

Coordination Dynamics of Semiotic Mediation: A Functional Dynamic Systems Perspective on Mathematics Teaching/Learning

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> Context • Radical embodied approaches to cognition propose a drastic alternative to representation-based models of the mind by way of theorizing and empirically demonstrating the constitutive roles of perception–action loops in human behavior. However, applying those approaches to higher-order processes – such as mathematical thinking and learning – remains one of the hottest debates within contemporary cognitive science. **> Problem** • How might a radical embodied perspective theoretically explain semiotic mediation? For example, how can we theorize the understanding of trigonometric relations expressed through the symbolic form of “ $\sin 2a = \sin a$ ”? **> Method** • Revisiting the nature of semiotic mediation, we pursue a historically grounded theoretical analysis that integrates perspectives from Lev Vygotsky and Nikolai Bernstein; to make the theoretical proposal accessible, we illustrate it by empirical data from a dual-eye-tracking study on teaching/learning trigonometry. **> Results** • We show how semiotic mediation of mathematical ideas is constituted as direct intercorporeal sensorimotor coordination between interlocutors. We treat semiotic actions as directly transforming an environment for the interlocutor, inviting new sensorimotor processes. New sensorimotor routines potentially lead to the emergence of pedagogically desired perception and action orientations, that is, new enactive capacity as the cognitive vehicle of mathematical reasoning. As such, Vygotskian cultural–historical ideas of semiotic mediation become a natural continuation of a radical embodied project developed by Bernstein, Kelso, Turvey, and others. Semiotic actions do not represent mind-independent reality as standalone tokens – rather, they present the environment itself, for the interlocutor, in a new way. **> Implications** • The proposed perspective avoids ontologically problematic views of pedagogical discourse as the negotiation of minds, instead focusing educators’ attention on transformations of the students’ environment that foster desired perceptions and actions. **> Constructivist content** • We develop an alternative to a social-constructivist reading of cultural-historical ideas, thus contributing to the understanding of higher-order cognition as direct extension of perception and action. **> Key words** • Bernstein, complex dynamic systems, coordination dynamics, functional systems, intercorporeal systems, mathematics education, multimodal joint attention, semiotic mediation, Vygotsky.

Introduction

« 1 » Scientific understanding of the human mind has been undergoing deep transformations with the rise of the embodied paradigm in the cognitive sciences (Newen, Bruin & Gallagher 2018). In turn, these embodied approaches have been transforming the learning sciences, as evidenced in embodiment theoreticians treating educational issues (Shapiro & Stolz 2019), learning scientists publishing special issues on embodiment and education (Hall & Nemirovsky

2012), and the International Society of the Learning Sciences thematizing embodiment approaches in its annual meeting (CSCL 2019). As educational researchers evaluate tenets and implications of embodiment, a particularly bold proposal comes from radical embodied approaches that deny the representationalist nature of cognition (e.g., Chemero 2009) and re-conceptualize the environment from an ecological perspective (Turvey 2019). These radical reformulations of cognition theory challenge widespread assumptions regarding the ontology of

mathematical concepts and the epistemology of mathematics students learning these concepts.

« 2 » Attempts to reformulate educational theories as drawing on embodiment perspectives fall short, as the 4E paradigm – the notion that cognition is embodied, embedded, extended, and enacted – has not yet satisfactorily theorized cognition beyond human–environment sensorimotor coupling (Hutto & Abrahamson 2022). On the one hand, non-representationalist accounts of motor actions and perceptual

processes are well grounded in the field of coordination dynamics (Turvey 1977; Kelso & Schöner 1988), which have provided detailed experimental demonstrations of complex-dynamic-systems processes in regulating behavioral change. Those accounts cohere (Baggs & Chemero 2021) with phenomenological and ecological-psychology perspectives on ontology that approach phenomena not from an objectivist definitive perspective but per their ad hoc, contextual, and goal-oriented subjective relevance for interaction (Turvey 2019). On the other hand, radical embodied approaches to higher-order processes are still embryonic (Sanches de Oliveira, Raja & Chemero 2021; van Dijk & Rietveld 2021; Veissière et al. 2020) and polemic (Goldinger et al. 2016; Pulvermüller 2013) and have not yet resolved the persisting explanatory dichotomies between physical and intellectual practices. The ontological gap between sensorimotor processes as functionally entangled with the material environment and higher-order processes that operate mental models impedes a holistic theoretical vision of the embodied mind engaged in mundane socio-cultural activity. As a result, educational researchers inspired by embodied approaches who aim to understand and facilitate mathematics acquisition choose a moderate embodied position when talking about the role of the body in educational design (e.g., Duijzer et al. 2019) or combine embodied and representational perspectives (e.g., Núñez, Edwards & Filipe Matos 1999; Bartolini Bussi & Mariotti 2015).

« 3 » An account provided in this article aims to seamlessly embroider cognitive processes as responsible for both *motor* and *mathematical* practice by theorizing skill – any human skill including semiotic activity – as the functional dynamic organization of perception and action. Experimental educational paradigms are creating new contexts in which to explore the ontological continuity from motor to mathematical skill, whether teaching–learning to flip a pancake or solve an equation for x . Our approach to conceptual learning endorses classical cultural–historical ideas on *semiotic mediation*, namely, involvement of words, formulas, visual notations, and other cultural artifacts, as an indispensable feature of higher-order

cognition (Vygotsky 1978; Leontiev 1978; Cole 1996; Roth & Radford 2011; Bartolini Bussi & Mariotti 2015). At the same time, our approach also endorses systemic analyses of physical movement performance (Kelso & Schöner 1988; Bernstein 1996; Mechsner et al. 2001; Sheets-Johnstone 2015) within the ecological environment (Gibson 1986; Heft 1989; Turvey 2019). *We propose to re-conceptualize semiotic mediation as a direct social extension of bodily dynamics in cultural ecologies.* We ask: *How can we describe semiotic mediation processes in consonance with the science of coordination dynamics?*

« 4 » In what follows, we will present a monistic approach to organism–environment interaction. This monistic approach proposes a coherent combination of, on the one hand, coordination dynamics’ findings on action regulation and, on the other hand, the cultural–historical view on artifacts’ mediating role in acquiring higher-order skills (see also Abrahamson 2021; Abrahamson & Trninic 2015). We elaborate this monistic theoretical conciliation as *intercorporeal bodies–artifacts functional dynamic systems* (Shvarts et al. 2021; Shvarts & Abrahamson in press). Our theoretical proposal is meant to resolve an ostensible ontological discontinuity between materiality and mental cognition, specifically looking to explain mathematical cognition as an extension of sensorimotor skills (Abrahamson 2021).

« 5 » We intend the article as a sketch for a possible theoretical vision of mathematical thinking, teaching, and learning phenomena. In the following sections, we provide conceptual historical ground for our ideas through an excursion into the historical origins of coordination dynamics and the cultural–historical approach (Section 2); outline our theoretical proposal (Section 3); and provide a brief illustration from empirical studies on hand and eye movements of a student and a tutor as they collaboratively solve mathematical problems (Section 4). These historical considerations and empirical excerpts are put forth not as providing strict evidence supporting our approach. Rather, they are intended to facilitate our presentation of the main theoretical proposal in Section 3 by providing a context for grasping those ideas.

Historical prelude: Lev Vygotsky and Nikolai Bernstein

« 6 » In early 20th century Russia, two researchers developed strong theoretical systems that later grounded large fields of studies. Vygotsky developed a cultural–historical approach to psychological functions (Vygotsky 1978), aiming to uncover the uniqueness of human cognition based on the historical and cultural development of the human species. His core idea of semiotic mediation and student–teacher collaboration as forming higher psychological functions grounded a vast range of cultural–historical studies in education, particularly in mathematics education (e.g., Radford & Roth 2011; Bartolini Bussi & Mariotti 2015; Cole 2016). Bernstein created a novel (and now widely accepted) methodology of recording movement dynamics, which led him to theorize the development of movement skills (Bernstein 1967). His physiological theory of movement construction kindled the development of the systemic field of coordination dynamics (e.g., Turvey 1977; Kelso & Schöner 1988). At first blush, the respective works of Vygotsky and Bernstein seem to address different aspects of cognition. However, we claim that these systems need to be understood as complementary and that their joint elaboration leads to important insights for a general theory of cognition and for educational design and practice (Abrahamson & Trninic 2015).

« 7 » Per Vygotsky, higher psychological functions, such as counting or writing, are social in origin; the relation between lower and higher functions is *complex*:

“Higher mental functions are not built up as a second story over elementary processes, but come as new psychological systems that include a complex merging of elementary functions that will be included in the new system, and themselves begin to act according to new laws.” (Vygotsky 1999: 43)

These new psychological entities are mediated by cultural tools and artifacts, including words; they are also *systemic*, as they coordinate multiple subsystems, such as motor actions, perception, and speech, in fulfilling some culturally relevant function.

« 8 » Bernstein is known less in educational literature, as the field of his studies was movement construction (Bernstein 1967). His contribution lies in a detailed account of motor actions' complex dynamics, which he developed in opposition to a then-dominant behavioristic model based on conditioned and unconditioned reflexes (Pavlov 1927). Bernstein focused on investigating human motor activity – for example, throwing a ball or playing the piano – as solving *motor problems* posed within complex environments. The main concern for Bernstein was the problem of too many degrees of freedom: How does the human cognitive system manage the performance of physical movements involving the coordinated micro-motor action of dozens of bodily muscles? How is this possible given that the environment is never quite the same? Just lifting a coffee mug – if controlled muscle-by-muscle as a system of direct responses – becomes an impossible task! Bernstein (1967) theorizes a complex and dynamic system comprising multiple *levels* – such as tackling gravity or orienting in space – that become coordinated in fulfilling a motor problem at hand. Bernstein's multilevel model of action is best known for the systemic idea of *synergies* (Kelso & Schöner 1988): higher (psychological) levels of action regulation propagate “down” as constraints facilitating self-organization of partially independent lower-level processes into coherent movement ensembles.

« 9 » Vygotsky and Bernstein developed their theories independently, and yet, we argue, these theories are complementary. Both scholars took holistic views on physiology and perception, developed earlier by Kurt Goldstein, Jakob von Uexküll, Kurt Koffka, and others, as fundamental principles and sources of inspiration (Vygotsky 1978; Sirotkina & Biryukova 2015: 271). Curiously enough, Vygotsky and Bernstein, both born in 1896, worked together in 1925–27 in the Institute of Psychology, where they assisted each other in their studies (Sirotkina 1996). Whereas their research foci diverged – Vygotsky concerned more with higher cognition, Bernstein more with the organization of bodily motion – already Alexander Luria, their mutual friend and collaborator, considered them as providing a unified theory of the human mind and enactment (Luria 1973: 247).

« 10 » Vygotsky speaks of higher mental processes as “meaningful functional systems,” which are to be understood based on the following assumptions:

“(a) the assumption of plastic, changeable inter-functional relations; (b) the assumption of *complex dynamic systems* which have to be considered as the result of integration of elementary functions; and (c) the assumption of a categorical reflexion [sic] of reality in the human mind.” (Vygotsky 1965: 382, our emphasis)

The first two assumptions are directly in line with Bernstein's idea that lower biomechanical levels (e.g., maintaining a posture) self-organize to solve a motor problem imposed at a higher level (e.g., throwing a ball). The third assumption departs from investigations of movement per se to treat categorization, the psychological process of recognizing classes of objects. Vygotsky clarifies that the higher cognitive process, such as categorization, relies on “extra-cerebral connections” (Vygotsky 1965: 385); thus he locates higher processes as distributed between brain and tools. In the thesis of extra-cerebral localization of cognitive functions, Vygotsky enriches Bernstein's ideas by an assumption that is now called extended cognition (Clark & Chalmers 1998; Kirchoff & Kiverstein 2019).

« 11 » Bernstein, in turn, while limiting the focus of his analysis to explicit physical movements, hypothesizes an additional level in human action regulation, which he addresses as the *symbolic level* (Bernstein 1967): particularly, this psychological level is involved in writing, speaking, and artistic performance. The symbolic level addresses motor problems requiring higher-level characteristics (e.g., in expressive theatrical or musical performance) and develops its own repertory for anticipating and correcting motor performance.

« 12 » Whereas Vygotsky and Bernstein drew from similar intellectual wells, cultural–historical activity theory and coordination dynamics developed as separate research fields, each bearing unique contributions to educational science. The contribution of Vygotsky's legacy to educational science barely needs introduction. Bernstein's ideas highly impacted key figures in the field

of complex dynamic systems (e.g., Turvey 1977; Kelso & Schöner 1988) – direct originators of contemporary radical embodied cognitive science. Indeed, a search conducted on 15 March 2021 in Scopus for papers citing “Vygotsky L. S.” and “Bernstein A. N.” (in relevant fields) reveals 38,399 and 2,638 quotations for each, respectively, and just 86 papers citing both authors. While the cultural–historical approach has incorporated ideas from embodied cognition (Bartolini Bussi & Mariotti 2015; Radford 2021), by and large it has not embraced coordination dynamics principles (but see, e.g., Tancredi, Abdu et al. 2022). Simultaneously, researchers within radical embodied approaches who have become aware of the fundamental roles of cultural artifacts in higher-order cognition (e.g., Malafouris 2019; Sanches de Oliveira, Raja & Chemero 2021) rarely cite the cultural–historical approach that has been promoting this idea for almost a century. It thus appears timely to develop a unified theoretical account that reconciles radical embodied approaches to cognition and cultural–historical ideas on higher-order functions.

Theoretical development: A functional dynamic systems perspective on semiotic mediation

Ecological ontology of culture

« 13 » The theoretical perspective of this article is based on reconsidering realistic ontology – the objective existence of things independently from a cognizing subject – towards a more subtle onto-epistemological view. In consonance with ecological psychology (Gibson 1986) and with cultural–historical calls for eliminating views of ontology and epistemology as separate presumptions (Stetsenko 2020), we adopt an *ecological ontology*, in which the existence of a quale for an individual is contingent on its relevance for an enactment within an organism–environment system (Gibson 1986; Turvey 2019). This onto-epistemological perspective considers the world as objectively revealing itself in different manners for different forms of life (Rietveld, Denys & van Westen 2018). Leaving behind details of

philosophical debates, we use the notion of *environment* as an ecological niche, which presents an organism with affordances – direct opportunities for enactment (Gibson 1986) – instead of cold physical qualities of mind-independent reality. Analysis of cultural environments as replete with artifacts suggests theorizing the environment as a system of *nested affordances* (Rietveld, Denys & van Westen 2018) that enable both naïve (material) and cultural (ideal) forms of enactment¹ (Cole 1996; Vygotsky 2001). For example, a cup holds an affordance for throwing as one would a stone but also for drinking tea. Moreover, a cultural environment includes multimodal semiotic means, such as a companion's exclamation, "Look, this cup can serve as a paperweight!" Yet our approach further complements the assumption of nested affordances in the environment with *brain-body potentialities* of the *organism*, that is, the organism's inherent and acquired counterparts for engaging with affordances (Shvarts et al. 2021). Highlighting the organism's role in exploiting organism–environment affordances becomes instrumental in educational research discourse, where we assume that the organism, i.e., a student, is to develop new potentialities. Then one can ask how these potentialities may develop and what roles other cultural agents may play in fostering this development, so that new affordances in the cultural environment come forth for the student. That is, the student learns.

A functional grip of a dynamic system and multilevel intentionality

« 14 » Scholars of radical embodied cognitive science have been seeking to bridge ideas from ecological psychology toward enactivism and complex dynamic systems theory to highlight cognitive activity as the continuous enactment of a *complex dynamic system* of sensorimotor processes (Baggs & Chemero 2021; Rietveld, Denys & van West-

1 | See more on merging the cultural-historical idea of naïve and ideal forms of action and the idea of nested affordances in an upcoming article currently under review, "Reifying actions into artifacts: An embodied perspective on process-object dialectics in higher-order mathematical thinking" by Anna Shvarts, Rogier Bos, Michiel Doorman and Paul Drijvers.

en 2018). Working largely in consonance with these ideas, we bring forth a historical version of the complex-dynamic systems approach that highlighted the *functionality* of its performance. Vygotsky's and Bernstein's ideas were developed synchronously with those of Pyotr Anokhin, whose research program addressed the plasticity of physiological processes (Anokhin 1975). The theory of *functional dynamic systems* highlighted that physiological processes spontaneously and flexibly re-assemble to fulfill a behavioral function (e.g., walking) within a changing environment vis-à-vis available neuronal and muscular resources, thus giving rise to "an adaptive effect in the organism–environment interaction achieved upon realization of that system" (Alexandrov et al. 2018: 2). A functional system bears an evolutionary advantage by exploiting the recurrence quality of the ecological environment and actions: based on previous partially repetitive experiences, the system anticipates how the environment will impact the organism at the completion of action (Anokhin 1962). This anticipatory mechanism has also been conceptualized as a *forward model*, namely anticipatory neural circuits pre-activated *ahead* of observable enactment, thus allowing one to attune the action to the environment on the basis of any detected discrepancy between anticipated and received sensations (Bernstein 1967). The future-directedness cognitive mechanism of anticipation can be conceptualized as *striving for a better functional grip* (q.v., a system's drive for relative equilibrium, Merleau-Ponty 2002, or its "tendency towards a grip on multiple affordances," Rietveld, Denys & van Westen 2018: 44).

« 15 » This fundamental idea of *striving for a better functional grip* propagates to semiotic mediation through the notion of *multilevel intentionality*. At the psychological level, behavior is organized as the goal-oriented, intentional activity of solving problems, including *motor problems* (Bernstein 1967, e.g., hammering a nail) or *mathematical problems* (solving a trigonometric equation). Yet, this psychological level is only one of multiple anticipatory levels acting simultaneously within a functional dynamic system.

« 16 » Multiple levels of intentionality – besides conscious goal-orientation – have

been theorized as *Ur-intentionality*: at the level of a bacterium's directedness along the chemical gradient (Hutto & Satne 2015); *motor intentionality* that orients a hand as it treats an object or greets a friend (Merleau-Ponty 2002); or *enactive intentionality* (shared motor intentionality), as when football players immediately read other players' directionality without mind-reading (Gallagher & Miyahara 2012). These philosophical ideas are supported by physiological and physical studies showing that a system's *ad hoc* state is best explained by its future functional outcome. In laboratory experiments, response times to a visually presented object are briefer when the response button resembles the shape of the object, as if the organism is ready to grasp the object (e.g., Tucker & Ellis 1998). The retina, a sensory receptor, is pre-activated to facilitate recognition of motion or simple configurations (Souihel & Cessac 2021; Zippora, Shimon & Ehud 2021), and eardrums oscillate in anticipation of visual sensory input to coordinate across modalities (Gruters et al. 2018). Some researchers discern end-directedness even in physical dissipative systems allegedly *seeking* states that increase entropy (Dixon et al. 2015). In a sense, functional dynamic systems are teleological at the level of neurons (Alexandrov et al. 2018) and bodily periphery: an entire body gets pre-activated as it anticipates the sensations of efficient enactment. Overall, a functional dynamic system develops to enhance a *functional grip* on the ecological environment at multiple levels, including sensorimotor and social interactions, by anticipating the environment, as it would appear when enactment is accomplished.

Sensorimotor dynamics and perceptual orientations

« 17 » Taken from a functional dynamic perspective, enactment can be understood as a multi-level phenomenon. *Sensorimotor* processes are activations of receptors and muscles – whether they are in the eyes, ears, fingers, or tongue – and brain structures; they are organized in multiple synergetic levels that are partially independent. *Perception and action* are higher-level emergent synergetic units that are phenomenologically given and, as such, meaningful for psychological and educational analysis.

« 18 » Critically, a possibility for action does not determine top-down the sensorimotor organization of a strictly unique response to environmental stimuli. Rather, functional dynamic systems are inherently contingent and flexible in achieving target outcomes (Bernstein 1967) through self-organizing into relative synergetic stability (Kelso & Schönner 1988). Complex environment–organism interaction systems “exhibit self-organization and emergent processes at multiple levels” (Thompson & Varela 2001: 421), each of them driven by their own intentionality yet impacting other levels, as “emergence involves both upward and downward causation” (ibid).

« 19 » The interplay of systemic levels is exploited by experienced sport educators (Chow et al. 2007; Newell & Ranganathan 2010). Swim coaches teaching the crawl style might tell you to “push the water back,” leaving you to figure out how to implement biomechanically that top-down imagistic instruction (Stoate & Wulf 2011; see also Hutto & Sánchez-García 2015). Reciprocally, investigators of problem-solving and mathematical learning have noted the bottom-up organization of sensorimotor activity into a well-formed perceptual structure, as, for example, coordinating speech, gestures, or specific eye-movements may bring about an insight moment or an efficient cognitive strategy (Church & Goldin-Meadow 1986; Grant & Spivey 2003; Stephen et al. 2009). Specifically, mixed-methods analyses of mathematics students’ hands-and-eye movements and their verbal–gestural utterances reveal the self-organization of sensorimotor processes into new, mathematically relevant, perceptual orientations known as *attentional anchors* (Tancredi, Abdu et al. 2022). Overall, as a new synergetic invariant of *sensorimotor processes* emerges at the physiological level, at the psychological level of analysis we see a new *form of action* and new *perceptual orientation*, thus bridging sensorimotor processes and direct ecological perception (Maturana & Varela 1992; Heft 2020). While motor synergies are critical for the analysis of coordination dynamics, the analysis of educational practice in mathematics foregrounds *perceptual orientations that emerge as the students’ means of organizing the performance of prescribed tasks*.

Cultural artifacts as stabilizers of functional systems’ dynamics

« 20 » Within the cultural-historical tradition (Leontiev 1978; Wenger 1998; Radford 2003), artifacts are conceptualized “as embodiments of certain cultural practices, crystallized templates of actions, schematized representations of certain ways of doing things as discovered in the collaborative history of humanity” (Vianna & Stetsenko 2006: 97). Think of a spoon that crystallizes the practice of eating. Appropriated and fashioned by subsequent generations, artifacts expand cognitive activity beyond skull, skin, and the individual, in consonance with extended cognition ideas (Vygotsky 1965; Clark & Chalmers 1998; Di Paolo 2009; Mernary & Gillett 2022).

« 21 » As physiological and phenomenological research shows, artifacts change body schema transform perception of reachable space (see Weser & Proffitt 2021, for an overview), allowing one to perceive and act with the world that lies on the other side of scissors or a guide cane (Gibson 1986: 40f; Merleau-Ponty 2002). From a functional dynamic systems approach, the cognitive system is extended when an artifact is appropriated into a functional system. An artifact becomes a direct extension of the body as it is incorporated into perception–action loops (Lockman 2000) per the task at hand (Bril 2015). A spoon extends the hand, thus partaking in a perception–action loop. We suggest calling the incorporation of an artifact into perception–action loops the genesis of a *body-artifacts functional system* that enables instrumented actions (Shvarts et al. 2021): artifacts now come to physically mediate perception and action.

« 22 » Vygotsky viewed cultural artifacts as playing constitutive roles in higher cognitive functions (Vygotsky 1978). Elaborating from embodiment perspectives, we suggest that artifacts, such as mathematical visualizations, partake in perception–action loops just like a cane or a pair of scissors do (Shvarts et al. 2021). For example, perceiving points on a Cartesian plane (Krichevets, Shvarts & Chumachenko 2014) and data on histograms (Boels, Bakker & Drijvers 2019) requires specific sensorimotor processes: educated perceivers move their eyes in a particular manner so that they make use of the diagrams’ cultur-

al affordances in reading relevant numeric data. An artifact itself preserves cultural practice – historically emerged stable dynamics – because its form specifies situated enactment as it is shaped within the practice. For example, a violin bow privileges a particular grasp of the hand at the nut, and the beads of a Soroban (Japanese abacus) match the counting hand (Monaghan, Trouche & Borwein 2016).

A functional dynamic system beyond one body

« 23 » In this section we extend principles of functional dynamic systems to situations of social collaboration. When two or more people collaborate on achieving a common goal of joint physical action within a shared space and time, they characteristically manifest tight intercorporeal coordination of sensorimotor processes. Coordination emerges spontaneously as people tap on a desk or swing on rocking chairs together (Oullier et al. 2008; Richardson et al. 2007), and people may achieve movement calibration at the level of milliseconds in dance or other motor performance (e.g., Sebanz, Bekkering & Knoblich 2011; Kimmel & Preuschl 2016). Intercorporeal coordination can be further found as two brains coordinate in wavelength and phases of neuronal activity (e.g., Liu et al. 2016; Fuchs & Kelso 2018) or in heart rhythms (Fusaroli et al. 2016). When people operate collaboratively on a common visual display, the quality of their collaboration is correlated with their dynamic gaze coordination (Schneider et al. 2018). Moreover, coordination increases as collaboration on a joint problem continues (Dale, Kirkham & Richardson 2011). Two bodies may couple their dynamics so tightly that the phenomenon can only be explained by assuming that the individuals are actively anticipating each other’s sensorimotor processes, thus manifesting motor intentionality at the social level.

« 24 » Granted, two bodies often reach joint aims without *full* congruency of sensorimotor processes. Yet, in situations such as carrying a table together or solving a puzzle together (Sebanz, Bekkering & Knoblich 2011), intercorporeal processes play a decisive role, where the bodies are coupled around a common piece of the environment. Collaborators jointly attend to the

environment and coordinate along one or more sensory modalities – such as looking and touching (Yu & Smith 2016; Shvarts & Abrahamson 2019; Pagnotta, Laland & Coco 2020), so that their perceptions of the environment may become sufficiently congruent to enable joint achievement of a mutually desirable outcome. In so doing, new features or relations within the environment may emerge as relevant for the task and themselves become objects of *multimodal joint attention* ready to be captured semiotically. Perception–action dynamics between people thus stabilize.

« 25 » Collaborating partners engaged in joint action do not build a representation of each other's minds (Gallagher & Miyahara 2012). Consider, for example, how time constraints would prohibit football-team players from doing so while in rapid action (Gallagher 2011). Rather, sensorimotor processes self-organize across the bodies into a system that exhibits *multilevel intentionality* towards a common action outcome: the bodies directly anticipate each other at the sensorimotor level, while collaborators are conscious only of their general action goal. The mirror-neuron theory, by which watching others' actions activates one's own corresponding motor neurons, might explain how *forward models* propagate beyond an individual, thus revealing an embodied level of anticipation within an intercorporeal system. Notably, mirror neurons would orient individuals toward the *outcomes* of each other's action, not towards spatial, morphological, or kinematic details of those motor processes (Rizzolatti et al. 2014). Collaborators directly couple their sensorimotor processes into a single *intercorporeal functional dynamic system* that subsume both bodies anticipating each other at multiple levels as they pursue a common goal (Newman, Griffin & Cole 1989; Shvarts & Abrahamson in press).

« 26 » While dynamic intercorporeal coupling of sensorimotor processes is evident in physical interactions, we propose a view of semiotic activity, too, as a sensorimotor process. To begin with, *speaking* consists of locutionary motor actions generating signals in the audial modality. Audial signals propagate through the environment causing direct coordination between bodies:

“[T]he air's molecules are made to vibrate following oscillatory patterns that can be controlled by articulators including the vocal cords, the tongue, the different parts of the oral tract, and, occasionally, the nasal cavity.” (Bottineau 2010: 272)

In this sense, speaking is remote-touching. Just as joint actors align their bodies in pragmatic actions, so they align their sensorimotor systems in semiotic action: accents, pause patterns, voice intensity, and so on increasingly align among interlocutors while communicating (see Dale et al. 2014). Yet, in order to explain how semiotic activity coordinates people not only at the sensorimotor level but at the level of common understanding, we will now extend the idea of bodies-artifacts functional systems to semiotic situations.

Semiotic activity capturing, preserving, and re-activating perceptual orientation

« 27 » Human capacity to expand collaboration between bodies beyond collocated synchronous moments provides an evolutionary advantage, such as in signaling the presence of a predator or hunting large prey. A theorization of language as extending human collaboration beyond joint action agrees with evolutionary anthropologists' thesis on the phylogenesis of speech (Donald 2010) and cognitive developmental psychologists' thesis on the ontogenesis of speech (O'Madagain & Tomasello 2021). Language, and, more generally, semiotic means, are human artifacts. Similar to other cultural artifacts, semiotic means are crystallized forms of stabilized cultural practice that emerge to mark and perpetuate stabilized perception–action dynamics between people. Unlike motor movements, which are continuously corrected in interaction with the environment (Bernstein 1967), semiotic means – words and symbols – are relatively fixed either as highly repetitive air vibrations or as written inscriptions independent from the context of enactment. As a result, semiotic means can serve as constraints for future dynamics (Rączaszek-Leonardi 2009), allowing an organism to extend situations at hand through time by leveraging non-local historical echoes (Cowley & Nash 2013). Semiotic means expand dynamic co-

ordinations beyond local perception–action practices through communicating with different partners and thus create ecological environments coordinated across different temporal scales (Steffensen & Harvey 2018). Overall, language can “freeze” – in the form of a new semiotic means – the overlapping perceptual orientations toward a shared domain of scrutiny of collaborating individuals in the moments of joint attention. Semiotic activity can then utilize this means to prospectively re-trigger those perceptual orientations even in the absence of temporal synchronization with an interlocutor or even in the absence of any interlocutor, like in reading. Moreover, semiotic stabilizations have consequences for individual cognition: they create a possibility for constraining a person's own dynamics (Abrahamson 2021) and for generalization beyond singular enactment (Bottineau 2010).

« 28 » During semiotic activity, one interlocutor physically alters the environment by sensorimotor production of words and symbols. In that sense, *words act as material tools*, thus unifying traditional distinctions between sensorimotor and semiotic practices. Imagine a friend handing you a spatula while you are cooking together. This motor act might signal to you that it is time to flip over the pancake. As you are familiar with this tool, you immediately appropriate the spatula into your body-artifacts functional system. Similarly, interlocutors “hand” each other words – composed of vibrating air or lines on the screen – to refocus the other's attention and reshape their perception of the environment by highlighting culturally relevant affordances (Goodwin 1994; Bottineau 2010; Van Den Herik 2018). A particular word or metaphor that has stabilized one person's effective perceptual orientation toward a situation can later bring forth target sensations – forward models – for another person operating in a similar situation (Abrahamson 2020). Whereas a precise perceptual re-orientation of one's interlocutor is impossible, interlocutors throw their words into the environment, in an act of *prolepsis*, namely anticipation of the other's ability to bootstrap themselves into a new understanding (Stone & Wertsch 1984). Coupled with the previously stabilized enactment, pre-established semiotic means solicit from the other interlocutor sensorimotor pro-

cesses *novel for the situation at hand*, and, therefore, potentially usher in contextually effective perceptual orientations.

« 29 » Collaborating peers share similar experiences, and so words can be expected to re-trigger similar-enough target perceptual orientations, thus mediating enactment. In educational situations, however, where, by definition, teachers and learners draw on asymmetric experiences and skills, verbal instruction may fail. Prior to understanding a new mathematical domain, learners lack experience in using domain-specific words and symbols in a culturally expected way. Moreover, they might not be able to establish target perceptual orientations towards shared domains of scrutiny, as their perception is not yet educated (Goodwin 1994; Radford 2010), and may thus fail to distinguish what their teachers say. For example, not only are the students unfamiliar with the “+” sign, they are also unable to identify conjoined segments on a sketch. We thus turn to examine how semiotic mediation operates in educational settings.

Teaching and learning motor skills and mathematics

« 30 » As we noted previously, semiotic means – including formulas and definitions – are cultural artifacts that mark stabilized perception–action dynamics, thus further stabilizing them. Yet, we further proposed, cultural perception–action dynamics related to mathematical concepts are not yet familiar to learners as they come to study the new notions, thus semiotic means *per se* cannot mediate students’ understanding. New functional systems of mathematically perceiving and naming the world need to be established.

« 31 » Functional systems are goal-oriented – they are about solving problems. Thus, solving problems lies at the core of any educational process, whether in mathematics or in sports. Sensorimotor problems of sport trainees naturally require developing new sensorimotor processes, which self-organize into stable perception–action loops. While teaching, coaches shape athletes’ perceptual orientations through multimodal feedback, commenting on the quality of performance, and feedforward, offering modes of engagement for subsequent trials (Chow et al. 2007; Newell & Ranganathan

2010). Problems in mathematics may seem symbolic or abstract; yet, we suggest, solving a problem comprises senso-motorically coordinating mathematical symbols with properties and structures of the environment, e.g., perceiving graphs of trigonometric functions by attending to the vertical and horizontal positions of the points. As students establish new sensorimotor synergies, new perceptual orientations emerge, which, in turn, enable meaningful (grounded) domain-specific discourse. Just like in sports, mathematics teachers have developed educational strategies to scaffold the students’ actions as they develop pedagogically desired sensorimotor synergies and perceptual orientations (see Shvarts & Bakker 2019 for a conceptualization of scaffolding as higher-level regulation in the student–teacher functional system).

« 32 » There are multiple examples in the literature of how teachers can foster target perceptual orientations. Educators might proleptically introduce target discourse (Stone & Wertsch 1984), establishing for the students’ sensorimotor system new structural potentialities to be fulfilled through engaging in the learning activities. To educate learners’ perception, teachers adjust their own sensorimotor skilled performance to resemble the students’ current performance level (e.g., Jermann, Nüssli & Li 2010). Furthermore, teachers exploit multiple modalities for organizing an otherwise highly ambiguous cultural environment, so that students come to distinguish mathematics in it: Teachers use gesture (e.g., Nathan & Alibali 2011; Maffia & Sabena 2020), rhythms (e.g., Radford 2010), and intonations (e.g., Roth 2008). Those direct material interventions *temporarily alter the students’ environment* to usher the perception of its problem-relevant mathematical structure. Teachers further support students in reifying their perception of this altered environment through introducing symbolic artifacts. Teachers’ sensorimotor communicative means are effective as they directly target students’ bodies intercorporeally. In an intercorporeal functional system, collaborators anticipate one another’s actions through shared motor intentionality (e.g., the case of football players); it is through such anticipation that students grasp the educators’ orienting hints, as they,

for example, anticipate the *target* of pointing gestures rather than simply following the gestures themselves (Shvarts & Abrahamson in press). Yet, educators’ multimodal expression can be ambiguous, and so learners need to actively search for meaningful coordination between different semiotic means (Shvarts 2018). As a result of teachers’ and students’ efforts, students finally come to share cultural forms of action and perception for problem-solving in the domain. They further stabilize these forms by way of the semiotic means that they re-establish together with the teachers.

« 33 » A complementary solution for providing mathematics education in line with functional dynamic systems conceptualization comes from the embodied design approach (Abrahamson 2014). In this case, new forms of perception and action are prompted through *the design* of temporarily altered environments – *fields of promoted actions* (Reed & Brill 1996; Abrahamson & Trninic 2015) – even before any mathematical discourse is introduced. In these cases, disciplinary discourse comes *post facto* to reify – mark and further stabilize – already established efficient sensorimotor synergies and mathematically relevant perceptual orientations: In embodied collaboration with teachers, students build semiotic forms on their sensorimotor experiences and emergent perceptual orientations (Flood 2018). Early evidence for the positive impact of this educational approach on classrooms (Alberto et al. 2021; Kosmas & Zaphiris 2023) and remote learners (Shvarts & van Helden 2021) has been encouraging, while pointing to the need for coherent integration into curricula, appropriate professional development, and compatible assessment protocols.

« 34 » Overall, educators prompt goal-oriented activity by organizing learning environments as temporary transformations – by extensive multimodal semiotic expressions or by educational designs – of the culturally normative ecological niche. The creation of such environments has been broadly conceptualized as a type of *prolepsis* (Cole 2016) – a social form of intentionality that teachers and adults exhibit as they pass cultural forms of perception and action over to future generations, thus handing down the culture, along with its material artifacts and their mediating capacity. In a sense,

those temporarily altered environments are a society's forward models – a large functional system that tends to reproduce itself unless new global aims are in place calling to revisit national educational agendas (Pettimengin 2021).

Empirical finale

« 35 » Let us consider the mathematical problem of finding a value a that satisfies the equation $\sin a = \sin 2a$. Before reading on, please consider for a moment how you would go about solving this problem. There could be multiple approaches. One could use a known formula for the double angle to solve the equation algebraically. Another approach could be to draw two graphs and calculate where they intersect. Yet a third alternative is to imagine two angles, a and $2a$, on a unit circle and figure out at which positions their sine values match. In our study, a tutor scaffolded a student in solving this equation using a unit circle, per this third approach. Dual-eye-tracking technology combined with videography allowed us to investigate intercorporeal coordinations of participants' sensorimotor processes (gazes and gestures).

« 36 » The analysis of empirical data builds on ethnomethodological conversation analysis of multimodal behavior as a tool to understand learning (Abrahamson et al. 2019) extended by the analysis of eye-movement patterns within and across two bodies. As our theoretical stand assumes any behavior is driven by multilevel intentionality, we describe behavior as such, thus relying, on the one hand, on our natural capacity as educators to make sense of what a student and a tutor do, and, on the other hand, interpreting it through the lens of our theory.

« 37 » Working in a technological environment, participants could move a point up and down along the y -axis, thus controlling at once two angles that always necessarily have the same sine value (corresponding to the green vertical projection, see Figure 1). Through manual exploration, one can determine a position where the two angle measures relate as 1:2. Analysis of eye-movements revealed specific gaze movements along the visual display performed toward arriving at the solution (see Figure 1).

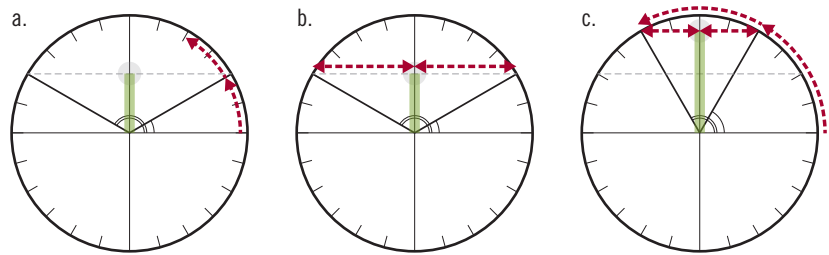


Figure 1 • Based on screenshots from an interactive technological environment for studying trigonometric functions on a unit circle. Red dashed arrows (not visible during teaching/learning) trace gaze movements that support noticing the two target mathematical relations: (a) doubling an angle; (b) attending to the equal sine values of two angles; (c) adjusting the position of the angles so that both mathematical relations are fulfilled at the same time.

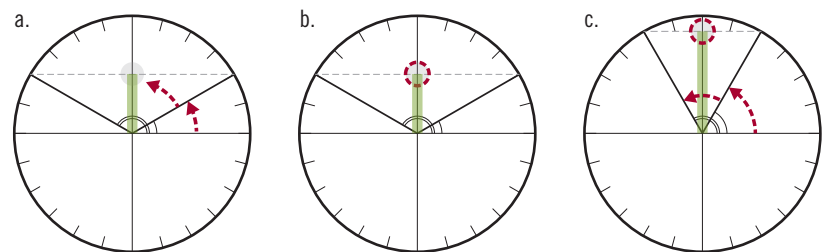


Figure 2 • Alternative way of attending to the same visual display. Red dashed arrows (not visible during teaching/learning) trace gaze movements that support noticing the two target mathematical relations: (a) doubling an angle; (b) attending to the equal sine values of two angles; (c) attending to the adjusted position of the angles without noticing that two mathematical relations are fulfilled at the same time.

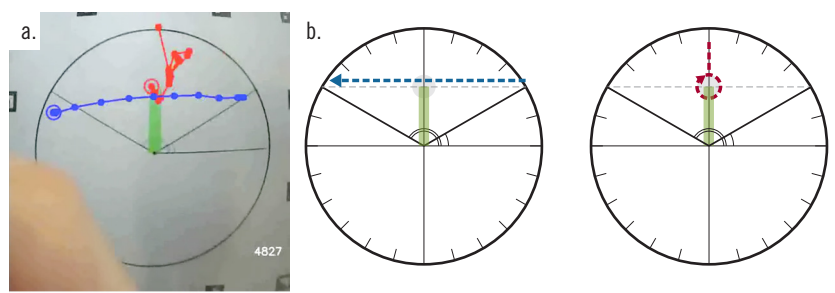


Figure 3 • Different sensorimotor actions on the same visual display while perceiving equal sine values (in all figures, the tutor's gaze is shown in blue and the student's gaze in red): (a) video screenshot with eye-gaze pathway overlays; (b) two schematic reproductions of the video screenshot, showing the tutor and student's respective gaze pathways. Here and further, the direction of the arrows signifies the order of attending to the display.

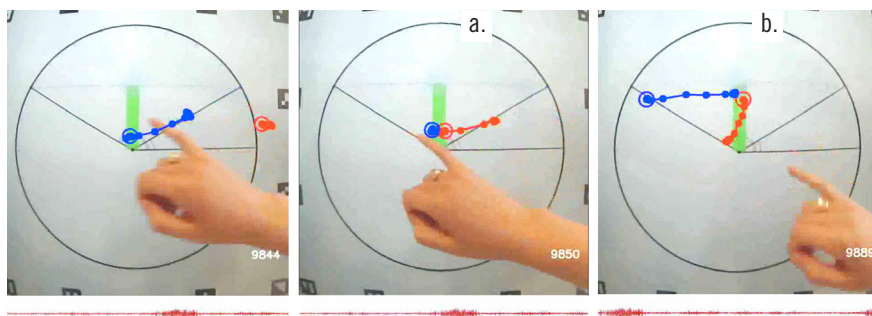


Figure 4 • The tutor gestures to the two angles in question, tracing their radii, while the student follows her attentively. However, their respective eye-gaze movements suggest they are interpreting differently how the diagram expresses the target mathematical relations the tutor is discussing.

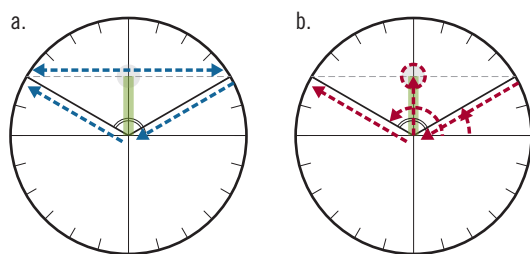


Figure 5 • Discoordination of perceptual structures despite the tutor’s verbal-gestural explanation: (a) the tutor attends to two radii as presenting angles and their vertical alignment (b) the student follows the tutor’s gesture along the radii yet then reverts to his prior, less-productive perceptual orientation.

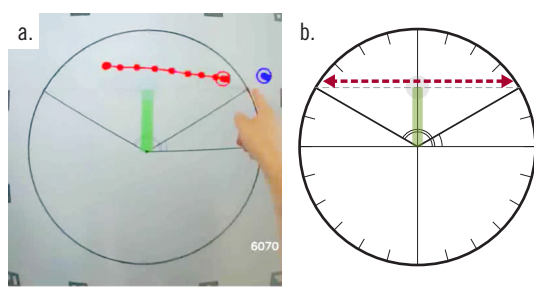


Figure 6 • The student attends to the vertical alignment of the angles for the first time.

« 38 » A pedagogical problem of using a unit circle, however, is that the visual display is ambiguous and the two target relations promoted by the visual display – equivalent sine values, a double angle – can be attended to differently, in a non-functional manner that does not lead to noticing the solution (Figure 2).

« 39 » In our study, the pedagogical problem was resolved through tutorial scaffolding. To begin with, the researcher pointed at the algebraic problem statement $\sin a = \sin 2a$ (a semiotic means placed above the unit circle on the interactive display) to orient the tutor and the student’s respective perception of the visual display.

However, the eye-tracking data suggest that they were attending to it differently (Figure 3).

« 40 » The tutor was apparently able to identify the student’s idiosyncratic orientation to the display by tightly coordinating her focus with his as she attentively followed his verbal-gestural explanation of the task (as eye-tracking data suggested, not provided here). Next, the tutor, made an attempt to modify the student’s attention so that it would match the task-effective perceptual orientation (see Figure 2 in contrast to Figure 1). To do so, she took semiotic measures to explicate her own perceptual orientation to the figure. She gestured along two radii composing an angle and asked whether the two angles in question correspond to the same “height” [sine values]. The student followed her gesture along the two angles (Figure 4a and 4b) but apparently still did not attend to the diagram as she did (Figure 4c). As a result, despite partial coordination with respect to the structures highlighted by their verbal-gestural discourse, the student and tutor perceived the display differently, attending to different diagrammatic features, as eye-tracking data reveals (Figure 5). So, ultimately the student did not foreground the relations that would be productive for solving the problem.

« 41 » Following a few minutes of further struggle the student begins to explicate why, in his view, the equation cannot possibly be solved. Yet, as he gestures at one angle, he briefly gazes toward the opposite angle (Figure 6), thus for the first time exhibiting the tutor’s attentional strategy. This brings him to an insight: “Like the mirrored spot!” Carefully supported by the tutor’s synchronized gaze and verbal confirmations in an episode of joint attention (Figure 7), the student practices this attentional strategy a few times and stabilizes it as a new sensorimotor synergy that engages the visual display in a new way. This new sensorimotor synergy enables him to perceive the diagram as coordinated with the algebraic equation, $\sin a = \sin 2a$, simultaneously attending to both the “2” (a double angle) and the “=” (equivalent sine functions). Immediately, he solves the problem.

« 42 » Soon afterwards, the student and the tutor tackle the problem $\sin a = \sin 3a$.

Their gazes are now tightly coordinated when the student refers to the diagram to explain his reading of the algebraic formula. Finding the solution, $\sin 45^\circ = \sin 135^\circ$, the student laughs,² immediately joined by the tutor. The participants attended to the multiplied angles and the horizontal alignment of the angles' vertical projections, connecting them by horizontal gaze saccades. This way, the sensorimotor processes, stabilized by the student in a previous task, was effectively adapted to the new task and matches the sensorimotor processes of the tutor. However, while congruent in *space*, the interlocutors' sensorimotor processes were not synchronized in *time*: they exhibit the same spatial articulation of eye-movements yet asynchronously. Nevertheless, the similar perceptual orientations enabled the participants to arrive at the mutual understanding that the target position was found, as their shared laughter evidenced. Now intercorporeally established as a consensual semiotic means, the formula orients the participants on the diagram in a similar manner at different time moments.

« 43 » The mathematical symbols have been mutually established as cueing shared perceptual orientations towards the diagrammatic environment. Within the established intercorporeal functional system, the tutor's semiotic activity influenced the student's sensorimotor processes yet could only partially re-orient the student's perception. It is through his active search for efficient and shared perceptual orientation that the student found an efficient sensorimotor synergy that happened to resemble the teacher's synergy. The interlocutors' shared experience of an efficient intercorporeal coordination becomes "frozen" in the formula, which carries forward this new perceptual orientation through time and contexts.

2 | We acknowledge the great role of the affective component in fulfilling multilevel intentionality through sensorimotor and intercorporeal interactions. However, careful elaboration of the affective dimension goes beyond the scope of this article.

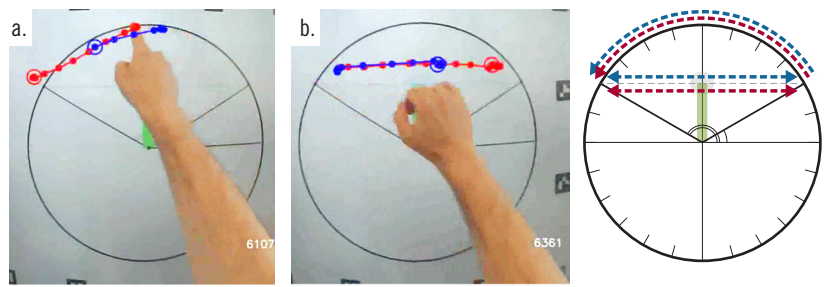


Figure 7 • Spatial-temporal coordination between two bodies, as they come to attend to the visual display in a coordinated manner.

Conclusion

« 44 » In this article we examined semiotic mediation – a higher-level cognitive process, which is often presented in the literature as beyond the reach of a radical embodied explanation in cognitive science. Our theoretical reconsideration of semiotic mediation unfolded over the following tenets:

- We adopt an ecological ontology, in which the presence of a quale is contingent on its relevance for an enactment within an organism–environment system.
- We consider cognitive activity as continuous goal-oriented assembling and exercise of a *functional dynamic system* of sensorimotor processes. The organism's cognitive system develops as the organism strives for a better functional grip on the environment driven by conscious goals and motor intentionality. New skills emerge that solve sensorimotor problems, including sports and mathematical tasks.
- Taken from a functional dynamic perspective, enactment can be understood as a multi-level phenomenon. Emergent stable sensorimotor synergies for organism–environment functional interactions bring forth new contingent forms of perceptual orientation to the environment. Learning achieves a better functional grip, thus coming to discern new environmental structures (affordances) that facilitate pedagogically desired forms of action.

- *Cultural artifacts* – such as spoons, sketches, formulas, words, or digital tools – come to reify effective sensorimotor processes and perceptual orientations. Further, they extend the organism's performance forming *body-artifacts functional systems*.
- Social collaboration is an intrinsically *intercorporeal* process, in which sensorimotor and perception–action processes run across different bodies. Striving for a better functional grip on the environment, collaborators with shared motor intentionality anticipate and influence each other's sensorimotor process. As the collaborators come to stable forms of perception–action practices, they may jointly distinguish new environmental aspects that become sources for semiotic reification.
- *Semiotic means* are a subtype of cultural artifacts that capture efficient intercorporeal coordination and may later be re-used, thus bridging different situations over space and time. *Semiotic activity* is treated as *physical temporal transformation of the cultural–material environment for the other*. As such, semiotic activity is a top-down introduction of constraints that impact the other's sensorimotor processes and possibly facilitate target perceptual orientation, thus transforming the ecological environment permanently.
- Mechanisms of teaching and learning are the same for physical and mathematical skills. Educators, including designers and teachers, offer temporar-



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ily transformed environments that solicit particular sensorimotor processes and thus invite pedagogically desired perceptual orientations. These shared perceptual orientations are semiotically captured and inducted into disciplinary discourse.

« 45 » Conceptually, our monistic reconsideration of semiotic mediation bridges the Vygotskian cultural-historical approach and the coordination dynamics research field, grounded in Bernstein's ideas. Our approach thus avoids the ontological gap between motor skills and mathematical or linguistic capabilities. The functional dynamic systems approach – as it is extended towards body-artifacts functional systems and intercorporeal functional systems – strives for a reconsideration of higher-order processes without appealing to the notion of mental representation, which is incompatible with contemporary ideas of cognition as emerging in and serving organism–environment interaction. Thus, we contribute to radical embodied cognitive science a dynamic-systems model of semiotic mediation. By stressing the functional constitution of dynamic systems, i.e., their multilevel intentionality, we highlight a perspective valuable for educational consider-

ations. We also highlight the importance of considering perception–action loops as emergent synergetic units of sensorimotor processes available for phenomenological inspection. Analysis of cognition's goal-directedness and of perceptual structures as emergent is critical for dialogue with students and teachers, the end-recipients of our theorization. As an implication for educational practice, our approach highlights and explains the limited efficiency of teaching semiotic activity. We stress the importance of creating problem-oriented material environments for sensorimotor interaction that facilitate the development of perceptual orientations and students' active semiosis. We see knowledge as inherently constituted within organism–environment interaction, where semiotic means – theorized as equally material as other cultural artifacts – come to take part.

« 46 » As we look to the potential futures of learning across the disciplines, we submit that onto-epistemological proposals – here, a theoretical consideration of semiotic activity as a subset of possible material transformations of environments that are constitutive for conceptual knowing – may usher in greater tolerance for how intersectionally diverse students come to grasp and

express new ideas (Abrahamson et al. 2019; Benally et al. 2022; Lambert et al. 2022; Tancredi, Wang et al. 2022; Liu & Takeuchi 2023). In particular, designing educational environments as fields of promoted action could bear axiological import by way of better serving students with a variety of grips on the environment. Mathematical knowledge is not in symbols alone (as Harnad's 1990 “symbol grounding problem” suggests). As Piaget maintained,

“[T]he formation of logical and mathematical structures in human thinking cannot be explained by language alone, but has its roots in the general coordination of actions.” (Piaget 1971: 19)

Yet neither is mathematical knowledge in perception–action loops alone. Rather, knowing mathematics emerges through guided coordination of sensorimotor and semiotic activity (Steffe & Kieren 1994), which are inseparable, from our perspective. As interactive digital resources and classroom epistemic climates come to attune to a functional-dynamic-systems view of mathematics teaching/learning, future education may acknowledge students' idiosyncratic sensorimotor processes and, thus, contribute to an equitable inclusive society.

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The authors declare that they have no competing interests.

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This target article is part of a bigger picture that encompasses several open peer commentaries and the response to these commentaries. You can read these accompanying texts on the following pages (or, if you have only the target article, follow the [embedded link](#) that takes you to the journal’s web page from which you can download these texts).

Open Peer Commentaries

on Anna Shvarts and Dor Abrahamson's "Coordination Dynamics of Semiotic Mediation: A Functional Dynamic Systems Perspective on Mathematics Teaching/Learning"

Prolepsis as a Coordinating Mechanism of Semiotic Mediation

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> Abstract • I focus on the concept of prolepsis, the representation or assumption of a future act or development as if presently existing or accomplished. It is suggested that prolepsis reveals the non-linear nature of semiotic mediation.

« 1 » Anna Shvarts and Dor Abrahamson's stimulating target article provided me with a more articulate language for thinking about semiotic mediation and left me pondering the fates of people suffering a brain lesion that leaves them stranded on one shore or the other of the flow of consciousness. In what I hope is a contribution to the authors' inquiry, in this brief commentary, I will focus on the key implication of their proposed perspective: the need to focus on "semiotic activity as a subset of possible material transformations of environments that are constitutive for conceptual knowing" (§46).

« 2 » Bringing together the work of Pyotr Anokhin, Nikolai Bernstein, Lev Vygotsky, and Alexander Luria, the authors propose that the developmental process that produces such transformations should be conceived of as a complex, dynamic, functional system encompassing both teacher and student, the activity of teaching/learn-

ing in its sociocultural context (§14). To this formulation, I want to add that the resulting complex dynamic is nonlinear. To illustrate how this nonlinearity is manifested in everyday as well as academic circumstances, I would like to draw attention to the concept of *prolepsis*, which is discussed at two points in the target article. In §28 Shvarts and Abrahamson describe the way that adults "hand" each other words" in the process of coordinating in joint activity, goal-directed action. The authors note that because perfect coordination of perception-action cycles is impossible, "interlocutors throw their words into the environment, in an act of *prolepsis*, namely anticipation of the other's ability to bootstrap themselves into a new understanding [...]" In §34 they reintroduce prolepsis to describe the learning environments that teachers create as temporary transformations that serve as "a society's forward models," i.e., large functional systems that reproduce themselves within a given educational regime.

« 3 » My concern is that the authors' characterization of prolepses as "anticipation of bootstrapping" or "a society's forward models," although perfectly appropriate in the context of their presentation, obscures the nonlinear, potentially transformative nature of prolepsis. To concretize my point, I turn to an example provided by the pediatrician, Aidan Macfarlane, in the 1970s. Macfarlane (1977) recorded conversations between obstetricians and parents at the moment of their children's birth. He found that the parents almost immediately started to talk about and to the child. A mother is recorded saying "It can't play rugby" of a

female baby, to which the father replies, "I shall be worried to death when she's eighteen."

« 4 » Putting aside our negative response to the sexism in these remarks, in the experience of English men and women living in the 1970s, it could be considered "common knowledge" that girls do not play rugby and that when they enter adolescence, they will be the object of boys' sexual attention, putting them at various kinds of risk. Using this knowledge derived from their cultural past and assuming cultural continuity (that the world will be very much for their daughter as it has been for them), parents project a probable future for the child and "pre-prepare" by enacting the gender-appropriate behavior in the present. For example, whether presented with a male or a female infant to interact with, adults bounce infants wearing blue diapers up and down and attribute "manly" virtues to them, while they treat infants wearing pink diapers gently and attribute beauty and sweet temperaments to them (Rubin, Provenzano & Luria 1974).

« 5 » The nonlinearity underlying prolepsis embodied in such interactions is illustrated in Figure 1. In Vygotskian terms, the horizontal lines represent timescales corresponding to the history of the physical universe, the history of life on earth (phylogeny), the history of human beings on earth (cultural-historical time), the life of the individual (ontogeny), and the history of moment-to-moment lived experience (micro genesis). The vertical ellipse represents the event of a child's birth. The dynamic of prolepsis is traced sequentially:

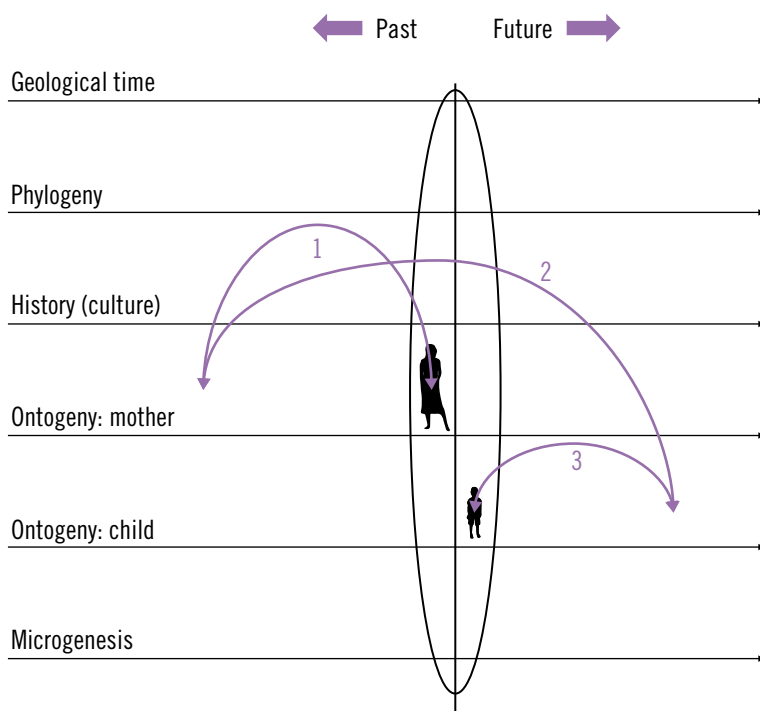


Figure 1 • The horizontal lines represent time scales corresponding to the history of the physical universe, the history of life on earth (phylogeny), the history of human beings on earth (cultural-historical time), the life of the individual (ontogeny), and the history of moment-to-moment lived experience (micro genesis). The vertical ellipse represents the event of a child's birth. The distribution of cognition in time is traced sequentially into (1) the mother's memory of her past, (2) the mother's imagination of the future of the child, and (3) the mother's subsequent behavior. (After Cole 1996).

- Step 1 is the mother's remembering her past,
- Step 2 is the mother's imagination of the child's future,
- Step 3 is the mother's proleptic organization of the child environment to embody its (imagined) future.

« 6 » Two features of the process of prolepsis are highlighted by this temporally extended process. First, there is the nonlinearity of the process. Second is the monism implied by the way in which a verbal exchange "imports" an (imagined/projected) future into material/embodied constraints on the child's experience in the present. This nonlinear process of transformation is what gives rise to the well-known phenomenon

that even adults totally ignorant of the biological gender of a newborn will treat a two-month-old baby quite differently depending upon its symbolic/cultural "gender."

« 7 » Clearly, the situations of the newborn child and a trigonometry student asked to find a value a that satisfies the equation $\sin a = \sin 2a$ differ in many respects. They share, however, incomprehension of the words they are hearing and are subject to a social environment that is actively imposing constraints on their experience that are intended to coordinate them with pre-existing norms and practices. Seen this way, proleptically designed environmental changes merge with the concept of scaffolding as developed in the target article and

earlier papers by the authors (e.g., Shvarts & Bakker 2019).

« 8 » Scaffolds embody patterns of constraints intended to coordinate the acting organism with the desired outcome. For the novice mathematician asked to solve $\sin a = \sin 2a$, the instructors' words may be insufficient to induce the appropriate change in the environment; however, armed with the kind of digital tools used by Shvarts and Abrahamson, the necessary constraints are creatable and the required coordination achievable.

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Bernstein, Vygotsky and the Field of Promoted Action: Some Insights from Motor Learning

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> Abstract • I aim to strengthen the connection between Bernstein and Vygotsky, the key theoretical pillars in the target article on semiotic mediation during mathematics teaching/learning interactions. The concept of the field of promoted action (FPA) as understood through a constraints-led approach is crucial to foster such connection. FPA heavily relies on Bernstein's characterization of (motor) problem solving in the learning process and relates to Vygotsky's notion of the zone of proximal development. The use of FPA provides a more parsimonious account of semiotic mediation. Instead of an active functional system comprising student-teacher intercorporeal interactions on the one hand and a passive functional system comprising body-artifacts interactions on the other, FPA comprises an integrated functional dynamic system. In this integrated system, agency is distributed across human and non-human elements, among which semiotic mediation takes place.

Introduction

« 1 » In their target article, Anna Shvarts and Dor Abrahamson deal with the topic of semiotic mediation (higher-order cognition) in mathematical teaching/learning. Their approach comes from an embodied, ecological dynamics perspective, bringing to the fore the fruitful theoretical collaboration between Nikolai Bernstein and Lev Vygotsky. In my commentary, I consider that such collaboration must be pursued further, connecting Bernstein's characterization of motor learning and Vygotsky's zone of proximal development (ZPD) through the concept of the field of promoted action (FPA).

« 2 » The structure of my commentary is as follows: I first briefly present the development of the concept of FPA in rela-

tion to ZPD. Then I consider the concept of FPA through the constraints-led approach to skill acquisition. Finally, I propose a definition of ZPD in connection to FPA and a specific consideration of agency, and revisit the empirical example of the target article in the light of this novel perspective.

Connecting ZPD and FPA through Bernstein's motor problems

« 3 » FPA is a term originally coined by Edward Reed in relation to Vygotsky's ZPD, which he defined as –

“the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers.” (Vygotsky 1978: 86)

Reed defined FPA as “[t]hose objects, places, events, and their affordances that an individual is encouraged to realize” (Reed 1993: 71). At first sight, there is little connection between these two concepts, but Reed conceived of the essential relation between FPA and ZPD from the very beginning through the work of Jaan Valsiner (1987).

« 4 » Valsiner's theory had added two more zones to Vygotsky's ZPD: the zone of free movement, which defines what is allowed within some defined limits; and the zone of promoted action, which defines what is promoted. Reed (1993) adapted Valsiner's concepts and defined ZPD as the difference between FPA (zone of promoted action by Valsiner) and the field of free action (zone of free action by Valsiner), which he defined as “[t]hose objects, places, events, and their affordances that an individual can realize on his or her own, and that the individual is allowed to use” (ibid: 71). Within ZPD, social scaffolding by adults facilitates affordance (opportunities for action) usage (ibid). With his emphasis on the concept of “affordance,” in this early characterization of FPA and ZPD, Reed seemed to be clearly concerned with the ecological psychology of James Gibson.

« 5 » Reed and Blandine Bril (1996) interpreted the concept of FPA from the perspective of Bernstein's theories on motor learning to compare motor development in different cultures. Even though they did

not provide a formal definition of FPA, they equated it to a subset of the ecological niche of the human species containing only “certain selected opportunities for experience and action” (Reed & Bril 1996: 439) that would pose only certain (promoted) motor problems to be solved during the learning process. When approaching motor learning, Bernstein (1996) avoided the classical assumption of the constant repetition of a technique (sequence of movements) to beat a neural pattern in the brain. He advocated a constant exposure of the learner to the same kind of motor problem, which demands the fulfilment of certain functional requirements through certain actions. Thus, on each occasion, the action (the way to fulfil the functional requirements) is the same, even though the movement pattern can vary each time. Bernstein (1996) captured such a process of motor learning in the elegant expression “repetition without repetition” in which the process of solving the motor problem was acquired by the learner.

« 6 » For Reed and Bril (1996), FPA referred to the kind of motor problems that different children in different cultures must face to fulfil the motor demands (e.g., posture, walking, cutting with a knife, or bladder control) of their respective cultures. In this FPA, children do not act on their own, but adults help via scaffolding.¹ For example, mothers are indispensable in collaborative interaction: Reed and Bril (1996) showed how West African mothers from ethnic groups such as the Bambara help the children's postural development from the very first months of life through a kind of “baby gymnastics,” manipulating the body of the baby in different patterned ways; they also help babies in toilet-training through touch and sound cueing.

« 7 » Reed and Bril's (1996) analysis of FPA based upon Bernstein's ideas is specifically mentioned in §33 of the target article and was applied to mathematics education by one of the target article authors in previous publications (Abrahamson & Trninic 2015; Abrahamson & Sánchez-García 2016). However, Reed and Bril's analysis did not

1 | See Shvarts & Bakker (2019) on the topic of scaffolding in Bernstein, Luria and Vygotsky and its later popularization by David Wood, Jerome Bruner and Gail Ross (1976).

deal with Vygotsky's ZPD anymore. So, in order to resume the Bernstein-Vygotsky dialogue and to improve understanding of semiotic mediation, another conception of FPA is needed. The next section provides precisely this, through the perspective of a constraints-led approach in skill acquisition.

Seeing FPA through a constraints-led approach

« 8 » In their article, Shvarts and Abrahamson subscribe to an “ecological ontology” (§13) and “dynamic systems theory” (§14), and endorse “systemic analyses of physical movement performance [...] within the ecological environment” (§3). In a nutshell, they agree with the essential tenets of the ecological dynamics approach that has been fruitfully applied mainly to the process of skill acquisition (Button et al. 2021: 48). Such an approach considers the nonlinearly coupled human-environment system as the unit of analysis. Such a dynamic relationship between humans and an environment within a goal-oriented activity is always affected by different types of constraints, which act as boundary conditions.

« 9 » The model of constraints currently used by ecological dynamics was originally developed by Karl Newell (1986). Newell differentiated between organismic constraints (body weight, height, shape, development of synaptic connections); task constraints (goals, rules, tools); and environmental constraints (gravity, natural temperature, natural light).

« 10 » The constraints-led approach to skill acquisition considers the teacher as a designer of learning situations (Woods et al. 2020). Her aim is to introduce organismic, task and/or environmental constraints to pose motor problems (Bernstein) for learners to explore, using relevant affordances from the environment (Gibson) to guide and coordinate appropriate actions to meet the functional requirements of the learning situation.

« 11 » Taking into account the preceding section, the teacher is basically a designer of FPAs. Nonetheless, she is not just an external observer but a part of FPA, actively collaborating with the student through “augmented information” (Newell & Valvano 1998) such as instructions or feedback. Such “augmented information” acts as constraints

(Newell & Ranganathan 2010) within FPA, helping to guide learners' exploration of the environment (Button et al. 2021: 181f). Instead of just considering “augmented information” as a constraint, I propose considering any kind of interaction between tutor and student (the scaffolding practices) also as constraints, channelling the process of learning in certain ways.

Revising the connection between ZPD and FPA

« 12 » As Shvarts and Abrahamson (2019: 2) remarked in a previous paper, Vygotsky's ideas on learning and development unfold via two lines of thought: in the first, the teacher educates the student through a careful design of the learning environment; the teacher acts as a gardener, indirectly affecting the growth of the plant (the development of the child); in the second, Vygotsky stressed the intensive collaborative interaction between child and teacher. For Shvarts and Abrahamson (2019), only the second option, referring to the social interaction between child and teacher (the scaffolding practices), enables the emergence of ZPD. In their target article, the same authors seem to equate FPA to the garden in which the student learns and develops. In particular, in §33, they present FPA precisely in this restricted view, connected to the “design of temporarily altered environments.” In this fashion, FPA and ZPD remain unrelated and disconnected. So how does FPA relate to the garden metaphor and ZPD in Vygotsky? **Q1**

Here, I want to put forward the claim that FPA refers to both the designed environment and the interactions between tutor-student. Thus, the whole FPA is crucial for ZPD to emerge. Following Vygotsky's (1978: 86) definition of ZPD (see above), we can now adapt it to include FPA: ZPD refers to the distance between the current developmental level as determined by independent problem solving and the level of potential development as determined through problem solving within FPA.

« 13 » Considering the role of the teacher as a designer of and a participant in FPA to foster learning, let me revisit the empirical finale of the target article (§§35–43). In §35 its authors pose the mathematical problem of finding a value a that satisfies the equation $\sin a = \sin 2a$ and they present the

video analysis of the teaching/learning interaction between tutor and student. To understand the emergence of ZPD in this case, the whole FPA must be taken into account, not just the scaffolding practices of the tutor interacting with the student. The case at hand about the resolution of $\sin a = \sin 2a$ includes a specific visual display of a circumference and lines. Both the specific display and tutor-student interactions (scaffolding practices) constitute a different FPA from another one in which the tutor and student were just talking about and imagining the problem without any material reference to a visual display on paper. Thus, it is necessary to consider the dynamics of FPA as a whole system, rather than as two subsystems (§45) composed of the student-teacher intercorporeal functional system (§§32, 43) on the one hand and body-artifacts functional system (§§21, 44) on the other.

« 14 » To avoid the separation between two different subsystems in which one seems to be active (student-teacher) and the other passive (body-artifacts) I propose a specific consideration of agency. An agent is something or someone that generates a difference (Mol 2010: 255); agency is the capacity of any agent to make a difference. Thus, the constraining actions produced by humans (e.g., a tutor providing “augmented information”) and non-humans (e.g., objects, visual displays) constitute some kind of agency that channels the modes of problem solving in certain ways. To be more precise, such constraining actions affect the student's agency during the learning process, enabling ZPD to emerge. Thus, human (tutor, teacher, peers, student) and non-human (objects, displays, etc.) agencies of different kinds compose a distributed agency (Enfield & Kockelman 2017) across the integrated functional system affecting the process of semiotic mediation. This leads to my concluding question: How do tutorial scaffolding and FPA relate to agency in the process of semiotic mediation? **Q2**

« 15 » In my commentary, I urged further pursuit of the theoretical collaboration between Nikolai Bernstein and Lev Vygotsky in order to understand semiotic mediation. I considered an integrated functional system (instead of two systems, as the authors in the target article propose) and spoke of distributed agency among human and non-human elements in such system.

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Varied Repetition in Embodied Learning of Mathematics

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> Abstract • Discussing the ideas of Bernstein, Shvarts, and Abrahamson, I suggest considering the more prominent role of Bernstein’s “repetition without repetition” principle in embodied learning. A vignette from an empirical study of co-construction activity illustrates the suggestion.

« 1 » When reading the target article by Anna Shvarts and Dor Abrahamson, I recalled a well-known quote from Kurt Lewin (1951: 169): “[T]here is nothing so practical as a good theory.” As a mathematics education design-based researcher conducting studies on learning and teaching in various embodied environments, I am interested in the connection between aspects of mathematical activity related to students’ interaction with the material environment and aspects related to mathematical formalism. Thus, bridging the ontological gap between sensorimotor and higher-order processes, which Shvarts and Abrahamson indicate as an impediment to the theory of embodied learning (§2), may facilitate overcoming a practical design challenge for learning where physical actions inevitably lead students to construct mathematical ideas.

« 2 » In order to bridge this theoretical gap, Shvarts and Abrahamson aim to present a “unified theoretical account that reconciles radical embodied approaches to cognition and cultural–historical ideas on higher-order functions” (§12). This account is rooted in the authors’ interpretation of and bridging between the work of Nikolai Bernstein and Lev Vygotsky – compatriots, colleagues, and collaborators¹ who worked in Soviet Russia in the 1920s (§9). The authors connect Bernstein’s ideas of multilevel self-organization of an organism’s action and Vygotsky’s cultural-historical approach,

1 | At least in one textbook (Artemov et al. 1927) they both are credited as contributors.

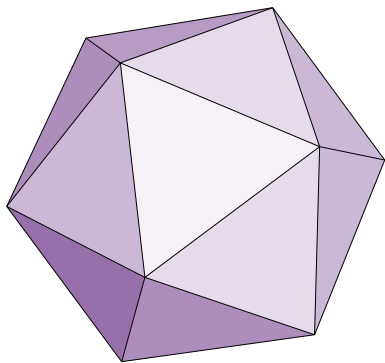


Figure 1 • The icosahedron.

including semiotic mediation and student–teacher collaboration (§15), also mobilizing the ideas from Pyotr Anokhin’s functional dynamic systems theory (§14).


« 3 » To demonstrate ontological continuity between motor and mathematical skills, the authors provided an account of a student solving a trigonometric equation $\sin a = \sin 2a$ while using a specifically designed technological tool (§37). By physically changing the height of one point along the Y axes of the unit circle, the student simultaneously manipulated the magnitudes of two angles and the corresponding magnitude of the sine function, which is equal for both. The problem was solved through the tutor’s actions, restructuring the student’s perception (§41) in concurrence with Shvarts and Abrahamson’s treatment of a semiotic activity as a “*physical transformation of the cultural–material environment for the other*” (§44, emphasis in the original).

« 4 » However, a notion central to Bernstein’s work seems to be missing from the provided account – “repetition without repetition” (Bernstein 1996: 204). Per Bernstein, through repetition, a learner of a new motor skill repeatedly solves a given motor problem. While doing so, she changes and improves the means for the solution and does not just reiterate them:

“Repetitions of a movement or action are necessary in order to solve a motor problem many times (better and better) and to find the best ways of solving it. Repetitive solutions of a problem are also necessary because, in natural conditions,

external conditions never repeat themselves and the course of the movement is never ideally reproduced. Consequently, it is necessary to gain experience relevant to all various modifications of a task, primarily, to all the impressions that underlie the sensory corrections of a movement.” (ibid: 176)

« 5 » Bernstein discerned two phases in learning the motor skill. Repetition plays an important yet different role in both phases. First, “[a] human starts learning a movement because he cannot do it” (Bernstein 1996: 204). Thus, when skill is not obtained, repeated efforts are explorational and are directed to establish some movement pattern leading to a goal. Second, Bernstein suggested that “secrets (of movement) are impossible to teach by demonstration” (ibid: 187), and after establishing a movement pattern, varying conditions leading to various experiences should be purposefully created (Newell 1996). Note a similarity of this idea from learning motor skills to the central tenets of the variation theory of learning (VTS) (Marton & Booth 2013). The VTS conceptualizes learning as a qualitative shift in perceiving the object of learning through experiencing difference (variation) between contrasting values, which leads to the separation of the value from the object of learning, and establishes a dimension of variation.

« 6 » In §14, Shvarts and Abrahamson point out that the ecological environment and actions within have a repetitive quality. This observation aligns with Dragan Trninić’s (2018) work, which examined the teaching methods from the physical disciplines, particularly martial arts, and provided a conceptualization of repetitive practice, also in mathematics education, as an exploratory activity integrating properties of direct instruction and discovery learning. Still, the authors came close to using the “repetition without repetition” principle in their analysis of embodied learning, yet fell short. Thus, I wonder: How does Bernstein’s principle of “repetition without repetition” manifest itself in the empirical example presented in the target article? 

« 7 » Reading the target article made me rethink some of the data I am currently analyzing. Drawing on the authors’ theo-

retical framework, I will extend Shvarts and Abrahamson’s argument to reflect on Bernstein’s “repetition without repetition” principle and illustrate this with an example from a mathematical embodied activity.

« 8 » In the ongoing research project (Benally et al. 2022; Palatnik 2022), students are given a 2D diagram and written instructions to construct an icosahedron – a polyhedron composed of twenty equilateral triangular faces. The instructions read: “Your team has to construct a model of the 3D solid polyhedron on the figure. The polyhedron has the following properties: All the faces are equilateral triangles, and the same number of edges meet at each vertex.” One of the mathematical ideas we want students to grasp is that exactly five edges converge at each vertex of the icosahedron. Being a construction problem, the task at the core of the activity has characteristics of both motor and mathematical problems. Each resource (a construction kit, a diagram, and printed instructions) can be considered a complementary means for the semiotic mediation of mathematical ideas necessary for learning about platonic solids. At some stages of construction, students’ actions resulted in model vertices with four and six edges. Thus, the mathematical idea that exactly five edges meet at each vertex of the platonic icosahedron was only partially mediated by semiotic means provided at the beginning of the task. The visibility of one vertex with five converging edges on the diagram (Figure 1) and the corresponding text was insufficient, at least for some participants.

« 9 » In order to advance the problem solution and repair the model, the participants conducted the following actions. They repeatedly read aloud the instructions, with an emphasis on the second parameter (number of edges); repeatedly inspected a 2D diagram (probably with attention shifted to the number of edges); repeatedly re-voiced the incorrect (four) and correct (five) number of edges converging at the vertex; and repeatedly checked the “fiveness” by counting edges (directly touching or pointing to them) on the emerging model (Figure 2). These repeating efforts can be considered the first – exploratory phase of motor-mathematical learning, where “repetition without repetition” occurs. Thus,

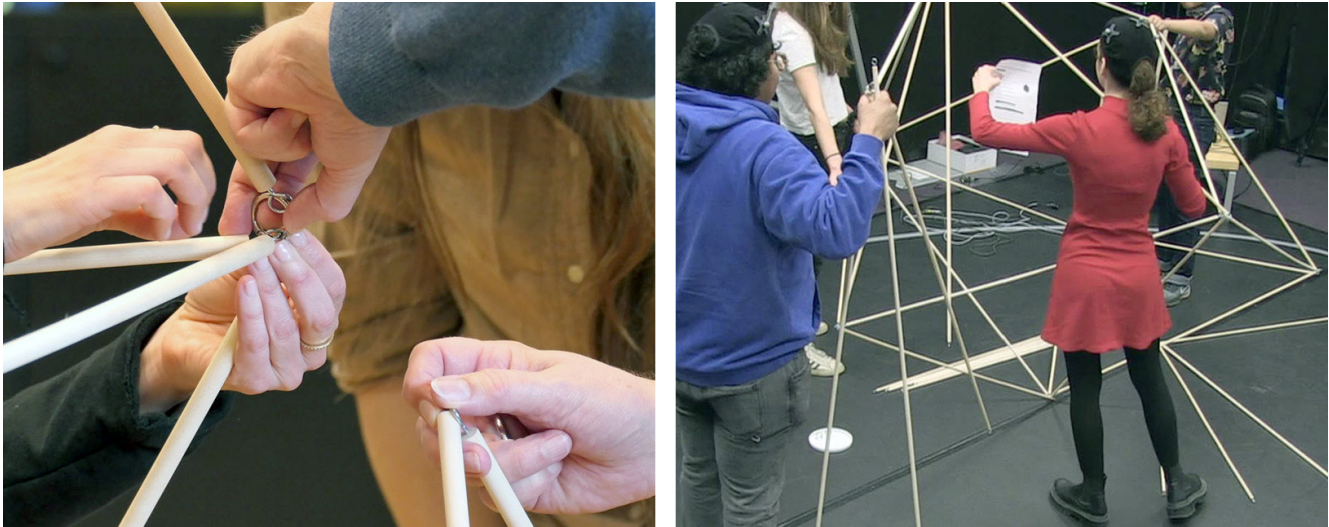


Figure 2 • Elements of the construction kit, and the process. (Left photo was taken by Alexander R. Jensenius).

to construct the model, participants strive for a *better functional grip* on personal and collective levels (§44). They must educate one another's perception (Goldstone et al. 2010) to achieve this improved functionality. Perception is educated through the participants' and model's movement in the environment, resulting in the collection and exchange of multiple visual, kinaesthetic, and tactile perspectives on the emerging model.

« 10 » The emerging model is an artifact with a special part in this complex functional dynamic system. As students construct the model, they construct their knowledge about a particular polyhedron and improve their skills in solving co-construction problems. This is not a mere metaphor. At the more advanced stages of construction, the students worked with, perceived, and named complex sub-structures of the model (“star,” “base,” “pentagon”), which hierarchically included more simple sub-structures – “triangular faces.” (These substructures can be found on the diagram and on the photo of the activity). At the later stages of the co-construction activity, the students utilized these sub-structures to complete the construction successfully and discover several properties of the icosahedron: symmetries, num-

ber of edges, faces, and vertices. In other instantiations, the students successfully constructed the same model on a different scale (Palatnik 2022). These more advanced efforts, in which students operate with more complex structures in a varying environment (for instance, by changing the position of the whole model in space), can be considered the second phase of motor-mathematical learning. In line with Bernstein, at this phase, “repetition without repetition” facilitates the discovery of properties that cannot be taught by demonstration alone or perceived through passive observation.

« 11 » Concluding the target article, Shvarts and Abrahamson suggest several tenets for theoretical reconsideration of semiotic mediation in embodied learning of mathematics (§44). In my commentary, I have argued for an additional consideration – varied repetition. The learning ecosystem should include carefully and purposefully varied opportunities for repeated problem solving involving sensorimotor and mathematical aspects. Learners' repetitions of collaborative attempts to solve these problems may be connected by semiotic means, which, in turn, can be transformed to allow better perception-guided action in the problematic situation.

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Could Education in the 21st Century Embrace Fuzziness, Ambiguity, and So On?

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> Abstract • While the target article articulates key concepts from enactivism and social perspectives very well, the framework the authors offer seems to obscure some epistemological tensions. One such tension concerns the goal of research and how the framework can contribute to education in the 21st century. The second is about the way in which the authors conceptualize the relationship between the theorizations they draw on. The third is in the adoption, at times, of a reifying perspective in contradiction with the “dynamical” approach the authors first adopt. All three can be seen as opportunities to embrace and promote fuzziness and ambiguity for 21st-century (mathematics) education.

« 1 » Anna Shvarts and Dor Abrahamson offer a target article in which they connect Lev Vygotsky's and Nikolai Bernstein's work in a coherent fashion. An enactivist perspective helps us understand how Bernstein's observations and conceptualization of bodily movement dynamics could hinge on (and perhaps support?) Vygotsky's higher cognitive functions through, for example, the use of artifacts. The authors also briefly illustrate the possibilities of this united framework for talking about mathematics education phenomena. More precisely, they draw on eye-tracking technology to examine how a student and a tutor work together on a simple mathematical problem.

« 2 » I deeply appreciate the idea of re-conceptualizing “semiotic mediation” from an enactivist perspective as a direct social extension of bodily dynamics. While the authors often do a very good job of articulating key concepts from enactivism and social perspectives, some epistemological tensions seem to be overlooked. For one, I feel that a broader framework to situate how they conceptualize research and this particular endeavour is necessary. Explicitly

presenting such a framework might also be an opportunity to explain how the authors see their work in regard to the theme of this special issue. And this makes me wonder: How does a “a functional dynamic systems perspective” connect with “Education in the 21st Century”? ① My intuition and understanding are that the authors should not (however tempting this may be) argue that this perspective leads to new (or worse: “better”) instructional or pedagogical directions (models, activities, etc.), and nor should they claim that it can change the face of mathematics education. For me, observations and interpretations such as the ones they offer illustrate many new pathways for research,¹ to develop *new understandings of what is going on* in teaching and learning. Perhaps, and I insist, *perhaps* it can inspire educators in how they work with students.

« 3 » In direct line with these remarks is my second question. On a few occasions, and this is key to their proposal (e.g., §§6, 9), the authors tell us that while works of Vygotsky and Bernstein seem to address different aspects of cognition, they “need to be understood as complementary” and lead to “important insights for a general theory of cognition and for educational design and practice.” The question of educational design surely touches upon Q1, which might also be interpreted as: What exactly does this framework contribute in terms of educational design and practice (e.g., for mathematics education), and how is it significant compared with what decades of research (e.g., in mathematics education) have already shown us? However, my second question is about *complementarity*.

« 4 » I wonder, from an epistemological perspective, what it means here to be complementary? Are the authors suggesting that Vygotsky's and Bernstein's work together “completely” explain or address the phe-

1 | I am thinking, for example, about the concerning trend to draw simplistic educational insights from neuroscience data (e.g., brain-scans of a handful of people performing simple mathematical tasks), with catchy inflated labels like “brain-based education,” “neuro-didactic,” “brain target teaching,” and so on. There is room for interdisciplinarity work between educators and neuroscience research, and the work presented here sounds to me like a promising pathway.

nomena they attend to? Q2 The question, “how can we describe semiotic mediation processes *in consonance* with the science of coordination dynamics” (§3, my emphasis) appears to point in a different direction. Consonance is about compatibility, about (harmoniously) “sounding together” (from the Latin *consonare*). It is not simply about playing the same note in a different octave, it is not saying the same thing in different ways. It can also mean “making a harmonious interval or chord” and, metaphorically, “being in agreement” (which can be quite different from repeating one another). There is also, for example, a claim that a radical embodied perspective *explains* mathematical cognition as an extension of sensorimotor skills (§4), and that the authors’ “theoretical proposal is meant to *resolve* an ostensible ontological discontinuity between materiality and mental cognition, specifically looking to *explain* mathematical cognition as an extension of sensorimotor skills” (ibid.). There are many ways of interpreting the relationship or the effect of two “theories.”

« 5 » While I will not develop this in detail here, I still want to suggest that the (simplistic) notion of “a theory” also needs to be questioned. Theoretical discourses are never unified, and never uniform. Even a single author’s perspective (fortunately) changes in the course of their career (take Wittgenstein, for example). Close to the constructivist discourse, one could look at the split between Humberto Maturana and Francisco Varela. I briefly mention this here because it might shed some light on ways to address my Q2: Were Maturana and Varela always developing the same theory, or compatible theories, or consonant theories? Other adjectives/metaphors could be “like-minded,” “consistent,” “conciliable,” “well-suited,” “correspondent,” “continuous,” and so on. Considering that the authors do not literally speak about two simple or well-defined “theories,” why not say something like “two perspectives,” “projects,” “directions,” two *theorizings* (which inevitably transform from utterance to utterance)?

« 6 » For me, these considerations regarding “theories” are crucial in relation to Q1 and to the question in the title of my commentary, i.e., whether education in the 21st century could embrace fuzziness, ambiguity, and so on. Is this “functional dy-

namic systems perspective” conceived of as (the beginning of) a definite answer to all our questions, and from which “objectively good/true” educational design or practices will be derived? And if not (which is to be hoped), what then is the broader scheme in which it participates? Could it be more like Paul Feyerabend’s¹ approach to science and philosophy, geared toward appreciating “the richness of being” instead of trying to control, exploit, and so on?

« 7 » I have asked these questions about mathematics education research on a few occasions (e.g., Maheux & Gandell 2019), and I am now developing them again in light of Feyerabend’s perspective, arguing in favour of approaches to research in mathematics education that open possibilities and create conversations, rather than attempts to provide answers and thus stop dialogues. So, I am curious about whether/how what the authors offer is open to *creating possibilities* as opposed to providing answers? This last question might directly concern Q2, because the way in which the authors conceived of the relationship between the two theoretical undertakings can go sharply in one direction or the other. Based on my (current!) understanding of enactivism,² there is no doubt as to how an enactivist perspective on research answers this question. However, this might be different from how the authors of the target article see it.

« 8 » I will complete this commentary with one last question. The authors give at-

1| In a nutshell, Feyerabend (e.g., 1999) argues that scientists are not merely rational minds and that science does not develop rationally either. Moreover, the rationality they value is also dangerous, because it dismisses idiosyncrasies and discrepancies, which are ubiquitous not only in everyday life but also in a scientific context.

2| For me, enactivism as a practice/theoretical effort in my domain culminates in Maturana and Varela’s observation that “Every human act takes place in language. Every act in language brings forth a world created with others in the act of coexistence which gives rise to what is human” (Maturana & Varela 1992: 274). What I do/write as a researcher contributes to bringing forth a certain world in which the acceptance of the other and of the conditions of existence is essential. This acceptance depends on curiosity, attention, dialogue, uncertainty, and so on.

tion to “tools,” “artifacts,” “instruments,” including, of course, language. Importantly, however, they seem to forget the notion of *activity*, which is absolutely central to some of the conceptualizations they draw on, such as those of Vygotsky, Aleksei Leontiev, Etienne Wenger or Luis Radford. Artifacts are not merely the embodiments of cultural practices. They are key components of *activities* in which their potential to embody cultural practices exists and can be expressed. An Odhner Arithmometer, state-of-the-art in 1907, now belongs in a museum and no longer functions as a computing instrument. It embodies knowledge and practices relating to preserving old technologies. Without such activity to give it meaning, the Arithmometer would end up in a landfill (if not taken apart).³ Teaching-learning activities are crucial to understanding artifacts and how they contribute to sustaining meaning among individuals in time and space, at all scales.

« 9 » All this is also essential to recognizing why the notion that meaning, practice, orientations, actions, and perceptions “stabilize,” “freeze” or even “crystallize” (e.g., §§27–29, but also §§30–34 and §§41f) is problematic to me. If we forget activity, the lived re-production of what we distinguish as an entity,⁴ we are left with *things* as so-called stabilizations. It comes to us naturally then to conceive that educators organize “learning environments as temporary transformations” while dealing with a “culturally normative ecological niche” (§34). Static states now dominate and norms rule, even though this is not coherent with the core epistemological stance supporting the work summoned by the authors. Of course, it makes it easier to later talk about a “stabilized” attentional strategy or sensorimotor routine (§§41f), but this takes us back to considering learning and knowing in terms

3| Archeologists are sometimes faced with this very problem, when an object, for example, the Phaistos Disc, seems to have served to embody knowledge and practices, but remains *unmeaning*, because the activities in which they were created and used are totally gone.

4| Be it socio-cultural or biological: for Maturana and Varela (1992), they function in the same way.

of acquisition and perhaps even as something standardizable, and so on.

« 10 » So, my last question is: Why do the authors adopt a reifying perspective instead of coherently arguing in favour of a fully “dynamical” approach? Q3 I see, in what they present up to that point, the possibility of embracing the world as flux (where patterns might emerge, but always from an observer’s perspective). Here, we do not have stabilized or statically organized environments (or perceptions, etc.) to transform, but rather moment-to-moment active coordination: un-resting reproduction of what we distinguish as activities at various, inseparable, scales. Such an approach was presented in relation to (mathematical) thinking, for example (Roth & Maheux 2015), or more broadly regarding mathematics education from an enactivist perspective (e.g., Proulx & Simmt 2013). In this thoroughly dynamical paradigm, what we might call “teaching and learning mathematics activities” are inseparable from the acts (every single one) in which these activities are constituted. Activities give special meaning to these acts (and artifacts, and so on), while, in return, the activities keep becoming what we make/call them.⁵

« 11 » For me, this last question also relates to embracing fuzziness and ambiguity, here in the case of language and other tools, attentional strategies and sensorimotor routines, meaning, practices, orientations, actions, perceptions, and so on. If a word *can* mean something, it is because it does not in itself mean that, or to paraphrase Jacques Derrida (e.g., 1998): there is only one way to say this, but that is not what I mean. Potential depends on openness and uncertainty as much as it draws on what is made possible by surrounding phenomena, but we tend to be so strongly shaped by modern thinking that we always push this to the background and see it as minorly important. Feyerabend (1990) attributes this attitude to dominant “scientific thinking” and it is imperative to get rid of idiosyncrasies, discrepancies, singu-

5| I am alluding to, for example, Edmund Husserl’s notion of geometry as being “on the way to” origin (Derrida 1989), or Gilles Deleuze’s (1994) idea of mathematics as becoming itself through circularity and/of repetitions (and differences).

larities, and so on. Openness and uncertainty are what we get when, for instance, individuals are seen as coordinating with/for one another over and over again with/in the socio-material setting created by the ongoing activity (e.g., Maheux & Roth 2011). The activity is something in which they *continuously* take part, of which they are *uninterruptedly* a part, to which they *unceasingly* contribute.

« 12 » Does Shvarts and Abrahamson’s framework add a lot to existing enactivist approaches to the phenomena they are interested in? In terms of depth or breadth, I doubt it. However, like the *n*-th iteration of an intricate fractal, their work adds details and exposes a fine-grained illustration of how the simple aphorism at the heart of enactivist thinking, “All doing is knowing and all knowing is doing” (Maturana & Varela 1992: 26), can be descried and ideated. I thank the authors for such an addition and look forward to seeing how this given iteration will move us on this path we lay down in/for 21st-century education. I sincerely hope it will develop in the spirit of celebrating and nourishing the richness of being in/for mathematics education and research.

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Theoretical and Educational Challenges with Enactivist Approaches to Mathematical Cognition

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> Abstract • Shvarts and Abrahamson bridge enactivism with Vygotsky's socio-cultural theory and Bernstein's coordination dynamics, and ground the proposed ideas with a case study. I question the interpretation of enactivism for mathematics educators, and the applicability of the proposed ideas to explain offline mathematics cognition.

« 1 » In their target article, Anna Shvarts and Dor Abrahamson interpret semiotic mediation of mathematical ideas as intercorporeal and sensorimotor coordination in a social and physical context. While they mainly argue for an enactivist epistemology to explain mathematical learning, they supplement enactivist theorizing with an amalgamation of Lev Vygotsky's (1978) cultural-historical view of artifacts' mediating role in acquiring higher-order skills and Nikolai Bernstein's (1967) movement and coordination dynamics. In what follows, I first interpret how the proposed approach can contribute to a wider understanding of embodied cognition (EC) in education, and, secondly, I question the applicability of the proposed ideas to mathematical cognition and learning in general, especially in the absence of a shared physical and semiotic context between two interlocutors.

Enactivism and education

« 2 » The authors argue that attempts to reformulate educational theories based on embodiment perspectives have not yet satisfactorily theorized cognition beyond human-environment sensorimotor coupling. The monistic approach proposed is intended to explain mathematical cognition as an extension of sensorimotor skills and resolve an ontological discontinuity between materiality and mental cognition. Combined with the authors' previous work (e.g., Abrahamson

2009; Shvarts et al. 2021), the proposed approach and the accompanying case study can be construed as a step toward developing an enactivist theory of mathematical thinking, learning, and teaching.

« 3 » Even though EC has been discussed widely in educational research, its impact on teacher training and practice has been limited. One reason for this is the theoretical gap between common notions about learning as mental activity alone and difficulties with making sense of the premises of the embodied views, particularly the radical and enactivist ones. Situated and bodily interactions with the environment supporting learning are not a foreign concept to teachers. For example, Jean Piaget's theories on the centrality of sensorimotor interactions in development, the use of math manipulatives, and constructivist learning design practices all share the common notion that bodily interactions have some form of relation with learning and cognitive development. It is perhaps due to this familiarity that embodied cognition is usually clumped with other approaches that assume a relationship between bodily activity and learning.

« 4 » EC is a mostly unfamiliar domain for teachers. Educational psychology textbooks and courses usually cover behaviorism, cognitivism, constructivism, and situated cognition as different paradigms of learning. Teachers' theorizing about learning involves a situation-specific and eclectic synthesis of these different paradigms. Even in learning design practices taking advantage of situated action in service of learning, bodily interactions are considered scaffolds for cognitive learning, helping shape "internal mental representations." Enactivism is a difficult approach to make sense of, both because it is orthogonal to common notions of learning as changes in mental structures and because its use to explain higher domains of cognition has been limited. My own experiences in teaching EC – to professional teachers in a master's level learning and cognition course over the last five years – have proved to me the challenges of breaking the rock-solid notions of learning and cognition as being "mental" and "representational."

« 5 » In the target article, the authors make enactivism and radical EC more accessible by representing a scenario where the learner makes sense of the mathemati-

cal content by exploiting the affordances of physical and cultural artifacts in the environment while coordinating the sensorimotor experience with a teacher. The environment is theorized as a system of *nested affordances* that enable both new and familiar forms of enactment. The learner can solve the trigonometric questions only by taking advantage of the artifacts and through corporeal coordination with an interlocutor. One question here is whether enactivism is only relevant when there are physical artifacts with affordances that facilitate the learning of conceptual content. In other terms, would it be possible to present an enactivist analysis of learning with a traditional method of learning trigonometry? Educators often get exposed to ideas of embodiment with learning-design examples that involve innovations with bodily interaction. This leads to the misconception that EC is a specific type of cognition that is only applicable when there is a direct relation between a designed corporeal experience and the semantic content. However, even when a student learns about the algebraic solution to $\sin a = \sin 2a$, principles of the enactivist theory proposed (e.g., affordances, action-perception loops, sensorimotor coordination) are equally applicable. It is even possible that teachers would have an easier time understanding that EC is not just a prescriptive learning-design theory, but an entirely different approach to learning and cognition, if it is presented with learning designs that are already familiar to them. So, to what extent should EC and embodied design (as a learning-design approach) be distinguished? And, can the proposed enactivist approach be applied to analyze learning in more traditional modes of learning that are taking place in classrooms? **Q1**

Mathematical cognition: Beyond enactivism?

« 6 » While theories of embodied cognition have high explanatory power in domains where there is a direct relation between the corporeal experience and the semantic content, there are "representation-hungry" (Clark & Toribio 1994) domains where cognition cannot be explained based on action-perception loops, dynamic assemblies of brain-body-environment systems, and affordances characterizing the organism's interactions with the environment. These

are domains where we, often unwillingly, resort to abstract mental representations, even if we espouse enactivist perspectives. Mathematics, being one of the most abstract domains of cognition, is one of them. This is mainly due to mathematical concepts not directly relating to daily, physical experiences and interactions. Early mathematical learning and development are relatively easy to explain from an enactivist perspective. For example, the development of number sense, and counting and arithmetic skills have been explained based on perceptuomotor abilities that allow object recognition, early tactile and motor experiences with fingers, and later finger-counting experiences (Berteletti & Booth 2016; Soylu, Lester & Newman 2018 for reviews). Theories on how finger-based interactions support development of mathematical abilities have embodied flavors, yet cannot be called fully enactivist. For instance, finger counting is used for addition and finger-counting experiences and habits seem to have an impact on later addition performance, even in adults. Yet, these have been explained as a sensorimotor simulation of finger-related sensorimotor circuitry during addition performance. In a sense, while early sensorimotor interactions are thought to support number development, accounts of adults' number processing performance still employ a cognitivist approach. Our reliance on concepts such as cognitive representations and mental structures equally applies to different domains of cognition (e.g., memory, language, visuospatial abilities), and has led to calls for alternative approaches, more in line with the EC principles (Anderson 2014).⁶

« 7 » What about offline math cognition? When an adult person does mental arithmetic – for example, multiplication of two two-digit numbers – there is no social or physical context and no perception–action loops. How can we interpret this situation from an enactivist standpoint? The mathematical experience during the complex multiplication operation is quite subjective. Some of the transformations can be explained relatively clearly (e.g., “first separate the tens and ones

6| See also my preprint “A new cognitive ontology for numerical cognition,” retrieved on 8 June 2022 from <https://doi.org/10.31219/osf.io/3vx56>

in the first number”), however even these descriptions do not capture the first-person experience during the procedure. In one of the few accounts of first-person mathematical experiences, Jacques Hadamard (1945) asked mathematicians to describe how they experienced mathematical insight. These descriptions show that mathematical insight involves, sometimes quite colorful, mixtures of internalized corporeal experiences (e.g., seeing numbers as faces or in colors). How we can use first-person experiences in studying mathematical cognition is not clear. Can we use perception–action loops or nested affordances when describing offline, internalized mathematical experiences? Q2 In experimental studies, first-person experiences are usually reduced to strategy differences. For example, when answering 49×6 one can follow $50 \times 6 = 300 \rightarrow 300 - 6 = 294$, or $40 \times 6 = 240 \rightarrow 9 \times 6 = 54 \rightarrow 240 + 54 = 294$. Different problem-solving strategies are assumed to have processing differences (e.g., differential working-memory load; Tronsky 2005) and involve different neural resources. So far, the best that can be done with offline cognition is to reduce the first-person accounts to strategy differences and associate different information-processing models with each strategy (even if this means sur-rendering to cognitivism).

« 8 » Reducing first-person experiences to strategy differences leads to talking about them in a disembodied manner. The multiplication strategies in the example do not constitute experiences, but different computational steps. We do not have mature research methodologies that guide us in capturing offline first-person experiences in a way that we can use in experimental studies, even though there have been some attempts at that. *Neurophenomenology* (see Stuart, Pierce & Beaton 2013 for a special issue on this topic) is one such attempt, where the reported first-person experiences guide the analysis of experimental data. Yet, even though they are promising, such methods are not yet widely used.

« 9 » According to the enactivist approach to mathematical cognition that the authors espouse, mathematical abilities, like other domains of cognition, rely on perception–action loops, sensorimotor coupling with the environment, and affordances formed as a result of corporeal experienc-

es. However, we are at an impasse when it comes to explaining higher mathematical skills. *Higher* here refers to any mathematical skill that we cannot directly relate to interactions within a physical and social context. Further, even when we can explain the acquisition of a new math concept from an enactivist standpoint, as Shvarts and Abrahamson did with trigonometry in their case study, it is not clear how we can explain the use of previous learning in a new context and at a future time point. That is, what happens when the student in the case study is given the same trigonometry problem a year after the initial learning experience, but this time without the same physical artifacts? Is there a process where the situated nature of learning in the physical and social context is transformed to create abstract and generalized representations and skills? Or, does the learner rely on internalized perception–action loops and affordances? Internal simulation of previously acquired skills with physical artifacts is well exemplified with the mental-abacus phenomenon, where experts report imagining the abacus during calculation (Stigler 1984) and with neural evidence (Hanakawa et al. 2003) collaborating the simulation account. Should the simulation view for offline math cognition be supported and to what extent is the simulation account compatible with enactivism? Q3

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Concepts, Material Anchors and Interactivity: A Dialectic Perspective

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> Abstract • My present reflections will center on a point the authors present as an afterthought, but that seems pivotal: mathematical knowledge is not comprised of perception–action loops alone. Instead, “guided coordination of sensorimotor and semiotic activity” is held to be essential. Shvarts and Abrahamson do not elaborate on how this happens. My aim is to sketch what an account giving equal weight to semiotic and embodied facets might look like, and to clarify why paying attention to the details of their interplay is crucial for evaluating ontological claims such as the monist position defended by the authors. I will presently address four questions: (a) why failing to tackle the semiotic pole explicitly is a risky methodological choice, (b) what literature we can draw on to address the embodied–semiotic relationship, (c) what empirical criteria ontological claims might hinge on, and (d) why a dialectic (and non-dualist) approach offers a credible alternative to monism.

« 1 » With a study of mathematical learning, a cultural paragon of higher cognition, Anna Shvarts and Dor Abrahamson illustrate a new analytic apparatus based on a “hard case.” This explicit focus is to be applauded as it is essential to a well-grounded embodiment-oriented rebuttal of old cognitivist assumptions. A careful micro-analysis is precisely the way to address this, since zooming in on processuality can highlight factors cognitivists scarcely bother with. Readers can follow how complex effects emerge from smaller local, embodied activities and see how process-embedded perceptual and manipulation skills contribute to overall learning. In addition, a focus on embodied coordination (with cognitive artifacts and between teacher and pupil) supports new ways of thinking about pedagogy; learning is seen as an interactive process that

runs through recursive iterations of micro-alignment, rather than being the result of a teacher just “imparting content.” The analysis bears witness to a deep interwovenness and causal continuity of sensorimotor and thought processes. Showing how this unfolds over time is the target article’s central achievement.

Risks of skipping the “thought” layer

« 2 » Shvarts and Abrahamson’s (§46) concessionary afterthought concerning the semiotic pole implies that mathematical reasoning processes *can* accompany the observable embodied process, and contribute to its causality. We may speculate on the presence of such semiotic processes that are left implicit in the data: There is, for instance, a point of the process where the student is guided to a perceptual pattern, but cannot stabilize it yet or exploit it for a conceptual breakthrough. The reason could well be a missing conceptual process that later crystallizes and allows perceptual noticing.

« 3 » To evaluate what might be involved here, we may differentiate two types of semiotic/conceptual processes, those which reside “behind” the embodied, public process, and instances of micro-cognition that unfold “in the cracks” of the process. Reasoning processes in the cracks may escape notice easily, because they are extremely short-lived, partial solutions, or (perhaps tentative) ideas for directed further exploration. In contrast, reasoning processes “behind” the embodied process can be more systematic, of longer duration, and greater explicitness in terms of (partial) mental models that get refined or revised. Both types of semiotic processes seem likely candidates in the case of mathematics, and each needs to be accounted for.

« 4 » Leaving the semiotic processes implicit may also backfire on the theory. Putative hidden semiotic elements may lessen the import of the claim that insight hinges on the “spatial–temporal coordination between two bodies” (Figure 7 in the target article). Playing devil’s advocate, such coordination could be interpreted as a consequence of conceptual alignment processes that precede embodied alignments, rather than the other way around. After all, would two people sharing similar cognitive mod-

els not also embrace similar sensorimotor exploratory or handling procedures to enact this model? Shvarts and Abrahamson's case-study does not fully bear out prior conceptual alignment, but it may well be that partial conceptual processes "take the lead" at points or operate as a background layer. It thus seems important to be explicit about how the semiotic and the embodied coalesce over time and what the causal contributions of each are.

Tracking semiotic-embodied integration

« 5 » How the semiotic and embodied poles work together has been addressed in micro-genetic studies that discuss similar mechanisms to those addressed in the target article. I would like to briefly point to some relevant methods and concepts they use.

« 6 » *Interactivity research* (Kirsh 2014; Steffensen, Vallée-Tourangeau & Vallée-Tourangeau 2016; Ross & Vallée-Tourangeau 2021) highlights the important role of sensorimotor engagement especially for insight problems. Studies on insight problems have shown that the "aha" moment is often preceded by adventitious-seeming material manipulations, for example in a Scrabble task, where shifting around the letter "tiles" facilitates the recognition of possible words. The breakthrough is reached through material fiddling and preceded by an act of perceptual noticing. Some of these studies seem to imply that the interactive process tells the whole story, but others point to complementarity of conceptual and embodied facets: A micro-phenomenological study entitled "What affords being creative? Opportunities for novelty in light of perception, bodily activity, and imaginative skill" (under review), which I conducted together with Camilla Groth, indicates that material serendipity can directly trigger matching imaginative processes in a pottery context. Many material "happy accidents" engage wider imaginings of further steps to be undertaken so that an aesthetically and functionally coherent object arises in the end.

« 7 » *Distributed cognition research* addresses the semiotic-embodied relationship very explicitly. Edwin Hutchins (1995) reconstructs how tools and artifacts, embodied skills, and group communication procedures work hand in hand with conceptual cogni-

tion: the ability of a navy aircraft carrier, for example, to take a bearing relative to the shore hinges on the precise interaction of these system elements. His analysis presents detailed *activity scores* using multiple cameras that allow plausible speculations on how external processes augment the thinking of individuals. The analysis makes explicit how conceptual protocols and navigational concepts coalesce with highly skilled sensorimotor procedures. Hutchins's (2010) micro-genetic study of a navigation-room troubleshooting process with a creative resolution shows another possible way of doing video-based analysis. Although this partly hides the conceptual work "behind" the observables, he explicitly speculates on what he calls (cognitive) *enactment of phenomenal objects* attached to each step in the observable process.

« 8 » Another key notion of distributed cognition would be *material anchors*, which facilitate cognitive activity. Hutchins (2005) illustrates the interplay of perceptual and conceptual processes with the simple example of what makes a "queue" different from a mere line-up. The cognitive difference is a directional trajectory that is imagined in addition to what is perceived. He goes on to discuss more complex material anchors used in clocks, dials, slide rules, finger calendars, and navigation aids, which all blend material anchors and conceptual activities. Hutchins extends the conceptual-integration framework from cognitive linguistics to explain how *blends* between perceptual and imaginative aspects arise; the analysis appeals to quite specific mechanisms such as co-projection and imagistic superimposition.

« 9 » Such phenomenological explicitness about the integration mechanisms is highly commendable in micro-genetic analyses. Applied to mathematical learning, this could mean following the trail of how imagery is gradually enriched in itself or better adapted to the perceptual ecology.

Empiricizing the ontology debate

« 10 » Shvarts and Abrahamson defend a strong form of ontological monism, rather than just stressing functional continuity, when they say: "knowing mathematics emerges from guided coordination of sensorimotor and semiotic activity [...],

which are inseparable, from our perspective" (§46). Claiming analytical inseparability would be somewhat self-defeating a strategy for micro-genetic research. Causal inseparability, however, is a more serious position to consider. This would apply when the interaction of parts creates a *constitutive emergent structure* rather than a merely additive effect. Sub-processes should be causally non-decomposable in a relevant sense. This being an empirical issue, I would have hoped for more explicit criteria.

« 11 » Criteria could come from complexity-informed metrics that decide whether a particular process is "interaction-dominant" (e.g., Van Orden, Holden & Turvey 2003). Qualitative research equally offers criteria. In Kimmel & van Alphen (2022), we use micro-genetic case comparisons to evaluate the contribution of interaction to creative cognition. This reveals that tango dancers sometimes rely on strong emergence as a source of joint creativity, but at other times are creative more in their encapsulated sub-spheres, or assist a partner in that "solo" function (i.e., no strong emergence, yet scaffolding). Thus, "private" processes contribute to improvisational creativity to varying degrees. In these cases, the action system's dispositional architecture remains the same (i.e., paths along which communication passes between individuals), but the operations that interconnect levels differ (i.e., emergence and downward causation).

« 12 » Equally in need of clarification is whether semiotic/conceptual processes just co-evolve with and reflect the sensorimotor scaffolds, or whether augmentative resources external to the sensorimotor situation are used – which might undercut the authors' "inseparability" claim. For example, as another study of dance improvisers (Kimmel & Hristova 2021) shows, conceptual inspirations frequently emerge from the immediate coupling situation. At other moments, however, external inspirations, notably from movement memories and everyday encounters, seep into the dance. While situated constraints affect how this happens, these resources augment the coupling whilst in progress.

« 13 » A general criterion for deciding how semiotic and embodied cognition are related lies in the putative differences between so-called "online" and "offline" cogni-

tion. Leon de Bruin and Lena Kästner (2011) propose that dynamic embodied cognition enlists “offline” resources as a function of situated coping. Individuals may move into memories or knowledge-based reasoning, but then re-enter the situated task using the resources generated there. Although the process is “perceptually unclamped” from the situation for a moment (Davis et al. 2015), “offline” processes still support “online” processes. This interesting view underscores functional continuity, but rejects monism. It seems that the relevant distinctness criterion here is the *provenance of the cognitive resource* used. I.e., it is a resource from the situation itself or from outside the situation, such as case memories or general knowledge? Trans-situational resources enlisted for a situated task are fundamental to embodied cognition. In mathematics this could be to draw an analogy to a strategy that has worked in a different problem context, or reason from some axiomatic principle towards the specific situation.

The benefits of a dialectic analysis

« 14 » More generally, two broad types of non-Cartesian strategies seem to be available in thinking about cognition. One is the monist position, which stresses essential ontological continuity between higher and lower cognition and *assimilates* semiotic activity to embodied-enactive principles. Julian Kiverstein and Erik Rietveld (2018), for instance, stress that “online” and “offline” cognition constitute fuzzy boundaries and suggest that notions such as imagining are embodied and fully situated “all the way up.” The alternative non-dualist interpretation is dialectic. It sees embodied and semiotic aspects as a process of ongoing (more or less effective) integration. This view allows for distinctness in kind, while investigating complementarity and processual work-sharing. It shifts the analytic focus to *mechanism meshing*, the procedural interplay of how embodied and semiotic processes trigger, scaffold, constrain, inform, and amplify one another. A dialectic view thus stresses functional coalescence, rather than ontological integration.

« 15 » In Kimmel & Irran (2021), we exemplify how a dialectic analysis can be implemented, by trying to unpick the complementary contributions of embodied and conceptual processes, but also investigating

their relative weight in bodywork therapists. Basically, when bodywork therapists try to find a good diagnosis and strategy, they draw on direct affordance perception and coupling-based resources of copresence and intercorporeality between therapist and client, i.e., situation-immanent resources. Yet they may frequently also decide to move into reasoning spaces that draw on general heuristics, mental computation tools, or inferences from general anatomy knowledge, resources that transcend the situation and require distinct skills. How therapists do this confirms two points de Bruin and Kästner had in mind:

- these resources are meant to inform the embodied coupling, and
- a recursive back-and-forth movement emerges in which embodied actions and conceptual reasoning augment and enrich each other.

For example, reasoning can direct attention to a diagnostically relevant body zone to explore in detail, but in return the embodied “here and now” must be consulted to confirm or specify a reasoning-based inference. Finally, Kimmel and Irran discovered that whether trans-situational resources are deployed follows a deliberate meta-cognitive decision. For simple client problems, following the surface affordances can be sufficient and add up to a good effect. However, there are other cases in which affordances, taken at face value, will not reveal the root causes of a problem. A more systemic evaluation is needed. In such cases an over-reliance on situated resources would lead to ineffective interventions or later problem relapse. These complex strategies increase functional leverage by dialectically interweaving the power of close embodied couplings/interactivity with inferences and concept use.

« 16 » One evident benefit of a dialectic view is that it facilitates ecumenical dialogue with conceptualist analyses in other branches of cognitive science. However, it has theoretical strengths to commend it as well: It has broad scope, never reifies process, and recognizes the contingent nature of full mind-body alignment (Giovine 2022), its nature as ongoing and effortful achievement, whereas a monistic interpretation tends to suggest that mind and body are inherently in sufficient correspondence. In other words, can Shvarts and Abraham-

son exclusively explain the pupil’s struggle as combined dis-attunement of mind and body to the task, or is there an element of dis-attunement between mind and body as well, and, if so, would this not sit uneasily with a monist perspective? Q1

Conclusion

« 17 » If mathematics learning indeed involves an interplay of sensorimotor and semiotic functions, the overall question in need of addressing is what relative explanatory weights should be accorded to each. Is sensorimotor activity an auxiliary scaffold for conceptual problem solving or do motor processes themselves produce the full solution? Is observable coordination a product of conceptual alignment, is it the other way around, or is there a more complex dialectical relationship at work between the two? And if the process evolves dialectically, which of the poles takes the lead and when? Q2

« 18 » Answering these tricky questions can benefit greatly from micro-genetic analyses. In view of the many possibly causal constellations in the semiotic-embodied interplay, it behooves us well to study cases on a broad basis before passing a final verdict on ontology. Asking whether thought processes occur “in between” or “behind” the interpersonally/materially mediated processes and whether they reflect trans-situational resources seems vital here.

« 19 » A premature monist diagnosis might encourage a neglect of how semiotic and embodied aspects coalesce or block the path towards a dialectic analysis, whose distinct strengths I hope to have illustrated. Either way, future research should endeavor not to under-problematize the many complex constellations in the semiotic-embodied interplay or overemphasize a prior integration at the expense of partial mind-body dis-alignment.

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Bridging the Ontological Gap?

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> Abstract • While deeply sympathising with Shvarts and Abrahamson’s thoughtful attempt at theoretically explaining semiotic mediation from a radical embodied perspective, we suggest that their metaphor of an “ontological gap” to be bridged deserves pondering. We also discuss their illustrative “empirical finale” from our enactivist viewpoint.

« 1 » In their landmark target article, Anna Shvarts and Dor Abrahamson address the fundamental question of how a radical embodied perspective might theoretically explain semiotic mediation, exemplifying this with a trigonometric equation. To this end, they theorise any human skill, and specifically sensorimotor or mathematical

skills, as the functional dynamic organisation of perception and action, including semiotic activity.

« 2 » We deeply sympathise with Shvarts and Abrahamson’s most commendable intent to avoid or to bridge the “ontological gap” between motor skills and mathematical/linguistic capabilities. We nevertheless have some misgivings regarding their “ontological gap” metaphor in §2. Indeed, from our radically enactivist perspective, we posit that this gap is rather a *hallucination* of ours, in the sense of Seth (2021). We experience it (better, *construct* it) as interacting observers, but it does not exist as a “thing-in-itself.” In our view, it is as “real” as the “ontological gap” between the “photon as wave” and the “photon as particle” in quantum physics, or – more modestly – between “the two sides of a coin.” To us, the “ontological gap” metaphor entails an unacknowledged *reification* (Sfard 2008; Soto-Andrade 2020) of the aforementioned processes, which could misleadingly suggest an essential difference between them, reified as things. At this level of theorising, we may have no choice but to metaphorise. In our view, though, a more adequate metaphor than the “ontological gap” metaphor is still wanted for the issue the authors intend to address.

« 3 » If we were to adhere, though, to the authors’ “ontological gap” metaphor, we might submit that a bridging of this ontological gap could be afforded by *metaphorising*, understood as a fundamental biological process of our species (Valdés-Zorrilla, Letelier & Soto-Andrade 2023). Importantly, we do not see metaphorising as merely substituting a name for another, or a mental image for another, but as a “hallucinatory” biological mechanism (in the sense pointed out above). This entails a dynamical system view of metaphorising as it is considered a process emerging from the sensorimotor dynamics and previous structural couplings of sentient beings with their Umwelt.

An enactive interpretation of Shvarts and Abrahamson’s “empirical finale”

« 4 » We comment now on Shvarts and Abrahamson’s intriguing “empirical finale” (§§35–43), where they “consider the mathematical problem of finding a value *a* that satisfies the equation $\sin a = \sin 2a$.” In our view,

however, the narrative does not begin with the *statement of the problem*, “already in the world, lying ‘out there’ somewhere, independent of us as knowers, waiting to be solved” (Proulx & Maheux 2017: §4). In this case, the given mathematical problem, which was soaring in the abstract realm, made landfall on a single student, helped by a tutor.

« 5 » Let us add two remarks. First, we notice that the Vygotskian setup of “a tutor scaffolded a student” in §35, while very interesting to investigate, is closer to a laboratory setup than to a classroom as it does not include the dynamics of a classroom. How could other didactic setups look that are more suited to classroom experiences, and which would include the dynamics of a classroom with 30 students, say, organised in small random groups engaged in a collective improvisation, akin to a musical improvisation, aiming at solving this unfriendly equation, under discrete monitoring by a teacher and an assistant? Q1

« 6 » Second, some “upstream” context is missing: We learn that “a tutor scaffolded a student in solving this equation using a unit circle,” but everything about the student (besides “his” gender: §41) is ignored, in particular, his age and background, the kind of teaching (or drilling...) he had previously been exposed to. However, it is the students’ previous history of interaction and structural couplings (Maturana & Varela 1980; Varela, Thompson & Rosch 1991) with mathematics that also matters. This makes us wonder how the cognitive dynamics described in §§39–42 may depend on hidden or uncontrolled variables such as the student’s educational level, his personal background, and his relationship to mathematics. How generic is the observed dynamics for students of this (unknown) cohort?

« 7 » Of particular interest is how the student first approached the given trigonometric equation, which is quite awkward-looking indeed, unrelated to more familiar linear or quadratic equations. Being confronted with it must elicit a certain “gut reaction” to the task: Does a disgust reaction emerge, or is it, rather, a feeling of uneasiness and stress, or even anxiety? It seems plausible that, in school, the student had automatised procedures for solving linear or quadratic equations, but this obnoxious equation needed to be tackled in a different

way. This makes us wonder what the role of the affective engagement of the student is in this sort of experience. How does the student’s engagement impinge on the interaction with the tutor? Q2 This complex affective-cognitive process would beg for a fine-grained micro-phenomenological exploration (Petitmengin, Remillieux & Valenzuela-Moguillansky 2019; Diaz-Rojas, Soto-Andrade & Videla 2021), which could zoom in on the lived experience and hitherto unnoticed aspects or moments of the process.

« 8 » How much does the technological tool (the visual display mentioned in §§37ff) help in solving the problem? The student could have also explored possible solutions with the help of free-hand drawings. The “pedagogical problem” of the visual display’s being “ambiguous” (§38) could arise due to the context in which the student receives or is exposed to the visual display showing everything at once. Like an answer to a question not asked, there are more details in the visual display than the student had time to ask himself about. In our view, this causes the student to be not active enough compared to his being engaged in intentional, deliberate exploration.

« 9 » The scaffolded approach presented in the target article seems somewhat top-down: it starts (§37) by exhibiting pairs of (supplementary) angles (between 0° and 180°), which turn out to have the same sinus (height). The student is then expected to figure out when the bigger angle in the pair would double the other. We suggest that a bottom-up approach could make more sense to the student, where he would try to figure out all by himself what happens to the height of the angles a and $2a$, when a is initially very small and increases slowly. This would amount to what we could call an enactive exploration. Hybrid options would be possible as well: using a suitable visual display, the student could define an angle by sliding a point on the unit circle with his *right hand* and at the same time sliding, with his *left hand*, a second point on the unit circle giving the angle $2a$. This sort of perceptually guided action could constitute a nice sensorimotor challenge for the student, familiar to the authors (e.g., Abrahamson & Sanchez-Garcia 2016), resulting in an Aha! experience for the student.

« 10 » In our view, the key metaphor underlying this enactive approach to the problem is: “sine is the height of a rotating point in a vertical unit circle.” This suggests a dynamic exploration, which starts with a point defining a tiny angle, which clearly does not afford a solution: $\sin 2a > \sin a$. Then, the angle is slowly increased. When the rotating point approaches the zenith, $\sin 2a < \sin a$, and it becomes intuitively obvious for the student that somewhere in the middle, between 0° and 90° , lies a solution. In this way, the student informally “proves” the existence (and unicity, for a between 0° and 360°) of the solution before being able to find it explicitly.

« 11 » After finding the solution by such a sensorimotor bottom-up exploration, the student may “see” that he got the same heights because of symmetry: the double angle $2a$ is just the *supplementary* angle of a . So, $2a = 180^\circ - a$, i.e., $3a = 180^\circ$, so $a = 60^\circ$. The student gets the solution for the double-angle equation by evenly slicing half of the unit circle into three pieces. Generalisation to the k -fold angle equation $\sin ka = \sin a$ is then straightforward: You just need to evenly slice half of the unit circle into $k + 1$ pieces, since you must have $ka = 180 - a$.

Precursors and turtles

« 12 » Let us conclude our commentary with two remarks. First, we fully share the authors’ conclusion in §45 about “the limited efficiency of teaching semiotic activity [and] the importance of creating problem-oriented material environments for sensorimotor interaction that facilitate the development of perceptual orientations and students’ active semiosis.” We would like to remark, though, that their conclusion, which is highly pertinent to education in the 21st century, closely converges with much older insights of some precursors of enactivism, who spoke different languages. They include:

- Johan Heinrich Pestalozzi’s notion of *Anschauung* (an active perception or intuition), which should constitute the first stage of the learning process (Herbart 1804). Pestalozzi emphasised the triad heart-mind-hand, integrating sensorimotor and, in contrast to Shvarts and Abrahamson, affective aspects in his approach to the teaching of mathematical concepts.

- Guy Brousseau's experimental epistemology of mathematics, whose dictum is: "We need to make mathematical notions exist before defining them, or trying to present their essence," inspired by Jean-Paul Sartre's (1943) "l'existence prime sur l'essence [existence precedes essence]."⁷ Brousseau presented a first draft of his ideas in the 1960s (Brousseau 1965), and further developed them in Brousseau (2002), emphasising an embodied approach through "lessons without words" but, contrary to Shvarts and Abrahamson, in a context of collective group work, taking advantage of the unfolding group dynamics in the classroom.

«13» Our second remark relates to §31, where the authors suggest that solving abstract mathematical problems consists of sensorimotorically coordinating mathematical symbols with properties and structures of the environment, e.g., in perceiving graphs of trigonometric functions. We agree, but we remark that this sensorimotor coordination is a multi-level process, which could be fathomed – metaphorically – as a *cascade* or *constellation* of metaphors. Indeed, a graph of a function relies first on metaphorising quantities as locations on a line, then on metaphorising input – output pairs as points on the plane and so on. In some sense, in these examples, as in many others, we have "metaphors (instead of turtles) all the way down," and also "all the way up"...⁸

7| Guy Brousseau, personal communication, 12 March 2023.

8| We refer here to "turtles all the way down" as a well-known metaphor for infinite regress, which appears in an anecdote told by Stephen Hawking (1988:3): "A well-known scientist (some say it was Bertrand Russell) once gave a public lecture on astronomy. He described how the earth orbits around the sun and how the sun, in turn, orbits around the centre of a vast collection of stars called our galaxy. At the end of the lecture, a little old lady at the back of the room got up and said: 'What you have told us is rubbish. The world is really a flat plate supported on the back of a giant tortoise.' The scientist gave a superior smile before replying, "What is the tortoise standing on?" 'You're very clever, young man, very clever,' said the old lady. 'But it's turtles all the way down!'"

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Authors' Response From 20th-Century Dialectics and Deconstruction to 21st-Century Transformative Monism

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> Abstract • In our response, we provide theoretical clarifications on how the functional dynamic systems approach allows a monist theorization of offline cognition and imagination; highlight the importance, for ontogenetic cultural development, of the system's sensorimotor dynamics as well as this dynamics' reification; elaborate on the notions of "field of promoted action," "zone of proximal development," "scaffolding," and "repetition without repetition" as well as purported relations among these notions. We stress the transformative potential both of culture's proleptic anticipation and, conversely, individuals' non-linear rebellion against it. We also address questions regarding educational applicability, design, and classroom practices. Finally, we unpack the ethical and transformative role of our theoretical work in 21st-century educational science.

« 1 » We are thrilled and honored by the community's animated open peer commentaries on our target article. The seven commentators' insightful comments, suggestions, questions, and critiques are invaluable for our research program, as we continuously strive to clearly explicate our ever-emerging theoretical propositions. For the most, we view the commentaries as reflecting our field's *zeitgeist* – a teeming confluence of epistemological commitments and competing interpretations inherent to current embodied, enactive, and related approaches to the theorization of human cognition in cultural contexts. At times, we were

asked to tune or hone our thesis given nuanced aspects of the very intellectual sources we already build upon. At other times, we faced a "Rosetta Stone" challenge of aligning keywords across apparently affiliated perspectives. And yet at other times, we wondered whether the current confluence that our field is witnessing is necessarily convergent or perhaps inevitably divergent.

« 2 » Among the comments, we distinguished three major themes: (a) theoretical inquiries and clarifications; (b) questions about educational applicability, design, and classroom practices; and (c) questions concerning the methodology and theoretical positioning of our research. Below, we begin with the first theme, which drew our commentators' greatest attention.

Theoretical inquiries

« 3 » Whereas some commentaries welcome our monist view, other commentaries question it. We explicate, below, how our monist position tackles such areas as offline cognition and imagination, traditionally difficult for embodied approaches in cognitive and educational science.

Monism: A constantly imagining body-brain system

« 4 » [Firat Soylu 02](#) and [Michael Kimmel \(§§14f\)](#) question the possibility of a monist approach to explaining offline mathematical cognition and thoughts more generally. Their challenge is important for theory, research, and practice, given that offline cognitive activity is not overtly observable as sensorimotor activity, at least not to the uninstrumented eye. [Kimmel](#) suggests that a dialectical relationship between sensorimotor activity and semiotic/conceptual/thinking activity is a preferred theoretical conceptualization. [Soylu](#) hypothesizes the existence of diverse cognitive strategies that, while operating in the solution of mathematical problems, are in principle unobservable.

« 5 » From our perspective of functional dynamic systems (and, more generally, enactivist accounts, see, e.g., [Thompson & Varela 2001](#)), sensorimotor activity is not necessarily observable. We define *sensorimotor processes* as "activations of receptors and muscles – whether they are in the eyes, ears, fingers, or tongue – and brain structures" that are "organized in multiple

synergetic levels that are partially independent" (§17). Brain activation participates in regulating any enactment, whether or not it is noticed by an external observer. From this perspective, in the course of learning, skills that are no longer overtly manifest to an observer have not been *internalized* (gone from outside to inside) but have *contracted* (shrunk; lost their observable constituents) ([Zaporozhets, Markova & Radina 1968](#); [Radford 2021](#)). This view is in consonance with the later [Lev Vygotsky's](#) ideas that reconsider interiorization ([Zavershneva & van der Veer 2018](#)), remarking that the "inner" realm is present from the very beginning of skill development. This reconsideration has attracted attention in the current discourse of mathematics-education research ([Roth & Jornet 2017](#); [Radford 2021](#)), because it re-frames the role of observable actions as that of initially establishing mathematical thinking; later, these actions fade from overt to covert, yet they still preserve the functionality of the dynamic system and use the same neuronal resources (cf. [neural reuse](#), [Anderson 2010](#)).

« 6 » This brings us to the commentators' concerns. [Kimmel](#) (§6) suggests that imagination is a necessarily dialectical counterpart to perception, and [Soylu 03](#) questions the compatibility of our enactivist account with the idea of simulation. As we point out in our article, the core of a functional dynamic system lies in "exploiting the recurrence quality of the ecological environment and actions: based on previous partially repetitive experiences, the system anticipates how the environment will impact the organism at the completion of action" (§14). This anticipation is organized in many levels, some of which we may notice introspectively as our intentions and imagination, some of which we do not, as they run in our bodies as motor intentionality or in society in the form of prolepsis. Perception itself is imagining a surrounding environment, as captured, for example, by notions of *retention* and *protention* in the Husserlian phenomenological tradition ([Nemirovsky, Kelton & Rhodehamel 2013](#); [Zagorianakos 2015](#)), possibly fulfilled by cascades of predicative and feedback processes between the neuronal levels ([Bastos et al. 2012](#)). In a sense, the entire functional dynamic system – which constantly strives for a better

functional grip – can be seen as constantly imagining the future (Merleau-Ponty 2002; Pelaprat & Cole 2011). This constant anticipation is accomplished by means of “a forward model, namely anticipatory neural circuits pre-activated *ahead* of observable enactment” (§14). More broadly, the relation between this pre-activation model and the various simulation models is important in marking theoretical divides. Let us focus on what exactly is being simulated. Lawrence Barsalou (1999, 2003) develops a neo-empiricist model, where the brain acts as an accumulation of diverse multimodal sensations, which can later be simulated and may form conceptual structures. We consider his account to be representationalist, where the brain tries to model the world. Instead, the notion of a forward model offers another type of simulation: it is not the world itself that is the source of simulation; instead, the forward model pre-activates those sensations that the organism had previously experienced while acting in functionally similar situations. Such anticipatory simulation cannot be said to “represent the world.” Instead, it contributes to the effective accomplishment of action.

« 7 » Just as anticipation is an indispensable part of any (online or offline, observable or unobservable) activity, so sensorimotor processes – activations of a system that activates *some parts* of brain structures, sensory organs, and muscles – are also always present. Kimmel Q1 wonders whether a pupil might struggle not only due to dis-attunement between their body and the environment but also between “mind and body.” We assume that Kimmel is referring to a situation where, although acting in similar-to-cultural ways, the learner might still struggle to express their action verbally in a culturally appropriate way. Such a situation would resemble, for example, the well-known phenomenon of gesture-speech mismatch (Church & Goldin-Meadow 1986). However, from our perspective, this dis-attunement means a discoordination not between mind and body, but between different levels of a functional dynamic system. Such a system consists of many partially independent levels (§7) that might struggle to function in attunement to one another. For example, a learner may be successful in pedaling a bicycle yet struggle to maintain balance.

« 8 » Overall, the construct of a functional dynamic system offers a monist view that does not limit all cognition to observable behavior organized as a unified and always-coordinated system. Instead, we talk about a functional system as a coordinated activation of multiple levels that jointly fulfill target behavior through constant anticipation of the environment and iterative attunement to it. We interpret higher cognition in line with Vygotsky’s notion of *higher psychological functions*, which are systemic dynamic entities. Higher psychological functions are initially constituted in a common environment through interaction with other people, mediated by semiotic means or cultural artifacts (Vygotsky 1978). So-called “offline” cognition can be seen as a possibility of the functional system to re-create those artifacts without *physically* embedding them in the environment. Yet, we might nevertheless detect rudimentary motor expressions that the system uses when running such imaginary interaction: movements of the fingers along an imaginary abacus (Kirsh, Caballero & Cuykendall 2012), micro-gestures when planning a dance (Kirsh 2010), gaze on an empty space for auxiliary constructions when solving geometry problems (Epelboim & Suppers 2001), and so on. Sensorimotor activity (defined as functional activation of the central and/or peripheral neuronal system, organs, and muscles) partakes in any problem-solving, including conceptual problems. Notwithstanding, Kimmel’s Q2 might be reformulated as, “What is the role of peripheral activations in solving conceptual problems?” This question can be answered from the perspective of ontogenesis: any brain-*without*-environment activity is a consequence of previous activity *in* the environment. This way, strategies for solving multiplication tasks – mentioned by Soylu – come from manipulations of numbers, at first on paper (by writing and/or looking) and later only through re-creating images that could have been perceived.

« 9 » A large agenda remains for empirical research in showing how and in which situations explicit motor components of actions contract, and how the brain re-activates itself using imaginary artifacts. The core paradigmatic shift in the cognitive sciences from representation-based models towards enactment-based models lies in shifting our theoretical conceptualization

from simulations as re-activating past experiences, to simulations as pre-activating anticipated sensations of future enactment.

Concepts as reifications of functional dynamics


« 10 » Now we turn to the commentaries that welcome our monist approach but question our fundamental theoretical choices. Daniela Diaz-Rojas & Jorge Soto-Andrade (§§3, 13) suggest that metaphorizing is an ultimate “hallucinatory” biological mechanism underlying mathematical conceptualization. From our perspective, “hallucinating” would capture a general mechanism of how perception is constituted: continuous anticipation of how the environment would be felt when we interact with it. This view is a natural continuation of Nikolai Bernstein’s (1967) view on movement as resolving the discrepancy between desired value (*Sollwert*) and measured value (*Istwert*), in which he relied on then-contemporary developments in control-system conceptualizations, in coordination with the ideas of Jakob von Uexküll and his notion of *Umwelt* (Sirotkina 2018).

« 11 » Based on this constant anticipation, the particularity of *conceptual* reasoning lies in the ability to use artifacts as perceptual “lenses” (Bryson 1988). For example, having read the proposition “ $\sin a = \sin b$,” one begins searching for angles with aligned vertical projections, as if looking at the world through an “equal sine values” lens and appraising everything for this property, thus *imagining it into the world*. We are not sure whether we would need the concept of metaphorizing to explain conceptualization, as we consider any perception to be constituted precisely as an ability to see more than is available to mere sensation.

« 12 » While metaphorizing might appear to be a redundant construct for our theorization, reification is a necessary component of any enactivist explanation. As Jean-François Maheux Q3 notices, we abandon a fully “dynamical” approach and introduce the idea of reification. Reification is a mechanism that enables structural transformations that are necessary for accumulating experience: such local structural transformations constantly happen in plastic interconnections between the neurons (Maturana & Varela 1992). From our perspective,


cultural artifacts are similar structural transformations but on a macro, social, level of the system, available to other individuals in the environment. They facilitate the practices that have appeared to be useful for a social group. Appropriating artifacts into body-artifacts functional systems renders learners into structurally different creatures capable of other actions and perceptions than individuals who never encountered similar education.

An elaboration of themes in the functional dynamics system approach

« 13 » Addressing our citation of Bernstein's neuro-physiological research on motor skill, Alik Palatnik  wonders how Bernstein's principle of "repetition without repetition" manifests in our empirical example. In that regard, Palatnik (§5) mentions Ference Marton's pedagogical methodology for systematically parametrizing students' practice items as a means of optimizing for the effect of variation on generalization. We concur with Palatnik on both the theoretical and practical cogency of Bernstein's insistence that motor problems are solved at the perceptual level of the system rather than piecemeal. Considering our task, we identify repetition-without-repetition both intra-item and inter-item: (a) In the specific task that serves to build our empirical case, *intra-item* variation is inherent to students themselves manipulating the angle (using the green "stem"; see our Figure 1), where each momentary geometrical configuration constitutes an opportunity for exercising the sensorimotor work of simultaneously monitoring two figural relations co-constitutive of the target trigonometric notion (doubling the angle and equalizing the sine values, which they monitor repeatedly, each time at a slightly different location and in a slightly different manner, as the eye-tracking data reveal); and (b) In the dyad's transition from the $\sin a = \sin 2a$ item to the subsequent $\sin a = \sin 3a$ item, the intervention achieves *inter-item* variation, generalizing toward $\sin a = \sin Ka$. In that respect, we perceive functional commonalities across the geometry activities discussed in our article and by Palatnik (§§8–10): in both tasks, students are to physically reconfigure available media resources to achieve a des-

ignated end-state (equivalence of two sine values and completion of the icosahedron structure, respectively), and in both tasks this challenge requires identifying independent constraints that emerge through manipulating the objects and then coordinating task-effective actions within these constraints.


« 14 » Theoretically, a critical question for Marton's variation theory, given its parallels – per Palatnik – with Bernstein's principle of repetition-without-repetition, lies in questioning *who* or *what* determines the limits of variability. Certain pedagogical issues would need to be addressed in order to productively align these two perspectives on how variability engenders generalization. Bernstein's maxim characterizes repetition with respect to the motor problem, where variability is sequestered within the limits of this problem. In a pedagogical situation, variability is taken to a new level by varying the problem types. So doing, however, we stand the risk of varying the environment along new parameters that extrapolate beyond the student's skill level, thus fostering an *empirical* generalization based on similarities, whose essential meaning is apparent to the teacher but obscure to the student (Davydov 1990).

« 15 » In an insightful historical elaboration, Raúl Sánchez-García (§§3f) locates the intellectual roots of Edward Reed's notion of a *field of promoted action* (FPA) – which we have been drawing on in our previous and current publications – in both Bernstein and Vygotsky. In that sense, Sánchez-García submits, the FPA should be conceived as an intrinsically and *a priori* systemic ontology that interleaves artifacts and humans in emergent and mutually elaborative dynamic interaction. From this stance, Sánchez-García queries our theoretical extraction of human agency from the FPA, as if the FPA is the inert stuff that is animated only once it is recruited into teaching/learning activity. Sánchez-García  further questions the relation between FPA and the zone of proximal development (ZPD) – he suggests that animating the FPA leads to activating the ZPD.

« 16 » Developing Vygotsky's teacher-as-gardener metaphor, we propose to analogize the learning environments we design to the earth she prepares and the fertilizers

that she adds in anticipation of energies that will bring forth flowers from seeds. These technological devices that we pack in a box and carry to our research sites are static artifacts that Anna Stetsenko (2002: 129) calls "crystallized templates of action and schematized representations of ways of doing things as discovered in the history of human civilization." It is here that we assimilate the historical notion of FPA into our practice of design-based research, and in the course of this assimilation we slightly accommodate the concept. As such, as design-based researchers, we note both the theoretical coherence and practical utility of discerning between the ecological environments that designers deliberately build, where interaction *might* happen, and their *in situ* animated embodiment stimulating the guided re-enactment of cultural practices.

« 17 » Inert templates of action are entities of *designers'* ecologies – designers initially only imagine how students are to interact with these templates of action. Those environments – FPA – become actuated only once our study participants arrive and engage in our experimental tasks, thus appropriating those FPA into *their own* ecologies. Whether the ZPD will eventually transpire cannot be predicted solely on the basis of a pre-given template (born in the designers' conversations). Rather, it is through the *ad hoc* collaborative efforts of learners and teachers that an FPA may support the emergence of the ZPD. We are grateful to Sánchez-García for the invitation to maintain our conceptualizations consistently ecological, which provoked further thinking on theory and methodology.

« 18 » To be sure, along with Sánchez-García  and other researchers espousing neo-materialist and extended-cognition approaches such as Lambros Malafouris (2010), and Elizabeth de Freitas and Nathalie Sinclair (2013), we embrace the idea that agency is distributed between humans and artifacts. Contrary to Sánchez-García's reading of our article (§14), we do not consider the body-artifacts functional system as a passive entity. *A fortiori*, any functional system is an intentionality-driven ensemble relaying complex agency. An intercorporeal student-teacher system is intertwined with artifacts, forming a complex system, where

agential centers shift throughout the process of learning. For example, in the course of scaffolding, agency in the student–teacher system gradually shifts towards the learner (van de Pol, Volman & Beishuizen 2010). As a learner appropriates a proposed artifact into her body–artifacts functional system, her agential boundary shifts, now including the artifact into the functional system (Shvarts et al. 2021). Narrowing it down to the situation of semiotic mediation, we might suggest that as a learner comes to use the new semiotic means (an artifact) suggested by a teacher, in the course of scaffolding, the agency of the teacher-artifact system is gradually transferred to the learner through this semiotic artifact. This way, a child might now command her own behavior in a culturally determined manner by intra-scaffolding.

« 19 » We note, in passing, that **Sánchez-García**'s contribution could serve as a response to **Maheux**'s (Q2) concerning the alleged *complementarity* of Bernstein and Vygotsky's approaches to human learning: by highlighting both the Vygotskian and Bernsteinian roots of the FPA and, in turn, interpreting the FPA as the designed learning environment, we intend to ground this would-be theoretical complementarity in the historical antecedents of the learning sciences. Overall, the focus of Bernstein's research was on motor skills and their development, while Vygotsky investigated higher psychological functions and their genesis; yet, they both implemented the principles of a complex and dynamic organization, thus we see their theoretical contributions as complementary and supporting us in sketching a general vision of all human skills as ontologically continuous.

« 20 » The affective dimension, which **Díaz-Rojas & Soto-Andrade** touch upon and question in their (Q2), shall play a critical part in our future theoretical project. According to Vygotsky (Zavershneva & van der Veer 2018), affective and intellectual aspects of cognition are inseparable: affect drives thinking, and perception transforms the environment in an affective manner. For Bernstein, too, what drives motor behavior is the discrepancy between received sensations and *desired* future. So, the affective dimension is the source and driving force of the functional system, as captured in our

theoretical elaborations by the concept of multilevel intentionality. However, we did not aim to address the affective dimension in the target article. Opening up this issue would require serious elaboration that goes far beyond the scope of this response.

« 21 » Finally, **Michael Cole**'s commentary pertains to the notion of prolepsis. Although prolepsis was mentioned only twice in our article, **Cole** submits, it is critical for understanding the approach. Indeed, it is only through interaction *between the levels* with different temporality that this anticipation can be transformative to the final outcome. **Cole** discusses the documented phenomenon of parents anticipating their newborn's gender behavior based on the child's sexual anatomy and their own historical experiences, thus shaping the child's future ontogenesis. Analogously, a verbally uttered anticipation of finding two angles with equal sine values shapes the future microgenesis of eye-movements in search of such angles. In both cases (ontogenesis, microgenesis), the recipient's consequent behavior appears pre-determined, anticipated, and transformed by sanctioned trajectories that had been preestablished in antecedent epochs. Notwithstanding, we submit, the non-linearity of dynamic systems permits the emergence of new, unexpected behavioral patterns that veer away from proleptic constructions, whether transformative or conservative. The role of 21st-century educators is to create environments where learners are free to travel their proleptically cast futures or rebel against them. As such, we align with what Stetsenko (2017: 4) calls the "transformative activist stance" for revising Vygotskian scholarship.

Teaching and design issues

« 22 » The empirical extract provided in the target article fell short of being a proper empirical contribution and, as our commentators noticed, lacks personal details about the participants of our case, careful description of the designs, details of the micro-genetic analysis of interaction, dialogues, or any other aspects that would help the reader dive into the participants' subjective experiences. We fully share the readers' willingness to know all these details and acknowledge the weakness caused

by the intended focus on theoretical and historical issues.

« 23 » **Soylu** (Q1) asks if a similar analysis and theoretical interpretation would be possible for mathematical learning with traditional teaching materials that do not provide an opportunity for direct motor interaction. We do not think that the technological possibility of moving points in the environment led to drastically different interactional patterns from those we would see in any other forms of one-to-one teaching of trigonometry. As a large body of literature shows, teachers' tracing and rhythmical gestures, voice intonations, and other sensorimotor resources are rendered to help students in revealing mathematical structures (e.g., Alibali & Nathan 2012). Yet, students' active involvement is critical to come to see the structures that are being shown: eye-tracking analysis reveals that gestures need to be anticipated, not just followed, to constitute shared understanding (Shvarts 2018a, 2018b), thus contributing to intercorporeal coordination.

« 24 » The question of how such coordinated ways of acting and perceiving can be achieved in large classrooms (**Díaz-Rojas & Soto-Andrade** (Q1)) is of great importance. We hypothesize that learning difficulties that many students experience in traditional classrooms are determined precisely by the incapacity of traditional classroom settings to involve the majority of students in mathematical forms of action and perception. As a result, students repeat observable components of action without uncovering their functionality and meaning. Our theoretical analysis suggests that fields of promoted actions, namely specially designed environments that support the development of (mathematical) forms of action and perception, could be a better option. However, as **Sánchez-García** points out in §12, promoting actions requires not only providing an interaction field and task prompts, but also collaborative activity within the zone of proximal development. Research on how classroom interaction can be organized so that each student gets an opportunity to constitute their mathematical forms of action and perception is only at its very beginning (Alberto, van Helden & Bakker 2022; Macrine & Fugate 2022). Yet, it is clear that teachers' behavior would need to undergo a

major transformation for them to attend to and appreciate the multiplicity of students' idiosyncratic strategies and support them in arriving at cultural–semiotic forms that express their experiences (Abrahamson et al. 2014). At the same time, new sensorimotor synergies that emerge with the support of technological environments can apparently be reproduced by students' bodily systems at a later moment and thus become available for further discussions that allow for establishing cultural forms of perceiving and describing in collaboration with their teacher and peers (Shvarts & van Helden 2021).

Between theory and axiology: An apology for the transformative activist stance in educational research

« 25 » Maheux's ^{Q1} questions the relation between the functional dynamic systems perspective, put forth in our article, and the notion of education in the 21st century, the theme of this special issue. We are heartened by Stetsenko's reading of Vygotsky (Stetsenko 2017). Mid-20th-century upper middle-class Americans, she writes, who by and large enjoyed life-style benefits of political stability, were amenable to a reactionary reading of Vygotsky, as though his ethos were to articulate how utopian regimes groom their young to perpetuate political nirvana. On the contrary, contends Stetsenko, Vygotsky's tumultuous early 20th-century Soviet Umwelt clanged with bells of transformation. Ultimately Vygotsky's intellectual quest was to pave an ideology of newly emerging social order; thus questioning order and authority was his duty (see also Guba & Lincoln 1994). As design-based researchers operating in academic institutions financed by our tax-paying neighbors, we take seriously our social responsibility to envision and initiate change. As Alan Kay reputedly said, "The best way to predict the future is to invent it."¹ We do not sit idly to observe "what will be, will be." Instead, we are mandated to ideate the future and bring about our ide-

1 | This idea can be attributed to others, even before Kay, see <https://quoteinvestigator.com/2012/09/27/invent-the-future>, consulted on 18 April 2023.

ation. It took half a century for Seymour Papert's vision of children programming computers – once ridiculed as ludicrous – to become mainstream household practice and school-fare. Embodied-design environments – those fields of promoted actions resulting from our theorizing – may likewise evolve into routine appliances. Or they may not. The point is to be ethical, proactive educational warriors. Change is possible – even 20th-century schools can change – and it is up to us to bring about this change. And changing education begins by thinking differently about what it means to learn and by re-thinking how we bring about change, not in the imposing manner of 20th-century positivist projects but as an invitation, as an open door.

« 26 » The functional dynamic systems perspective put forth in our article is our proposed conceptual foundation for operationalizing the embodied paradigm change in the cognitive and educational sciences. We do not aim at providing "a definite answer to all our questions [...] from which 'objectively good/true' educational design or practices will be derived," as Maheux (§6) wonders. Instead, we offer a perspective that may support creating, facilitating, and investigating new genres of educational design. This way, we fully agree with Diaz-Rojas & Soto-Andrade (§2) in their remark that the expression "bridging the ontological gap" might be misleading, as any theory we create necessarily has an onto-epistemological status: it is created as a conceptualization and yet, once put into action, it transforms the environments of future generations. As such, our perspective can be viewed as offering educational designers a forward model at the scale of our entire research field (cf. socio-technical imaginaries, Jasanoff & Sang-Hyun 2015) for educational designers. The question of what our theory has to do with future education is not so much an intellectual riddle as a transformative quest. By building a theory, we seek to anticipate the future – "anticipate" in the sense not of passively waiting to witness what unfolds, but in the sense of actively casting proleptic forward models of what, we believe, *should* unfold. Yet, the ultimate question is how to do it in a humble way, avoiding utopian futures of social engineering and welcoming other projects and visions to co-evolve.

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