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Publication Date

1964-09-08

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Berkeley, California

Contract No. W-7405-eng-48

VARIANTS OF CANONICAL FORMALISM

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8 September 1964

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ABSTRACT

Canonical formalism is presented in invariant notation as one of a family of solutions of the problem of finding infinite-dimensional subgroups of the group of all time-independent flows of points in a real n -dimensional space. When the problem is restricted to a local study with the flows restricted to such as are specified by a flow vector field with analytic components, and a geometry in the first-order neighborhood of each point specified by a real irreducible Lie group of n by n matrices is required to be preserved by the motions, then a limited number of classes of solutions appear (theorems of E. A. Ruh and Y. Matsushima). Ordinary canonical formalism is produced from one of these classes by an extra assumption of flatness. It is shown that, if the flatness assumption is replaced by one of maximum twist, no motions at all are possible.

* This work was supported by the U. S. Atomic Energy Commission.

1. Introduction

This work was prompted by two unrelated considerations.

The obvious motivation was to get some understanding of ordinary canonical formalism. I will argue that in a theory of motion wherein physics is pictured as a continuous flow of state points, two general requirements tend to lead to canonical formalism. One is a division of the theory into an incomplete kinematical phase and a final dynamical phase. The other is a requirement of sufficient flexibility at the kinematical stage for subsequent construction of the dynamics. Even though extra technical postulates of unknown strength will be assumed, material alternatives to canonical formalism will appear. The question of whether these cousins of canonical formalism may serve as outlines for new "classical limits" or new quantum theories will not be discussed.

The obscure motivation was to learn to do modern physics without either space variables or equivalent momentum variables, or any equivalent context in which Lorentz invariance is put in.¹ About this, I think the reader will agree that nothing has been learned. The abstract context is too sparse, yet it may also be too rich: An awkwardness of any continuous theory of motion is its clash with the spirit of the modern treatment of scattering theory on a purely asymptotic basis, which, aside from naive Lorentz invariance, makes a strong claim to have already done away with the naive conception of a space-time point of interaction.

So many of the notions introduced coincide with established concepts of differential geometry that it is doubtful whether any of the results are mathematically new, although I have not yet found a reference to the absolute rigidity of a most generally twisted symplectic structure (Section 10). The representation of the content of canonical formalism

free from the convention of canonical coordinates has been recently reviewed by R. Jost.²

2. The Problem of Canonical Formalism

Every student of mechanics is puzzled by the mathematical sense of canonical formalism. Sensible or not, it soon serves as an unsteady guide in learning quantum mechanics.

The mystery of canonical formalism is conventionally avoided by postulating Hamilton's action principle, $\delta \int L dt = 0$, with L a function of coordinates and velocities. This viewpoint and its generalizations will not be pursued here, for the following reasons: First, it is easier to think about differential laws than about integral principles.³ Second, extension of the action principle to quantum theories is a debatable matter.⁴ Feynman's⁵ simple picture relating the action integral to quantum mechanics violates the variational principle: all the possible classical paths contribute to the motion, the stationary-action path being distinguished only in that the paths near the stationary-action path contribute coherently. This is a fine explanation for Hamilton's principle in classical mechanics, but it suggests that there may ultimately be no place for it in quantum mechanics. Finally, a viewpoint other than the standard one may lead to new possibilities.

First-order differential equations will be assumed at the beginning:

$$\begin{aligned} \delta x^i &= \xi^i(x) \delta t, \\ x &= (x^1, \dots, x^n). \end{aligned} \quad (1)$$

The condition of the system is an element x of a manifold of n real dimensions, and is defined so exhaustively as to include complete initial-

value data, so as to reduce the differential equation system to first order. The motion field $\xi^i(x)$ is a real contravariant vector field taken to be independent of the real parameter t . The $\xi^i(x^1, \dots, x^n)$ will be assumed to be analytic in all the real variables for technical reasons; later, cases in which the variables may be paired into half as many complex variables in which half as many complex ξ^i 's are analytic will appear; this later "n/2-analyticity" should not be confused with the present assumption.

Notice that I am apparently being very naive about the concept of time, but that a reconsideration of time in which the "t" here is relegated to the position of a convenient parameter and "time" corresponds to some regularly increasing coordinate (or coordinates) within the system, is not precluded; see Section 12C.

In ordinary classical mechanics, x is a point of even-dimensional phase space, and

$$\xi^i(x) = \gamma^{ij} \frac{\partial H}{\partial x^j}, \text{ and } \gamma^{ij} + \gamma^{ji} = 0, \quad (2a)$$

where

$$\gamma^{ij} = \psi^{ij} = \text{diag}(\epsilon, \dots, \epsilon), \quad (2b)$$

$$\epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

the indicated matrix ϵ being repeated along the diagonal, and where H is the Hamiltonian.

(1) is at once the natural expression of Newton's method of fluxions, and is also a generalization of Hamilton's canonical equations, (2). The problem of canonical formalism is to understand why the form (1) ought to be restricted according to (2).

To point up the problem, let us ask, why not simply

$$\xi^1(x) = \frac{\partial H}{\partial x^1} ? \quad (3)$$

In order to answer this simple question, I find it necessary to adopt a group-theoretical outlook.

3. Group-Theoretical Formulation

It is conventional to present the content of a mechanical model in two stages, labelled "kinematics" and "dynamics". In the final stage, there is only one correct motion. However, at the kinematical stage, there is a group of possible motions. By writing (1) and (2) separately, I am breaking kinematics into at least two stages, and by questioning (2) I am proposing to cast away or to alter the second stage, but I will require that at each such stage of further limitation, a definite group of motions be defined.

The notion of defining the true motion in a series of stages, kinematical and dynamical, may be thought frivolous. It is however impossible to define a local kinematical feature of a single motion; the meaning of canonical formalism, in particular, is inherently bound up with the notion of an intermediate stage of dynamical ignorance. This is demonstrated by reduction of an arbitrary motion to the standard form $\xi^1 = \delta_n^1$, "solved Hamilton-Jacobi form", in Appendix 1.

Perhaps the point may be made less negatively by exploiting the possibility that the goal of one correct motion is too much of an oversimplification; that instead, a motion is only a partial view of the universe assuming significance only in its relationship to other kinematically admissible motions; that this way of thinking, more strongly evident in

quantum-mechanical measurement theory, yet leaves its mark on the classical limit in the form of kinematical stages.

The characterization of kinematical stages as groups is with one proviso: Since all arguments concern differential motions, all assertions are about infinitesimal elements of a group; i. e., about a Lie algebra, and often one that violates the usual definition in being of infinite dimension. I will therefore require only that the infinitesimal motions defined by various possible $\xi^i(x)$ in (1) satisfy the axioms for a Lie algebra, other than the one of finite dimension, and ignore the difficult question of whether the infinitesimal structure can really be integrated to a group. The difficulty is illustrated by the following one-dimensional example: $\xi(x) = -(x)^2$ integrates to $1/x(t) = t + (1/x(0))$, whence $x(0) = -1/a < 0$ gets sent to ∞ at $t = a$, and there is no positive t at which some finite point is not sent to ∞ .⁷

There is a reason aside from avoiding integration problems for arguing about the algebra and not the group: Although (1) is a generalization of Schrödinger's equation as well as of Hamilton's equations, I have classical mechanics in mind, which is in toto an approximation; even if we consider the Schrödinger equation, its linearity again suggests a smallness approximation. Any difficulties appearing in integration, especially before inserting the detailed dynamical model, might then constitute serious consideration of high orders of iteration of an inherently approximate scheme.

I offer the following remarks in justification of the assumption that the possible motions form a group (algebra):

1. We may require that kinematics delimit the configurations

achievable from any point set in state space, and that these fall into equivalence classes; namely, that any two configurations achievable from one be achievable from each other.

2. The operation of Poisson bracket satisfies the axioms for a Lie-algebra product. The corresponding group is the group of canonical transformations of phase space and functions on phase space. Quantum mechanics also has a kinematical group, that of unitary linear transformations of (pure) state space and linear operators defined on state space. Quantization is a reasoned transference of the structure constants from one group to the other. It is therefore in conformity with convention that a group is attached to a physical theory of motion.

The problem of canonical formalism can be restated as follows: Instead of obtaining the relevance of the Poisson bracket formalism to the motion from Hamilton's canonical equations, and those from the action principle, and noting that the equations correspond to a peculiar group, can the advantages of that peculiar group among other groups be directly recognized, or perhaps refuted?

4. Alternant

The alternant of the infinitesimal transformations $x^i \rightarrow x^i + \xi^i(x) \delta t$ and $x^i \rightarrow x^i + \eta^i(x) \delta t$, induced by the contravariant vector fields ξ^i and η^i , in the sense that first⁸ the ξ transformation is made with $\delta t = \epsilon_1$, then the η transformation with $\delta t = \epsilon_2$, then the ξ transformation with $\delta t = -\epsilon_1$, and finally the η transformation with $\delta t = -\epsilon_2$, is given to the second order of infinitesimals by

$$x^i \rightarrow x^i + [\xi, \eta]^i \epsilon_1 \epsilon_2,$$

with

$$[\xi, \eta]^i = \xi^j \frac{\partial \eta^i}{\partial x^j} - \eta^j \frac{\partial \xi^i}{\partial x^j}. \quad (4)$$

To obtain this result, care must be taken to refine $x \rightarrow x + \xi(x) \delta t$

to second-order accuracy: $x'^i = x^i + \int_0^\epsilon dt \xi^i(x(t)) =$

$$x^i + \xi^i(x) \epsilon + \frac{1}{2} \epsilon^2 \xi^j(x) \frac{\partial \xi^i(x)}{\partial x^j} + O(\epsilon^3).$$

If the specification of a subgroup is to be accomplished by imposing some condition $R(\xi)$ on the motion field $\xi^i(x)$ of (1), then inasmuch as the commutator of two group elements must belong to the group, R must be such that

$$R(\xi) \text{ and } R(\eta) \text{ imply } R([\xi, \eta]); \quad (5)$$

also, $R(\xi)$ and $R(\eta)$ must of course imply $R(a\xi + b\eta)$ for real constant a, b , this by a first-order estimate of the product of $x^i \rightarrow x^i + \xi^i(x) a$ with $x^i \rightarrow x^i + \eta^i(x) b$. Since we have determined not to follow out the finite consequences of infinitesimal postulates, the type of relationship R may be considered to be already defined by these postulates, which in mathematical terminology already assert that the fields satisfying R constitute a Lie algebra (of possibly infinite dimension) with bracket operation defined by (4), and ordinary linear combination with constant scalar coefficients of vectors for linear superposition.

The project envisaged, then, is the classification of such relations R . The actual discussion will unfortunately be much more limited. The Lie-algebra conditions (5) will be guaranteed if the ξ are restricted to preserve a quantity, for the succession of two operations which preserve a quantity preserves it, etc. Furthermore, only a brief sampling of

possible quantities will be covered. Out of this sampling, it will appear to be difficult to find substitutes for classical mechanics. Another approach would be to write arbitrary formulae restricting ξ , and then to cast out formulae which fail to meet (5). This general approach has not been used successfully by the author.

5. Transitivity

Conditions of local transitivity will also be considered.

If state space falls apart into submanifolds of dimension $n-1$ or less in which the motion is confined, then the kinematics is intransitive. The laws of motion could probably be restated for an individual submanifold, to advantage. This situation is to be distinguished from the decomposition of phase space into energy sheets in ordinary classical mechanics, because that decomposition occurs only at the dynamical phase of delimitation, the motion itself being known.

"First-order" intransitivity is a rule which confines $\xi^i(x)$ to a subspace of dimension $\leq n-1$ at the point x . Such a constraint must be holonomic to lead to the kinematical decomposition into submanifolds, envisaged above.

Ordinary canonical formalism possesses neither of the above attributes of intransitivity, because a Hamiltonian can be chosen so as to have $\frac{\partial H}{\partial x^i}(x)$ in an arbitrary direction; then $\xi^i = \gamma^{ij} \frac{\partial H}{\partial x^j}$ is equally arbitrary, because γ^{ij} is nonsingular.

What is ordinarily termed "kinematics" often does involve both holonomic and anholonomic constraints, but the detailed models which usually accompany such constraints are so close to the complete dynamics that such constraints will not be considered here.

6. Ordinary Classical Mechanics

As a Solution of (5), in Invariant Notation

It is easy to verify that (2) satisfies (5). More generally, if the only property of γ^{ij} used is the invariant condition

$$\gamma^{ij} + \gamma^{ji} = 0 \quad (6)$$

of antisymmetry, and if $\xi^i = \gamma^{ij} \frac{\partial H}{\partial x^j}$, and $\eta^i = \gamma^{ij} \frac{\partial K}{\partial x^j}$, then easy manipulation leads to

$$[\xi, \eta]^i = \gamma^{ij} \frac{\partial}{\partial x^j} (K, H)_{P.B.} + \gamma^{ijk} \frac{\partial K}{\partial x^j} \frac{\partial H}{\partial x^k}, \quad (7)$$

where

$$(K, H)_{P.B.} = \gamma^{ij} \frac{\partial K}{\partial x^i} \frac{\partial H}{\partial x^j} \quad (8)$$

and

$$\gamma^{ijk} = \gamma^{mi} \frac{\partial \gamma^{jk}}{\partial x^m} + \gamma^{mk} \frac{\partial \gamma^{ij}}{\partial x^m} + \gamma^{mj} \frac{\partial \gamma^{ki}}{\partial x^m}. \quad (9)$$

γ^{ijk} is totally antisymmetric in its three indices. It can be verified directly that it transforms as a third rank contravariant tensor under change of curvilinear coordinates x .

If

$$\gamma^{ijk} = 0, \quad (10)$$

then (7) shows that $[\xi, \eta]^i$ corresponds to use of $(K, H)_{P.B.}$ as a Hamiltonian. We therefore have the theorem:

$$(10, 2a) \text{ satisfies (5).} \quad (11)$$

The hypotheses of (11) are coordinate-invariant; (2b), not invariant, has been replaced by (10). (10) is only a slight generalization of (2b): It will be shown in Appendix 3 that if γ^{ij} is nonsingular, then there exist local coordinates in which (2b) is attained.

(8) reduces to the conventional definition of Poisson bracket

under (2b). If $x \rightarrow f(x)$ is a fixed function of the x^i , and $\dot{f}$ denotes the change of the value of $f(x)$ consequent on the motion (1), then

$$\dot{f}(x) = \frac{\partial f}{\partial x^i} \xi^i, \quad (12a)$$

which reduces to the usual expression

$$\dot{f} = (f, H)_{P.B.} \quad (12b)$$

under (2a).

(2a, 7, 10) imply a mapping of the Lie-algebra alternant of two infinitesimal flows into Poisson bracket of the corresponding Hamiltonians. This illuminates the fact that Poisson bracket satisfies the axioms for a Lie-algebra product.

If we assume that

$$\gamma^{ij} \text{ is nonsingular,} \quad (13a)$$

then we can define a reciprocal, covariant tensor by

$$\gamma_{ik} \gamma^{jk} = \delta_i^j, \quad (13b)$$

and verify that

$$\gamma_{ai} \gamma_{bj} \gamma_{ck} \gamma^{ijk} = -\gamma_{abc}, \quad \text{where} \quad (14a)$$

where

$$\gamma_{abc} \equiv \frac{\partial \gamma_{bc}}{\partial x^a} + \frac{\partial \gamma_{ca}}{\partial x^b} + \frac{\partial \gamma_{ab}}{\partial x^c}, \quad (14b)$$

short for $(\text{ext } \gamma)_{abc}$ (Appendix 2), is half of what is obtained from $\frac{\partial \gamma_{bc}}{\partial x^a}$ by antisymmetrizing. Under (13), the well-known invariance of exterior derivative (Appendix 2) and (14) provide an alternative proof of the tensor character of γ^{ijk} , and condition (10) may be restated as the vanishing of the exterior differential of $\gamma_{ab} dx^a dx^b$ or exterior derivative of γ_{ab} .

7. A First Answer to Question (3)

It is easy to verify that (3) generalized to

$$\xi^i(x) = g^{ij} \frac{\partial H}{\partial x^j}, \quad (3a)$$

where

$$g^{ij} = g^{ji}$$

is a nonsingular symmetric contravariant tensor and H is an arbitrary function of the x^i , does not satisfy (5), even if g^{ij} is assumed to be a constant matrix or the unit matrix. This provides a direct answer to the question (3).

The next two sections will suggest, however, that question (3) has not been properly posed. Namely, the solution (2) of the problem (5) will be obtained from a condition that motions leave invariant a nonsingular antisymmetric form. This suggests that the attempt to replace antisymmetry by symmetry, otherwise imitating the form of (2), is ill advised; that question (3) should be replaced by a study of the solutions of (5) obtained by substituting a symmetric form for the antisymmetric form in a statement of invariance.

8. Invariant Forms; Definition⁹ of the Displacement $\delta_\xi = \delta t$ ($\therefore \xi$)

(11) has been derived in virtue of the identity (7), with no reference to the concept of an invariant quantity. An obvious conjecture for relating ordinary classical mechanics to an invariant quantity is that canonical formalism will emerge from the postulate that $\delta_\xi \gamma^{ij} = 0$ or $\delta_\xi \gamma_{ij} = 0$ (under (13)) restrict the choice of $\xi^i(x)$ in (1). The conjecture is obvious because it sounds the same as the preservation of Hamilton's canonical equations by a canonical transformation, or a converse assertion

to the invariance of Poincare's antisymmetric bilinear form.¹⁰ In order to discuss the matter, it is of course necessary to define δ_ξ . In order to discuss other solutions of (5), it will be necessary to define δ_ξ of quantities other than antisymmetric forms.

If points move under (1), then the motion of infinitesimally close pairs of points is also determined. A contravariant vector field $x \rightarrow v^i(x)$ may be considered to be a field of infinitesimally close pairs of points $(x^i, x^i + v^i(x) \epsilon)$. This picture for a contravariant vector field leads to a definition of its ξ -induced displacement:

$$(x^i, x^i + v^i(x) \epsilon) \rightarrow (x^i + \xi^i(x) \delta t, x^i + v^i(x) \epsilon + \xi^i(x + v(x) \epsilon) \delta t) \\ = (x^i + \xi^i(x) \delta t, x^i + \xi^i(x) \delta t + \epsilon(v^i(x) + \frac{\partial \xi^i}{\partial x^j} v^j(x) \delta t)).$$

If we put $x^i + \xi^i(x) \delta t = y^i$, the field is equivalently given by

$$(y^i, y^i + \epsilon(v^i(y - \xi(y) \delta t) + \frac{\partial \xi^i}{\partial y^j}(y) v^j(y) \delta t)), \text{ or}$$

$$(y^i, y^i + \epsilon(v^i(y) - \frac{\partial v^i}{\partial y^j}(y) \xi^j(y) \delta t + \frac{\partial \xi^i}{\partial y^j}(y) v^j(y) \delta t)). \text{ The coefficient}$$

of ϵ is the new vector field at y . The increment to the v^i field at x is

$$\delta_\xi v^i(x) = \delta t (- \frac{\partial v^i}{\partial x^j}(x) \xi^j(x) + \frac{\partial \xi^i}{\partial x^j}(x) v^j(x)) = (v; \xi)^i(x) \delta t. \quad (15)$$

In this definition, we compare the altered field to the original field at the same manifold point, x . Therefore, the problem of comparing tensors at different points does not arise, and the introduction of a concept of covariant derivative would not modify (15). Consequently, in spite of the occurrence of explicit gradients of tensors, (15) defines a contravariant vector field $(v; \xi)^i(x)$. Similarly, δ_ξ and $(\cdot; \xi)$ as defined further below will not alter the tensorial character of the tensors acted upon. Of course, this can be explicitly verified.

By comparing (15) and (4), note that

$$(v; \xi)^i = [v, \xi]^i. \quad (15a)$$

Covariant vectors and tensors are defined as linear and multilinear real-valued functions of vectors. To go on, it is therefore necessary to consider the displacement of a real-valued scalar function, $x \rightarrow f(x)$. The picture here will be that as a point moves under (1), it bears its old function value; $(f + \delta f)(x + \delta x) = f(x)$;

$$\delta_\xi f(x) = - \delta t \xi^i \frac{\partial f}{\partial x^i} = (f; \xi)(x) \delta t. \quad (16)$$

One might say that f is here displaced as a constant of the motion. Note the contrast with (12a). For the present purpose, (16) is appropriate because $v^i(x)$ is sent (albeit deformed) to $x + \delta x$, and a covariant vector's action on it, $v^i(x) \rightarrow u_i(x) v^i(x) = f(x)$, will have to follow on to $x + \delta x$. (If in spite of this thought, the motion were considered not to alter a scalar function, so that $\delta_\xi f(x)$ were regarded zero, then definition of $\delta_\xi u_i(x)$ would be blocked by failure of $v^i(x)$ to factor.)

ξ -displacement of a covariant vector field $x \rightarrow u_i(x)$ will be defined by the conditions that its contraction $u_i v^i$ with an arbitrary contravariant vector field satisfy both (16) and $\delta_\xi(u_i v^i) = (\delta_\xi u_i) v^i + u_i (\delta_\xi v^i)$. By use of (15), it follows that

$$\delta_\xi u_i = \delta t \left(- u_j \frac{\partial \xi^j}{\partial x^i} - \xi^j \frac{\partial u_j}{\partial x^i} \right) \equiv (u; \xi)_i \delta t. \quad (17)$$

ξ -displacement of the volume element

$$\tau = \epsilon(i_1, \dots, i_n) v^{i_1} \dots v^{i_n} \quad (18)$$

spanned by n contravariant vector fields $v, \dots, v$, where

$\epsilon(i_1, \dots, i_n)$ is Kronecker's alternating symbol, is naturally defined by considering the volume element of the new vectors to be the new volume element. Analytically, $(\cdot; \xi)$ is applied to (18) by the rule for differentiating products, with $(\epsilon(i_1, \dots, i_n); \xi) = 0$. The result is

$$(\tau; \xi) = -\xi^j \frac{\partial \tau}{\partial x^j} + \frac{\partial \xi^j}{\partial x^j} \tau. \quad (19)$$

ξ -displacement of a relative tensor field of weight¹¹ w ,
 $x \rightarrow T^{a_1, \dots, a_r}_{b_1, \dots, b_s}(x)$, will be defined by requiring the scalar

$$T^{a_1, \dots, a_r}_{b_1, \dots, b_s}(\tau)^w \cdot {}^1u_{a_1} \dots {}^r u_{a_r} \cdot {}^1v^{b_1} \dots {}^s v^{b_s} \quad (20)$$

to displace according to (16), where τ is an arbitrary volume element (relative scalar of weight -1), w is an exponent, ${}^1u, \dots, {}^r u$ are arbitrary covariant vector fields, and ${}^1v, \dots, {}^s v$ are arbitrary contravariant vector fields, and also requiring that $(\cdot; \xi)$ distribute over the factors of (20) by the rule for differentiating products, including $((\tau)^w; \xi) = w (\tau)^{w-1} (\tau; \xi)$. If eq. (15, 17, 19) are used for the $(v; \xi)$, $(u; \xi)$, and $(\tau; \xi)$, it is found that

$$\begin{aligned} (\tau; \xi)^{a_1, \dots, a_r}_{b_1, \dots, b_s} = & -w \frac{\partial \xi^i}{\partial x^i} T^{a_1, \dots, a_r}_{b_1, \dots, b_s} \\ & - \xi^i \frac{\partial}{\partial x^i} (T^{a_1, \dots, a_r}_{b_1, \dots, b_s}) \\ & + T^{i, \dots, a_r}_{b_1, \dots, b_s} \frac{\partial \xi^{a_1}}{\partial x^i} + \dots \\ & - T^{a_1, \dots, a_r}_{i, \dots, b_s} \frac{\partial \xi^i}{\partial x^{b_1}} - \dots \end{aligned} \quad (21)$$

$(\cdot; \xi)$ can be extended to explicit derivatives of tensor fields by being defined to commute with $\frac{\partial}{\partial x^i}$. This is an obvious extension of linearity: $(T-U; \xi) = (T; \xi) - (U; \xi)$. Then, if

$$(\Gamma; \xi)^i_{jk} = \frac{\partial^2 \xi^i}{\partial x^k \partial x^j} - \xi^m \frac{\partial \Gamma^i_{jk}}{\partial x^m} + \Gamma^m_{jk} \frac{\partial \xi^i}{\partial x^m} - \Gamma^i_{mk} \frac{\partial \xi^m}{\partial x^j} - \Gamma^i_{jm} \frac{\partial \xi^m}{\partial x^k} \quad (22)$$

for an affine connection Γ^i_{jk} , $(\cdot; \xi)$ commutes with covariant differentiation¹² in the sense that $(\cdot; \xi)$ can be applied either directly to the covariant

derivative $v^i_k = \frac{\partial v^i}{\partial x^k} - \Gamma^i_{jk} v^j$ as a mixed second-rank tensor according to (21), or else to the separate terms on the right by using (22) and the laws for applying a derivative, to achieve the same result.

$(\cdot; \xi)$ also commutes with exterior derivative of antisymmetric covariant tensors and with divergence of antisymmetric contravariant tensor densities (Appendix 2). $(\cdot; \xi)$ commutes with linear combination with constant coefficients, and acts generally by the law for differentiating products, on products. From (21),¹³ $(\delta^i_j; \xi) = 0$, which entails that $(\cdot; \xi)$ commutes with contraction. $(\cdot; \xi)$ leaves invariant rank and weight of relative tensors, and symmetry or antisymmetry on any pair of subscripts or of superscripts.

9. Canonical Formalism and Variants, From $(\gamma_{ij}; \xi) = 0$

We are now prepared to relate invariance of an antisymmetric form to canonical formalism. If γ_{jk} is a covariant¹⁴ tensor, then

$$(\gamma_{jk}; \xi) = 0, \quad (23a)$$

as a special case of (21), can be written

$$\xi^i \left(\frac{\partial \gamma_{jk}}{\partial x^i} - \frac{\partial \gamma_{ik}}{\partial x^j} - \frac{\partial \gamma_{ji}}{\partial x^k} \right) + \frac{\partial}{\partial x^j} (\gamma_{ik} \xi^i) + \frac{\partial}{\partial x^k} (\gamma_{ji} \xi^i) = 0. \quad (23b)$$

If

$$\gamma_{ij} + \gamma_{ji} = 0, \quad (24)$$

then (23b) reduces to

$$\xi^i \gamma_{ijk} = \frac{\partial \xi_i}{\partial x^k} - \frac{\partial \xi_k}{\partial x^j}, \quad (25)$$

where γ_{ijk} is the exterior derivative (14b), and

$$\xi_j = \gamma_{ij} \xi^i. \quad (26)$$

In the case of vanishing exterior derivative,

$$\gamma_{ijk} = 0, \quad (27)$$

it is shown in Appendix 3 that "canonical" coordinates can be found in a neighborhood for which

$$\gamma_{ij} = \text{diag}(\epsilon, \dots, \epsilon, 0, \dots, 0), \quad (28)$$

$$\epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where the rank of the matrix is $2k$; i. e., the number of ϵ blocks is k .

(25, 27) alone imply that ξ_j is a gradient in a neighborhood,

$$\xi_j = \frac{\partial H}{\partial x^j}. \quad (29)$$

In canonical coordinates, (26, 28) then imply that $\xi_j = 0$ if $j > 2k$.

Therefore, H in (29) is a function of x^i for $i \leq 2k$ only. The equations (26) for $j \leq 2k$ can be solved to give

$$\xi^i = \sum_{j \leq 2k} \gamma^{ij} \frac{\partial H(x^1, \dots, x^{2k})}{\partial x^j}, \quad i \leq 2k, \quad (30a)$$

where γ^{ij} is also given by (28). There is

$$\text{no restriction on the } \xi^i, \quad i > 2k. \quad (30b)$$

We therefore have strict canonical formalism in the sense of (2b) for

some of the variables, x^i , $i \leq 2k$, so that these $x^i(t)$ are determined from the corresponding $x^i(0)$ and $H(x^1, \dots, x^{2k})$ only. Then the $x^i(t)$, $i > 2k$, are determined from free $\xi^i(x^1(t), \dots, x^{2k}(t), x^{2k+1}, \dots, x^n)$, $i > 2k$, where the solved motion is substituted for $x^1(t), \dots, x^{2k}(t)$.

To summarize, canonical formalism is characterized by the solution of (5) defined by $(\gamma_{ij}; \xi) = 0$ for a nonsingular antisymmetric covariant tensor γ_{ij} , of null exterior derivative.

10. $\gamma_{ijk} \neq 0$: Twisted Mechanics

The condition $\gamma_{ijk} = 0$ so far appears as a mysterious extra postulate, to facilitate the explicit expression in terms of a free Hamiltonian function, of all ξ such that $(\gamma_{ij}; \xi) = 0$. The rôle of this condition in other treatments of canonical formalism not based on Hamilton's principle is also as a postulate, usually introduced by starting with γ_{ij} as the exterior derivative of a covariant vector field, $\gamma_{ij} = \frac{\partial \gamma_i}{\partial x^j} - \frac{\partial \gamma_j}{\partial x^i}$. For example, the relationship of $\gamma_{ijk} = 0$ to Poincaré's integral invariants is described in Appendix 5.

It is interesting to ask whether $\gamma_{ijk} \neq 0$, "twist", entails defects. Some results in this direction will be presented for the case of nonsingular γ_{ij} .

$$\gamma^{im} \gamma_{ijk} \xi_m = \frac{\partial \xi_i}{\partial x^k} - \frac{\partial \xi_k}{\partial x^i}$$

shows ξ_j not to be exact provided we have no restriction on the direction of ξ^i (first-order transitivity). Under this hypothesis, it can also be shown that ξ_j is not even integrable, and therefore is not found to be tangent to a hypersurface of "constant energy" in the same way as when $\gamma_{ijk} = 0$. This "defect" is of uncertain meaning, especially in view of the triviality of a posteriori constants of motion (Appendix 1).

There is, however, a more definite defect. Namely,

$$u_i = \frac{1}{2} \gamma_{ijk} \gamma^{jk}, \quad (31)$$

is a covariant vector field independent of ξ which is preserved by the ξ -motion: $(u_i; \xi) = 0$. This follows from $(\gamma_{ij}; \xi) = 0$, because $(\cdot; \xi)$ can be applied to (31) and commuted through the various operations so as to act directly on γ_{ij} , by the remarks at the end of Section 8.

Furthermore,

$$u_{ij} = \frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i} \quad (32)$$

is "almost always" nonsingular; i. e., $\det u_{ij}$ need not vanish identically: This is established in Appendix 6 by evaluation of $\det u_{ij}$ in special cases. It follows from Section 12D that in this general situation, no analytic motion (except rest) is possible under $(\gamma_{ij}; \xi) = 0$.

Between $\gamma_{ijk} = 0$ and $\det u_{ij} \neq 0$ is a hierarchy of cases in which the Darboux class of u_{ij} , for example, can lie between the extreme values 0 and n (see Appendix 3), a hierarchy I have not studied.

The fact $(u^i; \xi) = 0$, where $u^i = \gamma^{ij} u_j$, provides us with an "internal time" whenever $u_j \neq 0$; see Section 12C.

I have sought in vain for non-constant scalars s , formed from γ_{ij} , γ_{ijk} , γ^{ij} , and the scalar density $\sigma = (\det \gamma_{ij})^{\frac{1}{2}}$. Such a scalar s would have $(s; \xi) = 0$, and would imply the defect of intransitive kinematics; see Section 12A. The following are identically zero: all contractions of form $\gamma_{...} \gamma_{...} \gamma^{...} \gamma^{...} \gamma^{...}$, and $\frac{1}{2} \gamma_{ijk} \gamma^{jk} \frac{1}{\sigma} \frac{\partial(\sigma \gamma^{im})}{\partial x^m}$, and $\gamma_{ij} \frac{1}{\sigma} \frac{\partial(\sigma \gamma^{ijk})}{\partial x^k}$.

$(\sigma; \xi) = 0$, incidentally, is a Liouville theorem¹⁵ even for twisted mechanics; see Section 12B.

It remains open whether there are mildly twisted variants of classical mechanics, in which $\gamma_{ijk} \neq 0$ but $\det u_{ij} = 0$, for which the motions leaving γ_{ij} invariant constitute "as large" a set of motions as that of flat mechanics. These would constitute an analogue of the homogeneous spaces of constant but nonvanishing curvature, in Riemannian geometry.

11. The First Rigidity Theorem: Motions Leaving a Symmetric Tensor Invariant

The consequences of replacing the invariance of a nonsingular antisymmetric tensor γ_{ij} by invariance of a nonsingular symmetric tensor g_{ij} , in the kinematical stage of a theory of motion, are described in this section.

Let g_{ij} , g^{ij} be nonsingular reciprocal symmetric tensors.

$(g_{ij}; \xi) = 0 \iff (g^{ij}; \xi) = 0$. The covariant form reads (see (21))

$$\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{kj} \frac{\partial \xi^k}{\partial x^i} + g_{ik} \frac{\partial \xi^k}{\partial x^j} = 0,$$

or

$$\xi^k \left(\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j} \right) + \frac{\partial \xi_j}{\partial x^i} + \frac{\partial \xi_i}{\partial x^j} = 0, \text{ where} \quad (33)$$

$$\xi_i = g_{ik} \xi^k,$$

which features the Christoffel symbol¹² belonging to the form g_{ij} .

$$2 g_{ka} \Gamma^a_{ij} = 2 \Gamma_{k,ij} = \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}. \text{ In the usual notation for}$$

covariant derivative corresponding to the use of Γ^i_{jk} as a parallel

displacement, (33) may be written simply $\xi_{j;i} + \xi_{i;j} = 0$. If the curvature,

defined by $v^i_{;j;k} - v^i_{;k;j} = R^i_{ajk} v^a$, namely

$$R^i_{ajk} = - \frac{\partial \Gamma^i_{aj}}{\partial x^k} + \frac{\partial \Gamma^i_{ak}}{\partial x^j} + \Gamma^b_{aj} \Gamma^i_{bk} - \Gamma^b_{ak} \Gamma^i_{bj}, \text{ vanishes, then}$$

local coordinates may be found in which g_{ij} is constant, and in these

(33) reduces to

$$\frac{\partial \xi_i}{\partial x^i} + \frac{\partial \xi_i}{\partial x^j} = 0. \quad (34)$$

If $\xi_i(x)$ is expanded in a MacLaurin series,

$$\begin{aligned} \xi_i(x^1, \dots, x^n) &= a_i + a_{i,j} x^j + \frac{1}{2} a_{i,j_1 j_2} x^{j_1} x^{j_2} + \dots \\ &= a_i + \sum_{r=1}^n \frac{1}{r!} a_{i,j_1 \dots j_r} x^{j_1} \dots x^{j_r}, \end{aligned} \quad (35)$$

where the j_r are dummies, not powers, and the coefficients $a_{1,j_1,\dots,j_r}$ are defined symmetric in the last r subscripts, then (34) may be expressed by

$$a_{1,j,j_2,\dots,j_r} + a_{j,1,j_2,\dots,j_r} = 0. \quad (36)$$

In words, the a 's are antisymmetric on the first two subscripts, but symmetric on the last r . However, if such a symbol has 3 or more subscripts, it must be zero: $a_{ijk\dots} = -a_{jik\dots} = -a_{jki\dots} = a_{kji\dots} = a_{kij\dots} = -a_{ikj\dots} = -a_{ijk\dots}$. (35) therefore simplifies to

$$\xi_1(x) = a_1 + a_{1,j} x^j, \quad (37)$$

with

$$a_{1,j} + a_{j,1} = 0. \quad (38)$$

This describes an infinitesimal rigid displacement, with the notion of the preserved distance given by use of g_{ij} as a metric. The significant thing is that (37) gives us a finite set of parameters $a_1, a_{1,j}$ to describe the general ξ which satisfies $(g_{1j}; \xi) = 0$; the number of independent parameters is in fact $n + \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n+1)$.

It may be argued directly from the equations (33) that they are at least this strong in limiting the number of free parameters, when the hypothesis of vanishing curvature is dropped. However, the arguments in Section 16 will encompass this result in a more general context.

Our experience of space is that the conditions in mutually distant neighborhoods may be widely different. The use of $\xi^i(x^1, \dots, x^n)$ analytic in the variables implies the possibility of extrapolation of the motion from a state-space neighborhood, but only by determining an infinite number of parameters in the neighborhood. On the other hand, a motion of the above type, when given in a neighborhood of state space, readily yields

the $\frac{1}{2}n(n+1)$ parameters needed to describe the entire motion of the state space. (Some of these motions may even fail to extend by not meeting matching conditions for coordinate patches, if we were to consider defining the motion in a truly global way, but we are avoiding global questions.) If motion based on an invariant symmetric form is to be considered for physical application, we are therefore met with two awkward conditions: 1, that ordinary space cannot be directly related to some of the coordinates in state space, and 2, that flexibility in defining the dynamics, that is, a large number of parameters left in the group of motions after the stage of kinematics, may be obtained only by making the dimensionality n of state space large. These awkward conditions suggest quantum mechanics, not classical mechanics. I therefore assume the attitude that a theory of motion, i. e., a solution of (5), is a (generalized) classical mechanics if and only if a rigidity phenomenon is not entailed; namely, that the group of possible motions in a finite-dimensional state space prior to dynamics have an infinite number of parameters. The free Hamiltonian function, $H(x^1, \dots, x^n)$, of course provides the infinite number of parameters in the ordinary classical case.

It is immediately suggested that various types of tensor T be used in place of γ and g , for study of the motions ξ allowed by $(T; \xi) = 0$, through Eq. (21).

12. Preservation of a Scalar, of a Scalar Density, of a Contravariant Vector, or of a Covariant Vector Field

A. Scalar. Intransitivity. - A scalar field $s(x)$ is left invariant by a motion if and only if the motion is confined to a set $s(x) = \text{constant}$.

If $s(x)$ is identically constant, this is no condition on the motion; if not, then the kinematics imposed by $(s; \xi) = 0$ is intransitive; state space is subdivided into $(n-1)$ -dimensional sheets in which arbitrary motions are allowed. By a transformation of coordinates, we can arrange that $s(x) = x^n$. Then $\xi^i(x) = \xi^i(x^1, \dots, x^{n-1})$, $i < n$; and $\xi^n(x) = 0$.

B. Scalar Density. Incompressibility. - The condition that a scalar density field $\sigma(x)$ be left invariant by the motion is

$$(\sigma; \xi) = - \frac{\partial(\sigma \xi^i)}{\partial x^i} = - \operatorname{div}(\sigma \xi) = 0. \quad (39)$$

This is incompressibility with respect to the volume element $1/\sigma$.

(39) is equivalent to the local representation of $\sigma \xi^i$ as a divergence:

$$\sigma \xi^i = \frac{\partial A^{ij}}{\partial x^j}, \quad (40a)$$

where A^{ij} is an arbitrary antisymmetric contravariant tensor density, or to

$$\xi^i = \frac{1}{\sigma} \frac{\partial(\sigma a^{ij})}{\partial x^j}, \quad (40b)$$

where a^{ij} is an arbitrary antisymmetric contravariant tensor. By a transformation of coordinates, $x \rightarrow x'$, we can arrange that $\sigma'(x') = 1$, and simplify the above formulae. For example, $x'^i = x^i$, $i < n$, $x'^n = f(x^1, \dots, x^n)$ with f arbitrary gives us the arbitrary Jacobian determinant needed for the purpose. All $\sigma = 1$ coordinates ("canonical with respect to a scalar density") are got from any one choice of such coordinates by all coordinate transformations with Jacobian determinant 1. The coordinates in which ξ^i is simply incompressible, $\xi^i = \frac{\partial a^{ij}}{\partial x^j}$, are mutually related by the transformations with constant Jacobian determinant.

C. Contravariant Vector. Time. - Owing to the circumstance that

$(v; \xi)^i = [v, \xi]^i$, the alternant, the condition that a contravariant vector

field $v^i(x)$ be left invariant by the ξ -motion coincides with the condition that the ξ -motion commute with the motion which would be produced by the $v^i(x)$ field, itself. This causes a state-space parameter to assume the rôle of a time variable. The point is most easily made if the $v^i(x)$ field is simplified to $v^i(x) = \delta_n^i$ by the transformation of Appendix 1. Then (15) becomes $\frac{\partial \xi^1}{\partial x^n} = 0$, or

$$\xi^1(x) = \xi^1(x^1, \dots, x^{n-1}). \quad (41)$$

If we use the word "time" for x^n , then (41) is an assertion of time-independence of the motion. x^n is not completely decoupled from the motion, however, because it is not entailed that ξ^n vanish.

D. Covariant Vector. Total Rigidity for n Even, and Maximal Class. - The

meaning of the condition that a covariant vector field $u_i(x)$ be left invariant under ξ -induced motion, $(u_i; \xi) = 0$, depends on the Darboux class of the field (see Appendix 3) and is best understood by first choosing coordinates so as to reduce the field to Darboux form,

$$u_i(x) dx^1 = x^1 dx^2 + x^3 dx^4 + \dots + x^{2c-1} dx^{2c} + \lambda dx^{2c+1}, \quad (42)$$

where $\lambda = 0$ if the Darboux class is $2c$, $\lambda = 1$ if the Darboux class is $2c + 1$. Then

$$u_{ij} = \frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}$$

is of form (28), with c copies of ϵ on the diagonal. Let u^{ij} have the same matrix as u_{ij} in special coordinates. Let

$$\xi_i = \xi^j u_{ji}, \text{ all } i; \quad (43a)$$

then

$$\xi^i = u^{ij} \xi_j, \text{ for } i \leq 2c. \quad (43b)$$

$(u_i; \xi) = 0$ can be written

$$\xi_i = \frac{\partial H}{\partial x^i}, \quad (44a)$$

with

$$H = - \xi_j^j u_j. \quad (44b)$$

From (43a) it follows that

$$\xi_i = 0, \quad i > 2c,$$

whence from (44a),

$$H = H(x^1, \dots, x^{2c}). \quad (45)$$

(43b) and the explicit u_j from (42) reduce (44b) to

$$H = - \xi_1 x^1 - \xi_3 x^3 - \dots - \xi_{2c-1} x^{2c-1} - \lambda \xi^{2c+1},$$

or, in virtue of (44a),

$$H + x^1 \frac{\partial H}{\partial x^1} + x^3 \frac{\partial H}{\partial x^3} + \dots + x^{2c-1} \frac{\partial H}{\partial x^{2c-1}} = -\lambda \xi^{2c+1}. \quad (46)$$

In the case of odd class, $\lambda = 1$, it follows from (46) that

$$\xi^{2c+1} = \xi^{2c+1}(x^1, \dots, x^{2c}). \quad (47a)$$

There is

$$\text{no restriction on } \xi^i, \quad i > 2c+1. \quad (47b)$$

If ξ^{2c+1} is an arbitrary function of the first $2c$ variables, H is determined from (46) and the requirement that it be analytic at $(0, \dots, 0)$.

$$\xi^i = u^{ij} \frac{\partial H}{\partial x^j}, \quad i \leq 2c, \quad (47c)$$

obtained from (43a) and (44a), determines the remaining ξ^i . Then $(u_i; \xi) = 0$ is satisfied with no further condition.

It is obvious that arbitrary analytic $H(x^1, \dots, x^{2c})$ determines analytic ξ^{2c+1} through (46). That arbitrary analytic ξ^{2c+1} determines no more than one analytic H in (46) follows by noting that

$1 + x^1 \frac{\partial}{\partial x^1} + \dots + x^{2c-1} \frac{\partial}{\partial x^{2c-1}}$ takes no nonzero monomial $(x^1)^{p_1} \dots (x^n)^{p_n}$ to zero (uniqueness). The action on the monomial is a factor

$1 + p_1 + p_3 + \dots + p_{2c-1}$, whence monomial ξ^{2c+1} is mapped to H by merely

dividing by this factor; it follows that a superposition of monomial ξ^{2c+1} 's converging in a polycylinder is mapped back to a convergent H in the same polycylinder (existence).

In the case of even class, $\lambda = 0$, the only analytic solution of (46) is $H = 0$; this is the above uniqueness theorem again. The case of even class is therefore very simply stated in Darboux coordinates:

$$\begin{aligned} \xi^1 &= 0, \quad 1 \leq 2c; \\ \xi^1 &\text{ arbitrary, } 1 > 2c. \end{aligned} \quad (48)$$

Corollary - There is no analytic motion (other than rest) which leaves invariant a covariant vector field of maximal Darboux class in an even even-dimensional space, $n = 2c$. The hypothesis of maximal Darboux class can be rephrased: $\det u_{..} \neq 0$.

To speak of motions which modify a covariant vector field $u_1(x)$ but only by a gradient, i. e., by an exterior derivative of a scalar, is to speak obliquely of motions which leave a covariant tensor field $\gamma_{ij}(x)$ invariant, where $\gamma_{ij}(x)$ is itself an exterior derivative (local equivalence). This latter is in turn locally equivalent to the statement that $(\text{ext } \gamma)_{ijk} = 0$. It is this way of speaking, expressed in Darboux coordinates, that occurs in the characterization of a motion $(q, p) \rightarrow (q', p')$ as equivalent to a canonical transformation, the latter defined by preservation of $\sum_{i=1}^{n/2} p_i dq_i$ up to the full-phase-space gradient of a generating function. Preservation of the covariant vector field in the sense of the present section is defined more strictly, in not allowing the gradient, whence the stricter result for $2c = n$.

The survey of motions leaving a few lowly inmates of the zoo of relative weighted tensors invariant is here abruptly terminated, because

the richness of the situation for tensors of the second rank is best described by special language which leads to an altered viewpoint, in which results of this section will however occasionally prove useful.

13. Relevance of Geometry in the Linear Neighborhood of a Point

The forms γ_{ij} and g_{ij} , nonsingular antisymmetric and symmetric forms, occur prominently in the classification of irreducible Lie algebras. The groups of linear transformations, $v^i = L^i_j v^j$, which preserve the forms $\gamma_{ij} u^i v^j$, $g_{ij} u^i v^j$ respectively, belong to this classification; they are named symplectic and orthogonal. This suggests that the study of (5) be confined to preservation of characteristics which figure in the deformation under the motion of the linear neighborhood of a point, and which also are described in terms of irreducible linear Lie algebras.

It is difficult to understand why any neighborhood of a state point should figure in conditioning the motion, unless the picture of a state point itself determining the motion is inadequate, i. e., unless the basic eq. (1) is regarded as approximate, in the sense in which in ordinary classical mechanics, the true propagation is viewed in terms of an underlying quantum-mechanical coherence of near-classical motions, superposed in a Feynman path integral. If such an underpinning is expressly adopted, then it is possible to deal with the ordinary action principles and to abandon entirely the present investigation. More generally, it is possible to anticipate the importance of antisymmetric geometric objects because these are the proper objects of the theory of integrands (exterior differential forms),¹⁶ which should enter somewhere in the writing of "superpositions". These arguments are not sufficiently

precise, however, to carry such weight as to preclude direct study of the consequences of geometric restrictions on the motion, stated in terms of neighborhoods of different types, and in particular of linear neighborhoods, of the state points.

14. Preservation of a G-Structure

The conditions $\delta_{\xi} \gamma_{ij} = 0$, $\delta_{\xi} g_{ij} = 0$, in leaving the coefficients of the forms $\gamma_{ij} u^i v^j$, $g_{ij} u^i v^j$ respectively, unchanged, may be said to leave the symplectic, orthogonal structures defined in terms of coordinate-gradient-basis components by these forms, invariant. These suggest an analogous study of the notion of a geometric "G-structure" defined by a group G of matrices acting on the linear neighborhood of each point, or space of contravariant vectors (tangent space) at each point.

By a (real) G-structure is meant an n -dimensional real manifold, wherein a particular set of bases of the tangent space at each point is distinguished in the following way: The distinguished bases at one point, called "frames", are such that the set of linear transformations for change of basis, restricted to the frames, is G , a given point-independent¹⁷ group of real n by n matrices. A G-structure may be minimally defined by specifying any one frame at each point, as all the others may be got by applying G ; relativists will recall the example of Vierbein formalism. A further condition of the definition of G-structure is that such a choice of one frame at each point be possible in a continuous and sufficiently differentiable way, in a neighborhood.

Suppose the ordered set of contravariant vectors v_k , $k = 1, \dots, n$, constitute a frame for a symplectic structure. The general

frame, of vectors k^f , is given by $k^f = M_k^m v^m$, where M_k^m is the general symplectic matrix,¹⁸

$$M_k^m M_l^j \psi^{kl} = \psi^{mj}. \quad (49)$$

In terms of components on the basis dual to coordinate gradients,

$$k^{fi} = M_k^m v^i.$$

It follows that

$$\psi_{mk} f^j k^{fi} = \psi_{mk} M_m^r M_k^s r^j v^i = \psi^{rs} r^j v^i = \gamma^{ji}, \quad (50)$$

and conversely, all k^{fi} satisfying (50) are frames. Preservation of the structure of frames is therefore equivalent to $(\gamma^{ji}; \xi) = 0$ or $(\gamma_{ji}; \xi) = 0$, for a nonsingular antisymmetric form.

If ψ is replaced by a diagonal matrix D with 1's, then -1's on its diagonal (built-in signature), then G is an appropriate "Lorentz" signature-modification of an orthogonal group, $O(n, D)$, and if the quantities defined in (50) are termed g^{ji} to avoid confusion, preservation of the $O(n, D)$ -structure is seen to be equivalent to $(g^{ji}; \xi) = 0$.

15. E. A. Ruh's Theorem A

E. A. Ruh¹⁹ has proved two rigidity theorems for motions preserving a real G -structure, generalizing work of H. Cartan, S. B. Myers and N. Steenrod, and others. In mathematical language, the theorems assert that the groups of automorphisms of certain real G -structures are finite-parameter Lie groups; I will say simply that the G -structures are rigid. Only one of the theorems is local; the other ("Theorem B") requires compactness of the manifold (state space), and is therefore inapplicable here, and will not be stated.

The local theorem asserts that, if the following real symbols

are all zero for some finite r , then a G -structure is rigid. The symbols are

$$a^1_{j_1, \dots, j_r}, \quad (51)$$

with indices running from 1 to n , symmetric on the subscripts, and with $(i, j_1) \rightarrow a^1_{j_1, j_2, \dots, j_r}$ required to be a matrix in the infinitesimal algebra g of G for each choice of $j_2, \dots, j_r$. If the symbols (51) vanish for some value of r , then they vanish for all greater r . The greatest r for which nonzero symbols (51) exist will be called the "type" of G , or of g .

A proof may be given which extends the proof already given for orthogonal structures with $R^1_{jkm} = 0$, in which the symbols (51) appear as coefficients in a Taylor expansion, the motions being restricted a priori to such as can be given locally by analytic functions:

Let

$$p \xi^1(x^1, \dots, x^n) = p a^1 + \sum_{r=1}^{\infty} \frac{1}{r} p a^1_{j_1, \dots, j_r} x^{j_1} \dots x^{j_r}, \quad (52)$$

$p = 1, 2$, express two infinitesimal motion fields ${}_1\xi, {}_2\xi$ by MacLaurin expansions, let G be of finite type s , and let

$${}_1 a^1_{j_1, \dots, j_r} = {}_2 a^1_{j_1, \dots, j_r}, \text{ all } r \leq s, \quad (53)$$

including ${}_1 a^1 = {}_2 a^1$. If ${}_1 T(t), {}_2 T(t)$ are the corresponding finite transformations for the parameter (time) t , and if $\xi^1(x)$ is the infinitesimal motion corresponding to the difference $({}_2 T(t))^{-1} ({}_1 T(t))$, then $\xi^1(x)$ will have no term of degree s or less in virtue of (53);

$$\xi^1(x^1, \dots, x^n) = \sum_{r>s} \frac{1}{r} a^1_{j_1, \dots, j_r} x^{j_1} \dots x^{j_r}. \quad (54)$$

If there is a nonvanishing coefficient in (54), then there is

one of minimal r , $r = m > s$. But the ensemble of leading $r = m$ coefficients will be shown to comprise matrices in g , with respect to their first two indices, and therefore to be zero under the hypothesis that g is of type s ; whence (54) will be known to be zero, and ${}_1T = {}_2T$. The motion will therefore be known to be determined by the ${}_1a^i_{j_1, \dots, j_r}$, $r \leq s$, including ${}_1a^i$, a finite number of parameters.

It remains only to show that the condition that the displaced of a frame be a frame entails that the leading coefficients of (54), considered as matrices in the first two indices, belong to g .

$$\xi^i(x^1, \dots, x^n) = \frac{1}{m} a^i_{j_1, \dots, j_m} x^{j_1} \dots x^{j_m} + O(x^{m+1}),$$

and

$$\frac{\partial}{\partial x^j} \xi^i(x^1, \dots, x^n) = a^i_{j, j_2, \dots, j_m} x^{j_2} \dots x^{j_m} + O(x^m), \quad (55)$$

where " x " in an order symbol indicates the magnitude of the largest of the x^i components.

Let frames be chosen in a neighborhood of the point $(0, \dots, 0)$ of expansion. The frame at x is an ordered set of n contravariant vectors, ${}_kv(x)$, $k = 1, \dots, n$, whose components on the coordinate-gradient bases are

$${}_kv^i(x^1, \dots, x^n) = {}_kv^i + {}_kv^i_{j_1} x^{j_1} + \dots$$

By choice of coordinates (a linear transformation will suffice), we may attain the simpler form

$${}_kv^i(x^1, \dots, x^n) = \delta^i_k + \sum_{r=1}^{\infty} \frac{1}{r} {}_kv^i_{j_1, \dots, j_r} x^{j_1} \dots x^{j_r}.$$

We will need only that

$${}_kv^i(x^1, \dots, x^n) = \delta^i_k + O(x),$$

$$\frac{\partial}{\partial x^j} ({}_kv^i(x^1, \dots, x^n)) = O(1). \quad (56)$$

To compute the infinitesimal motion of ${}_k v$ under ξ , we use (15), then (55) and (56):

$$\begin{aligned}({}_k v; \xi)^i(x^1, \dots, x^n) &= \frac{\partial \xi^i}{\partial x^j} {}_k v^j - \xi^j \frac{\partial {}_k v^i}{\partial x^j} \\ &= a^i_{j_1, j_2, \dots, j_m} x^{j_2} \dots x^{j_m} {}_k v^{j_1} + O(x^m),\end{aligned}$$

the term written being $O(x^{m-1})$, or

$$\begin{aligned}({}_k v; \xi)^i &= \lambda^i_{j_1}(x) {}_k v^{j_1} + O(x^m), \\ \lambda^i_{j_1}(x) &= a^i_{j_1, j_2, \dots, j_m} x^{j_2} \dots x^{j_m}.\end{aligned}$$

To $O(x^{m-1})$, and on the basis of increments of coordinates at x , the transformation of contravariant vectors at x induced by the motion is given by the matrix, $\lambda^i_{j_1}(x)$. Eq. (56) shows, however, that the transformation between the coordinate-increment basis and the basis $({}_1 v(x), \dots, {}_n v(x))$, which is a frame at x , is discrepant from the unit matrix by terms of order x . Therefore, to $O(x^{m-1})$, $\lambda^i_{j_1}(x)$ is also the matrix of the transformation for contravariant vectors on a frame at x . The condition of preservation of the G-structure requires that $\lambda^i_{j_1}(x)$ produce a new frame at x from the old one; i. e., that

$$\lambda^i_{j_1}(x) \in g,$$

or

$$a^i_{j_1, j_2, \dots, j_m} x^{j_2} \dots x^{j_m} \in g. \quad (57)$$

By linearity of the Lie algebra of matrices g , it follows that

$\lim_{\delta \rightarrow 0} (1/\delta) (\lambda^i_{j_1}(x^1, \dots, x^q + \delta, \dots, x^n) - \lambda^i_{j_1}(x))$ belongs to g also; i. e.,

we may apply $\frac{1}{m-1} \frac{\partial}{\partial x^{k_2}}$ to (57), then $\frac{1}{m-2} \frac{\partial}{\partial x^{k_3}}$ to the result, ..., to obtain

$$a^i_{j_1, k_2, \dots, k_m} \in g,$$

where (i, j) are the matrix indices, and $k_2, \dots, k_n$ are arbitrary fixed indices. This completes the proof.

Ruh's proof does not require motions (diffeomorphisms) to be a priori restricted to such as can be expanded in a Taylor series, but the transformation functions are required a priori to possess a finite number of derivatives, the number depending on the type, s .

16. Irreducible Real Lie Algebras of Infinite Type

If we limit the study of solutions of (5) to motions which preserve a real G -structure where G is a group of real matrices, then Ruh's theorem tells us that the only groups G which may provide non-rigid G -structures, or new types of classical kinematics, are those of infinite type. The argument of Section 11 shows that the real orthogonal groups, irrespective of signature of g_{ij} or of the nature of R^1_{jkm} , are all of type 1; rigidity of orthogonal structures in general is therefore immediate from Ruh's Theorem A. The freedom of a Hamiltonian if $\gamma_{ijk} = 0$, shows that symplectic structures are not necessarily rigid, and that the symplectic groups must be of infinite type.

Both these cases satisfy the additional condition that the algebra g be an irreducible set of matrices. Since irreducibility of g is thus, in particular, a feature of ordinary classical mechanics, it may not be too arbitrary to further limit (5) by imposing the restriction of irreducibility on g .

Y. Matsushima has classified all irreducible real matrix Lie algebras g of infinite type:²⁰

1. $g = gl(n)$: all real n by n matrices. $G = GL(n)$ is all

nonsingular n by n matrices.

2. $g = \mathfrak{sl}(n)$: those of trace zero; $G = \mathrm{SL}(n)$, those of determinant 1.

3. $g = \mathfrak{sp}(n, \Psi)$: G is all real n by n matrices which when applied to vectors u, v leave $\Psi_{ij} u^i v^j = \{u, v\}$ invariant, where Ψ_{ij} is a nonsingular antisymmetric form, in consequence of which n is even; g is all real n by n matrices M such that $\{Mu, v\} + \{u, Mv\} = 0$.

4. $g = \mathfrak{osp}(n, \Psi) = \mathfrak{sp}(n, \Psi) + r I$, the matrices of case 3 being augmented by real multiples of the unit matrix; i. e., the center of $\mathfrak{gl}(n)$ is thrown in, whence the "c" notation. Equivalently, $\Psi_{ij} u^i v^j \rightarrow \mathrm{ort.}(\Psi_{ij} u^i v^j)$, for the finite group G .

Cases 3 and 4 are comprised in the formula $g = \mathfrak{sp}(n, \Psi) + U$, where U is a vector subspace of the center of $\mathfrak{gl}(n)$, inasmuch as 0 and $r I$ are the only such subspaces.

5. $g = \mathfrak{sl}'(\frac{1}{2}n, C) + U$, where n is even.

$\mathfrak{sl}(\frac{1}{2}n, C)$ is comprised of the complex $\frac{1}{2}n$ by $\frac{1}{2}n$ matrices of trace zero, acting in the ordinary way on columns of $\frac{1}{2}n$ complex numbers. But such a matrix also defines a real linear transformation of the n real numbers obtained by regarding each complex number as the ordered pair of its real and imaginary parts, and therefore a real n by n matrix. The ensemble of these real matrices is $\mathfrak{sl}'(\frac{1}{2}n, C)$. Example, $\frac{1}{2}n = 1$:

$(a + bi)(x + yi) = ax - by + (bx + ay)i$ is written

$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax - by \\ bx + ay \end{bmatrix}$. Complex linearity appears dissected into real linearity and commutation with $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

U is any real vector subspace of the real-converted center Z' of $\mathfrak{gl}(\frac{1}{2}n, C)$, all complex $\frac{1}{2}n$ by $\frac{1}{2}n$ matrices. Z' is a space of 2 real

dimensions. $Z' = ((a + bi)I)' = \text{diag} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = (r e^{i\theta} I)'$
 $= \text{diag} \begin{bmatrix} r \cos \theta & r \sin \theta \\ -r \sin \theta & r \cos \theta \end{bmatrix}, \quad -\infty < a, b < \infty, \quad -\infty < r < \infty, \quad 0 \leq \theta < 2\pi.$

Besides the unique 0- and 2-dimensional cases for U , there are an infinitude of 1-dimensional cases obtained by fixing θ . I will accordingly speak of cases 5_0 , $5_{1\theta}$, and 5_2 .

6. $g = \text{sp}'(\frac{1}{2}n, G, \Psi) + U$, n a multiple of 4.

$\text{sp}(\frac{1}{2}n, G, \Psi)$ is comprised of the $\frac{1}{2}n$ by $\frac{1}{2}n$ complex matrices whose action on vectors leaves invariant the form $\Psi_{ij} u^i v^j$, in the infinitesimal sense, $\{Mu, v\} + \{u, Mv\} = 0$, where Ψ_{ij} is a nonsingular antisymmetric form: It is nonsingularity of Ψ_{ij} which entails that $\frac{1}{2}n$ be even. The prime denotes that these are considered as real n by n matrices as in the cases 5, and the cases for U are exactly as in 5, and lead to the obvious notation 6_0 , $6_{1\theta}$, 6_2 .

Under the hypothesis of compactness for the manifold, inadmissible from the strictly local viewpoint insisted on here, the cases under 5 and 6 of Matsushima's classification are also rigid. This follows from commutation with the real matrix i' representing multiplication by i , under a theorem of H. Cartan, or under Ruh's Theorem B. In ordinary classical mechanics under free real analytic Hamiltonians, extension to complex p 's and q 's leads to an already nonrigid subclass of motions preserving a 6_0 structure; the non-rigidity implies that even if it is thought worthwhile to prove global theorems, the hypothesis of compactness is at variance with the ordinary situation. It is indeed already at variance with the mechanical situation in the usual view of the unboundedness of the p 's.

17. Twist

A family of bases, one in an attached vector space at each point of a manifold (or of a patch of the manifold), defines a notion of parallelism (distant parallelism in the patch) by the condition that parallel vectors have the same components on the bases. In the case that the attached vector spaces are the tangent spaces, any parallel displacement defines a torsion tensor T ,

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}, \quad (58)$$

$$T^{(a)}_{(bc)} = c^{(a)}_{(bc)} - c^{(a)}_{(cb)} + f^i_{(b)} f^j_{(c)} \left(\frac{\partial f_i^{(a)}}{\partial x^j} - \frac{\partial f_j^{(a)}}{\partial x^i} \right), \quad (59)$$

where unparenthesized indices designate components on the coordinate-gradient bases and their duals, whereas the parenthesized indices designate components on a family of arbitrary mutually dual covariant and contravariant tangent-space bases, $f^{(a)}_i$ and $f^i_{(a)}$, "frames", one at each point,²¹ and where the $c^{(a)}_{(bc)}$ are the coefficients expressing the parallel displacement in frame components: $v^{(a)}_{||}(x, \delta) = v^{(a)}(x) + c^{(a)}_{(bc)}(x) v^{(b)}(x) \delta^{(c)}$.

In the present case, where the family of bases which induces the connection and that used to define parenthesized components coincide, $c^{(a)}_{(bc)} = 0$.

The torsion in this case, called the torsion of the family of frames, therefore vanishes if and only if

$$\frac{\partial f_j^{(a)}}{\partial x^i} - \frac{\partial f_i^{(a)}}{\partial x^j} = 0,$$

which is locally equivalent to

$$f^{(a)}_i = \frac{\partial \varphi^{(a)}}{\partial x^i}.$$

If new coordinates $x'^i = \varphi^{(i)}$ are introduced, then the new components of the same $f^{(a)}$ frame vectors are

$$f_j^{(a)} = \frac{\partial x^i}{\partial x'^j} f_i^{(a)} = \frac{\partial x^i}{\partial x'^j} \frac{\partial \varphi(a)}{\partial x^i} = \frac{\partial \varphi(a)}{\partial x'^j} = \delta_j^a. \quad (60)$$

In the new "flat" coordinates, the coordinate-bound bases coincide with the frames. Conversely, if there exist coordinates with $f_j^{(a)} = \delta_j^a$, a flat presentation of the frames, then the torsion of the frames vanishes.

Unfortunately, the above can be brought to bear on a G-structure only by choosing a family of frames, where frames are now to be chosen only from among the bases distinguished in the definition of the G-structure. Such a choice may involve torsion, even though another choice of frames distinguished in the same G-structure may be torsionless. A G-structure will therefore be said to be flat if there exists a family of frames proper to it, which is torsionless. The absence of this property will be called "twist".

Flatness is equivalent to the possibility of finding flat local coordinates, namely, coordinates for which the coordinate-gradient bases are frames in the given G-structure. For an $O(n, D)$ -structure (Section 14), flatness is well-known to be equivalent to vanishing of the Riemann curvature tensor R^i_{jkm} .²² For Matsushima class 3, flatness is shown to be equivalent to $\gamma_{ijk} = 0$ in Appendix 3. For Matsushima class 1, flatness is obviously automatic. For class 2, the contravariant frames are related by spanning a common volume element $\tau = 1/\sigma$, and then the possibility of transformation to coordinates where $\sigma = 1$ (Section 12B) demonstrates automatic flatness.

Except for the partial discussion of the consequences of twist for the class-3-structure-preserving motions, the consequences of twist will not be treated here, except to indicate where in the subsequent discussion of classes 4, 5, and 6, flatness conditions and flat

coordinates facilitate the treatment. There is consequently an extension to classes 4-6 of the open question, which $\gamma_{ijk} \neq 0$ modifications of class 3 are nonrigid, and how to rank them by flexibility.

Each flat class will be seen to realize the possibility of a nonrigid group (algebra) of motions.

18. The Motions Which Leave
the Flat Examples of Matsushima Classes Invariant

The motions which leave a flat G-structure invariant, where G is one of the infinite-type irreducible group representations listed by Matsushima, will be called motions of class 1, ... , class 6₂, according to the numbering of Section 16. The class-1 motions correspond to eq. (1), without kinematical restriction. The class-2 motions coincide with those of Section 12B. They may be termed, kinematics reduced to Liouville's theorem alone. The class-3 motions correspond to ordinary classical mechanics, provided the usual flatness condition, $\gamma_{ijk} = 0$, is assumed.

The following results are listed below, by class:
conditions on the ξ^i , formulae for $\xi^i(x^1, \dots, x^n)$ satisfying such conditions in terms of free analytic objects analogous to the Hamiltonian, and formulae for the alternant of two such, $_{12}\xi^i = [{}_1\xi, {}_2\xi]^i$, in terms of the correspondingly labelled free objects; these generalize the Poisson bracket of (flat) class 3.

Class 1. $\xi^i(x^1, \dots, x^n)$ are themselves free objects;

$$_{12}\xi^i = {}_1\xi^j \frac{\partial {}_2\xi^i}{\partial x^j} - {}_2\xi^j \frac{\partial {}_1\xi^i}{\partial x^j}$$

(eq. (4)).

Class 2, $\sigma = 1$ (flat) coordinates: $\frac{\partial \xi^i}{\partial x^1} = 0$, $\xi^1 = \frac{\partial a^{ij}}{\partial x^j}$, the

The free-object formulae are

$$\xi^j = -\frac{\lambda}{2} x^j + \psi^{jk} \frac{\partial H}{\partial x^k}, \quad (67)$$

$${}_{12}^\lambda = 0, \quad (68)$$

$${}_{12}^H = \psi^{ij} \frac{\partial {}_2^H}{\partial x^i} \frac{\partial {}_1^H}{\partial x^j} + \left(\frac{x^j}{2} \frac{\partial}{\partial x^j} - 1 \right) ({}_2^\lambda {}_1^H - {}_1^\lambda {}_2^H); \quad (69)$$

The algebra of alternants is that of class 3 motions only, in virtue of (68).

Class 5₂. All the subclasses under 5 and 6 belong to g , G constituted of matrices which commute with

$$i' = \text{diag} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (70)$$

It will be convenient, under assumption of sufficient flatness, to impress (restore) a complex notation on coördinate-bound components, before listing formulae for the separate cases.

The matrices of G interrelate frames. Under the flatness condition that the k^v with components $k^v_j = \delta^j_k$ relative to a coördinatization, constitute frames, the condition that the ξ -displaced of a frame be a frame and eq. (15) imply that the matrix of derivatives $(i, j) \rightarrow \frac{\partial \xi^i}{\partial x^j}$ belongs to g , and that consequently this matrix commutes with i' :

$$\frac{\partial \xi^{2j-1}}{\partial x^{2j-1}} = \frac{\partial \xi^{2j}}{\partial x^{2j}}; \quad \frac{\partial \xi^{2j-1}}{\partial x^{2j}} = -\frac{\partial \xi^{2j}}{\partial x^{2j-1}}. \quad (71)$$

These are precisely the Cauchy-Riemann equations corresponding to introduction of the complex variables

$$z^j = x^{2j-1} + i x^{2j}, \quad c\xi^j = \xi^{2j-1} + i \xi^{2j}. \quad (72)$$

The commutation with i' (in flat cases) is therefore succinctly summarized by the condition that the

$$c_{\xi^j}(z^1, \dots, z^{n/2}), \quad j = 1, \dots, n/2, \quad (73)$$

be analytic in all the z^k ; I will say "n/2-analytic", not to be confused with the general assumption that the real $\xi^k(x^1, \dots, x^n)$ are analytic in their n real arguments.

$$c(i \cdot \xi) = i \cdot c\xi.$$

The Cauchy-Riemann equations for ξ imply $(i \cdot v; \xi) = i \cdot (v; \xi)$. i -linearity in the second position follows, in turn, when the argument vector in the first position is n/2-analytic.

The alternant of two n/2-analytic infinitesimal motions is given by the expected analogue of (4):

$$c_{12}\xi^1 = c_1\xi^j \frac{\partial c_2\xi^1}{\partial z^j} - c_2\xi^j \frac{\partial c_1\xi^1}{\partial z^j}, \quad (74)$$

where the indices run from 1 to n/2.

The above formulae (73, 74) already describe the class 5_2 without further restriction, because $sl(\frac{1}{2}n, C) + Z$, with Z the whole center of $gl(\frac{1}{2}n, C)$, coincides with all of $gl(\frac{1}{2}n, C)$.

The results for the other classes 5 and 6 will also be quoted in terms of the $c\xi$. They are related to results for classes 1-4 by the following use of Schwartz reflection. The Schwartz-reflect $(c\xi(z^1, \dots, z^{n/2}))^* = C^c\xi C(z)$, of $c\xi(z)$, is n/2-analytic when $c\xi(z)$ is n/2-analytic, and so are $R^c\xi = \frac{1}{2}(c\xi + C^c\xi C)$ and $I^c = \frac{1}{2i}(c\xi - C^c\xi C)$; also,

$$c\xi = R^c\xi + i I^c\xi. \quad (75)$$

The fields $R^c\xi$ and $I^c\xi$ are their own Schwartz reflects; i. e., they are real in the sense that their Taylor expansions about a real point, all z^i real, have real coefficients. These fields therefore define real motions in the real n/2-dimensional subspace. For class 5_2 , these real motions are merely of class 1. For class 5_0 , these real motions are of class 2.

Class $5_{1\theta}$: g is real matrices M' corresponding to complex matrices M whose trace $r e^{i\theta} = \frac{2}{n} \rho e^{i\theta}$ has fixed phase θ . Let the volume element of $n/2$ complex $n/2$ -analytic contravariant vectors be their determinant. A $5_{1\theta}$ multiplies this determinant by $e^t \circ e^{i\theta}$, because the G -element (e^{tM}) corresponds to e^{tM} of determinant $e^t \rho e^{i\theta}$. Action of a general 5_2 -motion on an $n/2$ -analytic complex scalar density ${}^c\sigma$, the reciprocal of an $n/2$ -analytic complex volume element, is given by $({}^c\sigma; {}^c\xi) = -\frac{\partial({}^c\sigma {}^c\xi^i)}{\partial z^i}$. In particular, the density 1 is modified according to $(1; {}^c\xi) = -\frac{\partial {}^c\xi^i}{\partial z^i}$, or $\exp(-t \frac{\partial {}^c\xi^i}{\partial z^i})$ for a finite motion of time (real parameter) t . Therefore $\exp(-t \frac{\partial {}^c\xi^i}{\partial z^i}) = \exp(-\rho t e^{i\theta})$, and $\text{Im } \ln \ln (\det e^{tM})^{-1} = \theta$. $n/2$ -analyticity of ${}^c\xi$ and constancy of θ imply constancy of ρ also:

$$-\frac{\partial {}^c\xi^i}{\partial z^i} = \rho e^{i\theta} = -\lambda \text{ is constant.} \quad (76)$$

Two ${}^c\xi$'s of the same λ differ by a divergence. ${}^c\xi^i = \frac{2\lambda}{n} z^i$ satisfies (76). Therefore, the general $5_{1\theta}$ is

$${}^c\xi^i = h e^{i\theta} z^i + \frac{\partial c^{ij}}{\partial z^j}, \quad (77)$$

h a real constant associated with ξ and fixed by ξ , θ the constant angle mod π defining the class $5_{1\theta}$, and c^{ij} free $n/2$ -analytic antisymmetric, determined by ξ modulo an additive $n/2$ -divergence $\frac{\partial b^{ijk}}{\partial z^k}$.

$$12^h = 0$$

$$\begin{aligned} 12^{c^{ij}} &= \frac{\partial_2 c^{ik}}{\partial z^k} \frac{\partial_1 c^{jm}}{\partial z^m} - \frac{\partial_1 c^{ik}}{\partial z^k} \frac{\partial_2 c^{jm}}{\partial z^m} \\ &+ e^{i\theta} \left[2^h \left(z^i \frac{\partial_1 c^{jk}}{\partial z^k} - z^j \frac{\partial_1 c^{ik}}{\partial z^k} + \frac{n}{2} 1^{c^{ij}} \right) \right. \\ &\quad \left. - 1^h \left(z^i \frac{\partial_2 c^{jk}}{\partial z^k} - z^j \frac{\partial_2 c^{ik}}{\partial z^k} + \frac{n}{2} 2^{c^{ij}} \right) \right]. \end{aligned} \quad (78)$$

Class 5_0 . Put $h = 0$ in the formulae (77, 78) for $5_{1\theta}$.

For classes 6, I will take $\gamma_{ij} = \psi_{ij}$, $i, j = 1, \dots, n/2$.

Then 6_0 yields real motions of flat class 3 in canonical form, 6_2 yields real motions of flat class 4. The 6_{10} and the 6_0 motions are obvious subsets of the 6_2 motions.

Class 6_2 . The 6_2 formulae for ${}^c\xi$, restricted to the real subspace, are got from (75) and the class-4 formulae for the real motions. These formulae, analytic in the z^i , then apply generally, by analytic continuation:

$${}^c\xi^j = -\frac{{}^c\lambda}{2} z^j + \psi^{jk} \frac{\partial {}^cH}{\partial z^k},$$

where ${}^c\lambda$ is a free complex constant determined by the motion, ${}^cH$ is a free $n/2$ -analytic complex function determined modulo a complex additive constant.

Eq. (67-69) apply, with ${}^c\lambda$, ${}^cH$, z^j , and $j = 1, \dots, n/2$ instead of λ , H , x^j , and $j = 1, \dots, n$.

Class 6_{10} . ${}^c\lambda$ is constrained to be a real multiple of $e^{i\theta}$.

Class 6_0 . ${}^c\lambda = 0$.

19. Remarks on the Flat Matsushima-Class Motions

Every motion which leaves invariant a G_1 -structure leaves invariant a G -structure, if $G_1 \subset G$ and if the G -structure is obtained from the frames already present in the G_1 -structure by enlarging the set of frames to include all G -images of the former frames. The relationships of this type among the Matsushima classes are indicated in Fig. 1. Motions become more special as one reads down the diagram.

If one does not employ complex structure, one is restricted to classes 1-4, e. g., if n is even but not a multiple of 4. (If n is odd, there are, of course, only classes 1 and 2.) Except for doing nothing

(1) or fooling with Liouville's theorem (Liouville's theorem only, 2; Liouville's theorem repealed, 4), there is only the usual symplectic class (3), which is the most restrictive of the four non-complex classes.

6_0 is pictured as more special than 3, because

$$\psi'_{ij} = \text{diag} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \text{ and also } i' \psi'_{ij} = \text{diag} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \text{ are}$$

are invariant constant nonsingular antisymmetric forms; more general than 3 of half as many real dimensions because $\mathbb{C}H$ may have an imaginary part.

In Fig. 1, restrictions of motions involving the loss of only 1 real parameter are labelled "1", restrictions involving loss of half of the real parameters are marked "HALF".

One ubiquitous feature of the flexible irreducible real Lie-group representations is antisymmetry. Either there is an antisymmetric form of second rank, or one of n th rank (determinant), or a complex structure, preserved. The latter may be thought of as a vague kind of antisymmetry too, because the matrix (70) for i' is antisymmetric -- This is vague because it is a coordinate-dependent statement.

The irrelevance of components of connection to definition of the antisymmetrical derivative operations ext and div (Appendix 2) may perhaps be considered a tie between antisymmetry and "flexibility" in a different sense, if the specification of a connection is viewed as a locking together of the vector spaces at infinitesimally close points.²⁴

20. Motions Preserving a Flat $U'(\frac{1}{2}n, \mathbb{C})$ -Structure

This is a rigid case, of interest because of its relationship to quantum mechanics.

The group $U'(\frac{1}{2}n, C)$ acting on n -dimensional real vector space is obtained from the unitary matrices $U(\frac{1}{2}n, C)$ acting on $n/2$ -dimensional complex vector space by taking apart real and imaginary parts, in the manner of Section 19. $U'(\frac{1}{2}n, C) \subset O(n)$, the orthogonal n by n matrices, because of the preservation of the sum of squares of all the real and imaginary parts. Consequently, $O(n)$ rigidity applies to a $U'(\frac{1}{2}n, C)$ -structure, even a twisted one. In the flattest case, eq. (71) apply, as well as the explicit expression (37) for the ξ_k , $k = 1, \dots, n$. In coordinates for which the symmetric form of the $O(n)$ -structure reduces to δ_{ij} , the most general infinitesimal motion is given by

$$\xi^i(z^1, \dots, z^{n/2}) = A_i + A_{ij} z^j, \quad (79)$$

where A_i and A_{ij} are complex constants, and the A_{ij} constitute a skew-Hermitean matrix: $n(n+1)/2$ free real parameters. Expansion around an a priori fixed origin would eliminate the A_i , and yield the unitary kinematics of quantum mechanics, cut down to a finite-dimensional vector space. Flexibility can be attained only by letting $n \rightarrow \infty$.

Acknowledgments

The interest of Prof. T. T. Frankel, Prof. B. Harris, and Dr. M. Rosen of the Department of Mathematics at Brown University in this work is gratefully acknowledged; Prof. H. Federer provided me with the reference to Darboux' Theorem. Most crucial, however, was the following unexpected aid: Precisely when I was privately worried about the task of surveying all real irreducible matrix Lie algebras, I received E. A. Ruh's dissertation (sponsored by Prof. K. Nomizu) in the capacity of outside reader.

I would like to thank Prof. S. Kobayashi of the Department of Mathematics at Berkeley for an inspirational lecture on symmetric spaces, and Dr. J. Lepore, Prof. D. Judd, and Prof. G. Chew for hospitality at the Lawrence Radiation Laboratory this summer, where the latter part of this work was done.

Appendix 1

Constants of Motion and Solved Hamilton-Jacobi Form for a Single Motion

It is easy to construct arbitrary constants of motion in a neighborhood. Choose a neighborhood of x_0 small enough that ξ^i is nearly constant in it, to work in. Let the state space have n dimensions. Cut across the ξ^i field with an $(n-1)$ -flat F . Decompose the $(n-1)$ -flat into $(n-2)$ -flats F_p , indexed by one parameter, p . The motion curves satisfying eq. (1), through F_p , will form an $(n-1)$ -surface S_p . If all x in S_p are mapped to p , $x \rightarrow p(x)$, then $p(x)$ is a non-constant function, constant of the motion.

Furthermore, if a system of rectangular coordinates

$$p^1, \dots, p^{n-1} \quad (80)$$

is introduced into the $(n-1)$ -flat F , then each p^i can serve as a "p" while the rest index the points of F_p , whence $n-1$ independent constants of motion are attained.

If $p^1, \dots, p^{n-1}$ and the time at which the $(n-1)$ -flat F reaches the state point are used for coordinates, $x^i = p^i$, $i < n$, and $x^n = t$, then the motion is reduced to $\dot{x}^i = \delta_{ni}^i$. The motion is a Hamiltonian motion in the trivial solved Hamilton-Jacobi form. The only objection to this "cheating" construction that I know is that it requires knowledge of the particular motion, and is therefore foreign to the context of limitation of (1) to a large subgroup, which is then to again be materially restricted by dynamical assumptions to attain the 1-parameter motion group.

Appendix 2

Formulae from Exterior Differential Calculus¹⁶

The exterior derivative is the totally antisymmetrized gradient of a covariant tensor (of its totally antisymmetric part),

$$(\text{ext } t)_{a_1, \dots, a_r} = \frac{1}{(r-1)!} \sum_{\bar{w}} \text{sgn } \bar{w} \frac{\partial t_{\bar{w}(a_1), \dots, \bar{w}(a_{r-1})}}{\partial x^{a_r}},$$

where $\bar{w}$ runs over permutations of $(1, \dots, n)$. The result is an r th rank totally antisymmetric covariant tensor.

$$(\text{div } T)^{a_1, \dots, a_r} = \frac{\partial T^{a_1, \dots, a_r, a_{r+1}}}{\partial x^{a_{r+1}}}$$

produces an r th rank totally antisymmetric contravariant tensor density from one of $(r+1)$ st rank. A pair of tensors related by ext can be interchanged with a pair of densities related by div by means of contraction with the completely alternating Kronecker symbol $\epsilon(i_1, \dots, i_n)$ (dualization). $\text{ext ext } t = 0$, $\text{div div } T = 0$ (Poincaré's lemma). If $\text{ext } u = 0$ ($\text{div } U = 0$), then there is a neighborhood in which $u = \text{ext } t$ ($U = \text{div } T$) for some t (T) (converse to Poincaré's lemma).

Stokes' Theorem is

$$\int_V (\text{ext } t)_{a_1, \dots, a_r} dx^1 \dots dx^r = \int_S t_{a_1, \dots, a_{r-1}} dx^1 \dots dx^{r-1},$$

where the dx^i anticommute, V is an r -dimensional oriented subset of the n -space, and S is the oriented boundary of V .

Appendix 3

Canonical Coordinates from $\gamma_{ijk} = 0$; Darboux Class of a Covariant Vector Field

From $\gamma_{ijk} = 0$ it follows by the converse to Poincaré's lemma (Appendix 2) that locally,

$$\gamma_{ij}(x) = \frac{\partial \gamma_i(x)}{\partial x^j} - \frac{\partial \gamma_j(x)}{\partial x^i} \quad (81)$$

(81) subsumes $\gamma_{ijk} = 0$. The simplification of $\gamma_{ij}(x)$ now rests on seeking a simple form for the differential form

$$\gamma = \gamma_i(x) dx^i$$

by means of a coordinate transformation. For this a proposition of G. Darboux²⁵ suffices: The differential form γ can be brought to one of the two forms

$$z^1 dy^1 + \dots + z^c dy^c, \quad (82a)$$

$$z^1 dy^1 + \dots + z^c dy^c + dy^{c+1}, \quad (82b)$$

where the functions y^i, z^j are independent in the sense that their $2c$ or $2c + 1$ gradients are linearly independent at each point. The number $2c$ or $2c + 1$ of y 's and z 's in (82) is the Darboux class of the covariant vector field γ_i , its sole local invariant.

By adjoining $n - 2c$ or $n - 2c - 1$ of the original x^a to the list of y^i and z^j , we have new coordinates which will also be called x^i , in which γ takes the form (42). In either case, (81) reduces to (28), with $k = c$.

A rewording of Darboux' theorem which may make it more plausible is this: Let two forms be equivalent if they differ by an exact differential. Then try to reduce a given form γ to an equivalent form of the smallest number of terms, $\sum_{i=1}^c g^i(x) df^i(x)$, c minimum. Obviously one cannot do

better than to get the g^i and f^j independent, for then by rechoosing coordinates one gets (28), and to do better would be to reduce the rank of $[\gamma_{ij}]$. The theorem is that one can in fact do that well.

Appendix 4

Algebraic Lemmas

1. At any given point x_0 , there is a basis on which a covariant antisymmetric tensor assumes the form (28). There is of course a parallel statement for a contravariant antisymmetric tensor.

2. Given a covariant antisymmetric tensor γ_{ij} , there is a contravariant antisymmetric tensor γ^{ij} and a mixed tensor p_i^j (projection) with

$$\gamma_{ij} \gamma^{kj} = p_i^j, \quad (83a)$$

$$\gamma_{ij} p_k^j = \gamma_{ik}, \quad (83b)$$

$$\gamma^{ij} p_j^k = \gamma^{ik}, \quad (83c)$$

$$p_i^j p_j^k = p_i^k, \quad (83d)$$

with $\text{rank}[\gamma_{ij}] = \text{rank}[\gamma^{ij}] = \text{rank}[p_i^j]$.

Proof of 1. All quantities are at x_0 , which won't be mentioned further. Let $u^i \gamma_{ij} v^j = \{u, v\}$. Let the space of all contravariant vectors be U . If $\gamma_{ij} \neq 0$, pick ${}_1u$ so that $\gamma_{ij} {}_1u^j \neq 0$, and then pick v^i so that $v^i (\gamma_{ij} {}_1u^j) \neq 0$; i. e., $\{v, {}_1u\} \neq 0$. By rescaling v , convert it to ${}_2u$ with $\{{}_2u, {}_1u\} = -1$. ${}_2u$ is linearly independent of ${}_1u$ because $\{\lambda {}_1u, {}_1u\} = 0$, never -1 . The vectors u with $\{{}_2u, u\} = 0 = \{{}_1u, u\}$ include no ${}_1\lambda {}_1u + {}_2\lambda {}_2u$, and form a linear space S . Let ${}_1u, {}_2u$ followed by any basis ${}_3v, \dots, {}_nv$ of subspace S be used as a basis for U . Then ${}_1u^i = \delta_1^i$, ${}_2u^i = \delta_2^i$, and $\{{}_2u, {}_1u\} = \gamma_{21} = -1$; also $\{{}_3v, {}_1u\} = \{{}_3v, {}_2u\} = \dots =$

$\{n^v, 2^u\} = 0$ reduces to $\gamma_{ij} = 0$ if $i \leq 2$ and $j \geq 3$. Therefore,

$$[\gamma_{..}] = \begin{bmatrix} \epsilon & 0 \\ 0 & l\gamma_{..} \end{bmatrix}.$$

If $l\gamma_{..}$ is not zero, another ϵ can be extracted by specifying the basis of S further, etc. The process terminates only when $\gamma_{..}$ has the desired form (28), with the residual matrix zero; the cases (2b) and $\gamma_{ij} = 0$ are of course understood to be included.

Proof of 2. Again, all the work is at one point x_0 , which will be suppressed. Pick a basis at x_0 to effect (28). Let γ^{ij} be defined by the same matrix on the corresponding (dual) basis, and on the corresponding basis for mixed tensors, define p_i^k by (83a), or by $p_i^k = \text{diag}(1, 1, \dots, 1, 0, 0, \dots, 0)$, with the number of 1's equal to the (even) rank of $[\gamma_{ij}]$. All the relations (83) hold on these related special bases. Define γ^{ij} and p_i^j on general bases by extending them as tensors of the desired type. Then relations (83) become tensor equations which hold on a special family of related bases, and therefore hold generally.

Appendix 5

Relation to Poincaré's Integral Invariant. Time As $2k+1$ st Coordinate

The condition (27) of vanishing exterior derivative is part of the mystery of canonical formalism, in the present view, because $\delta_\xi \gamma_{ij} = 0$ already satisfies (5) without it. Accordingly, there is a limited discussion of twist, $\gamma_{ijk} \neq 0$, given in Section 10 and Appendix 6.

In conventional treatments of canonical formalism, (27) is postulated without misgivings, often in the implicit form (81). It may be useful to give a conventional treatment through (27) and the integral invariant of Poincaré, instead of $\delta_\xi \gamma_{ij} = 0$, in order to compare

precisely the statement $\delta_{\xi} \gamma_{ij} = 0$ to that of Poincaré.

Fig. 2. Volume bounded by $B + C - A$

From (27) alone, we

have by Stokes' Theorem

(Appendix 2) that

$$\int_{B+C-A} \gamma_{ij} dx^i dx^j = 0; \quad (84)$$

see Fig. 2. Poincaré's

condition is that if the

boundary of A is arbitrary,

but that of B is obtained

from that of A by the motion, then

$$\int_{B-A} \gamma_{ij} dx^i dx^j = 0. \quad (85)$$

Because of (84), this is equivalent to

$$\int_C \gamma_{ij} dx^i dx^j = 0, \quad (86)$$

where C is any surface generated from a loop ℓ by the motion. If the time is made short, (86) is seen to be equivalent to

$$\oint_{\ell} \gamma_{ij} \xi^i dx^j = 0, \quad (87)$$

or $\oint_{\ell} \xi_j dx^j = 0$, for all loops ℓ , which is locally equivalent to $\xi_j = \frac{\partial H}{\partial x^j}$. Nothing more need be said when γ_{ij} is nonsingular.

A cyclic coordinate makes a fine clock. It is however popular to elaborate a variant of this thought, as follows. Add the time to phase space to produce an odd-dimensional space, $n = 2k + 1$, and take γ_{ij} of maximal rank, i. e., $2k$. In this context, γ_{ij} annihilates a uniquely determined 1-dimensional space at each point x . The algebraic structure therefore already defines a flow pattern, and it is usual to²⁶ assume that this is already the motion:

$$\gamma_{ij} \xi^i = 0. \quad (88)$$

Under (88), $\delta_{\xi} \gamma_{ij} = 0$ and $\gamma_{ijk} = 0$ are equivalent.

Let (27) and (88) be assumed. Then eq. (26, 29, 30a) establish the reduction of (88) to

$$\xi^i = 0, \quad i \leq 2k, \quad (89)$$

while the rate of change of time with respect to the parameter, ξ^{2k+1} , is arbitrary. This is a Hamiltonian motion in the trivialized solved-Hamilton-Jacobi form. The tightness of (88) is clear in that no arbitrary function H appears in (89).²⁷ (88) is too tight for the viewpoint adopted here, in the same way that the use of the motion to introduce coördinates in Appendix 1 is objectionable: In either case, there is not really any intermediate stage of canonical formalism between (1) and the unique dynamical motion.

The tightness of (88) may be hidden. Let the reduction of Appendix 3 be taken only to stage (81). Write $\gamma = \gamma_i dx^i$ as

$$\gamma = \sum_{i=1}^{2k} \gamma_i(x) dx^i - H(x) dx^{2k+1}, \quad \text{with } x = (x^1, \dots, x^{2k+1}), \text{ and reduce}$$

only the front end of this to Darboux' canonical form. In the new coördinates, completed with "T" for the $2k+1^{\text{st}}$ one,

$$\gamma = x^1 dx^2 + \dots + x^{2k-1} dx^{2k} - H(x^1, \dots, x^{2k}, T) dT; \quad (90)$$

which in (88) produces a Hamiltonian motion with $H(x^1, \dots, x^{2k}, T)$ featured as a time-dependent Hamiltonian.²⁷

Appendix 6

det $u_{..} \neq 0$: Horribly Twisted Classical Mechanics Is Impossible

If $n = 2$, there is no twist. A discussion of twisted symplectic structures is therefore confined to $n = 4, 6, 8, \dots$. It is shown here that $\det u_{..} \neq 0$ where $u_{..}$ is the matrix of eq. (32), by construction of examples where $\det u_{..} \neq 0$. Then the argument of Section 10 shows that no flows exist for such horribly twisted structures; this total rigidity can be broken only by materially limiting the twist, so as to at least have $\det u_{..} = 0$.

Let

$$[\gamma_{..}] = \begin{bmatrix} 0 & 1^f & & & \\ -1^f & 0 & & & \\ & & 0 & 2^f & \\ & & -2^f & 0 & \\ & & & & \ddots & \ddots \end{bmatrix}.$$

By direct computation, the form of $u_{..}$ is seen to be

$$[u_{..}] = \begin{bmatrix} 0 & 12^A & 13^A & \dots \\ 21^A & 0 & 23^A & \dots \\ 31^A & 32^A & 0 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix},$$

where each entry is a 2 by 2 array,

$$IJ^A = \begin{bmatrix} \frac{\partial^2}{\partial x^{2I-1} \partial x^{2J-1}} & \frac{\partial^2}{\partial x^{2I-1} \partial x^{2J}} \\ \frac{\partial^2}{\partial x^{2I} \partial x^{2J-1}} & \frac{\partial^2}{\partial x^{2I} \partial x^{2J}} \end{bmatrix} (I^F - J^F),$$

where

$$I^F = \ln I^f.$$

Let

$$I^F = \frac{1}{2} I^F_{jk} x^j x^k$$

be quadratic forms with $I^F_{jk} = I^F_{kj}$ free symmetric. Then

$$I_J^A = \begin{bmatrix} I_J^F 2I-1, 2J-1 & I_J^F 2I-1, 2J \\ I_J^F 2I, 2J-1 & I_J^F 2I, 2J \end{bmatrix},$$

where

$$I_J^F jk = I^F_{jk} - J^F_{jk}.$$

A given K^F_{jk} appears only in those I_J^A elements where either $I = K$ or $J = K$, and where either $2I-1$ or $2I$ coincides with j , and where either $2J-1$ or $2J$ coincides with k . The restrictions on j, k show that only I_J^A and J_I^A share common free objects. Consequently, u_{ij} is free antisymmetric, except for the conditions $u_{12} = u_{34} = \dots = 0$; i. e., except for the diagonal zeros coming in 2 by 2 blocs.

n by n matrices ${}_nU$ of this form of determinant 1, not 0, are displayed:

$${}_4U = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$${}_6U = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$${}_{4k}U = \text{diag} ({}_4U, {}_4U, \dots),$$

$${}_{4k+2}U = \text{diag} ({}_6U, {}_4U, \dots).$$

so that the strength of (88) may be disguised by postulating (27) and (85) in the extended sense.

27. This approach is usually presented with (90) as a postulate, not a result of partial use of Darboux's reduction. Then since H is given, the tightness of (88) is not hidden, in the formal sense, but is only alternatively expressed.

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