1} = \left[\begin{array}{ccc||c} & \frac{1}{p}Y^{-1}_{1^{st}row} & & \frac{1}{p} \\ & Y^{-1}_{2^{nd}row} & & 0 \\ & \vdots & & \vdots \\ & Y^{-1}_{e^{th}row} & & 0 \\ \hline\hline \frac{1}{p} & \cdots & \frac{1}{p} & \frac{-1}{p} \end{array}\right].$$

Proof. (i) Note that

$$\begin{aligned} \frac{1}{p-1}(Y\overline{Y})_{m,n} &= \frac{1}{p-1}\sum_{i=1}^{p-1} Y_{m,i}\overline{Y}_{i,n} \\ &= \frac{1}{p-1}\sum_{i=0}^{e-1} \chi_m(b^i)\overline{\chi}_i(b^n) \\ &= \frac{1}{p-1}\sum_{i=0}^{e-1} \chi_m(b^i)\chi_i(b^{-n}) \\ &= \frac{1}{p-1}\sum_{i=0}^{e-1} \xi^{mi}\bar{\xi}^{in} \\ &= \frac{1}{p-1}\sum_{i=0}^{e-1} \chi_m(b^i)\overline{\chi_n(b^i)} \\ &= \delta_{m,n} \end{aligned} \tag{6.1}$$

where 6.1 follows from the first orthogonality relation.

(ii) By direct calculation, we have

$$(XX^{-1})_{1,1} = \frac{1}{p(p-1)} + \frac{p-2}{p-1} + \frac{1}{p} = 1 \tag{6.2}$$

$$(XX^{-1})_{p,p} = \frac{p-1}{p} + \frac{1}{p} = 1. \tag{6.3}$$

Furthermore, for $1 < i, j \leq p-1$, we have

$$\begin{aligned}
(XX^{-1})_{i,j} &= \sum_{k=1}^{p} \chi_{i,k}\chi_{k,j}^{-1} && (6.4)\\
&= \sum_{k=1}^{p-1} X_{i,k}X_{k,j}^{-1} + X_{i,p}X_{p,i}^{-1} && (6.5)\\
&= \sum_{k=2}^{p-1} X_{i,k}X_{k,j}^{-1} + X_{i,1}X_{1,j}^{-1} + X_{i,p}X_{p,i}^{-1} && (6.6)\\
&= \sum_{k=2}^{p-1} Y_{i,k}Y_{k,j}^{-1} + X_{i,1}X_{1,j}^{-1} + X_{i,p}X_{p,i}^{-1} && (6.7)\\
&= \delta_{i,j} - Y_{i,1}Y_{1,j}^{-1} + \frac{1}{p(p-1)} + \frac{1}{p} && (6.8)\\
&= \delta_{i,j} - \frac{1}{p-1} + \frac{1}{p(p-1)} + \frac{1}{p} && (6.9)\\
&= \delta_{i,j} && (6.10)
\end{aligned}$$

Also,

$$(XX^{-1})_{1,p} = \frac{1}{p} + \frac{-1}{p} = 0. \tag{6.11}$$

If $i = 1$ and $1 < j \leq p-1$, then

$$\begin{aligned}
(XX^{-1})_{1,j} &= \sum_{k=1}^{p} X_{1,k}X_{k,j}^{-1} && (6.12)\\
&= \sum_{k=1}^{p} X_{k,j}^{-1} && (6.13)\\
&= \frac{1}{p(p-1)} + \frac{1}{p-1}(\xi^{j} + \xi^{2j} + \cdots + \xi^{(e-1)j}) + \frac{1}{p} && (6.14)\\
&= \frac{1}{p(p-1)} - \frac{1}{p-1} + \frac{1}{p} = 0. && (6.15)
\end{aligned}$$

If $1 < i < p-1$ and $j = 1$, then

$$(XX^{-1})_{i,1} = \sum_{k=1}^{p} X_{i,k} X^{-1}_{k,1} \tag{6.16}$$

$$= \sum_{k=2}^{p-1} X_{i,k} X^{-1}_{k,1} + X_{i,1} X^{-1}_{1,1} + X_{i,p} X^{-1}_{p,1} \tag{6.17}$$

$$= \sum_{k=2}^{p-1} Y_{i,k} \frac{1}{p-1} + \frac{1}{p(p-1)} + \frac{1}{p} \tag{6.18}$$

$$= \frac{-1}{p-1} + \frac{1}{p(p-1)} + \frac{1}{p} = 0. \tag{6.19}$$

Also,

$$(XX^{-1})_{p,1} = \frac{p-1}{p(p-1)} + (-1)\frac{1}{p} = 0. \tag{6.20}$$

Hence, we completed the proof of (ii).

□

6.2.2 Separability and integrality condition in terms of matrices

For the rest of this chapter, we fix the following notation for the conjugacy class representatives of $G = C_p \rtimes C_{p-1}$: $x_1 := 1$, $x_2 := b, \cdots, x_{i+1} := b^i, \cdots, x_{p-1} = b^{p-2}, x_p := a$.

We let $\sigma \in S_p$ such that $\sigma(1) = 1$, $\sigma(p) = p$ and $\sigma(i+1) = p-i$ for $i \in \{1, 2, \cdots, p-1\}$. Note that in this way σ sends the index of $x_{i+1} = b^i$ to $x_{p-i} = b^{p-1-i} = b^{-i}$. We let P_σ denote the associated permutation matrix for σ.

Notation 82. *Let $\mu \in R(\mathbb{K}G, \mathbb{K}G)$ be such that $\mu = \sum_{\chi_i, \chi_j \in \mathrm{Irr}_{\mathbb{K}}(G)} a_{i,j} \cdot (\chi_i \times \chi_j^{\circ})$. Throughout we let $A = (a_{i,j})$ denote the coefficient matrix of μ.*

Note that for $m, n \in \{1, \cdots, p\}$, one has

$$\mu(x_m, x_n) = \sum_{i=1}^{p} \sum_{j=1}^{p} a_{i,j} \chi_i(x_m) \chi_j(x_n^{-1})$$

$$= \sum_{i=1}^{p} \chi_i(x_m) \sum_{j=1}^{p} a_{i,j} \chi_j(x_n^{-1})$$

$$
\begin{aligned}
&= \sum_{i=1}^{p} \chi_i(x_m) \sum_{j=1}^{p} a_{i,j} (XP_\sigma)_{j,n} \\
&= \sum_{i=1}^{p} \chi_i(x_m) (AXP_\sigma)_{i,n} \\
&= \sum_{i=1}^{p} X_{i,m} (AXP_\sigma)_{i,n} \\
&= \sum_{i=1}^{p} X^t_{m,i} (AXP_\sigma)_{i,n} \\
&= (X^t AXP_\sigma)_{m,n}.
\end{aligned}
$$

By Theorem 34 of Kiyota that we only need to assure that μ satisfies the separability condition for all elements in $G \times G$ and the integrality condition for only p-singular elements of $G \times G$ where $G = C_p \rtimes C_{p-1}$ to ensure that μ is perfect. Therefore μ being perfect implies that the matrix $(X^t AXP_\sigma)$ has the following shape with respect to the fixed conjugacy classes as above:

$$
X^t AXP_\sigma = \left[\begin{array}{c||ccc||c}
\alpha & \cdots & \star & \cdots & 0 \\
\hline\hline
\kappa_1 & \cdots & \star & \cdots & 0 \\
\vdots & & \vdots & & 0 \\
\kappa_{e-1} & \cdots & \star & \cdots & 0 \\
\hline\hline
0 & \cdots & 0 & \cdots & \gamma
\end{array}\right] =: S'
$$

where $\gamma \in p\mathcal{O}$ since $C_G(a) = C_p$. Thus $X^t AX = S' P_{\sigma^{-1}}$ and note that by the way it is defined, σ fixes 1 and p and so multiplying S' with $P_{\sigma^{-1}}$ from the right fixes the 1^{st} and p^{th}-columns of S' and permutes the other columns among themselves. We let S denote the matrix $S' P_{\sigma^{-1}}$ That is to say, we have

$$
X^t AX = \left[\begin{array}{c||ccc||c}
\alpha & \beta_1 & \cdots & \beta_{e-1} & 0 \\
\hline\hline
\kappa_1 & & & & 0 \\
\vdots & & M & & 0 \\
\kappa_{e-1} & & & & 0 \\
\hline\hline
0 & \cdots & 0 & \cdots & \gamma
\end{array}\right] =: S.
$$

Therefore, we can say that $A = (X^t)^{-1}SX^{-1} = (X^{-1})^t SX^{-1}$. Let us first describe the matrix SX^{-1} and its properties then describe the matrix A which we will use later.

$$SX^{-1} = \left[\begin{array}{ccc||c}
\frac{\alpha}{p(p-1)} + \frac{1}{p-1}\sum_{j=1}^{e-1} \beta_j & \cdots & \frac{\alpha}{p(p-1)} + \frac{1}{p-1}\sum_{j=1}^{e-1} \beta_j \bar{\xi}^{(e-1)j} & \frac{\alpha}{p} \\
\frac{\kappa_1}{p(p-1)} + \frac{1}{p-1}\sum_{j=1}^{e-1} m_{1j} & \cdots & \frac{\kappa_1}{p(p-1)} + \frac{1}{p-1}\sum_{j=1}^{e-1} m_{1j} \bar{\xi}^{(e-1)j} & \frac{\kappa_1}{p} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\kappa_i}{p(p-1)} + \frac{1}{p-1}\sum_{j=1}^{e-1} m_{ij} & \cdots & \frac{\kappa_i}{p(p-1)} + \frac{1}{p-1}\sum_{j=1}^{e-1} m_{ij} \bar{\xi}^{(e-1)j} & \frac{\kappa_i}{p} \\
\vdots & \cdots & \vdots & \vdots \\
\frac{\kappa_{e-1}}{p(p-1)} + \frac{1}{p-1}\sum_{j=1}^{e-1} m_{(e-1)j} & \cdots & \frac{\kappa_{e-1}}{p(p-1)} + \frac{1}{p-1}\sum_{j=1}^{e-1} m_{(e-1)j} \bar{\xi}^{(e-1)j} & \frac{\kappa_{e-1}}{p} \\
\hline\hline
\frac{\gamma}{p} & \cdots & \frac{\gamma}{p} & -\frac{\gamma}{p}
\end{array}\right].$$

For convenience, we will use the following notation for this matrix:

$$SX^{-1} = \left[\begin{array}{cccc||c}
s_{11} & s_{12} & \cdots & s_{1e} & \frac{\alpha}{p} \\
s_{21} & s_{22} & \cdots & s_{2e} & \frac{\kappa_1}{p} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
s_{i1} & s_{i2} & \cdots & s_{ie} & \frac{\kappa_i}{p} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
s_{e1} & s_{e2} & \cdots & s_{ee} & \frac{\kappa_{(e-1)}}{p} \\
\hline\hline
\frac{\gamma}{p} & \frac{\gamma}{p} & \cdots & \frac{\gamma}{p} & -\frac{\gamma}{p}
\end{array}\right]$$

Next, we observe the following relations:

$$\sum_{i=1}^{e} s_{1i} = (p-1)\frac{\alpha}{p(p-1)} = \frac{\alpha}{p} \tag{6.21}$$

$$\sum_{i=1}^{e} s_{2i} = (p-1)\frac{\kappa_1}{p(p-1)} = \frac{\kappa_1}{p} \tag{6.22}$$

$$\vdots \tag{6.23}$$

$$\sum_{i=1}^{e} s_{ei} = (p-1)\frac{\kappa_{e-1}}{p(p-1)} = \frac{\kappa_{e-1}}{p} \tag{6.24}$$

6.2.3 The coefficient matrix A

Now we are ready to describe the matrix A:

$$\left[\begin{array}{ccc||c}
\sum_{j=1}^{e} \frac{1}{p^{\delta_{j,1}}(p-1)} s_{j1} + \frac{\gamma}{p^2} & \cdots & \sum_{j=1}^{e} \frac{1}{p^{\delta_{j,1}}(p-1)} s_{je} + \frac{\gamma}{p^2} & a_{1,p} \\
\sum_{j=1}^{e} \frac{1}{p^{\delta_{j,1}}(p-1)} s_{j1}\bar{\xi}^{(j-1)} + \frac{\gamma}{p^2} & \cdots & \sum_{j=1}^{e} \frac{1}{p^{\delta_{j,1}}(p-1)} s_{je}\bar{\xi}^{(j-1)} + \frac{\gamma}{p^2} & a_{2,p} \\
\vdots & \vdots & \vdots & \vdots \\
\sum_{j=1}^{e} \frac{1}{p^{\delta_{j,1}}(p-1)} s_{j1}\bar{\xi}^{(e-1)(j-1)} + \frac{\gamma}{p^2} & \cdots & \sum_{j=1}^{e} \frac{1}{p^{\delta_{j,1}}(p-1)} s_{je}\bar{\xi}^{(e-1)(j-1)} + \frac{\gamma}{p^2} & a_{p-1,p} \\
\hline\hline
\frac{s_{11}}{p} + \frac{-\gamma}{p^2} & \cdots & \frac{s_{1e}}{p} + \frac{-\gamma}{p^2} & a_{p,p}
\end{array}\right] \tag{6.25}$$

where

$$a_{1,p} = \left(\frac{\alpha}{p^2(p-1)} + \sum_{i=1}^{e-1} \frac{\kappa_i}{p(p-1)} + \frac{-\gamma}{p^2} \right) \tag{6.26}$$

$$a_{2,p} = \left(\frac{\alpha}{p^2(p-1)} + \sum_{i=1}^{e-1} \frac{\kappa_i \xi^i}{p(p-1)} + \frac{-\gamma}{p^2} \right) \tag{6.27}$$

$$\vdots$$

$$a_{p-1,p} = \left(\frac{\alpha}{p^2(p-1)} + \sum_{i=1}^{e-1} \frac{\kappa_i \xi^{(p-2)i}}{p(p-1)} + \frac{-\gamma}{p^2} \right) \tag{6.28}$$

$$a_{p,p} = \frac{\alpha+\gamma}{p^2}. \tag{6.29}$$

Note that by the relations (6.26), $\cdots$, (6.29) we have

$$a_{1,p} + \cdots + a_{p-1,p} = \frac{\alpha}{p^2} + (p-1)\frac{-\gamma}{p^2} = a_{p,p} - \frac{\gamma}{p}. \tag{6.30}$$

We also note the following relations which follow from the relations (6.21) and (6.24) which will be referred to later:

$$\sum_{i=1}^{e} a_{1i} = a_{1p} + \frac{\gamma}{p} \tag{6.31}$$

$$\vdots \tag{6.32}$$

$$\sum_{i=1}^{e} a_{ei} = a_{ep} + \frac{\gamma}{p} \tag{6.33}$$

$$\sum_{i=1}^{e} a_{pi} = \frac{\alpha}{p^2} + (p-1)\frac{-\gamma}{p^2} = a_{pp} - \frac{\gamma}{p}. \tag{6.34}$$

6.2.4 Stable isometries of $\mathcal{O}(C_p \rtimes C_{p-1})$ in terms of matrices

Assumption: In addition to μ being perfect as above, now we let μ be also a stable isometry. Let $G = C_p \rtimes C_{p-1}$ as above. Thus, we have

(i) $\mu \cdot_G \mu^\circ = [\mathbb{K}G] + \pi$ for some $\pi \in \Pr(\mathcal{O}G, \mathcal{O}G)$,

(ii) $\mu^\circ \cdot_G \mu = [\mathbb{K}G] + \pi'$ for some $\pi' \in \Pr(\mathcal{O}G, \mathcal{O}G)$.

By 4.6, $AA^t = I + DVD^t$ and $A^tA = I + DTD^t$ for some $V, T \in \mathrm{Mat}_{e\times e}(\mathbb{Z})$ where $l := e = p-1$ and D is the decomposition matrix of $\mathcal{O}(C_p \rtimes C_{p-1})$, i.e.,

$$D = \left[\begin{array}{ccc} & I_{e\times e} & \\ \hline 1 & \cdots & 1 \end{array}\right].$$

Note that

$$A^tA = I + DTD^t = \left[\begin{array}{ccc||c} & & & \sum t_{1,i} \\ & T + I_{e\times e} & & \vdots \\ & & & \sum t_{e,i} \\ \hline\hline \sum t_{i,1} & \cdots & \sum t_{i,e} & \sum\limits_{i,j=1}^{e} t_{i,j} + 1 \end{array}\right]$$

where $T = (t_{i,j}) \in \mathrm{Mat}_{e\times e}(\mathbb{Z})$.

Note that this implies

$$(A^tA)_{p,p} = \sum_{i=1}^{e}(A^tA)_{i,p} + 1. \tag{6.35}$$

On the other hand, we have

$$(A^tA)_{p,p} = A^{\text{pth-col.}} \cdot A^{\text{pth-col.}} = \sum_{i=1}^{p} a_{i,p}^2. \tag{6.36}$$

The idea is to compare (6.35) and (6.36) and seek a result for γ. Firstly we will calculate $(A^tA)_{p,p}$ by using the equation (6.35) where we will use our observation regarding the shape of matrix A, (6.25).

$$\begin{aligned}
(A^tA)_{p,p} &= \sum_{i=1}^{e}(A^tA)_{i,p} + 1 && (6.37)\\
&= \sum_{i=1}^{e} A^{\text{ith col.}} \cdot A^{\text{pth col.}} + 1 && (6.38)\\
&= \sum_{i=1}^{e}\sum_{t=1}^{p} a_{t,i}a_{t,p} + 1 && (6.39)\\
&= \sum_{t=1}^{p}[\sum_{i=1}^{e} a_{t,i}]a_{t,p} + 1 && (6.40)\\
&= a_{1,p}^2 + \cdots + a_{e,p}^2 + (\frac{\gamma}{p} + a_{pp})(\frac{\alpha}{p^2} + (p-1)\frac{-\gamma}{p^2}) + 1 && (6.41)
\end{aligned}$$

where the last equality follows from the relations (6.31)– (6.33).

On the other hand, by (6.36), we have

$$(A^tA)_{p,p} = a_{1,p}^2 + \cdots + a_{e,p}^2 + a_{p,p}^2. \tag{6.42}$$

Now comparing (6.41) and (6.42), we obtain

$$(\frac{\gamma}{p} + a_{pp})(\frac{\alpha}{p^2} + (p-1)\frac{-\gamma}{p^2}) + 1 = a_{pp}^2 \tag{6.43}$$

which implies together with (6.34) that

$$\gamma = \epsilon p \text{ for } \epsilon = \pm 1. \tag{6.44}$$

Remark 83. *Note that we also showed that for a given stable isometry μ, the separability condition implies that μ satisfies the integrality condition for p-singular elements which then implies the perfectness by Kiyota's Theorem 34.*

6.2.5 Main theorem

Now, we are ready to prove the following theorem:

Theorem 84. *If $\mu \in \mathrm{SPI}(\mathcal{O}(C_p \rtimes C_{p-1}))$ with the coefficient matrix A then $A = \epsilon I + DWD^t$ for some $W \in \mathrm{Mat}_{l\times l}(\mathbb{Z})$ and $\epsilon = \pm 1$. In particular, the map*

$$\Phi : \mathrm{PI}(\mathcal{O}(C_p \rtimes C_{p-1})) \to \overline{\mathrm{SPI}}(\mathcal{O}(C_p \rtimes C_{p-1}))$$

is surjective.

Proof. By (6.44), we know that $\gamma = \epsilon p$ for $\epsilon = \pm 1$. Let $\tilde{A}$ denote the $e \times e$-matrix formed by removing the p^{th}-column and the p^{th}-row of the matrix A. Let $W = \tilde{A} - \epsilon I_{e\times e}$.

Consider

$$\epsilon I_{p\times p} + DWD^t = \epsilon I_{p\times p} + D(\tilde{A} - \epsilon I_{e\times e})D^t \tag{6.45}$$

$$= \left[\begin{array}{ccc||c} & & & \sum_{i=1}^{e} a_{1i} - \epsilon \\ & \tilde{A} & & \vdots \\ & & & \sum_{i=1}^{e} a_{ei} - \epsilon \\ \hline\hline \sum_{i=1}^{e} a_{i1} - \epsilon & \cdots & \sum_{i=1}^{e} a_{ie} - \epsilon & \sum_{i,j=1}^{e} a_{i,j} - \epsilon(p-1) \end{array}\right]. \tag{6.46}$$

On the other hand, we have

$$A = \left[\begin{array}{ccc||c} & & & a_{1p} \\ & \tilde{A} & & \vdots \\ & & & a_{ep} \\ \hline\hline a_{p1} & \cdots & a_{pe} & a_{pp} \end{array}\right].$$

We already showed from (6.31)–(6.33) that

$$a_{1p} = \sum_{i=1}^{e} a_{1i} - \epsilon \tag{6.47}$$

$$\vdots \tag{6.48}$$

$$a_{ep} = \sum_{i=1}^{e} a_{ei} - \epsilon. \tag{6.49}$$

By considering the matrix A, we can conclude

$$\sum_{i=1}^{e} a_{i1} = (p-1)\frac{s_{1,1}}{p(p-1)} + (p-1)\frac{\gamma}{p^2} \tag{6.50}$$

$$= \frac{s_{1,1}}{p} + \frac{-\gamma}{p^2} + \epsilon \tag{6.51}$$

$$= a_{p1} + \epsilon. \tag{6.52}$$

Hence, $a_{p1} = \sum_{i=1}^{e} a_{i1} - \epsilon$. In a similar way we can show that

$$a_{p2} = \sum_{i=1}^{e} a_{i2} - \epsilon \tag{6.53}$$

$$\vdots \tag{6.54}$$

$$a_{pe} = \sum_{i=1}^{e} a_{ie} - \epsilon. \tag{6.55}$$

Finally, it remains to show that $a_{pp} = \sum_{i,j=1}^{e} a_{ij} - (e-1)\epsilon$. Note that

$$\sum_{i,j=1}^{e} a_{ij} - (e-1)\epsilon = \sum_{j=1}^{e} a_{1,j} + \cdots + \sum_{j=1}^{e} a_{e,j} - (e-1)\epsilon \tag{6.56}$$

$$= (a_{1,p} + \epsilon) + \cdots + (a_{e,p} + \epsilon) - (e-1)\epsilon \tag{6.57}$$

$$= a_{1,p} + \cdots + a_{e,p} + e\epsilon - (e-1)\epsilon \tag{6.58}$$

$$= a_{1,p} + \cdots + a_{e,p} + \epsilon \tag{6.59}$$

$$= a_{p,p} - \epsilon + \epsilon \tag{6.60}$$

$$= a_{p,p} \tag{6.61}$$

where (6.57) follows from the relations in (6.31)–(6.33) and (6.60) follows from

$$a_{1p} + \cdots + a_{ep} = a_{pp} - \frac{\gamma}{p} = a_{pp} - \epsilon$$

by (6.30). This completes the proof. □

Chapter 7

Stable perfect isometries for blocks with Klein four defect group

In this chapter, we will prove the surjectivity of the map $\Phi : \mathrm{PI}(B) \to \overline{\mathrm{SPI}}(B)$ where B is a block algebra of $\mathcal{O}G$ with a Klein four defect group. Detailed information regarding the blocks with Klein four defect groups can be found in [15]. Throughout we assume that $(\mathbb{K}, \mathcal{O}, F)$ is large enough p-modular system where $p = 2$.

Remark 85. *([12, Corollary 1.4]) Let G be a finite group and B be a block algebra of $\mathcal{O}G$ having a Klein four defect group P. Then, B is Morita equivalent to either $\mathcal{O}P$ or $\mathcal{O}A_4$ or $\mathcal{O}A_5b_0$, the principal block algebra of $\mathcal{O}A_5$.*

Remark 86. [12, Corollary 1.5] *Let G be a finite group and B be a block algebra of $\mathcal{O}G$ having a Klein four defect group P. Then, B is derived equivalent to either $\mathcal{O}P$ or $\mathcal{O}A_4$.*

Note that Remark 86 implies that the block algebra B with a Klein four defect group P is perfectly isometric to either $\mathcal{O}P$ or $\mathcal{O}A_4$.

By Lemma 68 , to verify the surjectivity of

$$\Phi_B : \mathrm{PI}(B) \to \overline{\mathrm{SPI}}(B),$$

it suffices to consider the cases where $B = \mathcal{O}V_4$ and $B = \mathcal{O}A_4$.

Theorem 87. *The map* $\Phi_{V_4} : \mathrm{PI}(\mathcal{O}V_4) \to \overline{\mathrm{SPI}}(\mathcal{O}V_4)$ *is surjective.*

Proof. This follows from Corollary 75. □

Theorem 88. *The map* $\Phi_{A_4} : \mathrm{PI}(\mathcal{O}A_4) \to \overline{\mathrm{SPI}}(\mathcal{O}A_4)$ *is surjective.*

Proof. Recall that the character table of $A_4 \cong V_4 \rtimes C_3$ is as follows:

$V_4 \rtimes C_3$	1	(123)	(132)	(12)(34)
χ_1	1	1	1	1
χ_2	1	ξ	ξ^2	1
χ_3	1	ξ^2	ξ	1
$\chi_4 = \Phi_4$	3	0	0	-1

Let $X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \xi & \xi^2 & 1 \\ 1 & \xi^2 & \xi & 1 \\ 3 & 0 & 0 & -1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix}$. Note that Lemma 81 applies in which we replace p with $|V_4| = 4$. We note that the rest of the proof for this case follows exactly the same as the proof in Case 2 of Chapter 6 with the alteration of p with $|V_4| = 4$. Using the notation from Section 6.2, if μ is in $\mathrm{SPI}(\mathcal{O}A_4)$ and A is the coefficient matrix of μ, we have

(i) $\gamma = \epsilon \cdot 4$ where $\epsilon = \pm 1$.

(ii) $A = \epsilon I + DWD^t$ for some $W \in \mathrm{Mat}_{3\times 3}(\mathbb{Z})$.

(iii) In particular, $\Phi_{A_4} : \mathrm{PI}(\mathcal{O}A_4) \to \overline{\mathrm{SPI}}(\mathcal{O}A_4)$ is surjective.

□

Chapter 8

Stable p-permutation equivalences

8.1 Definition

Throughout we let e and f be blocks of $\mathcal{O}G$ and $\mathcal{O}H$, respectively. In this chapter, $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$ refers to the Grothendieck group of projective $(\mathcal{O}Ge, \mathcal{O}Hf)$-bimodules.

Definition 89. *A stable p-permutation equivalence between $\mathcal{O}Ge$ and $\mathcal{O}Hf$ is an element γ in $T^{\Delta}(\mathcal{O}Ge, \mathcal{O}Hf)$ satisfying the following conditions:*

(i) $\gamma \cdot_H \gamma^{\circ} = [\mathcal{O}Ge] + \pi_G$ *in* $T^{\Delta}(\mathcal{O}Ge, \mathcal{O}Ge)$ *for some* $\pi_G \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$,

(ii) $\gamma^{\circ} \cdot_G \gamma = [\mathcal{O}Hf] + \pi_H$ *in* $T^{\Delta}(\mathcal{O}Hf, \mathcal{O}Hf)$ *for some* $\pi_H \in \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Hf)$.

We start with some observations which are known to the experts. We will then use these observations to discuss a monoid structure on the set of stable p-permutation self-equivalences of a block.

Lemma 90. *Let $M \in T^{\Delta}(\mathcal{O}G, \mathcal{O}H)$, $W \in \mathrm{Pr}(\mathcal{O}H, \mathcal{O}K)$ and $N \in T^{\Delta}(\mathcal{O}K, \mathcal{O}L)$. Then, we have $M \otimes_{\mathcal{O}H} W \otimes_{\mathcal{O}K} N \in \mathrm{Pr}(\mathcal{O}G, \mathcal{O}L)$.*

Proof. It is sufficient to obtain this result for indecomposable modules. Let M be an indecomposable trivial source $\mathcal{O}(G \times H)$-module with vertex $\Delta(P, \phi, Q)$ for some $P \leq G, Q \leq H$ and a group isomorphism $\phi : Q \to P$, i.e., we have $M \mid \mathrm{Ind}_{\Delta(P,\phi,Q)}^{G\times H}(\mathcal{O})$, and let W be an indecomposable projective $\mathcal{O}(H \times K)$-module. By Theorem 47, we have

$$\mathrm{Ind}_{\Delta(P,\phi,Q)}^{G\times H}(\mathcal{O}) \otimes_{\mathcal{O}H} \mathrm{Ind}_1^{H\times K}(\mathcal{O}) \cong \bigoplus_{t\in[Q\backslash H/1]} \mathrm{Ind}_{\Delta(P,\phi,Q)\star 1}^{G\times K} \tag{8.1}$$

$$\cong \bigoplus_{t\in[Q\backslash H/1]} \mathrm{Ind}_1^{G\times K}(\mathcal{O}) \tag{8.2}$$

which follows from the fact that $\Delta(P,\phi,Q) \star 1 = 1$. Therefore, any indecomposable summand of $M \otimes_{\mathcal{O}H} W$ has a trivial source and trivial vertex, implying that $M \otimes_{\mathcal{O}H} W \in \mathrm{Pr}(\mathcal{O}G, \mathcal{O}K)$. By using the same idea, one can show that $W' \otimes_{\mathcal{O}K} N \in \mathrm{Pr}(\mathcal{O}H, \mathcal{O}L)$ for $W' \in \mathrm{Pr}(\mathcal{O}G, \mathcal{O}K)$ and indecomposable $N \in T^{\Delta}(\mathcal{O}K, \mathcal{O}L)$. Then it is enough to see that $M \otimes_{\mathcal{O}H} W \otimes_{\mathcal{O}K} N$ is in $\mathrm{Pr}(\mathcal{O}G, \mathcal{O}L)$. □

Lemma 91. *Let e, f, e_K and e_L be in $\mathrm{Bl}(\mathcal{O}G), \mathrm{Bl}(\mathcal{O}H), \mathrm{Bl}(\mathcal{O}K), \mathrm{Bl}(\mathcal{O}L)$, respectively. Let $M \in T^{\Delta}(\mathcal{O}Ge, \mathcal{O}Hf)$ and $N \in T^{\Delta}(\mathcal{O}Ke_K, \mathcal{O}Le_L)$ and let $P \in \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Ke_K)$. Then,*

$$M \otimes_H P \otimes_K N \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Le_L).$$

Proof. It follows from Lemma 90. □

Proposition 92. *Let $\gamma \in T^{\Delta}(\mathcal{O}Ge, \mathcal{O}Hf)$ be a stable p-permutation equivalence. Then,*

$$\mu = \kappa(\gamma) \in R(\mathbb{K}Ge, \mathbb{K}Hf)$$

is a stable perfect isometry between $\mathcal{O}Ge$ and $\mathcal{O}Hf$.

Proof. The proof is very similar to that of Proposition 51 which is stated in [1]. We note that the generalized character $\kappa(\gamma)$ is perfect by Remark 44 as $\gamma \in T^{\Delta}(\mathcal{O}Ge, \mathcal{O}Hf)$.

Next, since γ is a stable p-permutation equivalence, so (i)-(ii) holds in Definition 89, and by applying κ we obtain

$$\mu \cdot_H \mu^{\circ} = \kappa_{G\times G}(\gamma \cdot_H \gamma^{\circ}) = [\mathbb{K}Ge] + \kappa_{G\times G}(\pi_G) \tag{8.3}$$

$$\mu^{\circ} \cdot_G \mu = \kappa_{H\times H}(\gamma^{\circ} \cdot_G \gamma) = [\mathbb{K}Hf] + \kappa_{H\times H}(\pi_H) \tag{8.4}$$

where $\kappa_{G\times G}(\pi_G)$ and $\kappa_{H\times H}(\pi_H)$ are both generalized projective characters. This completes the proof. $\square$

Notation 93. *We let* $\mathrm{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf)$ *denote the set of stable p-permutation equivalences between* $\mathcal{O}Ge$ *and* $\mathcal{O}Hf$*.*

Definition 94. *Given* $\gamma \in \mathrm{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf)$*, we consider the set* $\gamma + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$*. By using Lemma 91, one can show that* $\gamma+\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf) \subseteq \mathrm{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf)$ *for all such* γ *in* $\mathrm{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf)$*.The set of such cosets* $\gamma+\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$ *is denoted by* $\overline{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf)$*. We let* $\mathrm{stab}_\circ T^\Delta(\mathcal{O}Ge) := \mathrm{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Ge)$ *and* $\overline{\mathrm{stab}_\circ} T^\Delta(\mathcal{O}Ge) := \overline{\mathrm{stab}_\circ} T^\Delta(\mathcal{O}Ge, \mathcal{O}Ge)$*. We consider the map* $\Psi_{G,H} : T^\Delta_\circ(\mathcal{O}Ge, \mathcal{O}Hf) \to \overline{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf)$ *which is defined by* $\gamma \mapsto \gamma + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf)$*.*

8.2 Properties

Lemma 95. *Let* e, f, e_K *be in* $\mathrm{Bl}(\mathcal{O}G), \mathrm{Bl}(\mathcal{O}H), \mathrm{Bl}(\mathcal{O}K)$*, respectively. Then,* $- \cdot_H -$ *induces the following bilinear maps*

$$\mathrm{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf) \times \mathrm{stab}_\circ T^\Delta(\mathcal{O}Hf, \mathcal{O}Ke_K) \to \mathrm{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Ke_K) \tag{8.5}$$

by $(\gamma, \gamma') \mapsto \gamma \cdot_H \gamma'$*, and*

$$\overline{\mathrm{stab}}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf) \times \overline{\mathrm{stab}}_\circ T^\Delta(\mathcal{O}Hf, \mathcal{O}Ke_K) \to \overline{\mathrm{stab}}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Ke_K) \tag{8.6}$$

by $(\gamma + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Hf), \gamma' + \mathrm{Pr}(\mathcal{O}Hf, \mathcal{O}Ke_K)) \mapsto (\gamma \cdot_H \gamma') + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ke_K)$*.*

In particular, one has the following commutative diagram

$$\begin{array}{ccc}
\mathrm{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf) \times \mathrm{stab}_\circ T^\Delta(\mathcal{O}Hf, \mathcal{O}Ke_K) & \longrightarrow & \mathrm{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Ke_K) \\
\downarrow & & \downarrow \\
\overline{\mathrm{stab}}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf) \times \overline{\mathrm{stab}}_\circ T^\Delta(\mathcal{O}Hf, \mathcal{O}Ke_K) & \longrightarrow & \overline{\mathrm{stab}}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Ke_K).
\end{array}$$

Proof. Firstly, given $\gamma \in T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf)$ and $\gamma' \in T^\Delta(\mathcal{O}Hf, \mathcal{O}Ke_K)$, then it follows that $\gamma \cdot_H \gamma' \in T^\Delta(\mathcal{O}Ge, \mathcal{O}Ke_K)$ by using the Lemma 48 applied to a special case for $X = G \times H$

and for $Y = H \times K$. Next, we let $\gamma \in \mathrm{stab}_\circ T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf)$ and $\gamma^{'} \in \mathrm{stab}_\circ T^\Delta(\mathcal{O}Hf, \mathcal{O}Ke_K)$. Since we know that $\gamma \cdot_H \gamma^{'} \in T^\Delta(\mathcal{O}Ge, \mathcal{O}Hf)$, it only remains to show (i)-(ii) in Definition 89 for $\gamma \cdot_H \gamma^{'}$.

$$
\begin{aligned}
(\gamma \cdot_H \gamma^{'}) \cdot_K (\gamma \cdot_H \gamma^{'})^\circ &= \gamma \cdot_H \gamma^{'} \cdot_K (\gamma^{'})^\circ \cdot_H \gamma^\circ && (8.7)\\
&= \gamma \cdot_H ([\mathcal{O}Hf] + \pi^{'}) \cdot_H \gamma^\circ && (8.8)\\
&= \gamma \cdot_H \gamma^\circ + \gamma \cdot_H \pi^{'} \cdot_H \gamma^\circ && (8.9)\\
&= [\mathcal{O}Ge] + \pi + \gamma \cdot_H \pi^{'} \cdot_H \gamma^\circ && (8.10)
\end{aligned}
$$

and we have $\pi + \gamma \cdot_H \pi^{'} \cdot_H \gamma^\circ \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ke_K)$ by using Lemma 91. This shows (i) of Definition 89 for $\gamma \cdot_H \gamma^{'}$, and (ii) follows very similarly.

For the second part, it suffices to show that

$$
(\gamma + \pi) \cdot_H (\gamma^{'} + \pi^{'}) + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ke_K) = (\gamma \cdot_H \gamma^{'}) + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ke_K). \qquad (8.11)
$$

Consider

$$
(\gamma + \pi) \cdot_H (\gamma^{'} + \pi^{'}) = \gamma \cdot_H \gamma^{'} + \gamma \cdot_H \pi^{'} + \pi \cdot_H \gamma^{'} + \pi \cdot_H \pi^{'}
$$

and by Lemma 91 we have $\gamma \cdot_H \pi^{'} + \pi \cdot_H \gamma^{'} + \pi \cdot_H \pi^{'}$ is in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ke_K)$ proving the claim. □

Lemma 96. *$\overline{stab}_\circ T^\Delta(\mathcal{O}Ge)$ has a group structure induced by $- \cdot_G -$. Then we have the group homomorphism $\Psi_G : T^\Delta_\circ(\mathcal{O}Ge, \mathcal{O}Ge) \to \overline{stab}_\circ T^\Delta(\mathcal{O}Ge)$ defined by $\gamma \mapsto \gamma + \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$.*

Proof. The first part follows from Lemma 95. Second part is straightforward. □

8.3 Surjectivity of $\Psi : T^\Delta_\circ(\mathcal{O}Ge, \mathcal{O}Ge) \to \overline{stab}_\circ T^\Delta(\mathcal{O}Ge)$

Just like for the map $\Phi : \mathrm{PI}(\mathcal{O}Ge) \to \overline{\mathrm{SPI}}(\mathcal{O}Ge)$, we are interested in the question of surjectivity of the map $\Psi : T^\Delta_\circ(\mathcal{O}Ge, \mathcal{O}Ge) \to \overline{stab}_\circ T^\Delta(\mathcal{O}Ge)$. The next result shows that in this case the surjectivity of Φ implies the surjectivity of Ψ.

Proposition 97. *Let $e \in \mathrm{Bl}(\mathcal{O}G)$. Then, the following commutative diagram commutes:*

$$\begin{array}{ccc} T^{\Delta}_{\circ}(\mathcal{O}Ge, \mathcal{O}Ge) & \xrightarrow{\Psi} & \overline{\mathrm{stab}_{\circ}}T^{\Delta}(\mathcal{O}Ge) \\ \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} \\ \mathrm{PI}(\mathcal{O}Ge) & \xrightarrow{\Phi} & \overline{SPI}(\mathcal{O}Ge). \end{array}$$

Moreover, if Φ is surjective, then so is Ψ.

Proof. The commutativity of the diagram is straightforward. For the final part, assume that Φ is surjective. Let $\gamma \in \mathrm{stab}_{\circ}T^{\Delta}(\mathcal{O}Ge)$ and let $\kappa(\gamma) = \mu$. Then, we have

$$\gamma \cdot_G \gamma^{\circ} = [\mathcal{O}Ge] + \pi_G \in T^{\Delta}(\mathcal{O}Ge, \mathcal{O}Ge) \tag{8.12}$$

$$\gamma^{\circ} \cdot_G \gamma = [\mathcal{O}Ge] + \pi_G^{'} \in T^{\Delta}(\mathcal{O}Ge, \mathcal{O}Ge) \tag{8.13}$$

for some π_G and $\pi_G^{'}$ in $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$.

By surjectivity of Φ, there exists $\pi^{''} \in \mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$ such that $\mu + \kappa(\pi^{''}) \in \mathrm{PI}(\mathcal{O}Ge)$. Therefore one has

$$\begin{aligned} [\mathbb{K}Ge] &= (\mu + \kappa(\pi^{''})) \cdot_G (\mu + \kappa(\pi^{''}))^{\circ} \\ &= \mu \cdot_G \mu^{\circ} + \kappa(\pi^{''}) \cdot_G \mu^{\circ} + \kappa(\pi^{''}) \cdot_G \kappa(\pi^{''})^{\circ} + \mu \cdot_G \kappa(\pi^{''})^{\circ} \\ &= [\mathbb{K}Ge] + \kappa(\pi_G) + \kappa_G(\pi^{''}) \cdot_G \mu^{\circ} + \kappa(\pi^{''}) \cdot_G \kappa(\pi^{''})^{\circ} + \mu \cdot_G \kappa(\pi^{''})^{\circ} \\ &= [\mathbb{K}Ge] + \kappa(\pi_G + \pi^{''} \cdot_G \gamma^{\circ} + \pi^{''} \cdot_G (\pi^{''})^{\circ} + \gamma \cdot_G (\pi^{''})^{\circ}). \end{aligned}$$

This implies that

$$\kappa(\pi_G + \pi^{''} \cdot_G \gamma^{\circ} + \pi^{''} \cdot_G (\pi^{''})^{\circ} + \gamma \cdot_G (\pi^{''})^{\circ}) = 0. \tag{8.14}$$

Note that $\kappa(\pi_G + \pi^{''} \cdot_G \gamma^{\circ} + \pi^{''} \cdot_G (\pi^{''})^{\circ} + \gamma \cdot_G (\pi^{''})^{\circ})$ is a generalized projective character by using a special case of Lemma 55 and the fact that π_G and $\pi^{''}$ is a generalized projective character. Since κ is injective on $\mathrm{Pr}(\mathcal{O}Ge, \mathcal{O}Ge)$, this implies that

$$\pi_G + \pi^{''} \cdot_G \gamma^{\circ} + \pi^{''} \cdot_G (\pi^{''})^{\circ} + \gamma \cdot_G (\pi^{''})^{\circ} = 0. \tag{8.15}$$

Next, we show that $\gamma + \pi^{''}$ is actually in $T_{\circ}^{\Delta}(\mathcal{O}Ge, \mathcal{O}Ge)$. Note that

$$(\gamma + \pi^{''}) \cdot_G (\gamma + \pi^{''})^{\circ} = \gamma \cdot_G \gamma^{\circ} + \gamma \cdot_G (\pi^{''})^{\circ} + \pi^{''} \cdot_G \gamma^{\circ} + \pi^{''} \cdot_G (\pi^{''})^{\circ} \tag{8.16}$$

$$= [\mathcal{O}Ge] + \pi_G + \gamma \cdot_G (\pi^{''})^{\circ} + \pi^{''} \cdot_G \gamma^{\circ} + \pi^{''} \cdot_G (\pi^{''})^{\circ} \tag{8.17}$$

$$= [\mathcal{O}Ge] \tag{8.18}$$

where the equality (8.18) follows from the equality (8.15). Note that $\Psi(\gamma + \pi^{''}) = \gamma$, showing the surjectivity of Ψ whenever the surjectivity of Φ is assumed. $\square$

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