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Journal
IEEE Transactions on Control of Network Systems, 7(2)

ISSN
2372-2533

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Publication Date
2020

DOI
10.1109/tcns.2019.2951672

Peer reviewed

Opinion Dynamics and Social Power Evolution: A Single-Timescale Model

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Abstract

This paper studies the evolution of self-appraisal and social power, for a group of individuals who discuss and form opinions. We consider a modification of the recently proposed DeGroot-Friedkin (DF) model, in which the opinion formation process takes place on the same timescale as the reflected appraisal process; we call this new model the single-timescale DF model. We provide a comprehensive analysis of the equilibria and convergence properties of the model for the settings of irreducible and reducible influence networks. For the setting of irreducible influence networks, the single-timescale DF model has the same behavior as the original DF model, that is, it predicts among other things that the social power ranking among individuals is asymptotically equal to their centrality ranking, that social power tends to accumulate at the top of the centrality ranking hierarchy, and that an autocratic (resp., democratic) power structure arises when the centrality scores are maximally nonuniform (resp., uniform). For the setting of reducible influence networks, the single-timescale DF model behaves differently from the original DF model in two ways. First, an individual, who corresponds to a reducible node in a reducible influence network, can keep all social power in the single-timescale DF model if the initial condition does so, whereas its social power asymptotically vanishes in the original DF model. Second, when the associated network has multiple sinks, the two models behave very differently: the original DF model has a single globally-attractive equilibrium, whereas any partition of social power among the sinks is allowable at equilibrium in the single-timescale DF model.

Index Terms

opinion dynamics, reflected appraisal, influence networks, mathematical sociology, network centrality, dynamical systems, coevolutionary networks

I. Introduction

Problem description and motivation: This article focuses on a model for the evolution of social power and self-appraisal in an influence network. The model combines an opinion dynamics process from network systems and a reflected appraisal process from applied psychology. The model is a variation of a recently-proposed dynamical system, called the DeGroot-Friedkin (DF) model, proposed and characterized in [19]; in this proposed variation, the

This material is based upon work supported by the U. S. Army Research Laboratory and the U. S. Army Research Office under grants number W911NF-15-1-0274 and W911NF-15-1-0577. The content of the information does not necessarily reflect the position or the policy of the Government, and no official endorsement should be inferred.

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opinion dynamics process takes place on the same timescale as the reflected appraisal process. In other words, in the (original) DF model reflected appraisal played out over an issue discussion sequence where opinion consensus was reached on each issue; here it plays out during the opinion influence process on a single issue. The model we study in this paper was also independently proposed and studied by Xu *et al.* [25]. The purpose of this article is to provide a rigorous and comprehensive analysis of the asymptotic behavior of the proposed model and to compare it with the DF model.

Literature review: Influence networks and opinion formation processes have been the subject of a rich literature, starting with the averaging model proposed by French in [9], studied also by Harary in [14] and DeGroot in [8], the Abelson model [1], the Friedkin-Johnsen model [11], [13], and the Hegselmann-Krause model [15] among others. Empirical evidence in support of the averaging model (including its variations) is described in [12], [5]. These models are now standard in surveys and textbooks such as [4], [16], [22], [3].

Recently, by combining the DeGroot model of opinion dynamics and a reflected appraisal mechanism, Jia *et al.* [19] proposed a DF model (the DF model) to describe the evolution of individuals' self-appraisal and social power in a network along an issue sequence. Empirical evidence in support of the reflected appraisal mechanism and other aspects of the DF model is provided in [10], which present a remarkable suite of issue-sequence effects on influence network structure consistent with theoretical predictions.

Building on the modeling ideas in [19], several extensions and variations have been proposed recently. For example, Mirtabatabaei *et al.* extended the DF model to include stubborn agents who have attachment to their initial opinions in [21]. A continuous-time self-appraisal model was introduced by Chen *et al.* in [7]. Considering time-varying doubly stochastic influence matrices, Xia *et al.* [24] investigated the convergence rate of the modified DF model, which was proven to converge exponentially fast. Very recent submissions (essentially simultaneous with and independent of this article) include [26], [6], [2]; specifically, the works [26], [6] deal with time-varying (deterministic or stochastic) influence networks and the article [2] provides novel stability analysis methods for nonlinear Markov chains (motivated by the DF model).

Finally and notably, motivated by [19], Xu *et al.* [25] proposed a *modified DF model* where the social power is updated without waiting for opinion consensus on each issue to take place, i.e., the local estimation of social power is truncated. In this sense, the time-constant of the opinion dynamic process is now the same as that for the reflected appraisal process. The analysis of the equilibrium points and their attractivity properties was given in [25] only for the setting where the interaction matrix is doubly stochastic. This is the model studied in this paper under the name "single-timescale DeGroot-Friedkin (DF) model."

Statement of contributions: Section II introduces the main modeling assumptions and the definition of the single-timescale DF model. Section III provides a comprehensive analysis of the proposed model for irreducible influence networks. Specifically, Theorem 3.1 characterizes the system behavior over influence networks with star topology and Theorem 3.2 treats the general case. The latter theorem subsumes the specific setting of doubly-stochastic influence networks. Lemma 3.3 characterizes the relationship with the DF model: the two models (over irreducible influence networks) converge to the same equilibria and therefore predict the same phenomena, e.g., social power ranking equal to an appropriate centrality ranking and social power accumulation at the top. Next, Section IV treats the setting of reducible influence networks. Theorem 4.1 shows that the single-timescale DF model behaviors similarly to the DF model over reducible influence networks with globally reachable nodes, but its set of equilibrium points contain all vertices of a simplex, including the cases that reducible nodes have all social power. In contrast, the reducible nodes loss their social power asymptotically in the DF model. Theorem 4.2 considers the most general case where the associated network has multiple sinks and the two models behave very differently: the DF model has a single globally-attractive equilibrium, whereas any partition of social power among the sinks is allowable at equilibrium in the single-timescale DF model. Finally, Section V contains some final remarks and all proofs are in the Appendices in the supplementary file.

In summary, we believe that these results are meaningful as they extend the validity and scope of the original analysis. It is important to establish the weakest possible conditions under which social power and self-appraisal evolve in a way comparable (or identical) to that predicted by the DF model. This paper, together with other efforts on time-varying influence networks, establishes some robustness in the dynamic behavior with respect to modeling uncertainties.

II. The single-timescale DF model

In this section we introduce and motivate the dynamical model for the evolution of the social influence network where social opinions and social power evolve simultaneously. This model combines the concepts of the DeGroot model for the dynamics of opinions over a single issue and of the Friedkin model for the dynamics of self-weight and social power over a sequence of issues.

We consider a group of $n \geq 2$ individuals who discuss an issue according to a DeGroot opinion formation model with an influence matrix W. Assume that individual opinions about the issue are described by a trajectory $t \mapsto y(t) \in \mathbb{R}^n$ that is determined by the DeGroot averaging model

$$y(t+1) = Wy(t), \quad t = 0, 1, 2, \dots, \tag{1}$$

with given initial conditions $y_i(0)$ for each individual i. Here, the influence matrix W is row-stochastic, i.e., each entry of W is non-negative and each row sum of W equals 1. By (1), each individual i updates its opinion according to the convex combination:

$$y_i(t+1) = w_{ii}y_i(t) + \sum\nolimits_{j=1, j\neq i}^{n} w_{ij}y_j(t).$$

From a psychological viewpoint, the diagonal and the off-diagonal entries of an influence matrix W play conceptually distinct roles. Specifically, the diagonal *self-weight* w_{ii} is the individual's self-appraisal (e.g., self-confidence, self-esteem, self-worth) and corresponds to the extent of closure to interpersonal influence of the ith individual. Instead, the off-diagonal entries w_{ij}, $j \neq i$, are *interpersonal weights* that the ith individual *accords* to other individuals.

For simplicity of notation, we adopt the shorthand $x_i \in [0, 1]$ to denote the self-weight w_{ii} of the ith individual. Because $1 - x_i$ is the aggregated influence on the ith individual of all other individuals, we may decompose the off-diagonal entries as $w_{ij} = (1 - x_i)c_{ij}$, where the coefficients c_{ij} are the *relative interpersonal weights* that the ith individual accords to other individuals. Given $c_{ii} = 0$, the matrix C, called the *relative interaction matrix* is row-stochastic with zero diagonal. Our construction assumes that the matrix C is constant. With these notations and assumptions, a time-dependent influence matrix is written as

$$W(x(t)) = \mathrm{diag}(x(t)) + (I_n - \mathrm{diag}(x(t)))C, \tag{2}$$

and the opinion dynamic process (1) is rewritten as

$$y(t+1) = W(x(t))y(t), \quad t = 0, 1, 2, \ldots.$$

If C is further assumed to be irreducible, the Perron-Frobenius Theorem for non-negative matrices implies that the influence matrix $W(x)$ with $x \geq 0$ admits a unique left eigenvector $w(x)^\top \geq 0$ associated with the eigenvalue 1, with non-negative entries. We may normalize $w(x)$ so that $w(x) \in \Delta_n$. We refer to this row vector $w(x)^\top$ the *dominant left eigenvector* of $W(x)$. If $W(x)$ is aperiodic additionally, then

$$\lim_{t\to\infty} W(x)^t = \mathbb{1}_n w(x)^\top.$$

Our model is completed by formulating how the self-weights $t \mapsto x(t)$ evolve during the opinion formation. By adopting to the psychological concept of *reflected appraisal*, we assume that individual social powers are adjusted along group discussions and the self-weight of an individual is set equal to the social power that the individual exercised over the influence network. We proposed a natural dynamical process [19] that allows each individual to accurately estimate her perceived power. The dynamical process is distributed in the sense that each individual only needs to interact with her influenced neighbors (i.e., those who accord positive interpersonal weights to the individual). By assuming that she is aware of the direct interpersonal weights accorded to her and the perceived powers of her influenced neighbors, each individual updates her perceived power as a convex combination of her own and her influenced neighbors' perceived powers. That is, in each discussion iteration, each individual i estimates her perceived power $p_i(t)$ according to

$$\begin{aligned} p_i(t+1) = w_{ii}(t)p_i(t) + \sum\nolimits_{j=1, j\neq i}^{n} w_{ji}(t)p_j(t),& \\ t = 0, 1, 2, \ldots,& \end{aligned} \tag{3}$$

or, equivalently, $p(t+1) = W(t)^\top p(t)$, where $W(t)$ represents the influence matrix associated to the issue discussion process. By assuming the self-weight of an individual is set equal to the social power that the individual exercised over the influence network, we have $p(t) = x(t)$ for all t. In short, the appraisal update mechanism "self-weight := relative control from the influence network" is written as

$$x(t+1) = W(x(t))^\top x(t), \quad t = 0, 1, 2, \ldots. \tag{4}$$

Because of the row stochastic $W(t)$, the sum of all elements of $x(t)$ is constant. Therefore, it is convenient to assume that the self-weight vector $x(t)$ takes value in Δ_n for all time t.

Given a vector $x = [x_1, \ldots, x_n]$, we denote $x^2 = [x_1^2, \ldots, x_n^2]$ with a slight abuse of notation and then $\mathbb{e}_i^2 = \mathbb{e}_i$. We conclude this modeling discussion with a summary definition.

Definition 2.1 (The single-timescale DF model for the evolution of social influence networks): Consider a group of $n \geq 2$ individuals discussing an issue. Let a row-stochastic zero-diagonal irreducible matrix C be the relative interaction matrix encoding the relative interpersonal weights among the individuals. The single-timescale DF model for the evolution of the self-weights $t \mapsto x(t) \in \Delta_n$ is defined as

$$\begin{aligned} x(t+1) = F(x(t)) &:= C^\top x(t) + (I - C^\top) x^2(t) \\ &= C^\top \left(x(t) - x^2(t) \right) + x^2(t). \end{aligned} \tag{5}$$

In this paper, we aim to (i) characterize the existence, stability, and region of attraction of the equilibria for the single-timescale DF model, and (ii) compare the behavior of the single-timescale DF model with the DF model. Based upon Definition 2.1 of the single-timescale DF model and the definition of the DF model in [19], both models try to describe and predict evolving social-power configures within a social network and try to explain when and why specific configures of self-weights (e.g., $x = \mathbb{e}_i$, namely autocratic configuration, or $x = \frac{1}{n}\mathbb{1}_n$, namely democratic configuration) are attractive. Nevertheless, the evolution of the single-timescale DF model is defined on a single issue discussion, that is, the process of opinion dynamics and the process of reflected appraisal take place over comparable timescales (in sense that the individual self-weight x_i is set equal to the individual perceived power p_i in (3) right after each opinion discussion iteration). Compared with that, the DF model is applied to group discussion on a sequence of issues, that is, the timescales for the two processes are separate: the opinion dynamics are faster than the reflected appraisal dynamics in the influence network. In other words, opinion consensus is achieved before individual self-weights are updated.

III. The single-timescale DF model over irreducible influence networks

In this section we begin the mathematical analysis of the single-timescale DF model. We consider two meaningful situations where the relative interaction matrix C has star topology and where the digraph associated to C is row-stochastic (including its special case where C is doubly-stochastic). We will show that the first situation leads to the emergence of an autocratic power structure with a single leader from all initial conditions, and the second situation leads to the general convergence of self-weight configures, including the emergence of a democratic power structure for doubly-stochastic C.

A. Interactions with star topology and autocratic influence networks

Consider the first case where the digraph associated to the relative interaction matrix has star topology. We assume $n \geq 3$ because the case $n = 2$ is trivial (where C is necessarily symmetric and doubly-stochastic).

Theorem 3.1 (Single-timescale DF model with star topology): For $n \geq 3$, consider the single-timescale DF dynamical system $x(t+1) = F(x(t))$ defined by a relative interaction matrix $C \in \mathbb{R}^{n \times n}$ that is row-stochastic, irreducible, and has zero diagonal. If C has star topology with center node 1, then

(i) (**Equilibria:**) the fixed points of F are the autocratic vertices $\{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, and

(ii) (**Convergence property:**) for all non-autocratic initial conditions $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, the self-weights $x(t)$ converges asymptotically to the autocratic configuration $\mathbb{e}_1$ as $t \to \infty$.

The result of Theorem 3.1 can be interpreted as follows. For the single-timescale DF model associated with star topology, the autocrat is predicted to appear on the center node along the opinion formation process – independently of the initial values in almost all scenarios (except those autocratic states corresponding to the equilibrium points of the system (5)). This is identical to the DF model.

B. Row-stochastic interactions and democratic influence networks

Now we consider the second case where the relative interaction matrix C is row-stochastic. Note that $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for $n = 2$ is such that, for any $(x_1, x_2) \in \Delta_2$ with strictly positive components, F in (5) always satisfies $F(x_1, x_2) = (x_1, x_2)$. We therefore discard this trivial case $n = 2$.

Theorem 3.2 (Single-timescale DF model with row-stochastic interactions): For $n \geq 3$, consider the single-timescale DF dynamical system $x(t+1) = W(x(t))^\top x(t)$ defined by a relative interaction matrix $C \in \mathbb{R}^{n \times n}$ that is row-stochastic, irreducible, and has zero diagonal. Assume that the digraph $G(C)$ associated to C does not have star topology and let $c^\top$ be the dominant left eigenvector of C. Then

(i) (**Equilibria:**) the set of fixed points of F is $\{\mathbb{e}_1, \ldots, \mathbb{e}_n, x^*\}$, where x^* lies in the interior of the simplex Δ_n and the ordering of the entries of x^* is equal to the ordering of the entries of c, and

(ii) (**Convergence property:**) for all non-autocratic initial conditions $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$, the self-weights $x(t)$ exponentially converges to the equilibrium configuration x^* as $t \to \infty$.

Based upon the proof of Theorem 3.1 (i) in Appendix A and the proof of Theorem 3.2 (i) in Appendix B , we immediately have the following extended results.

Lemma 3.3 (Relationship with the DF model over irreducible networks): Given the same C and the same non-autocratic initial state $x(0)$, the dynamical system (5) for the single-timescale DF model converges to the same equilibrium as the dynamical system for the DF model in [19]. Consequentially, the social power in the dynamical system (5) is accumulated to the individuals $\{i\}$ in the social network with high $\{c_i\}$ values.

The social power accumulation statement of Lemma 3.3 is directly from the same property of the DF system. (See details in [19].)

Although the opinion formulation and social power evolution timescales for the single-timescale DF model and the DF model are different, the equilibrium results of Theorem 3.2 and Lemma 3.3 are identical to those of the DF model with an identical C: the equilibrium properties from both models are uniquely determined by the dominant left eigenvector $c^\top$ of C (where c can be called *eigenvector centrality scores* as from [19]). In details, given an irreducible C without star topology, the vector of self-weights $x(s)$ in the single-timescale DF model converges to a unique equilibrium value x^* for all initial conditions, except the autocratic states. This equilibrium value x^* is uniquely determined by the eigenvector centrality score c. The entries of x^* are strictly positive and have the same ordering as that of c, that is, if the centrality scores satisfy $c_i > c_j$, then the equilibrium social power x^* satisfies $x_i^* > x_j^*$, and if $c_i = c_j$, then $x_i^* = x_j^*$. The model exhibits an interesting phenomenon similarly as from the DF model: an accumulation of social power in the central nodes of the network. The accumulation phenomenon is most evident for the star topology case: the center individual with $c_i = 0.5$ has a self-weight of 1, and all other individuals have 0 social powers even they may have strictly positive centrality scores. In contrast, if C is doubly-stochastic, Theorem 3.2 and Lemma 3.3 imply the self-weights of the single-timescale DF system exponentially converge to a democratic configure where the social power of each individual is uniform.

Numerical examples on irreducible networks

In this section we compare the dynamical behavior of the single-timescale DF model (5) with that of the DF model in [19] over an influence network with star topology and over a general irreducible influence network.

a) A network with star topology: We first simulate the self-weight evolution in a network with star topology C.

$$C = \begin{bmatrix} 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

In a network associated with C given in (6), the dynamical trajectories of the self-weights generated by the single-timescale DF model and by the DF model are illustrated in Figure 1. Two models converge to the same equilibrium. Specifically, individual 1 has $1/2$ eigenvector centrality score and her equilibrium self-weight (social power) is 1; the rest 9 individuals have $1/18$ eigenvector centrality score for each and their equilibrium self-weights (social powers) are 0. The social power accumulation phenomenon is most evident in such a network with star topology.

b) Reduced Krackhardt's advice network: Krackhardt's advice network, as illustrated in Figure 2, is based upon a US manufacturing organization, which represents 21 managers and a directed advice network C characterizing who sought advice from whom [20]. If individual i asks for advice from n_i different individuals, then we assume that $c_{ij} = 1/n_i$ for j in these n_i individuals, and $c_{ik} = 0$ for all other individuals k. (See a similar example in [16].) Moreover, self-weighting is not considered in C, that is, $c_{ii} = 0$ for all $i \in \{1, \dots, 21\}$.

The complete Krackhardt's network includes four managers (i.e., individuals 6, 13, 16 and 17) from whom no other individual requests advice. Hence, the complete Krackhardt's network is reducible. Here, we simulate the single-timescale DF model on a reduced Krackhardt's advice network (as shown in Figure 3) without these four nodes. The social power accumulation phenomenon within the reduced Krackhardt's advice network is demonstrated in Figure 4. We may also check from the simulation that the ordering of the vector components of x^* is consistent with that of c, that is, $x_i^* > x_j^*$ if and only if $c_i > c_j$ for $i, j \in \{1, \dots, 17\}$.

The dynamical trajectories of the self-weights generated by the single-timescale DF model (in dot lines) and by the DF model (in solid lines) are illustrated in Figure 5. Given non-autocratic initial conditions, both models converge to the same equilibrium, which is independent of initial conditions. These results are consistent with Theorem 3.2 and Lemma 3.3. Moreover, we observe from this and all following simulations that the single-timescale DF model has less monotonic behaviors and takes more iterations to converge, compared with the DF model.

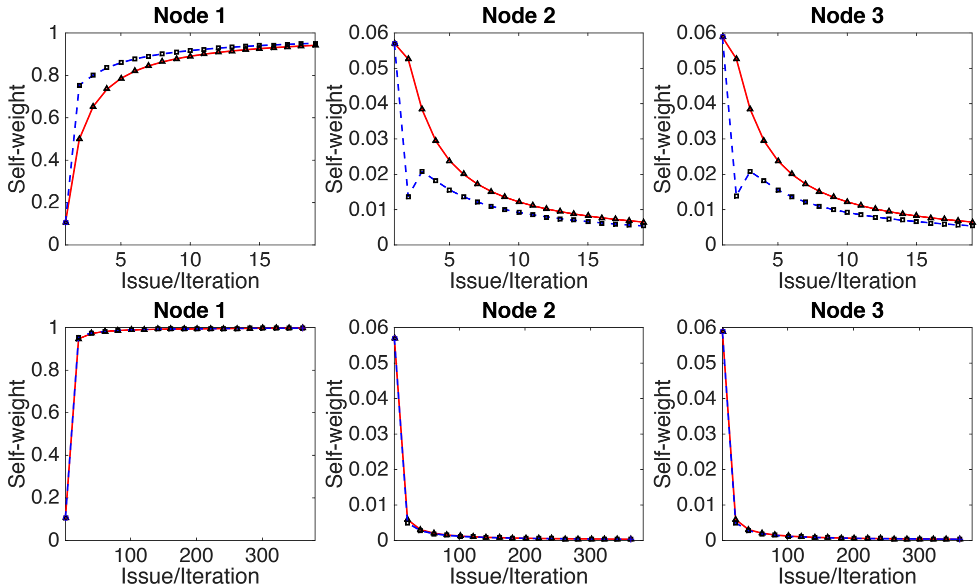

Fig. 1. Self-weight evolution for a network with star topology: we simulate both dynamics of the single-timescale DF model and of the DF model with the same initial conditions; we display the trajectories of 3 nodes. The dot lines are related to the single-timescale DF model and the solid lines are related to the DF model. The top figures show the short-term behaviors and the bottom figures show the long-term dynamics.

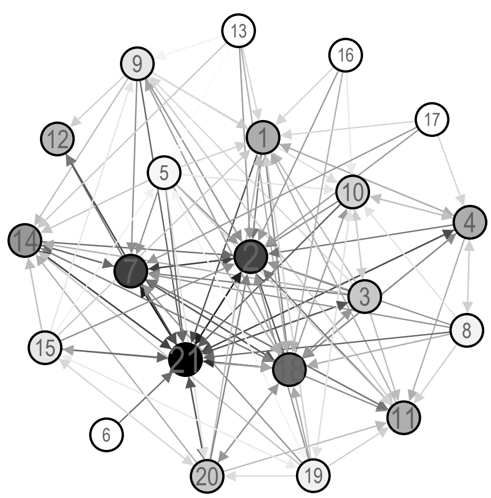

Fig. 2. Krackhardt's advice network with all 21 nodes. The color gradation of the nodes and the font size of the node labels represent c_i.

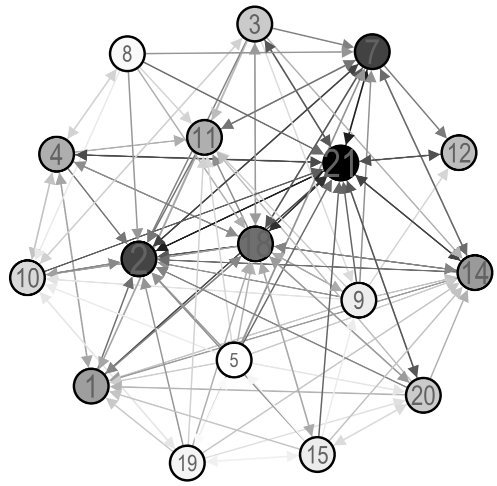

Fig. 3. Reduced Krackhardt's advice network with 17 nodes: the nodes 6, 13, 16 and 17 from the complete Krackhardt's advice network as in Figure 2 are excluded. The color gradation of the nodes and the font size of the node labels represent c_i.

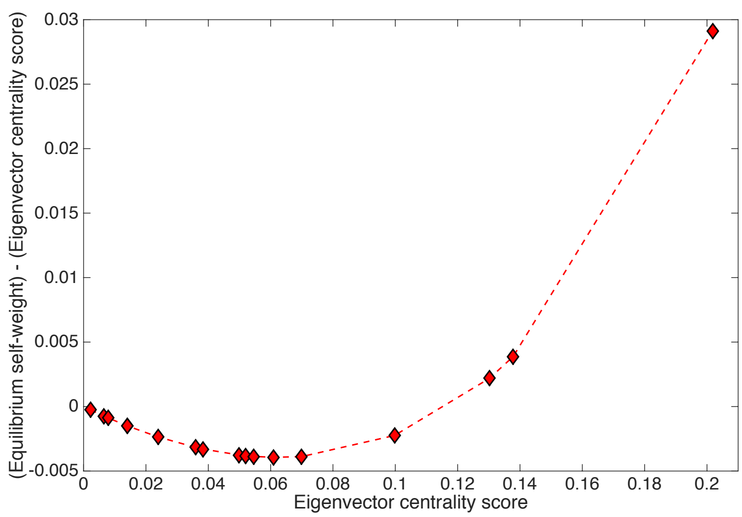

Fig. 4. Comparison between the eigenvector centrality scores and the equilibrium self-weights for the reduced Krackhardt's advice network: the social power accumulation.

IV. The single-timescale DF model over reducible influence networks

The analysis in the previous section assumes that the relative interaction matrix C is irreducible, i.e., the associated digraph is strongly connected and each node is reachable by any other node in the network. In this section we consider two different scenarios where the social influence network is not strongly connected as C is reducible. The part of work is comparable to the DF model analysis over reducible networks as in [18].

First, in Subsection IV-A the matrix C is assumed to be reducible and its associated digraph has globally reachable nodes. One can easily check that such a C admits a unique dominant left eigenvector. The analysis of the single-timescale DF model in this scenario is essentially similar to that for an irreducible matrix C. On one hand, given

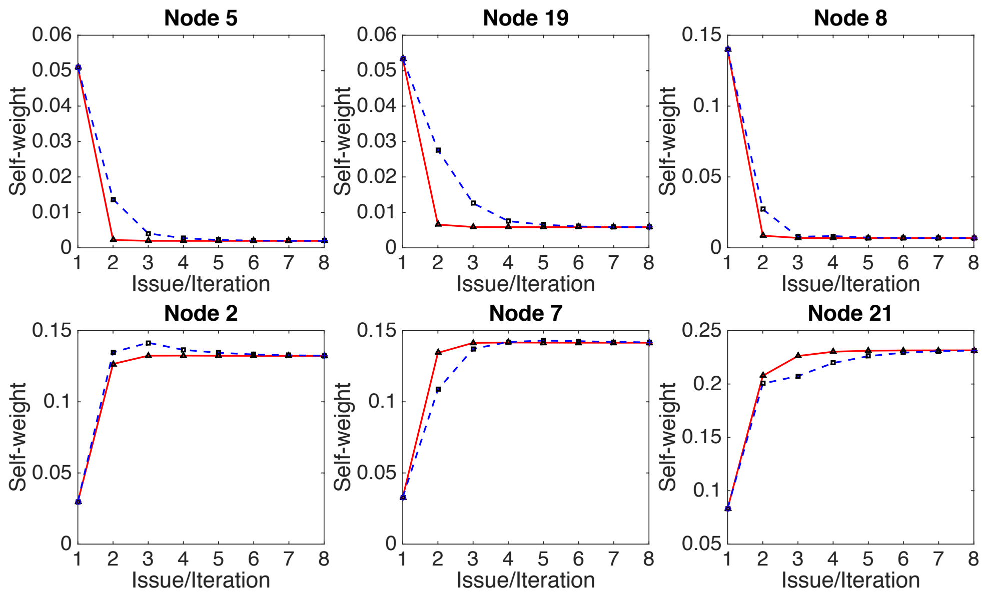

Fig. 5. Self-weight evolution for the reduced Krackhardt's advice network: We display the trajectories of 6 nodes with the same initial condition; the dot lines represent the single-timescale DF dynamics and the solid lines represent the DF dynamics.

non-autocratic initial conditions, the equilibrium of the single-timescale DF model is identical to that of the DF model with the same C; on the other hand, given autocratic initial conditions, the equilibrium of the single-timescale DF model is not necessarily the same as that of the DF model.

Second, in Subsection IV-B the matrix C is assumed to be reducible and its associated condensation digraph has multiple sinks. We then establish the existence and attractivity of the equilibria for the single-timescale DF dynamics with this most general setting. Different from the DF model which has a unique equilibrium, any partition of social power among the sinks is allowable at equilibrium of the single-timescale DF model here.

A. Reducible relative interactions with globally reachable nodes

In this subsection we consider the single-timescale DF model in the setting of reducible C with globally reachable nodes. Recall that C is reducible if and only if $G(C)$ is not strongly connected. Without loss of generality, assume that the globally reachable nodes are $\{1, \ldots, r\}$, for $r \leq n$, and let $G(C_r)$ be the subgraph induced by the globally reachable nodes. One can show that there does not exist a row-stochastic matrix C with zero diagonal and with only one globally reachable node. However, if $r = 1$, by assuming that node 1 is the only globally reachable node, it is necessary that $w_{11} = 1$ and then $x(0) = \mathbb{e}_1$ as W is row-stochastic by definition. The single-timescale DF dynamics then converge to $x^* = x(0)$ even if C is not well defined. We therefore assume $r \geq 2$ in the following.

Theorem 4.1 (Single-timescale DF behavior with reachable nodes): For $n \geq r \geq 2$, consider a single-timescale DF dynamical system $x(t+1) = F(x(t))$ as defined in (5) associated with a relative interaction matrix $C \in \mathbb{R}^{n \times n}$ which is row-stochastic, reducible and with zero diagonal. Let $\{1, \ldots, r\}$ be the globally reachable nodes of $G(C)$. Then the set of equilibrium points of F are $\{\mathbb{e}_1, \ldots, \mathbb{e}_n, x^*\}$, where $x^* \in \Delta_n$ has the following properties:

(i) if $r = 2$, then $x^* = \{(\alpha, 1-\alpha, 0, \cdots, 0)^\top\}$ for any $\alpha \in [0,1]$, and the self-weights $x(t)$ exponentially converge to x^* given a non-autocratic initial $x(0)$;

(ii) if $r \geq 3$ and $G(C_r)$ has star topology with the center node 1, then $x^* = \mathbb{e}_1$, and the self-weights $x(t)$ asymptotically converge to $\mathbb{e}_1$ given any non-autocratic initial $x(0)$;

(iii) if $r \geq 3$ and $G(C_r)$ does not have star topology, then $x^* \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ satisfies: 1) $x_i^* > 0$ for $i \in \{1, \dots, r\}$ and $x_j^* = 0$ for $j \in \{r+1, \dots, n\}$, and 2) the ranking of the entries of x^* is equal to the ranking of the eigenvector centrality scores c; moreover, the self-weights $x(t)$ exponentially converge to x^* given any non-autocratic initial $x(0)$.

Remark 1 (Comparison with the DF model): While the DF model and the single-timescale DF model have the same equilibrium set over irreducible networks, this is not true anymore for reducible networks with globally reachable nodes. By Theorem 4.1, all vertices of the simplex Δ_n, $\{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ are the equilibrium points of the single-timescale DF dynamical system, whereas only the vertices corresponding to globally reachable nodes, $\{\mathbb{e}_1, \dots, \mathbb{e}_r\}$, are the equilibrium points of the DF model. Nevertheless, the equilibrium point x^* in the interior of Δ_n for the single-timescale DF dynamics is identical to that associated with the DF model. In both models, x^* is almost globally attractive.

Numerical examples on reducible networks with globally reachable nodes

In the following, we simulate the single-timescale DF dynamics on the complete Krackhardt's advice network (as shown in Figure 2) and on a reducible network with star topology on its irreducible nodes.

a) Complete Krackhardt's advice network: The complete Krackhardt's network, as illustrated in Figure 2, includes four managers (i.e., individuals 6, 13, 16 and 17) from whom no other individual requests advice. Hence, this network is reducible but with globally reachable nodes (i.e., the rest 17 individuals). Similar to the reduced Krackhardt's network, if individual i asks for advice from n_i different individuals, then we assume that $c_{ij} = 1/n_i$ for j in these n_i individuals, and $c_{ik} = 0$ for all other individuals k. Moreover, self-weighting is not considered in C, that is, $c_{ii} = 0$ for all $i \in \{1, \dots, 21\}$. The corresponding vectors c and $x^* - c$ of the complete Krackhardt's advice network are demonstrated in Figure 6 to show the phenomenon of social power accumulation. Meanwhile, we can check that the ordering of the vector components of x^* is consistent with that of c, that is, $x_i^* > x_j^*$ if and only if $c_i > c_j$ for $i, j \in \{1, \dots, 21\}$.

The dynamical trajectories of the self-weights in the Krackhardt's advice network generated by the single-timescale DF model and the DF model are compared in Figure 7. For non-autocratic initial conditions, both models converge to the same equilibrium.

b) A reducible network with star topology on its irreducible subgraph: We additionally simulate the single-timescale DF dynamics on a reducible network with star topology on its irreducible subgraph. The single-timescale DF model and the DF model are compared in Figure 8 and Figure 9. We can observe that (i) given a non-autocratic initial condition, both dynamical systems converge to the same equilibrium $\mathbb{e}_1$, which implies all social power is accumulated on individual 1; (ii) given an autocratic initial condition on one reducible node, then the two systems converge to different equilibria. These statements are consistent with our discussion in Theorem 4.1.

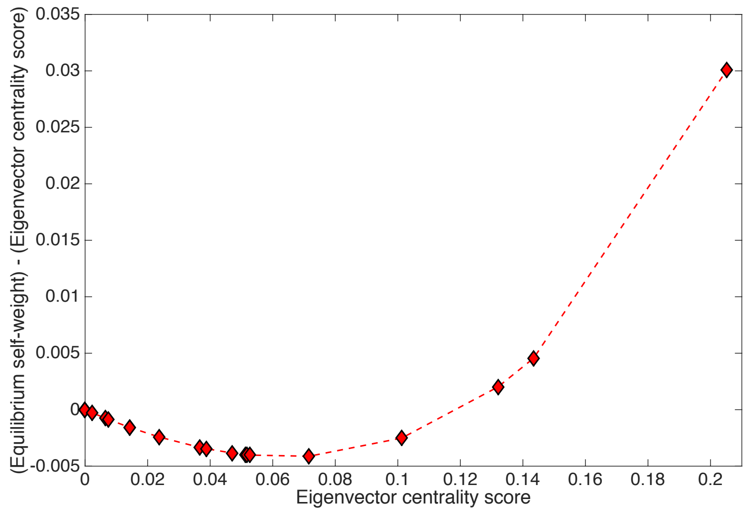

Fig. 6. Comparison between the eigenvector centrality scores and the equilibrium self-weights for the Krackhardt's advice network: the social power accumulation.

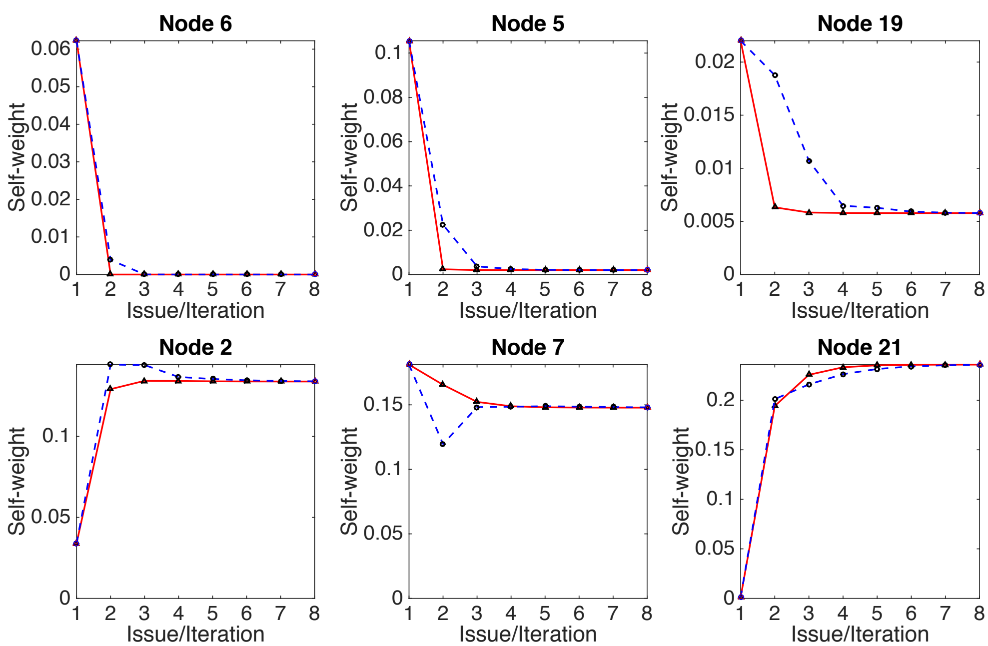

Fig. 7. Self-weight evolution for the Krackhardt's advice network: we display the trajectories of 6 nodes with the same initial conditions; the dot lines represent the single-timescale DF dynamics and the solid lines represent the DF dynamics.

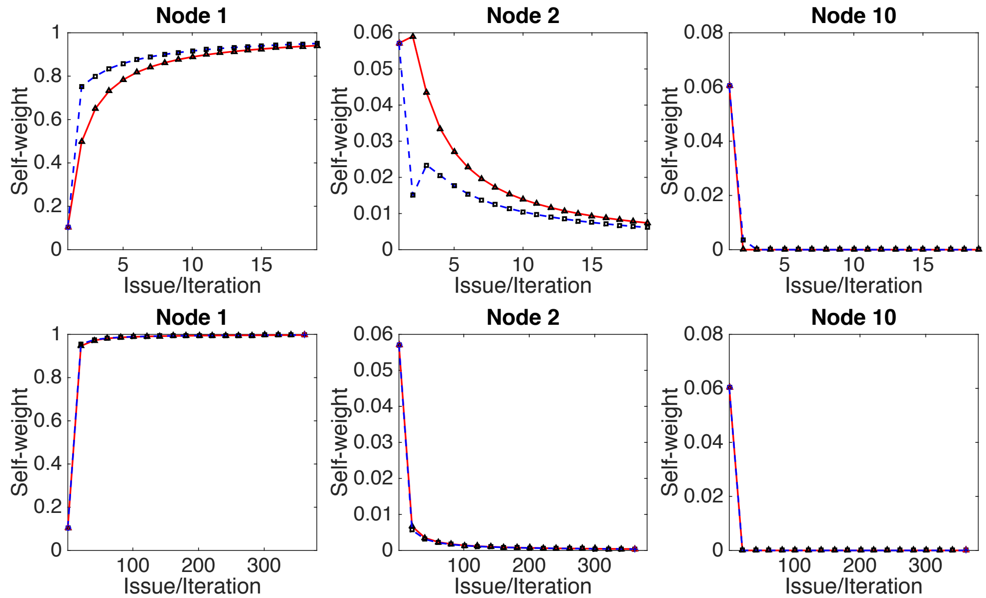

Fig. 8. Self-weight evolution for a network with star topology on its irreducible subgraph (that includes 10 nodes where node 10 is reducible and node 1 is the center): we simulate both dynamics of the single-timescale DF model and of the DF model with the same non-autocratic initial conditions. The dot lines represent the single-timescale DF dynamics and the solid lines represent the DF dynamics. The top subgraphs shows the short-term behaviors and the bottom subgraphs shows the long-term behaviors. Both systems converge to the same equilibrium $\mathbb{e}_1$.

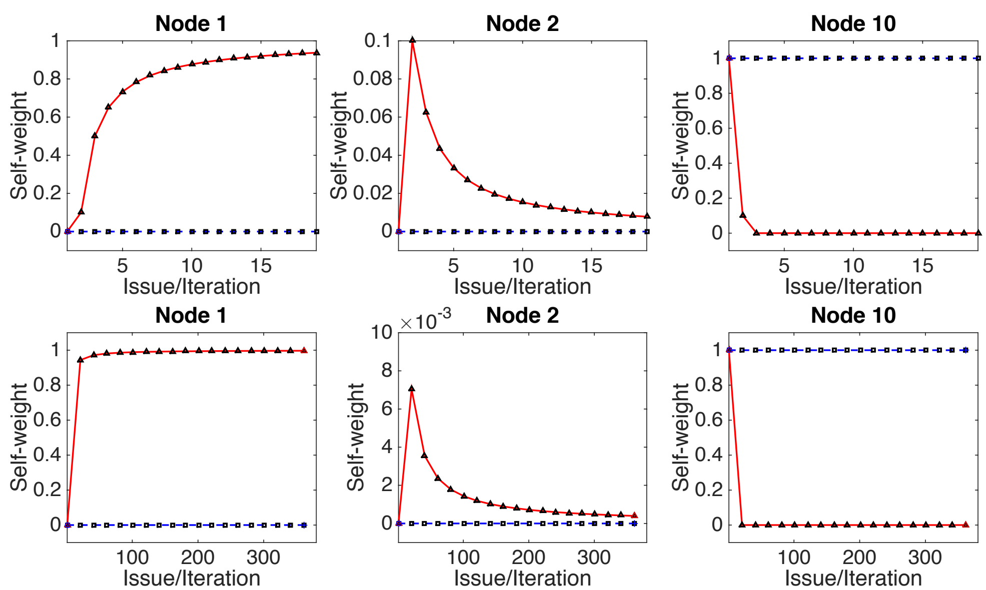

Fig. 9. Self-weight evolution for the same network as in Fig. 8: we simulate both dynamics of the single-timescale DF model and of the DF model with the same autocratic initial conditions $x(0) = \mathbb{e}_{10}$. The dot lines represent the single-timescale DF dynamics and the solid lines represent the DF dynamics. The top subgraphs shows the short-term behaviors and the bottom subgraphs shows the long-term behaviors. The two systems converge to two different equilibria $\mathbb{e}_{10}$ and $\mathbb{e}_1$, respectively.

B. Reducible relative interactions with multiple sink components

In this subsection we generalize the treatment of the single-timescale DF model to the setting of reducible C without globally reachable nodes. Such matrices C have an associated condensation digraph $D(G(C))$ with $K \geq 2$ sinks.

In what follows, n_k denotes the number of nodes in sink k, $k \in \{1, \dots, K\}$, of the condensation digraph; by construction $n_k \geq 1$. Assume that the number of nodes in $G(C)$, not belonging to any sink in $D(G(C))$, is m, that is, $\sum_{k=1}^{K} n_k + m = n$. After a permutation of rows and columns, C can be written as

$$C = \begin{bmatrix} C_{11} & 0 & \dots & 0 & 0 \\ 0 & C_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & C_{KK} & 0 \\ C_{M1} & C_{M2} & \dots & C_{MK} & C_{MM} \end{bmatrix}, \tag{7}$$

where the first $(n-m)$ nodes belong to the sinks of $D(G(C))$ and the remaining m nodes do not. By construction each $C_{kk} \in \mathbb{R}^{n_k \times n_k}$, $k \in \{1, \dots, K\}$, is row-stochastic and irreducible. The Perron-Frobenius Theorem for irreducible matrices implies that C_{kk} has a unique positive dominant left eigenvector $c_{kk}^{\top} = (c_{kk_1}, \dots, c_{kk_{n_k}})$, satisfying $c_{kk} \in \Delta_{n_k}$, independently of whether C_{kk} is aperiodic or periodic. Under these assumptions, the matrix C has the following properties [18]: 1) eigenvalue 1 has geometric multiplicity equal to K, the number of sinks in the condensation digraph $D(G(C))$; 2) C has K dominant left eigenvectors associated with eigenvalue 1, denoted by ${c^k}^{\top} \in \mathbb{R}^n$ for $k \in \{1, \dots, K\}$ and $c_i^k > 0$ if and only if node i belongs to sink k. We may check that $c_i^k = c_{kk_j}$ for $j = i - \sum_{l=1}^{k-1} n_\ell$. We also denote $x = (x_{11}^{\top}, x_{22}^{\top}, \dots, x_{KK}^{\top}, x_{MM}^{\top})^{\top}$, where $x_{kk} = (x_{kk_1}, \dots, x_{kk_{n_k}})^{\top} \in \mathbb{R}^{n_k}$ are the self-weights associated with sink k. Similarly, $x_i = x_{kk_j}$ for $j = i - \sum_{l=1}^{k-1} n_\ell$. Given x and C with the form (7), the corresponding W has the following form:

$$W = \begin{bmatrix} W_{11} & 0 & \dots & 0 & 0 \\ 0 & W_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & W_{KK} & 0 \\ W_{M1} & W_{M2} & \dots & W_{MK} & W_{MM} \end{bmatrix}, \tag{8}$$

where $W_{Mi} = (I_m - \operatorname{diag}(x_{MM}))\, C_{Mi}$ for $i < M$ and $W_{kk} = \operatorname{diag}(x_{kk}) + (I_{n_k} - \operatorname{diag}(x_{kk}))\, C_{kk}$ for $k\{1, \dots, K\}$.

Similar to the discussion on the single-timescale DF model with reducible C and with globally reachable nodes, for a social network with multiple sink components and with reducible nodes, the social power moves from the reducible nodes (by diminishing exponentially fast) to the sinks. The social power of each sink only increases or remains constant depending upon the initial conditions and the network structure. The social power dynamics in each sink are similar to those discussed in the irreducible case Theorem 3.2, though the total social power of the sink is neither equal to 1 nor constant in general.

Theorem 4.2 (Single-timescale DF behavior with multiple sinks): For $n \geq 3$, consider the single-timescale DF dynamical system $x(t+1) = F(x(t))$ as defined in (5) associated with a relative interaction matrix $C \in \mathbb{R}^{n \times n}$.

Assume that the condensation digraph $D(G(C))$ contains $K \geq 2$ sinks and that C is written as in equation (7). Then the following statements hold.

(i) (**Equilibrium:**) The set of equilibrium points of F is the union of the set of vertices $\{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ and of the set $\{x^* = x^*(\zeta^*) \in \Delta_n \mid \zeta^* \in \Delta_K\}$, where ζ_k^* is the total self-weight of sink k and where x^* is uniquely determined by ζ^* and has the following properties:

(i.1) if node i, $i \in \{1, \dots, n\}$, does not belong to any sink, then $x_i^* = 0$;

(i.2) if node i, $i \in \{1, \dots, n\}$, belongs to sink $k \in \{1, \dots, K\}$ and $n_k = 2$, then $x_i^* = \zeta_k^*/2$ if $\zeta_k^* < 1$, or $x_{kk}^* = (\alpha, 1-\alpha)^\top$ for some $\alpha \in [0, 1]$ if $\zeta_k^* = 1$;

(i.3) if node i, $i \in \{1, \dots, n\}$, belongs to sink $k \in \{1, \dots, K\}$ and $n_k \geq 3$, then $x_i^* > 0$ if $\zeta_k^* > 0$, or else $x_i^* = 0$ if $\zeta_k^* = 0$;

(i.4) for sinks with $n_k \geq 3$ and $\zeta_k^* > 0$, the ranking of the entries of the vector x_{kk}^* is equal to the ranking of the eigenvector centrality scores c_{kk}.

(ii) (**Monotonicity of sink social power:**) For all $t \geq 0$, the sink social power $\zeta_k(t)$, equal to the sum of the individual self-weights in each sink $k \in \{1, \dots, K\}$, is non-decreasing, i.e., $\zeta_k(t+1) \geq \zeta_k(t)$; if $\zeta_k(0) = 0$ for a sink k and $x_i(0) = 0$ for any reducible node i such that there exists a direct path from i to the sink k in the associated influence network, then $\zeta_k^* = \zeta_k(t) = 0$ for all $t \geq 0$.

(iii) (**Convergence of self-weights:**) For any initial $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, the self-weights $x(t)$ exponentially converge to an equilibrium point x^* as $t \to \infty$, where x^* is specified as in statement (i).

Remark 2 (Eigenvector centrality): Similar to the DF model on reducible networks with multiple sinks [18], we may regard $\zeta_k^* c_{kk}$ as the individual eigenvector centrality scores in sink k. A node has zero eigenvector centrality score if it does not belong to any sink. When the number of the sinks is $K \geq 2$ and $\zeta_k^* > 0$ for all $k \in \{1, \dots, K\}$, we have $\zeta_k^* c_{kk_i} < 0.5$ for any sink with at least two nodes. Consequently, the star topology in a sink does not correspond to an equilibrium point with all sink social power on the center node of the sink, as the eigenvector centrality score of the sink center is less than 0.5. Meanwhile, the social power accumulation is observed in each sink k: for any individuals $i, j \in \{1, \dots, n_k\}$ with centrality scores satisfying $c_{kk_i} > c_{kk_j} > 0$, the social power is increasingly accumulated in individual i compared to individual j, that is, $x_{kk_i}^*/c_{kk_i} > x_{kk_j}^*/c_{kk_j}$.

Remark 3 (Comparison with the DF model): For this most general case, the single-timescale DF model behaves very differently from the DF model: any partition of social power among the sinks is allowable at equilibrium of the single-timescale DF model, whereas the DF model has a single globally-attractive equilibrium, uniquely determined by C. In addition, all vertices of the simplex $\{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ are equilibrium points of the single-timescale DF model, but none of them is an equilibrium point of the DF model.

Numerical examples on the Sampson's monastery network

We demonstrate the single-timescale DF dynamics with a numerical application to the Sampson's monastery network [23]. We compare the single-timescale DF model with the DF model in terms of dynamical trajectory and equilibrium. The Sampson's monastery network and the corresponding C have been specified in our previous work [18] and we use the same setup of the network. In particular, C associated with Sampson's empirical data on esteem interpersonal relations is reducible. The condensation digraph associated with C includes two sinks: sink 1 consists of the nodes $\{1, 2\}$, and sink 2 consists of the nodes $\{3, \ldots, 15\}$, and the rest nodes are reducible; see Figure 10.

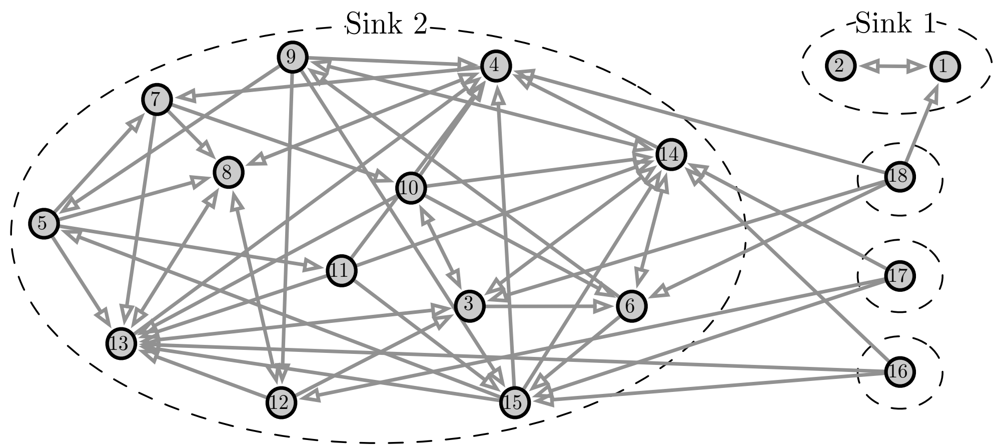

Fig. 10. Sampson's monastery network

We simulate both the single-timescale DF model and the DF model on this monastery network with the same randomly selected initial states $x(0) \in \Delta_{18}$.

The dynamical trajectories of 6 selected nodes in the Sampson's monastery network are illustrated in the first 6 subgraphs of Figure 11. The trajectories of the total self-weights in the two sinks under the same set of initial conditions are shown in the last two subgraphs of Figure 11.

In addition to the differences observed from Figure 11, we also note that, given different initial conditions and a constant C, the DF model always converges to the same equilibrium (see [18]), but the single-timescale DF model converges to different equilibria by simulation. Specifically, regarding the DF model, the reducible nodes have 0 self-weights after the second issue discussion iteration and the sum of the self-weights for each sink after the second iteration is uniquely determined by C but not $x(0)$. Moreover, the sink social power for each sink keeps constant afterwards. Regarding the single-timescale DF model, the social power on reducible nodes converges to 0 exponentially in general. Then at each iteration social power keeps migrating from reducible nodes to their connected sinks. Such dynamics depend not only upon C but also upon the self-weight profile $x(t)$. As a result, each sink social power keeps increasing. The simulations may illustrate how different $x(0)$ lead to different social power evolving processes and, therefore, different equilibria.

V. Conclusion

In this paper we have characterized the equilibrium and asymptotic behavior of a single-timescale DF model for the evolution of social power in a social influence network. Compared with the DF model, a fundamental assumption in this modified model is that individual social power evolves at the same timescale as the group opinion forms. That

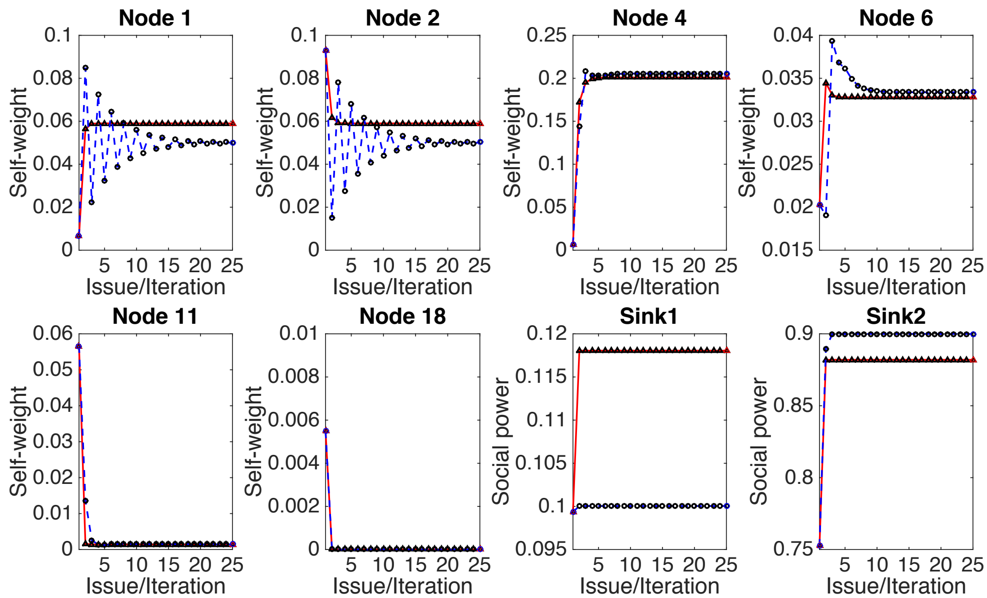

Fig. 11. Self-weight evolution for the Sampson's monastery network: we simulate both dynamics of the single-timescale DF model and of the DF model with the same initial conditions. The dot lines represent the single-timescale DF dynamics and the solid lines represent the DF dynamics. We observe the common points for the two systems, including 1) for two nodes $\{1, 2\}$ in sink 1 with $n_1 = 2$, the equilibrium self-weights are strictly positive and equal; 2) for the nodes in sink 2 with $n_2 = 13$, all equilibrium self-weights are strictly positive and $x_i^* > x_j^*$ if and only $c_i^2 > c_j^2$, in particular, node 4 has the max eigenvector centrality score in the sink, node 11 has the min score, and node 6 has a score in between; 3) the nodes $\{16, 17, 18\}$, which do not belong to any sink, have zero equilibrium self-weights. We also observe the differences between two systems, including 1) the convergence behaviors for a sink with two nodes are significantly different; 2) the equilibrium self-weight sums are different for each sink between two systems, even given the same initial conditions; 3) the convergence of the self-weight sum at each sink occurs in two steps for the DF model, but it may take more steps for the single-timescale DF model.

is to say, social power is updated without waiting for opinion consensus. We have derived a concise dynamical model for the single-timescale DF evolution and completely characterized its asymptotic properties on both irreducible and reducible networks; our results are consistent with the partial and independent analysis in [25]. We have also compared the new model with the DF model in terms of their dynamical behaviors. The analytical and numerical results show that (i) the single-timescale DF model has the same behavior as the DF model over irreducible networks; (ii) the single-timescale DF model behaves differently from the DF model over reducible networks: the new model has a broader equilibrium set including all autocratic points, and including equilibrium points corresponding to any partition of social power among the sinks if the underlying network has multiple sink components. Meanwhile, social power accumulation is also observed in the new model.

This paper completes the application of reflected appraisal mechanism to DeGroot's opinion dynamics model and extends the validity and scope of the original analysis on the DF model. This paper, together with other efforts on time-varying influence networks, establishes some robustness on social power and self-appraisal evolution predicted by the DF model with respect to modeling uncertainties. Much work remains to be done in order to understand social power evolution on various opinion formation processes. The potential examples include the Friedkin-Johnsen model [11], [12], where individuals tend to anchor their opinions on their initial values, and include influence networks with non-cooperative individuals (e.g., a preliminary work on existence of stubborn individuals [21]).

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APPENDIX A
PROOF OF THEOREM 3.1

Proof: Regarding fact (i), we first show that the set of vertices $\{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ are the fixed point of the dynamical system (5). Given $x(t) = \mathbb{e}_i$ and any C, it is clear that

$$x(t+1) = F(\mathbb{e}_i) = C^\top \mathbb{e}_i + (I - C^\top)\mathbb{e}_i^2 = C^\top \mathbb{e}_i + (I - C^\top)\mathbb{e}_i = \mathbb{e}_i.$$

Second, for C associated with star topology, we show that there does not exist a fixed point in the simplex except the vertices. By contradiction, assume that there exists a vector $x \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ such that $x = F(x)$. The fixed point equation $x = F(x)$ implies

$$x_i = \sum_{j=1, j\neq i}^{n} c_{ji}\left(x_j - x_j^2\right) + x_i^2, \quad \text{for all } i \in \{1, \dots, n\}. \tag{9}$$

If C is with star topology and the central node is 1, then $c_{ij} = 0$, $c_{j1} = 1$ and $c_{1j} > 0$ for all $j, i \in \{2, \dots, n\}$. Especially, $c_{1j} > 0$ for all $j \in \{2, \dots, n\}$ because, otherwise, C is reducible as $c_{ij} = 0$ for all $i \in \{1, \dots, n\}$ given j. Therefore, from (9),

$$\begin{aligned} x_j &= c_{1j}\left(x_1 - x_1^2\right) + x_j^2, \quad \text{for all } j \in \{2, \dots, n\}, \\ x_1 &= \sum_{j=2}^{n}\left(x_j - x_j^2\right) + x_1^2. \end{aligned}$$

That is to say,

$$x_1 - x_1^2 = \sum_{j=2}^{n}\left(x_j - x_j^2\right), \tag{10}$$

which implies $x_1 - x_1^2 > 0$ as $x \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, and hence $(x_j - x_j^2) = c_{1j}\left(x_1 - x_1^2\right) > 0$ for all $j \in \{1, \dots, n\}$ as $c_{1j} > 0$. Moreover, as $x_i(1 - x_i)$ is concave for $x_i \in [0, 1]$, given $n \geq 3$ and $x \in \text{interior}\,\Delta_n$, we have

$$\sum_{j=2}^{n} x_j(1 - x_j) > \sum_{j=2}^{n} \frac{x_j}{1 - x_1}(1 - x_1)x_1 = x_1(1 - x_1), \tag{11}$$

which contradicts equation (10). Overall, for C with star topology, all fixed points of the dynamical system (5) are the vertices of the simplex.

Regarding fact (ii), based upon the analysis above, for C with star topology, the dynamical system $x(t+1) = F(x(t))$ is specified as follows:

$$\begin{aligned} x_j(t+1) &= c_{1j}\left(x_1(t) - x_1(t)^2\right) + x_j(t)^2, \quad \text{for all } j \in \{2, \dots, n\}, \\ x_1(t+1) &= \sum_{j=2}^{n}\left(x_j(t) - x_j(t)^2\right) + x_1(t)^2. \end{aligned} \tag{12}$$

It is clear that the function $F(x)$ is continuous for $x \in \Delta_n$. If $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, then there exists a node j such that $1 > x_j(0) > 0$, which, together with (12), implies $x_1(1) > 0$. If $x_1(1) = 1$, then $x(t) = \mathbb{e}_1$ for all $t \geq 1$, and if $x_1(1) < 1$, then $x_j(2) > 0$ for all $j \in \{2, \dots, n\}$. Iteratively, we can show either $x(t) = \mathbb{e}_1$ or $x(t) > 0$ for all $t \geq 1$. Moreover, if $x(t) > 0$, then from (11),

$$x_1(t+1) - x_1(t) = \sum_{j=2}^{n} \left(x_j(t) - x_j(t)^2\right) - (x_1(t) - x_1(t)^2) > 0. \tag{13}$$

Define a Lyapunov function candidate $V(x) = 1 - x_1$ for $x \in \Delta_n$. A sublevel set of V is defined as $\{x \mid V(x) \leq \beta\}$ for a given constant β. It is clear that 1) any sublevel set of V is compact and invariant, 2) V is strictly decreasing anywhere along the trajectory of $x(t)$ in $\Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, and 3) V and F are continuous. Therefore, every trajectory starting in $\Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$ converges asymptotically to the equilibrium point $\mathbb{e}_1$ by the Lyapunov theorem for discrete-time dynamical systems. ■

APPENDIX B
PROOF OF THEOREM 3.2

Proof: Regarding fact (i), the equilibria of the influence evolution system (5) include all vertices of the simplex as we already demonstrate in Theorem 3.1. Now, for C irreducible and without star topology we show that there exists a unique $x^* \in \operatorname{interior} \Delta_n$ satisfying $x^* = F(x^*)$ and that the ordering of the elements of x^* is consistent with that of c. The fixed points of the dynamical system (5) shall satisfy

$$x^* - x^{*2} = C^\top (x^* - x^{*2}). \tag{14}$$

It is clear that if $x^* \notin \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, then $x^* - x^{*2} \neq 0$. Therefore, $(x^* - x^{*2})$ is a scalar multiple of the left eigenvector of C associated with eigenvalue 1. For C without star topology, we have

$$x^* - x^{*2} = \alpha^* c, \quad \text{or equivalently,} \quad x_i^* = \alpha^* \frac{c_i}{1 - x_i^*}, \quad \text{for all } i \in \{1, \dots, n\},$$

where the scalar α^* is such that $x^* \in \Delta_n$, that is to say,

$$\alpha^* = \frac{1}{\sum_{j=1}^{n} c_j / (1 - x_j^*)}.$$

It is clear that such an x^* is exactly the same as the non-vertex fixed point we obtained from the DF model. Therefore, the uniqueness of x^* is directly from Theorem 4.1 in [19].

Regarding fact (ii), from (4), we have

$$x(t+1) = \prod_{k=0}^{t} W(t-k)^\top x(0),$$

where $W(t-k) := W(x(t-k))$ for simplicity. If we can show the product $\prod_{k=0}^{t} W(k)$ converges, then $x(t)$ also converges. To do so, we claim:

(A1) for any $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \dots, \mathbb{e}_n\}$, $W(t)$ is aperiodic and irreducible for all $t \geq 0$ and $x(t) > 0$ for all $t \geq n - 1$;

(A2) the minimum positive entries of $W(t)$ are lower bounded uniformly for all t.

These two claims guarantee the exponential convergence to x^* for the dynamical system (4) and (5). (See Lemma D.1 in [17].)

Regarding the first claim (A1), as $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$, there exist $m \geq 2$ nodes with non-zero initial self-weights. Without loss of generality, we assume $x_i(0) > 0$ for $i \in \{1, \ldots, m\}$ and the rest $n - m$ nodes with zero initial self-weights. Then, we obtain

$$x(1) = C^\top \left(x(0) - x^2(0)\right) + x^2(0) = C^\top \begin{bmatrix} x_1(0) - x_1(0)^2 \\ \vdots \\ x_m(0) - x_m(0)^2 \\ 0 \\ \vdots \end{bmatrix} + \begin{bmatrix} x_1(0)^2 \\ \vdots \\ x_m(0)^2 \\ 0 \\ \vdots \end{bmatrix}. \tag{15}$$

Since $x_i(0) < 1$ for $i \in \{1, \ldots, m\}$, $x_i(0) - x_i(0)^2 > 1$. Moreover, since C is irreducible, there exist at least on edge from the last $n - m$ agents to the first m agents, which implies at least one $c_{ij} > 0$ for $i > m$ and $j \leq m$. Consequently, based upon (15), $x_i(1) > 0$ for such $i > m$ and $x_k(1) > 0$ for all $k \leq m$. By iteration, we obtain that $x(t) > 0$ for all $t \geq r$ given any non-vertex $x(0)$, where r is the diameter of the digraph associated to C (i.e., the maximum distance between any two nodes in $G(C)$).

Furthermore, consider $W(x(0)) = \operatorname{diag}(x(0)) + (I_n - \operatorname{diag}(x(0)))C$. Since $I_n - \operatorname{diag}(x(0))$ has all positive diagonal entries for non-vertex $x(0)$, $W(x(0))$ is irreducible. As $\operatorname{diag}(x(0)) \neq 0$, $W(x(0))$ is then aperiodic and primitive. The row stochasticity of $W(x)$ is directly from the row stochasticity assumption on C.

Regarding the second claim (A2), by the definition of $W(t)$ in (2) and the constant non-negative C, the minimum positive entries of $W(t)$ are lower bounded uniformly if there exists a finite time $\tau \geq 0$ such that all entries of $x(t)$ are lower bounded uniformly for all $t \geq \tau$.

First, we have proved above that $x(t) > 0$ for all time $t \geq r$ with r as the diameter of the digraph associated to C.

Second, we will show that all entries of $x(t)$ are uniformly lower bounded away from 0 for all $t \geq \tau$ with some $\tau \geq 0$. Let $\beta := \max_{1 \leq i,j \leq n} c_{ij}$ and $x_i(t) = 1 - \alpha$. It is clear that $0 < \alpha < 1$ and $\frac{1}{n-1} \leq \beta \leq 1$. Two cases (B1) $\beta < 1$ and (B2) $\beta = 1$ are considered in the following.

If (B1) $\beta < 1$, as

$$x_i(t+1) = x_i(t)^2 + \sum_{j=1, j\neq i}^{n} c_{ji} \left(x_j(t) - x_j(t)^2\right),$$

we have

$$\begin{aligned} x_i(t+1) &\leq x_i(t)^2 + (n-1)\beta \left(\frac{\alpha}{n-1} - \frac{\alpha^2}{(n-1)^2}\right) \\ &= (1-\alpha)^2 + \beta\alpha - \beta\frac{\alpha^2}{(n-1)}, \end{aligned} \tag{16}$$

where the inequality holds as $\beta \geq c_{ij}$ for all $1 \leq i, j \leq n$ and the scalar function $y - y^2$ is concave on $(0, 1)$. From (16), if $\alpha < \frac{1-\beta}{1-\beta/(n-1)}$ or equivalently $\beta < \frac{\alpha-\alpha^2}{\alpha-\alpha^2/(n-1)}$, by simple calculation, we have $x_i(t+1) < (1-\alpha)^2 + \alpha - \alpha^2 = 1 - \alpha = x_i(t)$. That is to say, if $x_i(t) > 1 - \frac{1-\beta}{1-\beta/(n-1)}$, then $x_i(t+1) < x_i(t)$. Moreover, $x_i(t) - x_i(t+1) \geq (\alpha - \alpha^2) - \beta\alpha + \beta\frac{\alpha^2}{(n-1)}$: when $\alpha < \frac{1-\beta}{1-\beta/(n-1)}$, the right hand of this inequality has the minimum positive value at the largest $x_i(t)$ (corresponding the smallest α) or at the point $x_i(t) = 1 - \frac{1-\beta}{1-\beta/(n-1)}$;

in both cases $x_i(t) - x_i(t+1)$ is strictly greater than 0. That implies the uniform decrease of $x_i(t)$ along t for $\alpha < \frac{1-\beta}{1-\beta/(n-1)}$.

Furthermore, if $x_i(t) = 1-\alpha < 1 - \frac{1-\beta}{1-\beta/(n-1)}$ with $\alpha > \frac{1-\beta}{1-\beta/(n-1)}$, from (16),

$$\begin{aligned} x_i(t+1) &\leq (1-\alpha)^2 + \beta\alpha - \beta\frac{\alpha^2}{(n-1)} \\ &= 1 + (\beta-2)\alpha + \Big(1 - \frac{\beta}{n-1}\Big)\alpha^2 \\ &= 1 + (\beta-2)\left(\frac{1-\beta}{1-\beta/(n-1)} + b\right) + \Big(1 - \frac{\beta}{n-1}\Big)\left(\frac{1-\beta}{1-\beta/(n-1)} + b\right)^2 \\ &= 1 - \frac{1-\beta}{1-\beta/(n-1)} - \beta b + \Big(1 - \frac{\beta}{n-1}\Big)b^2, \end{aligned} \tag{17}$$

where $b = \alpha - \frac{1-\beta}{1-\beta/(n-1)}$. It is clear that $0 < b < \frac{\beta}{1-\beta/(n-1)}$. Consequently, the part of the right hand side of (17) satisfies

$$-\beta b + \Big(1 - \frac{\beta}{n-1}\Big)b^2 = b\left(-\beta + \Big(1 - \frac{\beta}{n-1}\Big)b\right) < 0. \tag{18}$$

Hence, $x_i(t+1) < 1 - \frac{1-\beta}{1-\beta/(n-1)}$ from (17) and (18). Overall, if one entry of $x(t)$ is greater than $1 - \frac{1-\beta}{1-\beta/(n-1)}$, then via the single-timescale DF model (5), the value of the underlying entry is uniformly decreasing until it is less than $1 - \frac{1-\beta}{1-\beta/(n-1)}$. If one entry of $x(t)$ is less than $1 - \frac{1-\beta}{1-\beta/(n-1)}$, then it is less than $1 - \frac{1-\beta}{1-\beta/(n-1)}$ for all following iterations $(t+k)$, $k \in \mathbb{N}$. In other words, there exists a finite time τ such that all entries of $x(t)$ for all $t \geq \tau$ are bounded away from 1 uniformly. Consequently, from the equation (5), the facts $x(t) > 0$ and C irreducible, we have all entries of $x(t)$ are also bounded away from 0.

If (B2) $\beta = 1$, without loss of generality, assume $c_{i1} = \beta = 1$ for some $2 \leq i \leq n$. This implies that the i-th individual only accords relative interpersonal weight to the first individual in the group. As C is row-stochastic, $c_{ij} = 0$ for all $2 \leq j \leq n$. Moreover, as C is not with star topology, at least one individual j has $c_{j1} < 1$.In the following, we will show that, for a sufficiently large α satisfying $0 < \alpha < 1$, if $x_1(t) > \alpha$ for $t \geq n-1$, then $x_1(t+1) \leq x_1(t)$.

Here we first consider two exclusive and complete scenarios for the case (B2):

(C1) C satisfies $c_{n1} < 1$ and $c_{i1} = 1$ for all rest individuals $i \neq 1$; and

(C2) C satisfies $c_{j1} < 1$ for $n \geq j > m$ and $c_{i1} = 1$ for $m \geq i > 1$ where $n-1 > m > 1$.

Note that 1) in scenario (C1), $c_{in} = 0$ for all $i \neq 1$ and $0 < c_{1n} < 1$; 2) we can always re-arrange the indices of individuals such that scenario (C2) occurs for more that one individuals only accord interpersonal weights to the first individual.

Regarding the scenario (C1), by (5) and by the fact that C is not with star topology,

$$\begin{aligned} x_1(t+1) &= \sum_{i=2}^{n} c_{i1}(x_i(t) - x_i(t)^2) + x_1(t)^2 \\ &= \sum_{i=2}^{n-1}(x_i(t) - x_i(t)^2) + c_{n1}(x_n(t) - x_n(t)^2) + x_1(t)^2. \end{aligned} \tag{19}$$

Here we also assume $c_{n1} < 1$ without loss of generality.

We have proved that $x(t) > 0$ for $t \geq r$, where r is the diameter of the digraph associated to C and $r \leq n-1$. Following the equation (19), to prove $x_1(t+1) \leq x_1(t)$ for $x_1(t)$ sufficiently close to 1, it is sufficient to show that

$$\sum_{i=2}^{n-1}(x_i(t) - x_i(t)^2) + c_{n1}(x_n(t) - x_n(t)^2) + x_1(t)^2 \leq x_1(t). \tag{20}$$

From the dynamical system (5), we have

$$x(t+1) - x(t) = (C^\top - I)(x(t) - x(t)^2) \quad \text{for all } t \geq 0,$$

or equivalently for $x(t) > 0$ and $x(t) \neq x^*$,

$$x(t+1)\,\mathrm{diag}(x(t))^{-1} = (C^\top - I)(\mathbb{1}_n - x(t)) + \mathbb{1}_n = C^\top(\mathbb{1}_n - x(t)) + x(t).$$

Therefore, for $x_n(t) > 0$ and $x_n(t) \neq x_n^*$,

$$\frac{x_n(t+1)}{x_n(t)} = \sum_{i=1}^{n-1} c_{in}(1 - x_i(t)) + x_n(t) = c_{1n}(1 - x_1(t)) + x_n(t) < 1. \tag{21}$$

That is to say, $x_n(t+1) < x_n(t)$. Moreover, as $x_n(t+1) > c_{1n}(x_1(t) - x_1(t)^2)$, the following statement also holds:

$$x_n(t) > c_{1n}(x_1(t) - x_1(t)^2) \geq \gamma(1 - x_1(t)) = \gamma \sum_{i=2}^{n} x_i(t), \tag{22}$$

with $0 < \gamma \leq c_{1n}x_1(t) < 1$.

Moreover, based upon (22) and for a sufficient large $x_1(t) < 1$, we have the following statements related:

$$\begin{aligned}
&\sum_{i=2}^{n-1}(x_i(t) - x_i(t)^2) + c_{n1}(x_n(t) - x_n(t)^2) + x_1(t)^2 < x_1(t) \\
\iff &\sum_{i=2}^{n-1}(x_i(t) - x_i(t)^2) + c_{n1}(x_n(t) - x_n(t)^2) < \sum_{i=2}^{n} x_i(t) - (\sum_{i=2}^{n} x_i(t))^2 \\
\iff &(1 - c_{n1})(x_n(t) - x_n(t)^2) > (\sum_{i=2}^{n} x_i(t))^2 - \sum_{i=2}^{n} x_i(t)^2 \\
\impliedby\ &(1 - c_{n1})(\gamma \sum_{i=2}^{n} x_i(t) - \gamma^2(\sum_{i=2}^{n} x_i(t))^2) \geq (\sum_{i=2}^{n} x_i(t))^2 - \sum_{i=2}^{n} x_i(t)^2 \\
\iff &(1 - c_{n1})\gamma \sum_{i=2}^{n} x_i(t) \geq ((1 - c_{n1})\gamma^2 + 1)(\sum_{i=2}^{n} x_i(t))^2 - \sum_{i=2}^{n} x_i(t)^2 \\
\impliedby\ &(1 - c_{n1})\gamma \sum_{i=2}^{n} x_i(t) \geq ((1 - c_{n1})\gamma^2 + \frac{n-1}{n})(\sum_{i=2}^{n} x_i(t))^2 \\
\iff &\frac{(1 - c_{n1})\gamma}{(1 - c_{n1})\gamma^2 + \frac{n-1}{n}} \geq \sum_{i=2}^{n} x_i(t) = 1 - x_1(t).
\end{aligned} \tag{23}$$

The last statement holds for $x_1(t) \geq 1 - \frac{(1-c_{n1})\gamma}{(1-c_{n1})\gamma^2 + \frac{n-1}{n}}$, where $\gamma < (n-1)/n$ guarantees $0 < \frac{(1-c_{n1})\gamma}{(1-c_{n1})\gamma^2 + \frac{n-1}{n}} < 1$. Therefore, the inequality (20) holds. That is, $x_1(t+1) \leq x_1(t)$ for $x_1(t) \geq 1 - \frac{(1-c_{n1})\gamma}{(1-c_{n1})\gamma^2 + \frac{n-1}{n}}$. In addition, for the system (5), we have for all $x_1(t) \leq 1 - \frac{(1-c_{n1})\gamma}{(1-c_{n1})\gamma^2 + \frac{n-1}{n}} := \beta$,

$$x_1(t+1) = \sum_{i=2}^{n} c_{i1}(x_i(t) - x_i(t)^2) + x_1(t)^2 < \beta^2 + (1 - \beta)(1 - \frac{1 - \beta}{n - 1}) < 1. \tag{24}$$

This implies that there exists a finite time τ such that $x_1(t)$ for all $t \geq \tau$ are bounded away from 1 and bounded way from 0.

Regarding the scenario (C2), we may regard the set of individuals $(m+1, m+2, \cdots, n)$ as a single "node", as they are only directly connected to the first individual but not the rest set of individuals $(2, 3, \cdots, m)$. Similar arguments as for the scenario (C1) hold here to prove $x_1(t+1) \leq x_1(t)$. First, we have the similar statement to (21). For any $x_j(t) > 0, m \leq j \leq n$, and $x_j(t) \neq x_n^*$,

$$\begin{aligned}\frac{x_j(t+1)}{x_j(t)} &= \sum_{i=1}^{n-1} c_{ij}(1 - x_i(t)) + x_n(t) \\ &= \sum_{i \neq j, i=m+1}^{n} c_{ij}(1 - x_i(t)) + c_{1n}(1 - x_1(t)) + x_n(t) < n - m - 1 + 1 = n - m.\end{aligned}$$

That is to say, $x_j(t+1) < x_j(t)(n-m)$ for all $m+1 \leq j \leq n$. Moreover, as the digraph associated with C is irreducible, there exists at least one $m+1 \leq j \leq n$ such that $x_j(t+1) > c_{1j}(x_1(t) - x_1(t)^2)$, this implies $x_j(t) \geq \gamma_j(1 - x_1(t))$ for some $\gamma_j < 1$ and independent of time t. Consequently, there exists at least one different individual $m+1 \leq i \leq n, i \neq j$ such that $x_i(t+1) > c_{ji}(x_j(t) - x_j(t)^2) > c_{ji}x_j(t) > \gamma_i(1 - x_1(t))$. Similarly, we have all individuals $m+1 \leq j \leq n$ with $c_{j1} < 1$ satisfying $x_j(t) \geq \gamma_j(1 - x_1(t))$ for some $\gamma_j < 1$. Second, we have the similar statement to (23):

$$\begin{aligned}
& \sum_{i=2}^{m}(x_i(t) - x_i(t)^2) + \sum_{j=m+1}^{n} c_{j1}(x_j(t) - x_j(t)^2) + x_1(t)^2 \leq x_1(t) \\
\iff & \sum_{i=2}^{m}(x_i(t) - x_i(t)^2) + \sum_{j=m+1}^{n} c_{j1}(x_j(t) - x_j(t)^2) \leq \sum_{i=2}^{n} x_i(t) - (\sum_{i=2}^{n} x_i(t))^2 \\
\iff & \sum_{j=m+1}^{n} (1 - c_{j1})(x_j(t) - x_j(t)^2) \geq (\sum_{i=2}^{n} x_i(t))^2 - \sum_{i=2}^{n} x_i(t)^2 \\
\Longleftarrow & \ (n-m)(1 - c_{j1})(\gamma_j \sum_{i=2}^{n} x_i(t) - \gamma_j^2 (\sum_{i=2}^{n} x_i(t))^2) \geq (\sum_{i=2}^{n} x_i(t))^2 - \sum_{i=2}^{n} x_i(t)^2 \\
& \quad (\text{where } j = \operatorname{argmin}_{m < i \leq n}(1 - c_{i1})(x_i(t) - x_i(t)^2)) \\
\Longleftarrow & \ (1 - c_{j1})\gamma_j \sum_{i=2}^{n} x_i(t) \geq ((1 - c_{j1})\gamma_j^2 + 1)(\sum_{i=2}^{n} x_i(t))^2 - \sum_{i=2}^{n} x_i(t)^2 \\
\Longleftarrow & \ (1 - c_{j1})\gamma_j \sum_{i=2}^{n} x_i(t) \geq ((1 - c_{j1})\gamma_j^2 + \frac{n-1}{n})(\sum_{i=2}^{n} x_i(t))^2 \\
\iff & \frac{(1 - c_{j1})\gamma_j}{(1 - c_{j1})\gamma_j^2 + \frac{n-1}{n}} \geq \sum_{i=2}^{n} x_i(t) = 1 - x_1(t).
\end{aligned} \tag{25}$$

Hence, for $x_1(t) \geq 1 - \frac{(1 - c_{j1})\gamma_j}{(1 - c_{j1})\gamma_j^2 + \frac{n-1}{n}}$, from (25), we have

$$x_1(t+1) = \sum_{i=2}^{n} c_{i1}(x_i(t) - x_i(t)^2) + x_1(t)^2 \leq x_1(t).$$

As (23) always holds, we can prove that there exists a finite time τ_1 such that $x_1(t)$ for all $t \geq \tau_1$ are bounded away from 1 and from 0.

Overall, given any C irreducible and row-stochastic, each individual i in the network must satisfy one among the three cases (B1) (although we assume all non-zero $c_{ij} < 1$ in (B1), we only require $c_{ji} < 1$ for all $j \neq i$ given i in the proof), (C1) and (C2). That is, there always exists a finite time τ_1 such that $x_1(t)$ for all $t \geq \tau_1$ are bounded away from 1 and from 0, given non-vertex $x(0)$. Consequently, from the equation (5) and the facts C irreducible, there always exists a finite time τ_i such that all entries of $x_i(t)$ for all $t \geq \tau_i$ are bounded away from 1 and from 0 uniformly. Hence, there exists a finite time τ such that all entries of $x(t)$ for all $t \geq \tau$ are bounded away from 1 and from 0. As a result, the claim (A2) holds, which completes the proof of fact (ii). ■

Appendix C
Proof of Theorem 4.1

Proof: By definition,

$$x_i(t+1) = x_i(t)^2 + \sum_{j=1, j\neq i}^{n} c_{ji}\left(x_j(t) - x_j(t)^2\right).$$

As $x(0)$ is in a simplex, if $x_i(0) = 1$ then $x_j(0) = 0$ for all $j \neq i$. It is clear that $x_i^* = x_i(1) = 1$ and therefore, $x^* = \mathbb{e}_i$ given $x(0) = \mathbb{e}_i$ for all $i \in \{1, \ldots, n\}$. That is to say, $\{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$ are always the fixed points of the dynamical system (5).

Regarding fact (i), without loss of generality, we assume that node 1 and node 2 are globally reachable. The corresponding C has the following block matrix form

$$C = \begin{bmatrix} C_1 & 0 \\ C_{21} & C_{22} \end{bmatrix}, \tag{26}$$

where $C_1 \in \mathbb{R}^{2\times 2}$ is row stochastic, and $C_{22} \in \mathbb{R}^{n-2\times n-2}$ is substochastic as $C_{21} \geq 0$ and $C_{21} \neq 0$. Given $x(t) \in \Delta_n \setminus \{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$, the weight matrix $W(t)$ has the block matrix form via (2) as follows.

$$W(t) = \begin{bmatrix} W_1(t) & 0 \\ W_{21}(t) & W_{22}(t) \end{bmatrix}. \tag{27}$$

Here

$$W_1(t) := W_1(x_{(1,2)}(t)) = \operatorname{diag} x_{(1,2)}(t) + (I_2 - \operatorname{diag}(x_{(1,2)}(t)))C_1,$$

$$W_{21}(t) := W_{21}(x_{(3,\cdots,n)}(t)) = (I_{n-2} - \operatorname{diag}(x_{(3,\cdots,n)}(t)))C_{21},$$

$$W_{22}(t) := W_{22}(x_{(3,\cdots,n)}(t)) = \operatorname{diag} x_{(3,\cdots,n)}(t) + (I_{n-2} - \operatorname{diag}(x_{(3,\cdots,n)}(t)))C_{22},$$

given $x_{(1,2)} := \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$ and $x_{(3,\cdots,n)} := \begin{bmatrix} x_3 & \cdots & x_n \end{bmatrix}^\top$.

The single-timescale DF dynamics associated with C in (26) is as follows.

$$x_{(1,2)}(t+1) = W_1(t)^\top x_{(1,2)}(t) + W_{21}(t)^\top x_{(3,\cdots,n)}(t),$$

$$x_{(3,\cdots,n)}(t+1) = W_{22}(t)^\top x_{(3,\cdots,n)}(t), \qquad t = 0, 1, 2, \ldots.$$

As C_{22} is substochastic and $x(0) \in \Delta_n \setminus \{\mathbb{e}_1, \ldots, \mathbb{e}_n\}$, $W_{22}(0)$ is substochastic. That is, $\sum_{i=3}^n x_i(1) \leq \sum_{i=3}^n x_i(0)$ and $\max_{3\leq i\leq n} x_i(1) \leq \max_{3\leq i\leq n} x_i(0)$ for $\max_{3\leq i\leq n} x_i(0) \neq 0$. These statements hold for all $t \geq 0$ iteratively. In particular, for $x_{(3,\cdots,n)}(0) = 0$, $x_{(3,\cdots,n)}(t) = 0$ for all $t \geq 0$. Moreover, as C is reducible and has globally reachable nodes, given any initial conditions of $x_{(3,\cdots,n)}(0)$, the zero and non-zero pattern of $x_{(3,\cdots,n)}(t)$ shall keep constant for all $t \geq n-2$. That is, for $3 \leq i \leq n$ and $t \geq n-2$, if $x_i(t) > 0$ then $x_i(t+k) > 0$ for all finite $k \geq 0$, and if $x_i(t) = 0$ then $x_i(t+k) = 0$ for all $k \geq 0$.

Next, we will show that, given $x_{(3,\cdots,n)}(0) \neq 0$, $\lim_{t\to\infty} x_{(3,\cdots,n)}(t) = 0$ exponentially. By appropriately re-indexing all individual $3 \leq i \leq n$, we have C_{22} have the following normal form:

$$C_{22} = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & A_{m2} & \cdots & A_{mm} \end{bmatrix}.$$

If C_{22} is irreducible then $m = 1$; otherwise, each block matrix A_{ii} is irreducible for $i = \{1, \ldots, m\}$. Moreover, as C_{22} is substochastic and C is stochastic and have globally reachable nodes, each A_{ii} is substochastic with at least one row sum strictly less than 1. Consequently, from (27), we have

$$W_{22}(t) = \begin{bmatrix} B_{11}(t) & 0 & 0 & \cdots & 0 \\ B_{21}(t) & B_{22}(t) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ B_{m1}(t) & B_{m2}(t) & B_{m2}(t) & \cdots & B_{mm}(t) \end{bmatrix}.$$

where $B_{ii}(t) = \operatorname{diag}(x_{s_i}(t)) + \big(I_{|s_i|} - \operatorname{diag}(x_{s_i}(t))\big) A_{ii}$, given s_i is the set of individuals corresponding to the rows evolving in the block matrix A_{ii} and $|s_i|$ denotes the cardinality of the set s_i. It is clear that $B_{ii}(t)$ is irreducible, substochastic, and has at least one row sum strictly less than 1, for all $t \geq 0$. Moreover, as the maximum of the elements of $x_{s_i}(t)$ is less than or equal to the maximum of the elements of $x_{(3,\cdots,n)}(0)$, the elements of $B_{ii}(t)$ are upper bounded uniformly for all $t \geq 0$. Meantime, all $B_{ii}(t)$ for $t \geq n-2$ shall have the same zero and non-zero pattern on elements. As a result of all these facts and from [3, Corollary 4.11], $\prod_{k=0}^{t} B_{ii}(t)$ converges to $\mathbb{0}$ exponentially for each block matrix and hence, $\prod_{k=0}^{t} W_{22}(t)$ converges to $\mathbb{0}$ exponentially. From (5), $x_{(3,\cdots,n)}(t)$ converges to $\mathbb{0}_{n-2}$ exponentially.

As $x_{(1,2)}(t+1) = W_1(t)^\top x_{(1,2)}(t) + W_{21}(t)^\top x_{(3,\cdots,n)}(t)$ and $C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have

$$W_1(t) = \operatorname{diag} x_{(1,2)}(t) + (I_2 - \operatorname{diag}(x_{(1,2)}(t)))C_1 = \begin{bmatrix} x_1(t) & 1 - x_1(t) \\ 1 - x_2(t) & x_2(t) \end{bmatrix}.$$

Once $x_{(3,\cdots,n)}(t)$ converges to $\mathbb{0}_{n-2}$ exponentially, $x_{(1,2)}(t)$ simultaneously converges to an equilibrium $x^*_{(1,2)}$ satisfying

$$x^*_{(1,2)} := \lim_{t\to\infty} x_{(1,2)}(t) = \lim_{t\to\infty} W_1(t)^\top x_{(1,2)}(t).$$

That is

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} x_1^* & 1 - x_2^* \\ 1 - x_1^* & x_2^* \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} (x_1^*)^2 + x_2^* - (x_2^*)^2 \\ x_1^* - (x_1^*)^2 + (x_2^*)^2 \end{bmatrix} \iff x_2^* - (x_2^*)^2 = x_1^* - (x_1^*)^2.$$

As $\lim_{t\to\infty}(x_1(t) + x_2(t)) = 1$, $x_2^* - (x_2^*)^2 = x_1^* - (x_1^*)^2$ holds for any pair (x_1^*, x_2^*) satisfying $x_1^* + x_2^* = 1$.

Regarding fact (ii), the similar arguments in (i) can prove all $\{x_i(t)\}$ corresponding to reducible nodes converge to 0 exponentially. Consequently, if $\sum_{i=r+1}^{n} x_{(i)}(t) = \beta(t)$ sufficiently small, the following statement similar to (13) holds

$$x_1(t+1) - x_1(t) = \sum_{j=2}^{r} \left(x_j(t) - x_j(t)^2\right) - (x_1(t) - x_1(t)^2) > 0. \tag{28}$$

for all $x_1(t) \leq 1 - \sqrt{\frac{\beta(t)}{2}}$. It is true as

$$\begin{aligned}
&x_1(t) \leq 1 - \sqrt{\frac{\beta(t)}{2}} \iff \beta(t) \leq \frac{(1-x_1(t))^2}{2} \\
&\Longrightarrow \beta(t) < \frac{(1-x_1(t))^2}{2-x_1(t)} \iff \beta(t) < (1-x_1(t))^2 - \beta(t)(1-x_1(t)) \\
&\Longrightarrow \beta(t) < (1-x_1(t))^2 - \beta(t)(1-x_1(t)) + \sum_{j=2}^{r} x_j(t)^2 \iff \beta(t) < \sum_{j=2}^{r} x_j(t)(1-x_j(t)-x_1(t)) \\
&\iff \sum_{j=2}^{r} x_j(t)(1-x_j(t)) > \sum_{j=2}^{r} x_j(t)x_1(t) + \beta(t) = (\sum_{j=2}^{r} x_j(t) + \beta(t))(1 - \sum_{j=2}^{r} x_j(t)) > (1-x_1(t))x_1(t),
\end{aligned}$$

which implies (28). The asymptotic convergence of $x(t)$ to $\mathbb{e}_1$ is then established with the similar arguments in the proof of Theorem 3.1 (ii).

Regarding fact (iii), the existence and uniqueness of non-vertex equilibrium x^* is established in the same way as in Theorem 3.2 (i). x^* satisfies (14) as well. The convergence property is similar to that of Theorem 3.2 (ii). Specifically, $x_i(t) > 0$ for all $t \geq n$ and $1 \in \{1, \dots, r\}$. If we write $W(t)$ in the normal form as in (27), the statements (A1) and (A2) in the proof of Theorem 3.2 (ii) holds for $W_1(t)$ by the same arguments. That implies that $\prod_{k=0}^{t} W_1(t)$ converge exponentially to a rank–1 matrix with positive identical rows, which is equal to $\mathbb{1}_r (x^*_{(1,\cdots,r)})^\top$ and $x^*_{(1,\cdots,r)}$ is determined by (14). Denote $\prod_{k=0}^{t} W(t) = \begin{bmatrix} P_1(t) & 0 \\ P_{21}(t) & P_{22}(t) \end{bmatrix}$. It is clear that $P_1(t) = \prod_{k=0}^{t} W_1(t)$, $P_{22}(t) = \prod_{k=0}^{t} W_{22}(t)$ and $P_{21}(t) = P_{21}(t-1)W_1(t) + P_{22}(t-1)W_{22}(t)$. As $P_{22}(t)$ converges exponentially to $\mathbb{0}$, $P_{21}(t)$ then exponentially converges to $\mathbb{1}_{n-r}(x^*_{(1,\cdots,r)})^\top$ following the previous statement that $\prod_{k=0}^{t} W_1(t)$ exponentially converges to $\mathbb{1}_r (x^*_{(1,\cdots,r)})^\top$. Overall, $\prod_{k=0}^{t} W(t)$ converge exponentially to a rank–1 matrix with identical rows such that $x_{(1,\cdots,r)}$ converges exponentially to $x^*_{(1,\cdots,r)} > 0$, and $x_{(r+1,\cdots,n)}$ converges exponentially to $\mathbb{0}_{n-r}$, for any non-vertex $x(0)$. ■

Appendix D
Proof of Theorem 4.2

Proof: Regarding the first part of fact (i), the result is directly from the definition of the single-timescale DF model and has been proved in Theorem 3.1 and Theorem 4.1: $x(0) = \mathbb{e}_i$ implies $x(t) = x^* = \mathbb{e}_i$ for all $t \geq 0$ and $i \in \{1, \dots, n\}$.

Regarding fact (i.1), as we discussed in the proof of Theorem 4.1, $\prod_{\tau=0}^{t} W_{MM}(\tau)$ converges exponentially to $\mathbb{0}_{m\times m}$ as t goes to infinity, given $x_i(0) < 1$ for all reducible node i. That implies that $x_{MM}(t)$ converges exponentially to $\mathbb{0}_m$ as t goes to infinity.

Regarding fact (i.2), on an equilibrium x^* in a sink k with only two nodes, it shall satisfy from (5) that

$$x^*_{kk_1}(1-x^*_{kk_1}) = x^*_{kk_2}(1-x^*_{kk_2}). \tag{29}$$

If $\zeta^*_k < 1$, then the only solution to (29) is $x^*_{kk_1} = x^*_{kk_2} = \zeta^*_k/2$. If $\zeta^*_k = 1$, then any pair $(\alpha, 1-\alpha)^\top$ satisfies (29) and hence, $x^*_{kk} = (\alpha, 1-\alpha)^\top$ where $\alpha \in [0,1]$ depends upon the initial conditions and the topology of the network.

Regarding fact (i.3) and fact (i.4), the proof is similar to the analysis of Theorem 3.2 (i). For C_{kk} irreducible and $\max\{c_{kk_i}\} < 0.5$ we will show that there exists a unique $x^*_{kk} \in \operatorname{interior}\Delta_n$ satisfying $x^*_{kk} = W_{kk}(x^*_{kk})^\top x^*_{kk} + W_{Mk}(x^*_{MM})^\top x^*_{MM} = 0$. As $x^*_{MM} = 0$ from fact (i.1) above, the fix points shall satisfy $x^*_{kk} - {x^*_{kk}}^2 = C^\top_{kk}(x^*_{kk} - x^*_{kk})$. As x^*_{kk} shall be real valued and non-negative, given $\zeta^*_k > 0$, $(x^*_{kk} - {x^*_{kk}}^2)$ is a scalar multiple of the unique positive left eigenvector of C_{kk} associated with eigenvalue 1. As $\max\{c_{kk_i}\} < 0.5$ and $n_k \geq 3$,

$$x^*_{kk} - {x^*_{kk}}^2 = \alpha^*_{kk} c_{kk}, \text{ or equivalently, } x^*_{kk_i} = \alpha^*_{kk}\frac{c_{kk_i}}{1-x^*_{kk_i}}, \text{ for all } i \in \{1,\dots,n_k\}, \tag{30}$$

where the scalar α^*_{kk} is such that $\mathbb{1}^\top_{n_k} x^*_{kk} = \zeta^*_k$, that is to say,

$$\alpha^*_{kk} = \frac{\zeta^*_k}{\sum_{j=1}^n c_{kk_j}/(1-x^*_{kk_j})}.$$

One may check that this x^*_{kk} have the same form as the non-autocratic fixed point we obtained from the DF model [18]. Therefore, the uniqueness of x^*_{kk} is directly from Theorem 3.6 in [18]. Moreover, the ordering of the elements of x^*_{kk} is consistent with that of c_{kk} following (30).

Regarding fact (ii), as $x_{kk}(t+1) = W_{kk}(t)^\top x_{kk}(t) + W_{Mk}(t)^\top x_{MM}(t)$ with $W_{kk}(t)$ row stochastic and $W_{Mk}(t)^\top x_{MM}(t) \geq 0$, it is clear that $\mathbb{1}^\top_{n_k} x_{kk}(t+1) \geq \mathbb{1}^\top_{n_k} W_{kk}(t)^\top x_{kk}(t)$. That is $\zeta_k(t+1) \geq \zeta_k(t)$. For the second statement in fact (ii), subject to Assumption 1) $\zeta_k(0) = 0$ and Assumption 2) $x_i(0) = 0$ for any reducible node i such that there exists a directed path from i to the sink k in the network, we have $x_{kk}(1) = W_{kk}(0)^\top x_{kk}(0) + W_{Mk}(0)^\top x_{MM}(0) = 0$ as $W_{Mk}(0)^\top x_{MM}(0) > 0$ contradicts thef second assumption above. Iteratively, we have $x_{kk}(t+1) = W_{kk}(t)^\top x_{kk}(t) + W_{Mk}(t)^\top x_{MM}(t) = 0$ for all $t \geq 0$, where the second term shall be equal to $\mathbb{0}_m$ for all the time as, otherwise it contradicts the second assumption.

Regarding fact (iii), we will consider the convergence behaviors of self-weights in three different scenarios as described in facts (i.1)– (i.3).

Scenario 1: The exponential convergence of the self-weights on reducible nodes has been clarified in fact (i.1).

Scenario 2: The convergence of the self-weights on a sink with only two nodes is similar to that described in Theorem 4.1 fact (i) or fact (iii). The difference is that all self-weights are accumulated on the two irreducible nodes in Theorem 4.1 fact (i) but here ζ^*_k may be less than 1 depending upon the initial condition and the topology of the network. If $\zeta^*_k = 1$, then the convergence process here is exactly the same as Theorem 4.1 fact (i). If $\zeta^*_k < 1$, then the self-weights in the two-node sink here exponentially converge to a unique x^*_{kk}. The analysis is similarly to that in Theorem 4.1 fact (iii). As these two nodes have the same eigenvector centrality score, the unique equilibrium is $(\zeta^*_k/2, \zeta^*_k/2)^\top$ here.

Scenario 3: The convergence of the self-weights on a sink with three or more nodes is almost the same as that described in Theorem 4.1 fact (iii). The only difference is that all self-weights are accumulated on the irreducible nodes as in Theorem 4.1 fact (iii) but here ζ_k^* may be less than 1 depending upon the initial condition and the topology of the network. If $\zeta_k^* = 1$, then the analysis is the same to that of Theorem 4.1 fact (iii) or that of Theorem 3.2 fact (ii). If $\zeta_k^* < 1$, then we have $\zeta_k(t)$ are upper bounded away from 1 for all time t. As $\zeta_k(t)$ is non-decreasing, if $\zeta_k(t) > 0$ for $t = m$ (i.e., the max time for the social power migrating from a reducible node to the sink) then $\zeta_k(t)$ is uniformly bounded away from 0 for all $t \geq m$, otherwise, if $\zeta_k(m) = 0$ then $\zeta_k^* = 0$. Given $\zeta_k(t)$ bounded away from 1 and 0 uniformly, first we have $x_{kk}(t) > 0$ and each x_{kk_i} is bounded way from 1. Second, there exists a time τ such that any node in this sink has its self-weight $x_{kk_i}(t)$ lower bounded away from 0 for all $t \geq \tau$. If it is not true, then by the irreducible property of W_{kk} and the system definition (5), all its connected nodes (i.e., all nodes in the sink) shall be sufficiently close to 0 or 1 for infinite time instances (see the similar argument (22) in the proof of Theorem 3.2), that implies that $\zeta_k(t)$ is sufficiently close to 0 or 1 for infinite time instances, which is a contradiction. Third, the sum $\zeta_k(t)$ of the self-weights in this sink converges once all self-weights on reducible nodes exponentially converge to 0, and the self-weight dynamics in the sink are independent from the dynamics occurred in other sinks. Finally, we can conclude that the exponential convergence of the product of $W_{kk}(t)$ based upon all results above. Consequently, $x_{kk}(t)$ converges exponentially as we have shown similarly in Theorem 4.1 fact (iii). ∎