

UC Irvine

UC Irvine Previously Published Works

Title

Newton's law for a trajectory of concentration of solutions to nonlinear Schrodinger equation

Permalink

<https://escholarship.org/uc/item/2321j58f>

Journal

Communications on Pure & Applied Analysis, 13(5)

ISSN

1534-0392

Authors

Babin, Anatoli
Figotin, Alexander

Publication Date

2014

DOI

10.3934/cpaa.2014.13.1685

Peer reviewed

NEWTON'S LAW FOR A TRAJECTORY OF CONCENTRATION OF SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATION

ANATOLI BABIN AND ALEXANDER FIGOTIN

Dedicated to the memory of M.I. Vishik

ABSTRACT. One of important problems in mathematical physics concerns derivation of point dynamics from field equations. The most common approach to this problem is based on WKB method. Here we describe a different method based on the concept of trajectory of concentration. When we applied this method to nonlinear Klein-Gordon equation, we derived relativistic Newton's law and Einstein's formula for inertial mass. Here we apply the same approach to nonlinear Schrödinger equation and derive non-relativistic Newton's law for the trajectory of concentration.

1. INTRODUCTION

A. Einstein remarks in his letter to Ernst Cassirer of March 16, 1937, [50, pp. 393-394]: "One must always bear in mind that up to now we know absolutely nothing about the laws of motion of material points from the standpoint of "classical field theory." For the mastery of this problem, however, no special physical hypothesis is needed, but "only" the solution of certain mathematical problems". In this paper we treat this circle of problems. Namely, we derive here point dynamics governed by Newton's law from wave dynamics using concepts and methods of the classical field theory. That continues our study of a neoclassical model of electric charges which is designed to provide an accurate description of charge interaction with the electromagnetic field from macroscopic down to atomic scales, [7]-[11].

In this paper we derive Newtonian dynamics for localized solutions of Nonlinear Schrödinger equations (NLS) in the asymptotic limit when the solutions converge to delta-functions. The derivation is based on a method of concentrating solutions which was developed in our previous paper [11] where we derived the relativistic dynamics and Einstein's formula for localized solutions of the Nonlinear Klein-Gordon (KG) equation.

In many problems of physics and mathematics a common way to establish a relation between wave and point dynamics is by means of the WKB method, see for instance [41], [44, Sec. 7.1]. We remind that the WKB method is based on the quasiclassical ansatz for solutions to a hyperbolic partial differential equation and their asymptotic expansion. The leading term of the expansion satisfies the eikonal equation, and wavepackets and their energy propagate along the so-called characteristics of this equations. Consequently, these characteristics represent the

1991 *Mathematics Subject Classification.* 35Q55; 35Q60; 35Q70; 70S05; 78A35.

Key words and phrases. Nonlinear Schrödinger equation, Newton's law, Lorentz force, trajectory of concentration.

Supported by AFOSR grant number FA9550-11-1-0163.

point dynamics and are determined from the corresponding system of ODEs which can be interpreted as a law of motion or a law of propagation. The construction of the characteristics involves only local data. Asymptotic derivation of the Newtonian dynamics which does not rely on the fast oscillation of solutions as the WKB method does was developed for soliton-like solutions of the Nonlinear Schrodinger and Nonlinear Klein-Gordon equations in a number of papers, see [2], [25], [29], [40]. In the mentioned papers the derivation of the limit Newtonian dynamics of the soliton center relies on the asymptotic analysis of an ansatz structure of soliton-like solutions. *The proposed here approach also relates waves governed by certain PDE's in asymptotic regimes to the point dynamics but it differs fundamentally from the mentioned methods.* In particular, our approach is not based on any specific ansatz and it is not not entirely local but rather it is semi-local. Our semi-local method is in the spirit of the Ehrenfest theorem and is based on an analysis of a sequence of concentrating solutions which are not subjected to structural conditions. Compared to the mentioned approaches which use the WKB method or soliton-like structure of solutions, our arguments only use very mild assumptions of a localization of solutions in a small neighborhood of the trajectory combined with the systematic use there of integral identities derived from the energy-momentum and density conservation laws for the NLS. Our method allows to derive Newton's law under minimal restrictions on the time and spatial dependence of the variable coefficients and on the nonlinearity and allows to consider arbitrarily long time intervals.

The nonlinear Schrodinger (NLS) equation we study in this paper is of the form

$$(1.1) \quad i\tilde{\partial}_t\psi = \frac{\chi}{2m} \left[-\tilde{\nabla}^2\psi + G'_a(\psi^*\psi)\psi \right].$$

In the above equation $\psi = \psi(t, \mathbf{x})$ is a complex valued wave function, G'_a is a real valued nonlinearity, and the *covariant differentiation operators* $\tilde{\partial}_t$ and $\tilde{\nabla}$ are defined by

$$(1.2) \quad \tilde{\partial}_t = \partial_t + \frac{iq}{\chi}\varphi, \quad \tilde{\nabla} = \nabla - \frac{iq}{\chi\mathbf{c}}\mathbf{A},$$

where $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ are given twice differentiable functions of time t and spatial variables $\mathbf{x} \in \mathbb{R}^3$ interpreted as potentials of external electric and magnetic fields. The quantity $q|\psi|^2$ is naturally interpreted as the charge density with q being the value of the charge.

In the case when the potentials φ and \mathbf{A} are zero, the charge can be considered as "free" and the NLS equation (1.1) has localized solutions corresponding to resting and uniformly moving charges. Newton's law involves acceleration, and to produce an accelerated motion non-zero potentials are required. One may expect point-like dynamics for strongly localized solutions, and the nonlinearity G' provides existence of localized solutions of the NLS equation. The only condition imposed on the nonlinearity (in addition to natural continuity assumptions, see Condition 1) is the existence of a positive radial solution $\psi_1 = \dot{\psi}_1(|\mathbf{x}|)$ of the steady-state equation for a free charge:

$$(1.3) \quad -\nabla^2\dot{\psi}_1 + G'_1(|\dot{\psi}_1|^2)\dot{\psi}_1 = 0.$$

We assume that $\dot{\psi}(|\mathbf{x}|)$ is twice continuously differentiable for all $\mathbf{x} \in \mathbb{R}^3$ and is square integrable, that is

$$(1.4) \quad \int_{\mathbb{R}^3} |\dot{\psi}|^2 d\mathbf{x} = v_0 < \infty.$$

Importantly, we assume that the nonlinearity depends explicitly on the size parameter $a > 0$ as follows:

$$(1.5) \quad G'_a(s) = a^{-2}G'_1(a^3s), \quad s > 0.$$

Note that the steady-state equation with a general $a > 0$ takes the form

$$(1.6) \quad \nabla^2 \dot{\psi}_a + G'_a(|\dot{\psi}_a|^2) \dot{\psi}_a = 0$$

and has the following solution:

$$(1.7) \quad \dot{\psi}_a(r) = a^{-3/2} \dot{\psi}_1(a^{-1}r), \quad r = |x| \geq 0,$$

where $\dot{\psi}_1$ is the solution to equation (1.3) where $a = 1$. Obviously $\dot{\psi}_a$ satisfies the normalization condition (1.4) with the same v_0 . According to (1.7) a can be interpreted as a natural size parameter which describes $\dot{\psi}_a$.

A typical example of the ground state $\dot{\psi}_a$ is the Gaussian $\dot{\psi}_a = a^{-3/2}e^{-r^2/a^2}$ corresponding to a logarithmic nonlinearity G'_a discussed in Example 2. Evidently, $v_0^{-1}\dot{\psi}_a^2$ converges to Dirac's delta-function $\delta(\mathbf{x})$ as $a \rightarrow 0$. We study the behavior of localized solutions of the NLS equation in the asymptotic regime $a \rightarrow 0$.

The crucial role in our analysis is played by the key concept of concentrating solutions. Very roughly speaking, solutions $\psi = \psi_n$ concentrate at a given trajectory $\hat{\mathbf{r}}(t)$ if their charge densities $q|\psi|^2(\mathbf{x}, t)$ restricted to R_n -neighborhoods Ω_n of $\hat{\mathbf{r}}(t)$ locally converge to $q_\infty(t)\delta(\mathbf{x} - \hat{\mathbf{r}}(t))$ as

$$(1.8) \quad a_n \rightarrow 0, \quad R_n \rightarrow 0, \quad a_n/R_n \rightarrow 0, \quad n \rightarrow \infty.$$

Now we describe the concept of *concentrating* solutions in a little more detail. There are two relevant spatial scales for the NLS: the microscopic size parameter a which determines the size of a free charge and the macroscopic length scale R_{ex} of order 1 at which the potentials φ and \mathbf{A} vary significantly. We introduce the third intermediate spatial scale R_n which can be called the *confinement scale* and we assume that ψ_n asymptotically vanish at the boundary $\partial\Omega_n = \{|\mathbf{x} - \hat{\mathbf{r}}(t)| = R_n\}$ of the neighborhood Ω_n . According to (1.8) $R_n/a \rightarrow \infty$, therefore this assumption is quite natural for solutions with typical spatial scale a which are localized at $\hat{\mathbf{r}}$. At the same time, $R_n/R_{\text{ex}} \rightarrow 0$, therefore general potentials $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ can be replaced successfully in Ω_n by their linearizations at $\mathbf{x} = \hat{\mathbf{r}}(t)$. The exact definition of the concentrating solutions (or concentrating asymptotic solutions) involves two major assumptions:

- (1) volume integrals over the balls Ω_n of the charge density $q|\psi|^2$ and the momentum density \mathbf{P} should be bounded;
- (2) surface integrals over the spheres $\partial\Omega_n$ of certain quadratic expressions which involve ψ and its first order derivatives and originate from the elements of the energy-momentum tensor of the NLS equation tend to zero as $R_n \rightarrow 0$, $a_n/R_n \rightarrow 0$. Details of these assumptions can be found in Definitions 4 and 5.

A concise formulation of our main result, Theorem 6, is as follows. We prove that *if a sequence of asymptotic solutions of the NLS equation (1.1) concentrates at a trajectory $\hat{\mathbf{r}}(t)$ then this trajectory must satisfy the Newton equation*

$$(1.9) \quad m\partial_t^2 \hat{\mathbf{r}} = \mathbf{f}_{\text{Lor}}(t, \hat{\mathbf{r}}),$$

where \mathbf{f}_{Lor} is the Lorentz force which is defined by the classical formula

$$(1.10) \quad \mathbf{f}_{\text{Lor}}(t, \hat{\mathbf{r}}) = q\mathbf{E}(t, \hat{\mathbf{r}}) + \frac{q}{c}\partial_t \hat{\mathbf{r}} \times \mathbf{B}(t, \hat{\mathbf{r}}),$$

with the electric and magnetic fields \mathbf{E} and \mathbf{B} defined in terms of the potentials φ, \mathbf{A} in (1.2) according to the standard formula

$$(1.11) \quad \mathbf{E} = -\nabla\varphi - \frac{1}{c}\partial_t \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

Notice that that according to Newton's equation (1.9) the parameter m entering the NLS equation (1.1) can be interpreted as the mass of the charge.

Now we would like to comment on certain subjects related to our main Theorem 6. First of all, the assumptions imposed on the concentrating solutions are not restrictive. Indeed, the solutions are local and have only to be defined in a tubular neighborhood of the trajectory and no initial or boundary conditions are imposed on them. Second of all, only the charge and the momentum conservation laws are used in the derivation of the theorem. Since only conservation laws are used for the argument, the asymptotic solutions for the NLS equation are introduced as functions ψ for which the charge and the momentum conservation laws hold approximately. Namely, certain integrals which involve quantities that enter the conservation laws vanish in an asymptotic limit, see Definition 5 for details. Remarkably, such mild restrictions still completely determine all possible trajectories of concentration, that is the trajectory $\hat{\mathbf{r}}(t)$ is uniquely defined by the initial data $\hat{\mathbf{r}}(t_0)$ and $\partial_t \hat{\mathbf{r}}(t_0)$ as a solution of (1.9).

Equation (1.9) yields a *necessary condition* for solutions of the NLS equation to concentrate at a trajectory. To obtain this necessary condition we have to derive the point dynamics governed by an ordinary differential equation (1.9) from the dynamics of waves governed by a partial differential equation. The concept of "*concentration of functions at a given trajectory $\hat{\mathbf{r}}(t)$* ", see Definition 4, is the first step in relating spatially localized fields ψ to point trajectories. The definition of concentration of functions has a sufficient flexibility to allow for general regular trajectories $\hat{\mathbf{r}}(t)$ and plenty of functions are localized about the trajectory. But if a sequence of functions concentrating at a given trajectory *also satisfy or asymptotically satisfy the conservation laws for the NLS equation then, according to Theorems 3 and 6, the trajectory and the limit energy must satisfy Newton's equation.*

To derive Newton's equation (1.9), we consider $\psi(\mathbf{x}, t)$ restricted to a narrow tubular neighborhood of the trajectory of radius R_n and consider adjacent charge centers

$$(1.12) \quad \mathbf{r}_n(t) = \frac{1}{\rho_n} \int_{|\mathbf{x}-\hat{\mathbf{r}}|\leq R_n} \mathbf{x}q|\psi_n|^2 d\mathbf{x}, \quad \bar{\rho}_n = \int_{|\mathbf{x}-\hat{\mathbf{r}}|\leq R_n} q|\psi_n|^2 d\mathbf{x},$$

of the concentrating solutions. Then we infer integral equations for the restricted momentum and for the adjacent centers from the momentum conservation law and the continuity equation, and pass to the limit as $R_n \rightarrow 0, a_n \rightarrow 0$ as is shown in the proofs of Theorems 1-3. Note that the determination of the restricted charge density $\bar{\rho}$ and a similar momentum density involves integration over a large relative to a

spatial domain of radius R_n , $R_n \gg a$. Therefore it is natural to call our method of determination of the point trajectories *semi-local*. This semi-local feature applied to the nonlinear Klein-Gordon equation in [10], [11] allowed us to derive Einstein's relation between mass and energy with the energy being an integral quantity.

If the form factor $\check{\psi}_1(\theta)$ decays fast enough as $\theta \rightarrow \infty$ (faster than θ^{-2}), we also prove the converse statement (Theorem 8) to Theorem 6. Namely, *if $\hat{r}(t)$ is a solution of equation (1.9) then it is a trajectory of concentration for a sequence of asymptotic solutions to the NLS*. The proof of this theorem is based on an explicit construction of *wave-corpuscle solutions* of the NLS of the form $\psi(t, \mathbf{x}) = e^{iS} \check{\psi}(\mathbf{x} - \hat{\mathbf{r}})$ for a general enough class of potentials φ and \mathbf{A} with a certain phase function S . We choose auxiliary potentials φ_{aux} and \mathbf{A}_{aux} from this class to approximate φ and \mathbf{A} at $\hat{\mathbf{r}}(t)$. The wave-corpuscle solutions constructed for the auxiliary potentials φ_{aux} and \mathbf{A}_{aux} turn out to form a sequence of concentrating asymptotic solutions of the NLS. Therefore, for given twice continuously differentiable potentials φ and \mathbf{A} , a trajectory is a trajectory of concentration of asymptotic solutions of the NLS if and only if this trajectory satisfies Newton's law (1.9). The phase function S of the wave-corpuscle can be interpreted as the phase of de Broglie wave (see [7] for details). Hence, the described relation between Newtonian trajectories and concentrating asymptotic solutions can be interpreted as what is known as the wave-particle duality.

The paper's structure is as follows. In the following Subsection 1.1 we define the charge density, current density and momentum density for a solution of NLS and write down corresponding conservation laws. The customary field theory derivation of the conservation laws is given in Appendix 1. In Section 2 we describe the class of trajectories and in Subsection 2.2 we formulate restrictions on the fields φ and \mathbf{A} and give the definition of solutions concentrating to a trajectory (Definition 4). In Subsections 2.3 and 2.4 we prove our main result on the characterization of trajectories of concentration of solutions to the NLS equation (Theorem 3).

In Section 3 we consider "wave-corpuscles" that exactly preserve their shape in an accelerated motion and describe a class of potentials φ, \mathbf{A} which allow for such a motion. In Subsection 3.4 we prove that the wave corpuscles provide an example of concentrating solutions to the NLS equation (Theorem 5). In Section 4 we provide an extension of results of Section 2 to asymptotic solutions of the NLS equation, including the main Theorem 6. In Section 4.1 we prove that if a trajectory satisfies Newton's law then it is a trajectory of concentration of asymptotic solutions of the NLS equation.

1.1. Conservation laws for NLS. The *charge density* ρ and *current density* \mathbf{J} are defined by the formulas

$$(1.13) \quad \rho = q\psi\psi^*,$$

$$(1.14) \quad \mathbf{J} = -i\frac{q\chi}{2m} \left(\psi^* \tilde{\nabla} \psi - \psi \tilde{\nabla}^* \psi^* \right),$$

and for solutions of (1.1) they satisfy the *continuity equation*:

$$(1.15) \quad \partial_t \rho + \nabla \cdot \mathbf{J} = 0.$$

One can verify the validity of the continuity equation (1.15) directly by simply multiplying (1.1) by ψ^* and taking the imaginary part. An alternative derivation based on the Lagrangian field formalism is provided in Section 5. The *momentum*

density is defined by the formula

$$(1.16) \quad \mathbf{P} = i\frac{\chi}{2} \left[\psi \cdot \tilde{\nabla}^* \psi^* - \psi^* \cdot \tilde{\nabla} \psi \right],$$

and it is proportional to the current density, namely

$$(1.17) \quad \mathbf{P} = \frac{m}{q} \mathbf{J}.$$

The momentum density satisfies the *momentum equation* of the form

$$(1.18) \quad \partial_t \mathbf{P} + \partial_i T^{ij} = \mathbf{f},$$

where \mathbf{f} is the Lorentz force density defined by the formula

$$(1.19) \quad \mathbf{f} = \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B},$$

T^{ij} are the entries of the energy-momentum tensor defined by (5.21), (5.22), and the EM fields \mathbf{E} , \mathbf{B} are defined in terms of the potentials φ , \mathbf{A} by the standard formulas (1.11). The momentum equation can be derived from the NLS equation using multiplication by $\tilde{\nabla}^* \psi^*$ and some rather lengthy vector algebra manipulations. The derivation of the momentum equation based on the standard field theory formalism is given in Section 5.

2. DERIVATION OF NON-RELATIVISTIC POINT DYNAMICS FOR LOCALIZED SOLUTIONS OF NLS EQUATION

In this section we consider solutions of NLS equation (1.1) that are localized around a trajectory in the three dimensional space \mathbb{R}^3 and find the necessary condition for such a trajectory which coincides with Newton's law of motion.

2.1. Trajectories and their neighborhoods. The first step in introducing the concept of concentration to a trajectory is to describe the class of trajectories.

Definition 1 (trajectory). *A trajectory $\hat{\mathbf{r}}(t)$, $T_- \leq t \leq T_+$, is a twice continuously differentiable function with values in \mathbb{R}^3 satisfying*

$$(2.1) \quad |\partial_t \hat{\mathbf{r}}(t)| \leq C, \quad |\partial_t^2 \hat{\mathbf{r}}(t)| \leq C \quad \text{for } T_- \leq t \leq T_+.$$

Being given a trajectory $\hat{\mathbf{r}}(t)$, we consider a family of neighborhoods contracting to it. Namely, we introduce a ball of radius R centered at $\mathbf{x} = \hat{\mathbf{r}}(t)$:

$$(2.2) \quad \Omega(\hat{\mathbf{r}}(t), R) = \left\{ \mathbf{x} : |\mathbf{x} - \hat{\mathbf{r}}(t)|^2 \leq R^2 \right\} \subset \mathbb{R}^3, \quad R > 0.$$

Definition 2 (concentrating neighborhoods). *Concentrating neighborhoods $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n) \subset [T_-, T_+] \times \mathbb{R}^3$ of a trajectory $\hat{\mathbf{r}}(t)$ are defined as a family of tubular domains*

$$(2.3) \quad \hat{\Omega}(\hat{\mathbf{r}}(t), R_n) = \left\{ (t, \mathbf{x}) : T_- \leq t \leq T_+, \quad |\mathbf{x} - \hat{\mathbf{r}}(t)|^2 \leq R_n^2 \right\}$$

where R_n satisfy the contraction condition:

$$(2.4) \quad R_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The cross-section of the tubular domain at a fixed t is given by (2.2) and is denoted by Ω_n :

$$(2.5) \quad \Omega_n = \Omega_n(t) = \Omega(\hat{\mathbf{r}}(t), R_n) \subset \mathbb{R}^3.$$

2.2. Localized NLS equations. Let us consider the NLS equation (1.1) in a neighborhood of the trajectory $\hat{\mathbf{r}}(t)$. We remind that m is the mass parameter, q is the value of the charge, χ is a parameter similar to the Planck constant, c is the speed of light all of which are fixed. The size parameter a and potentials $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ form a sequence. We consider the NLS in a shrinking vicinity of the trajectory and make certain regularity assumptions on behavior of its coefficients. In the definitions below we use the following notations:

$$(2.6) \quad \partial_0 = c^{-1} \partial_t,$$

$$(2.7) \quad \begin{aligned} \nabla_{\mathbf{x}} \varphi &= \nabla \varphi = (\partial_1 \varphi, \partial_2 \varphi, \partial_3 \varphi), & |\nabla_{\mathbf{x}} \varphi|^2 &= |\nabla \varphi|^2 = |\partial_1 \varphi|^2 + |\partial_2 \varphi|^2 + |\partial_3 \varphi|^2, \\ \nabla_{0, \mathbf{x}} \varphi &= (\partial_0 \varphi, \partial_1 \varphi, \partial_2 \varphi, \partial_3 \varphi), & |\nabla_{0, \mathbf{x}} \varphi|^2 &= |\partial_0 \varphi|^2 + |\partial_1 \varphi|^2 + |\partial_2 \varphi|^2 + |\partial_3 \varphi|^2. \end{aligned}$$

Now we formulate the continuity assumptions we impose on the nonlinearity G'_1 in the NLS equation (1.1).

Condition 1. *The real-valued function $G'_1(s)$ is continuous for $s > 0$. It coincides with the derivative of the potential $G_1(s)$ which is differentiable for $s > 0$ and continuous for $s \geq 0$. We assume that the function $G_1(\psi^* \psi)$ of the complex variable $\psi \in \mathbb{C}$ is differentiable for all ψ and its differential has the form*

$$dG_1(\psi^* \psi) = g(\psi) d\psi + g^*(\psi) d\bar{\psi}$$

where

$$g(\psi) = \begin{cases} G'_1(\psi^* \psi) \psi & \text{for } \psi \in \mathbb{C}, \psi \neq 0, \\ 0 & \text{for } \psi = 0 \end{cases},$$

and $g(\psi)$ is continuous for $\psi \in \mathbb{C}$.

Note that the above condition allows a mild singularity of G'_1 at zero, for example the logarithmic nonlinearity (3.10) satisfies this condition.

Definition 3 (localized NLS equations). *Let $\hat{\mathbf{r}}(t)$ be a trajectory with its concentrating neighborhoods $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$, and let $a = a_n$, $\varphi = \varphi_n$, $\mathbf{A} = \mathbf{A}_n$ be a sequence of parameters and potentials entering the NLS equation (1.1). We say that NLS equations are localized in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ if the following conditions are satisfied. The parameters R_n and a_n as $n \rightarrow \infty$ become vanishingly small, that is*

$$(2.8) \quad a = a_n \rightarrow 0, \quad R_n \rightarrow 0$$

and the ratio $\theta = R_n/a$ grows to infinity:

$$(2.9) \quad \theta_n = \frac{R_n}{a_n} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

We introduce at the trajectory $\hat{\mathbf{r}}(t)$ the limit potentials $\varphi_\infty(t, \mathbf{x})$, $\mathbf{A}_\infty(t, \mathbf{x})$ which are linear in \mathbf{x} and are written in the form

$$(2.10) \quad \varphi_\infty(t, \mathbf{x}) = \varphi_\infty(t) + (\mathbf{x} - \hat{\mathbf{r}}) \cdot \nabla \varphi_\infty(t),$$

$$(2.11) \quad \mathbf{A}_\infty(t, \mathbf{x}) = \mathbf{A}_\infty(t) + (\mathbf{x} - \hat{\mathbf{r}}) \cdot \nabla \mathbf{A}_\infty(t),$$

where the coefficients $\varphi_\infty, \nabla \varphi_\infty, \mathbf{A}_\infty, \nabla \mathbf{A}_\infty$ satisfy the following boundedness conditions:

$$(2.12) \quad |\varphi_\infty| \leq C, \quad |\nabla_{0, \mathbf{x}} \varphi_\infty| \leq C \text{ for } T_- \leq t \leq T_+,$$

$$(2.13) \quad |\mathbf{A}_\infty| \leq C, \quad |\nabla_{0,\mathbf{x}}\mathbf{A}_\infty| \leq C \text{ for } T_- \leq t \leq T_+.$$

We suppose the EM potentials $\varphi_n(t, \mathbf{x}), \mathbf{A}_n(t, \mathbf{x})$ to be twice continuously differentiable in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$. The potentials $\varphi_n(t, \mathbf{x}), \mathbf{A}_n(t, \mathbf{x})$ locally converge to the limit linear potentials $\varphi_\infty(t, \mathbf{x}), \mathbf{A}_\infty(t, \mathbf{x})$, namely they satisfy the following relations:

(i) convergence as $n \rightarrow \infty$:

$$(2.14) \quad \max_{T_- \leq t \leq T_+, \mathbf{x} \in \Omega_n} (|\varphi_n(t, \mathbf{x}) - \varphi_\infty(t, \mathbf{x})| + |\nabla_{0,\mathbf{x}}\varphi_n(t, \mathbf{x}) - \nabla_{0,\mathbf{x}}\varphi_\infty(t, \mathbf{x})|) \rightarrow 0,$$

$$(2.15) \quad \max_{T_- \leq t \leq T_+, \mathbf{x} \in \Omega_n} (|\mathbf{A}_n(t, \mathbf{x}) - \mathbf{A}_\infty(t, \mathbf{x})| + |\nabla_{0,\mathbf{x}}\mathbf{A}_n(t, \mathbf{x}) - \nabla_{0,\mathbf{x}}\mathbf{A}_\infty(t, \mathbf{x})|) \rightarrow 0;$$

(ii) uniform in n estimates:

$$(2.16) \quad |\varphi_n(t, \mathbf{x})| \leq C, \quad |\nabla_{0,\mathbf{x}}\varphi_n(t, \mathbf{x})| \leq C \quad \text{for } (t, \mathbf{x}) \in \hat{\Omega}(\hat{\mathbf{r}}, R_n),$$

$$(2.17) \quad |\mathbf{A}_n(t, \mathbf{x})| \leq C, \quad |\nabla_{0,\mathbf{x}}\mathbf{A}_n(t, \mathbf{x})| \leq C \quad \text{for } (t, \mathbf{x}) \in \hat{\Omega}(\hat{\mathbf{r}}, R_n).$$

The limit EM fields $\mathbf{E}_\infty, \mathbf{B}_\infty$ at the trajectory are defined in terms of the linear potentials (2.10), (2.11) by (1.11), namely

$$(2.18) \quad \mathbf{E}_\infty = -\nabla\varphi_\infty(t, \hat{\mathbf{r}}) - \frac{1}{c}\partial_t\mathbf{A}_\infty(t, \hat{\mathbf{r}}), \quad \mathbf{B}_\infty = \nabla \times \mathbf{A}_\infty(t, \hat{\mathbf{r}}).$$

Note that according to (2.14), (2.15)

$$(2.19) \quad \mathbf{E} = \mathbf{E}_n \rightarrow \mathbf{E}_\infty, \quad \mathbf{B} = \mathbf{B}_n \rightarrow \mathbf{B}_\infty$$

in $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)$.

Throughout this paper we denote constants which do not depend on n by the letter C with different indices. Sometimes the same letter C with the same indices may denote in different formulas different constants. Below we often omit index n in a_n, φ_n etc.

The most important case where we apply the above definition is described in the following example.

Example 1. If the potentials φ_n, \mathbf{A}_n are the restrictions of fixed twice continuously differentiable potentials φ, \mathbf{A} to the domain $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)$, then $\varphi_\infty, \mathbf{A}_\infty$ is the linear part of φ, \mathbf{A} at $\hat{\mathbf{r}}$ and conditions (2.12)-(2.17) are satisfied with

$$(2.20) \quad \begin{aligned} \varphi_\infty(t, \mathbf{x}) &= \varphi(t, \hat{\mathbf{r}}) + (\mathbf{x} - \hat{\mathbf{r}}) \cdot \nabla\varphi(t, \hat{\mathbf{r}}), \\ \mathbf{A}_\infty(t, \mathbf{x}) &= \mathbf{A}(t, \hat{\mathbf{r}}) + (\mathbf{x} - \hat{\mathbf{r}}) \cdot \nabla\mathbf{A}(t, \hat{\mathbf{r}}), \end{aligned}$$

that is the coefficients in (2.10), (2.11) are defined as follows:

$$(2.21) \quad \varphi_\infty(t) = \varphi(t, \hat{\mathbf{r}}(t)), \quad \nabla\mathbf{A}_\infty(t) = \nabla\mathbf{A}(t, \hat{\mathbf{r}}),$$

and the EM fields $\mathbf{E}_\infty, \mathbf{B}_\infty$ in (2.18) are directly expressed in terms of φ, \mathbf{A} by (1.11), namely

$$(2.22) \quad \mathbf{E}_\infty = -\nabla\varphi(t, \hat{\mathbf{r}}) - \frac{1}{c}\partial_t\mathbf{A}(t, \hat{\mathbf{r}}), \quad \mathbf{B}_\infty = \nabla \times \mathbf{A}(t, \hat{\mathbf{r}}).$$

We introduce the local value of the charge restricted to domain $\Omega(\hat{\mathbf{r}}(t), R_n)$ by the formula

$$(2.23) \quad \bar{\rho}_n(t) = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \rho_n \, d\mathbf{x} = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} q |\psi_n|^2 \, d\mathbf{x},$$

with $\rho_n(t, \mathbf{x})$ being the charge density defined by (1.13), and we call $\bar{\rho}_n$ adjacent charge value.

Definition 4 (concentrating solutions). *Let $\hat{\mathbf{r}}(t)$ be a trajectory. We say that solutions ψ to the NLS equations concentrate at the trajectory $\hat{\mathbf{r}}(t)$ if Condition 1 is satisfied and the following conditions are fulfilled. First, a sequence of concentrating neighborhoods $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$, parameters $a = a_n$ and potentials $\varphi = \varphi_n$, $\mathbf{A} = \mathbf{A}_n$ are selected as in Definition 3. Second, there exists a sequence of functions $\psi = \psi_n$ which are twice continuously differentiable, that is $\psi_n \in C^2(\hat{\Omega}(\hat{\mathbf{r}}, R_n))$, and such that the charge density $\rho = q|\psi_n|^2$ and the momentum density \mathbf{P} defined by (1.16) for this sequence satisfy the following conditions:*

- (i) every function ψ_n is a solution to the NLS equation (1.1) in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$;
- (ii) the momentum density \mathbf{P} defined by (1.16) for this sequence is such that the following integrals are bounded:

$$(2.24) \quad \left| \int_{\Omega_n} \mathbf{P}_n(t) \, d\mathbf{x} \right| \leq C;$$

- (iii) the local charge value defined by (2.23) is bounded from above and below for sufficiently large n :

$$(2.25) \quad C \geq \bar{\rho}_n(t) \geq c_0 > 0 \text{ for } n \geq n_0, T_- \leq t \leq T_+;$$

- (iv) there exists $t_0 \in [T_-, T_+]$ such that the sequence of local charge values converges:

$$(2.26) \quad \lim_{n \rightarrow \infty} \bar{\rho}_n(t_0) = \bar{\rho}_\infty.$$

- (v) We also impose conditions on the following surface integrals over the spheres $\partial\Omega_n = \{|\mathbf{x} - \hat{\mathbf{r}}| = R_n\}$:

$$(2.27) \quad Q_0 = \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}}_i T^{ij} \, d\sigma dt',$$

where T^{ij} are given by (5.21), (5.22) and we make summation over repeating indices;

$$(2.28) \quad Q_{01} = \int_{t_0}^t \int_{\partial\Omega_n} \mathbf{P} \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \, d\sigma dt',$$

where \mathbf{P} is defined by (1.16);

$$(2.29) \quad Q_{20} = - \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho \, d\sigma,$$

where ρ is defined by (1.13);

$$(2.30) \quad Q_{22} = \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \mathbf{n} \cdot \mathbf{J} \, d\mathbf{x},$$

where \mathbf{J} is given by (1.14); and

$$(2.31) \quad Q_{23} = \int_{t_0}^t \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \mathbf{n} \rho \, d\mathbf{x} dt' - \int_{t_0}^t \int_{\partial\Omega_n} \mathbf{n} \cdot \mathbf{J} \, d\mathbf{x} dt'.$$

The integrals defined above are assumed to satisfy the following limit relations uniformly on the time interval $[T_-, T_+]$:

$$(2.32) \quad Q_0 \rightarrow 0,$$

$$(2.33) \quad Q_{01} \rightarrow 0,$$

$$(2.34) \quad Q_{20} \rightarrow 0,$$

$$(2.35) \quad Q_{22} \rightarrow 0,$$

$$(2.36) \quad Q_{23} \rightarrow 0.$$

(vi) We introduce the following integrals over Ω_n with vanishing at $\hat{\mathbf{r}}$ weights

$$(2.37) \quad Q_{30} = \int_{\Omega_n} (\mathbf{E} - \mathbf{E}_\infty) \rho \, d\mathbf{x},$$

$$(2.38) \quad Q_{31} = \int_{\Omega_n} \frac{1}{c} \mathbf{J} \times (\mathbf{B} - \mathbf{B}_\infty) \, d\mathbf{x},$$

and assume that

$$(2.39) \quad \int_{t_0}^t (Q_{30} + Q_{31}) \, dt' \rightarrow 0$$

uniformly on the time interval $[T_-, T_+]$. If all the above conditions are fulfilled, we call $\hat{\mathbf{r}}(t)$ a concentration trajectory of the NLS equation (1.1).

Obviously conditions (ii)-(iv) and (vi) provide boundedness and convergence of certain volume integrals over Ω_n , and condition (v) provides asymptotical vanishing of surface integrals over the boundary. Notice that condition (ii) provides for the boundedness of the momentum over domain Ω_n , condition, (v) provides for a proper confinement of ψ to Ω_n and estimate from below in condition (iii) ensures that the sequence is non-trivial. According to (2.25), $\bar{\rho}_n(t_0)$ is a bounded sequence, consequently it always contains a converging subsequence. Hence condition (iv) is not really an additional constraint but rather it assumes that such a subsequence is selected. The choice of a particular subsequence limit $\bar{\rho}_\infty$ is discussed in Remark 1. This condition describes the amount of charge which concentrates at the trajectory at the time t_0 . Condition (i) can be relaxed and replaced by the assumption that ψ_n is an asymptotic solution, see Definition 5 for details. We could also allow parameters χ, m, q to form sequences and depend on n , but for simplicity in this paper we assume them fixed.

The wave-corpuscles constructed in Section 3 provide a non-trivial example of solutions which concentrate at trajectories of accelerating charges.

2.3. Properties of concentrating solutions of NLS. We define the adjacent charge center \mathbf{r}_n by the formula

$$(2.40) \quad \mathbf{r}_n(t) = \frac{1}{\bar{\rho}_n} \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{x} \rho_n \, d\mathbf{x}.$$

Since $\psi_a(t, \mathbf{x})$ is a function of class C^1 with respect to (t, \mathbf{x}) , and $\hat{\mathbf{r}}(t)$ is differentiable, the vector $\mathbf{r}(t)$ is a differentiable function of time, and we denote by \mathbf{v} the velocity of the adjacent charge center:

$$(2.41) \quad \mathbf{v} = \mathbf{v}_n(t) = \partial_t \mathbf{r}.$$

Below we often make use of the following elementary identity:

$$(2.42) \quad \int_{\Omega_n} \partial_t f(t, \mathbf{x}) \, d\mathbf{x} = \partial_t \int_{\Omega_n} f(t, \mathbf{x}) \, d\mathbf{x} - \int_{\partial\Omega_n} f(t, \mathbf{x}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \, d\sigma,$$

where $\bar{\mathbf{n}}$ is the external normal to $\partial\Omega_n$, $\hat{\mathbf{v}} = \partial_t \hat{\mathbf{r}}$.

Lemma 1. *Let charge densities ρ_n satisfy (2.25). Then the adjacent ergocenters $\mathbf{r}_n(t)$ of the solutions converge to $\hat{\mathbf{r}}(t)$ uniformly on the time interval $[T_-, T_+]$.*

Proof. By (2.40)

$$(2.43) \quad \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \rho_n \, d\mathbf{x} = 0,$$

and according to (2.25)

$$(2.44) \quad \left| \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} (\mathbf{x} - \hat{\mathbf{r}}) \rho_n \, d\mathbf{x} \right| \leq R_n \int_{\Omega(\mathbf{0}, R_n)} \rho_n \, d\mathbf{y} \rightarrow 0.$$

Therefore

$$(\mathbf{r} - \hat{\mathbf{r}}) \bar{\rho} = \int_{\Omega_n} (\mathbf{x} - \hat{\mathbf{r}}) \rho_n \, d\mathbf{x} - \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \rho_n \, d\mathbf{x} \rightarrow 0.$$

Using (2.25) we conclude that

$$(2.45) \quad |\hat{\mathbf{r}} - \mathbf{r}_n| \rightarrow 0$$

uniformly on $[T_-, T_+]$. □

Lemma 2. *Let the current \mathbf{J} and the charge density ρ in (1.15) are such that Q_{23} defined by (2.31) satisfies (2.36). Then the local charge values converge uniformly on $[T_-, T_+]$ to a constant:*

$$(2.46) \quad \bar{\rho}_n(t) \rightarrow \bar{\rho}_\infty \quad \text{as } n \rightarrow \infty.$$

Proof. Integrating the continuity equation we obtain

$$(2.47) \quad \bar{\rho}(t) - \bar{\rho}(t_0) - \int_{t_0}^t \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \mathbf{n} \rho \, dx dt' + \int_{t_0}^t \int_{\partial\Omega_n} \mathbf{n} \cdot \mathbf{J} \, dx dt' = 0.$$

We use (2.36) and obtain (2.46). □

Lemma 3. *Assume that (2.24) holds. Then there is a subsequence of the concentrating sequence of solutions of the NLS such that*

$$(2.48) \quad \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{P}_n(t_0) \, d\mathbf{x} \rightarrow \mathbf{p}_\infty \quad \text{as } n \rightarrow \infty.$$

Assume also that boundary integrals (2.27) and (2.28) satisfy assumptions (2.32) and (2.33). Then for this subsequence

$$(2.49) \quad \int_{\Omega_n} \mathbf{P}_n(t) \, d\mathbf{x} = \int_{t_0}^t \int_{\Omega_n} \mathbf{f} \, dx dt' + \mathbf{p}_\infty + Q_{00}$$

where

$$(2.50) \quad Q_{00} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly on $[T_-, T_+]$.

Proof. According to (2.24) we can select a subsequence which has a limit \mathbf{p}_∞ and (2.48) holds. We use the momentum equation (1.18)

$$(2.51) \quad \partial_t \mathbf{P} + \partial_i T^{ij} = \mathbf{f}.$$

Integrating it over $\Omega(\hat{\mathbf{r}}(t), R_n) = \Omega_n$ and then with respect to time, we obtain the equation

$$(2.52) \quad \int_{t_0}^t \int_{\Omega_n} \partial_t \mathbf{P}(t') \, dx dt' - \int_{t_0}^t \int_{\Omega_n} \mathbf{f} \, dx dt' + Q_0 = 0$$

where Q_0 is defined by (2.27). Using (2.42) we rewrite the equation in the form

$$(2.53) \quad \int_{\Omega_n} \mathbf{P}(t) \, dx - \int_{\Omega_n} \mathbf{P}(t_0) \, dx - Q_{01} - \int_{t_0}^t \int_{\Omega_n} \mathbf{f} \, dx dt' + Q_0 = 0,$$

where Q_{01} is defined by (2.28), $\Omega_n = \Omega(\hat{\mathbf{r}}(t), R_n)$. Using (2.32), (2.33) and (2.48) we obtain (2.49) and (2.50) from (2.53). \square

Lemma 4. *In addition to the assumptions of Lemmas 1, 2, 3 assume that the boundary integrals (2.29) and (2.30) vanish as $n \rightarrow \infty$, namely (2.34) and (2.35) are fulfilled uniformly on the time interval $[T_-, T_+]$. Then*

$$(2.54) \quad \int_{\Omega_n} \mathbf{J} \, dx = \mathbf{v} \bar{\rho}_\infty + Q_{200},$$

$$(2.55) \quad \int_{\Omega_n} \mathbf{P} \, dx = m \mathbf{v} \frac{\bar{\rho}_\infty}{q} + \frac{m}{q} Q_{200},$$

where $\mathbf{v} = \partial_t \mathbf{r}$ and

$$(2.56) \quad Q_{200} \rightarrow 0$$

uniformly on the time interval $[T_-, T_+]$.

Proof. According to the continuity equation, we obtain the identity

$$\int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \partial_t \rho \, dx + \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \nabla \cdot \mathbf{J} \, dx = 0.$$

Using the commutation relation

$$(2.57) \quad \partial_j (x_i f) - x_i \partial_j f = \delta_{ij} f$$

to transform the second integral, we obtain the following equation

$$(2.58) \quad \int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) \rho) \, dx + \partial_t \mathbf{r} \int_{\partial \Omega_n} \rho \, dx + \int_{\partial \Omega_n} (\mathbf{x} - \mathbf{r}) \mathbf{n} \cdot \mathbf{J} \, dx = \int_{\Omega_n} \mathbf{J} \, dx.$$

Using the definition of the charge center \mathbf{r} and (2.42) we infer that the first term in the above equation has the following form:

$$(2.59) \quad \int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) \rho) \, dx = - \int_{\partial \Omega_n} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho \, d\sigma.$$

where $\hat{\mathbf{v}} = \partial_t \hat{\mathbf{r}}$. We can express then $\mathbf{v} = \partial_t \mathbf{r}$ from (2.58), and using (2.24), (2.30), (2.45) and (2.25) can estimate the integrals which enter (2.58) concluding that

$$(2.60) \quad |\mathbf{v}| \leq C \quad \text{for} \quad -T \leq t \leq T.$$

We rewrite (2.58) in the form

$$(2.61) \quad \int_{\Omega_n} \mathbf{J} d\mathbf{x} = \mathbf{v} \bar{\rho}_\infty + Q_{20} + Q_{21} + Q_{22},$$

where $\bar{\rho}_\infty$ is the same as in (2.26),

$$Q_{21} = \mathbf{v} \left(\int_{\Omega_n} \rho_n d\mathbf{x} - \bar{\rho}_\infty \right).$$

According to (2.26) and (2.60) $Q_{21} \rightarrow 0$. Using (2.29) and (2.30) we conclude that (2.54) and (2.56) hold with $Q_{200} = Q_{20} + Q_{21} + Q_{22}$. Using (1.17) we deduce (2.55) from (2.54). \square

2.4. Derivation of Newton's equation for the trajectory of concentration.

Theorem 1. *For a concentrating sequence of solutions of the NLS equation the adjacent center velocities $\mathbf{v} = \mathbf{v}_n = \partial_t \mathbf{r}_n$ satisfy the equation*

$$(2.62) \quad m\mathbf{v} \frac{1}{q} \bar{\rho}_\infty = \int_{t_0}^t \left(\bar{\rho}_\infty \mathbf{E}_\infty(t') + \bar{\rho}_\infty \mathbf{v} \times \frac{1}{c} \mathbf{B}_\infty(t') \right) dt' + \mathbf{p}_\infty + Q_6$$

where $Q_6 \rightarrow 0$ uniformly on $[T_-, T_+]$, \mathbf{p}_∞ is the same as in (2.48), and $\mathbf{E}_\infty, \mathbf{B}_\infty$ are the same as in (2.18).

Proof. We substitute (2.55) into (2.49) and obtain the following equation:

$$(2.63) \quad m\mathbf{v} \frac{1}{q} \bar{\rho}_\infty + \frac{m}{q} Q_{200} = \int_{t_0}^t \int_{\Omega_n} \mathbf{f} d\mathbf{x} dt' + \mathbf{p}_\infty + Q_{00}.$$

where the Lorentz force density \mathbf{f} is given by (1.19). To evaluate the terms involving the Lorentz force density, we use the limit EM fields defined in accordance with (2.10), (2.11) by formula (2.18). We obtain

$$(2.64) \quad \int_{\Omega_n} \mathbf{f} d\mathbf{x} = \mathbf{E}_\infty \int_{\Omega_n} \rho d\mathbf{x} + \int_{\Omega_n} \frac{1}{c} \mathbf{J} d\mathbf{x} \times \mathbf{B}_\infty + Q_3$$

where $Q_3 = Q_{30} + Q_{31}$ is expressed in terms of (2.37), (2.38). Therefore, using (2.54) we obtain from (2.64) that

$$(2.65) \quad \int_{\Omega_n} \mathbf{f} d\mathbf{x} = \bar{\rho}_\infty \mathbf{E}_\infty + \bar{\rho}_\infty \mathbf{v} \times \frac{1}{c} \mathbf{B}_\infty + Q_3 + Q_4 + Q_5,$$

$$(2.66) \quad Q_4 = \frac{1}{c} Q_{200} \times \mathbf{B}_\infty, \quad Q_5 = \mathbf{E}_\infty (\bar{\rho}_n - \bar{\rho}_\infty).$$

Using (2.39), (2.46), (2.56), (2.12), and (2.13), we obtain (2.62) with

$$Q_6 = Q_{00} - \frac{m}{q} Q_{200} + \int_{t_0}^t (Q_3 + Q_4 + Q_5) dt'.$$

\square

The following statement provides an explicit necessary condition for a trajectory of concentration.

Theorem 2 (Trajectory of concentration criterion). *Let solutions ψ of the NLS equation (1.1) concentrate at $\hat{\mathbf{r}}(t)$. Then the trajectory $\hat{\mathbf{r}}(t)$ satisfies the equation*

$$(2.67) \quad \partial_t^2 \hat{\mathbf{r}} = \mathbf{f}_\infty$$

with the Lorentz force $\mathbf{f}_\infty(t)$ expressed in terms of the potentials (2.10), (2.11) by the formula

$$(2.68) \quad \mathbf{f}_\infty(t) = -q\mathbf{E}_\infty + \frac{q}{c}\partial_t\hat{\mathbf{r}} \times \mathbf{B}_\infty.$$

Proof. The function $\mathbf{v}(t)$ can be considered as a solution of the integral equation (2.62) on the interval $[T_-, T_+]$, and this equation is evidently linear. Since the term $Q_6 \rightarrow 0$ uniformly, the sequence $\mathbf{v}_n(t)$ of its solutions converges uniformly to the solution of the equation

$$(2.69) \quad m\mathbf{v}_\infty \frac{1}{q}\bar{\rho}_\infty = \bar{\rho}_\infty \int_{t_0}^t \left(\mathbf{E}_\infty(t') + \mathbf{v}_\infty \times \frac{1}{c}\mathbf{B}_\infty(t') \right) dt' + \mathbf{p}_\infty.$$

Now we want to prove that $\mathbf{v}_\infty = \hat{\mathbf{v}} = \partial_t\hat{\mathbf{r}}$. Note that according to Lemma 1

$$\int_{t_0}^t \mathbf{v}_n dt' = \mathbf{r}_n(t) - \mathbf{r}_n(t_0) \rightarrow \hat{\mathbf{r}}(t) - \hat{\mathbf{r}}(t_0),$$

and, hence,

$$\int_{t_0}^t \mathbf{v}_\infty dt' = \hat{\mathbf{r}}(t) - \hat{\mathbf{r}}(t_0).$$

The above identity implies $\partial_r\hat{\mathbf{r}} = \hat{\mathbf{v}}(t) = \mathbf{v}_\infty(t)$ and consequently $\hat{\mathbf{r}}(t)$ satisfies (2.69):

$$(2.70) \quad m\partial_t\hat{\mathbf{r}} \frac{1}{q}\bar{\rho}_\infty = \bar{\rho}_\infty \int_{t_0}^t \left(\mathbf{E}_\infty(t') + \partial_r\hat{\mathbf{r}} \times \frac{1}{c}\mathbf{B}_\infty(t') \right) dt' + \mathbf{p}_\infty.$$

The derivative of the above equation yields (2.67). \square

As a corollary of Theorem 2 we obtain the following theorem describing the whole class of trajectories of concentration of NLS equation (1.1).

Theorem 3 (Non-relativistic Newton's law). *Assume that (i) potentials $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ are defined and regular in a domain $D \subset \mathbb{R} \times \mathbb{R}^3$; (ii) the trajectory $(t, \hat{\mathbf{r}}(t))$ lies in this domain and the limit potentials φ_∞ , \mathbf{A}_∞ are the restriction of the potentials φ and \mathbf{A} as in (2.20). Let EM fields $\mathbf{E}(t, \mathbf{x})$, $\mathbf{B}(t, \mathbf{x})$ be defined by the potentials as in the formula (1.11). Let solutions ψ of the NLS equation (1.1) concentrate at $\hat{\mathbf{r}}(t)$. Then equation (2.67) for the trajectory $\hat{\mathbf{r}}$ takes the form of Newton's law of motion with the Lorentz force corresponding to the external EM fields $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$, that is*

$$(2.71) \quad m\partial_t^2\hat{\mathbf{r}} = q\mathbf{E}(t, \hat{\mathbf{r}}) + \frac{q}{c}\partial_t\hat{\mathbf{r}} \times \mathbf{B}(t, \hat{\mathbf{r}}).$$

Therefore, for the NLS equation with given potentials φ , \mathbf{A} any concentration trajectory must coincide with the solution of the equation (2.71). For given potentials φ , \mathbf{A} the concentration trajectory through a point $\hat{\mathbf{r}}(t_0)$ is uniquely determined by the velocity $\partial_t\hat{\mathbf{r}}(t_0)$. In particular, it does not depend on the parameter χ which enters the NLS equation. All possible concentration trajectories are solutions of the equation (2.71).

Remark 1. *Consider the situation of Example 1 and Theorem 3. The sequences a_n , R_n , θ_n , φ_n , \mathbf{A}_n , ψ_n enter the definition of a concentrating solution. If we take two different sequences which fit the definition, we obtain the same equation (2.71). More than that, the trajectory $\hat{\mathbf{r}}(t)$ is uniquely defined by the initial data $\hat{\mathbf{r}}(t_0)$, $\partial_t\hat{\mathbf{r}}(t_0)$ and consequently it does not depend on the particular sequence. Remarkably,*

the equation for the trajectory is independent of the value of $\bar{\rho}_\infty$, and this property does not hold in the relativistic case, see [11].

Remark 2. We can modify the definition of concentration at a trajectory by allowing parameter χ to be not fixed but form a sequence. The statements of Theorem 2 and Theorem 3 continue to hold in this case.

3. WAVE-CORPUSCLES IN ACCELERATED MOTION

An example of concentrating solutions is provided by what we call wave-corpuscles. We define the wave-corpuscle ψ by the formula

$$(3.1) \quad \psi(t, \mathbf{x}) = e^{iS} \mathring{\psi}(|\mathbf{y}|),$$

$$(3.2) \quad S = S(\mathbf{y}, t), \quad \mathbf{y} = \mathbf{x} - \mathbf{r}(t),$$

where the form factor $\mathring{\psi}$ satisfies the steady-state equation (1.6) and the normalization condition (1.4).

To give non-trivial examples of localized form factors, it is convenient to start with the form factor $\mathring{\psi}(r)$ and to describe the nonlinearity which produces the form factor as a solution of (1.6). To define the nonlinearity, we impose first our requirements on the ground state $\mathring{\psi}(r)$ of the charge distribution. A ground state is a positive function $\mathring{\psi}(r)$, $r = |x|$, which is twice differentiable, satisfies the charge normalization condition (1.4) and is monotonically decreasing:

$$(3.3) \quad \partial_r \mathring{\psi}(r) < 0 \quad \text{for } r > 0.$$

If $\mathring{\psi}(r)$ is a ground state, we can determine the nonlinearity G' from the following equation obtained from (1.6):

$$(3.4) \quad \nabla^2 \mathring{\psi} = G'(|\mathring{\psi}|^2) \mathring{\psi}.$$

For a radial $\mathring{\psi}$ we obtain then an expression for the nonlinearity G' :

$$(3.5) \quad G'(\mathring{\psi}^2(r)) = G'(\mathring{\psi}^2(r)) = \frac{(\nabla^2 \mathring{\psi})(r)}{\mathring{\psi}(r)}.$$

Since $\mathring{\psi}^2(r)$ is a *monotonic* function, we can find its inverse $r = r(\mathring{\psi}^2)$, yielding

$$(3.6) \quad G'(s) = \frac{\nabla^2 \mathring{\psi}(r(s))}{\mathring{\psi}(r(s))}, \quad 0 = \mathring{\psi}^2(\infty) \leq s \leq \mathring{\psi}^2(0).$$

Since $\mathring{\psi}(r)$ is smooth and $\partial_r \mathring{\psi} < 0$, $G'(|\mathring{\psi}|^2)$ is smooth for $0 < |\mathring{\psi}|^2 < \mathring{\psi}^2(0)$; we extend $G'(s)$ for $s \geq \mathring{\psi}^2(0)$ as a smooth function for all $s > 0$. To describe the localization of the ground state $\mathring{\psi}$, we introduce an explicit dependence on the *size parameter* $a > 0$ as in (1.7):

$$(3.7) \quad \mathring{\psi}(r) = \mathring{\psi}_a(r) = a^{-3/2} \mathring{\psi}_1(a^{-1}r), \quad r = |x| \geq 0.$$

This corresponds to the dependence of the nonlinearity on the size parameter described by (1.5). Note that the antiderivative $G(s)$ is defined for $s \geq 0$ according to Condition 1 and is given by the formula.

$$(3.8) \quad G(s) = \int_0^s G'(s') ds'$$

Example 2. We define a Gaussian form factor by the formula

$$(3.9) \quad \mathring{\psi}(r) = C_g e^{-r^2/2}$$

where C_g is a normalization factor, $C_g = \pi^{-3/4}$ if $v_0 = 1$ in (1.4). Such a ground state is called *gausson* in [17], [18]. Elementary computation shows that

$$\frac{\nabla^2 \mathring{\psi}(r)}{\mathring{\psi}(r)} = r^2 - 3 = -\ln \left(\mathring{\psi}^2(r) / C_g^2 \right) - 3.$$

Hence, we define the nonlinearity corresponding to the Gaussian by the formula

$$(3.10) \quad G'(|\psi|^2) = -\ln(|\psi|^2 / C_g^2) - 3,$$

and refer to it as the logarithmic nonlinearity. The nonlinear potential function has the form

$$(3.11) \quad G(s) = \int_0^s G'(s') ds' = -s \ln s + s \left(\ln \frac{1}{\pi^{3/2}} - 2 \right).$$

Dependence on the size parameter $a > 0$ is given by the formula

$$(3.12) \quad G'_a(|\psi|^2) = -a^{-2} \ln(a^3 |\psi|^2 / C_g^2) - 3a^{-2}.$$

Note that according to Gross inequality the Gaussian provides the minimum of energy

$$E = \int u \, d\mathbf{x}$$

subjected to the normalization condition (1.4), where the energy density u is defined by (5.19).

More examples of nonlinearities can be found in [7]–[11], and many facts from the theory of the NLS equations can be found in [20], [51]. The NLS equations with logarithmic nonlinearity are studied in [21], [19], [20].

Below we find conditions on the external fields φ and \mathbf{A} which allow the wave-corpucle of the form (3.1) to preserve exactly its shape $|\psi|$ in accelerated motion governed by the NLS equation along a trajectory $\mathbf{r}(t)$. In particular, Newton's law of motion emerges as the necessary condition for such a motion.

3.1. Criterion for shape preservation. It is convenient to rewrite the NLS equation (1.1) in the moving frame using a change of variables $\mathbf{x} - \mathbf{r}(t) = \mathbf{y}$. In y -coordinates the NLS equation (1.1) takes the form

$$(3.13) \quad \chi i \partial_t \psi - \chi i \mathbf{v} \cdot \nabla \psi - q \varphi \psi = \frac{\chi^2}{2m} \left[- \left(\nabla - \frac{i q}{\chi c} \mathbf{A} \right)^2 \psi + G'(\psi^* \psi) \psi \right],$$

where \mathbf{v} is the center velocity:

$$(3.14) \quad \mathbf{v}(t) = \partial_t \mathbf{r}(t).$$

We substitute (3.1) in (3.13), and, canceling the factor e^{iS} , we arrive to the following equivalent equation for the phase S :

$$(3.15) \quad -\chi \mathring{\psi} \partial_t S - \chi i \mathbf{v} \cdot \nabla \mathring{\psi} + \chi \mathbf{v} \cdot \nabla S \mathring{\psi} - q \varphi \mathring{\psi} = \frac{\chi^2}{2m} \left[- \left(\nabla - \frac{i q}{\chi c} \mathbf{A} + i \nabla S \right)^2 \mathring{\psi} + G'(\mathring{\psi}^2) \mathring{\psi} \right].$$

Expanding $\left(\nabla - \frac{iq\mathbf{A}}{\chi c} + i\nabla S\right)^2 \dot{\psi}$ and using (1.6) to exclude the nonlinearity G' , we rewrite (3.15) in the form

$$(3.16) \quad \begin{aligned} & -\dot{\psi}\chi\partial_t S - \chi i\mathbf{v} \cdot \nabla \dot{\psi} + \chi \mathbf{v} \cdot \nabla S \dot{\psi} - q\varphi \dot{\psi} = \\ & = \frac{\chi^2}{2m} \left[2 \left(\frac{iq}{\chi c} \mathbf{A} - i\nabla S \right) \nabla \dot{\psi} - \dot{\psi} \nabla \cdot \left(i\nabla S - \frac{iq}{\chi c} \mathbf{A} \right) + \left(\nabla S - \frac{q}{\chi c} \mathbf{A} \right)^2 \dot{\psi} \right]. \end{aligned}$$

Every term in the above equation has either the factor $\dot{\psi}$ or $\nabla \dot{\psi}$. Collecting terms at $\nabla \dot{\psi}$ and $\dot{\psi}$ respectively, we obtain two equations:

$$(3.17) \quad \left(\chi \nabla S - \frac{q}{c} \mathbf{A} - m\mathbf{v} \right) \cdot \nabla \dot{\psi} = 0,$$

$$(3.18) \quad -\chi \partial_t S + \chi \mathbf{v} \cdot \nabla S - q\varphi = -i \frac{\chi}{2m} \nabla \cdot \left(\chi \nabla S - \frac{q}{c} \mathbf{A} \right) + \frac{1}{2m} \left(\chi \nabla S - \frac{q}{c} \mathbf{A} \right)^2.$$

Now we would like to find conditions on the potentials φ , \mathbf{A} under which there exists the phase S which satisfies the above system of equations (3.17), (3.18). Such a phase provides for the wave-corpucle solution to the NLS equation (3.13). We begin with equation (3.17) first. Since

$$\nabla \dot{\psi} = \dot{\psi}'(|\mathbf{y}|) \frac{\mathbf{y}}{|\mathbf{y}|},$$

we conclude that (3.17) is equivalent to the following orthogonality condition:

$$(3.19) \quad \left(\frac{q}{c} \mathbf{A} - \chi \nabla S + m\mathbf{v} \right) \cdot \mathbf{y} = \mathbf{0}.$$

To treat the above equation we use the concept of a *sphere-tangent field*. We call a vector field $\check{\mathbf{V}}(\mathbf{y})$ sphere-tangent if it satisfies the orthogonality condition

$$(3.20) \quad \check{\mathbf{V}}(\mathbf{y}) \cdot \mathbf{y} = 0 \quad \text{for all } \mathbf{y}.$$

Obviously, a sphere-tangent vector field is tangent to spheres centered at the origin. Any vector field $\mathbf{V}(\mathbf{y})$ of class $C^1(\mathbb{R}^3)$ can be uniquely splitted into a potential field ∇P and a sphere-tangent field $\check{\mathbf{V}}$ which satisfies the orthogonality condition (3.20), namely

$$(3.21) \quad \mathbf{V} = \nabla P + \check{\mathbf{V}}$$

(see Section 6 for details and explicit formulas). We split the vector potential \mathbf{A} into its sphere-tangent and gradient parts, namely

$$(3.22) \quad \mathbf{A}(t, \mathbf{y}) = \check{\mathbf{A}}(t, \mathbf{y}) + \mathbf{A}_\nabla(t, \mathbf{y})$$

with

$$(3.23) \quad \check{\mathbf{A}}(t, \mathbf{y}) \cdot \mathbf{y} = 0,$$

$$(3.24) \quad \mathbf{A}_\nabla(t, \mathbf{y}) = \nabla P(t, \mathbf{y}).$$

Equation (3.19) implies that the field $\frac{q}{c} \mathbf{A} - \nabla S + m\mathbf{v}$ is purely sphere-tangent, therefore its gradient part is zero and we conclude that the following equation is equivalent to (3.19):

$$(3.25) \quad \chi \nabla S = m\mathbf{v} + \frac{q}{c} \mathbf{A}_\nabla.$$

Now let us consider equation (3.18). Expressing ∇S from (3.25) we write (3.18) in the form

$$(3.26) \quad -\chi \partial_t S + \mathbf{v} \cdot \left(m\mathbf{v} + \frac{q}{c} \mathbf{A}_\nabla \right) - q\varphi = i \frac{\chi}{2m} \frac{q}{c} \nabla \cdot \check{\mathbf{A}} + \frac{1}{2m} \left(m\mathbf{v} - \frac{q}{c} \check{\mathbf{A}} \right)^2.$$

Taking the imaginary part of (3.26), we see that the sphere-tangent part of the vector potential must satisfy the following zero divergency condition:

$$(3.27) \quad \nabla \cdot \check{\mathbf{A}} = 0.$$

The real part of (3.26) yields equation

$$(3.28) \quad \chi \partial_t S = m\mathbf{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}_\nabla - \frac{1}{2m} \left(m\mathbf{v} - \frac{q}{c} \check{\mathbf{A}} \right)^2 - q\varphi.$$

Hence, (3.28) and (3.25) take the form

$$(3.29) \quad \chi \partial_t S = \frac{1}{2} m\mathbf{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A} - \frac{1}{2m} \frac{q^2}{c^2} \check{\mathbf{A}}^2 - q\varphi,$$

$$(3.30) \quad \chi \nabla S = m\mathbf{v} + \frac{q}{c} \mathbf{A}_\nabla.$$

The above relations give an expression for the 4-gradient of χS . The right-hand sides of (3.29), (3.30) can be considered as the coefficients of a differential 1-form. Hence, if the phase S which solves (3.16) exists, the form must be exact, and consequently it must be closed. Conversely, according to Poincaré's lemma, the form on \mathbb{R}^4 is exact if it is closed on \mathbb{R}^4 , and then the phase S can be found by the integration of the differential form. Since the right-hand side of (3.30) is the gradient, the form is closed if the following condition is satisfied:

$$(3.31) \quad \nabla \left(\frac{1}{2} m\mathbf{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A} - \frac{1}{2m} \frac{q^2}{c^2} \check{\mathbf{A}}^2 - q\varphi \right) = \partial_t \left(m\mathbf{v} + \frac{q}{c} \mathbf{A}_\nabla \right).$$

This equation can be interpreted as a balance of forces which allows to exactly preserve the shape of the wave-corpucle. *Equation (3.31) together with condition (3.27) constitutes the criterion for the preservation of the shape $|\psi|$.*

3.2. Trajectory and phase of a wave-corpucle. If equation (3.31) holds, the phase function S can be found by integrating the exact 1-form:

$$(3.32) \quad S(t, \mathbf{y}) = \frac{1}{\chi} \int_\Gamma \left(\frac{1}{2} m\mathbf{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A} - \frac{1}{2m} \frac{q^2}{c^2} \check{\mathbf{A}}^2 - q\varphi \right) dt + \left(m\mathbf{v} + \frac{q}{c} \mathbf{A}_\nabla \right) \cdot d\mathbf{y}$$

where Γ is a curve in time-space connecting $(0, \mathbf{0})$ with (t, \mathbf{y}) . Since equation (3.31) is fulfilled, the integral does not depend on the curve. In particular, we take as $\Gamma = \Gamma_0$ a curve formed by two straight-line segments: the first segment from $(0, \mathbf{0})$ to $(t, \mathbf{0})$ and the second from $(t, \mathbf{0})$ to (t, \mathbf{y}) . That yields

$$(3.33) \quad \begin{aligned} S(t, \mathbf{y}) &= \frac{1}{\chi} \int_{\Gamma_0} \left(\frac{1}{2} m\mathbf{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A} - \frac{1}{2m} \frac{q^2}{c^2} \check{\mathbf{A}}^2 - q\varphi \right) dt + \left(m\mathbf{v} + \frac{q}{c} \mathbf{A}_\nabla \right) \cdot d\mathbf{y} \\ &= m\mathbf{v} \cdot \mathbf{y} + \frac{q}{c} \mathbf{y} \cdot \mathbf{A}(t, \mathbf{0}) + s_p(t) + s_{p2}(t, \mathbf{y}), \end{aligned}$$

where

$$(3.34) \quad s_p(t) = \frac{1}{\chi} \int_0^t \left(\frac{1}{2} m \mathbf{v}^2(t) + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}(t, \mathbf{0}) - q\varphi(t, 0) \right) dt,$$

$$(3.35) \quad s_{p2}(t, \mathbf{y}) = \frac{q}{\chi c} \int_0^1 \mathbf{y} \cdot (\mathbf{A}(t, s\mathbf{y}) - \mathbf{A}(t, \mathbf{0})) ds,$$

and $s_{p2}(t, \mathbf{y})$ is at least quadratic in \mathbf{y} . In the above derivation we used that $\check{\mathbf{A}}(t, \mathbf{0}) = 0$, $\mathbf{y} \cdot \check{\mathbf{A}}(t, \mathbf{y}) = 0$, and $\mathbf{A}_\nabla(t, \mathbf{0}) = \mathbf{A}(t, \mathbf{0})$.

Singling out the linear part of the phase we can write above formulas in the form

$$(3.36) \quad S(t, \mathbf{y}) = \frac{1}{\chi} m \tilde{\mathbf{v}} \cdot \mathbf{y} + s_p(t) + s_{p2}(t, \mathbf{y}),$$

where

$$(3.37) \quad \tilde{\mathbf{v}} = \mathbf{v} + \frac{q}{mc} \mathbf{A}(t, \mathbf{0}),$$

$$(3.38) \quad s_p(t) = \frac{1}{\chi} \int_0^t \left(\frac{1}{2} m \tilde{\mathbf{v}}^2(t) - \frac{q^2}{2mc^2} (\mathbf{A}(t, \mathbf{0}))^2 - q\varphi(t, 0) \right) dt.$$

The integrability condition (3.31) can be written now in the form

$$(3.39) \quad m \partial_t^2 \mathbf{r} = \frac{q}{c} \nabla (\mathbf{v} \cdot \mathbf{A}) - \frac{q^2}{c^2} \frac{1}{2m} \nabla \check{\mathbf{A}}^2 - q \nabla \varphi - \frac{q}{c} \partial_t \mathbf{A}_\nabla.$$

The above equation together with (3.27) is a system of equations which involves the EM potentials φ, \mathbf{A} and the corpuscle center trajectory $\mathbf{r}(t)$. Its fulfillment guarantees that the wave-corpuscle preserves its shape in the dynamics described by the NLS equation. We refer to equations (3.39), (3.27) *non-relativistic wave-corpuscle dynamic balance conditions*. Obviously the conditions do not depend on the nonlinearity G .

By setting $\mathbf{y} = 0$ in (3.39) and taking into account that $\check{\mathbf{A}}_{\text{ex}}(t, 0) = 0$ we obtain the *point balance condition*:

$$(3.40) \quad m \partial_t^2 \mathbf{r} = \frac{q}{c} \nabla (\mathbf{v} \cdot \mathbf{A})(t, 0) - q \nabla \varphi(t, 0) - \frac{q}{c} \partial_t \mathbf{A}(t, 0).$$

To interpret this condition, we note that $\mathbf{y} = 0$ in the moving frame in the above equation corresponds to $\mathbf{x} = \mathbf{r}$ in the resting frame and

$$(3.41) \quad \partial_t \mathbf{A}(t, \mathbf{x})_{\mathbf{x}=\mathbf{r}} = \partial_t \mathbf{A}(t, \mathbf{y})_{\mathbf{y}=\mathbf{0}} - \mathbf{v} \cdot \nabla \mathbf{A}(t, \mathbf{y})_{\mathbf{y}=\mathbf{0}},$$

hence the point balance condition (3.40) takes in the resting frame the form

$$(3.42) \quad m \partial_t^2 \mathbf{r} = \frac{q}{c} \nabla (\mathbf{v} \cdot \mathbf{A})(t, \mathbf{r}) - q \nabla \varphi(t, \mathbf{r}) - \frac{q}{c} \partial_t \mathbf{A}(t, \mathbf{r}) - \frac{q}{c} \mathbf{v} \cdot \nabla \mathbf{A}(t, \mathbf{r}).$$

Recall that the expression for the Lorentz force in terms of the EM potentials is given by the formula

$$\mathbf{f}_{\text{Lor}} = -q \nabla \varphi - \frac{q}{c} \partial_t \mathbf{A} + \frac{q}{c} \nabla (\mathbf{v} \cdot \mathbf{A}) - \frac{q}{c} (\mathbf{v} \cdot \nabla) \mathbf{A},$$

therefore the right-hand side of (3.42) coincides with the Lorentz force (1.10). Hence, the point balance condition (3.40) in \mathbf{x} -coordinates has the form

$$(3.43) \quad m \partial_t^2 \mathbf{r} = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}),$$

$$(3.44) \quad \mathbf{f}_{\text{Lor}}(t, \mathbf{r}) = q \mathbf{E}(t, \mathbf{r}) + \frac{q}{c} \mathbf{v} \times \mathbf{B}(t, \mathbf{r})$$

where EM fields \mathbf{E} and \mathbf{B} are given by (1.11). Hence the point balance condition coincides with Newton's law of motion (1.9) of a charged point subjected to the Lorentz force \mathbf{f}_{Lor} .

We always assume that the trajectory $\mathbf{r}(t)$ satisfies the point balance condition (3.43), therefore the integrability condition (3.39) after taking into account (3.40) can be written in the form

$$(3.45) \quad \begin{aligned} & \frac{q}{c} (\nabla(\mathbf{v} \cdot \mathbf{A}(t, \mathbf{y}) - \nabla(\mathbf{v} \cdot \mathbf{A})(t, \mathbf{0}))) - \frac{q^2}{c^2} \frac{1}{2m} \nabla \dot{\mathbf{A}}^2 - q (\nabla \varphi(t, \mathbf{y}) - \nabla \varphi(t, \mathbf{0})) \\ & - \frac{q}{c} \partial_t (\mathbf{A}_{\nabla}(t, \mathbf{y}) - \mathbf{A}_{\nabla}(t, \mathbf{0})) = \mathbf{0} \end{aligned}$$

where the left hand side explicitly vanishes for spatially constant fields \mathbf{A} and constant $\nabla \varphi$.

Now let us discuss the universality of the non-relativistic dynamic balance conditions. The conditions (3.27), (3.45) of preservation of the shape $|\psi|$ are *universal*, namely they do not depend on the nonlinearity G or on the form factor $\dot{\psi}$. Recall that when deriving the conditions we set coefficients at $\nabla \dot{\psi}$ and $\dot{\psi}$ in (3.16) to be zero. We would like to show that this is a necessary requirement for the conditions to be universal.

Equation (3.16) has the form

$$Q_0 \dot{\psi}(r) + \dot{\psi}'(r) Q_1 \cdot \mathbf{y} / r = 0,$$

where $\dot{\psi}(r)$ is a form factor, and coefficients Q_0, Q_1 obviously do not depend on $\dot{\psi}$. If $Q_1 \cdot \mathbf{y}$ is not identically zero, we can separate the variables:

$$\frac{Q_0}{r Q_1 \cdot \mathbf{y}} = -\partial_r \ln \dot{\psi}(r),$$

where the left-hand side is $\dot{\psi}$ -independent and the right-hand side depends on $\dot{\psi}$. It is impossible, and hence $Q_1 \cdot \mathbf{y}$ must be zero as well Q_0 , leading to equations (3.17), (3.18).

3.3. Wave-corpuscles in the EM field. The simplest example of fields φ, \mathbf{A} for which dynamic balance conditions are fulfilled are spatially constant $\varphi(t), \mathbf{A}(t)$; equations (3.27), (3.45) are obviously fulfilled. Now we construct a more general example of the fulfillment of the condition (3.45). In this example we prescribe arbitrary EM potentials which are linear in \mathbf{x} , take a trajectory which satisfies Newton's law (3.43), and assume that second and higher order components of the EM potentials expansion at $\mathbf{r}(t)$ must satisfy certain restrictions. Namely, we assume that \mathbf{A} involves a linear in $\mathbf{y} = \mathbf{x} - \mathbf{r}$ part which can be given by an arbitrary 3×3 matrix $\mathbf{A}_1(t)$ with time-dependent elements:

$$(3.46) \quad \mathbf{A}_1(t, \mathbf{y}) = \mathbf{A}_1(t) \mathbf{y} = \mathbf{y} \cdot \nabla \mathbf{A}(t, \mathbf{0}).$$

We assume that quadratic and higher order components of the sphere-tangent part of \mathbf{A} are set to zero, but allow an arbitrary potential part

$$(3.47) \quad \mathbf{A}(t, \mathbf{y}) = \mathbf{A}(t, \mathbf{0}) + \mathbf{A}_1(t, \mathbf{y}) + \mathbf{A}_{\nabla 2}(t, \mathbf{y}),$$

where $\mathbf{A}_{\nabla 2}$ has at least the second order zero at the origin and is potential:

$$\mathbf{A}_{\nabla 2}(t, \mathbf{y}) = \nabla P_3$$

with an arbitrary P_3 . For such a field, its potential and sphere-tangent parts are given respectively by formulas

$$(3.48) \quad \mathbf{A}_\nabla(t, \mathbf{y}) = \nabla P_1 + \nabla P_2 + \nabla P_3,$$

$$(3.49) \quad \check{\mathbf{A}}(t, \mathbf{y}) = \frac{1}{2} \mathbf{A}_1(t) \mathbf{y} - \frac{1}{2} \mathbf{A}_1^\top(t) \mathbf{y},$$

where

$$(3.50) \quad P_1 = \mathbf{y} \cdot \mathbf{A}(t, \mathbf{0}), \quad P_2(t, \mathbf{y}) = \frac{1}{2} (\mathbf{y} \cdot \mathbf{A}_1(t) \mathbf{y}),$$

\mathbf{A}_1^\top stands for \mathbf{A}_1 transposed, and P_3 is a function with zero of at least the third degree at the origin. Note that an action of an anti-symmetric matrix can be written using the cross product, therefore (3.49) can be written as follows:

$$(3.51) \quad \check{\mathbf{A}}(t, \mathbf{y}) = \frac{1}{2} \check{\mathbf{B}}(t) \times \mathbf{y},$$

where $\check{\mathbf{B}} = \nabla \times \check{\mathbf{A}}$. The electric potential φ has the form

$$(3.52) \quad \varphi = \varphi(t, \mathbf{0}) + \mathbf{y} \cdot \nabla \varphi(t, \mathbf{0}) + \varphi_2(t, \mathbf{y}),$$

where $\varphi(t, \mathbf{0})$ and $\nabla \varphi(t, \mathbf{0})$ are given continuously differentiable functions of t , and $\varphi_2(t, \mathbf{y})$ has order two or higher in \mathbf{y} and is subject to the condition (3.54) we formulate below.

Let us verify the conditions which guarantee that the wave-corpuscle is an exact solution. Notice that condition (3.27) for $\check{\mathbf{A}}$ defined by (3.49) is fulfilled. Hence, it is sufficient to satisfy (3.45), which takes the form

$$(3.53) \quad \frac{q}{c} \nabla (\mathbf{v} \cdot \nabla P_3) - \frac{q^2}{c^2} \frac{1}{2m} \nabla \check{\mathbf{A}}^2 - q \nabla \varphi_2(t, \mathbf{y}) - \frac{q}{c} \partial_t (\nabla P_2(t, \mathbf{y}) + \nabla P_3(t, \mathbf{y})) = \mathbf{0}.$$

To satisfy (3.53), we set

$$(3.54) \quad \varphi_2(t, \mathbf{y}) = -\frac{q}{c^2} \frac{1}{2m} \check{\mathbf{A}}^2 - \frac{1}{c} \partial_t (P_2(t, \mathbf{y}) + P_3(t, \mathbf{y})) + \frac{1}{c} \mathbf{v} \cdot \nabla P_3.$$

We want now to determine the phase S of the wave-corpuscle using (3.34), (3.35). If P_3 is homogenous of third degree, we obtain

$$(3.55) \quad s_p(t) = \frac{1}{\chi} \int_0^t \left(\frac{1}{2} m \mathbf{v}^2(t) + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}(t, \mathbf{0}) - q \varphi(t, \mathbf{0}) \right) dt,$$

$$(3.56) \quad \begin{aligned} s_{p2}(t, \mathbf{y}) &= \frac{q}{c} \frac{1}{\chi} \int_0^1 \mathbf{y} \cdot \mathbf{A}_1(t, \mathbf{y}) s ds + \frac{q}{c} \frac{1}{\chi} \int_0^1 \mathbf{y} \cdot \mathbf{A}_{\nabla 2}(t, \mathbf{y}) s^2 ds \\ &= \frac{q}{2c} \frac{1}{\chi} \mathbf{y} \cdot \mathbf{A}_1(t, \mathbf{y}) + \frac{q}{3c} \frac{1}{\chi} \mathbf{y} \cdot \mathbf{A}_{\nabla 2}(t, \mathbf{y}). \end{aligned}$$

In this case the phase function of the wave-corpuscle involves a term $s_{p2}(t, \mathbf{y})$.

Formula (3.1) takes the form

$$(3.57) \quad \psi(t, \mathbf{x}) = e^{iS} \psi^\circ(|\mathbf{x} - \mathbf{r}(t)|),$$

$$(3.58) \quad S(t, \mathbf{y}) = m \frac{1}{\chi} \mathbf{v} \cdot \mathbf{y} + \frac{q}{c} \frac{1}{\chi} \mathbf{y} \cdot \mathbf{A}(t, \mathbf{0}) + s_p(t) + s_{p2}(t, \mathbf{y}).$$

where $\mathbf{y} = \mathbf{x} - \mathbf{r}(t)$, $\mathbf{v} = \partial_t \mathbf{r}$.

If the external magnetic field satisfies the following anti-symmetry conditions

$$(3.59) \quad \begin{aligned} \mathbf{y} \cdot \mathbf{A}_1(t, \mathbf{y}) &= 0, \\ \mathbf{y} \cdot \mathbf{A}_{\nabla^2}(t, \mathbf{y}) &= 0 \end{aligned}$$

then $s_{p2} = 0$, and the phase function is linear in \mathbf{y} .

The above calculations can be summarized in the following statement

Theorem 4. *Let the potentials φ, \mathbf{A} have the form (3.52), (3.46), (3.47) where $\varphi(t, \mathbf{0})$ and $\nabla\varphi(t, \mathbf{0})$ are given continuously differentiable functions of t . Suppose also that $\mathbf{A}(t, \mathbf{0})$ is a given continuously differentiable function of t , $\mathbf{A}_1(t)$ is an arbitrary 3×3 matrix which is continuously differentiable depends on t , ∇P_3 is a continuously differentiable function of t and \mathbf{y} . Let the quadratic part φ_2 of the potential φ satisfy (3.54). Let also trajectory $\mathbf{r}(t)$ satisfies Newton's equation (3.43)-(3.44). Then the wave-corpuscle defined by formula (3.57)-(3.58) is a solution to NLS equation (1.1).*

Remark 3. *The above construction does not depend on the nonlinearity $G' = G'_a$ as long as (1.6) is satisfied. It is also uniform with respect to $a > 0$, and the dependence on a in (3.57) is only through $\check{\psi}(|\mathbf{x} - \mathbf{r}|) = a^{-3/2}\check{\psi}_1(a^{-1}|\mathbf{x} - \mathbf{r}|)$. Obviously, if $\psi(t, \mathbf{x})$ is defined by (3.1) then $|\psi(t, \mathbf{x})|^2 \rightarrow \delta(\mathbf{x} - \mathbf{r})$ as $a \rightarrow 0$.*

Remark 4. *The form (3.1) of exact solutions is the same as the WKB ansatz in the quasi-classical approach, [41]. The trajectories of the charge center coincide with trajectories that can be found by applying well-known quasiclassical asymptotics to solutions of (1.1) if one neglects the nonlinearity. Note though that there are two important effects of the nonlinearity not presented in the standard quasiclassical approach. First of all, due to the nonlinearity the charge preserves its shape in the course of evolution on unbounded time intervals whereas in the linear model any wavepacket disperses over time. Second of all, the quasiclassical asymptotic expansions produce infinite asymptotic series which provide for a formal solution, whereas the properly introduced nonlinearity as in (1.6), (3.6) allows one to obtain an exact solution. For a treatment of mathematical aspects of the approach to nonlinear wave mechanics based on the WKB asymptotic expansions we refer the reader to [41], [35] and references therein.*

Remark 5. *We can use in the definition of a wave-corpuscle (3.1) a radial form factor $\check{\psi}$ which instead of (1.6) satisfies the eigenvalue problem*

$$(3.60) \quad -\nabla^2 \check{\psi} + G'(|\check{\psi}|^2) \check{\psi} = \lambda \check{\psi}.$$

Obviously the above equation (3.60) can be considered as the steady state equation (1.6) with a modified nonlinearity $G' - \lambda$. Note that ψ is a solution of the NLS equation (1.1) with the nonlinearity G' if and only if the function $e^{-i\frac{\lambda}{2m}\lambda t}\psi$ is a solution of the NLS equation with the nonlinearity $G' - \lambda$. Therefore the wave-corpuscle solutions constructed in Theorem 4 based on $\check{\psi}$ with the nonlinearity $G' - \lambda$ provide solutions to the NLS equation with the original nonlinearity, one has only to add to the phase function S an additional term $\frac{\lambda}{2m}\lambda t$. Existence of many solutions of the nonlinear eigenvalue problems of the form (3.60) was proved in many papers, see [14], [15], [18], [30]. Therefore there are many wave-corpuscle solutions of a given NLS equation with the same trajectory of motion $\mathbf{r}(t)$ and the same EM fields which correspond to different form factors.

3.4. Wave-corpuscles as concentrating solutions. The wave-corpuscles constructed in Theorem 4 provide an example of concentrating solutions. In order to see that we have to verify all the requirements of Definition 4. It is important to note that the dependence on a in (3.1) is only through $\dot{\psi}(|\mathbf{x} - \mathbf{r}|) = a^{-3/2}\dot{\psi}_1(a^{-1}|\mathbf{x} - \mathbf{r}|)$ where $\dot{\psi}_1$ is a given smooth function. The verification of conditions imposed in the Definition 4 is straightforward. For instance,

$$(3.61) \quad \begin{aligned} \bar{\rho}_n &= q \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} a^{-3} \left| \dot{\psi}_1(a^{-1}|\mathbf{x} - \mathbf{r}|) \right|^2 d\mathbf{x} \\ &= q \int_{\Omega(\mathbf{0}, R_n/a)} \dot{\psi}_1^2(|\mathbf{y}|) d\mathbf{y} \rightarrow q \int_{\mathbb{R}^3} \dot{\psi}_1^2(|\mathbf{y}|) d\mathbf{y}, \end{aligned}$$

where $R_n/a \rightarrow \infty$ according to (2.9) and using that we obtain (2.25) and (2.26). To estimate integral (2.24) we note that according to the definition of momentum density (1.16), (1.2)

$$(3.62) \quad \mathbf{P} = i\chi \frac{1}{2} \left[\frac{\nabla \psi^*}{\psi^*} - \frac{\nabla \psi}{\psi} + 2 \frac{i\mathbf{q}}{\chi c} \mathbf{A} \right] \psi^* \psi,$$

and for the wave-corpuscles

$$(3.63) \quad \mathbf{P} = \chi \left[\nabla S - \frac{q}{\chi c} \mathbf{A} \right] \dot{\psi}^2.$$

Since the phase S in (3.58) is a smooth function which does not depend on a , and \mathbf{A} satisfies (2.17), the estimate (2.24) can be obtained using (3.61). Using (3.63) and (3.61) we obtain (2.33):

$$(3.64) \quad |Q_{01}| \leq |t - t_0| \max_{T_- \leq s \leq T_+} |\partial_t \hat{\mathbf{r}}| \max_{T_- \leq s \leq T_+} \int_{\partial\Omega_n} |\mathbf{P}| d\sigma \rightarrow 0.$$

Similarly we obtain (2.34), (2.35) and (2.36). Note that $|\mathbf{E}_n(0) - \mathbf{E}_\infty(0)| = |\mathbf{E}(0) - \mathbf{E}_\infty(0)| = 0$ and $|\mathbf{B}_n(0) - \mathbf{B}_\infty(0)| = |\mathbf{B}(0) - \mathbf{B}_\infty(0)| = 0$ according to (2.19). Using continuity of $\mathbf{E}(t, \mathbf{y})$ and $\mathbf{B}(t, \mathbf{y})$ we conclude that $|\mathbf{E}_n - \mathbf{E}_\infty| \rightarrow 0$ and $|\mathbf{B}_n - \mathbf{B}| \rightarrow 0$ in Ω_n since $R_n \rightarrow 0$. Therefore fulfillment of (2.39) follows from (3.61) and (3.63).

To obtain (2.32) we split the tensor T^{ij} into diagonal and non-diagonal parts given by (5.21) and (5.22):

$$(3.65) \quad T^{ij} = T_{i \neq j}^{ij} + T_{i=j}^{ij}.$$

According to (2.27)

$$Q_0 = Q_{0, \neq} + Q_{0, =}$$

where

$$Q_{0, \neq} = \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}}_i T_{i \neq j}^{ij} d\sigma dt', \quad Q_{0, =} = \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}}_i T_{i=j}^{ij} d\sigma dt'.$$

According to (5.22) to estimate $|Q_{0, \neq}|$ it is sufficient to prove that

$$(3.66) \quad \max_{T_- \leq t \leq T_+} \int_{\partial\Omega_n} \left(\frac{\chi^2}{2m} |\nabla \psi(t, \mathbf{x})|^2 + |\psi(t, \mathbf{x})|^2 \right) d\sigma \rightarrow 0.$$

Using this estimate and (5.22) we obtain

$$(3.67) \quad |Q_{0, \neq}| \leq |t - t_0| \max_{T_- \leq s \leq T_+} \int_{\partial\Omega_n} |T_{i \neq j}^{ij}| d\sigma \rightarrow 0.$$

To estimate $Q_{0,=}$ we use (5.21)

$$(3.68) \quad \int_{\partial\Omega_n} \bar{\mathbf{n}}_i T^{ii} d\sigma = - \int_{\partial\Omega_n} \frac{\chi^2}{m} \bar{\mathbf{n}}_i \tilde{\partial}_i \psi \tilde{\partial}_i^* \psi^* d\sigma \\ + \int_{\partial\Omega_n} \left[\frac{\chi^2}{2m} \left(G(|\psi|^2) + |\tilde{\nabla}\psi|^2 \right) + i\frac{\chi}{2} \left(\psi \tilde{\partial}_i^* \psi^* - \psi^* \tilde{\partial}_i \psi \right) \right] \sum_i \bar{\mathbf{n}}_i d\sigma$$

Since $|\psi|^2 = |\dot{\psi}|^2$ is a radial function and $\bar{\mathbf{n}}_i = y_i/|\mathbf{y}|$ is odd with respect to y_i -reflection, we see that in the above integral

$$(3.69) \quad \int_{\partial\Omega_n} G(|\psi|^2) \bar{\mathbf{n}}_i d\sigma = 0.$$

We also note that

$$(3.70) \quad i\frac{\chi}{2} \left(\psi \tilde{\partial}_i^* \psi^* - \psi^* \tilde{\partial}_i \psi \right) = i\frac{\chi}{2} (\psi \partial_t \psi^* - \psi^* \partial_t \psi) + q\varphi |\psi|^2 \\ = \chi |\psi|^2 \partial_t S + q\varphi |\psi|^2$$

where $\partial_t S$ and φ are bounded functions in Ω_n . Under the assumption (3.66) straightforward estimates produce that

$$i\frac{\chi}{2} \int_{\partial\Omega_n} \left(\psi \tilde{\partial}_i^* \psi^* - \psi^* \tilde{\partial}_i \psi \right) d\sigma \rightarrow 0, \\ \int_{\partial\Omega_n} \frac{\chi^2}{m} \bar{\mathbf{n}}_i \tilde{\partial}_i \psi \tilde{\partial}_i^* \psi^* d\sigma \rightarrow 0, \\ \int_{\partial\Omega_n} \frac{\chi^2}{2m} |\tilde{\nabla}\psi|^2 d\sigma \rightarrow 0$$

uniformly on $[T_-, T_+]$. Hence, if (3.66) holds, we obtain that

$$(3.71) \quad Q_{0,=} \rightarrow 0,$$

and taking into account (3.67) we conclude that (2.32) holds.

Now let us prove that (3.66) holds under certain decay conditions. Since the phase S in (3.58) is bounded and have bounded derivatives in $\Omega(\hat{\mathbf{r}}(t), R_n)$, to obtain (3.66) it is sufficient to estimate surface integrals of $|\nabla\dot{\psi}|^2$ and $|\dot{\psi}|^2$. Obviously,

$$(3.72) \quad \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} a^{-3} \left| \nabla \dot{\psi}_1(a^{-1}|\mathbf{x} - \mathbf{r}|) \right|^2 d\sigma \\ = 4\pi R_n^2 a^{-5} \left| \dot{\psi}'_1(R_n/a) \right|^2 = 4\pi a R_n^{-4} \theta^6 \left| \dot{\psi}'_1(\theta) \right|^2,$$

$$(3.73) \quad \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} a^{-3} \left| \dot{\psi}_1(a^{-1}|\mathbf{x} - \mathbf{r}|) \right|^2 d\sigma = \\ = 4\pi a^{-3} R_n^2 \left| \dot{\psi}_1(R_n/a) \right|^2 = 4\pi a R_n^{-2} \theta^4 \left| \dot{\psi}_1(\theta) \right|^2,$$

where $R_n/a = \theta \rightarrow \infty$, $R_n \rightarrow 0$. We assume that the form factor $\dot{\psi}_1(r)$ and its derivative $\dot{\psi}'_1(r)$ satisfy the following decay conditions:

$$(3.74) \quad \theta^2 \left| \dot{\psi}_1(\theta) \right| \leq C \text{ as } \theta \rightarrow \infty,$$

$$(3.75) \quad \theta^3 \left| \dot{\psi}'_1(\theta) \right| \leq C \text{ as } \theta \rightarrow \infty,$$

and we take such a_n, R_n that

$$(3.76) \quad a_n R_n^{-4} \rightarrow 0, \quad a_n \rightarrow 0, \quad R_n \rightarrow 0.$$

Under these assumptions we conclude that (3.66) is fulfilled.

Summing up the above arguments we obtain the following statement:

Theorem 5. *Let $\dot{\psi}'_1$ satisfy (3.74), (3.75) and a_n, R_n satisfy (3.76). Let $\psi = \psi_n$ be wave-corpucle solutions of the NLS equation constructed in Theorem 4. Then the sequence ψ_n concentrates at the trajectory $\hat{\mathbf{r}}(t) = \mathbf{r}(t)$.*

4. CONCENTRATION OF ASYMPTOTIC SOLUTIONS

From the proofs of Section 2 one can see that in the derivation of the Newtonian dynamics we use only the conservation laws (1.15) and (1.18). Since we use only asymptotic properties, it is natural to consider fields which satisfy NLS equations and the conservation laws not exactly, but approximately. Namely, we assume now that the conservation laws (1.15) and (1.18) are replaced by

$$(4.1) \quad \partial_t \rho + \partial_t \rho' + \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}' = 0,$$

$$(4.2) \quad \partial_t \mathbf{P} + \partial_t \mathbf{P}' + \partial_i T^{ij} + \partial_i T'^{ij} = \mathbf{f} + \mathbf{f}'$$

where charge density ρ , current density \mathbf{J} and tensor elements T^{ij} are defined by (1.13), (1.14) and (5.22) in terms of given functions $\psi, \varphi, \mathbf{A}$ and $\rho', \mathbf{J}', \mathbf{P}', T'^{ij}$, and quantities \mathbf{f}' are perturbation terms which vanish as $n \rightarrow \infty$. We show now how to modify the proofs of statements in Section 2 on concentrating solutions to obtain similar statements on concentrating asymptotic solutions.

In the proof of Lemma 3 equation (2.51) is replaced by (4.2). This leads to replacement of the equation (2.53) by the equation

$$(4.3) \quad \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{P}(t) \, d^3x - \int_{\Omega(\hat{\mathbf{r}}(t_0), R_n)} \mathbf{P}(t_0) \, d^3x - Q_{01} \\ - \int_{t_0}^t \int_{\Omega_n} \mathbf{f} \, dx dt' + Q_0 + Q'_0 = 0,$$

with

$$(4.4) \quad Q'_0 = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{P}'(t) \, d^3x - \int_{\Omega(\hat{\mathbf{r}}(t_0), R_n)} \mathbf{P}'(t_0) \, d^3x \\ - \int_{t_0}^t \int_{\partial\Omega_n} \mathbf{P}' \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \, d\sigma dt' + \int_{t_0}^t \int_{\Omega_n} \partial_i T'^{ij} \, dx dt' - \int_{t_0}^t \int_{\Omega_n} \mathbf{f}' \, dx dt'.$$

If

$$(4.5) \quad Q'_0 \rightarrow 0$$

the proof of Lemma 3 remains valid with $Q_{00} = Q_0 + Q'_0 - Q_{01}$, and we obtain the following lemma:

Lemma 5. *Let the conservation law (1.18) be replaced by (4.2) with condition (4.5) satisfied. Then the statement of Lemma 3 remains true:*

$$(4.6) \quad \int_{\Omega_n} \mathbf{P}_n(t) \, d\mathbf{x} = \int_{t_0}^t \int_{\Omega_n} \mathbf{f} \, dx dt' + \mathbf{p}_\infty + Q_{00}$$

where

$$(4.7) \quad Q_{00} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly on $[T_-, T_+]$.

Sufficient conditions for fulfillment of (4.5) are the following limit relations

$$(4.8) \quad \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{P}'(t) \, d\mathbf{x} \rightarrow 0,$$

$$(4.9) \quad \int_{t_0}^t \int_{\partial\Omega_n} \mathbf{P}' \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \, d\sigma dt' \rightarrow 0,$$

$$(4.10) \quad \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}}_i T^{ij'} \, d\sigma dt' \rightarrow 0,$$

$$(4.11) \quad \int_{t_0}^t \int_{\Omega_n} \mathbf{f}' \, d\mathbf{x} dt' \rightarrow 0$$

uniformly on the time interval $[T_-, T_+]$.

Let us take a look at the proof of Lemma 4. Since we replace the continuity equation (1.15) by (4.1), the equation (2.58) involves now additional terms:

$$(4.12) \quad \int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) \rho) \, d\mathbf{x} + \partial_t \mathbf{r} \int_{\Omega_n} \rho \, d\mathbf{x} + \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \mathbf{n} \cdot \mathbf{J} \, d\mathbf{x} + Q'_2 = \int_{\Omega_n} \mathbf{J} \, d\mathbf{x},$$

with

$$(4.13) \quad Q'_2 = \int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) \rho') \, d\mathbf{x} + \partial_t \mathbf{r} \int_{\Omega_n} \rho' \, d\mathbf{x} + \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \mathbf{n} \cdot \mathbf{J}' \, d\mathbf{x} - \int_{\Omega_n} \mathbf{J}' \, d\mathbf{x}.$$

We assume that

$$(4.14) \quad Q'_2 \rightarrow 0$$

and conclude that the statement of Lemma 4 holds:

Lemma 6. *Let the continuity equation (1.15) be replaced by (4.1) with condition (4.14) fulfilled. Then the statement of Lemma 4 is true.*

Note that according to (2.59) sufficient conditions for fulfillment of (4.14) are as follows:

$$(4.15) \quad \int_{\Omega_n} \rho' \, d\mathbf{x} \rightarrow 0,$$

$$(4.16) \quad \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \mathbf{n} \cdot \mathbf{J}' \, d\mathbf{x} \rightarrow 0,$$

$$(4.17) \quad \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho' \, d\sigma \rightarrow 0.$$

Now we consider the proof of Lemma 2. Equation (2.47) is replaced by

$$(4.18) \quad \bar{\rho}(t) - \bar{\rho}(t_0) - \int_{t_0}^t \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \mathbf{n} \rho \, d\mathbf{x} dt' + \int_{t_0}^t \int_{\partial\Omega_n} \mathbf{n} \cdot \mathbf{J} \, d\mathbf{x} dt' + Q'_3 = 0,$$

where

$$(4.19) \quad Q'_3 = \bar{\rho}'(t) - \bar{\rho}'(t_0) - \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \mathbf{n} \rho' d\mathbf{x} + \int_{\partial\Omega_n} \mathbf{n} \cdot \mathbf{J}' d\mathbf{x}.$$

The proof of Lemma 2 is preserved if we assume that

$$(4.20) \quad Q'_3 \rightarrow 0.$$

Lemma 7. *Let the continuity equation (1.15) be replaced by (4.1) with condition (4.20) fulfilled. Then the statement of Lemma 2 is true.*

Now we collect all the assumptions required for exact solutions in Section 2 that remain valid for asymptotic solutions in the following definition:

Definition 5 (Concentrating asymptotic solutions). *We assume all conditions of Definition 4 except condition (i), which is replaced by the following weaker condition: conservation laws (4.2) and (4.1) are fulfilled and conditions (4.14), (4.5), (4.20) hold. Then we say that asymptotic solutions of the NLS equation (1.1) concentrate at $\hat{\mathbf{r}}(t)$. We call $\hat{\mathbf{r}}(t)$ a concentration trajectory of the NLS equation in asymptotic sense if there exists a sequence of asymptotic solutions which concentrates at $\hat{\mathbf{r}}(t)$.*

Using Lemmas 5 and 6 instead of Lemmas 3 and 4 we obtain statements of Theorems 2 and 3 under the assumption that asymptotic solutions ψ of the NLS equation (1.1) concentrate at $\hat{\mathbf{r}}(t)$. In particular, we obtain

Theorem 6. *Assume that potentials $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ are defined and twice continuously differentiable in a domain $D \subset \mathbb{R} \times \mathbb{R}^3$, the trajectory $(t, \hat{\mathbf{r}}(t))$ lies in this domain and the limit potentials φ_∞ , \mathbf{A}_∞ are the restriction of fixed potentials φ, \mathbf{A} as in (2.20). Let EM fields $\mathbf{E}(t, \mathbf{x}), \mathbf{B}(t, \mathbf{x})$ be determined in terms of the potentials by formula (1.11). Let asymptotic solutions ψ of the NLS equation (1.1) asymptotically concentrate at $\hat{\mathbf{r}}(t)$. Then the trajectory $\hat{\mathbf{r}}$ satisfies Newton's law of motion (2.71).*

4.1. Point trajectories as trajectories of asymptotic concentration . Let us consider NLS equation (1.1) in a domain $D \subset \mathbb{R} \times \mathbb{R}^3$ and assume that potentials $\varphi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$ are defined and twice continuously differentiable in domain D . Let us consider equations (3.43)-(3.44) describing Newtonian dynamics of a point charge in EM field. Let us consider a solution $\mathbf{r}(t)$ of equations (3.43)-(3.44), and assume that the trajectory $(t, \mathbf{r}(t))$ lies in D on the time interval $T_- \leq t \leq T_+$. Now we construct asymptotic solutions of the NLS which concentrate at $\mathbf{r}(t)$.

As a first step we find the linear part of $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ at $\mathbf{r}(t)$ as in (2.20)

$$(4.21) \quad \begin{aligned} \varphi_\infty(t, \mathbf{x}) &= \varphi(t, \mathbf{r}) + (\mathbf{x} - \mathbf{r}) \nabla \varphi(t, \mathbf{r}), \\ \mathbf{A}_\infty(t, \mathbf{x}) &= \mathbf{A}(t, \mathbf{r}) + (\mathbf{x} - \mathbf{r}) \nabla \mathbf{A}(t, \mathbf{r}), \end{aligned}$$

As a second step we construct the auxiliary potentials

$$(4.22) \quad \begin{aligned} \varphi_{\text{aux}}(t, \mathbf{x}) &= \varphi(t, \mathbf{r}) + (\mathbf{x} - \mathbf{r}) \nabla \varphi(t, \mathbf{r}) + \varphi_2(t, \mathbf{x} - \mathbf{r}), \\ \mathbf{A}_{\text{aux}}(t, \mathbf{x}) &= \mathbf{A}(t, \mathbf{r}) + (\mathbf{x} - \mathbf{r}) \nabla \mathbf{A}(t, \mathbf{r}), \end{aligned}$$

where φ_2 is determined by (3.54) and $\varphi_2(t, \mathbf{x} - \mathbf{r})$ is quadratic with respect to $(\mathbf{x} - \mathbf{r})$.

The wave-corpuscle ψ described in Theorem 4 is a solution of the NLS equation with the potentials $(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}})$, and hence it exactly satisfies the corresponding conservation laws. From fulfillment of (1.15) and (1.18) for $\mathbf{P}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}})$, $\mathbf{J}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}})$, $T^{ij}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}})$, $\mathbf{f}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}})$ we obtain fulfillment of (4.1), (4.2) with

$$(4.23) \quad \mathbf{P}' = \mathbf{P}(\varphi, \mathbf{A}) - \mathbf{P}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}),$$

$$(4.24) \quad \mathbf{J}' = \mathbf{J}(\varphi, \mathbf{A}) - \mathbf{J}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}),$$

$$(4.25) \quad \mathbf{f}' = \mathbf{f}(\varphi, \mathbf{A}) - \mathbf{f}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}),$$

$$(4.26) \quad T^{ij'} = T^{ij}(\varphi, \mathbf{A}) - T^{ij}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}).$$

The expression for ρ does not depend on the potentials, therefore

$$(4.27) \quad \rho' = 0.$$

Now we need to verify that conditions (4.14), (4.5), (4.20) hold for the wave-corpuscle ψ .

From (1.16), (3.63) we see that

$$(4.28) \quad \mathbf{P}(\varphi, \mathbf{A}) - \mathbf{P}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}) = -\frac{q}{c}(\mathbf{A} - \mathbf{A}_{\text{aux}})\psi^*\psi,$$

and according to (1.17)

$$(4.29) \quad \mathbf{J}(\varphi, \mathbf{A}) - \mathbf{J}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}) = -\frac{q^2}{mc}(\mathbf{A} - \mathbf{A}_{\text{aux}})\psi^*\psi.$$

From the expression for the Lorentz density (1.19) we obtain

$$\begin{aligned} \mathbf{f}(\varphi, \mathbf{A}) - \mathbf{f}(\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}) &= \rho(\mathbf{E} - \mathbf{E}_{\text{aux}}) + \frac{1}{c}(\mathbf{J} \times \mathbf{B} - \mathbf{J}_{\text{aux}} \times \mathbf{B}_{\text{aux}}) = \\ &= \rho(\mathbf{E} - \mathbf{E}_{\text{aux}}) + \frac{1}{c}((\mathbf{J} - \mathbf{J}_{\text{aux}}) \times \mathbf{B} + \mathbf{J}_{\text{aux}} \times (\mathbf{B} - \mathbf{B}_{\text{aux}})). \end{aligned}$$

Using (3.63) and (1.17) we rewrite expression for \mathbf{f}' in the form

$$(4.30) \quad \begin{aligned} \mathbf{f}' &= \rho(\mathbf{E} - \mathbf{E}_{\text{aux}}) \\ &+ \frac{1}{c} \left(-\frac{q^2}{mc} \psi^2 (\mathbf{A} - \mathbf{A}_{\text{aux}}) \times \mathbf{B} + \chi \frac{q}{m} \psi^2 \left[\nabla S - \frac{q}{\chi c} \mathbf{A}_{\text{aux}} \right] \times (\mathbf{B} - \mathbf{B}_{\text{aux}}) \right). \end{aligned}$$

To estimate the difference of tensor elements (4.26) we use (5.22) and (5.21). Note that according to the construction of φ_{aux} and \mathbf{A}_{aux} the differences $\varphi - \varphi_{\text{aux}}$ and $\mathbf{A} - \mathbf{A}_{\text{aux}}$ have the second order zero at $\mathbf{r}(t)$. Hence the differences $\mathbf{E} - \mathbf{E}_{\text{aux}}$ and $\mathbf{B} - \mathbf{B}_{\text{aux}}$ have a zero of the first order at $\mathbf{x} = \mathbf{r}$:

$$(4.31) \quad |\mathbf{A} - \mathbf{A}_{\text{aux}}| \leq C|\mathbf{x} - \mathbf{r}|^2, \quad |\varphi - \varphi_{\text{aux}}| \leq C|\mathbf{x} - \mathbf{r}|^2,$$

$$(4.32) \quad |\mathbf{E} - \mathbf{E}_{\text{aux}}| \leq C|\mathbf{x} - \mathbf{r}|, \quad |\mathbf{B} - \mathbf{B}_{\text{aux}}| \leq C|\mathbf{x} - \mathbf{r}|,$$

and they are vanishingly small in $\Omega(\mathbf{r}(t), R_n)$. Now we estimate terms which enter (4.4), (4.13), (4.19).

Lemma 8. *Let $\mathbf{P}', \mathbf{J}', \mathbf{f}', T^{ij'}$ be defined by (4.23)-(4.26), conditions (3.74), (3.75) and (3.76) fulfilled. Then (4.8)-(4.11) and (4.15)-(4.17) and (4.20) hold.*

Proof. The proof is based on inequalities (4.32) and (4.31). To obtain (4.8) we use (4.28) and (3.61):

$$\begin{aligned} \left| \int_{\Omega(\mathbf{r}(t), R_n)} \mathbf{P}'(t) \, d\mathbf{x} \right| &\leq C \int_{\Omega(\mathbf{r}(t), R_n)} |\mathbf{x} - \mathbf{r}|^2 \dot{\psi}_a^2(|\mathbf{x} - \mathbf{r}|) \, d\mathbf{x} \\ &\leq CR_n^2 \int_{\Omega(\mathbf{0}, R_n/a)} \dot{\psi}_a^2(|\mathbf{y}|) \, d\mathbf{y} \leq C_1 R_n^2, \end{aligned}$$

and we obtain (4.8) for $R_n \rightarrow 0$. To obtain (4.9) we observe that according to (4.28)

$$\begin{aligned} (4.33) \quad &\left| \int_{\partial\Omega(\mathbf{r}(t), R_n)} \mathbf{P}' \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \, d\sigma \right| \\ &\leq C \int_{\partial\Omega(\mathbf{r}(t), R_n)} |\mathbf{x} - \mathbf{r}|^2 \dot{\psi}_a^2(|\mathbf{x} - \mathbf{r}|) \, d\sigma = 4\pi CR_n^4 a^{-3} \dot{\psi}_a^2(R_n/a) \\ &= 4\pi CR_n \theta^3 \dot{\psi}_a^2(\theta) \rightarrow 0 \end{aligned}$$

Similarly we obtain (4.29). A straightforward estimate of (4.30) yields (4.11). Note that according to (5.21) and (5.19) the terms in T^{ij} involving G do not depend on (φ, \mathbf{A}) , and hence G does not enter T^{ij} . According to (5.22), (5.21), (3.70) and (1.2) $T^{ij}(\varphi, \mathbf{A})$ is a quadratic function of potentials and T^{ij} equals sum of terms, every of which involves factors $\mathbf{A} - \mathbf{A}_{\text{aux}}$ or $\varphi - \varphi_{\text{aux}}$ which satisfy (4.31) and also factors $\psi \nabla \psi^*$ or $\nabla \psi \psi^*$ or $\psi \psi^*$. The factors $\mathbf{A} - \mathbf{A}_{\text{aux}}$ or $\varphi - \varphi_{\text{aux}}$ in $\Omega(\mathbf{r}(t), R_n)$ produce coefficient CR_n^2 . All the terms are easily estimated, for example an elementary inequality

$$\int_{\partial\Omega(\mathbf{r}(t), R_n)} \left| \dot{\psi}_a \right| \left| \nabla \dot{\psi}_a \right| \, d\sigma \leq \frac{1}{2} \int_{\partial\Omega(\mathbf{r}(t), R_n)} \left| \dot{\psi}_a \right|^2 + \left| \nabla \dot{\psi}_a \right|^2 \, d\sigma$$

allows to use (3.72)-(3.73), and we obtain that

$$\int_{\partial\Omega(\mathbf{r}(t), R_n)} \left| \dot{\psi}_a \right| \left| \nabla \dot{\psi}_a \right| \, d\sigma \rightarrow 0.$$

Therefore (4.10) is fulfilled. Since $\rho' = 0$ (4.15) and (4.17) are fulfilled, (4.16) follows from 4.8) and (1.17). To check condition (4.20) we note that $\rho' = 0$ and after using (1.17) we estimate the surface integral in (4.19) involving \mathbf{J}' similarly to (4.33). \square

Theorem 7. *Let potentials $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ be twice continuously differentiable in a domain D . Let form factor $\dot{\psi}_1$ satisfy conditions (3.74), (3.75). Let sequences a_n, R_n satisfy (3.76). Let $\mathbf{r}(t)$ be a solution of equations (3.43)-(3.44) with trajectory in D . Then wave-corpuscles constructed in Theorem 5 based on $\mathbf{r}(t)$ and potentials $\varphi_{\text{aux}}, \mathbf{A}_{\text{aux}}$ given by (4.22) provide asymptotic solutions in the sense of Definition 5 that concentrate at $\mathbf{r}(t)$.*

As a corollary we obtain the following theorem.

Theorem 8. *Let $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ are defined and twice continuously differentiable in a domain D . Then a trajectory $\mathbf{r}(t)$ which lies in D is a concentration trajectory of asymptotic solutions of NLS equation (1.1) if and only if it satisfies Newton's law (3.43) with the Lorentz force defined by (3.44).*

Since asymptotic solutions in the sense of Definition 5 in Theorem 7 are constructed as wave-corpuscles, we obtain the following corollary.

Corollary 1. *If a sequence of concentrating asymptotic solutions concentrates at a trajectory $\mathbf{r}(t)$, then there exists a sequence of concentrating wave-corpuscle asymptotic solutions which concentrate at the same trajectory.*

5. APPENDIX 1: LAGRANGIAN FIELD FORMALISM FOR NONLINEAR SCHRÖDINGER EQUATION

To derive the conservation laws for the NLS equation it is convenient to use relativistic notation. We introduce the following 4-differential operators

$$(5.1) \quad \partial_\mu = \left(\frac{1}{c} \partial_t, \nabla \right), \quad \partial^\mu = \left(\frac{1}{c} \partial_t, -\nabla \right),$$

where the indices μ take four values $\mu = 0, 1, 2, 3$ and count components of the right-hand sides; in particular $\partial_0 = \frac{1}{c} \partial_t$. We also denote

$$(5.2) \quad \psi^{;\mu} = \partial^\mu \psi, \quad \psi_{;\mu} = \partial_\mu \psi.$$

The EM 4-potential is given by the formula

$$(5.3) \quad A = A^\mu = (\varphi, \mathbf{A}),$$

where $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ are given potentials, and the EM power tensor is defined as follows:

$$(5.4) \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

The covariant derivatives which involve the EM fields are defined by (1.2), and corresponding 4-differential operators have the form

$$(5.5) \quad \tilde{\partial}_\mu = \left(\frac{1}{c} \tilde{\partial}_t, \tilde{\nabla} \right), \quad \tilde{\partial}^\mu = \left(\frac{1}{c} \tilde{\partial}_t, -\tilde{\nabla} \right).$$

We also use the following notation for covariant derivatives

$$(5.6) \quad \psi^{;\mu} = \tilde{\partial}^\mu \psi, \quad \psi^{;\mu*} = \tilde{\partial}^{\mu*} \psi^*, \quad \psi_{;\mu} = \tilde{\partial}_\mu \psi, \quad \psi_{\ell;\mu}^* = \tilde{\partial}_\mu^* \psi^*.$$

Obviously,

$$(5.7) \quad \psi^{;\mu} = \psi^{;\mu} + \frac{iq}{\chi c} A^\mu \psi.$$

The NLS equation (1.1) is the Euler-Lagrange field equation for the following Lagrangian density:

$$(5.8) \quad L = i \frac{\chi}{2} \left[\psi^* \tilde{\partial}_t \psi - \psi \tilde{\partial}_t^* \psi^* \right] - \frac{\chi^2}{2m} \left[\tilde{\nabla} \psi \tilde{\nabla}^* \psi^* + G(\psi^* \psi) \right]$$

where $G(s)$ is given by (3.8). The 4-current J^ν for the Lagrangian is defined by the formula

$$(5.9) \quad J^\nu = -i \frac{q}{\chi} \left(\frac{\partial L}{\partial \psi_{;\nu}} \psi - \frac{\partial L}{\partial \psi_{;\nu}^*} \psi^* \right),$$

it can be written in the form

$$(5.10) \quad J = J^\nu = (c\rho, \mathbf{J})$$

where ρ, \mathbf{J} are given by (1.13),(1.14). The Lagrangian (5.8) and the NLS equation (1.1) are gauge invariant, that is invariant with respect to the multiplication of ψ by $e^{i\gamma}$ with real γ :

$$(5.11) \quad L(e^{i\gamma}\psi, e^{i\gamma}\psi_{;\mu}, e^{-i\gamma}\psi^*, e^{-i\gamma}\psi_{\ell;\mu}^*) = L(\psi, \psi_{;\mu}, \psi^*, \psi_{\ell;\mu}^*).$$

If we take derivative of the above condition (5.11) with respect to γ at $\gamma = 0$, we obtain the following structural restriction on the Lagrangian L :

$$(5.12) \quad \frac{\partial L}{\partial \psi_{;\mu}} \psi_{;\mu} - \frac{\partial L}{\partial \psi_{;\mu}^*} \psi_{;\mu}^* + \frac{\partial L}{\partial \psi} \psi_{\ell} - \frac{\partial L}{\partial \psi^*} \psi_{\ell}^* = 0.$$

Direct verification shows that if a Lagrangian L satisfies the above structural condition, then the current defined by (5.9) for a solution of the Euler-Lagrange field equation (NLS in our case) satisfies the continuity equation

$$(5.13) \quad \partial_{\nu} J^{\nu} = 0$$

which can be written in the form (1.15).

We introduce the *energy-momentum tensor* (EnMT) (see [13] for the general theory) for the NLS by the following formula:

$$(5.14) \quad T^{\mu\nu} = \frac{\partial L}{\partial \psi_{;\mu}} \psi^{;\nu} + \frac{\partial L}{\partial \psi_{;\mu}^*} \psi^{;\nu*} - g^{\mu\nu} L,$$

where $g^{\mu\nu}$ is the Minkowski metric tensor, that is

$$(5.15) \quad g^{\mu\nu} = 0 \text{ for } \mu \neq \nu, \quad g^{00} = 1, \quad \text{and } g^{jj} = -1 \text{ for } j = 1, 2, 3.$$

Proposition 1. *Let EnMT $T^{\mu\nu}$ be defined by formula (5.14) for a solution of the NLS equation (1.1). Then $T^{\mu\nu}$ satisfies the following EnMT conservation law*

$$(5.16) \quad \partial_{\mu} T^{\mu\nu} = f^{\nu}$$

where f^{ν} is the Lorentz force density defined by the formula

$$(5.17) \quad f^{\nu} = \frac{1}{c} J_{\mu} F^{\nu\mu}$$

with J_{μ} and $F^{\nu\mu}$ defined by respectively by (5.9) and (5.4).

Proof. According to (5.7) and (5.9) the definition (5.14) can be written in the form

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial L}{\partial \psi_{;\mu}} \psi^{;\nu} + \frac{\partial L}{\partial \psi_{;\mu}} \frac{iq}{\chi c} A^{\nu} \psi + \frac{\partial L}{\partial \psi_{;\mu}^*} \psi^{;\nu*} - \frac{\partial L}{\partial \psi_{;\mu}^*} \frac{iq}{\chi c} A^{\nu} \psi^* - g^{\mu\nu} L \\ &= \frac{\partial L}{\partial \psi_{;\mu}} \psi^{;\nu} + \frac{\partial L}{\partial \psi_{;\mu}^*} \psi^{;\nu*} - \frac{1}{c} J^{\mu} A^{\nu} - g^{\mu\nu} L. \end{aligned}$$

where J^{ν} is defined by (5.9). We differentiate the above expression and use the continuity equation (5.13):

$$\begin{aligned} \partial_{\mu} T^{\mu\nu} &= \partial_{\mu} \left(\frac{\partial L}{\partial \psi_{;\mu}} \psi^{;\nu} \right) + \partial_{\mu} \left(\frac{\partial L}{\partial \psi_{;\mu}^*} \psi^{;\nu*} \right) - \partial_{\mu} (J^{\mu} A^{\nu}) - \partial_{\mu} (g^{\mu\nu} L) \\ &= \partial_{\mu} \frac{\partial L}{\partial \psi_{;\mu}} \psi^{;\nu} + \frac{\partial L}{\partial \psi_{;\mu}} \partial_{\mu} \psi^{;\nu} + \partial_{\mu} \frac{\partial L}{\partial \psi_{;\mu}^*} \psi^{;\nu*} + \frac{\partial L}{\partial \psi_{;\mu}^*} \partial_{\mu} \psi^{;\nu*} - \frac{1}{c} J^{\mu} \partial_{\mu} A^{\nu} - g^{\mu\nu} \partial_{\mu} (L). \end{aligned}$$

The NLS equation (1.1) can be written in the form

$$(5.18) \quad \frac{\partial L}{\partial \psi^*} - \partial_{\mu} \frac{\partial L}{\partial \psi_{;\mu}^*} = 0,$$

where the Lagrangian L is considered as a function of $\psi, \psi^*, \psi_{,\mu}^*, \psi_{,\mu}^*$, and of the spatial and time variables which enter through A^ν . Using (5.18) together with its conjugate we obtain that

$$\partial_\mu T^{\mu\nu} = -\frac{1}{c} J^\mu \partial_\mu A^\nu + \left[\frac{\partial L}{\partial \psi} \partial^\nu \psi + \frac{\partial L}{\partial \psi^*} \partial^\nu \psi^* + \frac{\partial L}{\partial \psi_{,\mu}^*} \partial^\nu \psi_{,\mu}^* + \frac{\partial L}{\partial \psi_{,\mu}} \partial^\nu \psi_{,\mu} - \partial^\nu (L) \right]$$

Note that the expression in brackets equals the partial derivative $\partial^\nu L$ (we denote $\partial^\nu (\mathcal{L})$ the complete derivative of \mathcal{L} and $\partial^\nu \mathcal{L}$ the partial one). Therefore

$$\partial_\mu T^{\mu\nu} = -\frac{1}{c} J^\mu \partial_\mu A^\nu + \partial^\nu L,$$

Note that the partial derivative $\partial^\nu L$ can be evaluated as follows:

$$\partial^\nu L = \frac{iq}{\chi c} \left(\frac{\partial L}{\partial \psi_{,\mu}} \psi - \frac{\partial L}{\partial \psi_{,\mu}^*} \psi^* \right) \partial^\nu A_\mu = -\frac{1}{c} J^\mu \partial^\nu A_\mu = -\frac{1}{c} J_\mu \partial^\nu A^\mu.$$

Hence,

$$\partial_\mu T^{\mu\nu} = -\frac{1}{c} J_\mu \partial^\mu A^\nu + \frac{1}{c} J_\mu \partial^\nu A^\mu,$$

yielding the EnMT conservation law (5.16). \square

The entries of the EnMT can be interpreted as energy and momentum densities u, p^j respectively, namely

$$(5.19) \quad u = T^{00} = \frac{\chi^2}{2m} \left[|\tilde{\nabla} \psi|^2 + G(|\psi|^2) \right],$$

$$(5.20) \quad p^j = T^{0j} = i \frac{\chi}{2} \left(\psi \tilde{\partial}_j^* \psi^* - \psi^* \tilde{\partial}_j \psi \right).$$

The formula for the momentum density can be written in the form (1.16). The proportionality (1.17) is a specific property of the NLS Lagrangian and does not hold in a general case. Remaining entries of the EnMT take the form

$$T^{j0} = -\frac{\chi^2}{2m} \left(\tilde{\partial}_t \psi \tilde{\partial}_j^* \psi^* + \tilde{\partial}_t^* \psi^* \tilde{\partial}_j \psi \right), \quad j = 1, 2, 3,$$

$$(5.21) \quad T^{ii} = u - \frac{\chi^2}{m} \tilde{\partial}_i \psi \tilde{\partial}_i^* \psi^* + i \frac{\chi}{2} \left(\psi \tilde{\partial}_t^* \psi^* - \psi^* \tilde{\partial}_t \psi \right),$$

and for $i \neq j$, $i, j = 1, 2, 3$,

$$(5.22) \quad T^{ij} = -\frac{\chi^2}{2m} \left(\tilde{\partial}_i \psi \tilde{\partial}_j^* \psi^* + \tilde{\partial}_j \psi \tilde{\partial}_i^* \psi^* \right).$$

Note that the Lorentz force density f^ν in (5.16) can be written in the form

$$(5.23) \quad f^\nu = \frac{1}{c} J_\mu F^{\nu\mu} = (f^0, \mathbf{f}) = \left(\frac{1}{c} \mathbf{J} \cdot \mathbf{E}, \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right)$$

where, in particular, the momentum equation has the form (1.18).

Remark 6. Note that the derivation of the conservation law (5.16) is valid for any local solution ψ of the NLS equation which has continuous second derivatives, G is continuously differentiable on the range of $|\psi|^2$ and the nonlinear term $G(\psi\psi^*)$ which enters (5.8) has continuous first derivatives. Note that the derivatives $G(\psi\psi^*)$

6. APPENDIX 2: SPLITTING OF A FIELD INTO POTENTIAL AND
SPHERE-TANGENT PARTS

Here we describe splitting of a general vector field into sphere-tangent and gradient components which was used in derivation of the properties of wave-corpuscles.

Lemma 9. *Any continuously differentiable vector field $\mathbf{V}(\mathbf{y})$ can be uniquely split into a gradient field ∇P and a sphere-tangent field $\check{\mathbf{V}}$*

$$(6.1) \quad \mathbf{V} = \nabla P + \check{\mathbf{V}}.$$

where P is continuously differentiable, $\check{\mathbf{V}}$ is continuous and satisfies the orthogonality condition (3.20), namely $\check{\mathbf{V}} \cdot \mathbf{y} = \mathbf{0}$.

Proof. To determine P we multiply (6.1) by \mathbf{y} and obtain

$$\mathbf{y} \cdot \mathbf{V} = \mathbf{y} \cdot \nabla P + \mathbf{y} \cdot \check{\mathbf{V}} = \mathbf{y} \cdot \nabla P$$

The directional derivative $\mathbf{y} \cdot \nabla$ can be written as $r\partial_r$ and using notation $\mathbf{y} = \mathbf{\Omega}r$ where $|\mathbf{\Omega}| = 1$, $r = |\mathbf{y}|$ we rewrite the above equation in the form

$$\mathbf{\Omega} \cdot \mathbf{V} = \partial_r P(\mathbf{\Omega}r).$$

Note that P is defined from the above equation up to a function $C(\mathbf{\Omega})$; since the only function of this form which is continuous at the origin must be a constant, P is defined uniquely modulo constants. We can find P by integration:

$$(6.2) \quad P(\mathbf{y}) = \int_0^{|\mathbf{y}|} \mathbf{\Omega} \cdot \mathbf{V}(\mathbf{\Omega}r) dr = \int_0^{|\mathbf{y}|} \frac{1}{|\mathbf{y}|} \mathbf{y} \cdot \mathbf{V} \left(\frac{1}{|\mathbf{y}|} \mathbf{y}r \right) dr.$$

If the potential P is defined by (6.2) then $\mathbf{y} \cdot \check{\mathbf{V}} = \mathbf{y} \cdot \mathbf{V} - \mathbf{y} \cdot \nabla P$ satisfies (3.20). \square

Now we provide some explicit formulas for polynomial fields \mathbf{V} .

To obtain an explicit expression for P , we assume that \mathbf{V} and P are expanded into series of homogenous expressions:

$$(6.3) \quad \mathbf{V}(\mathbf{y}) = \sum_j \mathbf{V}_j(\mathbf{y}), \quad P = \sum_j P_j(\mathbf{y}), \quad \check{\mathbf{V}}(\mathbf{y}) = \sum_j \check{\mathbf{V}}_j(\mathbf{y}),$$

where

$$\mathbf{V}_j(\zeta\mathbf{y}) = \zeta^j \mathbf{A}_j(\mathbf{y}), \quad P_j(\zeta\mathbf{y}) = \zeta^j \varphi_j(\mathbf{y}).$$

For a j -homogenous \mathbf{V}_j we have

$$P_{j+1}(\mathbf{y}) = \int_0^{|\mathbf{y}|} \frac{1}{|\mathbf{y}|} \mathbf{y} \cdot \mathbf{V}_j \left(\frac{1}{|\mathbf{y}|} \mathbf{y}r \right) dr = \int_0^{|\mathbf{y}|} \frac{1}{|\mathbf{y}|} \frac{1}{|\mathbf{y}|^j} \mathbf{y} \cdot \mathbf{V}_j(\mathbf{y}) r^j dr,$$

implying

$$(6.4) \quad P_{j+1}(\mathbf{y}) = \frac{1}{(j+1)|\mathbf{y}|^{j+1}} \mathbf{y} \cdot \mathbf{V}_j(\mathbf{y}) |\mathbf{y}|^{j+1} = \frac{1}{j+1} \mathbf{y} \cdot \mathbf{V}_j(\mathbf{y}).$$

In particular, the zero order term \mathbf{V}_0 corresponds to

$$(6.5) \quad P_1(\mathbf{y}) = \mathbf{V}(0) \cdot \mathbf{y}, \quad \check{\mathbf{V}}_0(\mathbf{y}) = 0.$$

The first order one corresponds to

$$(6.6) \quad P_2(\mathbf{y}) = \frac{1}{2} \mathbf{V}_1(\mathbf{y}) \cdot \mathbf{y}$$

and

$$(6.7) \quad \nabla P_2(\mathbf{y}) = \frac{1}{2}\mathbf{V}_1(\mathbf{y}) + \frac{1}{2}\mathbf{V}_1^T(\mathbf{y}).$$

Obviously, $\nabla P_2(\mathbf{y})$ coincides with the symmetric part of the linear transformation $\mathbf{V}_1(\mathbf{y})$, and

$$(6.8) \quad \check{\mathbf{V}}_1(\mathbf{y}) = \frac{1}{2}\mathbf{V}_1(\mathbf{y}) - \frac{1}{2}\mathbf{V}_1^T(\mathbf{y})$$

coincides with the anti-symmetric part. For higher values of j we have

$$\nabla P_{j+1}(\mathbf{y}) = \frac{1}{j+1}\nabla(\mathbf{y} \cdot \mathbf{V}_j(\mathbf{y})).$$

Using vector calculus we obtain

$$\begin{aligned} \nabla(\mathbf{y} \cdot \mathbf{V}_j(\mathbf{y})) &= (\mathbf{y} \cdot \nabla)\mathbf{V}_j + (\mathbf{V}_j \cdot \nabla)\mathbf{y} + \mathbf{y} \times (\nabla \times \mathbf{V}_j) + \mathbf{V}_j \times (\nabla \times \mathbf{y}) \\ &= (\mathbf{y} \cdot \nabla)\mathbf{V}_j + \mathbf{V}_j + \mathbf{y} \times (\nabla \times \mathbf{V}_j) \end{aligned}$$

where, by Euler's identity for homogenous functions,

$$(\mathbf{y} \cdot \nabla)\mathbf{V}_j(\mathbf{y}) = j\mathbf{V}_j(\mathbf{y}).$$

Hence

$$\begin{aligned} \nabla P_{j+1}(\mathbf{y}) &= \frac{1}{j+1}\nabla(\mathbf{y} \cdot \mathbf{V}_j(\mathbf{y})) = \frac{1}{j+1}((\mathbf{y} \cdot \nabla)\mathbf{V}_j + \mathbf{V}_j + \mathbf{y} \times (\nabla \times \mathbf{V}_j)) = \\ &= \frac{1}{j+1}(j\mathbf{V}_j + \mathbf{V}_j + \mathbf{y} \times (\nabla \times \mathbf{V}_j)) = \mathbf{V}_j + \frac{1}{j+1}\mathbf{y} \times (\nabla \times \mathbf{V}_j), \end{aligned}$$

$$(6.9) \quad \check{\mathbf{V}}_j(\mathbf{y}) = \mathbf{V}_j(\mathbf{y}) - \nabla P_{j+1}(\mathbf{y}) = -\frac{1}{j+1}\mathbf{y} \times (\nabla \times \mathbf{V}_j).$$

Acknowledgment. The research was supported through Dr. A. Nachman of the U.S. Air Force Office of Scientific Research (AFOSR), under grant number FA9550-11-1-0163.

REFERENCES

- [1] T. D'Aprile and D. Mugnai, Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A 134(5), 893–906 (2004).
- [2] J. C. Bronski and R. L. Jerrard, *Soliton dynamics in a potential*, Math. Res. Lett. 7, 329–342, (2000).
- [3] W. Appel and M. Kiessling, *Mass and Spin Renormalization in Lorentz Electrodynamics*, Ann. Phys., **289**, 24–83, (2001).
- [4] A. Babin and A. Figotin, *Linear superposition in nonlinear wave dynamics*, Reviews Math. Phys. 18 (2006), no. 9, 971–1053.
- [5] A. Babin and A. Figotin, *Wavepacket Preservation under Nonlinear Evolution*, Commun. Math. Phys. 278, 329–384 (2008).
- [6] A. Babin and A. Figotin, *Nonlinear Dynamics of a System of Particle-Like Wavepackets*, Instability in Models Connected with Fluid Flows, Ed. C. Bardos and A. Fursikov, International Mathematical Series, Vol. 6, Springer, 2008.
- [7] A. Babin and A. Figotin, *Wave-corpucle mechanics for electric charges*, J. Stat. Phys., **138**: 912–954, (2010).
- [8] Babin A. and Figotin A., *Some mathematical problems in a neoclassical theory of electric charges*, Discrete and Continuous Dynamical Systems A, **27**(4), 1283–1326 (2010).
- [9] Babin A. and Figotin A., *Electrodynamics of balanced charges*, Found. Phys., **41**: 242–260, (2011).
- [10] Babin A. and Figotin A., *Relativistic dynamics of accelerating particles derived from field equations*, Found. Phys., **42**: 996–1014, (2012).

- [11] Babin A. and Figotin A., *Relativistic Point Dynamics and Einstein Formula as a Property of Localized Solutions of a Nonlinear Klein-Gordon Equation*, Comm. Math. Phys. 322, 453-499, (2013).
- [12] D. Bambusi, L. Galgani, *Some Rigorous Results on the Pauli-Fierz Model of Classical Electrodynamics*, Ann. Inst. H. Poincaré, Phys. Théor. 58 155-171 (1993).
- [13] A. Barut, *Electrodynamics and Classical Theory of Fields and Particles*, Dover, 1980.
- [14] H. Berestycki; P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. 82 (4) (1983) 313–345.
- [15] H. Berestycki; P.-L. Lions, *Nonlinear scalar field equations. II. Existence of infinitely many solutions*, Arch. Rational Mech. Anal. 82 (1983), no. 4, 347–375.
- [16] M. Del Pino; J. Dolbeault, *The optimal Euclidean L_p -Sobolev logarithmic inequality*, J. Funct. Anal. 197 (2003), no. 1, 151–161.
- [17] I. Białynicki-Birula; J. Mycielski, *Nonlinear Wave Mechanics*, Annals of Physics, **100**, 62-93, (1976).
- [18] I. Białynicki-Birula; J. Mycielski, *Gaussons: Solitons of the Logarithmic Schrödinger Equation*, Physica Scripta 20, 539-544 (1979).
- [19] T. Cazenave, *Stable solutions of the logarithmic Schrödinger equation*, Nonlinear Anal. 7 (1983), 1127-1140.
- [20] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, 10 AMS, Providence RI, 2003.
- [21] T. Cazenave and A. Haraux, *Équations d'évolution avec non linéarité logarithmique*, Ann. Fac. Sci. Toulouse Math. (5) 2 (1980), no. 1, 21–51.
- [22] T. Cazenave; P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. 85 (1982), no. 4, 549–561.
- [23] Einstein, Albert (1905). "Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?". Annalen der Physik 18 (13): 639–641. English translation: "Does the Inertia of a Body Depend Upon Its Energy Content?". Translation by George Barker Jeffery and Wilfrid Perrett in *The Principle of Relativity*, London: Methuen and Company, Ltd. (1923).
- [24] Feynman R., Leighton R. and Sands M., *The Feynman Lectures on Physics*, Addison-Wesley, Reading, Vol. III, 1965.
- [25] J. Frohlich, T.-P. Tsai, and H.-T. Yau, *On the point-particle (Newtonian) limit of the nonlinear Hartree equation*, Comm. Math. Phys., 225, 223–274 (2002).
- [26] Goldstein H., Poole C., Safko J., *Classical Mechanics*, 3rd ed., Addison-Wesley, 2000.
- [27] V. Imaikin, A. Komech, B. Vainberg, *On scattering of solitons for the Klein-Gordon equation coupled to a particle*, Comm. Math. Phys. 268 (2006), no. 2, 321-367. arXiv:math.AP/0609205
- [28] Itzykson C. and Zuber J., *Quantum Field Theory*, McGraw-Hill, 1980.
- [29] B. Jonsson, J. Fröhlich, S. Gustafson and I. M. Sigal, *Long time motion of NLS solitary waves in a confining potential*, Ann. Henri Poincaré 7, 621–660 (2006).
- [30] M. Heid, H. Heinz and T. Weth, *Nonlinear Eigenvalue Problems of Schrödinger Type Admitting Eigenfunctions with Given Spectral Characteristics*, Math. Nachr. **242** (2002), 91 – 118.
- [31] J. Jackson, *Classical Electrodynamics*, 3rd Edition, Wiley, 1999.
- [32] T. Kato, *Nonlinear Schrödinger equations*, in "Schrödinger operators", (H. Holden and A. Jensen, eds.), Lecture Notes in Physics 345, Springer Verlag 1989.
- [33] M. Kiessling, *Electromagnetic Field Theory without Divergence Problems I. The Born Legacy*, J. Stat. Physics, **116**: 1057-1122 (2004).
- [34] M. Kiessling, *Quantum Abraham models with de Broglie-Bohm laws of quantum motion*, e-print available online at arXiv:physics/0604069v2.
- [35] A. Komech, *Lectures on Quantum Mechanics (nonlinear PDE point of view)*, Lecture Notes of the Max Planck Institute for Mathematics in the Sciences, LN 25/2005, Leipzig, 2005. <http://www.mis.mpg.de/preprints/ln/lecturenote-2505-abstr.html>.
- [36] A.I. Komech, A.A. Komech, *Global Attraction to Solitary Waves in Models Based on the Klein-Gordon Equation*, SIGMA, Symmetry Integrability Geom. Methods Appl. 4 (2008), Paper 010, 23 pages, electronic only. <http://www.emis.de/journals/SIGMA/2008/>
- [37] A. Komech, M. Kunze, H. Spohn, *Effective Dynamics for a mechanical particle coupled to a wave field*, Comm. Math. Phys. 203 (1999), 1-19.
- [38] Lanczos C. *The Variational Principles of Mechanics*, 4th ed., Dover, 1986.
- [39] Landau L. and Lifshitz E., *The classical theory of fields*, Pergamon, Oxford, 1975.

- [40] E. Long and D. Stuart, *Effective dynamics for solitons in the nonlinear Klein-Gordon-Maxwell system and the Lorentz force law*, Rev. Math. Phys., 21, 459–510 (2009).
- [41] Maslov V. P., Fedoriuk M. V., *Semi-Classical Approximation in Quantum Mechanics*, Reidel, Boston (1981).
- [42] Møller C., *The Theory of Relativity*, 2nd edition, Oxford, 1982.
- [43] Morse P. and Feshbach H., *Methods of Theoretical Physics*, Vol. I, McGraw-Hill, 1953.
- [44] Nayfeh A., *Perturbation methods*, Wiley, 1973.
- [45] P. Pearle *Classical Electron Models*, in *Electromagnetism Paths to Research*, D. Teplitz, ed. (Plenum, New York, 1982), pp. 211–295.
- [46] H. Poincaré, Comptes Rend. 140, 1504 (1905); Rendiconti del Circolo Matematico di Palermo, **21**, 129–176, 1906.
- [47] F. Rohrlich, *Classical Charged Particles*, Addison-Wesley, 3d ed., 2007.
- [48] J. Schwinger, *Electromagnetic Mass Revisited*, Foundations of Physics, Vol. 13, No. 3, pp. 373–383, 1983.
- [49] H. Spohn, *Dynamics of Charged Particles and Their Radiation Field*, Cambridge Univ. Press, 2004.
- [50] Stachel J., *Einstein from B to Z*, Burkhouer, 2002.
- [51] C. Sulem and P. Sulem, *The nonlinear Schrödinger equation. Self-focusing and wave collapse*, Springer, 1999.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT IRVINE, IRVINE, CA 92697-3875, U.S.A.

E-mail address: `ababine@math.uci.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT IRVINE, IRVINE, CA 92697-3875, U.S.A.

E-mail address: `afigotin@math.uci.edu`