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Journal
The Annals of Probability, 14(4)

ISSN
0091-1798

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Publication Date
1986-10-01

DOI
10.1214/aop/1176992375

Peer reviewed

The Annals of Probability
1986, Vol. 14, No. 4, 1347–1358

LIMIT THEOREMS FOR A TRIANGULAR SCHEME OF *U*-STATISTICS WITH APPLICATIONS TO INTER-POINT DISTANCES

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The asymptotic distribution of a "triangular" scheme of U-statistics is studied. Two limit theorems, applicable in different situations, are given. One theorem yields convergence to a normal distribution; the other includes Poisson limits and other limit laws. Applications to statistics based on small interpoint distances in a sample are given.

1. Introduction. There has been a renewed interest in the limit theory relating to U-statistics (Hoeffding, 1948) especially focussing on degenerate kernels and nonnormal limits. See, for instance, Rubin and Vitale (1980), Dynkin and Mandelbaum (1983), and Berman and Eagleson (1983). In Section 2 of this paper we study a "triangular" scheme of U-statistics and establish their asymptotic normality under suitable conditions. It is worth noting that one can obtain normal limit laws even with degenerate kernels. Other infinitely divisible limit laws for the triangular scheme, including the Poisson limits, are examined in the next section. Applications to limit distributions of statistics based on interpoint distances are discussed in Section 4. These applications, which were the source of motivation for the results derived here, were suggested by the studies one of us made about statistics based on spacings. [See, for instance, Holst and Rao (1981).] They were also sparked by the recent papers of Bickel and Breiman (1983) and Onoyama et al. (1983). Theorems yielding asymptotic normality in this setting have also been proved by Weber (1983) using a martingale approach. His theorems yield the same conclusion as Theorem 2.1, but under different (nonequivalent) conditions.

A few words about notation: We write "$\to_d$" to denote convergence in distribution and "$\to_c$" to denote complete convergence [cf. Loève (1963), page 178]. We write $\mathrm{Po}(\lambda)$ to denote the Poisson distribution with mean λ and $N_p(\mu, \Sigma)$ to denote a p-dimensional normal distribution with mean vector μ and covariance matrix Σ.

2. Asymptotic normality of a triangular scheme of U-statistics. Let $X_1, X_2, \ldots$ be a sequence of independently and identically distributed (i.i.d.) random variables. We will use X and Y to denote two independent random

Received October 1984; revised April 1985.

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AMS 1980 *subject classifications*. Primary 60F05, 60E07; secondary 62E20.

Key words and phrases. U-statistics, triangular scheme, multigraph, limit laws, infinitely divisible distributions, inter-point distances.

variables with this common distribution. Let further, for each $n = 2, 3, \dots$ $f_n(x, y)$ be a measurable symmetric function of two variables. We will study the "triangular scheme" of U-statistics defined by

$$U_n = \sum_{1 \le i < j \le n} f_n(X_i, X_j) = \tfrac{1}{2} \sum_{i \ne j} f_n(X_i, X_j). \tag{2.1}$$

We will only consider bounded functions $\{f_n\}$, although the theorems can be extended to functions satisfying certain integrability conditions (see Remark 2.4).

We divide f_n into four parts by defining

$$\begin{aligned} \mu_n &= Ef_n(X, Y), \\ g_n(x) &= Ef_n(x, Y) - \mu_n, \\ h_n(x, y) &= f_n(x, y) - g_n(x) - g_n(y) - \mu_n. \end{aligned} \tag{2.2}$$

Note that

$$Eg_n(X) = 0 = Eh_n(x, Y) = Eh_n(X, y) \tag{2.3}$$

and that h_n is symmetric.

We further write

$$V_n = \sum_1^n g_n(X_i) \quad \text{and} \quad W_n = \sum_{1 \le i < j \le n} h_n(X_i, X_j). \tag{2.4}$$

Thus

$$\begin{aligned} U_n &= \sum_{i<j} \left(h_n(X_i, X_j) + g_n(X_i) + g_n(X_j) + \mu_n\right) \\ &= W_n + (n-1)V_n + \binom{n}{2}\mu_n. \end{aligned} \tag{2.5}$$

It is easily seen, using (2.3), that

$$\begin{aligned} EV_n^2 &= nE(g_n(X))^2, \\ EW_n^2 &= \binom{n}{2}E(h_n(X, Y))^2, \\ EV_nW_n &= 0 \end{aligned} \tag{2.6}$$

and thus

$$\operatorname{Var}(U_n) = \frac{n(n-1)}{2} Eh_n^2 + n(n-1)^2 Eg_n^2 \sim \sigma_n^2, \tag{2.7}$$

where we put

$$\sigma_n^2 = \tfrac{1}{2}n^2(Ef_n^2 - \mu_n^2) + n^3Eg_n^2 = \tfrac{1}{2}n^2Eh_n^2 + (n^3 + n^2)Eg_n^2. \tag{2.8}$$

We assume that $\sigma_n > 0$, i.e., f_n is not constant.

THEOREM 2.1. *With the above notation, suppose that as $n \to \infty$*

$$\text{(2.9)}\qquad \text{(i)}\quad \sup_{x,y} |f_n(x, y)| = o(\sigma_n), \qquad \text{(ii)}\quad \sup_{x} E|f_n(x, Y)| = o(\sigma_n/n).$$

Then

$$\text{(2.10)}\qquad \frac{U_n - \binom{n}{2}\mu_n}{\sigma_n} \to_d N(0,1).$$

PROOF. Replacing f_n by $(f_n - \mu_n)/\sigma_n$, we may assume without any loss of generality that $\mu_n = 0$ and $\sigma_n = 1$. In this case

$$\text{(2.11)}\qquad \tfrac{1}{2}n^2 Eh_n^2 + n^3 Eg_n^2 \to 1,$$

whence for any subsequence of $\{U_n\}$, we may select a further subsequence such that $n^3 Eg_n^2$ converges, say to α^2, where $0 \le \alpha^2 \le 1$. Then $\{n^2 Eh_n^2\}$ converges along this latter subsequence, to $\beta^2 = 2 - 2\alpha^2$. Further, $\sup|h_n(x, y)| \le 4\sup|f_n(x, y)| = o(1)$, $|g_n(x)| \le 2E|f_n(x, Y)| = o(1/n)$, and $\sup E|h_n(x, Y)| \le 4\sup E|f_n(x, Y)| = o(1/n)$. Hence, Theorem 2.2 which follows, applies to the sequences $\{(n-1)g_n\}$ and $\{h_n\}$, whence

$$U_n = (n-1)V_n + W_n \to_d N(0, \alpha^2 + \beta^2/2) = N(0,1)$$

along this latter subsequence. Hence $U_n \to N(0,1)$. □

THEOREM 2.2. *Suppose that $h_n(X, Y)$ is a symmetric function and that as $n \to \infty$*

$$\text{(2.12)}\qquad \begin{aligned}
&\text{(i) } Eg_n(X) = Eh_n(x, Y) = 0,\\
&\text{(ii) } \sup_x |g_n(x)| \to 0,\\
&\text{(iii) } nE\big(g_n(X)^2\big) \to \alpha^2, \qquad 0 \le \alpha^2 < \infty,\\
&\text{(iv) } n^2 E\big(h_n(X, Y)^2\big) \to \beta^2, \qquad 0 \le \beta^2 < \infty,\\
&\text{(v) } \sup_{x,y} |h_n(x, y)| \to 0,\\
&\text{(vi) } n \sup_x E|h_n(x, Y)| \to 0.
\end{aligned}$$

Then, with V_n and W_n defined in (2.4),

$$\text{(2.13)}\qquad (V_n, W_n) \to_d N\left(0, \begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2/2 \end{pmatrix}\right).$$

PROOF. We will use the method of moments and show that $E(V_n^l W_n^m)$ converges to the corresponding product of moments of $N(0, \alpha^2)$ and $N(0, \beta^2/2)$ for every $l, m \ge 0$.

We employ the language of graph theory: A weighted multigraph consists of a set $\{v_i\}$ of vertices, a number $e_{ij} \geq 0$ of (undirected) edges between each pair (v_i, v_j) of vertices $(i \neq j)$, and an integer $w_i \geq 0$ for each vertex. If Γ is a weighted multigraph, then $e(\Gamma)$ denotes $\sum_{i<j} e_{ij}$, the total number of edges, $v(\Gamma)$ denotes the number of vertices v_i such that either $w_i \neq 0$ or $e_{ij} \neq 0$ for some j, $w(\Gamma)$ denotes $\sum w_i$, and $\Gamma!$ denotes $\prod_{i<j} e_{ij}! \prod_i w_i!$.

Let G_N be the set of all weighted multigraphs with $\{1, \dots, N\}$ as the set of vertices, and let $G_{N, v, e, w} = \{\Gamma \in G_N\colon v(\Gamma) = v,\ e(\Gamma) = e \text{ and } w(\Gamma) = w\}$.

Finally we define, for $\Gamma \in G_n$,

$$Z_n(\Gamma) = \prod_1^n g_n(X_i)^{w_i} \prod_{i<j\leq n} h_n(X_i, X_j)^{e_{ij}}. \tag{2.14}$$

Hence

$$\begin{aligned} E(V_n^l W_n^m) &= \sum_{\substack{i_p < j_p \leq n \\ k_q \leq n}} E\left(\prod_{p=1}^m h_n(X_{i_p}, X_{j_p}) \prod_{q=1}^l g_n(X_{k_q}) \right) \\ &= \sum_{v=1}^n \sum_{G_{n,v,m,l}} \frac{m!\,l!}{\Gamma!} EZ_n(\Gamma). \end{aligned} \tag{2.15}$$

By symmetry, we obtain

$$E(V_n^l W_n^m) = \sum_v \binom{n}{v} \sum_{G_{v,v,m,l}} \frac{m!\,l!}{\Gamma!} EZ_n(\Gamma). \tag{2.16}$$

We will study convergence of $\binom{n}{v} EZ_n(\Gamma)$ as $n \to \infty$, for every Γ, thus obtaining the convergence of $E(V_n^l W_m^m)$. First we treat the case where Γ is connected.

LEMMA 2.3. *If Γ is connected and either*

(a) $e_{ij} \geq 2$ *for some i and j,*
(b) Γ *contains a cycle $i_0, i_1, \dots, i_k = i_0$ with $e_{i_j, i_{j+1}} \geq 1$, $k \geq 3$, or*
(c) $w(\Gamma) \geq 2$,

then $E|Z_n(\Gamma)| = O(n^{-v(\Gamma)})$ and $E|Z_n(\Gamma)| = o(n^{-v(\Gamma)})$ unless $v(\Gamma) = 1$, $w(\Gamma) = 2$, or $v(\Gamma) = 2$, $e(\Gamma) = 2$, and $w(\Gamma) = 0$. (See Figure 1.)

PROOF OF LEMMA 2.3. First we show that if $v(\Gamma) > 1$ and Γ' is the graph obtained from Γ by deleting one vertex and all its adjoining edges from Γ, then

$$E|Z_n(\Gamma)| = o\big(n^{-1} E|Z_n(\Gamma')|\big). \tag{2.17}$$

Let i be the removed vertex and let Γ'' be Γ' with one edge connecting i to the remainder of Γ restored, say the edge from i to j.

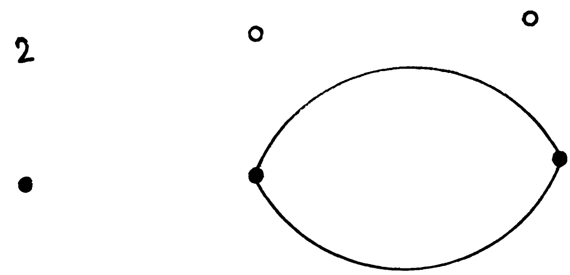

FIG. 1. *The two special connected graphs.*

If we assume, as we may, that $\sup|g_n|, \sup|h_n| \le 1$, then $|Z_n(\Gamma)| \le |Z_n(\Gamma'')|$. Further, $Z_n(\Gamma'') = Z_n(\Gamma') \cdot h_n(X_j, X_i)$, and since $Z_n(\Gamma')$ is independent of X_i,

$$E|Z_n(\Gamma'')| = E\big(|Z_n(\Gamma')|E_{X_i}|h_n(X_j, X_i)|\big) \le E|Z_n(\Gamma')|\sup_x E|h_n(x, Y)|. \tag{2.18}$$

Hence (2.17) follows by the assumption (2.12)(vi).

CASE a. By (2.17) we may delete the vertices one by one until only two remain. It is always possible to do this while keeping the remainder connected. In this case

$$\begin{aligned} |Z_n(\Gamma)| &= |h_n(X_i, X_j)|^{e_{ij}}|g_n(X_i)|^{w_i}|g_n(X_j)|^{w_j} \\ &\le |h_n(X_i, X_j)|^2(\sup|h_n|)^{e_{ij}-2}\sup|g_n|^{w_i+w_j}. \end{aligned} \tag{2.19}$$

Hence $E|Z_n(\Gamma)| = O(n^{-2})$ and $o(n^{-2})$ unless $e_{ij} = 2$ and $w_i = w_j = 0$, by (2.12)(ii), (2.12)(iv), and (2.12)(v). Note that in the latter case, $v(\Gamma) = 2$, $e(\Gamma) = 2$, and $w(\Gamma) = 0$, which corresponds to one of our exceptional cases.

CASE b. Let Γ_1 be the graph obtained from Γ by moving one edge from the pair (i_1, i_2) to the pair (i_2, i_3). Let Γ_2 be the graph obtained by the reverse procedure. Then $Z_n(\Gamma)^2 = Z_n(\Gamma_1)Z_n(\Gamma_2)$, so by Hölder's inequality

$$E|Z_n(\Gamma)| \le \big(E|Z_n(\Gamma_1)|E|Z_n(\Gamma_2)|\big)^{1/2}. \tag{2.20}$$

Since Γ_1 and Γ_2 are connected and covered by Case a, this proves that

$$E|Z_n(\Gamma)| = o(n^{-v(\Gamma)}).$$

CASE c. We study two subcases. If $w_i \ge 2$ for some vertex, we delete all other vertices one by one, using (2.17). Thus it suffices to consider graphs with a single vertex. Then $Z_n(\Gamma) = g_n(X_i)^{w_i}$, and the estimate follows by (2.12)(ii) and (2.12)(iii).

In the second subcase, there exist two vertices i and j with $w_i, w_j \geq 1$. Let Γ_1 and Γ_2 be the graphs obtained from Γ by changing the pair of weights w_i, w_j to $w_i + 1, w_j - 1$ and $w_i - 1, w_j + 1$.

The first subcase applies to Γ_1 and Γ_2 and the proof is completed by Hölder's inequality as in Case b. □

CONCLUSION OF PROOF OF THEOREM 2.2. If Γ is connected and none of the cases of the lemma applies, the graph must be a tree with at most one vertex with a weight $w_i \neq 0$. Either $v(\Gamma) = 1$, in which case $Z_n(\Gamma) = g_n(X_i)$, or Γ contains a vertex i with only one edge, to j say, and $w_i = 0$. In the latter case, let Γ' be Γ with this edge deleted. Then $Z_n(\Gamma) = Z_n(\Gamma')h_n(X_j, X_i)$, and since $Z_n(\Gamma')$ is independent of X_i, $E_{X_i}(Z_n(\Gamma)) = 0$. Thus, in both cases $EZ_n(\Gamma) = 0$.

Returning to a general graph Γ, we separate it into its connected components $\Gamma_1, \dots, \Gamma_k$. Then $Z_n(\Gamma) = \prod_1^k Z_n(\Gamma_i)$ and the terms are independent. Hence $EZ_n(\Gamma) = O(n^{-v(\Gamma)})$ for any graph Γ, and $n^{v(\Gamma)}EZ_n(\Gamma) \to 0$ as $n \to \infty$ unless each component of Γ is either one of the two types of graphs in Figure 1. If Γ consists of p subgraphs of the first type and q of the second type, $v(\Gamma) = p + 2q$, $e(\Gamma) = 2q$, $w(\Gamma) = 2p$, $\Gamma! = 2^{p+q}$, and

$$n^{v(\Gamma)}EZ_n(\Gamma) = \prod_1^p nEg_n\left(X_{i_s}\right)^2 \prod_1^q n^2 Eh_n\left(X_{j_r}, X_{k_r}\right)^2 \tag{2.21}$$
$$\to \alpha^{2p}\beta^{2q} \quad \text{as } n \to \infty.$$

Consequently,

$$\binom{n}{v} \sum_{G_{v,v,m,l}} \frac{m!\,l!}{\Gamma!} EZ_n(\Gamma) \to 0 \quad \text{as } n \to \infty, \tag{2.22}$$

unless m and l are even, $m = 2q$ and $l = 2p$, and $v = p + 2q$, in which case there are $v!/(p!2^q q!)$ significant graphs, each contributing $(1/v!)(m!l!/2^{p+q})\alpha^{2p}\beta^{2q}$ to the limit. Hence, by (2.16),

$$E\left(V_n^l W_n^m\right) \to \frac{m!}{2^p p!} \frac{l!}{2^q q!} \alpha^{2p}\left(\beta^2/2\right)^q \tag{2.23}$$

if $l = 2p$, $m = 2q$ (and 0 if l or m is odd). □

REMARK 2.4. By truncation, in Theorem 2.2 it suffices to assume (i), (iii), (iv), and (vi) of (2.12) and $nE(g_n^2(X)I(|g_n(X)| > \varepsilon)) \to 0$,

$$n^2E\left(h_n^2(X,Y)I\left(|h_n(X,Y)| > \varepsilon\right)\right) \to 0 \quad \text{as } n \to \infty$$

for every $\varepsilon > 0$.

PROOF. There exists a sequence $\varepsilon_n \to 0$ such that $nE(g_n^2(X)I(|g_n(X)| > \varepsilon_n)) \to 0$ and $n^2E(h_n^2 I(|h_n| > \varepsilon_n)) \to 0$. Define $g_n' = g_n I(|g_n| \leq \varepsilon_n)$ and $h_n' = h_n I(|h_n| \leq \varepsilon_n)$ and put $g_n'' = g_n' - Eg_n$, $h_n''(x, y) = h_n'(x, y) - Eh_n'(x, Y) - Eh_n'(X, y) + Eh_n'(X, Y)$. The theorem applies to g_n'' and h_n'', and

$$E\left(\sum (g_n - g_n'')(X_i)\right)^2, \qquad E\left(\sum \left(h_n(X_i, X_j) - h_n''(X_i, X_j)\right)\right)^2 \to 0. \qquad \square$$

3. Poisson and other limits. The following theorem overlaps partly with Theorem 2.1, but it includes also nonnormal limits, in particular Poisson limits. The theorem can be interpreted as saying essentially that U_n behaves as the sum of $\binom{n}{2}$ independent random variables, provided (3.1) is satisfied.

THEOREM 3.1. *Suppose that $E|f_n|$ is finite for every n and that for any three i.i.d. random variables (X, Y, Z) from the sequence,*

$$(3.1)\quad n^3E|f_n(X,Y)f_n(X,Z)| = n^3E\Big(E\big(|f_n(X,Y)|\big|X\big)\Big)^2 \to 0 \quad \text{as } n \to \infty.$$

If α, a_n are real numbers and $d\Psi$ a finite positive measure on R such that, with F_n the distribution function of $f_n(X,Y)$,

$$(3.2)\qquad \tfrac{1}{2}n^2\frac{t^2}{1+t^2}\,dF_n(t) \to_c d\Psi(t)$$

and

$$(3.3)\qquad \tfrac{1}{2}n^2E\frac{f_n(X,Y)}{1+f_n^2(X,Y)} - a_n \to \alpha,$$

then

$$(3.4)\qquad U_n - a_n \to_d V,$$

where V has the infinitely divisible characteristic function

$$(3.5)\qquad \varphi_V(t) = \exp\left(it\alpha + \int\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}\,d\Psi(x)\right).$$

PROOF. We will prove this theorem using the technique of a random sample size. [Silverman and Brown (1978) and Berman and Eagleson (1983) use different methods. Cf. also Dynkin and Mandelbaum (1983) where another application of this method to U-statistics is given.]

Thus, let $N \sim \mathrm{Po}(n)$ be independent of $X_1, X_2, \ldots$ and define

$$(3.6)\qquad U_n' = \sum_{i<j\le N} f_n(X_i, X_j).$$

If $N > n$, then $U_n' - U_n = \sum_{n<j\le N,\, i<j} f_n(X_i, X_j)$, and thus

$$E\big(|U_n' - U_n|\big|N\big) \le N(N-n)E|f_n(X,Y)|.$$

Similarly, if $N < n$, $E(|U_n' - U_n||N) \le n(n-N)E|f_n|$. Hence

$$\begin{aligned} E|U_n' - U_n| &\le E\big((n + |N-n|)|N-n|\big)E|f_n| \\ &= \big(nE|N-n| + E(N-n)^2\big)E|f_n| \\ &\le (n\sqrt{n} + n)E|f_n|. \end{aligned}$$

Since $E|f_n| = EE(|f_n(X,Y)||X) \le (E(E|f_n(X,Y)||X)^2)^{1/2} = o(n^{-3/2})$ by (3.1),

$$(3.7)\qquad E|U_n' - U_n| \to 0.$$

Hence it suffices to prove that $U_n' - a_n \to_d V$.

The random set $\{X_i\}_{i=1}^N$ is a Poisson process with intensity $n\,dP$. Hence we reformulate (and generalize) the problem as follows.

Let (Ω, μ) be a σ-finite measure space and let $f_\lambda(x, y)$ be a symmetric function defined on $\Omega \times \Omega$ for each $\lambda > 0$ (or each λ in some sequence tending to infinity.) Let Ξ be a Poisson process on Ω with intensity $\lambda\, d\mu$. We regard Ξ as a random set $\{\xi_i\}$ (the numbering of the points in Ξ is arbitrary) and define

$$U_\lambda = U_\lambda(\Xi) = \sum_{i<j} f_\lambda(\xi_i, \xi_j). \tag{3.8}$$

(The sum converges absolutely a.s. provided $\int |f_\lambda| < \infty$.)

In view of the above discussion, Theorem 3.1 is a consequence of the following more general theorem.

THEOREM 3.2. *Suppose that* $\int\int |f_\lambda| < \infty$ *for every* λ *and that*

$$\lambda^3 \int\int\int |f_\lambda(x, y) f_\lambda(x, z)|\, d\mu(x)\, d\mu(y)\, d\mu(z) \to 0 \quad \text{as } \lambda \to \infty. \tag{3.9}$$

Define

$$\Psi_\lambda(t) = \tfrac{1}{2}\lambda^2 \int\int \frac{f_\lambda^2}{1 + f_\lambda^2} I(f_\lambda \le t)\, d\mu \times d\mu \tag{3.10}$$

(*thus* $d\Psi_\lambda$ *is a finite positive measure*) *and*

$$\alpha_\lambda = \tfrac{1}{2}\lambda^2 \int \frac{f_\lambda}{1 + f_\lambda^2}\, d\mu \times d\mu. \tag{3.11}$$

If α, a_λ *are real numbers and* $d\Psi$ *a finite positive measure on* R *such that*

$$d\Psi_\lambda \to_c d\Psi \quad \text{and} \quad \alpha_\lambda - a_\lambda \to \alpha, \tag{3.12}$$

then

$$U_\lambda - a_\lambda \to_d V, \tag{3.13}$$

where V *has the infinitely divisible characteristic function* φ_V *given by* (3.5).

PROOF. We put

$$\begin{aligned} \psi_\lambda(t) &= \tfrac{1}{2}\lambda^2 \int (e^{itf_\lambda(x,y)} - 1)\, d\mu(x)\, d\mu(y) \\ &= \tfrac{1}{2}\lambda^2 \int (e^{its} - 1) \frac{2}{\lambda^2} \frac{1 + s^2}{s^2}\, d\Psi_\lambda(s) \\ &= ita_\lambda + \int \left(e^{its} - 1 - \frac{its}{1 + s^2} \right) \frac{1 + s^2}{s^2}\, d\Psi_\lambda(s). \end{aligned} \tag{3.14}$$

By the theory of infinitely divisible distributions [Loève (1963), Section 22.1],

$$\exp(\psi_\lambda(t) - ita_\lambda) \to \varphi_V(t) \quad \text{(uniformly on compact intervals)}. \tag{3.15}$$

Let φ_λ be the characteristic function of U_λ. We differentiate $\varphi_\lambda(t) = E\exp(itU_\lambda)$:

$$\frac{d}{dt}\varphi_\lambda(t) = E(iU_\lambda e^{itU_\lambda}) = iE\tfrac{1}{2}\sum_{j\neq k} f_\lambda(\xi_j, \xi_k)e^{itU_\lambda(\Xi)}.$$

Expectations of such sums are easily computed by integrals [cf. e.g., Janson (1983), Lemma 2.1] and we obtain

$$\frac{d}{dt}\varphi_\lambda(t) = i\tfrac{1}{2}\lambda^2\int\int f_\lambda(x, y)Ee^{itU_\lambda(\Xi\cup\{x, y\})}\,d\mu(x)\,d\mu(y). \tag{3.16}$$

However, if we put $S_x(\Xi) = \Sigma f_\lambda(x, \xi_i)$,

$$U_\lambda(\Xi\cup\{x, y\}) = U_\lambda(\Xi) + S_x(\Xi) + S_y(\Xi) + f_\lambda(x, y),$$

and thus, if we write for simplicity dx for $d\mu(x)$,

$$\frac{d}{dt}\varphi_\lambda(t) = \frac{i}{2}\lambda^2\int\int f_\lambda(x, y)e^{itf_\lambda(x, y)}Ee^{it(U_\lambda(\Xi)+S_x(\Xi)+S_y(\Xi))}\,dx\,dy. \tag{3.17}$$

We note that $E|S_x(\Xi)| \le E\Sigma|f_\lambda(x, \xi_i)| = \lambda\int|f_\lambda(x, y)|\,dy$.

By the definition of ψ_λ,

$$\frac{d}{dt}\psi_\lambda(t) = \frac{i}{2}\lambda^2\int\int f_\lambda(x, y)e^{itf_\lambda(x, y)}\,dx\,dy.$$

Consequently,

$$\begin{aligned}
&\left|\frac{d}{dt}\varphi_\lambda(t) - \varphi_\lambda(t)\frac{d}{dt}\psi_\lambda(t)\right| \\
&\quad= \left|\frac{i}{2}\lambda^2\int\int f_\lambda(x, y)e^{itf_\lambda(x, y)}Ee^{itU_\lambda(\Xi)}\left(e^{it(S_x(\Xi)+S_y(\Xi))} - 1\right)dx\,dy\right| \\
&\quad\le \tfrac{1}{2}\lambda^2\int\int|f_\lambda(x, y)|\cdot E|tS_x(\Xi) + tS_y(\Xi)|\,dx\,dy \\
&\quad\le \tfrac{1}{2}\lambda^2\int\int|f_\lambda(x, y)|\,|t|\left(E|S_x(\Xi)| + E|S_y(\Xi)|\right)dx\,dy \\
&\quad\le t\lambda^2\int\int|f_\lambda(x, y)|\lambda|f_\lambda(x, z)|\,dz\,dx\,dy \to 0,
\end{aligned} \tag{3.18}$$

by assumption (3.9) as $\lambda\to\infty$ (uniformly for t in a bounded interval). Hence

$$\left|\frac{d}{dt}\left(e^{-\psi_\lambda(t)}\varphi_\lambda(t)\right)\right| = e^{-\operatorname{Re}\psi_\lambda(t)}\left|\frac{d}{dt}\varphi_\lambda(t) - \varphi_\lambda(t)\frac{d}{dt}\psi_\lambda(t)\right| \to 0 \tag{3.19}$$

and integrating, since $\varphi_\lambda(0) = 1$ and $\psi_\lambda(0) = 0$,

$$e^{-\psi_\lambda(t)}\varphi_\lambda(t) \to 1. \tag{3.20}$$

Combining this with (3.15), we obtain the required result

$$\varphi_\lambda(t)e^{-ita_\lambda} \to \varphi_V(t). \qquad \square$$

REMARK 3.3. We note that if $\sup|f_n| \to 0$, the only possible measure $d\Psi$ satisfying (3.2) is a point mass at the origin. Hence, if the theorem applies at all, it yields in this case convergence to a normal distribution. There is a large overlap here with the case $\sigma_n^2 \to 1$, $n^3 E g_n^2 \to 0$ of Theorem 2.1.

REMARK 3.4. The other main example of an infinitely divisible distribution is a Poisson distribution which we obtain if $d\Psi$ is a point mass at 1. For example, if f_n is an indicator function for every n, $U_n \to \mathrm{Po}(\lambda)$ provided (3.1) holds along with $\frac{1}{2}n^2 E f_n \to \lambda$ [cf. Silverman and Brown (1978)].

REMARK 3.5. The conditions (3.2) and (3.3) can be replaced by

$$\tfrac{1}{2}n^2\big(Ee^{itf_n(X,Y)} - 1\big) - ita_n \to \psi(t),$$

where $\psi(t)$ is continuous at 0. Then $\varphi_V(t) = \exp \psi(t)$ [cf. Loève (1963) 22.1.D].

COROLLARY 3.6. *Suppose that $f_n^{(j)}$, $j = 1, \dots, l$, satisfy the conditions of Theorem 3.1 and that $f_n^{(i)} f_n^{(j)} = 0$ when $i \neq j$. Then the corresponding sums $U_n^{(j)}$ are asymptotically independent.*

This corollary may be proved using the Cramér–Wold device.

4. Applications to small inter-point distances in a sample. Let X have an absolutely continuous distribution in R^d with density function $p(x)$, let $\{r_n\}$ be a sequence of positive numbers and put

$$f_n(x, y) = I(|x - y| < r_n). \tag{4.1}$$

Thus U_n is the number of pairs of points with a distance less than r_n. Since $X - Y$ has the density function $p * \check{p}$, where $\check{p}(x) = p(-x)$,

$$\mu_n = E f_n^2 = \int_{|x| < r_n} p * \check{p}(x)\, dx. \tag{4.2}$$

We assume henceforth that $p \in L^2$. Then $p * \check{p}$ is continuous and thus, if $c_d = \pi^{d/2}/\Gamma((d/2) + 1)$ is the volume of the unit sphere in R^d,

$$r_n^{-d}\mu_n \to c_d p * \check{p}(0) = c_d \int p^2, \tag{4.3}$$

provided $r_n \to 0$.

We obtain different behaviour depending on the rate of decrease of r_n.

(a) If $n^2 r_n^d \to 0$, $EU_n \to 0$, and thus $U_n \to_p 0$.
(b) If $n^2 r_n^d \to \alpha$, $0 < \alpha < \infty$, $U_n \to_d \mathrm{Po}(\frac{1}{2}c_d \alpha \int p^2)$.

This is an easy consequence of Theorem 3.1 (and Remark 3.4); see Silverman and Brown (1978) for details. This connects to the work of Onoyama et al. (1983) on the minimum distance in the following way. Let $Y_n = \min_{1 \le i < j \le n} \|X_i - X_j\|$. Let $r_n = t^{1/d} n^{-2/d}$ for a fixed $t > 0$ and let $U_n = \sum_{i<j} I(\|X_i - X_j\| < r_n)$ be the corresponding U-statistic. Then

$$P\big(n^2 Y_n^d \ge t\big) = P(Y_n \ge r_n) = P(U_n = 0) \to e^{-1/2 c_d t \int p^2} \quad \text{as } n \to \infty.$$

Thus $n^2 Y_n^d$ has a limiting exponential distribution with parameter $\frac{1}{2}c_d \int p^2$.

More generally, we may study the smallest, second smallest, etc., of the inter-point distances. Denoting these by $Y_n^{(1)} = Y_n$, $Y_n^{(2)}, \dots$ it follows that $\{n^2 Y_n^{(i)d}\}_{i=1}^{\infty}$ converge weakly to a Poisson process with intensity $\frac{1}{2}c_d \int p^2$, whence $n^2(Y_n^{(k)})^d \to_d \Gamma(k, \frac{1}{2}c_d \int p^2)$; see Silverman and Brown (1979) and Onoyama et al. (1983).

(c) If $n^2 r_n^d \to \infty$ and $r_n \to 0$, then $(U_n - \binom{n}{2}\mu_n)/\sigma_n \to_d N(0,1)$ where μ_n and σ_n are as defined in (2.2) and (2.8).

PROOF. Observe that $\sigma_n^2 \geq \frac{1}{2}n^2(\mu_n - \mu_n^2) \sim cn^2 r_n^d \to \infty$. Thus $\sup|f_n| = 1 = o(\sigma_n)$. Further, by Hölder's inequality,

$$\begin{aligned} E|f_n(x,Y)| &= \int_{|y-x|<r_n} p(y)\,dy \\ &\leq \left(\int_{|y-x|<r_n} p^2(y)\,dy \int_{|y-x|<r_n} dy \right)^{1/2} \\ &= c_d^{1/2} \cdot r_n^{d/2} \left(\int_{|y-x|<r_n} p^2(y)\,dy \right)^{1/2}. \end{aligned}$$

Since $\int p^2\,dy < \infty$, the last integral tends to 0 uniformly in x as $r_n \to 0$. Thus

$$\sup_x E|f_n(x,Y)| = o\left(r_n^{d/2}\right) = o(\sigma_n/n).$$

Hence the conclusion of case (c) follows from Theorem 2.1. □

Let us now impose the somewhat stronger hypothesis that $p \in L^3$ on the density function of X. Then, with $I_n(x) = I(|x| < r_n)$, we have $Ef_n(x,Y) = I_n * p(x)$ and $g_n = I_n * p - \mu_n$. Since translation is continuous in L^3, $(c_d r_n^d)^{-1} I_n * p \to p$ in L^3 as $n \to \infty$, and consequently,

$$r_n^{-2d} E g_n^2 = r_n^{-2d} \int \left(I_n * p(x)\right)^2 p(x)\,dx - r_n^{-2d}\mu_n^2 \to c_d^2 \left(\int p^3 - \left(\int p^2 \right)^2 \right). \tag{4.4}$$

Note that $(\int p^2)^2 \leq \int p^3 \int p = \int p^3$ with equality iff p is constant. Hence the right-hand side of (4.4) is positive unless X is uniformly distributed on some set $E \subset R^d$.

Excluding this case, (2.8), (4.3), and (4.4) yield

$$\sigma_n^2 \sim \tfrac{1}{2}c_d \int p^2 n^2 r_n^d + c_d^2 \left(\int p^3 - \left(\int p^2 \right)^2 \right) n^3 r_n^{2d}.$$

In particular, if $nr_n^d \to 0$, then $\sigma_n^2 \sim \frac{1}{2}c_d \int p^2 n^2 r_n^d$ and if $nr_n^d \to \infty$,

$$\sigma_n^2 \sim c_d^2 \left(\int p^3 - \left(\int p^2 \right)^2 \right) n^3 r_n^{2d}.$$

On the other hand, if X is uniformly distributed on a set E, $\sigma_n^2 \sim \frac{1}{2}c_d \int p^2 n^2 r_n^d$ when $\sup nr_n^d < \infty$ but the behaviour when r_n decreases slower than $n^{-1/d}$ depends on the structure of the boundary of E. If the boundary is piecewise continuously differentiable, it is not difficult to show that $Eg_n^2 \sim Cr_n^{2d+1}$ for some constant C, $0 < C < \infty$. Hence the above estimate of σ_n^2 is valid for

$r_n = o(n^{-1/(d+1)})$ and $\sigma_n^2 \sim Cn^3 r_n^{2d+1}$ when $r_n n^{1/(d+1)} \to \infty$. See also Weber (1983).

(d) If $r_n = r$ is constant, U_n is a standard U-statistic. In this case $g_n = g$ is independent of n and, except in trivial cases, nonzero. Hence $n^{-3/2}(U_n - \binom{n}{2}\mu) \to_d N(0, \sigma^2)$ for some σ^2.

In the case of a uniform distribution of X, the variance of g_n arises from edge effects only. If we instead let X be uniformly distributed on a sphere or a torus, g_n vanishes completely. Hence, if $r_n \to 0$ and $n^2 r_n^d \to \infty$, $\sigma_n^2 \sim Cn^2 r_n^d$ with $0 < C < \infty$ and U_n is asymptotically normal. See, e.g., Silverman (1978). Note that if r_n is constant, this is an example of a degenerate case where U_n is not asymptotically normal. [In fact, $n^{-1}(U_n - \binom{n}{2}\mu_n)$ converges to a linear combination of exponential distributions. See, for instance, Giné (1975) and Rao (1972).]

In the other extreme case, where $n^2 r_n^d$ tends to a constant, U_n converges to a Poisson distribution. Hence we have a large range of cases with asymptotic normal limits, as well as two nonnormal distributions in the extreme cases.

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