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THE CONNECTION BETWEEN SUPERSYMMETRY AND ORDINARY LIE SYMMETRY GROUPS*

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ABSTRACT

The mathematical structure of supersymmetry groups and their representations is discussed. It is shown that corresponding ordinary Lie symmetry groups may be used instead. O'Raifeartaigh's theorem applies and these groups realize one of the possibilities it permits, namely, the Lie algebra is the semidirect product of a semisimple Lie algebra and a solvable non-Abelian Lie algebra. The theorem of Coleman and Mandula is not directly applicable because the Hilbert space of physical states is not invariant under the action of the group. It is possible to consider an extended space of states and to define an 'inner product' which is preserved by the action of the group, but this product is not positive definite.

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1. INTRODUCTION

One of the techniques, developed in the investigation of dual models, which seems to have other applications is that of supersymmetry groups. Whereas ordinary Lie groups are generated by algebras involving only commutator (Lie) products, these groups are generated by a (graded) algebra involving commutators and anticommutators. It has the form:

$$[L_i, L_j] = c_{ij}{}^k L_k$$

$$[L_i, G_r] = f_{ir}{}^s G_s$$

$$\{G_r, G_s\} = d_{rs}{}^k L_k \ . \qquad (1.1)$$

When exponentiated to form a group the anticommuting elements of the algebra, G_r, have anticommuting "parameters" associated with them. The object of this paper is to examine the structure of this group and its parameter space, and to show that it may easily be regarded as an ordinary Lie group, with no loss of structure.

An algebra of the form of eqs. (1.1) was first discussed by Ramond and Neveu and Schwarz [1] in the investigation of a dual model for pions. The algebra is related to coordinate transformations in an internal space [2]. Wess and Zumino [3] showed how a similar algebra, having the Poincaré algebra of four-dimensional space-time as a subalgebra, might be constructed. Subsequently a number of Lagrangian models possessing the corresponding symmetry have been constructed [4,5] and shown to have remarkable renormalization properties [5,6]. Thus, apparently, a symmetry group containing the Lorentz group as a subgroup in a nontrivial way (i.e. other than as a direct factor of the algebra) has been found and, secondly, interacting field theories having this symmetry to all orders of perturbation theory have been constructed. The first of these results would appear to be at variance with O'Raifeartaigh's theorem [7] and the second with that of Coleman and Mandula [8]. It has often been suggested [4,5,9] that these potential paradoxes are avoided because a supersymmetry algebra, such as that defined by eqs. (1.1), is not an ordinary Lie algebra--it involves anticommutators. One of the main points of the present paper is to show that this is not the case. The presence of anticommutators is not an essential factor in understanding the relationship of supersymmetries to these theorems (although it may be important in other aspects of their application).

The algebra introduced by Wess and Zumino and subsequently studied by other authors including Salam and Strathdee [10] takes the form

$$\{Q_\alpha, Q_\beta\} = -2(\gamma^\mu C)_{\alpha\beta} P_\mu$$
$$[P^\mu, P^\nu] = 0, \qquad [P^\mu, Q_\alpha] = 0 . \tag{1.2}$$

Here **C** denotes the charge conjugation operator and Q satisfies the Majorana condition:

$$Q = Q^C , \quad \equiv C\bar{Q}^T . \tag{1.3}$$

Additionally the operators Q_α and the momentum operators, P_μ, transform as a Dirac spinor and a vector, respectively, under the generators $M_{\mu\nu}$ of the homogeneous Lorentz group. In order to form finite elements of the corresponding supersymmetry group which involve the Q_α, these operators are multiplied by parameters ϵ_α, which are regarded as anticommuting amongst themselves:

$$\{\epsilon_\alpha, \ \epsilon_\beta\} = 0 \tag{1.4}$$

and exponentiated to form $\exp(\bar{\epsilon}\, Q)$.

The quantities thus obtained have been said to form an "extended group." It is certainly possible to regard this structure as a generalization of a Lie group, involving anticommuting parameters [11]. Meanwhile, it is still a group in that it satisfies the appropriate axioms. Moreover the parameters have a manifold structure irrespective of how they are multiplied, and this structure is respected by the group operations. Thus it must also be a Lie group in the usual sense. Thus one can ask what is the corresponding Lie algebra and whether this algebra may be used equally well as that of eqs. (1.2).

To obtain the corresponding Lie algebra we take a basis for the space of anticommuting parameters ϵ_α and then consider products of these basis elements with the basis of the generalized Lie algebra. Thus precisely which Lie algebra is obtained depends on the choice of the space of parameters, ϵ. Clearly, one would try to choose this in the most economical way consistent with retaining all of the structure of the original generalized Lie algebra. In general, this results in an increase in the dimension of the algebra, but for the specific case of interest here, eqs. (1.2), it is possible to avoid any increase. The details of the construction are discussed in Section 2. The result is to replace eqs. (1.2) by

$$\begin{aligned} [S_\alpha, \ S_\beta] &= -2(\gamma_5\, \gamma_\mu\, C)_{\alpha\beta}\, \tilde{P}^\mu \\ [\tilde{P}^\mu, \ \tilde{P}^\nu] &= 0, \qquad [\tilde{P}^\mu, \ S_\alpha] = 0\,. \end{aligned} \tag{1.5}$$

The commutation relations of S_α and $\tilde{P}_\mu$ with the homogeneous Lorentz generators $M_{\mu\nu}$ are again those appropriate for a Dirac spinor and a vector, but $\tilde{P}_\mu$ is not, strictly speaking, the usual momentum operator. However the hypotheses of O'Raifeartaigh's theorem [7] are still satisfied, as we have a Lie algebra $(S_\alpha, \tilde{P}_\mu, M_{\mu\nu})$ containing an algebra isomorphic to the Poincaré algebra as a subalgebra, and one may enquire how the resulting algebra eqs. (1.5) relates to his conclusions. In these conclusions O'Raifeartaigh classified the possibilities for Lie algebras containing the Poincaré algebra into four cases. This algebra falls into case (iii), that of a semidirect sum of a semisimple Lie algebra (which in this case is just the algebra M of the homogeneous Lorentz group) and a solvable non-Abelian algebra (which here is the algebra of the Q_α and the $\tilde{P}_\mu$). O'Raifeartaigh argued that case (iii) was unlikely to be interesting mainly because hermitian conjugation could not be defined in the usual way, and that such algebras were unfamiliar in physical applications. In section 3 we will discuss further why the first of these reasons is not a hinderance in supersymmetries, showing that they have finite dimensional representations that respect generalized inner products.

The theorem of Coleman and Mandula [8] is more stringent than that of O'Raifeartaigh. Once its hypotheses are satisfied no possibility is left except that the algebra be the direct sum of the Poincaré algebra and a semisimple Lie algebra. It might not be reasonable to apply the theorem to the group generated by $(S_\alpha, \tilde{P}_\mu, M_{\mu\nu})$ since the subgroup generated by $(\tilde{P}_\mu, M_{\mu\nu})$ does not have the physical significance of the Poincaré algebra in the theories recently constructed; it is merely isomorphic to it. However we could consider group generated by $(S_\alpha, \tilde{P}_\mu, P_\mu, M_{\mu\nu})$, where P_μ genuinely are the translation generators. P_μ commutes with $\tilde{P}_\nu$ and Q_α and has the usual commutation relations with the Lorentz generators $M_{\mu\nu}$. The problem with this group is that it takes us out of the space of physical states. S_α and $\tilde{P}_\mu$ have the effect of taking physical states to states which have been multiplied by anticommuting numbers and so are no longer 'physical'.

One could try to enlarge the space of states, for the purpose of applying this theorem to include all those obtained by the action of the group on physical states. However then the 'inner-product' respected by the group action is not positive definite. This positive definiteness is crucial in the proof of Coleman and Mandula. (See ref. 8, proof of lemma 6.) Thus, however considered, the applications of supersymmetry groups fail to meet the hypotheses of this theorem. Information is here being obtained about physical states by symmetries realized in a larger space. This is not an unusual situation in quantum physics (cf. isotopic spin, gauge invariance in Yang-Mills theories, etc.), but the precise way it is done here is somewhat novel.

The relation to O'Raifeartaigh's theorem is discussed somewhat further in section 2, which is devoted to constructing the Lie algebra for the supersymmetry group of Wess and Zumino [4] and understanding its use, with reference to the theorems mentioned above. Section 3 discusses the construction of inner products respected by the action of the group, and the hermitian conjugation of operators involving the anticommuting parameters. Section 4 deals briefly with constructions for a general algebra of the type of eqs. (1.1). Section 5 contains some concluding comments.

2. THE LIE ALGEBRA OF A SUPERSYMMETRY GROUP

The supersymmetry transformations introduced by Wess and Zumino [4] may be regarded as acting on an eight-dimensional space with coordinates [10] (x_μ, θ_α). (The labels μ and α are Lorentz vector and Dirac spinor indices respectively and refer to the action of the homogeneous Lorentz group on this space.) The coordinates x_μ, θ_α are not ordinary complex numbers, but rather the elements of a Grassmann algebra. This algebra is constructed as follows. (Cf. the discussion of representations of generalized Lie groups given in [11].) Introduce N anticommuting elements $\omega_1, \omega_2, \cdots, \omega_N$:

$$\{\omega_i, \omega_j\} = 0 . \tag{2.1}$$

With these we can construct 2^N independent products, 1, ω_i, $\omega_i \omega_j$ $(i < j)$, $\cdots$, $\omega_1 \omega_2 \cdots \omega_N$, and take linear combinations of these with complex numbers. In this way we have defined an algebra, Ω, say, over the complex numbers in which we can add, multiply two elements, and multiply by complex numbers. We can define a 'parity' operation on this algebra by sending $\omega_i \to -\omega_i$ and so on consistently with multiplication. Only the linear combinations of products of even numbers of the ω_i are left invariant under this operation. These form a linear subspace $\Omega^{(+)}$. Linear combinations of products of odd numbers of the ω_i change sign and form a linear subspace $\Omega^{(-)}$. Clearly

$$\Omega = \Omega^{(+)} + \Omega^{(-)} . \tag{2.2}$$

The x_μ are elements of $\Omega^{(+)}$; the θ_α are elements of $\Omega^{(-)}$. Clearly the division (2.2) gives Ω a graded structure similar to that of the algebras defined by eqs. (1.1), (1.2) and (1.5) in that

$$\Omega^{(-)}\,\Omega^{(-)} \subseteq \Omega^{(+)}; \quad \Omega^{(-)}\,\Omega^{(+)} \subseteq \Omega^{(-)};$$
$$\Omega^{(+)}\,\Omega^{(+)} \subseteq \Omega^{(+)} . \tag{2.3}$$

In order to have reality conditions we define an operation of complex conjugation on Ω. Under this operation complex numbers z go to their ordinary complex conjugates z^*, and products of elements in Ω satisfy

$$(\zeta_1 \zeta_2)^* = \zeta_2^* \zeta_1^*, \qquad \zeta_i \in \Omega . \tag{2.4}$$

Then to determine the operation we need only specify ω_i^*. The simplest prescription is $\omega_i^* = \omega_i$; this means we have chosen a real basis for Ω as an algebra. However for notational purposes below it will be convenient to allow $\omega_i^* = A_{ij}\,\omega_j$ where A is a fixed matrix satisfying $A^* = A^{-1}$.

We divide the ω_i into two sets, the first m of them and their products are to span the parameter space Ω_0. The remaining $n = N - m$ of them and their products span a space Ω_1.

The infinitesimal element of the group corresponding to $\bar{\epsilon}Q$ has an action on the coordinate space given by

$$x_\mu \to x_\mu + i\,\bar{\epsilon}\,\gamma_\mu\,\theta$$
$$\theta_\alpha \to \theta_\alpha + \epsilon_\alpha . \tag{2.5}$$

This map preserves $x_\mu \in \Omega^{(+)}$, $\theta_\alpha \in \Omega^{(-)}$ if $\epsilon_\alpha \in \Omega_0^{(-)}$ and further will preserve reality conditions on x_μ and θ_α

$$x_\mu = x_\mu^* \quad \text{and} \quad \theta^C \equiv C\,\gamma_0^T\,\theta^* = \theta \tag{2.6}$$

provided that ϵ satisfies a similar Majorana condition: $\epsilon^C = \epsilon$.

To discuss the corresponding Lie group structure we must specify m. We must force m to be at least 2; otherwise every parameter ϵ_α will be proportional to ω_1 and so any two infinitesimal elements $\bar{\epsilon}Q$, $\bar{\epsilon}'Q$ will commute: $[\bar{\epsilon}Q, \bar{\epsilon}'Q] = 0$. This will result in the Q algebra losing its structure and becoming Abelian. If we take $m = 2$, varying the real degrees of freedom contained in ϵ would seem to single out the products $\omega_i\, Q_\alpha$, $i = 1,2$, implying that we would have to consider two four-component spinor operators rather than one. We can take advantage of the special structure of eqs. (1.2) to avoid this doubling. It is easily seen from eqs. (1.2) and the relation $\gamma_5 C = C\,\gamma_5^T$ that the two independent components of $(1 + \gamma_5)Q$ anticommute. (We are taking $\gamma_5^2 = 1$.) Thus we will lose no structure if we multiply the components of $(1 + \gamma_5)Q$ by the same anticommuting element ω_1 . Similarly the two independent components of $(1 - \gamma_5)Q$ anticommute and may be considered to be multiplied always by ω_2. Because of Majorana condition and the fact that

$$\Big((1 \pm \gamma_5)\epsilon\Big)^C = (1 \mp \gamma_5)\epsilon^C \tag{2.7}$$

we need ω_1^* to be proportional to ω_2 . So define complex conjugation so that $\omega_1^* = \omega_2$, $\omega_2^* = \omega_1$. We then place the condition on the parameters ϵ that they are of the form

$$\epsilon = \omega\,\xi \tag{2.8}$$

where the matrix ω is defined by

$$\omega = \frac{1}{2}(1+\gamma_5)\omega_1 + \frac{1}{2}(1-\gamma_5)\omega_2 , \tag{2.9}$$

and the ξ_α are ordinary complex numbers. Because

$$\gamma_0{}^T \omega^* = \omega^T \gamma_0{}^T \quad \text{and} \quad C\,\omega^T = \omega C , \tag{2.10}$$

following from eq. (2.9), the Majorana condition ϵ is equivalent to the same condition on ξ. If we define

$$S = \omega\, Q \tag{2.11}$$

$$\bar{\epsilon} Q = \bar{\xi}\, \omega\, Q = \bar{\xi} S \tag{2.12}$$

and we may regard $\bar{\xi}S$ as replacing $\bar{\epsilon}Q$. We can calculate the algebra of the S_α from eqs. (1.2), using $\{\epsilon_\alpha, Q_\beta\} = 0$,

$$\begin{aligned} [S_\alpha, S_\beta] &= \omega_{\alpha\gamma} \{Q_\gamma, Q_\delta\}\, \omega_{\beta\delta} \\ &= -2(\omega\, \gamma^\mu\, C\, \omega^T)_{\alpha\beta}\, P_\mu \\ &= -2(\gamma_5\, \gamma^\mu\, C)_{\alpha\beta}\, \tilde{P}_\mu \end{aligned} \tag{2.13}$$

where $\tilde{P}_\mu = \omega_1 \omega_2\, P_\mu$.

Thus we have an algebra consisting of S_α, $\tilde{P}_\mu$, P_μ, $M_{\mu\nu}$ and it is an ordinary Lie algebra. The subalgebras consisting of $(\tilde{P}_\mu, M_{\mu\nu})$ and $(P_\mu, M_{\mu\nu})$ are isomorphic to the Poincaré algebra. The closed algebra consisting of $(S_\alpha, M_{\mu\nu}, \tilde{P}_\mu)$ is particularly interesting because it contains the Poincaré algebra $(M_{\mu\nu}, \tilde{P}_\mu)$ in a nontrivial way. As we discussed in the Introduction, O'Raifeartaigh's theorem limits the ways in which this can be done. O'Raifeartaigh started from Levi's theorem which states that every Lie algebra E can be written as the semidirect sum of a semisimple Lie algebra and a solvable algebra which we will call Λ and Σ respectively:

$$E = \Lambda + \Sigma . \tag{2.14}$$

(To define a semidirect sum replace the anticommutator in eqs. (1.1) by a commutator; the algebra is then the semidirect sum $L + G$. The algebra Σ being solvable means that if we define inductively $\Sigma^{(1)} = \Sigma$, $\Sigma^{(n)} = [\Sigma^{(n-1)}, \Sigma^{(n-1)}]$ then $\Sigma^{(n)} = 0$ for some n.) The Poincaré algebra is $\mathcal{G}$ of the form of a semidirect sum $\mathcal{G} = M + P$ where M is the algebra of the homogeneous Lorentz group and P the translation algebra. O'Raifeartaigh showed that if $M + P \subset E = \Lambda + \Sigma$ one could by redefinition take $M \subset \Lambda$ and then either $P \subset \Sigma$ or $P \cap \Sigma = 0$. The latter possibility involves embedding $\mathcal{G}$ as a subalgebra of a simple algebra. The former possibility was split into three cases. In case (i) $P = \Sigma$ and then E is the direct sum of $\mathcal{G}$ and a semisimple algebra. In case (ii) Σ was Abelian but larger than P. Case (iii) is the one realized here. In the present example $\Sigma = (S_\alpha, \tilde{P}_\mu)$ is solvable but non-Abelian. Actually it satisfies the stronger condition of being nilpotent (that is, if we define $\Sigma_1 = \Sigma$, $\Sigma_n = [\Sigma, \Sigma_{n-1}]$, then $\Sigma_n = 0$ for some n).

The main reason for regarding this possibility as uninteresting before was the difficulty in defining hermitian conjugation for the representations and this will be discussed for the case at hand in the next section.

The representations of the supersymmetry algebra of eqs. (1.1) have been found using superfields [10]. These are functions $\phi(x, \theta)$ defined on the coordinate space taking values in the algebra Ω. They may be taken to have prescribed Lorentz transformation properties and the simples possibility to consider is a scalar superfield. The action of an infinitesimal element of the group $\overline{\epsilon}Q = \overline{\xi}S$ is given by

$$\phi(x_\mu, \theta_\alpha) \rightarrow \phi(x_\mu - i\,\overline{\epsilon}\,\gamma_\mu\,\theta, \;\; \theta_\alpha - \epsilon_\alpha) . \tag{2.15}$$

In order to obtain a representation in which $P_\mu = -i\,\partial/\partial x^\mu$ we need to take n to be at least four. We may write

$$\begin{aligned}\phi(x_\mu - i\,\overline{\epsilon}\,\gamma_\mu\theta, \;\; \theta_\alpha - \epsilon_\alpha) &= \phi(x_\mu, \theta_\alpha) - i\overline{\epsilon}\,\gamma_\lambda\theta\,\partial^\lambda\,\phi(x_\mu, \theta_\alpha) \\ &\quad - \epsilon_\alpha\,\delta_\alpha\,\phi(x_\mu, \theta_\alpha) + O(\epsilon^2)\end{aligned} \tag{2.16}$$

where $\partial^\lambda \equiv \partial/\partial x_\lambda$ and the differential operator δ is defined on a function of an anticommuting variable χ, $\phi(\chi) = \phi_0 + \chi\phi_1$ by $\delta\phi(\chi) = \phi_1$. The corresponding operator for θ_α is denoted by δ_α. Before evaluating the action of δ_α all quantities $\overline{\theta}$ must be replaced by $C^{-1}\theta$. Then we may verify directly from this definition

$$\{\delta_\alpha, \;\; \delta_\beta\} = 0 \qquad\qquad \{\delta_\alpha, \;\; \theta_\beta\} = \delta_{\alpha\beta} . \tag{2.17}$$

In this way we see that $\overline{\xi}S$ is represented by theoperator

$$\overline{\epsilon}\,C\,\delta \; - \; i\,\overline{\epsilon}\,\gamma_\mu\,\theta\,\partial^\mu = \overline{\xi}\,\omega(C\delta - i\,\gamma_\mu\,\theta\,\partial^\mu) \tag{2.18}$$

or

$$S_\alpha = \Big(\omega[C\delta - i\,\gamma_\mu\,\theta\,\partial^\mu]\Big)_\alpha . \tag{2.19}$$

Following Salam and Strathdee [10] further we may perform a decomposition of the superfield $\phi(x, \theta)$ into component fields by expanding in θ

$$\phi(x, \theta) = A(x) + \bar{\theta}\,\psi(x) + \frac{1}{4}\{\bar{\theta}\,\theta\, F(x) + i\,\bar{\theta}\,\gamma_5\,\theta G(x) + \frac{1}{4}\,\bar{\theta}\,\gamma_5\gamma_\mu\theta\, A^\mu(x)\} + \frac{1}{4}\,(\bar{\theta}\theta)\,\bar{\theta}\,\chi(x) + \frac{1}{32}\,(\bar{\theta}\theta)^2\, D(x)\ . \tag{2.20}$$

The action of the group on the component fields A, ψ, etc., still involves the anticommuting quantities ω_1, ω_2; e.g.

$$\delta A = \bar{\xi}\,\omega\,\psi\ . \tag{2.21}$$

This means that even if we remove the variables θ by the expansion of eq. (2.20) the fields A, ψ, etc., still contain anticommuting elements. Thus the realization of the group generated by S_α is in a space where the coordinates are elements of the algebra $\mathfrak{L}_0$. We use the fields to create a Fock apace, $\mathcal{F}$, in the usual way but take combinations of the states using elements of $\mathfrak{L}_0$ instead of complex numbers as coefficients. The physical states are those with complex numbers as coordinates only, because it is only such states which will give complex numbers for probability amplitudes. We can define an 'inner product' on the whole of the Fock space in the usual way starting from the orthonormal basis of states and using the complex conjugation operation on $\mathfrak{L}_0$. This inner product will take values in $\mathfrak{L}_0$, but will reduce to the usual inner product on the physical states.

S_α satisfies the hermiticity condition

$$\langle\, \Psi_1,\ \bar{\xi}\, S\, \Psi_2 \,\rangle = \langle\, \bar{\xi}\, S\, \Psi_1,\ \Psi_2 \,\rangle\ . \tag{2.22}$$

The action of the group is thus unitary in the sense of preserving this generalized inner product. The theorem of Coleman and Mandula [8] is not applicable because we have a representation in something more general than a Hilbert space. It has an inner product satisfying the hermiticity condition

$$\langle \Psi_1, \Psi_2 \rangle = \langle \Psi_2, \Psi_1 \rangle^* \tag{2.23}$$

but this product takes values in Ω_0. The physical states do form a conventional Hilbert space but this is not invariant under the action of the group.

In the next section we will discuss a method of defining a complex valued inner product on $\mathcal{H}$ and regarding it as an ordinary Hilbert space by expanding each state in terms of ω_1, ω_2. This will result in a quadrupling of the states and we will find that the appropriate inner product is not positive definite.

3. CONSTRUCTION OF INNER PRODUCTS

To construct inner products on the space of functions $\phi(x, \theta)$ we use the technique of integrating over the anticommuting variables which has previously been used in similar contexts by Berezin and Kac [11], and by Montonen [12]. The integral of such a function is to be thought of as a linear functional on the space of functions, attaching to a function a quantity as follows. If $\phi(\chi)$ is a function of the single anticommuting variable χ, $\phi(\chi) = \phi_0 + \chi\phi_1$, we define the integral

$$\int [\phi] d\chi = \phi_1 . \tag{3.1}$$

ϕ_1 may contain other anticommuting quantities. For a function of several anticommuting quantities $\chi_1, \chi_2, \cdots, \chi_n$, we define multiple integration with respect to them by repeated application of this rule. Thus only the coefficient of $\chi_1, \chi_2, \cdots, \chi_n$ survives.

Suppose complex conjugation is defined so that the χ_i are real: $\chi_i^* = \chi_i$. We may define a bilinear form on the functions $\phi_{(r)}(\chi)$ by

$$\langle\phi_{(1)}, \phi_{(2)}\rangle = \int [\phi^*_{(1)} \phi_{(2)}] d\chi_1 \cdots d\chi_n . \tag{3.2}$$

(For even n. For odd we replace $\phi^*_{(1)}$ by $\tilde{\phi}^*_{(1)}$ where $\tilde{\phi}_{(1)}$ is obtained from $\phi_{(1)}$ by applying the parity operation to all the coefficients if these are not just complex numbers.) The inner product satisfies the hermiticity condition

$$\langle\phi_{(1)}, \phi_{(2)}\rangle^* = \eta\langle\phi_{(2)}, \phi_{(1)}\rangle \tag{3.3}$$

where

$$\eta = (-1)^{\frac{1}{2}n(n-1)} .$$

Thus for $n = 2, 3, 6, 7, \cdots$ we have a symplectic inner product, while for $n = 1, 4, 5, \cdots$ it is truly hermitian. Even in this latter case and when the coefficients in the $\phi_{(i)}$ are complex numbers the resulting inner product is not positive definite for any n. Because they are real, the χ_i are hermitian with respect to the inner product:

$$\langle\chi_i \phi_{(1)}, \phi_{(2)}\rangle = \langle\phi_{(1)}, \chi_i \phi_{(2)}\rangle . \tag{3.4}$$

Further, the differential operator δ_i corresponding to X_i is easily shown to be hermitian $\delta_i^\dagger = \delta_i$. Note that it has the important property

$$\int [\delta_i \, \phi] dX_1 \cdots dX_n = 0 . \qquad (3.5)$$

For the specific case of the superfield $\phi(x, \theta)$ we integrate with respect to the four real degrees of freedom contained in the Majorana spinor θ and further integrate with respect to the four real variables x_μ to obtain an inner product taking values in Ω_0.

$$\langle \phi_{(1)}, \phi_{(2)} \rangle = \int [\phi^*_{(1)} \, \phi_{(2)}] d^4\theta_\alpha \, d^4x_\mu . \qquad (3.6)$$

For the θ_α the operation of hermitian conjugation coincides with complex conjugation and the δ_α also satisfy

$$\delta^C \equiv C \gamma_0^T \delta^\dagger = \delta . \qquad (3.7)$$

So with respect to this inner product the S_α satisfy the Majorana condition and the group action is unitary. Note however that we have had to generalize this concept to apply to a space with an inner product taking values in Ω_0 and even then it is not "positive definite" in any sense.

We could get a conventional inner product by integrating out the dependence on ω_1 and ω_2 but this would be antihermitian rather than hermitian. Thus we can not make the Fock space $\mathcal{F}$ of the last section into a conventional Hilbert space in this way.

By considering the subspace of fixed momentum we get a representation of the group which is finite dimensional and unitary in this generalized sense.

4. GENERAL SUPERSYMMETRY ALGEBRAS

For a general algebra of the form of eqs. (1.1) we can always construct a corresponding Lie algebra by introducing two anticommuting elements, ω_1 and ω_2. Define $G_{ra} = \omega_a G_r$ and $\tilde{L}_i = \omega_1 \omega_2 L_i$. Then L_i, G_{ra}, $\tilde{L}_j$ close to form a Lie algebra of twice the size.

$$[L_i, L_j] = c_{ij}{}^k L_k \qquad [\tilde{L}_i, L_j] = c_{ij}{}^k \tilde{L}_k$$

$$[L_i, G_{ra}] = f_{ir}{}^s G_{sa}$$

$$[G_{ra}, G_{sb}] = \epsilon_{ab} d_{rs}{}^k \tilde{L}_k \tag{4.1}$$

and $[\tilde{L}_i, G_{ra}] = [\tilde{L}_i, \tilde{L}_j] = 0$. Here $\epsilon_{12} = -\epsilon_{21} = 1$; $\epsilon_{11} = \epsilon_{22} = 0$.

The Jacobi identities for the original supersymmetry algebra will imply them for this ordinary Lie algebra. If we had introduced more anticommuting quantities the resulting algebra would have a more complicated structure of grading. Clearly one can introduce concepts of solvability and nilpotency for supersymmetry algebras. The associated Lie algebras will possess the corresponding properties when the supersymmetry algebras do. The converse is not true; it is possible for (4.1) to be nilpotent while (1.1) is not. However the specific case discussed in section 2 is not of this type; both algebras (1.2) and (1.5) are nilpotent.

5. COMMENTS

We have discussed how supersymmetry groups and algebras may be replaced by conventional Lie groups and algebras. In general, as we saw in section 4, this results in a loss of economy; we need a larger algebra to contain the same structure. However for the algebra of

Wess and Zumino this is not the case: the Lie algebra has the same dimension as the original supersymmetry algebra.

Rephrasing the symmetry operations in terms of Lie groups has the advantage that the language is more familiar, even if it needs a larger algebra in general. For the specific case discussed it clarifies the relation of supersymmetric theories to the theorems limiting the way internal and space-time symmetries may be combined. It is seen that rather than avoiding O'Raifeartaigh's theorem [7] it exploits one of the possibilities it admits: the use of solvable non-Abelian subgroups. It might be thought that the solvability or, in this case, the nilpotency, "results" from insisting on writing the algebra as an ordinary Lie algebra. It is clear that this is in no sense the case because the ordinary supersymmetry algebra is nilpotent.

Such groups were previously discarded because of the difficulty of construcing useful finite dimensional representations. The other novel feature of supersymmetries is that they avoid this problem by using representations in an extended Hilbert space, $\mathfrak{H}$, with coefficients in a Grassmann algebra Ω_0. Again this feature does not result from rewriting the representations in Lie group terms, but is inherent in the use of the original generalized Lie algebra. The inner product takes values in Ω_0. We can regard this space as an ordinary complex space by regarding Ψ, $\omega_1\Psi$, $\omega_2\Psi$ and $\omega_1\omega_2\Psi$ as independent. This gives a quadrupling of the dimension. As we discussed in section 3 a nontrivial complex inner product can be defined but it is not positive definite. The physical states form a subspace of $\mathfrak{H}$ regarded as a complex vector space but not in the sense that we can take combinations of them with elements of Ω_0. We can regard $\mathfrak{H}$ as a complex vector space with representations of both the symmetry group and the algebra $\mathcal{S}_0$ defined on it in such a way that any representation matrix of the group commutes with any element of the algebra. However it seems more natural to regard the structure associated with $\mathcal{S}_0$ as intrinsic to $\mathcal{H}$, that is to regard $\mathcal{H}$ as a sort of vector space with coordinates in $\mathcal{S}_0$. Such an object is called an $\mathcal{S}_0$-module. (The concept of a module is essentially the same as that of a representation space.) We then have a representation of the group in a $\mathcal{S}_0$-module (such that the action of the group commutes with the module structure). Thus the physical states from a complex vector subspace but not a submodule. It is the fact that the physical states are not left invariant by the group that avoids the theorem of Coleman and Mandula [8].

The idea of considering the physical states to be a subspace of the representation space (that is of using a subsidiary condition) has been discussed before in the context of combining space and internal symmetries. Typically there are difficulties with unitarity [13]. With supersymmetries these are avoided because the physical unitarity is a vestige of unitarity with respect to the generalized inner product on (the $\mathcal{S}_0$-module) $\mathcal{H}$ rather than unitarity with respect to an ordinary complex inner product on a larger space.

The purpose of the present discussion has not been to arrive at a more elegant description of supersymmetries, but a more familiar one. Rephrasing their use in terms of Lie groups has the advantage of making clear their relation to previous theorems on Lie algebra and Lie groups. It shows that solvable groups may have applications in particle physics, at least if suitably represented.

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