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SUPERSTRING AMPLITUDES AND CONTACT INTERACTIONS

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ABSTRACT

We show that scattering amplitudes computed from light-cone superstring field theory are divergent at tree level. The divergences can be eliminated, and supersymmetry restored, by the addition of certain counter terms to the light-cone Hamiltonian. These counter terms have the form of local contact interactions, whose existence we had previously deduced on grounds of vacuum stability, and closure of the super-Poincaré algebra. The quartic contact interactions required in Type I and Type IIB superstring theories are constructed in detail.

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Introduction

Superstring field theory was originally constructed in light-cone gauge by Green and Schwarz [1], who argued that the theory of closed superstrings is cubic with only three-string vertices, while the only four-string vertex of open superstrings is of the Kaku-Kikkawa form (apart from a possible term in the purely forward direction). In two previous articles [2,3] we took issue with that claim, and showed that supersymmetry (i.e. the closure of the super-Poincaré algebra) required the existence of new, local, contact interactions, of quartic order in the string fields. In other words, the string theory constructed by Green and Schwarz is incomplete, and not supersymmetric as it stands.

This leads to a puzzle. If the Green-Schwarz field theory is not supersymmetric, then scattering amplitudes constructed from that theory should not be supersymmetric either, even at tree level. But tree level amplitudes for arbitrary numbers of incoming and outgoing massless superstrings have been computed, and these amplitudes are manifestly supersymmetric. There would seem to be no need (and indeed no room) for any additional interaction terms.

The purpose of this article is therefore to closely reexamine the superstring tree amplitude calculation, concentrating on the open and closed 2 string $\rightarrow$ 2 string process. We will show that, contrary to common belief, a careful calculation based entirely on the existing theory does *not* yield a dual, relativistic, supersymmetric amplitude. In fact, the resulting amplitude is not even finite! A finite amplitude can be obtained by performing certain partial integrations [4] in the Koba-Nielsen variables, which leaves a finite supersymmetric Koba-Nielsen integral, plus divergent and unwelcome boundary terms. In existing calculations of tree amplitudes, these divergent surface terms have simply been dropped by hand. It is clear that the only systematic way of dropping contributions to scattering amplitudes is to introduce counter terms in the Hamiltonian to cancel them. The main result of our paper is to construct the required quartic counter terms of open and closed superstrings in detail. We will show that these terms are local contact interactions of the form [2] shown pictorially in Fig. 1.

The existence of superstring contact interactions had been previously deduced on the following grounds. Consider, for example, the type IIB theory. The super-Poincaré algebra contains, in the notation of Ref. [1], the following relations

$$\{Q_I^{-A}, Q_I^{-\bar{B}}\} = 2\delta^{A\bar{B}} H \quad (1.1a)$$

$$\{Q_I^{-\bar{A}}, Q_I^{-\bar{B}}\} = \{Q_I^{-A}, Q_I^{-B}\} = 0, \quad (1.1b)$$

where Q_I^- is one of the ($N = 2$) supercharges and similar relations hold for Q_{II}^- . The supercharges and the Hamiltonian were constructed in [1] to $O(\lambda)$

$$Q_I^- = Q_{2,I}^- + \lambda Q_{3,I}^- \quad (1.2a)$$

$$H = H_2 + \lambda H_3, \quad (1.2b)$$

(where suffixes 2,3 indicate that the operator is quadratic or cubic in the string fields; λ is the coupling) and shown to satisfy (1.1) also to $O(\lambda)$. Green and Schwarz in Ref. [1] argued that contributions to the matrix element of the lhs of (1.1) cancel at $O(\lambda^2)$; however, the main point of our previous papers [2,3] was that this cancellation is not complete*. As a result there must be a further term in the supercharge $\lambda^2 Q_{4,I}^-$, and also a quartic term in the hamiltonian $\lambda^2 H_4$ given by

$$2\delta^{A\bar{B}} H_4 = \{Q_{3,I}^{-A}, Q_{3,I}^{-\bar{B}}\} + \{Q_{2,I}^{-A}, Q_{4,I}^{-\bar{B}}\} + \{Q_{4,I}^{-A}, Q_{2,I}^{-\bar{B}}\}. \quad (1.3)$$

For details we refer the reader to our previous work. One point that should be noted, however, is that the cubic part of the supercharge Q_3^- has the form of a 3-string vertex, and the non-vanishing piece of $\{Q_3^-, Q_3^-\}$ comes from contributions where the string join/split points of each vertex coincide. This is why H_4 has the form of a local contact interaction. Divergences which arise when superstring vertices coincide is a theme that will reappear in the tree amplitude calculations carried out in the next section.

From Eq. (1.3) we see that the computation of H_4 requires knowledge of Q_4^- , which in principle could be constructed from the super-Poincaré algebra, but is at present unknown. In this paper our approach is, instead, to construct H_4 from the requirement that tree amplitudes be finite and supersymmetric. This approach has the great advantage that we are able to construct H_4 explicitly, without detailed knowledge of higher order terms in the supercharge Q^- .

Our motivation for finding the counter terms of the light-cone superstring is not simply to redo tree amplitudes, but rather to understand the structure of the interaction terms in the superstring Hamiltonian. If, for example, we uncover a Higgs-like structure in the light-cone Hamiltonian, then it may be possible to address non-perturbative questions of vacuum structure in superstring field theory. There will be some preliminary discussion of this issue in Section 4.

Before plunging into the tree amplitude calculation, it may be helpful to comment briefly on the following "a priori" objections to superstring contact interactions:

1) *The usual string interactions of the bosonic theory cover moduli space completely, where is the room for further interactions?* The contact interactions we will discuss do not cover a finite region of the parameter space, as does, e.g. the Kaku-Kikkawa 4-string interaction [5]. Rather these

* It was noted in [1] that cancellation might fail in a zero-measure set of states where incoming and outgoing string-lengths match exactly. In [2,3], however, it was shown that the cancellation in fact fails for *arbitrary* string lengths.

terms act like delta-functions in the moduli, contributing only at singular points where light-cone vertices coincide.

2) *Why do the fermionic string theories require contact terms, and apparently not the bosonic string theories?* In a fermionic string theory, as opposed to the bosonic case, there are local operators acting at the light-cone points $\rho = \tau + i\sigma$ where strings join or split. In a light-cone diagram the operators at two different vertices ρ_1, ρ_2 give rise to Green's functions $K(\rho_1, \rho_2)$, which are divergent as the vertices approach one another. This results in a divergent integral over the Koba-Nielsen variables, which requires an infinite subtraction.

3) *Why is the need for contact interactions not apparent in the first-quantized approach?* In the first-quantized approach, the p^+ coordinate of all but two external strings is transformed to zero. In that case the interaction vertices coincide only when Koba-Nielsen variables coincide. This means that the divergence due to interaction points coinciding, which is momentum-independent, is lumped together with the usual momentum-dependent singularities, and is, in a sense, "swept under the rug" by the procedure of analytic continuation.

The outline of our paper is as follows: In Section 2 we calculate the scattering amplitude of four massless particles, first for the case of open superstrings, and then for closed Type IIB superstrings. We thereby determine the matrix element of H_4 , which is required to cancel the divergent boundary terms. In Section 3 we establish the general form of H_4 , not just its matrix element between massless states, and equations (3.4-3.6) are the main results of this article. In Section 4 we give a very preliminary discussion of issues relating to vacuum stability, and non-perturbative string-field condensation. Section 5 contains some concluding remarks and speculations. There is also an Appendix containing tree amplitude calculations for the Neveu-Schwarz-Ramond model; these calculations are analogous to those in Section 2.

2. Boundary terms in scattering amplitudes

2.1 Open superstring

In this section we calculate the amplitude for scattering of two massless open superstring states into two other such states. First we discuss the conformal map used and introduce some notation. The string light-cone diagram (τ, σ) is mapped into the upperhalf complex z -plane by

$$\tau + i\sigma = \alpha_1 \ln(1 - z) + \alpha_3 \ln z + \alpha_4 \ln(z - x) + \text{const} , \quad (2.1)$$

where by conformal invariance we choose the Koba-Nielsen variables Z_1, Z_2, Z_3 fixed at $1, \infty, 0$ and the fourth $Z_4 = x \in R$ essentially describes the time difference between the two 3-vertices, over which we have to integrate to get the amplitude. The two interaction points (labeled 1 and 2) of the light-cone diagram correspond to

$$z_{1,2} = \frac{1}{2} \{a(1-x) + cx + 1 \pm \sqrt{D}\}, \quad (2.2)$$

where D is the quadratic

$$\begin{aligned} D(x) &= (A + C - 2B)x^2 + 2x(B - A) + A \\ &= (A + C - 2B)(x - x_-)(x - x_+) \end{aligned} \quad (2.3)$$

with zeros

$$x_{\pm} = (A + C - 2B)^{-1} \left\{ A - B \pm \sqrt{B^2 - AC} \right\}$$

and where we have defined the following parameters in terms of ratios of the string lengths ($\pi|\alpha|$)

$$\begin{aligned} a &\equiv \alpha_1/\alpha_2 & b &\equiv -\alpha_4/\alpha_2 & c &\equiv -\alpha_3/\alpha_2 \\ A &\equiv (1+a)^2 & B &\equiv b+ac & C &\equiv (1-c)^2. \end{aligned}$$

Green and Schwarz [1] have already calculated the st part of the four particle scattering amplitude, so we can be relatively brief in reporting our calculation of the full amplitude. The map (2.1) is the same as that of Ref. [1] and note that their quadratic $\Delta = D/A$. We will start with the st amplitude, from which the su and tu parts follow by duality. There are three steps:

1. Using two generalized 3-vertices with parameters $\xi_{1,2}^I, \phi_{1,2}^{\bar{A}}$, see (2.6) below, the mode sums over the intermediate string states are performed giving T_{1234} ;
2. In order to have the vertex insertions corresponding to the cubic Hamiltonian we have to take appropriate derivatives *w.r.t.* ξ and ϕ of T_{1234} , setting $\xi = \phi = 0$ afterwards;
3. The resulting expression can be simplified by partial integrations; it is in this step that we will pick up divergent boundary terms.

Step 1 of the intermediate string mode sums gives

$$\begin{aligned} T_{1234}^{st} &= f \frac{1}{2} (\alpha_1 + \alpha_2) \int_0^1 dx |x|^{-s/2} |1-x|^{-t/2} \frac{\sqrt{D/A}}{x(1-x)} P_B E_F \\ &\quad \delta^8 \left(\sum_{r=1}^4 p_r^I \right) \delta \left(\sum_r \alpha_r \right) \delta^4 \left(\sum_r \alpha_r \theta_r \right), \end{aligned} \quad (2.4)$$

which up to the prefactor agrees with (5.39) of Green and Schwarz [1]. The factor $f \equiv (\alpha_1 \alpha_2 \alpha_3 \alpha_4)^{-5/2}$ results from contractions with the external fields (giving $1/\alpha_1 \alpha_2 \alpha_3 \alpha_4$) and from the factors in the bosonic and fermionic measure of the 3-vertex (giving $(\alpha_1 \alpha_2 \alpha_3 \alpha_4)^{1/2-2}$), see [3]. The bosonic string modes give in (2.4) the Koba-Nielsen factor and the expression $P_B = P_B(\xi, \xi_2)$,

which is given in (5.35) of [1]. The fermionic modes yield the exponential $E_F = E_F(\phi_1, \phi_2)$ given in (5.37) of [1], but the minus signs there in front of the factors $(1 + c_1 A_{01})$ should be corrected to plus signs [2,3]. The s -channel fills out the integral over $(0, x_0)$ in (2.4), while t -channel gives the $(x_0, 1)$ part.

Step 2 will give us the correct vertex insertions which are

$$\begin{aligned} |H_1\rangle &= \left(\sqrt{\frac{1}{2}} Z_1^L - Z_1^i \rho_{AB}^i Y_1^{\bar{A}} Y_1^{\bar{B}} + \frac{1}{3} \sqrt{2} Z_1^R \epsilon^{ABCD} Y_1^{\bar{A}} Y_1^{\bar{B}} Y_1^{\bar{C}} Y_1^{\bar{D}} \right) |V\rangle_1 \\ \langle H_2| &= {}_2\langle V| \left(\sqrt{\frac{1}{2}} Z_2^L + Z_2^i \rho_{AB}^i Y_2^{\bar{A}} Y_2^{\bar{B}} + \frac{1}{3} \sqrt{2} Z_2^R \epsilon^{ABCD} Y_2^{\bar{A}} Y_2^{\bar{B}} Y_2^{\bar{C}} Y_2^{\bar{D}} \right), \end{aligned} \quad (2.5)$$

where 1 and 2 label the two vertices and the expressions for Z and Y are given in (4-25,26) of [1]. Since the generalized 3-vertex (for strings 1 and 2 combining in 3) used in step 1 was

$$|V_{gen}\rangle = \exp \left[|2\alpha_1\alpha_2/\alpha_3|^{1/2} \xi^I Z^I + \frac{1}{2} \alpha_1\alpha_2\alpha_3^{-1/2} \phi^A Y^{\bar{A}} \right] |V\rangle \quad (2.6)$$

appropriate derivatives *w.r.t.* $\xi_{1,2}$ and $\phi_{1,2}$ will give the required vertex insertions (2.5). These derivatives operating on T_{1234} will give us the amplitude, which consists of 9 terms. Six of these terms (LL , RR , iL , Li , iR , Ri) can be written as the corresponding terms of the JJ parts below in (2.7). The remaining three terms (ii , LR , RL) are more complicated, part of them give the corresponding terms of JJ and the rest combine nicely in the parts i_M and i_N below. Including an open string coupling constant g for each 3-vertex we have after these manipulations:

$$\begin{aligned} A^{st} &= \frac{f}{8\sqrt{\alpha_1\alpha_2\alpha_3\alpha_4}} \delta^{14} g^2 \int_0^1 dx \frac{|x|^{-s/2} |1-x|^{-t/2}}{x(1-x)} \\ &\quad \times [x J_{23} J_{41} + (1-x) J_{12} J_{34} + M i_M + N i_N] \end{aligned} \quad (2.7)$$

with

$$\begin{aligned} i_M &= - \frac{\alpha_1\alpha_2\alpha_3\alpha_4}{C} \left\{ \frac{t}{2} [ACx^2(1-x) - Bx(D - x^2C)] D^{-1} \right. \\ &\quad + \frac{s}{2} [BCx^2(1-x) - Cx(D - x^2C)] D^{-1} \\ &\quad \left. + x(1-x)D^{-2} [ACD + 2Cx^2(B^2 - AC)] \right\} \end{aligned} \quad (2.8a)$$

$$\begin{aligned} i_N &= 2 \frac{\alpha_1\alpha_2\alpha_3\alpha_4}{C} \alpha_2^2 \left\{ \frac{t}{2} [Cx(A + (B-A)x)^2 - xB^2D] D^{-1} \right. \\ &\quad + \frac{s}{2} [Cx(A + (B-A)x)(B + (C-B)x) - xBCD] D^{-1} \\ &\quad \left. + x(1-x)D^{-2}(B^2 - AC)2Cx(A + (B-A)x) \right\} \end{aligned} \quad (2.8b)$$

where M and N are

$$M \equiv \epsilon^{ABCD} \left\{ \frac{1}{6} \alpha_1^2 \alpha_2^2 \Theta_{12}^{\bar{A}} \Theta_{12}^{\bar{B}} \Theta_{12}^{\bar{C}} \Theta_{12}^{\bar{D}} + \frac{1}{6} \alpha_3^2 \alpha_4^2 \Theta_{34}^{\bar{A}} \Theta_{34}^{\bar{B}} \Theta_{34}^{\bar{C}} \Theta_{34}^{\bar{D}} + \alpha_1 \alpha_2 \alpha_3 \alpha_4 \Theta_{12}^{\bar{A}} \Theta_{12}^{\bar{B}} \Theta_{34}^{\bar{C}} \Theta_{34}^{\bar{D}} \right\} \quad (2.9a)$$

$$N \equiv \epsilon^{ABCD} \left\{ \frac{1}{6} \alpha_1 \alpha_2 \Theta_{34}^{\bar{A}} \Theta_{12}^{\bar{B}} \Theta_{12}^{\bar{C}} \Theta_{12}^{\bar{D}} + \frac{1}{6} \alpha_3 \alpha_4 \Theta_{12}^{\bar{A}} \Theta_{34}^{\bar{B}} \Theta_{34}^{\bar{C}} \Theta_{34}^{\bar{D}} \right\} \quad (2.9b)$$

and J_{rs} is the same as defined in [1]

$$J_{rs} \equiv P_{rs}^L + \sqrt{1/2} \alpha_r P_{rs}^i \rho_{AB}^i \Theta_{rs}^{\bar{A}} \Theta_{rs}^{\bar{B}} + \frac{1}{6} \alpha_{rs}^2 P_{rs}^R \epsilon^{ABCD} \Theta_{rs}^{\bar{A}} \Theta_{rs}^{\bar{B}} \Theta_{rs}^{\bar{C}} \Theta_{rs}^{\bar{D}} \quad (2.9c)$$

with

$$\alpha_{rs} \equiv -\alpha_r \alpha_s (\alpha_r + \alpha_s)$$

$$P_{rs}^I \equiv \alpha_r p_s^I - \alpha_s p_r^I$$

$$\Theta_{rs}^{\bar{A}} \equiv \frac{\theta_s^{\bar{A}} - \theta_r^{\bar{A}}}{\alpha_s + \alpha_r}$$

Step 3 simplifies the part $M i_M + N i_N$ in (2.7) by use of the following equations

$$\begin{aligned} & \frac{(\alpha_2 + \alpha_3)^2}{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \int_0^1 dx x^{-1-s/2} (1-x)^{-1-t/2} i_M = \\ & (\alpha_1 \alpha_3 + \alpha_2 \alpha_4) \left(1 - \frac{s}{2} - \frac{t}{2} \right) \int_0^1 dx |x|^{-s/2} |1-x|^{-t/2} \\ & - \alpha_2^2 \left[|x|^{-s/2} |1-x|^{-t/2} x(1-x) D^{-1} \{ B(x(A+C-2B) + B-A) + AC - B^2 \} \right]_0^1 \end{aligned} \quad (2.10a)$$

$$\begin{aligned} & \frac{(\alpha_2 + \alpha_3)^2}{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \int_0^1 dx x^{-1-s/2} (1-x)^{-1-t/2} i_N = \\ & 8 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \left(1 - \frac{s}{2} - \frac{t}{2} \right) \int_0^1 dx |x|^{-s/2} |1-x|^{-t/2} \\ & + 2 \alpha_2^4 \left[|x|^{-s/2} (1-x)^{-t/2} x(1-x) (B^2 - AC) D^{-1} \{ x(A+C-2B) - A \} \right]_0^1 \end{aligned} \quad (2.10b)$$

These equations can be checked, with some effort, using the partial integration formulae [4]

$$\begin{aligned}
\left\{ \frac{s}{2} \text{ or } \frac{t}{2} \right\} \int_a^b dx \frac{|x|^{-s/2} |1-x|^{-t/2}}{x(1-x)} f(x) \\
= \frac{s+t}{2} \int_a^b dx \{x \text{ or } 1-x\} \frac{|x|^{-s/2} |1-x|^{-t/2}}{x(1-x)} f \\
\{+ \text{ or } -\} \int_a^b dx |x|^{-s/2} |1-x|^{-t/2} f' \\
\{- \text{ or } +\} \left[|x|^{-s/2} |1-x|^{-t/2} f \right]_a^b.
\end{aligned}$$

Hence, using (2.10) in (2.7) the final expression for the st amplitude is:

$$\begin{aligned}
A^{st} = \frac{f}{8\sqrt{\alpha_1\alpha_2\alpha_3\alpha_4}} \delta^{14} g^2 \int_0^1 dx \frac{|x|^{-s/2} |1-x|^{-t/2}}{x(1-x)} \left\{ x J_{23} J_{41} + (1-x) J_{12} J_{34} \right. \\
\left. + x(1-x) \left(1 + \frac{u}{2} \right) (-2F_{1234}) \right\} + \left[A_b \right]_{x=0}^{x=1} \quad (2.11a)
\end{aligned}$$

$$\propto \frac{\Gamma(1-s/2)\Gamma(1-t/2)}{\Gamma(1-s/2-t/2)} \left[\frac{J_{23}J_{41}}{t} + \frac{J_{12}J_{34}}{s} + F_{1234} \right] \delta^{14}, \quad (2.11b)$$

where δ^{14} denotes the delta functions in (2.4), including one for $\sum p_r^-$, and F_{1234} is defined as in Ref. 1 (we also define G_{1234} to be used shortly)

$$\begin{aligned}
F_{1234} &\equiv - \frac{\alpha_1\alpha_2\alpha_3\alpha_4}{(\alpha_2 + \alpha_3)^2} \frac{1}{2} [(\alpha_1\alpha_3 + \alpha_2\alpha_4)M + 8\alpha_1\alpha_2\alpha_3\alpha_4 N] \\
G_{1234} &\equiv - \frac{(\alpha_1\alpha_2\alpha_3\alpha_4)^{3/2}}{(\alpha_2 + \alpha_3)^2} [M + 2(\alpha_1\alpha_3 + \alpha_2\alpha_4)N].
\end{aligned}$$

In (2.11a) the boundary term follows from the expression

$$\begin{aligned}
A_b = \frac{f}{8\sqrt{\alpha_1\alpha_2\alpha_3\alpha_4}} g^2 \delta^{14} |x|^{-s/2} |1-x|^{-t/2} x(1-x)^{-2} \\
\left\{ F_{1234}(x(A+C-2B) + B-A) + G_{1234}\sqrt{B^2-AC} \right\}, \quad (2.12)
\end{aligned}$$

but in the st channel it is evaluated at $x=0,1$ and vanishes by analytic continuation in s and t (the argument is that in (part of) the physical region it vanishes and elsewhere it is defined to be zero by analytic continuation). The integral form (2.11a) of the amplitude agrees with (5-48) of Green and Schwarz [1] up to the factor -2 in the F_{1234} term, which is also necessary to obtain (2.11b).

The st part of the amplitude is simply (2.11b). So far we are in complete agreement with the result of the tree amplitude calculation in Ref. [1], which was carried out only for the planar s -channel contribution. By duality the su part follows by having the integral in (2.11a) run over $(-\infty, 0)$ and again the boundary terms at $x = -\infty, 0$ vanish by analytic continuation. However, the tu part is more subtle. It is here, in the non-planar contribution to the amplitude, that we see the

need for counter terms. As is well known, the amplitude calculated from the two 3-vertices covers only the interval $(1, x_-) \cup (x_+, \infty)$ of the integral in (2.11a). The finite Kaku-Kikkawa exchange vertex [5] covers the remaining (x_-, x_+) part of the integral, so that

$$A^{tu} = \int_1^\infty dx \dots + \left[A_b \right]_1^{x_-} + \left[A_b \right]_{x_+}^\infty, \quad (2.13)$$

where the boundary terms result as above, from partial integrations in the 3-vertices calculation. There is no problem with boundary terms at $x = 1, \infty$, which again vanish by analytic continuation. However, the boundary term A_b is *divergent* at the points $x = x_\pm$, resulting in an *infinite* non-planar scattering amplitude. The reason for this divergence was already mentioned in the Introduction. The points $x_\pm$ have a special role in light-cone diagrams, since at these values the join/split points of each of the 3-string vertices coincide. The local vertex insertions (2.5), which have no analogue in the bosonic theory, are responsible for the factor $D(x)^{-1}$ in (2.8), which diverges as the vertices approach one another, i.e., as $x \rightarrow x_\pm$.

To explicate the form of the divergent boundary terms, we will use a time-split regularization, i.e. $x_- \rightarrow x_- - \delta_-$, $x_+ \rightarrow x_+ + \delta_+$ with $0 < \delta_\pm \ll 1$. We find the boundary term in the tu -channel to be

$$\left[A_b \right]_{x_+ + \delta_+}^{x_- - \delta_-} = -fg^2 \delta^{14} \sum_{\pm} (\pm) |x_\pm|^{2-s/2} |1 - x_\pm|^{-t/2} \frac{\sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4}}{16\delta_\pm} (4\sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} N \pm M). \quad (2.14)$$

As explained in the introduction this non-relativistic, non-dual, non-supersymmetric piece (2.14) of the amplitude must be subtracted by a counter term, which we will discuss in Section 3.*

We end this subsection with a technical remark. The above results are for general values of $\alpha_1, \alpha_3, \alpha_4$ in which case the interaction points can only coincide in the tu -channel, i.e. when $x = x_\pm$. There are, however, two special cases; namely $\alpha_1 = -\alpha_4$ and $\alpha_1 = -\alpha_3$, in which the interaction points can coincide in the s -channel [1], at $x = 1$ and $-\infty$ respectively. As in the 1st quantized calculation, the boundary terms can then be formally eliminated by analytic continuation, but in these special cases the validity of that procedure is questionable, as discussed in section 5 of Ref. [3]. It is possible that additional contact terms must be introduced to eliminate s -channel boundary terms for the zero measure set of final states $\alpha_1 = -\alpha_4$ or $-\alpha_3$. (In Ref. [1] these zero-measure cases

* Of course the Kaku-Kikkawa term could be redefined to include the divergent counter term acting at $x_\pm$, in addition to the finite exchange vertex acting in the region $[x_-, x_+]$, but this is obviously a matter of semantics. In any case, divergent counter terms are also required for the closed string, which has no Kaku-Kikkawa term.

were the *only* string configurations considered relevant for new interaction terms, and were believed to act in the purely forward direction. Interactions of the form shown in Fig. 1 were not considered there.)

2.2 Closed superstring

For definiteness we only consider the four massless particle scattering amplitude for type IIB closed superstrings, for the heterotic string the calculation is no doubt analogous. The closed superstring 3-vertex in the formulation of Green and Schwarz [1] is simply two “stuttered” open string vertices: one 3-vertex in terms of tilde variables and another in terms of tilde-less variables corresponding to the left and right-movers on the closed string, see Ref. [1,3] for details. The amplitude is simply the product of one open string amplitude with Koba-Nielsen variable $Z \in C$ and θ variables and another with $\bar{Z}$ and $\tilde{\theta}$ where in each sector the momenta p^I are multiplied by a factor 1/2, which henceforth we absorb in the mass-scale M . From (2.7) the 4 massless particle closed string amplitude is then

$$\begin{aligned}
 A = & \frac{F}{64\alpha_1\alpha_2\alpha_3\alpha_4} \lambda^2 \delta^8(\sum p_r) \delta(\sum \alpha_r) \delta^4(\sum \alpha_r \theta_r) \delta^4(\sum \alpha_r \tilde{\theta}_r) \\
 & \times \int \frac{d\bar{Z}dZ}{2i} Z^{-1-s/2} (1-Z)^{-1-t/2} [Z J_{23} J_{41} + (1-Z) J_{12} J_{34} + M i_M + N i_N] \\
 & \times \bar{Z}^{-1-s/2} (1-\bar{Z})^{-1-t/2} [\bar{Z} \tilde{J}_{23} \tilde{J}_{41} + (1-\bar{Z}) \tilde{J}_{12} \tilde{J}_{34} + \tilde{M} \tilde{i}_M + \tilde{N} \tilde{i}_N] , \quad (2.15)
 \end{aligned}$$

where $i_{M,N}$ and $\tilde{i}_{M,N}$ are given in (2.8) with x replaced by Z and $\bar{Z}$, respectively, and $F \equiv (\alpha_1\alpha_2\alpha_3\alpha_4)^{-9/2}$ is a factor from the external field contractions and measure factors of the 3-vertices [3] and $\tilde{J}$ is as in (2.9c) but with $\tilde{\theta}$ variables instead. Step 3 of the partial integrations is the only thing we have to do. We will take the integration in (2.15) over the complex Z plane with, as a time-split regularization, little disks with radii $\delta_{\pm}$ around $x_{\pm}$ excised. Now we do first partial integration *w.r.t.* Z and then *w.r.t.* $\bar{Z}$ (of course, the end result is independent of the order) to get, cf. (2.11a, 2.12),

$$\begin{aligned}
& \int \frac{d\bar{Z}dZ}{2i} |Z|^{-2-s}|1-Z|^{-2-t} \\
& \quad \times \left[ZJ_{23}J_{41} + (1-Z)J_{12}J_{34} + Z(1-Z) \left(1 + \frac{u}{2}\right) (-2F_{1234}) \right] \\
& \quad \times \left[\bar{Z}\tilde{J}_{23}\tilde{J}_{41} + (1-\bar{Z})\tilde{J}_{12}\tilde{J}_{34} + \bar{Z}(1-\bar{Z}) \left(1 + \frac{u}{2}\right) (-2\tilde{F}_{1234}) \right] \\
& + \oint dZ Z^{-1-s/2}(1-Z)^{-1-t/2} \\
& \quad \times \left[ZJ_{23}J_{41} + (1-Z)J_{12}J_{34} + Z(1-Z) \left(1 + \frac{u}{2}\right) (-2F_{1234}) \right] \\
& \quad \times \bar{Z}^{-1-s/2}(1-\bar{Z})^{-1-t/2} \frac{-2}{D(\bar{Z})} \\
& \quad \times \left[\tilde{F}_{1234}(\bar{Z}(A+C-2B) + B-A) + \tilde{G}_{1234}\sqrt{B^2-AC} \right] \\
& + \oint d\bar{Z} Z^{-1-s/2}(1-Z)^{-1-t/2} \frac{-2}{D(Z)} \\
& \quad \times \left[F_{1234}(Z(A+C-2B) + B-A) + G_{1234}\sqrt{B^2-AC} \right] \\
& \quad \times \bar{Z}^{-1-s/2}(1-\bar{Z})^{-1-t/2} \left[\bar{Z}\tilde{J}_{23}\tilde{J}_{41} + (1-\bar{Z})\tilde{J}_{12}\tilde{J}_{34} + \tilde{M}\tilde{i}_M + \tilde{N}\tilde{i}_N \right], \tag{2.16}
\end{aligned}$$

where the contours of the second and third integral run counter-clockwise around $x_{\pm}$ in their respective complex planes. In (2.16) we have clearly indicated the Z or $\bar{Z}$ dependence of D , see (2.3) for its definition. Inspection shows that the Z contour integrals around $x_{\pm}$ vanish. The same would be true of the $\bar{Z}$ contour integral, were it not for the strongly singular part with $D(\bar{Z})^{-2}$ in $\tilde{i}_M$ and $\tilde{i}_N$, cf. (2.8). Hence in the limit $\delta_{\pm} \rightarrow 0$ we have the first integral of (2.16) over the entire complex plane together with the divergent boundary terms

$$\begin{aligned}
A_b^{closed} = & -F \lambda^2 \delta^{18} \pi \sum_{\pm} |x_{\pm}|^{4-s} |1-x_{\pm}|^{-t} \frac{1}{16^2 \delta_{\pm}^2} \alpha_1 \alpha_2 \alpha_3 \alpha_4 \\
& \left(4\sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} N \pm M \right) \left(4\sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \tilde{N} \pm \tilde{M} \right). \tag{2.17}
\end{aligned}$$

Once again the boundary terms at $x_{\pm}$ correspond to coinciding 3-string vertices, in the configuration shown in Fig. 1b. Note that (2.17) is basically the square of the open string term (2.14), which makes sense for a local interaction.

We have not carried out the corresponding calculation for the heterotic string, but it is probably very similar. The amplitude would be as in (2.15), but with the last square bracket replaced by some $i_h(\bar{Z})$ from the bosonic sector. Doing the required partial integrations, it is again expected that a (possibly finite) residue is picked up at $x_{\pm}$.

In conclusion, we have seen in this section that a careful evaluation of the 4 particle scattering amplitude for open and closed superstrings from a Feynman diagram with two 3-vertices (for the

open string there is also one with the Kaku-Kikkawa-like quartic vertex) gives the usual amplitudes plus divergent boundary terms (2.14, 2.17), which exist for arbitrary string lengths. These undesired terms, which spoil duality, supersymmetry and Lorentz-invariance, must be cancelled by some new counter terms, if the light-cone superstring is to make sense. Those new terms are discussed in the next section.

3. Quartic contact interactions

As explained in the introduction we have shown previously [2,3] that supersymmetry required new quartic contact interaction H_{4c} , however their precise form was unknown since we do not have Q_4^- explicitly. In this section we follow another path towards H_{4c} . From our calculation of the scattering amplitude we demand that the matrix element of H_{4c} must cancel the extra boundary terms (2.14) or (2.17), so that

$$\langle\langle 0_3, 0_4 \parallel H_{4c} \parallel 0_1, 0_2 \rangle\rangle = -A_b \Big|_{x_{\pm}} \quad (3.1)$$

To get the full operator expression for H_{4c} we proceed as follows: We note that for open superstrings the matrix element of $\{Q_3^{-A}, Q_3^{-\bar{A}}\}$ has *precisely* the same form as $-A_b$ up to a numerical factor. This means that we don't have to search for a 4-vertex with the correct overlap structure, it is already contained in $\{Q_3^{-A}, Q_3^{-\bar{A}}\}$. Adjusting the numerical factor we then have H_{4c} as a general operator expression.

Let us start with the calculation of the open superstring anticommutator

$$M^{A,\bar{A}} \equiv \langle\langle 0_3, 0_4 \parallel \{Q_3^{-A}, Q_3^{-\bar{A}}\} \parallel 0_1, 0_2 \rangle\rangle \quad (3.2)$$

between massless on-shell states. This was done in [2] and is similar to, but much simpler than, the amplitude calculation of Section 2.1. For (3.2) we only need to evaluate

$$\frac{2}{3} \epsilon^{ACDE} \left\{ - \frac{\partial}{\partial \phi_2^C} \frac{\partial}{\partial \phi_2^D} \frac{\partial}{\partial \phi_2^E} \frac{\partial}{\partial \phi_1^A} + \frac{\partial}{\partial \phi_2^A} \frac{\partial}{\partial \phi_1^C} \frac{\partial}{\partial \phi_1^D} \frac{\partial}{\partial \phi_1^E} \right\} E_F \Big|_{\phi_1 = \phi_2 = 0}$$

where the minus sign* is from the order reversal under Hermitian conjugation, cf. (2.5). We find for the anticommutator matrix element

$$M^{A,\bar{A}} = f g^2 \delta^{14} \sum_{\pm} (\pm) |x_{\pm}|^{2-s/2} |1 - x_{\pm}|^{-1/2} \frac{1}{2 \delta_{\pm}} \sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \left(4 \sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} N \pm M \right), \quad (3.3)$$

* We missed this minus in $D^{(a,b)}$ of (3.16) in [2], which entails a change to plus of the sign in front of $\alpha_3 \alpha_4$ in the curly bracket of $D^{A\bar{B}}$ of (3.23) there.

which is for general values of α_r , for some special α_r the discussion at the end of Section 2.1 applies. Remarkably, (3.3) is the same as (2.14) up to a factor $-1/8$. As noted above, this immediately gives us the form of the general quartic contact Hamiltonian for open superstrings (Fig. 1a)

$$H_{4c}^{open} = \int d\mu_4^{open} (\alpha_1 \alpha_2 \alpha_3 \alpha_4)^{-3/2} \langle N_4, N_3 | h_{4c}^{open} | N_1, N_2 \rangle \text{Tr}\{\Phi(4)\Phi(3)\Phi(2)\Phi(1)\} \quad (3.4)$$

with

$$d\mu_4^{open} \equiv \prod_{r=1}^4 d\alpha_r d^3 p_r d^4 \theta_r \delta(\sum \alpha_r) \delta(\sum p_r) \delta(\sum \alpha_r \theta_r),$$

where $\Phi(r)$ is the open superstring field creation-annihilation operator (with occupation numbers N_r) and h_{4c} in terms of first quantized modes is (no sum over A)

$$h_{4c}^{open} = \frac{1}{8} \sum_{\text{contractions}} \sum_{\{N\}} \left\{ \langle \dots, N | Q_3^{-A} \rangle \langle Q_3^{-\bar{A}} | N, \dots \rangle \right. \\ \left. + \langle \dots, N | Q_3^{-\bar{A}} \rangle \langle Q_3^{-A} | N, \dots \rangle \right\}, \quad (3.5)$$

The notation in (3.5) is as follows: The kets $|Q_3^- \rangle$ are the three-string overlap vertices defined in [1]. In the bra-ket product above, one of the string states in the ket vertex $|Q_3^- \rangle$ is identified with one of the string states in the bra vector $\langle Q_3^- |$; all possible occupation numbers $\{N\}$ of the identified string state are summed over. The possible pairings (contractions) indentifying one string state in the ket vertex and one string state in the bra vertex must also be summed over; in the 4-point amplitude these pairings correspond to channels. In fact, only those pairings corresponding to the non-planar tu channel give a non-zero result, cf. (3.3). Note that if we take the matrix element of (3.4), the external field contractions give a factor $(\alpha_1 \alpha_2 \alpha_3 \alpha_4)^{-1}$ so that we end up with a factor $f = (\alpha_1 \alpha_2 \alpha_3 \alpha_4)^{-5/2}$ and that the trace in (3.4) gives for the non-planar tu contractions the Chan-Paton factor $\text{Tr} \lambda_1 \lambda_3 \lambda_2 \lambda_4$. [As an aside we note that had we *not* used plus signs in front of $(1 + c_1 A_{01})$ in E_F , see Section 2.1, the monomial N would pick up a minus sign in our amplitude calculation, e.g., (2.11, 14), whereas in the anticommutator matrix element (3.3) M would, so that the factor of (3.5) would change to $-1/8$. However the new F_{1234} in the amplitude would no longer agree with [1]. Anyway, our calculation [2,3] of E_F did give us the plus signs.]

For open superstrings the supersymmetry relation (1.1) to $O(g^2)$ is

$$H_{4c} + H_{4KK} = \frac{1}{2} \left\{ Q_3^{-A}, Q_3^{-\bar{A}} \right\} + \frac{1}{2} \left\{ Q_2^{-A}, Q_4^{-\bar{A}} \right\} + \frac{1}{2} \left\{ Q_4^{-\bar{A}}, Q_2^{-A} \right\}$$

and from the fact that $H_{4c} = \frac{1}{8} \left\{ Q_3^{-A}, Q_3^{-\bar{A}} \right\}$ we see that the contact part of $\left\{ Q_2^{-A}, Q_4^{-\bar{A}} \right\} + \left\{ Q_4^{-\bar{A}}, Q_2^{-A} \right\}$ must equal $-\frac{3}{4} \left\{ Q_3^{-A}, Q_3^{-\bar{A}} \right\}$, the rest generating the Kaku-Kikkawa exchange interaction H_{4KK} . It would be interesting to know the precise form of this omnipotent Q_4^- .

Turning to the contact interaction for the closed superstring the situation is now rather simple. As we remarked below (2.17) it just is the “square” of the open string term, so that we find (Fig. 1b)

$$H_{4c}^{closed} = \pi \int d\mu_4^{closed} (\alpha_1 \alpha_2 \alpha_3 \alpha_4)^{-7/2} \langle N_4, \tilde{N}_4, N_3, \tilde{N}_3 | h_{4c}^{open} \tilde{h}_{4c}^{open} | N_1, \tilde{N}_1, N_2, \tilde{N}_2 \rangle \Psi(4) \Psi(3) \Psi(2) \Psi(1) \quad (3.6)$$

with

$$d\mu_4^{closed} \equiv d\mu_4^{open} d^4 \tilde{\theta} \delta^4 \left(\sum \alpha_r \tilde{\theta}_r \right),$$

where in $h_{4c} \tilde{h}_{4c}$ the contractions in the tilde and tildeless sector are the same (this is also the case in the scattering amplitude calculation $\langle \langle 4, 3 \parallel H_3^{\epsilon l} H_3^{\epsilon l} \parallel 1, 2 \rangle \rangle$ of Section 2.2).

As the two (I,II) cubic supersymmetry generators split “unevenly” over the two sectors, i.e. $|Q_{3,I}^{closed} = |Q_3^- \rangle \otimes |\tilde{H} \rangle$ and $|Q_{3,II}^{closed} = |H \rangle \otimes |\tilde{Q}_3^- \rangle$, we cannot write H_{4c}^{closed} as simply proportional to $\{Q_{3,a}^{A,closed}, Q_{3,a}^{\bar{A},closed}\}$ with $a = I$ or II as was the case for the open superstring and, probably, for the heterotic string. The form of Q_4^{closed} is again unknown. We have bypassed these difficulties by constructing the quartic contact interactions (3.4) and (3.6) from the requirement that they cancel the divergent boundary terms of the tree amplitude in Section 2. To construct $Q_{4,I,II}$ explicitly and check the super-Poincaré algebra to $O(\lambda^2)$ is a task which is no doubt feasible, but which may be very laborious.

4. Vacuum condensates

For the cubic closed superstring theory, we found in Ref. [3] a trial state $|| F \rangle\rangle$ for which the energy expectation value could be arbitrarily negative; from this we concluded that the theory, as it stood, was not supersymmetric and that the cure came probably from new higher order contact interactions. Having derived the quartic contact interaction (3.6) we revisit this issue. For our specific ansatz

$$|| F \rangle\rangle = \exp \left[\beta \int_0^\infty d\alpha \alpha \int d^8 p d^4 \theta d^4 \tilde{\theta} f_\alpha(p^I)(t + \tilde{t}) \Psi_{-\alpha}^{N=\tilde{N}=0}(p, \theta, \tilde{\theta}) \right] || 0 \rangle\rangle, \quad (4.1)$$

where $|| 0 \rangle\rangle$ is the perturbative vacuum and $t \equiv \frac{\epsilon^{ABCD}}{24} \theta^A \bar{\theta}^B \bar{\theta}^C \bar{\theta}^D$, the energy expectation value is

$$E_F(\beta) \equiv \langle \langle F || H_2 + \lambda H_3 + \lambda^2 H_{4c} || F \rangle \rangle / \langle \langle F || F \rangle \rangle \\ = \beta^2 E_2 + \beta^3 E_3 + \beta^4 E_4, \quad (4.2)$$

where, after the Grassmann integrals are performed, the coefficients E_2 and E_3 are integrals of the $f_\alpha(p)$ given in [3] and E_4 from H_{4c} will be given shortly. Choosing, for example, $f_\alpha(p) = \exp(-|\vec{p}|^2 - \alpha^2)$ the constants E_2 and E_3 are finite and positive, so that, without the quartic term, the energy $\rightarrow -\infty$ as the parameter $\beta \rightarrow -\infty$. In order to calculate $\langle\langle F || H_{4c} || F \rangle\rangle$ we need the following Grassmann integral

$$\begin{aligned} & \int \prod_{r=1}^4 d^4\theta_r d^4\tilde{\theta}_r \delta(\sum \alpha_r \tilde{\theta}_r) \delta(\sum \alpha_r \theta_r) \\ & \quad \times \left(4 \sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} N \pm M \right) \left(4 \sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \tilde{N} \pm \tilde{M} \right) \\ & \quad \times (t_1 + \tilde{t}_1)(t_2 + \tilde{t}_2)(t_3 + \tilde{t}_3)(t_4 + \tilde{t}_4) \\ & = 32 \alpha_1^2 \alpha_2^2 \alpha_3^2 \alpha_4^2 K_\pm(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \end{aligned} \quad (4.3)$$

with

$$\begin{aligned} K_\pm \equiv & \left\{ \left[4 \sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} (\alpha_1 \alpha_3 + \alpha_2 \alpha_4) \mp (4 \alpha_1 \alpha_2 \alpha_3 \alpha_4 + (\alpha_1 \alpha_3 + \alpha_2 \alpha_4)^2) \right]^2 \right. \\ & + \left. \left[4 \sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} (\alpha_1 \alpha_4 + \alpha_2 \alpha_3) \pm (4 \alpha_1 \alpha_2 \alpha_3 \alpha_4 + (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)^2) \right]^2 \right. \\ & \left. + (\alpha_1 + \alpha_2)^8 \right] (\alpha_1 + \alpha_2)^{-8} \end{aligned}$$

and we see that the fermionic integral is non-zero and positive. With this result we get for the quartic coefficient of (4.2)

$$\begin{aligned} E_4 = \frac{\pi}{32} \lambda^2 \int d\mu_{4B} \sum_{\pm} & \left[\frac{1}{\sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4}} \frac{1}{\delta_{\pm}^2} |x_{\pm}|^{4-s} |1 - x_{\pm}|^{-t} K_{\pm} \right. \\ & \left. (f_{\alpha_1}(p_2) f_{\alpha_2}(p_2) f_{\alpha_3}^*(p_3) f_{\alpha_4}^*(p_4) + \text{c.c.}) \right] \end{aligned} \quad (4.4)$$

with

$$d\mu_{4B} \equiv \prod_{r=1}^4 d\alpha_r d^8 p_r \delta(\sum \alpha_r) \delta^8(\sum p_r) \left(\theta(\alpha_1) \theta(\alpha_2) \theta(-\alpha_3) + \theta(-\alpha_1) \theta(-\alpha_2) \theta(\alpha_3) \right),$$

where the step functions indicate that strings 1 and 2 are incoming ($\alpha_1, \alpha_2 > 0$) and 3 and 4 outgoing and *vice versa*. In (4.4) we read for s and t , cf. Ref. 1,

$$s = \overline{P}_{12} \cdot \overline{P}_{12} / \alpha_1 \alpha_2 = \overline{P}_{34} \cdot \overline{P}_{34} / \alpha_3 \alpha_4$$

$$t = \left((\alpha_1 \alpha_3 + \alpha_2 \alpha_4) \overline{P}_{23} \overline{P}_{41} + (\alpha_1 + \alpha_3)^2 \overline{P}_{12} \overline{P}_{34} \right) / 2 \alpha_1 \alpha_2 \alpha_3 \alpha_4,$$

where $\overline{P}$ is defined below (2.9). For positive $f_\alpha(p)$ the coefficient E_4 is positive definite, just as E_2 and E_3 , and will prevent runaway of $\beta \rightarrow -\infty$, confirming our expectations [3].

For completeness we give the coefficients E_2 and E_3 calculated in [3]

$$E_2 = \int_0^\infty d\alpha \int d^8 p \, 8 f_\alpha^*(p) |\vec{p}|^2 f_\alpha(p) \quad (4.5a)$$

$$E_3 = \lambda \int d\mu_{3B} \left\{ 2 |\alpha_1 \alpha_2 \alpha_3|^{-5/2} (\alpha_1^4 + \alpha_2^4 + \alpha_3^4) \exp \left(-\tau_0 |\overline{P}|^2 / \alpha_1 \alpha_2 \alpha_3 \right) \right. \\ \left. P^L P^R (f_{\alpha_3}^*(p_3) f_{\alpha_2}(p_2) f_{\alpha_1}(p_1) + c.c.) \right\} \quad (4.5b)$$

with

$$d\mu_{3B} \equiv \prod_{r=1}^3 d\alpha_r d^8 p_r \delta \left(\sum \alpha_r \right) \delta^8 \left(\sum p_r \right) (\theta(\alpha_1) \theta(\alpha_2) + \theta(-\alpha_1) \theta(-\alpha_2))$$

We now show that λ has mass dimension $[\lambda] = -4$, which is appropriate for a 10-d gravitational coupling constant ($[O]$ means $\dim[O]$). An overall factor of the mass scale M ($=1$ for us) gives (4.2) the correct dimension as long as the coefficients $E_{2,3,4}$ are dimensionless (in (3.5) $|\vec{p}|^2$, $|\overline{P}|^2$ and $P^L P^R$ are multiplied by M^{-2}). With α dimensionless, and $[\theta] = -[d\theta] = -1/2$ we have* $[\Psi] = -6$, so that from (4.1) $[f_\alpha(p)] = -4$ and $E_{2,3,4}$ are indeed dimensionless for $[\lambda] = -4$.

The question now is what condensate, within our ansatz (4.1), is preferred energetically, or which function $f_\alpha(p)$ has the lowest minimum in energy (4.2)? A careful regulation is required, but here we will limit ourselves to a heuristic discussion. Suppose we have a lattice cutoff [6], where the smallest possible α is α_0 . The number of modes for a string of length $2p^+ = \alpha$ is then $N = \alpha/\alpha_0$. We do not know precisely the relation between the timesplit cutoff $\delta_\pm$ and the mode number cutoff N , but heuristically we can write for the singularity $1/\delta \approx G(1^-)$, where $G(z) \equiv (1-z)^{-1} = \exp(\sum_{n=1}^\infty z^n/n!)$. A mode number cutoff N would replace $G(z)$ by $(1-ze^{-1/N})^{-1}$, which evaluated at $z = 1^-$ gives the relation

$$\frac{1}{\delta} \approx N = \alpha/\alpha_0. \quad (4.6)$$

With this relation and the α integrals in the coefficients $E_{2,3,4}$ cutoff at α_0 we find for more or less arbitrary $f_\alpha(p)$, being non-zero and $O(1)$ over range $|\alpha| < \alpha_{max} \equiv R\alpha_0$ and $|\vec{p}| < O(1)$, the energy expectation value (4.2)

$$E(\beta) = \frac{\alpha_0}{R} \left(C_2 (\beta R)^2 + C_3 \lambda (\beta R)^3 \alpha_0^{-1/2} + C_4 \lambda^2 (\beta R)^4 \right), \quad (4.7)$$

* For any first quantized operator o the quadratic field operator [1] is $O_2 = \int d\alpha \alpha \int d^8 p d^4 \theta d^4 \tilde{\theta} \Psi o \Psi$. Demanding O_2 to have the same dimension as o gives $[\Psi] = -6$.

where C_2, C_3, C_4 are positive constants of order 1. This has a minimum at

$$\begin{aligned}\beta_{min} &= -\frac{3C_3}{4C_4} \frac{\sqrt{\alpha_0}}{\alpha_{max}} \frac{1}{\lambda} \\ E_{min} &= -\frac{C_3}{4} \left(\frac{3C_3}{4C_2}\right)^3 \frac{1}{\lambda^2} \frac{1}{\alpha_{max}},\end{aligned}\tag{4.8}$$

Clearly for $\alpha_{max} \rightarrow \alpha_0 \rightarrow 0$ the energy is lowered. We see that, within this ansatz, a string-field condensate is preferred with vanishing α but, as it stands, 8 transverse momenta still of order 1 ($= M$). This would imply that condensates are non-uniform in different directions, suggesting that the non-perturbative vacuum might be different from flat 10-dimensional Minkowski space.

However, we are still missing something, and the estimates above should not be taken seriously. For one thing, E_{min} is still negative, which is clearly wrong for a supersymmetric theory. The origin of this error is first of all that, even at $O(\lambda^2)$, there is almost certainly an additional *quadratic* counter term $\lambda^2 H_{2c}$ which we have not yet calculated or taken into account. The need for H_{2c} is apparent when one considers the one-loop correction to the superstring propagator. Here again there is the possibility of join/split points coinciding, and strong divergences can be expected to arise as in the (non-planar) tree amplitudes. In the super-Poincaré algebra, we also see that $\{Q_3^-, Q_3^-\}$ produces, in addition to a quartic interaction, also a divergent quadratic term when there are two internal field contractions. H_{2c} could probably be determined from a calculation of the appropriate one-loop light-cone diagram. To our knowledge, this calculation has never been performed for any fermionic string field theory. In addition, if the condensate is large enough, terms in the Hamiltonian of order higher than λ^2 (see below) would become significant.

5. Discussion

In our two previous papers it was shown that the super-Poincaré algebra demands quartic contact terms in the light-cone superstring field theory; and in Section 2 of this paper we have shown that, without such terms, superstring tree amplitudes are neither finite nor supersymmetric. Finally, in Section 3, the quartic terms required to restore these desirable properties to $O(\lambda^2)$ were constructed in detail.

However, our work does not complete the construction of light-cone superstring field theory. The reason is that in scattering amplitudes involving three or more 3-string vertices, there are points in moduli space corresponding to three or more light-cone vertices coinciding. By the same reasoning as in Section 2, there will again be divergent boundary terms in the integration over Koba-Nielsen variables, which must be cancelled by further contact terms of higher order in string fields and coupling constant λ . In short, superstring field theory in light-cone gauge is very likely non-polynomial. This leads to two questions. First, if the light-cone field theory is non-polynomial,

is it at all useful? Second, are covariant formulations of superstring field theory also afflicted with endless counter terms?

In answer to the first question, it should be noted that there are many complicated potentials in the real world which are fairly well represented by the first few terms in their Taylor expansion. If this is also the case for light-cone superstrings, then it may be possible to calculate, for example, the dilation condensate, along the lines of Section 4, with a light-cone Hamiltonian truncated to $O(\lambda^2)$. An advantage of the light-cone formulation is that it lends itself to the variational methods outlined in Ref. [3] and Section 4. For this purpose we still need the $\lambda^2 H_{2c}$ term mentioned at the end of Section 4. The construction of this term requires a calculation of the 1-loop correction to the superstring propagator, which is under study.

Regarding the question of whether covariant formulations of superstring field theory also require contact terms, the answer is that we do not know. The covariant fermionic string theory of Witten [7] does have local operators at the join/split vertex, as in the light-cone case; whether this leads to analogous problems when vertices coincide remains to be seen. If the covariant theory does terminate at cubic order in the string fields, then the contact interactions we have found in light-cone gauge must be generated when unphysical fields are integrated out. This is in fact what happens in supersymmetric point field theories, where a ϕ^3 Lagrangian generates a ϕ^4 Hamiltonian upon integrating out the auxiliary field. However the other possibility of having genuine higher order terms in addition to the cubic action of Ref. 7 is still not ruled out.

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Appendix: Neveu-Schwarz-Ramond String

In this Appendix we revisit Mandelstam's calculation [4] of the scattering amplitude of four tachyons ("pions") of the open Neveu-Schwarz string, in particular the tu -part of the amplitude. This illustrates, in simplified form, the same ingredients as the superstring results described in Section 2. We will refer to Equation (6.5), say, of Ref. 4, as M(6.5).

Mandelstam found the standard Lovelace-Shapiro st -amplitude $A(s,t) = (s+t+1)B(-s,-t)$, but did not consider the tu -channel. To get the tu -amplitude we can use M(6.5) with, by duality, the Koba-Nielsen variable Z in the interval $I \equiv [-\infty, Z_-] \cup [Z_+, 0]$ instead of the interval $[0, 1]$ appropriate for the st -amplitude. Taking the naive short string limit $\alpha_3 = 0, \alpha_2 \rightarrow 0$ gives then, cf. M(6.6),

$$A(t, u) = (1+t) \int_{-\infty}^0 dZ |Z|^{-2-t} |1-Z|^{-1-s} = (u+t+1)B(-u,-t), \quad (A1)$$

where we used the change of variables $Y = (1-Z)^{-1}$ and $s+t+u = -2$ to get the Euler beta-function. However for finite string lengths we will pick up divergent boundary terms at

$$Z_{\pm} = (c-a)^{-2} \left[-(c+ab) \pm 2\sqrt{abc} \right], \quad (A2)$$

which are the zeros of the quadratic $D(Z) = (c-a)^2(Z-Z_-)(Z-Z_+)$ and $a \equiv -\alpha_1/\alpha_4, b \equiv -\alpha_2/\alpha_4, c \equiv \alpha_3/\alpha_4$, see [4] for more details. For the tu -amplitude we have from M(6.10)

$$\begin{aligned} A(t, u) = - \int_{Z \in I} dZ |1-Z|^{-1-s} |Z|^{-1-t} \left\{ (t-1)CZ + tB(1-Z) + sBZ \right. \\ \left. + (s-1)A(1-Z) + 2CZ + 2A(1-Z) + (d-2)abc(1-Z)Z D^{-1} \right\} D^{-1}. \end{aligned} \quad (A3)$$

where $A = (1-a)^2, B = b+ac, C = (1+c)^2$. Using partial integration formulae as below (2.10) the integrand simplifies for $d = 10$ to $-(s+t+1) |Z|^{-1-t} |1-Z|^{-1-s}$, but we pick up boundary terms at $Z = -\infty, Z_-, Z_+$ and 0. The boundary terms at $-\infty$ and 0 vanish by analytic continuation in u and t , but not those at $Z_{\pm}$. The exchange interaction [5] will add the integration interval $[Z_-, Z_+]$, so that we have finally

$$\begin{aligned} A(t, u) = - (s+t+1) \int_{-\infty}^0 dZ |Z|^{-1-t} |1-Z|^{-1-s} \\ + \left[|Z|^{-t} |1-Z|^{-s} (Z(C+A-2B) + B-A) D(Z)^{-1} \right]_{Z_--\delta_-}^{Z_++\delta_+} \end{aligned} \quad (A4)$$

$$\begin{aligned}
&= (u+t+1)B(-u, -t) \\
&\quad + |Z_+|^{-t} |1-Z_+|^{-s} \frac{1}{2\delta_+} + |Z_-|^{-t} |1-Z_-|^{-s} \frac{1}{2\delta_-} ,
\end{aligned} \tag{A5}$$

where $\delta_{\pm} \rightarrow 0$ provides a time split regularization. The divergent boundary terms in (A5) must be subtracted by an appropriate quartic contact interaction.

We will not give in detail the calculation of the scattering amplitude of four tachyons of the closed Neveu-Schwarz string. The amplitude is an integral over $dZd\bar{Z}$ of a term depending on Z as given by (A3) multiplied by the same one depending on $\bar{Z}$. The standard amplitude [8] follows together with boundary terms due to the D^{-2} singularities in (A3), regulated by excising small disks with radii $\delta_{\pm}$,

$$\begin{aligned}
A_{\text{closed}} &= -\pi \frac{\Gamma(-s)}{\Gamma(1+s)} \frac{\Gamma(-t)}{\Gamma(1+t)} \frac{\Gamma(-u)}{\Gamma(1+u)} \\
&\quad + \pi \sum_{Z=Z_{\pm}} |Z|^{-2t} |1-Z|^{-2s} \frac{1}{4\delta_{\pm}^2} .
\end{aligned} \tag{A6}$$

Again a quartic contact interaction must subtract the divergent, non-dual, non-Lorentz invariant boundary terms in (A6). Note that these closed string contact interactions are like the square of the open string ones as might be expected for local interactions.

Two final remarks: First, we repeat that a first-quantized calculation, with $p^+ = 0$ for all but two external strings, immediately gives (A1), whereas the boundary terms (A5) only show up for finite p^+ momenta. Indeed we can check that these boundary terms in (A5) vanish by analytic continuation in t for $\alpha_3, \alpha_2 \rightarrow 0$ and $\alpha_3/\alpha_2 \rightarrow 0$, since in that case $Z_{\pm} \rightarrow 0$ so that the boundary terms go as $|\delta_{\pm}|^{-1-t}$, which vanish for $\text{Re}(t) < -1$. The same holds for the superstring calculation in Section 2. Second, it might be that our calculation has some relevance for the gauge invariant NSR theory of Witten [7], which has similar vertex insertions as Mandelstam's [4], and perhaps that theory needs counter terms also, see Section 5. If so, these terms may be proportional (with a divergent coefficient perhaps, cf. (3.4)) to the higher order Chern-Simons forms ($n \geq 2$)

$$I_{2n+1} = (n+1)g^{n-1} \oint Y\left(\frac{\pi}{2}\right)^{2n-2} \int_0^1 dt A \star (tQA + gt^2 A \star A)^n , \tag{A7}$$

where the n -th power involves the product $\star$ ($n-1$)-times and the functional integral $\oint$ and the product $\star$ are defined in equations (34) and (37) of Ref. 7. Obviously I_{2n+1} has zero ghost number and is gauge invariant, it is also supersymmetric as can be checked following Ref. 7. Gauge symmetry may thus greatly constrain the form of higher order “contact” terms, but explicit calculations are needed to obtain their coefficients and, first of all, to see if these coefficients are zero or not.

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Fig. 1. Quartic superstring contact interactions in the light-cone gauge: (a) non-planar contact interactions for open superstrings (b) contact interactions for closed type II superstrings (and also for heterotic strings).

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