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Planar projections of spatial Pythagorean-hodograph curves

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Abstract

Although the orthogonal projection of a spatial Pythagorean–hodograph (PH) curve on to a plane is not (in general) a planar PH curve, it is possible to construct spatial PH curves so as to ensure that their orthogonal projections on to planes of a prescribed orientation are planar PH curves. The construction employs an analysis of the root structure of the components of the quaternion polynomials that generate spatial PH curves, and it encompasses both helical and non–helical spatial PH curves. An initial characterization for orthogonal projections of spatial PH curves on to the coordinate planes provides the basis for a generalization to projections of arbitrary direction, based on unit quaternion rotation transformations of $\mathbb{R}^3$.

Key words: Pythagorean–hodograph curves; helical curves; planar orthogonal projections; polynomial factorizations; quaternions; spatial rotations.

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1. Introduction

Planar projections of three–dimensional shapes play a key role in their specification and comprehension, whether as figures in a book, images on a computer screen, or orthographic views in engineering drawings. The first systematic study of the geometry of space curves appears in the 1731 treatise *Recherche sur les courbes à double courbure* [4] by the French mathematician Alexis Claude Clairault (1713–1765). Clairault's approach was to define space curves by their projections on to two of the coordinate planes: the curvatures of these two planar projections determine the shapes of space curves, and this motivated him to call them "curves of double curvature" [2]. The polynomial Pythagorean–hodograph (PH) curves are an important subclass of spatial curves. A polynomial PH curve $\mathbf{r}(\xi)$ is characterized by the fact that its *parametric speed* (the derivative $\mathrm{d}s/\mathrm{d}\xi$ of arc length s with respect to the curve parameter) is a polynomial in ξ [6], a feature that endows PH curves with advantageous computational properties.

It may be desirable, in certain applications, to ensure that these properties hold for both a spatial PH curve $\mathbf{r}(\xi)$ and its orthogonal projection $\mathbf{p}(\xi)$ on to a prescribed plane — i.e., that $\mathbf{p}(\xi)$ is a planar PH curve. This would, for example, ensure the ratio of arc lengths along $\mathbf{r}(\xi)$ and $\mathbf{p}(\xi)$ is a rational function of ξ, and allow the determination of points of equal fractional arc length along the two curves by solving a polynomial equation. Also, the rational offsets [6] to planar PH curves allow construction of a rationally parameterized planar domain in tangential/normal coordinates about $\mathbf{p}(\xi)$, which may prove useful in designing rational surfaces containing $\mathbf{r}(\xi)$ by specifying a "height function" above the plane, over this domain. A more detailed analysis of the applications of spatial PH curves with planar PH projections is deferred to a future study.

The constructions of planar and spatial Pythagorean–hodograph (PH) curves are based on models that employ the complex–number and quaternions algebras, respectively [6]. These models yield constructions that are invariant with respect to the chosen orientation of the coordinate axes in $\mathbb{R}^2$ and $\mathbb{R}^3$ and reflect the fact that, whereas rotations in $\mathbb{R}^2$ are commutative, rotations about distinct axes in $\mathbb{R}^3$ are (in general) non–commutative. A further indicator of the disparate nature of planar and spatial PH curve representations is the fact that an orthogonal projection[1] of a spatial PH curve on to a given plane does not, in general, yield a planar PH curve. However, it is possible to construct spatial PH curves such that their parallel planar projections in a general direction are guaranteed to be planar PH curves. The intent of the present study is to formulate a comprehensive characterization of these constructions.

The spatial PH curves encompass, as a special subset, the family of all *helical* polynomial space curves. A helical curve is characterized by the fact [20] that its unit tangent $\mathbf{t}$ maintains a constant angle ψ (the helix angle) with respect to a fixed unit vector $\mathbf{a}$ (the helix axis): $\mathbf{a} \cdot \mathbf{t} = \cos\psi$. Equivalently, a helical curve exhibits a constant ratio of curvature and torsion: $\kappa/\tau = \tan\psi$. The helix axis $\mathbf{a}$ can also be interpreted

[1] We focus here on *orthogonal projections* — i.e., parallel projections on to planes orthogonal to the projection direction.

as the unitization of the Darboux vector $\mathbf{d} = \kappa\,\mathbf{b} + \tau\,\mathbf{t}$, which describes the rate of rotation of the Frenet frame (where $\mathbf{t}$ and $\mathbf{b}$ are the unit tangent and binormal vectors). The projection of a helical PH curve on to a plane with normal vector $\pm\mathbf{a}$ is obviously a planar PH curve. However, there also exist non–helical PH curves that admit planar PH curves as their projections on to a given plane.

In terms of the quaternion representation, it is shown that the problem of identifying when the projection of a spatial PH curve on to one of the coordinate planes yields a planar PH curve is equivalent to determining when the product of the sums–of–squares of two pairs of real polynomials $a(\xi), b(\xi)$ and $c(\xi), d(\xi)$ is equal to the perfect square of a real polynomial $w(\xi)$ — i.e., $[\,a^2(\xi) + b^2(\xi)\,]\,[\,c^2(\xi) + d^2(\xi)\,] = w^2(\xi)$. Using unit quaternions as spatial rotation operators, this result is then generalized to planar projections of spatial PH curves in general direction.

The remainder of this paper is organized as follows. We commence with a brief review of the complex–number and quaternion representations of planar and spatial PH curves in Section 2, and identify some key properties of the helical and "double" PH (DPH) curves. Since the quaternion form of spatial PH curves incorporates planar PH curves — residing in any plane in $\mathbb{R}^3$ — as special instances, Section 3 discusses the identification of these degenerate instances. Section 4 then develops the characterization of spatial PH curves that have planar PH curve projections on to one of the coordinate planes, and provides both helical and non–helical examples of these curves. The generalization for projections on to planes with normals of general direction is then developed in Section 5, through a quaternion mapping of the standard Cartesian frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to a general orthonormal frame $(\mathbf{l}, \mathbf{m}, \mathbf{n})$. Finally, Section 6 summarizes the key results of this study, and identifies issues that deserve further investigation.

2. Planar and spatial Pythagorean-hodograph curves

We focus on the hodograph (parametric derivative) $\mathbf{r}'(\xi)$ of a curve $\mathbf{r}(\xi)$, since integration of the hodograph only introduces a constant that defines the initial curve point $\mathbf{r}(0)$. Henceforth we denote by $|\cdot|$ the absolute value of a real or complex number, the Euclidean norm of a vector in $\mathbb{R}^2$ or in $\mathbb{R}^3$, or the norm of a quaternion.

A planar PH curve $\mathbf{r}(\xi) = (x(\xi), y(\xi))$ may be constructed [5] from a complex polynomial $\mathbf{w}(\xi) = f(\xi) + \mathrm{i}\,g(\xi)$ by identifying $x'(\xi)$ and $y'(\xi)$ with the real and imaginary part of the expression

$$\mathbf{r}'(\xi) \,=\, \mathbf{w}^2(\xi)\,, \tag{1}$$

and integrating with respect to ξ. The resulting curve satisfies

$$x'^2(\xi) + y'^2(\xi) \,=\, \sigma^2(\xi)\,,$$

where

$$x'(\xi) \,=\, f^2(\xi) - g^2(\xi)\,, \quad y'(\xi) \,=\, 2\,f(\xi)g(\xi)\,, \quad \sigma(\xi) \,=\, f^2(\xi) + g^2(\xi)\,. \tag{2}$$

The polynomial $\sigma(\xi)$ defines the *parametric speed* of the curve $\mathbf{r}(\xi)$, i.e., the derivative $\mathrm{d}s/\mathrm{d}\xi$ of the curve arc length s with respect to the parameter ξ.

A spatial PH curve may be constructed [3] from a polynomial

$$\mathcal{A}(\xi) \;=\; u(\xi) + v(\xi)\,\mathbf{i} + p(\xi)\,\mathbf{j} + q(\xi)\,\mathbf{k}\,, \tag{3}$$

expressed in the quaternion basis $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ through the product

$$\mathbf{r}'(\xi) \;=\; \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)\,, \tag{4}$$

where $\mathcal{A}^*(\xi) = u(\xi) - v(\xi)\,\mathbf{i} - p(\xi)\,\mathbf{j} - q(\xi)\,\mathbf{k}$ is the quaternion conjugate of $\mathcal{A}(\xi)$. Integration yields a PH space curve $\mathbf{r}(\xi) = (x(\xi), y(\xi), z(\xi))$ satisfying

$$x'^2(\xi) + y'^2(\xi) + z'^2(\xi) \;=\; \sigma^2(\xi)\,, \tag{5}$$

where

$$\begin{aligned}
x'(\xi) \;&=\; u^2(\xi) + v^2(\xi) - p^2(\xi) - q^2(\xi)\,, \\
y'(\xi) \;&=\; 2\,[\,u(\xi)q(\xi) + v(\xi)p(\xi)\,]\,, \\
z'(\xi) \;&=\; 2\,[\,v(\xi)q(\xi) - u(\xi)p(\xi)\,]\,, \\
\sigma(\xi) \;&=\; u^2(\xi) + v^2(\xi) + p^2(\xi) + q^2(\xi)\,,
\end{aligned} \tag{6}$$

and $\sigma(\xi)$ is again the parametric speed of $\mathbf{r}(\xi)$.

When the complex polynomial $\mathbf{w}(\xi)$ in (1) and quaternion polynomial $\mathcal{A}(\xi)$ in (4) are of degree m, integration yields planar and spatial PH curves of degree $n = 2m + 1$. Note that the planar PH curves are subsumed as a proper subset of the spatial PH curves, since choosing

$$u(\xi) \;=\; \frac{f(\xi)}{\sqrt{2}}\,, \quad v(\xi) \;=\; \frac{f(\xi)}{\sqrt{2}}\,, \quad p(\xi) \;=\; \frac{g(\xi)}{\sqrt{2}}\,, \quad q(\xi) \;=\; \frac{g(\xi)}{\sqrt{2}}\,,$$

in (6) yields

$$x'(\xi) = f^2(\xi) - g^2(\xi)\,, \;\; y'(\xi) = 2\,f(\xi)g(\xi)\,, \;\; z'(\xi) = 0\,, \;\; \sigma(\xi) = f^2(\xi) + g^2(\xi)\,.$$

Many algorithms for the construction of planar and spatial PH curves that satisfy prescribed geometrical constraints have been developed, for example [7, 8, 12, 14, 16, 17, 19, 23].

One class of spatial PH curves that admit planar PH curve projections is the family of *polynomial helical curves*, also known as *curves of constant slope* [20, 24]. The tangent to such a curve $\mathbf{r}(\xi) = (x(\xi), y(\xi), z(\xi))$ makes a constant angle $\psi \in [\,0, \pi\,]$ (the *helix angle*) with a fixed direction $\mathbf{a}$ (the *helix axis*) in space. Choosing a coordinate system in which $\mathbf{a}$ is parallel to the z–axis, we have

$$\frac{\mathrm{d}z}{\mathrm{d}s} \;=\; \frac{\mathrm{d}z}{\mathrm{d}\xi}\,\frac{\mathrm{d}\xi}{\mathrm{d}s} \;=\; \frac{z'}{\sigma} \;=\; \cos\psi\,,$$

and consequently $z'(\xi) = \cos\psi\,\sigma(\xi)$. Hence, the projection $\mathbf{p}(\xi) = (x(\xi), y(\xi))$ of $\mathbf{r}(\xi)$ on to the (x, y) plane satisfies

$$x'^2(\xi) + y'^2(\xi) \;=\; (1 - \cos^2\psi)\,\sigma^2(\xi)\,,$$

and $\mathbf{p}(\xi)$ is a planar PH curve with the parametric speed $\sigma_{\mathbf{p}}(\xi) = \sin\psi\,\sigma(\xi)$. It is known [15] that all spatial PH cubics are helical, and that every *polynomial* helical curve is necessarily a spatial PH curve [13].

Helical curves are also characterized [20, 24] by a constant curvature/torsion ratio, $\kappa(\xi)/\tau(\xi) = \tan\psi$. Since the curvature and torsion are defined by

$$\kappa(\xi) \;=\; \frac{|\,\mathbf{r}'(\xi)\times\mathbf{r}''(\xi)\,|}{\sigma^3(\xi)}\,, \quad \tau(\xi) \;=\; \frac{(\mathbf{r}'(\xi)\times\mathbf{r}''(\xi))\cdot\mathbf{r}'''(\xi)}{|\,\mathbf{r}'(\xi)\times\mathbf{r}''(\xi)\,|^2}\,,$$

this means that the ratio

$$\frac{|\,\mathbf{r}'(\xi)\times\mathbf{r}''(\xi)\,|^3}{\sigma^3(\xi)\,(\mathbf{r}'(\xi)\times\mathbf{r}''(\xi))\cdot\mathbf{r}'''(\xi)}$$

must be constant — which implies that $|\,\mathbf{r}'(\xi)\times\mathbf{r}''(\xi)\,|$ must be a polynomial. This is the distinctive feature of the "double" PH (DPH) curves [1, 10, 11, 22], for which $|\,\mathbf{r}'(\xi)\,|$ and $|\,\mathbf{r}'(\xi)\times\mathbf{r}''(\xi)\,|$ are *both* polynomials in ξ. Now every spatial PH curve satisfies [10] the condition

$$|\,\mathbf{r}'(\xi)\times\mathbf{r}''(\xi)\,|^2 \;=\; \sigma^2(\xi)\,\rho(\xi)\,,$$

where the polynomial $\rho(\xi)$ may be defined in terms of the form (6) by

$$\rho \;=\; 4\,[\,(up' - u'p + vq' - v'q)^2 + (uq' - u'q - vp' + v'p)^2\,]\,.$$

Hence, for a DPH curve, the polynomial $\rho(\xi)$ must be a perfect square.

3. Degeneration to a planar PH curve

In special circumstances, it is possible for a spatial PH curve generated by a quaternion polynomial $\mathcal{A}(\xi)$ to degenerate into a planar curve. The simplest instance is that of the spatial PH cubics $\mathbf{r}(\xi)$, generated by linear polynomials $\mathcal{A}(\xi) = \mathcal{A}_0(1-\xi) + \mathcal{A}_1\xi$. According to Proposition 22.1 in [6], $\mathbf{r}(\xi)$ is planar if and only if the coefficients of $\mathcal{A}(t)$ satisfy the relation

$$\mathcal{A}_1 \;=\; \mathcal{A}_0(\alpha + \beta\,\mathbf{i} + \gamma\,\mathbf{j} + \delta\,\mathbf{k})$$

with $\beta = 0$ and[2] $(\gamma, \delta) \neq (0, 0)$. Setting $\mathcal{A}_r = u_r + v_r\mathbf{i} + p_r\mathbf{j} + q_r\mathbf{k}$ for $r = 0, 1$, the plane Π in which $\mathbf{r}'(\xi)$ resides is defined by the (non–unit) normal vector $\mathbf{n} = (\lambda, \mu, \nu)$ with components

$$\begin{aligned}
\lambda &= 2\,(v_0p_0 - u_0q_0)\,\gamma + 2\,(u_0p_0 + v_0q_0)\,\delta\,, \\
\mu &= (u_0^2 - v_0^2 + p_0^2 - q_0^2)\,\gamma - 2\,(u_0v_0 - p_0q_0)\,\delta\,, \\
\nu &= 2\,(u_0v_0 + p_0q_0)\,\gamma + (u_0^2 - v_0^2 - p_0^2 + q_0^2)\,\delta\,.
\end{aligned}$$

[2] When $(\gamma, \delta) = (0, 0)$ the curve degenerates to a straight line.

The parametric speed of this planar PH curve is

$$\sigma(\xi) \,=\, |\mathcal{A}_0|^2 \,(\,[\,(1-\alpha)^2+\gamma^2+\delta^2)\,]\,\xi^2 - 2\,(1-\alpha)\,\xi + 1\,)\,,$$

which can be written as $\sigma(\xi) = |\mathcal{A}_0|^2(f^2(\xi)+g^2(\xi))$, where $f(\xi) = (1-\alpha)\,\xi - 1$, $g(\xi) = \sqrt{\gamma^2+\delta^2}\,\xi$. Thus, by a suitable choice of orthonormal basis vectors in the plane Π, the planar PH curve $\mathbf{r}(\xi)$ is consistent with the form (2).

Analogous results hold for degeneration of spatial PH quintic curves generated by quadratic polynomials $\mathcal{A}(\xi) = \mathcal{A}_0(1-\xi)^2 + \mathcal{A}_1 2(1-\xi)\xi + \mathcal{A}_2\xi^2$. If we write

$$\mathcal{A}_r \,=\, \mathcal{A}_0(\alpha_r + \beta_r\,\mathbf{i} + \gamma_r\,\mathbf{j} + \delta_r\,\mathbf{k})\,, \quad r = 1, 2\,,$$

then according to Proposition 22.3 in [6] the spatial PH curve $\mathbf{r}(\xi)$ generated by (4) is planar if and only if $\beta_1 = \beta_2 = \gamma_1\delta_2 - \gamma_2\delta_1 = 0$ with[3] $(\gamma_1, \gamma_2, \delta_1, \delta_2) \neq (0,0,0,0)$ provided that the hodograph is primitive, i.e., $\gcd(x', y', z') = 1$. The methodology can be extended to develop characterizations for the degeneration of higher–order spatial PH curves, but the resulting conditions on the coefficients of $\mathcal{A}(t)$ become more involved.

4. Projection on to the coordinate planes

The helical polynomial curves are not the only spatial PH curves that admit planar PH curve projections. We begin by considering projections of spatial PH curves on to the Cartesian coordinate planes, and study projections of general direction in Section 5. From (6), one can verify that

$$y'^2 + z'^2 \;=\; 4\,(u^2+v^2)(p^2+q^2)\,, \tag{7}$$

$$z'^2 + x'^2 \;=\; [\,(u+q)^2+(v+p)^2\,]\,[\,(u-q)^2+(v-p)^2\,]\,, \tag{8}$$

$$x'^2 + y'^2 \;=\; [\,(u+p)^2+(v-q)^2\,]\,[\,(u-p)^2+(v+q)^2\,]\,. \tag{9}$$

Thus, for the projection of $\mathbf{r}(\xi)$ on to the (y,z), (z,x), or (x,y) planes to be a planar PH curve, the product of two sums of squares of certain combinations of the polynomials $u(\xi), v(\xi), p(\xi), q(\xi)$ must be the perfect square of a polynomial. Equations (7)–(9) may be regarded as instances — for real polynomials — of the Diophantus (or Brahmagupta–Fibonacci) identity, which states that if two positive integers are sums of squares, then their product is also a sum of squares. The following examples illustrate a number of different circumstances under which the right–hand side of (7) corresponds to the perfect square of a polynomial.

Example 1. We begin with an example of a non–helical space curve for which the right–hand side of (7) is a perfect square because u^2+v^2 and p^2+q^2 are both perfect squares. Consider the quadratic polynomials

$$u(\xi) \,=\, -\,3\,\xi^2 - 6\,\xi\,, \quad v(\xi) \,=\, 4\,\xi^2 - 2\,\xi - 2\,,$$

[3]When $(\gamma_1, \gamma_2, \delta_1, \delta_2) = (0,0,0,0)$ the curve degenerates to a straight line.

$$p(\xi) \,=\, 2\,\xi - 5\,, \quad q(\xi) \,=\, -\,2\,\xi^2 + 10\,\xi - 12\,,$$

which generate the hodograph

$$\begin{aligned} x'(\xi) &= 21\,\xi^4 + 60\,\xi^3 - 128\,\xi^2 + 268\,\xi - 165\,, \\ y'(\xi) &= 12\,\xi^4 - 20\,\xi^3 - 96\,\xi^2 + 156\,\xi + 20\,, \\ z'(\xi) &= -\,16\,\xi^4 + 100\,\xi^3 - 134\,\xi^2 - 52\,\xi + 48\,, \end{aligned}$$

of a spatial PH quintic curve $\mathbf{r}(\xi) = (x(\xi), y(\xi), z(\xi))$. The parametric speed of this curve is

$$\sigma(\xi) \,=\, 29\,\xi^4 - 20\,\xi^3 + 176\,\xi^2 - 252\,\xi + 173\,,$$

and it is a true space curve, since $(\mathbf{r}'(\xi) \times \mathbf{r}''(\xi)) \cdot \mathbf{r}'''(\xi) \not\equiv 0$. Furthermore, $|\,\mathbf{r}'(\xi) \times \mathbf{r}''(\xi)\,|$ is not a polynomial, so $\mathbf{r}(\xi)$ is not a DPH curve and is therefore not helical. In this case, we have

$$u^2(\xi) + v^2(\xi) \,=\, (5\,\xi^2 + 2\,\xi + 2)^2\,, \quad p^2(\xi) + q^2(\xi) \,=\, (2\,\xi^2 - 10\,\xi + 13)^2\,,$$

and $(u^2 + v^2)(p^2 + q^2)$ is a perfect square since $u^2 + v^2$ and $p^2 + q^2$ are both individually perfect squares. Consequently, by (7), the projection $\mathbf{p}(\xi) = (y(\xi), z(\xi))$ of $\mathbf{r}(\xi)$ on to the (y, z) plane is a planar PH curve, satisfying $y'^2(\xi) + z'^2(\xi) = \sigma_{\mathbf{p}}^2(\xi)$, with $\sigma_{\mathbf{p}}(\xi) = 2(5\,\xi^2 + 2\,\xi + 2)(2\,\xi^2 - 10\,\xi + 13)$. Straightforward computations reveal that this is not true for the projections of $\mathbf{r}(\xi)$ on to the other two coordinate planes.

Example 2. Consider now an example of a helical space curve, for which the right–hand side of (7) is a perfect square since $u^2 + v^2$ and $p^2 + q^2$ are proportional to each other. For the quadratic polynomials

$$u(\xi) \,=\, 3\,\xi^2 + 2\,\xi - 1\,, \quad v(\xi) \,=\, \xi^2 - \xi + 3\,,$$

$$p(\xi) \,=\, 2\,\xi^2 + 3\,\xi - 4\,, \quad q(\xi) \,=\, 4\,\xi^2 + \xi + 2\,,$$

we obtain the hodograph

$$\begin{aligned} x'(\xi) &= -\,10\,\xi^4 - 10\,\xi^3 - 5\,\xi^2 + 10\,\xi - 10\,, \\ y'(\xi) &= 28\,\xi^4 + 24\,\xi^3 + 6\,\xi^2 + 32\,\xi - 28\,, \\ z'(\xi) &= -\,4\,\xi^4 - 32\,\xi^3 + 42\,\xi^2 + 24\,\xi + 4\,, \end{aligned}$$

of a spatial PH quintic curve $\mathbf{r}(\xi) = (x(\xi), y(\xi), z(\xi))$. The parametric speed of this curve is

$$\sigma(\xi) \,=\, 15\,(2\,\xi^4 + 2\,\xi^3 + \xi^2 - 2\,\xi + 2)\,,$$

and it is again a true space curve, since $(\mathbf{r}'(\xi) \times \mathbf{r}''(\xi)) \cdot \mathbf{r}'''(\xi) \not\equiv 0$. Moreover, $|\,\mathbf{r}'(\xi) \times \mathbf{r}''(\xi)\,|$ is a polynomial, so $\mathbf{r}(\xi)$ is a DPH curve, and it is helical (with axis in the x–direction) since $\mathrm{d}x/\mathrm{d}s = x'/\sigma = -\,1/3$. For this example, we have

$$u^2(\xi) + v^2(\xi) \,=\, 5\,(2\,\xi^4 + 2\,\xi^3 + \xi^2 - 2\,\xi + 2)\,,$$

$$p^2(\xi) + q^2(\xi) \;=\; 10\,(2\,\xi^4 + 2\,\xi^3 + \xi^2 - 2\,\xi + 2)\,.$$

We see that $(u^2 + v^2)(p^2 + q^2)$ is a perfect square since $u^2 + v^2$ and $p^2 + q^2$ are proportional to each other. Consequently, the projection $\mathbf{p}(\xi) = (y(\xi), z(\xi))$ of $\mathbf{r}(\xi)$ on to the (y, z) plane is a planar PH curve, satisfying $y'^2(\xi) + z'^2(\xi) = \sigma^2_{\mathbf{p}}(\xi)$ with $\sigma_{\mathbf{p}}(\xi) = 10\sqrt{2}(2\,\xi^4 + 2\,\xi^3 + \xi^2 - 2\,\xi + 2)$.

Figure 1 illustrates the spatial PH quintic curves constructed in Examples 1 and 2, together with their planar PH quintic orthogonal projections onto the (y, z) plane.

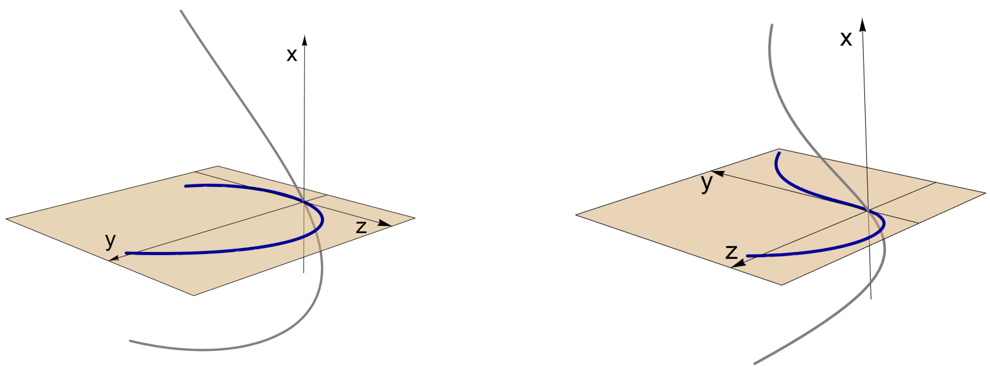

Figure 1: The spatial PH quintic curves (gray) constructed in Example 1 (left) and Example 2 (right), that possess planar PH quintic curves (blue) as their projections on to the (y, z) plane. The curves on the left and right are plotted over the parameter intervals $[\,-0.5, 1\,]$ and $[\,-1, 1\,]$ respectively.

Example 3. As indicated by the following example, neither of the conditions on $u^2 + v^2$ and $p^2 + q^2$ in the two preceding examples is necessary for the right–hand side of (7) to be a perfect square. Consider the cubic polynomials

$$u(\xi) \;=\; \xi^3 - 5\,\xi^2 + 8\,\xi - 2\,, \quad v(\xi) \;=\; -\,\xi^2 + 6\,\xi - 6\,,$$
$$p(\xi) \;=\; \xi^3 - 5\,\xi^2 - \xi + 13\,, \quad q(\xi) \;=\; 5\,\xi^2 - 18\,\xi + 9\,,$$

with $\gcd(u(\xi), v(\xi)) = \gcd(p(\xi), q(\xi)) = 1$, for which we have

$$\begin{aligned} u^2(\xi) + v^2(\xi) \;&=\; \xi^6 - 10\,\xi^5 + 42\,\xi^4 - 96\,\xi^3 + 132\,\xi^2 - 104\,\xi + 40 \\ &=\; (\xi^2 - 6\,\xi + 10)(\xi^2 - 2\,\xi + 2)^2\,, \end{aligned}$$

$$\begin{aligned} p^2(\xi) + q^2(\xi) \;&=\; \xi^6 - 10\,\xi^5 + 48\,\xi^4 - 144\,\xi^3 + 285\,\xi^2 - 350\,\xi + 250 \\ &=\; (\xi^2 - 6\,\xi + 10)(\xi^2 - 2\,\xi + 5)^2\,. \end{aligned}$$

Now $u^2(\xi) + v^2(\xi)$ and $p^2(\xi) + q^2(\xi)$ are not perfect squares, and are also not proportional to each other, but they satisfy

$$[\,u^2(\xi) + v^2(\xi)\,]\,[\,p^2(\xi) + q^2(\xi)\,] \;\equiv\; w^2(\xi) \tag{10}$$

where

$$
\begin{aligned}
w(\xi) &= \xi^6 - 10\,\xi^5 + 45\,\xi^4 - 120\,\xi^3 + 204\,\xi^2 - 200\,\xi + 100 \\
&= (\xi^2 - 2\,\xi + 2)(\xi^2 - 6\,\xi + 10)(\xi^2 - 2\,\xi + 5)\,.
\end{aligned} \tag{11}
$$

From (6) the polynomials $u(\xi), v(\xi), p(\xi), q(\xi)$ generate the spatial hodograph

$$
\begin{aligned}
x'(\xi) &= -6\,\xi^4 + 48\,\xi^3 - 153\,\xi^2 + 246\,\xi - 210\,, \\
y'(\xi) &= 8\,\xi^5 - 64\,\xi^4 + 208\,\xi^3 - 376\,\xi^2 + 384\,\xi - 192\,, \\
z'(\xi) &= -2\,\xi^6 + 20\,\xi^5 - 74\,\xi^4 + 144\,\xi^3 - 168\,\xi^2 + 112\,\xi - 56\,, \\
\sigma(\xi) &= 2\,\xi^6 - 20\,\xi^5 + 90\,\xi^4 - 240\,\xi^3 + 417\,\xi^2 - 454\,\xi + 290\,.
\end{aligned} \tag{12}
$$

However, since $|\,\mathbf{r}'(\xi) \times \mathbf{r}''(\xi)\,|^2$ has the factorization

$$
\begin{aligned}
& 16\,(\xi^2 - 6\,\xi + 10)^2 (2\,\xi^4 - 8\,\xi^3 + 22\,\xi^2 - 28\,\xi + 29)^2 \\
& \times (\xi^2 - 2\,\xi + 2)(\xi^2 - 2\,\xi + 5)(4\,\xi^4 - 32\,\xi^3 + 109\,\xi^2 - 162\,\xi + 90)\,,
\end{aligned}
$$

it is evidently not a perfect square, and consequently the degree 7 spatial PH curve $\mathbf{r}(\xi) = (x(\xi), y(\xi), z(\xi))$ is not a DPH curve, and is not helical. The projection $\mathbf{p}(\xi) = (y(\xi), z(\xi))$ of $\mathbf{r}(\xi)$ on to the (y, z) plane is a PH curve, satisfying $y'^2(\xi) + z'^2(\xi) = \sigma_{\mathbf{p}}^2(\xi)$ with parametric speed

$$
\sigma_{\mathbf{p}}(\xi) \,=\, 2\,\xi^6 - 20\,\xi^5 + 90\,\xi^4 - 240\,\xi^3 + 408\,\xi^2 - 400\,\xi + 200\,.
$$

We now study the circumstances under which the product of two sums of squares of certain combinations of the polynomials $u(\xi), v(\xi), p(\xi), q(\xi)$ coincides with the perfect square of a polynomial.

Proposition 1. *Real polynomials $a(\xi), b(\xi), c(\xi), d(\xi)$ of degree m satisfy*

$$
[\,a^2(\xi) + b^2(\xi)\,]\,[\,c^2(\xi) + d^2(\xi)\,] \,=\, w^2(\xi) \tag{13}
$$

for a real polynomial $w(\xi)$ of degree $2\,m$ with highest–order coefficient w_{2m}, if and only if they are expressible in the form

$$
a(\xi) \,=\, k_0 f_0(\xi)\,\mathrm{Re}\left[\mathrm{e}^{\mathrm{i}\,\varphi_0} \prod_{i=1}^{r} (\xi - \boldsymbol{\xi}_i)^{\mu_i}\right], \qquad b(\xi) \,=\, \sigma_0 k_0 f_0(\xi)\,\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}\,\varphi_0} \prod_{i=1}^{r} (\xi - \boldsymbol{\xi}_i)^{\mu_i}\right], \tag{14a}
$$

$$
c(\xi) \,=\, k_1 f_1(\xi)\,\mathrm{Re}\left[\mathrm{e}^{\mathrm{i}\,\varphi_1} \prod_{i=1}^{r} (\xi - \boldsymbol{\xi}_i)^{\nu_i}\right], \qquad d(\xi) \,=\, \sigma_1 k_1 f_1(\xi)\,\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}\,\varphi_1} \prod_{i=1}^{r} (\xi - \boldsymbol{\xi}_i)^{\nu_i}\right], \tag{14b}
$$

where k_0 and k_1 are non–zero real values, such that

$$
k_0\,k_1 \,=\, \pm\,w_{2m}\,; \tag{15}
$$

$\sigma_0, \sigma_1 \in \{-1, 1\}$ are independent signs; $f_0(\xi)$ and $f_1(\xi)$ are real monic polynomials of degree α and β with real roots; $\varphi_0, \varphi_1 \in [0, \pi)$ are the arguments of two unit complex numbers; r is a positive[4] *integer; and $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r$ are complex values with associated integers $\mu_1, \ldots, \mu_r \geq 0$ and $\nu_1, \ldots, \nu_r \geq 0$ that satisfy*

$$\alpha + \mu_1 + \cdots + \mu_r \;=\; \beta + \nu_1 + \cdots + \nu_r \;=\; m\,, \tag{16}$$

the sum $\mu_i + \nu_i$ being even for $i = 1, 2, \ldots, r$.

Proof : It is straightforward to verify that the polynomials defined by (14) satisfy equation (13). To show the necessity of conditions (14), we first note that any *real* root of $a^2(\xi) + b^2(\xi)$ or $c^2(\xi) + d^2(\xi)$ must be a root of the (monic) greatest common divisors $f_0(\xi) := \gcd(a(\xi), b(\xi))$ or $f_1(\xi) := \gcd(c(\xi), d(\xi))$, respectively. Thus, writing

$$a(\xi) \;=\; f_0(\xi)\,\tilde{a}(\xi)\,,\; b(\xi) \;=\; f_0(\xi)\,\tilde{b}(\xi)\,,\; c(\xi) \;=\; f_1(\xi)\,\tilde{c}(\xi)\,,\; d(\xi) \;=\; f_1(\xi)\,\tilde{d}(\xi)\,, \tag{17}$$

we have

$$f_0^2(\xi) f_1^2(\xi)\,[\,\tilde{a}^2(\xi) + \tilde{b}^2(\xi)\,]\,[\,\tilde{c}^2(\xi) + \tilde{d}^2(\xi)\,] \;=\; w^2(\xi)\,.$$

Consequently, $f_0(\xi)\,f_1(\xi)$ must be a factor of $w(\xi)$, so that $w(\xi) = f_0(\xi)\,f_1(\xi)\,\tilde{w}(\xi)$ where $\tilde{w}(\xi)$ has only complex conjugate roots and is of even degree $2m - \alpha - \beta$. Thus it can be factorized as $\tilde{w}(\xi) = w_{2m}\,\mathbf{z}(\xi)\,\bar{\mathbf{z}}(\xi)$, where

$$\mathbf{z}(\xi) = \prod_{i=1}^{r} (\xi - \boldsymbol{\xi}_i)^{\lambda_i}\,, \tag{18}$$

i.e., $\mathbf{z}(\xi)$ has r distinct complex roots $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r$ with multiplicities $\lambda_1, \ldots, \lambda_r > 0$ such that

$$2\,(\lambda_1 + \cdots + \lambda_r) \;=\; 2\,m - \alpha - \beta\,.$$

Then $[\,\tilde{a}^2(\xi) + \tilde{b}^2(\xi)\,]\,[\,\tilde{c}^2(\xi) + \tilde{d}^2(\xi)\,] = w_{2n}^2\,\mathbf{z}^2(\xi)\,\bar{\mathbf{z}}^2(\xi)$ and from the factorizations

$$\tilde{a}^2(\xi) + \tilde{b}^2(\xi) \;=\; [\,\tilde{a}(\xi) + \mathrm{i}\,\tilde{b}(\xi)\,]\,[\,\tilde{a}(\xi) - \mathrm{i}\,\tilde{b}(\xi)\,]\,, \quad \tilde{c}^2(\xi) + \tilde{d}^2(\xi) \;=\; [\,\tilde{c}(\xi) + \mathrm{i}\,\tilde{d}(\xi)\,]\,[\,\tilde{c}(\xi) - \mathrm{i}\,\tilde{d}(\xi)\,]\,,$$

we may write

$$[\,\tilde{a}(\xi) + \mathrm{i}\,\sigma_0\,\tilde{b}(\xi)\,]\,[\,\tilde{c}(\xi) + \mathrm{i}\,\sigma_1\,\tilde{d}(\xi)\,] \;=\; \pm w_{2n}\,\mathrm{e}^{\mathrm{i}\,\varphi}\,\mathbf{z}^2(\xi)\,, \quad [\,\tilde{a}(\xi) - \mathrm{i}\,\sigma_0\,\tilde{b}(\xi)\,]\,[\,\tilde{c}(\xi) - \mathrm{i}\,\sigma_1\,\tilde{d}(\xi)\,] \;=\; \pm w_{2n}\,\mathrm{e}^{-\mathrm{i}\,\varphi}\,\bar{\mathbf{z}}^2(\xi)$$

for any $\varphi \in [0, 2\pi)$. Now the roots of $\mathbf{z}^2(\xi)$ may be apportioned to $\tilde{a}(\xi) + \mathrm{i}\,\sigma_0\,\tilde{b}(\xi)$ and $\tilde{c}(\xi) + \mathrm{i}\,\sigma_1\,\tilde{d}(\xi)$ with multiplicities $\mu_1, \ldots, \mu_r$ and $\nu_1, \ldots, \nu_r$ that satisfy

$$\mu_i + \nu_i \;=\; 2\,\lambda_i \quad \text{for} \quad i = 1, \ldots, r\,, \tag{19a}$$

[4]Except that, in the case $r = 0$, it is understood that the products in (14a)–(14b) are set equal to 1.

$$\mu_1 + \cdots + \mu_r \;=\; m - \alpha\,, \qquad \nu_1 + \cdots + \nu_r \;=\; m - \beta\,. \tag{19b}$$

Furthermore, splitting the leading coefficient w_{2n} as in (15) and the angle φ as $\varphi_0 + \varphi_1$, we obtain

$$\tilde{a}(\xi) + \mathrm{i}\,\sigma_0\,\tilde{b}(\xi) \;=\; k_0\,\mathrm{e}^{\mathrm{i}\,\varphi_0} \prod_{i=1}^{r} (\xi - \boldsymbol{\xi}_i)^{\mu_i}\,, \quad \tilde{c}(\xi) + \mathrm{i}\,\sigma_1\,\tilde{d}(\xi) \;=\; k_1\,\mathrm{e}^{\mathrm{i}\,\varphi_1} \prod_{i=1}^{r} (\xi - \boldsymbol{\xi}_i)^{\nu_i}\,. \tag{20}$$

Since k_0, k_1 can be positive or negative, we can limit the arguments φ_0, φ_1 to $[0,\pi)$. Taking the real and imaginary parts of (20), and using (17), we obtain the stated forms (14) — subject to conditions (15)–(16) and (19) — for real polynomials $a(\xi), b(\xi), c(\xi), d(\xi)$ of degree m to satisfy (13) for a real polynomial $w(\xi)$ of degree $2\,m$. ∎

Remark 1. Concerning equations (19), it is of interest to study the cardinality of sets of the form

$$\left\{ (\mu_1, \mu_2, \ldots, \mu_r) \,\middle|\, \sum_{j=1}^{r} \mu_j = m - \alpha,\;\; 0 \le \mu_j \le 2\lambda_j\,,\; j = 1, 2, \ldots, r \right\}.$$

In the particular case $\lambda_1 = \cdots = \lambda_r = \lambda$, the cardinality is equal [18] to

$$\sum_{k=1}^{r} \binom{r}{k} C(m - \alpha, k, 2\,\lambda)\,,$$

where $C(m - \alpha, k, 2\,\lambda)$ denotes the number of restricted integer compositions of $m - \alpha$ into k parts, all bounded by $2\,\lambda$.

Example 4. Consider an $m = 3$ case with $f_0(\xi) = f_1(\xi) = 1$, and let $w(\xi)$ be as specified in (11). Then $w_{2m} = 1$ and

$$w(\xi) \;=\; (\xi - \boldsymbol{\xi}_1)(\xi - \bar{\boldsymbol{\xi}}_1)(\xi - \boldsymbol{\xi}_2)(\xi - \bar{\boldsymbol{\xi}}_2)(\xi - \boldsymbol{\xi}_3)(\xi - \bar{\boldsymbol{\xi}}_3)\,, \qquad \boldsymbol{\xi}_1 = 1 + \mathrm{i},\; \boldsymbol{\xi}_2 = 3 - \mathrm{i},\; \boldsymbol{\xi}_3 = 1 - 2\,\mathrm{i}\,,$$

which implies that $r = 3$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$. According to (19) and Remark 1 there are 7 different possibilities for the choices of μ_1, μ_2, μ_3 — namely,

$$(\mu_1, \mu_2, \mu_3) \;\in\; \{\,(2,1,0),\, (2,0,1),\, (1,2,0),\, (1,1,1),\, (1,0,2),\, (0,2,1),\, (0,1,2)\,\}\,. \tag{21}$$

Then from (19a) we have $\nu_i = 2 - \mu_i$, $i = 1, 2, 3$. Consider first the case $\mu_1 = \mu_2 = \mu_3 = 1$, and hence $\nu_1 = \nu_2 = \nu_3 = 1$. From (14) it follows that

$$\begin{bmatrix} a(\xi) \\ b(\xi) \end{bmatrix} \;=\; k_0 \begin{bmatrix} 1 & 0 \\ 0 & \sigma_0 \end{bmatrix} \begin{bmatrix} \cos\varphi_0 & -\sin\varphi_0 \\ \sin\varphi_0 & \cos\varphi_0 \end{bmatrix} \begin{bmatrix} \xi^3 - 5\,\xi^2 + 8\,\xi - 8 \\ 2\,\xi^2 - 6\,\xi + 6 \end{bmatrix},$$

$$\begin{bmatrix} c(\xi) \\ d(\xi) \end{bmatrix} \;=\; k_1 \begin{bmatrix} 1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} \begin{bmatrix} \xi^3 - 5\,\xi^2 + 8\,\xi - 8 \\ 2\,\xi^2 - 6\,\xi + 6 \end{bmatrix}$$

for any non–zero real numbers k_0, k_1 with product ± 1, any $\varphi_0, \varphi_1 \in [0, \pi)$, and any $\sigma_0, \sigma_1 \in \{-1, 1\}$. This implies that

$$a^2(\xi) + b^2(\xi) \;=\; k_0^2\, w(\xi)\,, \quad c^2(\xi) + d^2(\xi) \;=\; k_1^2\, w(\xi)\,.$$

Since $k_0^2\, k_1^2 = 1$, equation (13) is clearly satisfied.

Consider now the case $\mu_1 = 2$, $\mu_2 = 1$, $\mu_3 = 0$ (and $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 2$). From (14) we obtain

$$\begin{aligned}
\begin{bmatrix} a(\xi) \\ b(\xi) \end{bmatrix} &= k_0 \begin{bmatrix} 1 & 0 \\ 0 & \sigma_0 \end{bmatrix} \begin{bmatrix} \cos\varphi_0 & -\sin\varphi_0 \\ \sin\varphi_0 & \cos\varphi_0 \end{bmatrix} \begin{bmatrix} \xi^3 - 5\,\xi^2 + 8\,\xi - 2 \\ -\xi^2 + 6\,\xi - 6 \end{bmatrix}, \\
\begin{bmatrix} c(\xi) \\ d(\xi) \end{bmatrix} &= k_1 \begin{bmatrix} 1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} \begin{bmatrix} \xi^3 - 5\,\xi^2 - \xi + 13 \\ 5\,\xi^2 - 18\,\xi + 9 \end{bmatrix}.
\end{aligned} \tag{22}$$

Hence

$$\begin{aligned}
a^2(\xi) + b^2(\xi) &= k_0^2\,(\xi - \boldsymbol{\xi}_1)^2(\xi - \bar{\boldsymbol{\xi}}_1)^2(\xi - \boldsymbol{\xi}_2)(\xi - \bar{\boldsymbol{\xi}}_2) \\
&= k_0^2\,(\xi^2 - 2\,\xi + 2)^2(\xi^2 - 6\,\xi + 10)\,,
\end{aligned}$$

$$\begin{aligned}
c^2(\xi) + d^2(\xi) &= k_1^2\,(\xi - \boldsymbol{\xi}_2)(\xi - \bar{\boldsymbol{\xi}}_2)(\xi - \boldsymbol{\xi}_3)^2(\xi - \bar{\boldsymbol{\xi}}_3)^2 \\
&= k_1^2\,(\xi^2 - 6\,\xi + 10)(\xi^2 - 2\,\xi + 5)^2\,,
\end{aligned}$$

and equation (13) is satisfied. Note that $\gcd(a(\xi), b(\xi)) = \gcd(c(\xi), d(\xi)) = 1$, and that $a^2(\xi) + b^2(\xi)$ and $c^2(\xi) + d^2(\xi)$ are neither perfect squares nor constant multiples of each other. For all the remaining cases in (21) the observations are similar to this last case.

Given polynomials $a(\xi), b(\xi), c(\xi), d(\xi)$ that satisfy (13), it is straightforward to see from (7)–(9) that one can construct spatial PH curves which have planar PH curves as their projections on to the (y, z), (z, x), (x, y) planes through the choices

$$(u(\xi), v(\xi), p(\xi), q(\xi)) \;=\; \frac{(a(\xi), b(\xi), c(\xi), d(\xi))}{\sqrt{2}}\,, \tag{23}$$

$$(u(\xi), v(\xi), p(\xi), q(\xi)) \;=\; \frac{(a(\xi) + c(\xi), b(\xi) + d(\xi), b(\xi) - d(\xi), a(\xi) - c(\xi))}{2}\,, \tag{24}$$

$$(u(\xi), v(\xi), p(\xi), q(\xi)) \;=\; \frac{(a(\xi) + c(\xi), b(\xi) + d(\xi), a(\xi) - c(\xi), d(\xi) - b(\xi))}{2}\,, \tag{25}$$

respectively, for the components $u(\xi), v(\xi), p(\xi), q(\xi)$ of $\mathcal{A}(\xi)$ in (6). In particular, from (23) it follows that the hodograph of a spatial PH curve $\mathbf{r}(\xi) = (x(\xi), y(\xi), z(\xi))$ is equal to

$$
\begin{aligned}
x'(\xi) &= \tfrac{1}{2}\, k_0^2 f_0^2(\xi) \prod_{i=1}^{r} \left(\xi^2 - 2\,\mathrm{Re}(\boldsymbol{\xi}_i)\,\xi + |\boldsymbol{\xi}_i|^2\right)^{\mu_i} - \tfrac{1}{2}\, k_1^2 f_1^2(\xi) \prod_{i=1}^{r} \left(\xi^2 - 2\,\mathrm{Re}(\boldsymbol{\xi}_i)\,\xi + |\boldsymbol{\xi}_i|^2\right)^{\nu_i}, \\
y'(\xi) &= \sigma_0\, k_0\, k_1 f_0(\xi) f_1(\xi) \begin{cases} \mathrm{Im}\big[\mathrm{e}^{\mathrm{i}\,(\varphi_0+\varphi_1)}\, \mathbf{z}_0(\xi)\, \mathbf{z}_1(\xi)\big], & \sigma_0\,\sigma_1 = 1 \\ \mathrm{Im}\big[\mathrm{e}^{\mathrm{i}\,(\varphi_0-\varphi_1)}\, \mathbf{z}_0(\xi)\, \bar{\mathbf{z}}_1(\xi)\big], & \sigma_0\,\sigma_1 = -1 \end{cases}, \\
z'(\xi) &= -\, k_0\, k_1 f_0(\xi) f_1(\xi) \begin{cases} \mathrm{Re}\big[\mathrm{e}^{\mathrm{i}\,(\varphi_0+\varphi_1)}\, \mathbf{z}_0(\xi)\, \mathbf{z}_1(\xi)\big], & \sigma_0\,\sigma_1 = 1 \\ \mathrm{Re}\big[\mathrm{e}^{\mathrm{i}\,(\varphi_0-\varphi_1)}\, \mathbf{z}_0(\xi)\, \bar{\mathbf{z}}_1(\xi)\big], & \sigma_0\,\sigma_1 = -1 \end{cases},
\end{aligned}
\tag{26}
$$

where

$$
\mathbf{z}_0(\xi) := \prod_{i=1}^{r} (\xi - \boldsymbol{\xi}_i)^{\mu_i}, \quad \mathbf{z}_1(\xi) := \prod_{i=1}^{r} (\xi - \boldsymbol{\xi}_i)^{\nu_i}.
$$

Note that $\mathbf{z}_0(\xi)\, \mathbf{z}_1(\xi) = \mathbf{z}^2(\xi)$, where $\mathbf{z}(\xi)$ is given by (18). The projection $\mathbf{p}'(\xi) = (y'(\xi), z'(\xi))$ of $\mathbf{r}'(\xi)$ on to the (y, z) plane is a hodograph of a PH curve with the parametric speed

$$
\sigma_{\mathbf{p}}(\xi) = \sqrt{y'^2(\xi) + z'^2(\xi)} = |k_0\, k_1 f_0(\xi) f_1(\xi)|\ |\mathbf{z}(\xi)|^2. \tag{27}
$$

Thus, any real zero of $f_0(\xi)$ or $f_1(\xi)$ incurs an irregular point on the curve $\mathbf{p}(\xi)$. Note also that changing the sign of both σ_0 and σ_1 implies only a reflection of $\mathbf{p}'(\xi)$ in the (y, z) plane across the z–axis. Moreover, the hodograph $\mathbf{p}'(\xi)$ involves only the product of two constants k_0, k_1 and the sum or difference of arguments φ_0, φ_1. The complex product in (26) that involves $\varphi = \varphi_0 + \varphi_1$ or $\varphi = \varphi_0 - \varphi_1$ determines an anti–clockwise rotation of $\mathbf{z}_0(\xi)\, \mathbf{z}_1(\xi)$ (or $\mathbf{z}_0(\xi)\, \bar{\mathbf{z}}_1(\xi)$) in the complex plane by the angle φ, which results in the same rotation of the hodograph in the (y, z) plane, as is demonstrated in the following example.

Example 5. Employing the polynomials $a(\xi), b(\xi), c(\xi), d(\xi)$ in Example 4, given by (22) with $\sigma_0 = \sigma_1 = 1$, and the expressions (23) we obtain the hodograph $(x'(\xi), y'(\xi), z'(\xi))$ with

$$
\begin{aligned}
x'(\xi) \ &= \ \tfrac{1}{2}\left(\xi^2 - 6\xi + 10\right)\left[k_0^2\left(\xi^2 - 2\xi + 2\right)^2 - k_1^2\left(\xi^2 - 2\xi + 5\right)^2\right], \\
\begin{bmatrix} y'(\xi) \\ z'(\xi) \end{bmatrix} \ &= \ k_0\, k_1 \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} 4\,\xi^5 - 32\,\xi^4 + 104\,\xi^3 - 188\,\xi^2 + 192\,\xi - 96 \\ -\xi^6 + 10\,\xi^5 - 37\,\xi^4 + 72\,\xi^3 - 84\,\xi^2 + 56\,\xi - 28 \end{bmatrix},
\end{aligned}
\tag{28}
$$

where $\varphi = \varphi_0 + \varphi_1$. Figure 2 shows the corresponding PH curves, with $\mathbf{r}(0) = (0, 0, 0)$, for different choices of φ, k_0, k_1. On the left we fix $\varphi = 0$ and choose $k_0 \in \{0.5, 0.75, 1, 2, 3, 4, 5\}$ and $k_1 = 1/k_0$. The spatial curves are plotted in shades of blue varying from dark to light, while the projected curve is shown in black and is the same for all the choices. On the right we fix $k_0 = k_1 = 1$ and choose $\varphi \in \{0, \pi/5, 2\pi/5, 3\pi/5, 4\pi/5, \pi\}$. The spatial PH curves are plotted as dashed curves and their projections as solid curves, with different φ

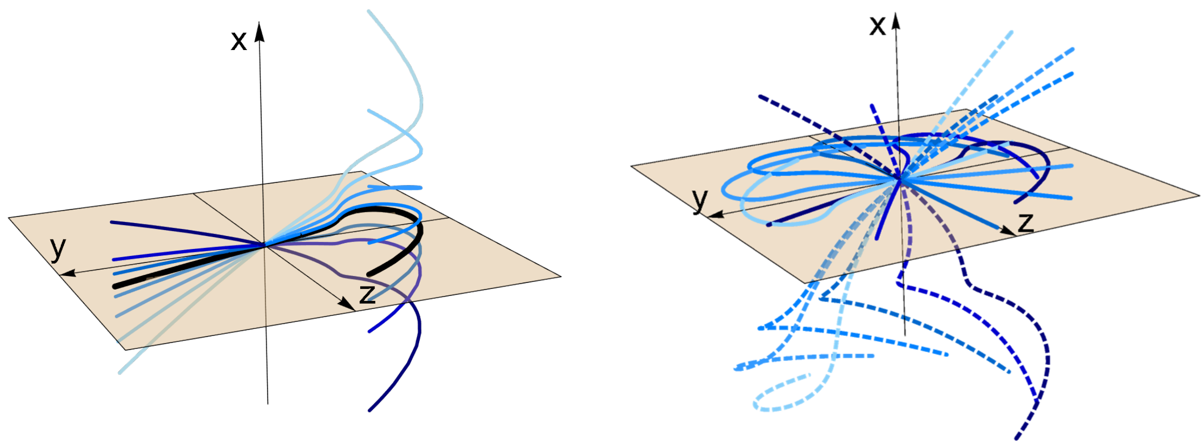

Figure 2: The spatial PH curves in Example 5 for different choices of k_0, k_1, φ and their projections on to (y, z) plane (see the text for a detailed explanation of these plots). The curves are plotted over the parameter interval $[-0.5, 4]$.

values represented by shades of blue varying from dark to light. The planar projections are rotations of the instance with $\varphi = 0$ (the dark blue curve).

Changing the signs of σ_0, σ_1 in (22) to $\sigma_0 = 1$ and $\sigma_1 = -1$, the second and the third component of the hodograph (28) become

$$\begin{bmatrix} y'(\xi) \\ z'(\xi) \end{bmatrix} = k_0\, k_1 \left(\xi^2 - 6\xi + 10\right) \begin{bmatrix} \cos(\varphi_0 - \varphi_1) & -\sin(\varphi_0 - \varphi_1) \\ \sin(\varphi_0 - \varphi_1) & \cos(\varphi_0 - \varphi_1) \end{bmatrix} \begin{bmatrix} -6\left(\xi^3 - 3\xi^2 + \xi + 1\right) \\ -\xi^4 + 4\xi^3 + 7\xi^2 - 22\xi + 8 \end{bmatrix}, \tag{29}$$

while $x'(\xi)$ remains the same. An example of the PH curve obtained from (29) for $\varphi_0 = \varphi_1 = 0$ and $k_0 = k_1 = 1$ is shown in Figure 3 (left) together with its planar PH projection. Comparing it with the PH curves derived from (28) it is clearly evident that we obtain a different curve.

Modifying the polynomials $a(\xi), b(\xi), c(\xi), d(\xi)$ from Example 4 by taking $f_0(\xi) = \xi - \frac{1}{2}$, $f_1(\xi) = \xi + \frac{1}{4}$, we obtain, using (26) with $\varphi_0 = \varphi_1 = 0$, $k_0 = k_1 = 1$ and $\sigma_0 = -\sigma_1 = 1$, the PH curve shown as the gray curve in Figure 3 (right). Its projection on to the (y, z) plane is a PH curve with a parametric speed that is zero at the irregular points $\xi = \frac{1}{2}$ and $\xi = -\frac{1}{4}$ (indicated as red dots in Figure 3).

The spatial PH curves obtained from (28) or (29) do not have planar PH projections on to the (z, x) or (x, y) plane. Such curves can be constructed using expressions (24)–(25) to compute the hodograph. Using

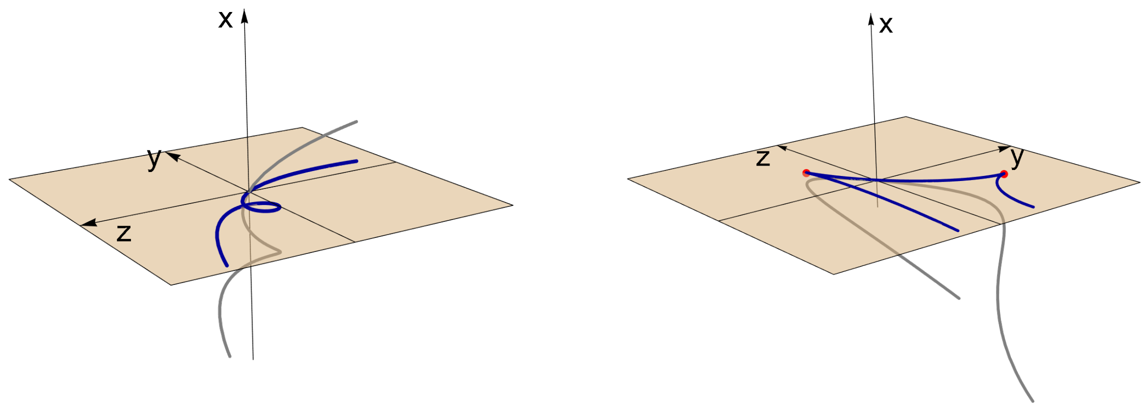

Figure 3: Left: the spatial PH curve obtained from (29) with $\sigma_0 = 1$, $\sigma_1 = -1$, $\varphi_0 = \varphi_1 = 0$ and $k_0 = k_1 = 1$ together with its projection on to the (y, z) plane, for $\xi \in [\,-0.5, 3.85\,]$. Right: the spatial PH curve with non–constant $f_0(\xi)$, $f_1\xi)$ and its planar PH projection, with irregular points identified by red dots, for $\xi \in [\,-0.5, 1\,]$.

the polynomials (22) with $\sigma_0 = \sigma_1 = 1$ we compute from (24) the hodograph $(\tilde{x}'(\xi), \tilde{y}'(\xi), \tilde{z}'(\xi))$ with

$$\begin{aligned} &\tilde{y}'(\xi) = \tfrac{1}{2}\,\left(\xi^2 - 6\xi + 10\right)\left[\,k_0^2\left(\xi^2 - 2\xi + 2\right)^2 - k_1^2\left(\xi^2 - 2\xi + 5\right)^2\,\right], \\ &\begin{bmatrix} \tilde{x}'(\xi) \\ \tilde{z}'(\xi) \end{bmatrix} = k_0\,k_1\left(\xi^2 - 6\xi + 10\right)\begin{bmatrix} \cos\left(\varphi_1 - \varphi_0\right) & -\sin\left(\varphi_1 - \varphi_0\right) \\ \sin\left(\varphi_1 - \varphi_0\right) & \cos\left(\varphi_1 - \varphi_0\right) \end{bmatrix}\begin{bmatrix} \xi^4 - 4\xi^3 - 7\xi^2 + 22\xi - 8 \\ 6\left(\xi^3 - 3\xi^2 + \xi + 1\right) \end{bmatrix}. \end{aligned} \tag{30}$$

Comparing (30) with the hodograph (29), derived from the polynomials (22) with the choice $\sigma_0 = 1$, $\sigma_1 = -1$, we observe that

$$\begin{bmatrix} \tilde{x}'(\xi) \\ \tilde{y}'(\xi) \\ \tilde{z}'(\xi) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x'(\xi) \\ y'(\xi) \\ z'(\xi) \end{bmatrix},$$

i.e., (30) is just a rotation of the hodograph (29).

As noted in Example 5, we now show that using (24) to compute the hodograph, which preserves the PH property when projected on to the (z, x) plane, yields a hodograph that is just a particular rotation of the hodograph obtained from (23). Specifically, setting

$$\hat{\mathcal{A}}(\xi) \,=\, \mathcal{V}\mathcal{U}\,\frac{a(\xi) + b(\xi)\,\mathbf{i} + c(\xi)\,\mathbf{j} + d(\xi)\,\mathbf{k}}{\sqrt{2}}, \quad \mathcal{V} \,=\, \frac{-1 - \mathbf{j}}{\sqrt{2}}, \quad \mathcal{U} \,=\, \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$$

we observe that

$$\tilde{\mathbf{r}}' \,=\, \hat{\mathcal{A}}\,\mathbf{i}\,\hat{\mathcal{A}}^* \,=\, (ac - bd)\,\mathbf{i} + \tfrac{1}{2}(a^2 + b^2 - c^2 - d^2)\,\mathbf{j} - (bc + ad)\,\mathbf{k}\,,$$

which is the hodograph obtained from (24) by changing the sign of $d(\xi)$. By Proposition 1, this sign change corresponds to changing the sign of σ_1. Note further that multiplication by the unit quaternion $\mathcal{U}$ implies a rotation of the (y,z) plane on to the (z,x) plane while multiplication by $\mathcal{V}$ just incurs a rotation in the (z,x) plane. Since such rotations are handled by changing the angles φ_0, φ_1 it suffices to consider only the multiplication by $\mathcal{U}$. This idea is further generalized in Section 5. Similar observations hold true for the hodograph obtained from (25). More precisely, one can easily verify that (25) generates a hodograph that is simply a 90° clockwise rotation in the (y,z) plane of the hodograph obtained from (24). However, the parametric speeds of the spatial PH curves corresponding to the hodographs derived from (23)–(25) all have the same parametric speed, namely

$$\sigma(\xi) \,=\, \tfrac{1}{2}\,[\, k_0^2 f_0^2(\xi)\,|\mathbf{z}_0(\xi)|^2 + k_1^2 f_1^2(\xi)\,|\mathbf{z}_1(\xi)|^2\,]\,.$$

Moreover, the projections of $\mathbf{p}(\xi)$ on to the (y,z), (z,x), or (x,y) plane also have the same parametric speed, given by (27).

The following example illustrates the simplest instance ($m=1$) of Proposition 1 — corresponding to the spatial PH cubics, which are known [15] to be all helical curves (with planar cubics as degenerate cases).

Example 6. For $m=1$, we can either have (i) $f_0(\xi)=\xi-\zeta$ and $f_1(\xi)=\xi-\eta$ for real values $\zeta\neq\eta$ with the products involving complex values in (14a)–(14b) set equal to 1, or (ii) $f_0(\xi)=f_1(\xi)=1$ with only a single complex value $\boldsymbol{\xi}_1$ in the products. For both cases, we fix $\varphi_0=\varphi_1=0$. Then case (i) yields $a(\xi)=\sigma_0\,b(\xi)=k_0(\xi-\zeta)$, $c(\xi)=\sigma_1\,d(\xi)=k_1(\xi-\eta)$ and

$$x'(\xi) \,=\, k_0^2(\xi-\zeta)^2-k_1^2(\xi-\eta)^2\,, \quad y'(\xi) \,=\, (\sigma_0+\sigma_1)\,k_0\,k_1\,(\xi-\zeta)(\xi-\eta)\,, \quad z'(\xi) \,=\, (\sigma_0\,\sigma_1-1)\,k_0\,k_1\,(\xi-\zeta)(\xi-\eta)$$

from (6) and (23). Since $y'(\xi)$ and $z'(\xi)$ are linearly dependent, $\mathbf{r}(\xi)$ degenerates to a planar cubic PH curve. Setting $\boldsymbol{\xi}_1=\zeta+\mathrm{i}\,\eta$ in case (ii) we obtain $a(\xi)=k_0(\xi-\zeta)$, $b(\xi)=-\,\sigma_0\,k_0\,\eta$, $c(\xi)=k_1(\xi-\zeta)$, $d(\xi)=-\,\sigma_1\,k_1\,\eta$ and hence

$$x'(\xi) \,=\, \tfrac{1}{2}(k_0^2-k_1^2)\,[\,(\xi-\zeta)^2+\eta^2\,]\,, \quad y'(\xi) \,=\, -\,(\sigma_0+\sigma_1)\,k_0\,k_1\,(\xi-\zeta)\eta\,, \quad z'(\xi) \,=\, -\,k_0\,k_1\,[\,(\xi-\zeta)^2-\sigma_0\,\sigma_1\,\eta^2\,]\,,$$

with the parametric speed $\sigma(\xi)=\frac{1}{2}(k_0^2+k_1^2)\,[\,(\xi-\zeta)^2+\eta^2\,]$. Since the unit tangent $\mathbf{t}(\xi)=\mathbf{r}'(\xi)/\sigma(\xi)$ satisfies $\mathbf{a}\cdot\mathbf{t}(\xi)=\cos\psi$ with $\mathbf{a}=\mathbf{i}$ and $\cos\psi=(k_0^2-k_1^2)/(k_0^2+k_1^2)$, the curve is evidently a helical cubic, with a PH projection on to the (y,z) plane. For $\sigma_0\,\sigma_1=-1$ it reduces to a planar curve.

Remark 2. There are two notable special cases in the satisfaction of equation (13):

(1) $a^2(\xi)+b^2(\xi)$ and $c^2(\xi)+d^2(\xi)$ are individually perfect squares;

(2) $a^2(\xi)+b^2(\xi)$ and $c^2(\xi)+d^2(\xi)$ are constant multiples of each other.

These special cases encompass all PH cubics ($m=1$) and PH quintics ($m=2$) — see Examples 1 and 2. However, for PH curves of degree ≥ 7 ($m\geq 3$), most of the solutions are such that $a^2(\xi)+b^2(\xi)$ and $c^2(\xi)+d^2(\xi)$ are neither both perfect squares, nor proportional — see Examples 3–5.

Remark 3. For a PH curve of degree $n = 2m+1$, at least one of the polynomials $u(\xi), v(\xi), p(\xi), q(\xi)$ in (6) must be of degree m. Proposition 1 requires a minor modification when $a(\xi), b(\xi), c(\xi), d(\xi)$ are not all of degree m, as follows. Suppose that $\max(\deg(a), \deg(b)) = d_1$, $\max(\deg(c), \deg(d)) = d_2$ where either $d_1 = m$, $d_2 < m$ or $d_1 < m$, $d_2 = m$. Then $\deg((a^2+b^2)(c^2+d^2)) = 2\,(d_1+d_2)$, and consequently $\deg(w) = d_1+d_2$ in (13). The polynomials $a(\xi), b(\xi), c(\xi), d(\xi)$ have the form specified in Proposition 1, except that the relations (15) and (16) now become

$$k_0\, k_1 \;=\; \pm\, w_{2(d_1+d_2)}\,,$$

$$\alpha + \mu_1 + \cdots + \mu_r \;=\; d_1\,, \quad \beta + \nu_1 + \cdots + \nu_r \;=\; d_2\,.$$

Consider, for example, the case $\sigma_0 = 1$, $\varphi_0 = 0$, $k_0 = 1$, $f_0(\xi) = \xi$ and $\sigma_1 = 1$, $\varphi_1 = 0$, $k_1 = 2$, $f_1(\xi) = 1$ of (14a)–(14b), with $r = 1$, $\boldsymbol{\xi}_1 = \mathrm{i}$, and $\mu_1 = \nu_1 = 1$, which yields

$$a(\xi) \;=\; \xi^2\,,\;\; b(\xi) \;=\; -\,\xi\,,\;\; c(\xi) \;=\; 2\,\xi\,,\;\; d(\xi) \;=\; -\,2\,,$$

with $d_1 = 2 = m$ and $d_2 = 1 < m$. These polynomials satisfy (13) with $w(\xi) = 2\,\xi(\xi^2+1)$, and defining $u(\xi), v(\xi), p(\xi), q(\xi)$ in terms of them through (23) yields a spatial PH quintic with a planar PH quintic projection onto the (y,z) plane.

5. Planar projections of any direction

We now examine the circumstances under which the projection of spatial PH curves on to planes with a general normal direction in $\mathbb{R}^3$ yield planar PH curves. We first consider the relationship between the spatial hodograph $\mathbf{r}'(\xi)$ and its projection $\mathbf{p}'(\xi)$ on to a plane with a general normal.

Lemma 1. *Let Π be a plane through the origin with a unit normal vector $\mathbf{l} = \lambda\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k}$ of general direction, with $\lambda^2+\mu^2+\nu^2 = 1$, and let $\mathbf{r}'(\xi)$ be the spatial Pythagorean hodograph (6) generated by the quaternion polynomial (3). Then the orthogonal projection $\mathbf{p}'(\xi)$ of $\mathbf{r}'(\xi)$ on to Π is*

$$\mathbf{p}'(\xi) \;=\; \mathbf{r}'(\xi) - (\mathbf{l}\cdot\mathbf{r}'(\xi))\,\mathbf{l}\,, \tag{31}$$

and in terms of the quaternion form (4), this may be expressed as

$$\mathbf{p}'(\xi) \;=\; \frac{1}{2}\,[\,\mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi) \;-\; \tilde{\mathcal{A}}(\xi)\,\mathbf{i}\,\tilde{\mathcal{A}}^*(\xi)\,]\,,$$

where $\tilde{\mathcal{A}}(\xi) = \mathbf{l}\,\mathcal{A}(\xi)$.

Proof: To verify this, note [9] that the quaternion polynomial $\tilde{\mathcal{A}}(\xi) = \mathcal{U}\,\mathcal{A}(\xi)$ obtained by multiplying $\mathcal{A}(\xi)$ with a unit quaternion of the form $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{l})$ generates a hodograph $\tilde{\mathbf{r}}'(\xi) = \tilde{\mathcal{A}}(\xi)\,\mathbf{i}\,\tilde{\mathcal{A}}^*(\xi)$ that corresponds to a rotation of $\mathbf{r}'(\xi)$ through angle θ about $\mathbf{l}$. In particular, the pure vector unit quaternion

$\mathcal{U} = (0, \pm\mathbf{l})$ defines a rotation by an odd integer multiple of π about $\mathbf{l}$. Now let $(\mathbf{l}, \mathbf{m}, \mathbf{n})$ be an orthonormal frame such that $\mathbf{m}$ and $\mathbf{n}$ span the plane Π, and $\mathbf{l}$ is orthogonal to it. Then the rotation of $\mathbf{r}'(\xi)$ by an odd multiple of π about $\mathbf{l}$ reverses its $\mathbf{m}, \mathbf{n}$ components and leaves the $\mathbf{l}$ component unchanged. Thus, in the expression $\frac{1}{2}[\,\mathbf{r}'(\xi) - \tilde{\mathbf{r}}'(\xi)\,]$ the $\mathbf{m}$ and $\mathbf{n}$ components of $\mathbf{r}'(\xi)$ are preserved, while the $\mathbf{l}$ component is cancelled out. This defines the projection $\mathbf{p}'(\xi)$ of $\mathbf{r}'(\xi)$ on to the plane Π. ∎

From (31) we have

$$\sigma^2(\xi) - [\,\mathbf{l}\cdot\mathbf{r}'(\xi)\,]^2 \,=\, |\mathbf{p}'(\xi)|^2\,, \tag{32}$$

where $\sigma(\xi) = |\mathcal{A}(\xi)|^2$ is the parametric speed of the spatial PH curve $\mathbf{r}(\xi)$. Thus, if $\mathbf{p}(\xi)$ is to be a planar PH curve with parametric speed $|\mathbf{p}'(\xi)| = \sigma_{\mathbf{p}}(\xi)$ for some polynomial $\sigma_{\mathbf{p}}(\xi)$, the three polynomials $\sigma(\xi), \mathbf{l}\cdot\mathbf{r}'(\xi), \sigma_{\mathbf{p}}(\xi)$ must (assuming that $\gcd(\sigma(\xi), \mathbf{l}\cdot\mathbf{r}'(\xi), \sigma_{\mathbf{p}}(\xi)) = 1$) be [21] of the form

$$\sigma(\xi) \,=\, r^2(\xi) + s^2(\xi)\,, \quad \mathbf{l}\cdot\mathbf{r}'(\xi) \,=\, 2\,r(\xi)s(\xi)\,, \quad \sigma_{\mathbf{p}}(\xi) \,=\, r^2(\xi) - s^2(\xi)$$

for relatively prime polynomials $r(\xi), s(\xi)$. Setting $\mathbf{w}(\xi) = r(\xi) + \mathrm{i}\,s(\xi)$, these relations imply that $\mathbf{p}'(\xi)$ is a planar Pythagorean hodograph if and only if a complex polynomial $\mathbf{w}(\xi)$ exists, such that $\sigma(\xi) = |\mathcal{A}(\xi)|^2$ and $\mathbf{r}'(\xi) = \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)$ satisfy

$$|\mathcal{A}(\xi)|^2 \,=\, |\mathbf{w}(\xi)|^2 \quad \text{and} \quad \mathbf{l}\cdot(\mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)) \,=\, \mathrm{Im}(\mathbf{w}^2(\xi))\,, \tag{33}$$

and we than have $\sigma_{\mathbf{p}}(\xi) = \mathrm{Re}(\mathbf{w}^2(\xi))$. Note also from (32) that $\sigma_{\mathbf{p}}(\xi)$ satisfies

$$[\,\sigma(\xi) + \mathbf{l}\cdot\mathbf{r}'(\xi)\,]\,[\,\sigma(\xi) - \mathbf{l}\cdot\mathbf{r}'(\xi)\,] \,=\, \sigma_{\mathbf{p}}^2(\xi)\,.$$

This reasoning can, in principle, be used to determine constraints on the coefficients of $\mathcal{A}(\xi)$ that depend on the normal vector $\mathbf{l}$ to the projection plane. However, we adopt a simpler geometrical approach based on noting that the standard Cartesian basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ can be mapped to the basis $(\mathbf{l}, \mathbf{m}, \mathbf{n})$ associated with the plane Π through a spatial rotation, specified by a unit quaternion.

Proposition 2. *Let $\mathbf{r}'(\xi)$ be a spatial Pythagorean hodograph generated from a quaternion polynomial $\mathcal{A}(\xi) = u(\xi) + v(\xi)\,\mathbf{i} + p(\xi)\,\mathbf{j} + q(\xi)\,\mathbf{k}$ by expression* (4)*. Then the projection $\mathbf{p}'(\xi)$ of $\mathbf{r}'(\xi)$ on to the plane Π with unit normal vector $\mathbf{l} = \lambda\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k}$ (with $\mathbf{l} \neq -\mathbf{i}$) is a planar Pythagorean hodograph if and only if $\mathcal{A}(\xi)$ is of the form*

$$\mathcal{A}(\xi) \,=\, \mathcal{U}\,\mathcal{Q}(\xi)\,, \tag{34}$$

where $\mathcal{U}$ is the unit pure vector quaternion defined by

$$\mathcal{U} \,=\, \frac{\mathbf{l} + \mathbf{i}}{|\,\mathbf{l} + \mathbf{i}\,|} \,=\, \frac{(1+\lambda)\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k}}{\sqrt{2(1+\lambda)}}\,, \tag{35}$$

and $\mathcal{Q}(\xi)$ is defined in terms of the polynomials (14*a*)–(14*b*) *that satisfy* (13) *as*

$$\mathcal{Q}(\xi) \,=\, \frac{a(\xi) + b(\xi)\,\mathbf{i} + c(\xi)\,\mathbf{j} + d(\xi)\,\mathbf{k}}{\sqrt{2}}\,. \tag{36}$$

Proof : For $\mathbf{l} \neq -\mathbf{i}$ we can generate an orthonormal basis $(\mathbf{l},\mathbf{m},\mathbf{n})$ that incorporates $\mathbf{l}$ through the spatial rotation operations defined by

$$\mathbf{l} = \mathcal{U}\,\mathbf{i}\,\mathcal{U}^*\,, \quad \mathbf{m} = \mathcal{U}\,\mathbf{j}\,\mathcal{U}^*\,, \quad \mathbf{n} = \mathcal{U}\,\mathbf{k}\,\mathcal{U}^*\,, \tag{37}$$

where $\mathcal{U}$ is the unit pure vector quaternion specified by (35). This yields $\mathbf{l} = \lambda\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k}$ as the normal to Π, and the basis vectors spanning Π are given by

$$\mathbf{m} = \frac{(1+\lambda)\,\mu\,\mathbf{i} + (\mu^2 - (1+\lambda))\,\mathbf{j} + \mu\,\nu\,\mathbf{k}}{1+\lambda}\,,$$

$$\mathbf{n} = \frac{(1+\lambda)\,\nu\,\mathbf{i} + \mu\,\nu\,\mathbf{j} + (\nu^2 - (1+\lambda))\,\mathbf{k}}{1+\lambda}\,.$$

Let $\hat{\mathbf{r}}'(\xi) = \mathcal{Q}(\xi)\,\mathbf{i}\,\mathcal{Q}^*(\xi)$ be the Pythagorean hodograph generated by the quaternion polynomial (36), and let $\mathcal{A}(\xi) = \mathcal{U}\,\mathcal{Q}(\xi)$. Then

$$\mathbf{r}'(\xi) = \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi) = \mathcal{U}\,\mathcal{Q}(\xi)\,\mathbf{i}\,\mathcal{Q}^*(\xi)\,\mathcal{U}^* \tag{38}$$

corresponds [9] to performing on $\hat{\mathbf{r}}'(\xi)$ the rotation that maps $(\mathbf{i},\mathbf{j},\mathbf{k})$ on to $(\mathbf{l},\mathbf{m},\mathbf{n})$. For the projection of $\hat{\mathbf{r}}'(\xi)$ on to the $(\mathbf{j},\mathbf{k})$ plane to be a planar Pythagorean hodograph, the components of $\mathcal{Q}(\xi)$ must be of the form (36), in terms of the polynomials defined by (14a)–(14b). Since the rotation specified by $\mathcal{U}$ maps $\hat{\mathbf{r}}'(\xi)$ to $\mathbf{r}'(\xi)$, and $(\mathbf{i},\mathbf{j},\mathbf{k})$ to $(\mathbf{l},\mathbf{m},\mathbf{n})$, the projections $\hat{\mathbf{p}}'(\xi)$ and $\mathbf{p}'(\xi)$ on to the $(\mathbf{j},\mathbf{k})$ and $(\mathbf{m},\mathbf{n})$ planes satisfy

$$|\hat{\mathbf{p}}'(\xi)|^2 = [\,\mathbf{j}\cdot\hat{\mathbf{r}}'(\xi)\,]^2 + [\,\mathbf{k}\cdot\hat{\mathbf{r}}'(\xi)\,]^2 = [\,\mathbf{m}\cdot\mathbf{r}'(\xi)\,]^2 + [\,\mathbf{n}\cdot\mathbf{r}'(\xi)\,]^2 = |\mathbf{p}'(\xi)|^2\,,$$

and consequently $\mathbf{p}'(\xi)$ is a Pythagorean hodograph in the $(\mathbf{m},\mathbf{n})$ plane if and only if $\hat{\mathbf{p}}'(\xi)$ is a Pythagorean hodograph in the $(\mathbf{j},\mathbf{k})$ plane. ∎

It is also possible to express $\mathbf{r}'(\xi)$ in terms of $(\mathbf{l},\mathbf{m},\mathbf{n})$ rather than $(\mathbf{i},\mathbf{j},\mathbf{k})$, as follows. Inverting $\mathbf{l} = \mathcal{U}\,\mathbf{i}\,\mathcal{U}^*$ gives $\mathbf{i} = \mathcal{U}^*\,\mathbf{l}\,\mathcal{U}$, and upon inserting this into $\mathbf{r}'(\xi) = \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)$ with $\mathcal{A}(\xi) = \mathcal{U}\,\mathcal{Q}(\xi)$ we obtain

$$\mathbf{r}'(\xi) = \mathcal{U}\,\mathcal{Q}(\xi)\,\mathcal{U}^*\,\mathbf{l}\,\mathcal{U}\,\mathcal{Q}^*(\xi)\,\mathcal{U}^* = \hat{\mathcal{A}}(\xi)\,\mathbf{l}\,\hat{\mathcal{A}}^*(\xi)\,,$$

where

$$\hat{\mathcal{A}}(\xi) := \mathcal{U}\,\mathcal{Q}(\xi)\,\mathcal{U}^*\,.$$

By expanding this product, we can express the components of $\hat{\mathcal{A}}(\xi) = \hat{u}(\xi) + \hat{v}(\xi)\,\mathbf{l} + \hat{p}(\xi)\,\mathbf{m} + \hat{q}(\xi)\,\mathbf{n}$ in terms of the components of $\mathcal{Q}(\xi)$ as

$$\hat{u}(\xi) = \frac{a(\xi)}{\sqrt{2}}\,, \quad \hat{v}(\xi) = \frac{b(\xi)}{\sqrt{2}}\,, \quad \hat{p}(\xi) = \frac{c(\xi)}{\sqrt{2}}\,, \quad \hat{q}(\xi) = \frac{d(\xi)}{\sqrt{2}}\,.$$

The $(\mathbf{l},\mathbf{m},\mathbf{n})$ components of $\mathbf{r}'(\xi)$ are then given by

$$\begin{aligned}
\mathbf{l}\cdot\mathbf{r}'(\xi) &= \hat{u}^2(\xi) + \hat{v}^2(\xi) - \hat{p}^2(\xi) - \hat{q}^2(\xi)\,, \\
\mathbf{m}\cdot\mathbf{r}'(\xi) &= 2\,[\,\hat{u}(\xi)\hat{q}(\xi) + \hat{v}(\xi)\hat{p}(\xi)\,]\,, \\
\mathbf{n}\cdot\mathbf{r}'(\xi) &= 2\,[\,\hat{v}(\xi)\hat{q}(\xi) - \hat{u}(\xi)\hat{p}(\xi)\,]\,.
\end{aligned}$$

Remark 4. From equations (34)–(36), we see that quaternion polynomials $\mathcal{A}(\xi) = \mathcal{U}\,\mathcal{Q}(\xi) = u(\xi) + v(\xi)\,\mathbf{i} + p(\xi)\,\mathbf{j} + q(\xi)\,\mathbf{k}$ generating spatial Pythagorean hodographs, whose projections on to the plane with unit normal vector $\mathbf{l} = \lambda\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k}$ are planar Pythagorean hodographs, have components of the form

$$\begin{aligned}
u(\xi) &= \frac{-\,(1+\lambda)\,b(\xi) - \mu\,c(\xi) - \nu\,d(\xi)}{2\sqrt{1+\lambda}}\,, \\
v(\xi) &= \frac{(1+\lambda)\,a(\xi) + \mu\,d(\xi) - \nu\,c(\xi)}{2\sqrt{1+\lambda}}\,, \\
p(\xi) &= \frac{-\,(1+\lambda)\,d(\xi) + \mu\,a(\xi) + \nu\,b(\xi)}{2\sqrt{1+\lambda}}\,, \\
q(\xi) &= \frac{(1+\lambda)\,c(\xi) - \mu\,b(\xi) + \nu\,a(\xi)}{2\sqrt{1+\lambda}}\,.
\end{aligned}$$

Example 7. Consider the plane Π with unit normal vector with $(\lambda,\mu,\nu) = (2,-2,1)/3$ and the polynomials $a(\xi), b(\xi), c(\xi), d(\xi)$ specified in Example 4 by (22) with $k_0 = k_1 = 1$, $\varphi_0 = \varphi_1 = 0$, and $\sigma_0 = \sigma_1 = 1$. Then the unit vectors spanning Π are

$$\mathbf{m} \,=\, \frac{-\,10\,\mathbf{i} - 11\,\mathbf{j} - 2\,\mathbf{k}}{15}\,, \quad \mathbf{n} \,=\, \frac{5\,\mathbf{i} - 2\,\mathbf{j} - 14\,\mathbf{k}}{15}\,,$$

and the quaternion polynomial $\mathcal{A}(\xi) = \mathcal{U}\,\mathcal{Q}(\xi)$ has components

$$\begin{aligned}
u(\xi) &= \frac{2\,\xi^3 - 10\,\xi^2 - 14\,\xi + 47}{2\sqrt{15}}\,, \quad v(\xi) \,=\, \frac{4\,\xi^3 - 30\,\xi^2 + 77\,\xi - 41}{2\sqrt{15}}\,, \\
p(\xi) &= \frac{-\,2\,\xi^3 - 16\,\xi^2 + 80\,\xi - 47}{2\sqrt{15}}\,, \quad q(\xi) \,=\, \frac{6\,\xi^3 - 32\,\xi^2 + 15\,\xi + 51}{2\sqrt{15}}\,.
\end{aligned}$$

The components of the hodograph $\mathbf{r}'(\xi) = \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)$ and its parametric speed are then determined from (6) as

$$\begin{aligned}
x'(\xi) &= \frac{-\,\xi^6 + 2\,\xi^5 + 21\,\xi^4 - 88\,\xi^3 + 139\,\xi^2 - 82\,\xi - 46}{3}\,, \\
y'(\xi) &= \frac{2\,\xi^6 - 64\,\xi^5 + 456\,\xi^4 - 1528\,\xi^3 + 3001\,\xi^2 - 3454\,\xi + 2162}{15}\,, \\
z'(\xi) &= \frac{28\,\xi^6 - 296\,\xi^5 + 1134\,\xi^4 - 2192\,\xi^3 + 2339\,\xi^2 - 1106\,\xi + 118}{30}\,, \\
\sigma(\xi) &= \frac{(\xi^2 - 6\,\xi + 10)(2\,\xi^4 - 8\,\xi^3 + 22\,\xi^2 - 28\,\xi + 29)}{2}\,.
\end{aligned}$$

Now the components of $\mathbf{r}'(\xi)$ in the plane Π spanned by $\mathbf{m}$ and $\mathbf{n}$ are

$$\begin{aligned}
\mathbf{m}\cdot\mathbf{r}'(\xi) &= 4\,\xi^5 - 32\,\xi^4 + 104\,\xi^3 - 188\,\xi^2 + 192\,\xi - 96\,, \\
\mathbf{n}\cdot\mathbf{r}'(\xi) &= -\,\xi^6 + 10\,\xi^5 - 37\,\xi^4 + 72\,\xi^3 - 84\,\xi^2 + 56\,\xi - 28\,,
\end{aligned}$$

and since

$$[\,\mathbf{m}\cdot\mathbf{r}'(\xi)\,]^2 + [\,\mathbf{n}\cdot\mathbf{r}'(\xi)\,]^2 \,=\, [\,(\xi^2 - 2\,\xi + 5)(\xi^2 - 2\,\xi + 2)(\xi^2 - 6\,\xi + 10)\,]^2\,,$$

the projection $\mathbf{p}'(\xi)$ of $\mathbf{r}'(\xi)$ on to Π is a planar Pythagorean hodograph. The spatial PH curve $\mathbf{r}(\xi)$ and its PH projection $\mathbf{p}(\xi)$ on to the plane Π are shown in Figure 4.

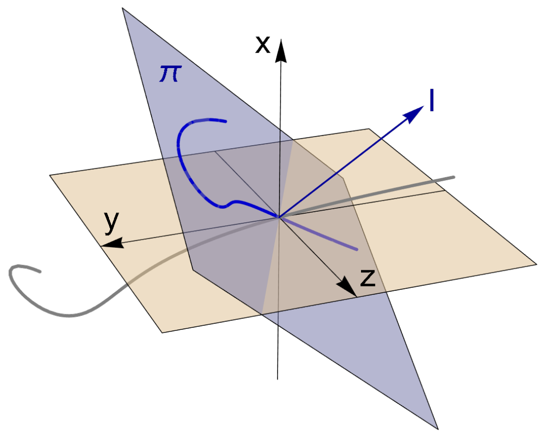

Figure 4: The spatial PH curve $\mathbf{r}(\xi)$ for $\xi \in [-0.5, 4]$ (gray curve) in Example 7 and its PH projection (blue curve) on to the plane Π.

Remark 5. The spatial PH curves of given degree that admit planar PH curve projections on to a prescribed plane Π form a proper subset of all spatial PH curves of the same degree. A degree m quaternion polynomial $\mathcal{A}(\xi)$ generates a spatial hodograph $\mathbf{r}'(\xi)$ of degree $2\,m$. Since $\mathcal{A}(\xi)$ has four quaternion components, $\mathbf{r}'(\xi)$ depends on $4\,(m+1)$ free parameters. On the other hand, if $\mathbf{r}'(\xi)$ has a PH projection onto Π, it depends on at most $2\,(m+2)$ free parameters, since the polynomials (14a)–(14b) are determined by the real constants k_0, k_1 and φ_0, φ_1, the polynomials $f_0(\xi)$ and $f_1(\xi)$ with real roots, and the complex values $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r$. These embody the maximum number of freedoms when $f_0(\xi), f_1(\xi)$ have only simple roots with $\gcd(f_0(\xi), f_1(\xi)) = 1$, and the complex values are all simple, i.e., $\mu_1 = \cdots = \mu_r = \nu_1 = \cdots = \nu_r = 1$ in (16), which implies that $\alpha + r = \beta + r = m$ (and thus $\alpha = \beta$). Since the polynomials (14a)–(14b) then depend on $2\,\alpha$ real roots and $r = m - \alpha$ complex values, and four real constants k_0, k_1, φ_0, φ_1 we deduce that $\mathbf{r}'(\xi)$ depends on $2(m+2)$ real parameters. When $f_0(\xi), f_1(\xi)$ have multiple roots or common roots, or when complex values $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r$ with multiplicities greater than 1 are employed, the number of real parameters embodied in the spatial Pythagorean hodographs $\mathbf{r}'(\xi)$ that admit planar PH projections is reduced accordingly.

Remark 6. It is of interest to consider the following related question: for any given spatial PH curve, can we find *some* plane on to which its orthogonal projection is a planar PH curve? The answer does not follow from the preceding analysis, but we can use Remark 5 to count the number of free parameters. Not prescribing a particular projection plane increases the number of free parameters for the unit quaternion (35) by two (a plane is identified by its unit normal vector), but in general this is still not sufficient to ensure that a plane can be found on to which the projection of the spatial PH curve will be a planar PH curve.

6. Closure

Although the complex–number and quaternion models are well–established approaches to the construction and analysis of planar and spatial Pythagorean–hodograph (PH) curves, the fact that projections of spatial PH curves on to planes with general normal directions do not ordinarily yield planar PH curves has not previously been remarked upon. The present study has developed a comprehensive characterization of the conditions on the quaternion polynomials generating spatial PH curves, that ensure they have planar PH curve projections on to any given plane. These conditions identify the spatial PH curves that have PH planar projections as a proper subset of the complete family of spatial PH curves.

It was shown that, in the generic case, such spatial PH curves incorporate approximately half the number of free parameters embodied in unrestricted spatial PH curves of given degree, and consequently higher degrees must be employed to satisfy geometric design constraints — e.g., interpolation of end–point Hermite data or satisfaction of total arc length constraints [8]. These are non–trivial problems that deserve a separate in–depth investigation. A further topic of interest concerns other types of lower–dimension projections of spatial PH curves that generate PH projected curves — for example, central (perspective) projections from the origin on to planes of any orientation in $\mathbb{R}^3$, or on to the unit sphere in $\mathbb{R}^2$.

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