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UNIVERSITY OF CALIFORNIA,  
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Arithmetic Stability of Higher Rank Artin–Schreier–Witt Towers

DISSERTATION

submitted in partial satisfaction of the requirements  
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Shichen Tang

Dissertation Committee:  
Professor Daqing Wan, Chair  
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2022



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# ABSTRACT OF THE DISSERTATION

Arithmetic Stability of Higher Rank Artin–Schreier–Witt Towers

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Let  $p$  be a prime and  $q = p^a$  be a power of  $p$ . Consider a tower of finite Galois covers  $\cdots \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 = \mathbb{P}^1$ , whose total Galois group  $G$  is a compact  $p$ -adic Lie group, where  $C_0, C_1, \cdots$  are smooth projective geometrically irreducible curves over  $\mathbb{F}_q$ . It is conjectured by Daqing Wan that if this tower is naturally arising from algebraic geometry, then the zeta functions  $Z(C_n, s)$  must have a stable pattern when  $n \rightarrow \infty$ . The most general form of this conjecture is wide open, and it is already very difficult even when  $G$  is abelian. In this dissertation, we study the case where the tower is a higher rank Artin–Schreier–Witt tower totally ramified at infinity and unramified at other points. We prove that if this tower is strongly genus stable, then the slopes of the Newton polygon of  $Z(C_n, s)$  are uniformly distributed on the interval  $[0, 1]$  when  $n \rightarrow \infty$ . Furthermore, if the tower satisfies a slightly stronger condition than being strongly genus stable, we prove that the slopes of  $Z(C_n, s)$  are given by a finite union of arithmetic progressions. This generalizes the main results of [5] from rank one case  $d = 1$  to higher rank case  $d > 1$ . We also prove a spectral halo property for Artin–Schreier–Witt eigenvarieties corresponding to *overconvergent* higher rank Artin–Schreier–Witt towers.

# Chapter 1

## Introduction

Fix a prime  $p$  and let  $q = p^a$ . Let  $C$  be a smooth projective geometrically irreducible curve over  $\mathbb{F}_q$ . The zeta function of  $C$  is

$$Z(C, s) = \exp\left(\sum_{n \geq 1} \frac{s^n}{n} \cdot \#C(\mathbb{F}_{q^n})\right) = \prod_{x \in |C_n|} \frac{1}{1 - s^{\deg(x)}} \in 1 + s\mathbb{Z}[[s]].$$

It is known to Weil[16] no later than 1949 that

$$Z(C, s) = \frac{P(C, s)}{(1-s)(1-qs)},$$

where  $P(C, s) \in 1 + s\mathbb{Z}[s]$  is a polynomial of degree  $2g(C)$ , pure of  $q$  weight one, and  $g(C)$  denotes the genus of  $C$ . One can factor  $P(C, s)$  over  $\overline{\mathbb{Q}}$  as

$$P(C, s) = \prod_{i=1}^{2g} (1 - \alpha_i s).$$

The algebraic integers  $\alpha_i \in \overline{\mathbb{Q}}$  are the reciprocal roots of  $P(C, s)$ . An important topic in arithmetic geometry is to understand these algebraic integers. Since  $P(C, s)$  is pure of  $q$  weight one, we know  $|\alpha_i| = \sqrt{q}$  for any  $i$  and any Archimedean absolute value  $|\cdot|$ . For any

$\ell$ -adic absolute value  $|\cdot|_\ell$  with  $\ell \neq p$ , the functional equation of  $Z(C, s)$  implies  $|\alpha_i|_\ell = 1$  for any  $i$ . But the  $p$ -adic absolute values depend on the curve  $C$ . In other words, the  $p$ -adic Newton polygon of  $P(C, s)$  depends on the curve  $C$ , and is in general much more complicated and much less predictable.

In many cases there is a Hodge lower bound for the Newton polygon improving the classical Hodge bound, and a criterion for when the Newton polygon coincides with the improved Hodge lower bound is established, see for example [14], [10], [7] and [8]. Also see the recent paper [6] by Kramer-Miller for the most recent results in this direction. However, when the Newton polygon is strictly above the improved Hodge lower bound, it is much harder to find a pattern. In a very special kind of  $\mathbb{Z}_p$ -tower of covers, Davis, Wan and Xiao [1] found that the  $p$ -adic absolute value of the reciprocal roots exhibits some asymptotic stable behavior. Their results were then generalized in [9], [5], and generalized to  $\mathbb{Z}_{p^a}$ -towers in [12]. In this dissertation, we will dig further into the geometric and arithmetic asymptotic patterns of  $\mathbb{Z}_{p^a}$ -towers.

## 1.1 Higher rank Artin–Schreier–Witt towers

Let us first define the tower of covers we are interested in. We fix some notations. Let  $p$  be a prime, and let  $q = p^a$  be a power of  $p$ .  $K = \mathbb{F}_q(X)$  is the field of rational functions on the projective line  $\mathbb{P}^1$  over  $\mathbb{F}_q$ ,  $K^{\text{sep}}$  is the separable closure of  $K$ . For a field  $k$ ,  $W(k)$  is the ring of Witt vectors over  $k$ . For a Witt vector  $w = (a_1, a_2, \dots) \in W(k)$ , its  $n$ -th truncation is the vector  $w_n = (a_1, \dots, a_n)$ . The ring of all  $n$ -th truncations of Witt vectors in  $W(k)$  is denoted by  $W_n(k)$ . The ring endomorphism  $\cdot^F$  on  $W(k)$  (or  $W_n(k)$ ) is such that  $(a_1, a_2, \dots)^F = (a_1^p, a_2^p, \dots)$  (or the  $n$ -th truncation of this equation, respectively.)

For a positive integer  $d$  dividing  $a$ , the  $\mathbb{Z}_{p^a}$ -Artin–Schreier–Witt tower (or simply  $\mathbb{Z}_{p^a}$ -tower)

over  $\mathbb{A}_{\mathbb{F}_q}^1$  associated to  $f \in W(K)$  is a sequence of finite Galois covers

$$\mathcal{P} : \cdots \rightarrow C_n \rightarrow \cdots C_1 \rightarrow C_0 = \mathbb{P}^1,$$

constructed via Artin–Schreier–Witt theory from  $f = (f_1, f_2, \dots) \in W(K)$ . For each  $m$ , the affine curve  $C_n$  is defined by the Witt vector equation

$$\underline{Y}_n^{F^d} - \underline{Y}_n = (f_1, \dots, f_n),$$

where  $\underline{Y} = (Y_1, Y_2, \dots) \in W(K^{\text{sep}})$ . For example,  $C_1$  is defined by the equation

$$Y_1^{p^d} - Y_1 = f_1,$$

$C_2$  is defined by the equation above and an additional equation

$$Y_2^{p^d} - Y_2 + \frac{Y_1^{p^{d+1}} - Y_1^p - (Y_1^{p^d} - Y_1)^p}{p} = f_2,$$

and similar for  $C_3, C_4, \dots$ .

The total Galois group of this tower  $\mathcal{P}$  is  $W(\mathbb{F}_{p^d}) \cong \mathbb{Z}_{p^d}$ , the unramified extension of  $\mathbb{Z}_p$  of degree  $d$ , which is isomorphic to  $(\mathbb{Z}_p)^d$  as topological groups. We refer to  $d$  as the *rank* of this tower, since  $d$  is the rank of its total Galois group as a free  $\mathbb{Z}_p$ -module.

In this dissertation, we restrict our attention to towers that are totally ramified at one rational point, and unramified at other points, with no constant subextension. Up to a change of coordinates, we can assume the tower is totally ramified at  $\infty$ . In other words, we consider étale covers

$$\mathcal{P} : \cdots \rightarrow C_n^{\text{affine}} \rightarrow \cdots C_1^{\text{affine}} \rightarrow C_0^{\text{affine}} = \mathbb{A}^1,$$

where  $C_n^{\text{affine}}$  is  $C_n$  removing the unique point at infinity. In this case, by [3], one can write

$$f = f([X]) = b_0 + \sum_{i:(i,p)=1} b_i [X]^i \in W(K),$$

where  $[X] = (X, 0, 0, \dots) \in W(K)$ ,  $b_i \in \mathbb{Z}_q$ ,  $v_p(b_i) \rightarrow \infty$  as  $i \rightarrow \infty$  and  $\min\{v_p(b_i) : (i, p) = 1\} = 0$ . If  $f$  is a polynomial with each  $b_i$  being either 0 or a  $(q-1)$ -st root of unity, then  $f$  is called a *unit root polynomial*, and the corresponding  $\mathbb{Z}_{p^d}$ -tower is called a *unit root tower*. In literature, the patterns of slopes are first discovered for unit root towers, as in [1] and [12].

## 1.2 Asymptotic patterns

In general, for a  $\mathbb{Z}_{p^d}$ -tower  $\mathcal{P} : \dots \rightarrow C_n \rightarrow \dots \rightarrow C_1 \rightarrow C_0 = \mathbb{P}^1$  associated to  $f$ , the genus  $g(C_n)$  can grow arbitrarily fast when  $n \rightarrow \infty$ . By [3], the growth of  $g(C_n)$  is at least in a degree  $d+1$  polynomial of  $p^n$ . We say  $\mathcal{P}$  is *genus stable* if  $g(C_n)$  is actually a degree  $d+1$  polynomial of  $p^n$  for large  $n$ . Genus stable towers are natural objects to be studied, as it is conjectured by Wan [15] that any  $\mathbb{Z}_{p^d}$ -tower coming from geometry is genus stable.

By [3] and [4], whether or not the tower  $\mathcal{P}$  associated to  $f$  is genus stable can be determined by an explicit condition on  $f$ .

Let  $v_q$  denote the  $q$ -adic valuation, normalized such that  $v_q(q) = 1$ . The study of  $p$ -adic absolute values is equivalent to the study of  $q$ -adic slopes of the Newton polygon of  $P(C_n, s)$ . In a genus stable  $\mathbb{Z}_{p^d}$ -tower, we expect to observe two different types of asymptotic behavior about slopes.

**Definition 1.1** (slope uniformity for zeta function). We say the  $\mathbb{Z}_{p^d}$ -tower  $\mathcal{P}$  is *slope uniform* if the  $q$ -adic Newton slopes of  $P(C_n, s)$  approaches a uniform distribution on the interval  $[0, 1]$  when  $n$  approaches infinity. In other words, let  $\alpha_{n,1}, \dots, \alpha_{n,2g_n}$  denote reciprocal roots

of  $P(C_n, s)$ , then  $\mathcal{P}$  is slope uniform if for all  $0 \leq a < b \leq 1$ ,

$$\lim_{n \rightarrow \infty} \frac{\#\{i : v_q(\alpha_{n,i}) \in [a, b]\}}{2g(C_n)} = b - a.$$

**Definition 1.2** (Slope stability for zeta function). We say the  $\mathbb{Z}_p^d$ -tower  $\mathcal{P}$  is slope stable if there exists positive integers  $m'$  and  $r$ , and exists  $r$  many rational numbers

$$0 < \alpha_1 \leq \cdots \leq \alpha_r < 1,$$

such that for  $n \geq m'$ , the multiset of slopes of  $\frac{P(C_n, s)}{P(C_{n-1}, s)}$  is give by the truncation of  $r + 1$  many arithmetic progressions, explicitly expressed in terms of  $\alpha_1, \dots, \alpha_r$ .

REMARK 1.3. For the slope stable towers considered in this thesis, the multiset of slopes of  $\frac{P(C_n, s)}{P(C_{n-1}, s)}$  can be obtained by the following procedure: first scale the set  $\{\alpha_1, \dots, \alpha_r\}$  by a factor of  $\frac{1}{p^{n-m'}}$ , then shift the scaled set to the right for  $0, \frac{1}{p^{n-m'}}, \frac{2}{p^{n-m'}}, \dots, 1 - \frac{1}{p^{n-m'}}$  units, then take the union of all the shifted sets, make several copies and take union again, and finally union with another several copies of the set  $\left\{ \frac{1}{p^{n-m'}}, \frac{2}{p^{n-m'}}, \dots, 1 - \frac{1}{p^{n-m'}} \right\}$ . When the slopes can be described in this way, it is clear that they approaches a uniform distribution on  $[0, 1]$  when  $n \rightarrow \infty$ . For this reason, we view slope stability as a stronger property than slope uniformity. Also note that by counting degrees of the polynomials  $P(C_n, s)$ , it can be easily shown that slope stability implies genus stability.

In literature, Davis–Wan–Xiao [1] first proved that a unit root  $\mathbb{Z}_p$ -tower is slope stable. After that, Kosters–Zhu [5] studied *strongly genus stable*  $\mathbb{Z}_p$ -towers.

**Definition 1.4** (strong genus stability). Recall that  $f([X]) = b_0 + \sum_{i \geq 1, (i,p)=1} b_i [X]^i \in W(K)$ . We say the tower  $\mathcal{P}$  is *strongly genus stable* if

$$\delta = \delta(\mathcal{P}) := \max \left\{ \frac{i}{p^{v_p(b_i)}} : i = 0, 1, 2, \dots \right\}$$

exists. When  $\delta$  exists, put  $m = m(\mathcal{P}) := -v_p(\delta)$ .

REMARK 1.5. Since  $(i, p) = 1$  when  $i \geq 1$ , there must be a unique  $i$  such that  $\delta = \frac{i}{p^m}$ . We prove in Chapter 2 that the existence of  $\delta$  implies genus stability, and a treatment in a more general setting is given in [3]. But note that genus stability cannot imply existence of  $\delta$  in general. In fact, it was originally claimed in [3] that the existence of  $\delta$  is equivalent to genus stability. However, Kusters and Wan discovered that the existence of  $\delta$  is actually slightly stronger than genus stability, see [4].

In their paper, Kusters–Zhu proved that strongly genus stable  $\mathbb{Z}_p$ -towers are slope uniform. They also proved that a slightly stronger condition than strong genus stability on  $\mathbb{Z}_p$ -towers (but much weaker than being a unit root tower) can imply slope stability, which we will discuss in detail later.

In [12], the result of Davis–Wan–Xiao was generalized in another direction, by considering more general unit root  $\mathbb{Z}_{p^d}$ -towers instead of just  $d = 1$  case, but they only proved slope uniformity, because they were not able to prove slope stability. In this dissertation, we further generalize [1] by considering strongly genus stable  $\mathbb{Z}_{p^d}$ -towers, for general  $d$ . Our first main result is:

**Theorem 1.6.**

- (1) *If  $\mathcal{P}$  is a strongly genus stable  $\mathbb{Z}_{p^d}$ -tower, then it is slope uniform.*
- (2) *Suppose  $\mathcal{P}$  is strongly genus stable with  $\delta(\mathcal{P}) = \delta$ . There exists a constant  $C = C(\delta, q)$ , such that if  $v_p(b_j) > i$  for all  $j > \delta p^i - C$  when  $i$  is large enough, then we have  $\mathcal{P}$  is slope stable.*

We refer to Theorem 3.10 in Section 3.3 for an explicit description of the multiset of slopes of  $\frac{P(C_{n,s})}{P(C_{n-1,s})}$  when  $n$  is large.

### 1.3 Slope of L-function

Theorem 1.6 can be reduced to a question on  $L$ -functions. Let  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$  be a continuous character of the Galois group of  $\mathcal{P}$ , the  $L$ -function  $L_f(\chi, s)$  over  $\mathbb{A}_{\mathbb{F}_q}^1$  is defined by

$$L_f(\chi, s) = \prod_{x \in |\mathbb{A}^1|} \frac{1}{1 - \chi(\text{Frob}_x) s^{\deg(x)}} \in 1 + s\mathbb{C}_p[[s]],$$

where  $|\mathbb{A}^1|$  denotes the set of closed points in  $\mathbb{A}_{\mathbb{F}_q}^1$ . When  $\chi = 1$  is the trivial character,  $L_f(\chi, s)$  is the same as the zeta function of  $\mathbb{A}_{\mathbb{F}_q}^1$ , which is  $\frac{1}{1-qs}$ . When  $\chi$  is a nontrivial finite character,  $L_f(\chi, s)$  is a polynomial whose degree can be easily calculated, see Theorem 2.12. Understanding the slopes of  $L(\chi, s)$  for all finite characters  $\chi$  can help us understand slopes of  $Z(C_n, s)$ . In fact,  $C_n$  has one unique point at infinity, let  $C_n^{\text{affine}} = C_n - \{\infty\}$ , then

$$Z(C_n^{\text{affine}}, s) = \frac{P(C_n, s)}{1 - qs},$$

and we have the decomposition

$$Z(C_n^{\text{affine}}, s) = \prod_{\chi \text{ factors through } p^n \mathbb{Z}_{p^d}} L_f(\chi, s).$$

We refer to Section 2.2 for the proof. Now, to understand the asymptotic behavior of  $L(\chi, s)$ , we will first prove:

**Theorem 1.7** (Slope uniformity for  $L$ -functions). *Let  $\mathcal{P}$  be a strongly genus stable  $\mathbb{Z}_{p^d}$ -tower associated to  $f([X]) = b_0 + \sum_{i:(i,p)=1} b_i [X]^i$ , with  $\delta = \max \left\{ \frac{i}{p^{v_p(b_i)}} : i = 0, 1, 2, \dots \right\}$ , and let  $m$  be such that  $\delta = \frac{d_m}{p^m}$ . Let  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$  be a nontrivial finite character of order  $p^{m_\chi} > p^m$ . Then  $L_f(\chi, s)$  is a polynomial of degree  $\delta p^{m_\chi - 1} - 1$ . Write*

$$L_f(\chi, s) = \sum_{k=0}^{\delta p^{m_\chi - 1} - 1} a_k s^k.$$

We have the following:

(1)  $a_0 = 1$ . For any  $0 < n < p^{m_\chi - m - 1}$ , we have

$$v_q(a_{nd_m - 1}) = \frac{n(nd_m - 1)}{2p^{m_\chi - m - 1}} \text{ and } v_q(a_{nd_m}) = \frac{n(nd_m + 1)}{2p^{m_\chi - m - 1}},$$

and we have

$$v_q(a_{\delta p^{m_\chi - 1} - 1}) = \frac{\delta p^{m_\chi - 1} - 1}{2}.$$

(2) The  $q$ -adic Newton polygon of  $L_f(\chi, s)$  starts from  $(0, 0)$ , ends at  $(\delta p^{m_\chi - 1} - 1, \frac{\delta p^{m_\chi - 1} - 1}{2})$ , and passes through the points

$$\left( nd_m - 1, \frac{n(nd_m - 1)}{2p^{m_\chi - m - 1}} \right) \text{ and } \left( nd_m, \frac{n(nd_m + 1)}{2p^{m_\chi - m - 1}} \right),$$

for any  $0 < n < p^{m_\chi - 1}$ .

(3) The  $q$ -adic Newton polygon of  $L_f(\chi, s)$  has slopes (in multiset notation and in increasing order)

$$\bigcup_{i=1}^{p^{m_\chi - m - 1}} \{\alpha_{i1}, \dots, \alpha_{id_m}\} - \{0\},$$

where

$$\begin{cases} \alpha_{ij} = \frac{i-1}{p^{m_\chi - m - 1}} & \text{if } j = 1, \\ \frac{i-1}{p^{m_\chi - m - 1}} < \alpha_{ij} < \frac{i}{p^{m_\chi - m - 1}} & \text{if } j > 1. \end{cases}$$

REMARK 1.8. The  $d = 1$  case of the previous theorem is proved by Kusters–Zhu in [5].

Theorem 1.7 directly implies that the  $q$ -adic slopes of  $L_f(\chi, s)$  approach a uniform distribution on  $[0, 1]$  when  $m_\chi \rightarrow \infty$ , and hence implies part (1) of Theorem 1.6.

In order to prove the second part of Theorem 1.6, we seek a similar stable pattern for  $L$ -functions. In  $d = 1$  case, for nice  $\mathbb{Z}_p$ -towers, the Newton polygon of  $L_f(\chi, s)$  is completely determined by the order of  $\chi$  when this order is large, and can be written down explicitly using some arithmetic progressions. This is the idea of Kosters–Zhu’s work, which implies the  $d = 1$  case of Theorem 1.6. For  $d > 1$ , it is unclear whether the Newton polygon of  $L_f(\chi, s)$  is completely determined by the order of  $\chi$ . To our best knowledge, this is still an open problem. Therefore, we need to start from dividing the set of finite characters into several parts such that in each part the Newton polygon is completely determined by the order of the character.

Let us denote the set of all finite characters of  $\mathbb{Z}_{p^d}$  by  $W$ . For a character  $\chi_0 \in W$  of order  $p^{m_0}$ , we denote the multi-set of the  $q$ -adic slopes of  $L_f(\chi_0, s)$  by

$$\{\alpha_1, \dots, \alpha_r\},$$

where  $r = \deg L_f(\chi_0, s)$ . For a subset  $V \subseteq W$ , we say  $V$  is an  $f$ -stable family of  $\chi_0$  if every character in  $V$  has order at least  $p^{m_0}$ , and for any character  $\chi \in V$  of order  $p^n$ , the multi-set of the  $q$ -adic slopes of  $L_f(\chi, s)$  is

$$\bigcup_{i=0}^{p^n - p^{m_0} - 1} \left\{ \frac{i}{p^{n-m_0}}, \frac{\alpha_1 + i}{p^{n-m_0}}, \dots, \frac{\alpha_r + i}{p^{n-m_0}} \right\} - \{0\}.$$

It is a union of  $r$  many arithmetic progressions. For any nontrivial character  $\chi_0$ , its  $f$ -stable family at least contains  $\chi_0$  itself. If  $W - \{1\}$  (where 1 denotes the trivial character) consists of only finitely many stable families, then the data of the Newton polygons for *all* finite characters is completely determined by the data of the Newton polygons for *finitely many* characters. We show that this is the case when the condition in the second part of Theorem 1.6 is satisfied.

**Theorem 1.9** (Slope stability for  $L$ -functions). *Suppose  $\mathcal{P}$  is strongly genus stable with*

$\delta(\mathcal{P}) = \delta$ . There exists a constant  $C = C(\delta, q)$ , such that if  $v_p(b_j) > i$  for all  $j > \delta p^i - C$  when  $i$  is large enough, then we have:

There are an integer  $m' > m$  and a set  $\mathfrak{C}$  consisting of  $\frac{(p^d-1)p^{(d-1)(m'-1)}}{p-1}$  many characters of order  $p^{m'}$ , such that any character  $\chi$  of order  $p^{m_x} \geq p^{m'}$  belongs to the  $f$ -stable family of some  $\chi_0 \in \mathfrak{C}$ . Furthermore,  $\chi$  is in the  $f$ -stable family of  $\chi_0 \in \mathfrak{C}$  if and only if  $\ker \chi \subseteq \ker \chi_0$ .

Part (2) of Theorem 1.6 is proved by considering all the characters of order  $p^n$ , where  $n > m'$  and  $m'$  as in Theorem 1.9, and apply Theorem 1.9 to all the  $L$ -functions  $L(\chi, s)$ .

The key idea of the proofs of Theorems 1.7 and 1.9 is to reduce to a one parameter family of  $d=1$  cases with the help of the universal  $L$ -series that was introduced in [10] and generalized in [12]. It is more convenient to consider the  $L$ -function for  $\chi$  over the torus  $\mathbb{G}_m = \mathbb{A}_{\mathbb{F}_q}^1 - \{0\}$ , defined by:

$$L_f^*(\chi, s) = \prod_{x \in |\mathbb{G}_m|} \frac{1}{1 - \chi(\text{Frob}_x) s^{\deg(x)}} \in 1 + s\mathbb{C}_p[[s]].$$

Fix a  $\mathbb{Z}_{p^d}$ -tower  $\mathcal{P}$  associated to  $f$ , then there is a universal  $L$ -series

$$L_f^*(\underline{T}, s) \in 1 + s\mathbb{Z}_p[[\underline{T}]][[s]],$$

where  $\underline{T} = (T_1, \dots, T_d)$ , such that for every continuous character  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$ , the  $L$ -function  $L_f^*(\chi, s)$  can be recovered from  $L_f^*(\underline{T}, s)$  by evaluating  $T_i$  at some numbers in  $\mathbb{C}_p$  determined by  $\chi$ . The details about this universal  $L$ -function is discussed in 3.2. Using the universal  $L$ -series, we will show that  $L_f(\chi, s)$  for a  $\mathbb{Z}_{p^d}$ -tower is the same as  $L_{cf}(\bar{\chi}, s)$  for a  $\mathbb{Z}_p$ -tower, where  $c \in \mathbb{Z}_{p^d}^*$  and  $\bar{\chi}$  is a restriction of  $\chi$ , in the sense that there is some  $\eta : \mathbb{Z}_{p^d} \rightarrow \mathbb{Z}_p$  such that  $\chi = \bar{\chi} \circ \eta$ . Then the problem is successfully reduced to the same question on  $\mathbb{Z}_p$ -towers generated by  $cf$ , which is already proved in [5].

The complete proof of Theorems 1.7 and 1.9 as well as the proof for Theorem 1.6 can be found in Section 3.3.

## 1.4 Infinite characters and the Artin–Schreier–Witt eigenvariety

Note that all the theorems above are about finite characters. It would be interesting to study infinite characters as well. For a continuous character  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$ , define the characteristic function by

$$C_f^*(\chi, s) = L_f^*(\chi, s)L_f^*(\chi, qs)L_f^*(\chi, q^2s) \cdots \in 1 + s\mathbb{C}_p[[s]].$$

We hope to understand the  $q$ -adic Newton polygon of  $C_f^*(\chi, s)$ . We express our results in terms of the Artin–Schreier–Witt eigenvariety defined below.

Similar to the universal  $L$ -series, we also have a universal characteristic series

$$C_f^*(\underline{T}, s) \in 1 + s\mathbb{Z}_p[[\underline{T}]][[s]],$$

such that for every continuous character  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$ , the characteristic function  $C_f^*(\chi, s)$  can be recovered from  $C_f^*(\underline{T}, s)$  by evaluating  $T_i$ . Analogous to the Coleman–Mazur eigen-curve for igusa tower of modular curves, the *Artin–Schreier–Witt eigenvariety*, denoted by  $\mathcal{E}_f$ , is defined to be the zero locus of the universal characteristic series  $C_f^*(\underline{T}, s)$ , inside the rigid analytic space  $\mathbb{G}_m^{\text{rig}} \times (\mathcal{W} - \{0\})$ , where  $\mathcal{W}$  is the *weight space* defined by

$$\mathcal{W} = \{(t_1, \dots, t_d) \in \mathbb{C}_p^d : v_p(t_k) > 0, 1 \leq k \leq d\}.$$

Consider a point  $(\underline{T}, s)$  on  $\mathcal{E}_f$ . The “ $a_p$ -map” sends this point to  $s^{-1} \in \mathbb{C}_p^*$ . The “slope map” is the  $q$ -adic valuation of the image of the “ $a_p$ -map”. The projection map sending  $(\underline{T}, s)$  to  $\underline{T}$  is called the *weight map*, denoted by  $\text{wt}$ . We have the following diagram:

$$\begin{array}{ccccc}
 & & \text{slope} & & \\
 & & \curvearrowright & & \\
 \mathcal{E}_f & \xrightarrow{s \mapsto s^{-1}} & \mathbb{C}_p^* & \xrightarrow{v_q} & \mathbb{Q} \\
 \text{wt} \downarrow & & & & \\
 \mathcal{W} - \{0\} & & & & 
 \end{array}$$

In other words, this eigenvariety  $\mathcal{E}_f$  is a  $d$ -dimensional rigid analytic variety that parameterizes all the Newton slopes of  $C_f^*(\chi, s)$  for all continuous characters  $\chi$ . Analogous to the halo conjecture for Coleman–Mazur eigencurves, we expect the Artin–Schreier–Witt eigenvariety  $\mathcal{E}_f$  to have a nice decomposition into disjoint union of subspaces near the boundary of the weight space.

In the spirit of this analogy, in Chapter 7, we prove our second main result:

**Theorem 1.10.** *The Artin–Schreier–Witt eigenvariety constructed from an overconvergent power series  $f$ , when restricted to the admissible locus and the boundary of the weight space, is an infinite disjoint union of subspaces which are finite and flat over the weight annulus. The slopes on each component lies in an interval determined by the weight parameters.*

Theorem 7.1 in Chapter 7 is a more precise statement of this theorem. We explain some necessary definitions and backgrounds in Chapters 4, 5, 6.

Finally, we can also interpret our main theorem in terms of the Artin–Schreier–Witt eigenvariety for a unit root polynomial  $f$ . We show that there is a stronger decomposition of the eigenvariety over some rigid lines on the weight space  $\mathcal{W}$ . This is Theorem 7.4.

# Chapter 2

## Preliminaries

### 2.1 Rigid analytic geometry

In this section, we briefly review the theory of rigid analytic geometry. We use [2] and [13] as our references. Our purpose is to properly define varieties over  $\mathbb{C}_p$  such that each variety has a sheaf of “holomorphic functions” and a nice Čech cohomology theory of sheaves.

We put  $K = \mathbb{C}_p$ . The subalgebra

$$T_{n,K} = T_n = K\langle z_1, \dots, z_n \rangle := \left\{ f = \sum_{\alpha} c_{\alpha} z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in K[[z_1, \dots, z_n]] : \lim_{\alpha_1 + \dots + \alpha_n \rightarrow \infty} c_{\alpha} = 0 \right\},$$

of  $K[[z_1, \dots, z_n]]$  is called a *standard affinoid algebra* or *Tate algebra* over  $K$ , where  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi-index.  $T_n$  is a  $K$ -Banach algebra with respect to the multiplicative norm

$$\left\| \sum_{\alpha} c_{\alpha} z_1^{\alpha_1} \cdots z_n^{\alpha_n} \right\| = \max_{\alpha} c_{\alpha}.$$

The valuation ring of  $K$  is  $K^\circ := \{a \in K : |a| \leq 1\}$ . The unique maximal ideal of  $K^\circ$  is  $K^{\circ\circ} := \{a \in K : |a| < 1\}$ . The residue field of  $K$  is  $k := K^\circ/K^{\circ\circ}$ . Let  $T_n^\circ := \{f \in T_n : \|f\| \leq 1\}$ , then there is a surjective map

$$T_n^\circ \rightarrow k[z_1, \dots, z_n],$$

given by  $f \mapsto \bar{f}$ , where  $\bar{f}$  is the reduction of  $f$  modulo  $K^{\circ\circ}$ . An element  $f \in T_n$  is called *regular with respect to  $z_n$  of degree  $d$*  if  $\bar{f}$  has the form  $\lambda z_n^d + \sum_{i=0}^{d-1} c_i z_n^i$  with  $\lambda \in k^*$  and all  $c_i \in k[z_1, \dots, z_{n-1}]$ . We view  $T_{n-1} = K\langle z_1, \dots, z_{n-1} \rangle$  as a  $K$ -subalgebra of  $T_n$ .

**Theorem 2.1** (Weierstrass preparation and division).

- (1) *Division: Let  $f \in T_n$  be regular in  $z_n$  of degree  $d$  and let  $g$  be any element of  $T_n$ . There are unique elements  $r \in T_{n-1}[z_n]$  of degree  $< d$  in the variable  $z_n$  and  $q \in T_n$  such that  $g = qf + r$ . Moreover  $\|g\| = \max(\|q\|, \|r\|)$ .*
- (2) *Preparation: Let  $f \in T_n$  has norm 1. There exists a  $k$ -algebra automorphism  $\sigma$  of  $T_n$  such that  $\sigma(f)$  is regular in  $z_n$ .*

*Proof.* This is [2, Theorem 3.1.1]. □

An *affinoid algebra over  $K$*  is a  $K$ -algebra which is a finite extension of  $T_{n,K}$  for some  $n \geq 0$ . The below properties of affinoid algebras are consequences of the Weierstrass preparation and division theorem.

**Theorem 2.2.**

- (1) *Any affinoid algebra is noetherian.*
- (2)  *$T_n$  is a unique factorization domain (of Krull dimension  $n$ ).*

- (3) Let  $\|\cdot\|$  be a norm on some affinoid algebra  $A$ , which makes  $A$  into a Banach space over  $K$ . Then every ideal of  $A$  is closed with respect to this norm.
- (4) For every ideal  $I \subseteq T_n$  there exists an integer  $d$  and an injective, finite morphism of  $K$ -algebras  $T_d \rightarrow T_n/I$  (i.e. which makes  $T_n/I$  into a finitely generated  $T_d$ -module). The Krull dimension of  $T_n/I$  is  $d$ .
- (5) For any maximal ideal  $M \subseteq T_n$ , the field  $T_n/M$  is isomorphic to  $K$ .
- (6) Any affinoid algebra  $A$  has the form  $T_n/I$  (for some  $n$  and some ideal  $I$ ). The norm on  $T_n$  induces a norm on  $A$  for which  $A$  is a Banach algebra over  $K$ .
- (7) All norms on an affinoid algebra  $A$ , which makes  $A$  into a Banach algebra over  $K$ , are equivalent.

*Proof.* This is [2, Theorem 3.2.1]. □

One associates to an affinoid algebra  $A$  over  $K$  the set  $X = \text{Max}(A)$  of its maximal ideals. There is a canonical topology on  $X$ , which is the one generated by all sets  $\{x \in X : |f(x)| \leq 1\}$ , with  $f$  varying in  $A$  and  $|\cdot|$  being the  $p$ -adic absolute value on  $K$ . However, this topology does not lead to a good notion of analytic function. In particular, the presheaf of holomorphic functions is not a sheaf, see for example [2, Exercice 2.4.4]. To remedy this defect Tate introduced a Grothendieck topology for  $X$ . The easy form of Grothendieck topology, used here, is called a  $G$ -topology.

**Definition 2.3.** Let  $X$  be a set. A  $G$ -topology  $T$  on  $X$  is given by the following data and requirements:

- (1) A family  $\mathcal{F}$  of subsets of  $X$  with the properties:  $\emptyset, X \in \mathcal{F}$  and if  $U, V \in \mathcal{F}$ , then  $U \cap V \in \mathcal{F}$ .

(2) For each  $U \in \mathcal{F}$  a set  $\text{Cov}(U)$  of coverings of  $U$  by elements of  $\mathcal{F}$ .

(3) For each  $U \in \mathcal{F}$ ,  $\{U\} \in \text{Cov}(U)$ .

(4) For  $V, U \in \mathcal{F}$  with  $V \subseteq U$  and  $\mathcal{U} \in \text{Cov}(U)$  the covering  $\mathcal{U} \cap V := \{U' \cap V : U' \in \mathcal{U}\}$  belongs to  $\text{Cov}(V)$ .

(5) Let  $U \in \mathcal{F}$ ,  $\{U_i\}_{i \in I} \in \text{Cov}(U)$  and  $\mathcal{U}_i \in \text{Cov}(U_i)$ , then

$$\bigcup_{i \in I} \mathcal{U}_i := \{U' : U' \text{ belongs to some } \mathcal{U}_i\}$$

is an element of  $\text{Cov}(U)$ .

The  $U \in \mathcal{F}$  are called *admissible sets* or *T-open* and the elements of  $\text{Cov}(U)$  are called *admissible coverings* or *T-coverings*.

We start defining admissible sets on  $X = \text{Max}(A)$  from *rational subdomains*.

**Definition 2.4.** A subset  $R \subseteq X$  is called a *rational subdomain* if there exists  $f_0, f_1, \dots, f_s \in A$ , generating the unit ideal of  $A$  (i.e. has no common zero) such that

$$R = \{x \in X : |f_i(x)| \leq |f_0(x)| \text{ for } i = 1, \dots, s\}.$$

The affinoid algebra  $O_X(R) = A\langle Z_1, \dots, Z_s \rangle / (f_1 - f_0 Z_1, \dots, f_s - f_0 Z_s)$  is associated to  $R$ .

The admissible sets for the *very weak G-topology* on  $X$  are the rational subdomains of  $X$ . The sheaf of  $K$ -algebras  $O_X$  is the sheaf of “holomorphic functions” on  $X$ . The very weak  $G$ -topology can be improved by considering slightly finer  $G$ -topologies instead. We say a  $G$ -topology  $T'$  is *slightly finer than* the  $G$ -topology  $T$  if all of the following hold:

- (1) Every  $T$ -open set is  $T'$ -open and every  $T$ -covering is a  $T'$ -covering.
- (2) Every  $T'$ -open  $U$  has a  $T'$ -covering by  $T$ -open sets.
- (3) Every  $T'$ -covering of a  $T$ -open set can be refined by a  $T$ -covering.

Consider the collection of all  $G$ -topologies slightly finer than the very weak  $G$ -topology. It can be shown [2] that there is a finest  $G$ -topology in this collection. It turns out that this  $G$ -topology is the *strong  $G$ -topology*  $T^*$  on  $X$ , defined by the following[13].

**Definition 2.5** (Strong  $G$ -topology). A subset  $U \subseteq X$  is called  $T^*$ -open if there are rational subdomains  $U_i \subseteq X$  for  $i \in I$  such that

- (1)  $U \in \bigcup_{i \in I} U_i$  (in particular  $U$  is open in the canonical topology) and
- (2) for any map  $\alpha : \text{Max}(B) \rightarrow \text{Max}(A)$  induced by a homomorphism of affinoid algebras  $A \rightarrow B$  with  $\text{im}(\alpha) \subseteq U$  the covering  $\text{Max}(B) = \bigcup_{i \in I} \alpha^{-1}(U_i)$  has a finite subcovering.

Let  $V$  and  $V_j$  for  $j \in J$  be  $T^*$ -open subsets of  $X$  such that  $V = \bigcup_{j \in J} V_j$ ; this covering of  $V$  is an admissible  $T^*$ -covering if for any map  $\alpha : \text{Max}(B) \rightarrow \text{Max}(A)$  with  $\text{im}(\alpha) \subseteq V$  the covering  $\text{Max}(B) = \bigcup_{j \in J} \alpha^{-1}(V_j)$  can be refined into a finite covering by rational subdomains.

**Definition 2.6.** An *affinoid variety* or *affinoid space* over  $K$  is a triple  $(X, T, O_{X,T})$  where  $X = \text{Max}(A)$  for some  $K$ -affinoid algebra,  $T$  is a  $G$ -topology equal to, or slightly finer than, the very weak  $G$ -topology. The sheaf of  $K$ -algebras  $O_{X,T}$  for the given  $G$ -topology is uniquely determined by the requirement that for any  $R = \{x \in X : |f_i(x)| \leq |f_0(x)| \text{ for } i = 1, \dots, s\}$  one has  $O_{X,T}(R) = A\langle Z_1, \dots, Z_s \rangle / (f_1 - f_0 Z_1, \dots, f_s - f_0 Z_s)$ .

**EXAMPLE 2.7.** The maximum spectrum  $\text{Max}(T_n)$  of the Tate algebra  $T_n$  is an affinoid space. By the Weierstrass preparation and division theorems, its underlying set is

$$\text{Max}(T_n) = \mathbb{B}^n := \{(t_1, \dots, t_d) \in \mathbb{C}_p^d : |t_k| \leq 1, 1 \leq k \leq d\}.$$

The structure sheaf  $O_{X,T}$  is “the same” for the very weak  $G$ -topology and any slightly finer  $G$ -topology. In fact, let  $(\text{Affinoid}_K, T)$  denote the category of affinoid spaces with the  $G$ -topology  $T$ , then the obvious functor  $(\text{Affinoid}_K, \text{very weak}) \rightarrow (\text{Affinoid}_K, \text{strong})$  is an equivalence of categories. For the rest of this thesis, unless otherwise stated, we only consider the strong  $G$ -topology on  $X$  and use  $O_X$  to denote the structure sheaf  $O_{X,T^*}$ .

We note that the product of two affinoid spaces is again an affinoid space.

**Lemma 2.8.** *The complete tensor product  $A_1 \hat{\otimes}_K A_2$  of two affinoid algebras  $A_1, A_2$  over  $K$  is again an affinoid algebra over  $K$ . Moreover, their complete tensor product is the coproduct in the category of the affinoid algebras over  $K$ .*

*Proof.* This is [2, Lemma 3.7.1]. □

With this lemma in hand, we define the product of two affinoid spaces  $X_i = \text{Max}(A_i)$  as  $X_1 \times_K X_2 := \text{Max}(A_1 \hat{\otimes}_K A_2)$ . This product is indeed the product in the category of affinoid spaces over  $K$ .

**Definition 2.9.** A *rigid analytic variety* or *rigid space* over  $K$  is a set  $X$  equipped with a  $G$ -topology and a sheaf of  $K$ -algebras  $O_X$  such that there is an admissible covering  $X = \bigcup_{i \in I} U_i$  where each  $(U_i, O_X|_{U_i})$  is isomorphic to an affinoid space over  $K$ .

REMARK 2.10. let  $(\text{Rigid}_K, T)$  denote the category of rigid analytic spaces with the  $G$ -topology  $T$  on each  $U_i$ . It is pointed out in [2] that the obvious functor  $(\text{Rigid}_K, \text{weak}) \rightarrow (\text{Affinoid}_K, \text{strong})$  is *not* an equivalence of categories.

In practice, rigid analytic varieties are often constructed by gluing affinoid spaces (or even rigid spaces over  $K$ ). This procedure is quite similar to constructions in algebraic geometry.

Given are the gluing data:

- (a) A family  $\{X_i\}_{i \in I}$  of affinoid spaces over  $K$  (or rigid spaces over  $K$ ).

- (b) For every  $(i, j) \in I^2$  admissible subsets  $X_{i,j} \subseteq X_i$  and an isomorphism  $\phi_{j,i} : X_{i,j} \rightarrow X_{j,i}$  of rigid spaces over  $K$ .

These data are supposed to satisfy:

- (1)  $X_{i,i} = X_i$  and  $\phi_{i,i} = id$  for all  $i$ .
- (2)  $\phi_{i,j}$  is the inverse of the morphism  $\phi_{j,i}$  for all  $i, j$ .
- (3) For all  $i, j, k$  one has  $\phi_{j,i}(X_{i,j} \cap X_{i,k}) = X_{j,i} \cap X_{j,k}$  and  $\phi_{k,i} = \phi_{k,j} \circ \phi_{j,i}$  holds on  $X_{i,j} \cap X_{i,k}$ .

For the gluing data satisfying (1)–(3), there exists a unique rigid space  $(Y, T_Y, O_Y)$  over  $K$  having an admissible covering  $\{Y_i\}_{i \in I}$  such that each  $Y_i$  is an affinoid space over  $K$  (or an open rigid subspace), and provided with an isomorphism  $\psi_i : X_i \rightarrow Y_i$  of rigid spaces, such that for every  $(i, j) \in I^2$ , the restriction  $\psi_i|_{X_{i,j}} = \psi_j|_{X_{j,i}} \circ \phi_{j,i}$ . The  $G$ -topology  $T_Y$  is given by  $U \in T_Y$  if and only if  $U \cap X_i \in T_{X_i}$  for all  $i$ . Further a covering  $\{U_j\}$  of  $U \in T_Y$  with all  $U_j \in T_Y$  is admissible if and only if  $\{U_j \cap X_i\}$  is an admissible covering of  $U \cap X_i$  for all  $i$ .

We also remark that the existence of products  $X_1 \times_K X_2$  for affinoid spaces over  $K$  leads, by gluing, to the existence of fiber products for rigid spaces over  $K$ .

EXAMPLE 2.11. The weight space  $\mathcal{W} = \{(t_1, \dots, t_d) \in \mathbb{C}_p^d : v_p(t_k) > 0, 1 \leq k \leq d\} \subseteq \mathbb{B}^n$  has an admissible covering  $\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}^{\geq 1/p^{1/n}}$ , where

$$\mathcal{W}^{\geq 1/p^{1/n}} = \{(t_1, \dots, t_d) \in \mathbb{C}_p^d : |t_k| \leq |p^{1/n}| = 1/p^{1/n}, 1 \leq k \leq d\}.$$

By definition  $\mathcal{W}^{\geq 1/p^{1/n}}$  is a rational subdomain in  $\mathbb{B}^n$ . So  $\mathcal{W}$  is a rigid analytic variety and a subspace of  $\mathbb{B}^n$ .

## 2.2 L-function

In this section, we review the basic properties of  $L$ -functions. We fix a  $\mathbb{Z}_{p^d}$ -tower  $\mathcal{P} : \cdots \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 = \mathbb{A}_{\mathbb{F}_q}^1$ , defined by the Witt vector equation  $\underline{Y}^{F^d} - \underline{Y} = f([x])$ . Throughout this chapter, we assume  $\mathcal{P}$  is totally ramified at infinity and unramified at other points.

Let  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$  be a finite continuous character. Recall that the  $L$ -function over  $\mathbb{A}_{\mathbb{F}_q}^1$  is

$$L_f(\chi, s) = \prod_{\mathfrak{p} \in |\mathbb{A}_{\mathbb{F}_q}^1|} \frac{1}{1 - \chi(\text{Frob}_{\mathfrak{p}})s^{\deg \mathfrak{p}}} \in 1 + s\overline{\mathbb{Z}}_p[[s]],$$

where  $\text{Frob}_{\mathfrak{p}} \in \mathbb{Z}_{p^d} = W(\mathbb{F}_{p^d})$  is the Frobenius element associated to  $\mathfrak{p}$ .

Since  $\mathfrak{p} \neq \infty$ , the tower is unramified at  $\mathfrak{p}$ . Let  $x \in (\overline{\mathbb{F}}_q)^*$  be any of the conjugate geometry points in the closed point  $\mathfrak{p}$ . When there is no ambiguity, we use  $\mathfrak{p}$  and  $x$  interchangeably.

By definition the Frobenius element satisfies

$$\text{Frob}_{\mathfrak{p}}(Y_i) \equiv Y_i^{p^{a \deg \mathfrak{p}}} \pmod{X - x}.$$

Note that in finite extension of  $K = \mathbb{F}_q[X]$ , modulo  $X - x$  is same as setting  $X = x$ . Let  $\underline{y} = (y_1, y_2, \dots) \in W(\overline{\mathbb{F}}_p)$  be such that  $\underline{y}^{F^d} - \underline{y} = f([x])$ , where  $[\cdot]$  denotes the Teichmüller lifting. Then, we have

$$\begin{aligned} \text{Frob}_{\mathfrak{p}} &= \underline{y}^{F^{a \deg \mathfrak{p}}} - \underline{y} \\ &= \sum_{n=1}^{(a/d) \deg \mathfrak{p}} \underline{y}^{F^{nd}} - \underline{y}^{F^{(n-1)d}} \\ &= \sum_{n=1}^{(a/d) \deg \mathfrak{p}} f([x])^{F^{(n-1)d}} \\ &= \text{Tr}_{\mathbb{Q}_{q^{\deg \mathfrak{p}}} / \mathbb{Q}_{p^d}} f([x]). \end{aligned}$$

Therefore, we have a second formula for the  $L$ -function:

$$L_f(\chi, s) = \prod_{x \in |\mathbb{A}_{\mathbb{F}_q}^1|} \frac{1}{1 - \chi \left( \text{Tr}_{\mathbb{Q}_q^{\deg(x)}/\mathbb{Q}_{p^d}} f([x]) \right) s^{\deg(x)}}.$$

The zeta function has a decomposition as a product of  $L$ -functions. Suppose  $x \in |\mathbb{A}_{\mathbb{F}_q}^1|$  and  $y_1, \dots, y_r$  are closed points in  $|C_n|$  lying over  $x$ , let  $F$  be the subgroup generated by  $\text{Frob}_x$  in  $\text{Gal}(C_n/\mathbb{A}^1) = \mathbb{Z}_{p^d}/(p^n)$ . Then  $\deg(y_i) = |F| \cdot \deg(x)$  for all  $i$  and  $r = \frac{p^nd}{|F|}$ , which means

$$\begin{aligned} \prod_{i=1}^r \frac{1}{1 - s^{\deg(y_i)}} &= \prod_{i=1}^r \prod_{\zeta^{|F|=1}} \frac{1}{1 - \zeta s^{\deg(x)}} \\ &= \prod_{\chi: \mathbb{Z}_{p^d}/(p^n) \rightarrow C_p^*} \frac{1}{1 - \chi(\text{Frob}_x) s^{\deg(x)}}, \end{aligned}$$

$$\begin{aligned} Z(C_n^{\text{affine}}, s) &= \prod_{y \in |C_n^{\text{affine}}|} \frac{1}{1 - s^{\deg(y)}} \\ &= \prod_{\chi \text{ factors through } (p^n)} \prod_{x \in |\mathbb{A}^1|} \frac{1}{1 - \chi(\text{Frob}_x) s^{\deg(x)}} \\ &= \prod_{\chi \text{ factors through } (p^n)} L_f(\chi, s). \end{aligned}$$

In this thesis, we use two other functions to help us study the  $L$ -function over  $\mathbb{A}_{\mathbb{F}_q}^1$ . The first one is the  $L$ -function over the torus  $\mathbb{G}_m = \mathbb{A}_{\mathbb{F}_q}^1 - \{0\}$ . As mentioned in the introduction, this is

$$L_f^*(\chi, s) = \prod_{x \in |\mathbb{G}_m|} \frac{1}{1 - \chi(\text{Frob}_x) s^{\deg(x)}} = \prod_{x \in |\mathbb{G}_m|} \frac{1}{1 - \chi \left( \text{Tr}_{\mathbb{Q}_q^{\deg(x)}/\mathbb{Q}_{p^d}} f([x]) \right) s^{\deg(x)}} \in 1 + s\overline{\mathbb{Z}}_p[[s]].$$

Since  $x = 0$  is a point of degree one, we have

$$L_f^*(\chi, s) = (1 - \chi(\text{Frob}_0)s)L_f(\chi, s),$$

so the Newton polygon of  $L_f^*(\chi, s)$  always has one more line segment of slope zero than the Newton polygon of  $L_f(\chi, s)$ .

The second one is the characteristic function (over  $\mathbb{G}_m$ ):

$$C_f^*(\chi, s) = L_f^*(\chi, s)L_f^*(\chi, qs)L_f^*(\chi, q^2s)\cdots \in 1 + s\overline{\mathbb{Z}}_p[[s]].$$

This function is studied in Chapter 4.

## 2.3 Genus stability

For the purpose of this section, fix a  $\mathbb{Z}_p^d$ -tower  $\mathcal{P}$  associated to  $f$ , where  $f = f([X]) = b_0 + \sum_{i \geq 1, (i,p)=1} b_i$  is a power series in  $[X] = (X, 0, 0, \dots)$  such that  $\lim_{i \rightarrow \infty} b_i = 0$ . Suppose  $\mathcal{P}$  is a *strongly genus stable* tower in the sense of Definition 1.4, and recall from this definition that  $\delta = \max \left\{ \frac{i}{p^{v_p(b_i)}} : i = 0, 1, 2, \dots \right\}$ , and  $m = -v_p(\delta)$ . The goal of this section is to justify the terminology by showing that the genus growth in this tower is indeed in a degree  $d + 1$  polynomial of  $p^n$ . We use the factorization of zeta function proved in the previous section and a formula for the degrees of  $L$ -functions.

We write

$$f = \sum_{i=0}^{\infty} p^i f_i,$$

where

$$f_i = \sum_{j=0}^{\infty} [a_{ij}][X]^j,$$

with  $a_{ij} \in \mathbb{F}_q$  and  $a_{ij} = 0$  if  $p|j$ . Note that  $\lim_{i \rightarrow \infty} b_i = 0$  implies that each  $f_i$  is a polynomial

in  $[X]$ . Denote  $d_i = \deg f_i$ , then

$$\delta = \max \left\{ \frac{d_i}{p^i} : i = 0, 1, 2, \dots \right\}.$$

Furthermore,  $\delta = \frac{d_m}{p^m}$  and since  $p \nmid d_i$  for all  $i \in \mathbb{Z}_{\geq 0}$ , we have  $\delta > \frac{d_i}{p^i}$  for all  $i \neq m$ .

Following the treatment of [11], for each finite character  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$  of order  $p^{m_\chi}$ , we associate  $f$  with a Newton polyhedron  $\Delta = \Delta(f, \chi)$ , which is the convex hull in  $\mathbb{Q}$  of the set  $\{p^{m_\chi-1-i}d_i : 0 \leq i \leq m-1\} \cup \{0\}$ . The degree of  $L_f^*(\chi, s)$  is calculated by:

**Theorem 2.12** (Liu–Wei, 2007). *The L-function  $L_f^*(\chi, s)$  is a polynomial of degree  $\text{Vol}(\Delta)$ .*

*Proof.* This is a special case ( $n = 1$ ) of [11, Theorem 1.3]. □

**Corollary 2.13.** *The L-function  $L_f^*(\chi, s)$  is a polynomial of degree at most  $\delta p^{m_\chi-1}$ . Furthermore, when  $m_\chi \geq m + 1$ , the degree is equal to  $\delta p^{m_\chi-1}$ .*

*Proof.* The Newton polyhedron  $\Delta$  is a line segment on  $\mathbb{Q}$ . Its length is

$$\max_{0 \leq i \leq m-1} \{p^{m_\chi-1-i}d_i\} = p^{m_\chi-1} \cdot \max_{0 \leq i \leq m-1} \left\{ \frac{d_i}{p^i} \right\} \leq p^{m_\chi-1} \delta.$$

Furthermore, when  $m_\chi \geq m + 1$ , we have

$$\max_{0 \leq i \leq m-1} \left\{ \frac{d_i}{p^i} \right\} = \frac{d_m}{p^m} = \delta.$$

□

**Corollary 2.14.** *The quotient  $\frac{P(C_{n+1}, s)}{P(C_n, s)}$  is a polynomial of degree at most  $(\delta p^n - 1)(p^d - 1)p^{dn}$ .*

*Furthermore, the degree is equal to  $(\delta p^n - 1)(p^d - 1)p^{dn}$  if  $n \geq m$ .*

*Proof.* Since  $L_f^*(\chi, s) = (1 - \chi(\text{Frob}_0)s)L_f(\chi, s)$ , the L-function over  $\mathbb{A}^1$ ,  $L_f(\chi, s)$ , has degree

at most  $\delta p^{m_\chi-1} - 1$ , and equal to  $\delta p^{m_\chi-1} - 1$  if  $m_\chi \geq m$ . By the decomposition of  $Z(C_n^{\text{affine}}, s)$ ,

$$\frac{P(C_{n+1}, s)}{P(C_n, s)} = \frac{Z(C_{n+1}^{\text{affine}}, s)}{Z(C_n^{\text{affine}}, s)} = \prod_{\chi: m_\chi = n+1} L_f(\chi, s).$$

There are  $(p^d - 1)p^{dn}$  many characters with  $m_\chi = n + 1$  in total, and we have  $\deg L_f(\chi, s) \leq \delta p^{(n+1)-1} - 1$ . So  $\frac{P(C_{n+1}, s)}{P(C_n, s)}$  is a polynomial with  $\deg \frac{P(C_{n+1}, s)}{P(C_n, s)} \leq (\delta p^n - 1)(p^d - 1)p^{dn}$ . If  $n \geq m$ , we have  $\deg L_f(\chi, s) = \delta p^{(n+1)-1} - 1$ , so the degree is equal to  $(\delta p^n - 1)(p^d - 1)p^{dn}$ .  $\square$

**Corollary 2.15.** *Let  $g_n$  denote the genus of  $C_n$ . Then*

$$g_n \leq \frac{\delta(p^d - 1)}{2(p^{d+1} - 1)} ((p^n)^{d+1} - 1) - \frac{(p^n)^d - 1}{2}.$$

Furthermore, there are rational numbers  $a, b, c$  determined by  $f$  and  $d$  such that for all  $n \geq m$ ,

$$g_n = a(p^n)^{d+1} + b(p^n)^d + c.$$

In particular, the tower  $\mathcal{P} : \cdots \rightarrow C_1 \rightarrow C_0 = \mathbb{A}^1$  is genus stable.

*Proof.* Recall that  $2g_n = \deg P(C_n, s)$ . So  $g_0 = 0$ , and

$$\begin{aligned} 2g_n &= (2g_n - 2g_{n-1}) + (2g_{n-1} - 2g_{n-2}) + \cdots + (2g_1 - 2g_0) \\ &\leq \sum_{i=0}^{n-1} (\delta p^i - 1)(p^d - 1)p^{di} \\ &= \frac{\delta(p^d - 1)}{p^{d+1} - 1} ((p^n)^{d+1} - 1) - (p^n)^d + 1. \end{aligned}$$

The upper bound for  $g_n$  is proved. Furthermore,

$$\begin{aligned}
2g_n &= (2g_n - 2g_{n-1}) + (2g_{n-1} - 2g_{n-2}) + \cdots + (2g_{m+1} - 2g_m) + 2g_m \\
&= \sum_{i=m}^{n-1} (\delta p^i - 1)(p^d - 1)p^{di} + 2g_m \\
&= \frac{\delta(p^d - 1)}{p^{d+1} - 1} ((p^n)^{d+1} - (p^m)^{d+1}) - (p^n)^d + (p^m)^d + 2g_m \\
&= \frac{\delta(p^d - 1)}{p^{d+1} - 1} (p^n)^{d+1} - (p^n)^d + \left( (p^m)^d + 2g_m - \frac{\delta(p^d - 1)(p^m)^{d+1}}{p^{d+1} - 1} \right).
\end{aligned}$$

We can take

$$a = \frac{\delta(p^d - 1)}{2(p^{d+1} - 1)}, \quad b = -\frac{1}{2}, \quad c = \frac{1}{2} \left( (p^m)^d + 2g_m - \frac{\delta(p^d - 1)(p^m)^{d+1}}{p^{d+1} - 1} \right).$$

□

# Chapter 3

## Slopes of the zeta functions

The goal of this chapter is to prove Theorem 1.6. We fix  $f = f([X]) = b_0 + \sum_{i:(i,p)=1} b_i [X]^i \in W(K)$  and let  $d|a$ . Consider the  $\mathbb{Z}_p^d$ -tower associated to  $f$ ,

$$\mathcal{P} : \cdots \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 = \mathbb{P}^1.$$

As we mentioned in the introduction, the zeta function of  $C_n$  has a decomposition into product of  $L$ -function, thus reducing the problem to the study of  $L$ -functions. Therefore it would be helpful if all such  $L$ -function belongs to a parameterized family. We build the theory by first defining the universal character, which parametrizes all the finite characters.

### 3.1 The universal character

Let  $\beta = \{c_1, \dots, c_d\}$  be a basis for  $\mathbb{Z}_p^d$  as a free  $\mathbb{Z}_p$ -module, and let  $c_1^*, \dots, c_d^*$  be such that  $\text{Tr}_{\mathbb{Q}_p^d/\mathbb{Q}_p}(c_k c_k^*) = 1$  and  $\text{Tr}_{\mathbb{Q}_p^d/\mathbb{Q}_p}(c_i c_j^*) = 0$  if  $i \neq j$ . In other words, the dual basis of  $\beta$  is given by  $\{\text{Tr}(c_k^* \cdot -)\}_{k=1}^d$ . Note that  $c_1, \dots, c_d$  uniquely determines an isomorphism between

the  $\mathbb{Z}_p$ -modules  $\mathbb{Z}_{p^d}$  and  $\mathbb{Z}_p^d$ ,

$$\begin{array}{ccc} \mathbb{Z}_{p^d} & \xrightarrow{\cong} & \mathbb{Z}_p^d \\ x & \longmapsto & \left( \mathrm{Tr}_{\mathbb{Q}_{p^d}/\mathbb{Q}_p}(xc_1), \dots, \mathrm{Tr}_{\mathbb{Q}_{p^d}/\mathbb{Q}_p}(xc_d) \right) \\ y_1c_1^* + \dots + y_dc_d^* & \longleftarrow & (y_1, \dots, y_d). \end{array}$$

Using this correspondence, the *universal character* of  $\mathbb{Z}_{p^d}$  is defined by

$$\begin{array}{ccc} \chi_{\mathrm{univ}} : \mathbb{Z}_{p^d} & \longrightarrow & \mathbb{Z}_p[[T_1, \dots, T_d]]^* \\ x & \longmapsto & (1 + T_1)^{\mathrm{Tr}_{\mathbb{Q}_{p^d}/\mathbb{Q}_p}(xc_1)} \dots (1 + T_d)^{\mathrm{Tr}_{\mathbb{Q}_{p^d}/\mathbb{Q}_p}(xc_d)}. \end{array}$$

Suppose  $x \in |\mathbb{A}^1|$  is a closed point, then

$$\begin{aligned} \chi_{\mathrm{univ}}(\mathrm{Frob}_x) &= \chi_{\mathrm{univ}}\left(\mathrm{Tr}_{\mathbb{Q}_{q^{\mathrm{deg}} x}/\mathbb{Q}_{p^d}}(f[x])\right) \\ &= \prod_{k=1}^d (1 + T_k)^{\mathrm{Tr}_{\mathbb{Q}_{p^d}/\mathbb{Q}_p}\left(\mathrm{Tr}_{\mathbb{Q}_{q^{\mathrm{deg}} x}/\mathbb{Q}_{p^d}}(f[x])\right)} \\ &= \prod_{k=1}^d (1 + T_k)^{\mathrm{Tr}_{\mathbb{Q}_{q^{\mathrm{deg}} x}/\mathbb{Q}_p}(f[x])}. \end{aligned}$$

Any continuous character  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$  can be recovered from  $\chi_{\mathrm{univ}}$  by evaluating  $T_k$  at  $\chi(c_k^*) - 1$  for all  $k$ ,

$$\chi : \mathbb{Z}_{p^d} \xrightarrow{\chi_{\mathrm{univ}}} \mathbb{Z}_p[[T_1, \dots, T_d]]^* \xrightarrow{T_k \mapsto \chi(c_k^*) - 1 \text{ for all } k} \mathbb{C}_p^*.$$

## 3.2 The $T$ -adic $L$ -series and characteristic series

Now we describe how to parameterize the family of  $L$ -function. Since any continuous character is a specialization of the universal character, if we define a universal  $L$ -series as the  $L$ -series attached to the universal character, then we can recover any  $L$ -function from it by the same specialization. This idea leads to the definition of the The  $T$ -adic  $L$ -series. Note that the definition of the universal character depend on the choice of basis  $\beta = \{c_1, \dots, c_d\}$ , so this  $T$ -adic  $L$ -series should also implicitly dependent on the choice of basis.

**Definition 3.1.** The (rank  $d$ )  $T$ -adic  $L$ -series of  $f$  over  $\mathbb{G}_m$  is defined by

$$L_f^*(\underline{T}, s) = L_{f,\beta}^*(\underline{T}, s) := \prod_{x \in |\mathbb{G}_m|} \frac{1}{1 - \chi_{\text{univ}}(\text{Frob}_x) s^{\deg(x)}} \in 1 + s\mathbb{Z}_p[[T_1, \dots, T_d]][[s]].$$

This definition implicitly depends on the basis  $\beta$  because the definition of the universal character depends on  $\beta$ . We include  $\beta$  in the subscript when we want to emphasize the choice of basis. Otherwise,  $\beta$  is omitted.

Let  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$  be a continuous character. By evaluating  $T_k$  at  $\chi(c_k^*) - 1$  for all  $k$ , we immediately get

$$L_f^*(\chi, s) = \prod_{x \in |\mathbb{G}_m|} \frac{1}{1 - \chi(\text{Frob}_x) s^{\deg(x)}} = L_f^*(\underline{T}, s) \Big|_{T_k = \chi(c_k^*) - 1 \text{ for all } k}.$$

Therefore, the  $T$ -adic  $L$ -series  $L_f^*(\underline{T}, s)$  is universal in the sense that the  $L$ -function  $L_f^*(\chi, s)$  can be recovered from the  $T$ -adic  $L$ -series.

We also have the characteristic series:

**Definition 3.2.** The (rank  $d$ )  $T$ -adic characteristic series associated to  $f$  over  $\mathbb{G}_m$  is defined

by

$$C_f^*(\underline{T}, s) = C_{f,\beta}^*(\underline{T}, s) := L_f^*(\underline{T}, s)L_f^*(\underline{T}, qs)L_f^*(\underline{T}, q^2s) \cdots \in 1 + s\mathbb{Z}_p[[T_1, \dots, T_d]][[s]].$$

The *characteristic function of a continuous character*  $\chi$  (over  $\mathbb{G}_m$ ), as mentioned in the introduction, is

$$C_f^*(\chi, s) = L_f^*(\chi, s)L_f^*(\chi, qs)L_f^*(\chi, q^2s) \cdots \in 1 + s\overline{\mathbb{Z}}_p[[s]].$$

By the relation between  $L_f^*(\underline{T}, s)$  and  $L_f^*(\chi, s)$ , it is clear that

$$C_f^*(\chi, s) = C_f^*(\underline{T}, s) \Big|_{T_k = \chi(c_k^*) - 1 \text{ for all } k}.$$

$L_f^*(\underline{T}, s)$  and  $C_f^*(\underline{T}, s)$  also have generating series expressions, which are in fact more convenient to deal with than the Euler product expression. To express them as generating series, we need to use the  $T$ -adic exponential sums.

**Definition 3.3.** Let  $d$  be a positive integer dividing  $a$ . For a positive integer  $n$  and a basis  $\beta = \{c_1, \dots, c_d\}$  for  $\mathbb{Z}_p^d$  as a free  $\mathbb{Z}_p$ -module, the (rank  $d$ )  $n$ -th  $T$ -adic exponential sum of  $f$  over  $\mathbb{F}_{q^n}^*$  is defined by

$$S_f^*(n, \underline{T}) = S_{f,\beta}^*(n, \underline{T}) := \sum_{x \in \mathbb{F}_{q^n}^*} \prod_{1 \leq k \leq d} (1 + T_k)^{\text{Tr}_{\mathbb{Q}_q^n/\mathbb{Q}_p}(c_k f([x]))},$$

where  $\underline{T} = (T_1, \dots, T_d)$ . We include  $\beta$  in the subscript of  $S$  if we want to emphasize the choice of basis, or if we want to emphasize the rank  $d$ , which is also the cardinality of the basis  $\beta$ .

**Proposition 3.4.**

(1)  $L_f^*(\underline{T}, s)$  is the generating series

$$L_f^*(\underline{T}, s) = L_{f,\beta}^*(\underline{T}, s) := \exp \left( \sum_{n=1}^{\infty} S_{f,\beta}^*(n, \underline{T}) \frac{s^n}{n} \right).$$

(2)  $C_f^*(\underline{T}, s)$  is the generating series

$$C_f^*(\underline{T}, s) = C_{f,\beta}^*(\underline{T}, s) := \exp \left( \sum_{n=1}^{\infty} \frac{1}{1 - q^n} S_{f,\beta}^*(n, \underline{T}) \frac{s^n}{n} \right).$$

*Proof.*

$$\begin{aligned} L_f^*(\chi_{\text{univ}}, s) &= \prod_{x \in |\mathbb{A}_{\mathbb{F}_q}^1| - \{0\}} \exp(-\log(1 - \chi(\text{Frob}_x) s^{\deg x})) \\ &= \prod_{x \in |\mathbb{A}_{\mathbb{F}_q}^1| - \{0\}} \exp \left( \sum_{r=1}^{\infty} \frac{1}{r} (\chi(\text{Frob}_x) s^{\deg x})^r \right) \\ &= \exp \left( \sum_{x \in |\mathbb{A}_{\mathbb{F}_q}^1| - \{0\}} \sum_{r=1}^{\infty} \frac{1}{r} \left( \prod_{k=1}^d (1 + T_k)^{\text{Tr}_{\mathbb{Q}_q^{\deg x}/\mathbb{Q}_p}(c_k f([x]))} s^{\deg x} \right)^r \right) \\ &= \exp \left( \sum_{x \in |\mathbb{A}_{\mathbb{F}_q}^1| - \{0\}} \sum_{r=1}^{\infty} \prod_{k=1}^d (1 + T_k)^{r \text{Tr}_{\mathbb{Q}_q^{\deg x}/\mathbb{Q}_p}(c_k f([x]))} \frac{1}{r} s^{r \deg x} \right) \\ &= \exp \left( \sum_{x \in (\mathbb{F}_q)^*} \sum_{r=1}^{\infty} \prod_{k=1}^d (1 + T_k)^{\text{Tr}_{\mathbb{Q}_q^{r \deg x}/\mathbb{Q}_p}(c_k f([x]))} \frac{1}{r \deg x} s^{r \deg x} \right) \\ &= \exp \left( \sum_{n=1}^{\infty} \sum_{x \in \mathbb{F}_{q^n}^*} \prod_{k=1}^d (1 + T_k)^{\text{Tr}_{\mathbb{Q}_{q^n}/\mathbb{Q}_p}(c_k f([x]))} \frac{1}{n} s^n \right). \end{aligned}$$

The second last equality holds because a close point on the affine line splits into  $\deg x$  many

conjugates in  $(\overline{\mathbb{F}}_q)^*$ . The equation for  $L_f^*(\underline{T}, s)$  is proved. As for the characteristic series,

$$\begin{aligned}
C_f^*(\underline{T}, s) &= \prod_{k=0}^{\infty} L_f^*(\underline{T}, q^k s) \\
&= \prod_{k=0}^{\infty} \exp \left( \sum_{n=1}^{\infty} S_f^*(n, \underline{T}) \frac{(q^k s)^n}{n} \right) \\
&= \exp \left( \sum_{n=1}^{\infty} S_f^*(n, \underline{T}) \left( \sum_{k=0}^{\infty} \frac{(q^k s)^n}{n} \right) \right) \\
&= \exp \left( \sum_{n=1}^{\infty} \frac{1}{1 - q^n} S_f^*(n, \underline{T}) \frac{s^n}{n} \right).
\end{aligned}$$

□

### 3.3 Generalization to higher rank

The goal of this section is to prove the higher rank case of Theorems 1.7 and 1.9. We make use of the rank one case of these two theorems, which are proved by Kusters–Zhu in [5] and stated below.

**Theorem 3.5.** *Let  $\chi$  be a finite character of  $\mathbb{Z}_p$ . We can write*

$$L_f(\chi, s) = \sum_{k=0}^{\delta p^{m_\chi - 1} - 1} a_k s^k,$$

such that:

(1)  $a_0 = 1$ . For any  $0 < n < p^{m_\chi - m - 1}$ , we have<sup>1</sup>

$$v_q(a_{nd_{m-1}}) = \frac{n(nd_m - 1)}{2p^{m_\chi - m - 1}} \text{ and } v_q(a_{nd_m}) = \frac{n(nd_m + 1)}{2p^{m_\chi - m - 1}},$$

---

<sup>1</sup>We fix a typo of Kusters–Zhu here. Note that we cannot take  $n = p^{m_\chi - m - 1}$ , because otherwise  $nd_m > \delta p^{m_\chi - 1} - 1$ .

and we have

$$v_q(a_{\delta p^{m_\chi-1}-1}) = \frac{\delta p^{m_\chi-1} - 1}{2}.$$

- (2) The  $q$ -adic Newton polygon of  $L_f(\chi, s)$  starts from  $(0, 0)$ , ends at  $(\delta p^{m_\chi-1}-1, \frac{\delta p^{m_\chi-1}-1}{2})$ , and passes through the points

$$\left( nd_m - 1, \frac{n(nd_m - 1)}{2p^{m_\chi-m-1}} \right) \text{ and } \left( nd_m, \frac{n(nd_m + 1)}{2p^{m_\chi-m-1}} \right),$$

for any  $0 < n < p^{m_\chi-1}$ .

- (3) The  $q$ -adic Newton polygon of  $L_f(\chi, s)$  has slopes (in multiset notation and in increasing order)

$$\bigcup_{i=1}^{p^{m_\chi-m-1}} \{\alpha_{i1}, \dots, \alpha_{id_m}\} - \{0\},$$

where

$$\begin{cases} \alpha_{ij} = \frac{i-1}{p^{m_\chi-m-1}} & \text{if } j = 1, \\ \frac{i-1}{p^{m_\chi-m-1}} < \alpha_{ij} < \frac{i}{p^{m_\chi-m-1}} & \text{if } j > 1. \end{cases}$$

*Proof.* This is [5, Theorem 7.1]. □

**Theorem 3.6.** Suppose  $\mathcal{P}$  is strongly genus stable with  $\delta(\mathcal{P}) = \delta$ . Put  $C = \frac{a(p-1)(d_m-1)^2}{8}$ .

Suppose  $v_p(b_j) > i$  for all  $j > \delta p^i - C$  when  $i > N$ , and put

$$m' = 1 + \left\lceil \log_p \left( \frac{p^N + C/\delta}{p-1} \right) \right\rceil.$$

Then there is character  $\chi_0$  of order  $p^{m'}$ , such that any character  $\chi$  of order  $p^{m_\chi} \geq p^{m'}$  belongs to the  $f$ -stable family of  $\chi_0$ .

*Proof.* See the proof for Theorem B and Remark 8.6 in [5]. □

Fix a finite continuous character  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$ , and let  $p^{m_\chi}$  be the order of  $\chi$ . The first step is to find a corresponding  $\mathbb{Z}_p$ -tower and a character of  $\mathbb{Z}_p$ , so that we can apply the rank one case.

Let  $c_1^* \in \mathbb{Z}_{p^d}$  be such that  $\chi(c_1^*)$  is a primitive  $p^{m_\chi}$ -th root of unity, and extend it to a basis  $c_1^*, c_2^*, \dots, c_d^*$  of  $\mathbb{Z}_{p^d}$  as a free  $\mathbb{Z}_p$ -module, such that  $\chi(c_k^*) = 1$  for  $k \geq 2$ . This is always possible because otherwise we can replace  $c_k^*$  by  $c_k^* - nc_1^*$  for some integer  $n$ . Let  $c_k \in \mathbb{Z}_{p^d}$  be such that  $\text{Tr}_{\mathbb{Q}_{p^d}/\mathbb{Q}_p}(c_k c_k^*) = 1$  and  $\text{Tr}_{\mathbb{Q}_{p^d}/\mathbb{Q}_p}(c_i c_j^*) = 0$  if  $i \neq j$ .

Now denote  $\beta = \{c_1, \dots, c_d\}$ . We have

$$\begin{aligned} L_f^*(\chi, s) &= L_{f,\beta}^*(\underline{T}, s) \Big|_{T_1=\chi(c_1^*)-1, T_k=0 \text{ for } k \geq 2} \\ &= \exp \left( \sum_{n=1}^{\infty} S_{f,\beta}^*(n, \underline{T}) \frac{s^n}{n} \right) \Big|_{T_1=\chi(c_1^*)-1, T_k=0 \text{ for } k \geq 2} \\ &= \exp \left( \sum_{n=1}^{\infty} S_{c_1 f, \{1\}}^*(n, T) \frac{s^n}{n} \right) \Big|_{T=\chi(c_1^*)-1}, \end{aligned} \tag{3.1}$$

Let  $\bar{\chi} : \mathbb{Z}_p \rightarrow \mathbb{C}_p^*$  be the restriction of  $\chi$  to the subgroup  $c_1^* \mathbb{Z}_p$  of  $\mathbb{Z}_{p^d}$ , and identify  $c_1^* \mathbb{Z}_p$  with  $\mathbb{Z}_p$  in the canonical way. Equivalently, this means

$$\bar{\chi}(1) = \chi(c_1^*).$$

For the  $\mathbb{Z}_p$ -tower associated to  $c_1 f$  and for the character  $\bar{\chi}$ , we have the  $L$ -function

$$\begin{aligned} L_{c_1 f}^*(\bar{\chi}, s) &= L_{c_1 f, \{1\}}^*(T, s) \Big|_{T=\chi(c_1^*)-1} \\ &= \exp \left( \sum_{n=1}^{\infty} S_{c_1 f, \{1\}}^*(n, T) \frac{s^n}{n} \right) \Big|_{T=\chi(c_1^*)-1}. \end{aligned} \tag{3.2}$$

By comparing (3.1) and (3.2), we get:

**Lemma 3.7.**

$$L_f^*(\chi, s) = L_{c_1f}^*(\bar{\chi}, s).$$

Recall that the Newton polygon of  $L_f^*(\chi, s)$  always has one more line segment of slope zero than the Newton polygon of  $L_f(\chi, s)$ . As a consequence, we have:

**Corollary 3.8.** *The  $q$ -adic Newton polygon of  $L_f(\chi, s)$  is the same as that of  $L_{c_1f}(\bar{\chi}, s)$ .*

This corollary says that if we know the Newton polygon of the rank one  $L$ -function, we immediately get the Newton polygon of the rank  $d$   $L$ -function. Now we use this idea to divide characters into stable families, and eventually prove Theorems 1.7 and 1.9.

*Proof of Theorem 1.7.* Note that

$$c_1f([X]) = c_1b_0 + \sum_{i:(i,p)=1} c_1b_i[X]^i,$$

so

$$\max \left\{ \frac{i}{p^{v_p(b_i)}} : i = 0, 1, 2, \dots \right\} = \delta = \max \left\{ \frac{i}{p^{v_p(c_1b_i)}} : i = 0, 1, 2, \dots \right\}.$$

Apply Theorem 3.5, we can write

$$L_{c_1f}(\bar{\chi}, s) = \sum_{k=0}^{\delta p^{m\chi-1}-1} a_k s^k,$$

such that:

(1)  $a_0 = 1$ . For any  $0 < n < p^{m_\chi - m - 1}$ , we have

$$v_q(a_{nd_{m-1}}) = \frac{n(nd_m - 1)}{2p^{m_\chi - m - 1}} \text{ and } v_q(a_{nd_m}) = \frac{n(nd_m + 1)}{2p^{m_\chi - m - 1}},$$

and we have

$$v_q(a_{\delta p^{m_\chi - 1} - 1}) = \frac{\delta p^{m_\chi - 1} - 1}{2}.$$

(2) The  $q$ -adic Newton polygon of  $L_{c1f}(\bar{\chi}, s)$  starts from  $(0, 0)$ , ends at  $(\delta p^{m_\chi - 1} - 1, \frac{\delta p^{m_\chi - 1} - 1}{2})$ , and passes through the points

$$\left( nd_m - 1, \frac{n(nd_m - 1)}{2p^{m_\chi - m - 1}} \right) \text{ and } \left( nd_m, \frac{n(nd_m + 1)}{2p^{m_\chi - m - 1}} \right),$$

for any  $0 < n < p^{m_\chi - 1}$ .

(3) The  $q$ -adic Newton polygon of  $L_{c1f}(\bar{\chi}, s)$  has slopes (in multiset notation and in increasing order)

$$\bigcup_{i=1}^{p^{m_\chi - m - 1}} \{ \alpha_{i1}, \dots, \alpha_{id_m} \} - \{0\},$$

where

$$\begin{cases} \alpha_{ij} = \frac{i-1}{p^{m_\chi - m - 1}} & \text{if } j = 1, \\ \frac{i-1}{p^{m_\chi - m - 1}} < \alpha_{ij} < \frac{i}{p^{m_\chi - m - 1}} & \text{if } j > 1. \end{cases}$$

Therefore, by Corollary 3.8, the same is true for  $L_f(\chi, s)$ , for the same  $\delta$ . □

*Proof of Theorem 1.9.* Let  $f([X]) = b_0 + \sum_{i:(i,p)=1} b_i [X]^i$  be such that

$$\delta = \max \left\{ \frac{i}{p^{v_p(b_i)}} : i = 0, 1, 2, \dots \right\} = \frac{d_m}{p^m},$$

where  $m, d_m \in \mathbb{Z}_{\geq 0}$ ,  $p \nmid d_m$ . We take

$$C = \frac{a(p-1)(d_m-1)^2}{8}.$$

Note that  $d_m$  is uniquely determined by  $\delta$  because  $p \nmid d_m$ . Also note that the numbers  $a$  and  $p$  are uniquely determined by  $q$ , so  $C = C(\delta, q)$ . Suppose there exists some positive integer  $N$  such that  $v_p(b_j) > i$  for all  $i > N$  and  $j > \delta p^i - C$ . Note that for such  $i, j$  and for any unit  $c_1 \in \mathbb{Z}_{p^d}^*$ , we automatically have  $v_p(c_1 b_j) > i$ . Now take

$$m' = 1 + \left\lceil \log_p \left( \frac{p^N + C/\delta}{p-1} \right) \right\rceil.$$

Let  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$  be a finite continuous character of order  $p^{m_x} \geq p^{m'}$ . Chose a basis  $\beta^* = \{c_1^*, c_2^*, \dots, c_d^*\}$  of  $\mathbb{Z}_p^d$  such that  $\chi(c_1^*)$  is a primitive  $p^{m_x}$ -th root of unity and  $\chi(c_2^*) = \dots = \chi(c_d^*) = 1$ . Let  $c_1, \dots, c_d$  and  $\bar{\chi}$  be as defined in this section. Define  $\chi_0 : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$  to be the character such that

$$\chi_0(c_1^*) = \zeta, \chi_0(c_2^*) = \dots = \chi_0(c_d^*) = 1,$$

where  $\zeta$  is *any* primitive  $p^{m'}$ -th root of unity. We will show that  $\chi$  is in the  $c_1 f$ -stable family of  $\chi_0$ .

Consider the  $\mathbb{Z}_p$ -tower associated to  $c_1 f$ . Let  $\bar{\chi}_0$  be a character of  $\mathbb{Z}_p$ , such that

$$\bar{\chi}_0(1) = \chi_0(c_1^*).$$

By Theorem 3.6,  $\bar{\chi}$  is in the  $c_1 f$ -stable family of  $\bar{\chi}_0$ . By Lemma 3.7, we have

$$L_f(\chi, s) = L_{c_1 f}(\bar{\chi}, s) \text{ and } L_f(\chi_0, s) = L_{c_1 f}(\bar{\chi}_0, s),$$

so  $\chi$  is in the  $f$ -stable family of  $\chi_0$ .

Note that in the construction above,  $\chi_0(c_1^*)$  can be *any* primitive  $p^{m'}$ -th root of unity. This means for any  $n \in (\mathbb{Z}/p^{m'}\mathbb{Z})^*$ , the  $f$ -stable families of  $\chi_0$  and  $\chi_0^n$  are the same. Also note that  $\ker\chi_0 = \ker\chi_0^n$ . The sets in the form  $\{\chi_0^n : n \in (\mathbb{Z}/p^{m'}\mathbb{Z})^*\}$  where  $\chi_0$  is a character of order  $p^{m'}$  form a partition of the set of all characters of order  $p^{m'}$ . To construct the set  $\mathfrak{C}$ , we just need to take one representative in each of the sets  $\{\chi_0^n : n \in (\mathbb{Z}/p^{m'}\mathbb{Z})^*\}$ , which has  $(p-1)p^{m'-1}$  many elements. There are in total  $(p^d-1)p^{d(m'-1)}$  many characters of order  $p^{m'}$ , so

$$\#\mathfrak{C} = \frac{(p^d-1)p^{d(m'-1)}}{(p-1)p^{m'-1}} = \frac{(p^d-1)p^{(d-1)(m'-1)}}{p-1}.$$

Furthermore, suppose  $\chi_0, \chi_1 \in \mathfrak{C}$ ,  $\chi_0 \neq \chi_1$  and  $\chi$  is in the  $f$ -stable family of  $\chi_0$ . Then clearly

$$\ker\chi = p^{m \times} c_1^* \mathbb{Z}_p + c_2^* \mathbb{Z}_p + \cdots + c_d^* \mathbb{Z}_p \subseteq \ker\chi_0.$$

Since  $\chi_0, \chi_1 \in \mathfrak{C}$ ,  $\chi_0 \neq \chi_1$  by our construction there is at least one  $k$ ,  $2 \leq k \leq d$ , such that  $\chi_1(c_k^*) \neq 1$ . So  $c_k^*$  is not in  $\ker\chi_1$ , but it is in  $\ker\chi$ . Therefore,  $\chi$  is in the  $f$ -stable family of  $\chi_0$  if and only if  $\ker\chi \subseteq \ker\chi_0$ .  $\square$

REMARK 3.9. If  $d=1$ , then  $\mathfrak{C}$  contains only one character, so Theorem 1.9 reduces exactly to 3.6.

Now we are ready to proof Theorem 1.6. We state and prove a more precise version of Theorem 1.6:

**Theorem 3.10.**

- (1) *If  $\mathcal{P}$  is a strongly genus stable  $\mathbb{Z}_{p^d}$ -tower, then it is slope uniform.*

(2) Suppose  $\mathcal{P}$  is strongly genus stable with  $\delta(\mathcal{P}) = \delta$ . There exists a constant  $C = C(\delta, q)$ , such that if  $v_p(b_j) > i$  for all  $j > \delta p^i - C$  when  $i$  is large enough, then we have  $\mathcal{P}$  is slope stable. Moreover, let  $m'$  be as in the Slope Stability Theorem for  $L$ -functions, then the multiset of slopes of  $\frac{P(C_n, s)}{P(C_{n-1}, s)}$  is

$$\bigcup_{i=0}^{p^{n-m'}-1} \left\{ \frac{\alpha_1 + i}{p^{n-m'}}, \dots, \frac{\alpha_r + i}{p^{n-m'}} \right\}^{\times (p-1)p^{d(n-m')+m'-1}} \cup \bigcup_{i=1}^{p^{n-m'}-1} \left\{ \frac{i}{p^{n-m'}} \right\}^{\times (p^d-1)p^{dn-d}},$$

where  $r = (\delta p^{m'-1} - 1) \cdot \frac{p^d - 1}{p - 1} \cdot p^{(m'-1)(d-1)}$ .

*Proof.* Recall that

$$\frac{P(C_n, s)}{P(C_{n-1}, s)} = \prod_{\chi \text{ has order } p^n} L_f(\chi, s).$$

By the Slope Uniformity Theorem for  $L$ -functions, taking  $n \rightarrow \infty$ , the slopes of  $L_f(\chi, s)$  approach the uniform distribution on  $[0, 1]$ . So  $\frac{P(C_n, s)}{P(C_{n-1}, s)}$  is a polynomial, and its slopes approach the uniform distribution on  $[0, 1]$ . So the product  $P(C_n, s) = \prod_{k=1}^n \frac{P(C_k, s)}{P(C_{k-1}, s)}$  has slopes approaching the uniform distribution on  $[0, 1]$ . This proves sentence (1).

By Theorem 1.7, for any  $\chi_0 \in \mathfrak{C}$ , we have

$$\deg L_f(\chi_0, s) = \delta p^{m'-1} - 1.$$

Let

$$r = (\delta p^{m'-1} - 1) \cdot \#\mathfrak{C} = (\delta p^{m'-1} - 1) \cdot \frac{p^d - 1}{p - 1} \cdot p^{(m'-1)(d-1)},$$

and let

$$\{\alpha_1, \dots, \alpha_r\} = \bigcup_{\chi_0 \in \mathfrak{C}} \{q\text{-adic slopes of } L_f(\chi_0, s)\}.$$

For  $n \geq m'$ , any character of order  $p^n$  is in the  $f$ -stable family of some  $\chi_0 \in \mathfrak{C}$ . For any  $\chi_0 \in \mathfrak{C}$ , without loss of generality suppose the multi-set of slopes for  $L_f(\chi_0, s)$  is  $\{\alpha_1, \dots, \alpha_{\delta p^{m'-1}-1}\}$ , then by Theorem 1.9, for any  $\chi$  in the  $f$ -stable family of  $\chi_0$  and has order  $p^n$ , the multi-set of slopes for  $L_f(\chi, s)$  is

$$\bigcup_{i=0}^{p^{n-m'}-1} \left\{ \frac{i}{p^{n-m'}}, \frac{\alpha_1 + i}{p^{n-m'}}, \dots, \frac{\alpha_{\delta p^{m'-1}-1} + i}{p^{n-m'}} \right\} - \{0\}.$$

The number of characters of order  $p^n$  does not depend on  $\chi_0$ , because the construction of all such characters does not depend on the choice of basis  $\beta = \{c_1, \dots, c_d\}$ . Therefore, each  $f$ -stable family has

$$\frac{(p^d - 1)p^{d(n-1)}}{\#\mathfrak{C}} = (p-1)p^{d(n-m')+m'-1}$$

many characters of order  $p^n$ . Taking disjoint union of multi-sets of slopes over  $f$ -stable family of  $\chi_0$ , we get the multi-set

$$\bigcup_{i=0}^{p^{n-m'}-1} \left\{ \frac{i}{p^{n-m'}}, \frac{\alpha_1 + i}{p^{n-m'}}, \dots, \frac{\alpha_{\delta p^{m'-1}-1} + i}{p^{n-m'}} \right\}^{\times (p-1)p^{d(n-m')+m'-1}} - \{0\}^{\times (p-1)p^{d(n-m')+m'-1}},$$

where the superscript  $\cdot^{\times k}$  means each element in the underlying multi-set has multiplicity  $k$ .

Then by taking disjoint union over  $\mathfrak{C}$ , we get the multi-set

$$\bigcup_{i=0}^{p^{n-m'}-1} \left\{ \frac{\alpha_1 + i}{p^{n-m'}}, \dots, \frac{\alpha_r + i}{p^{n-m'}} \right\}^{\times (p-1)p^{d(n-m')+m'-1}} \cup \bigcup_{i=1}^{p^{n-m'}-1} \left\{ \frac{i}{p^{n-m'}} \right\}^{\times (p^d-1)p^{dn-d}}.$$

Number of elements in this set is

$$\begin{aligned}
& p^{n-m'} \cdot (r) \cdot (p-1)p^{d(n-m')+m'-1} + (p^{n-m'} - 1)(p^d - 1)p^{dn-d} \\
= & p^{n-m'} \cdot (\delta p^{m'-1} - 1) \cdot \#\mathfrak{C} \cdot (p-1)p^{d(n-m')+m'-1} + (p^{n-m'} - 1)(p^d - 1)p^{dn-d} \\
= & (\delta p^{n-1} - p^{n-m'}) \cdot (p^d - 1)p^{dn-d} + (p^{n-m'} - 1)(p^d - 1)p^{dn-d} \\
= & (\delta p^{n-1} - 1) \cdot (p^d - 1)p^{dn-d}.
\end{aligned}$$

□

# Chapter 4

## Infinite characters

Chapter 3 deals with finite characters. But in order to study the eigenvariety, we also need to consider infinite characters. Note that in Section 3.3, we are able to reduce to rank one case exactly because any finite continuous character  $\chi : \mathbb{Z}_p^d \rightarrow \mathbb{C}^*$  has a cyclic image, so  $\chi$  can always be viewed as the induction of a character of  $\mathbb{Z}_p$ . However this is no longer true for infinite characters. It turns out that we need to put a stronger restriction on the  $\mathbb{Z}_p^d$ -tower and use a different method to treat infinite characters.

### 4.1 Overconvergent $\mathbb{Z}_p^d$ -towers

Let  $\mathcal{P}$  be a strongly genus stable  $\mathbb{Z}_p^d$ -tower associated to  $f([X]) = b_0 + \sum_{i:(i,p)=1} b_i [X]^i \in W(K)$ , totally ramified at infinity and unramified at other points. Let  $\delta$ ,  $m$ ,  $d_i$ , be as in Section 2.3, and note that  $\delta = \frac{d_m}{p^m}$ .

**Definition 4.1.** We say  $f$  is *overconvergent* if there is a positive rational number  $D$  and an integer  $m_D > m$  such that  $d_i < D\delta i$  for all  $i \geq m_D$ ,  $i \in \mathbb{Z}$ . We also say the  $\mathbb{Z}_p^d$ -tower associated to such  $f$  is overconvergent.

REMARK 4.2. Consider  $f$  as a power series in  $\mathbb{Z}_q[[x, 0, 0, \dots]] \subsetneq W(K)$ . The generic radius of convergence is one. When such  $D$  exists, however, the radius of convergence is at least  $p^{D\delta}$ . In fact, one can check that the existence of such  $D$  is equivalent to the radius of convergence of  $f$  being strictly greater than one. In other words,  $f$  is overconvergent means  $v_p(b_i)$  has a linear growth when  $i \rightarrow \infty$ , whereas  $f$  is strongly genus stable means  $v_p(b_i)$  has at least a logarithmic growth when  $i \rightarrow \infty$ , where  $b_i \in \mathbb{Z}_q$  is the  $i$ -th coefficient of  $f$ .

Note that for overconvergent  $f$ , the numbers  $D$  and  $m_D$  are usually not unique. When we want to make  $m_D$  smaller, we may need to increase  $D$ . On the other hand, if we want an almost optimal  $D$ , we may need to set  $m_D$  to be very large. The strength of our main result on overconvergent  $\mathbb{Z}_{p^d}$ -towers depends on both  $D$  and  $m_D$ , therefore we need to “balance between  $D$  and  $m_D$ ” to get the strongest result. For this reason, we let  $(D, m_D)$  to be the tuple with smallest  $m_D$ , in the set

$$\{(D, m_D) : d_i < D\delta i, \forall i \geq m_D; m_D \geq 2; \frac{p^{m_D/2}}{m_D} \geq D\}.$$

**Assumption 4.3.** From now to the end of Chapter 6, we fix a strongly genus stable  $\mathbb{Z}_{p^d}$ -tower associated to  $f$ , assume  $f$  is overconvergent, and let  $\delta, D, m, m_D, d_m$  be as defined before. Note that  $m_D > m$  and  $\delta = \frac{d_m}{p^m}$ . We also fix a basis  $\{c_1, \dots, c_d\}$  of  $\mathbb{Z}_{p^d}$  as a free  $\mathbb{Z}_p$ -module such that  $c_k$  is a  $(p^d - 1)$ -st root of unity. Put  $c_k^* \in \mathbb{Z}_{p^d}$  to be such that  $\text{Tr}_{\mathbb{Q}_{p^d}/\mathbb{Q}_p}(c_k c_k^*) = 1$  and  $\text{Tr}_{\mathbb{Q}_{p^d}/\mathbb{Q}_p}(c_i c_j^*) = 0$  if  $i \neq j$ .

## 4.2 The Artin–Hasse exponential series

Recall that the Artin–Hasse exponential series is defined by

$$E(T) = \exp\left(\sum_{i=0}^{\infty} \frac{T^{p^i}}{p^i}\right) = \prod_{p \nmid i, i \geq 1} (1 - T^i)^{-\mu(i)/i} \in 1 + T + T^2 \mathbb{Z}_p[[T]].$$

**Lemma 4.4.** For each  $i \in \mathbb{Z}_{\geq 1}$ , the map

$$E(\cdot) : T^i \mathbb{Z}_p[[T]] \rightarrow 1 + T^i \mathbb{Z}_p[[T]]$$

is a bijection, such that

$$E(T + T^2 \mathbb{Z}_p[[T]]) = 1 + T + T^2 \mathbb{Z}_p[[T]].$$

In particular, for any  $r \in T \mathbb{Z}_p[[T]]$ , we have  $v_T(r) = v_T(E(r) - 1)$ , where  $v_T$  denotes the  $T$ -adic valuation on  $\mathbb{Z}_p[[T]]$  with  $v_T(T) = 1$ .

*Proof.* See [5, Lemma 2.1]. □

For  $i \in \mathbb{Z}_{\geq 0}$ ,  $k \in \mathbb{Z}$  and  $1 \leq k \leq d$ , define  $\pi_i(T) \in T \mathbb{Z}_p[[T]]$  by

$$\pi_i(T) = E^{-1} \left( (1 + T)^{p^i} \right).$$

**Lemma 4.5.** Recall that  $m_D \geq 2$  and  $\frac{p^{m_D/2}}{m_D} \geq D$ . For any  $t \in \mathbb{C}_p$  with  $0 < v_p(t) \leq 1/p^{m_D}$  and for any  $i \in \mathbb{Z}$ , we have

$$v_p(\pi_i(t)) \geq \begin{cases} p^i v_p(t) & \text{if } i \leq m_D, \\ D \cdot i \cdot v_p(t) & \text{if } i > m_D, \end{cases}$$

where we use the convention that  $v_p(0) = +\infty$ .

*Proof.* By Lemma 4.4, there is  $g(T) \in T + T^2 \mathbb{Z}_p[[T]]$  such that

$$E(g(T)) = 1 + T.$$

In fact,  $g$  is the preimage of  $1 + T \in 1 + T \mathbb{Z}_p[[T]]$  under the bijection  $E$ , so  $E^{-1}(1 + T) = g(T)$ .

We can express  $\pi_i(t)$  in terms of  $g$ ,

$$\pi_i(t) = E^{-1} \left( (1+t)^{p^i} \right) = g \left( (1+t)^{p^i} - 1 \right) \in \left( (1+t)^{p^i} - 1 \right) + \left( (1+t)^{p^i} - 1 \right)^2 \mathbb{Z}_p[[T]].$$

Note that  $v_p \left( (1+t)^{p^i} - 1 \right) > 0$ . Hence the term with smallest valuation in  $\pi_i(t)$  (i.e. the “leading term”) has to be from  $(1+t)^{p^i} - 1$ , so

$$v_p(\pi_i(t)) = v_p \left( t^{p^i} + \sum_{j=1}^{p^i-1} \binom{p^i}{j} t^j \right).$$

When  $i \leq m_D$ , we have  $v_p(t^{p^i}) = p^i v_p(t) \leq 1$ , and  $p | \binom{p^i}{j}$  for  $1 \leq j \leq p^i - 1$ , so  $t^{p^i}$  is the leading term. Hence

$$v_p(\pi_i(t)) \geq v_p(t^{p^i}) = p^i v_p(t).$$

When  $i > m_D$ , it is harder to find the leading term. Write  $j = p^\alpha u$  with  $p \nmid u$ , it is well known that  $p^{i-\alpha} | \binom{p^i}{j}$ . This implies

$$v_p \left( \binom{p^i}{j} t^j \right) \geq i - \alpha + j v_p(t) \geq i - \alpha + p^\alpha v_p(t).$$

We separate into two cases.

*Case 1.*  $i - \alpha \geq \frac{Di}{p^{m_D}}$ . Then since  $v_p(t) \leq 1/p^{m_D}$ , we have

$$v_p \left( \binom{p^i}{j} t^j \right) \geq i - \alpha \geq \frac{Di}{p^{m_D}} \geq D \cdot i \cdot v_p(t).$$

*Case 2.*  $i - \alpha < \frac{Di}{p^{m_D}}$ . Then since  $\frac{p^{m_D/2}}{m_D} \geq D$ , we have  $\frac{D}{p^{m_D}} \leq \frac{1}{p^{m_D/2} m_D}$ , so

$$\alpha > i - \frac{Di}{p^{m_D}} \geq i - \frac{i}{p^{m_D/2} m_D} \geq \frac{i}{2}.$$

Because  $i > m_D$ , we have  $\alpha > \frac{m_D}{2}$ . The function  $\frac{p^x}{2^x}$  is increasing when  $x > 1$ , so  $\frac{p^\alpha}{2^\alpha} > \frac{p^{m_D/2}}{2^{m_D/2}} \geq D$ . Therefore,

$$v_p \left( \binom{p^i}{j} t^j \right) \geq p^\alpha v_p(t) \geq 2D\alpha v_p(t) > D \cdot i \cdot v_p(t).$$

□

### 4.3 The characteristic series

**Notation 4.6.**

(1) Define

$$\mathfrak{X} = \left\{ (i, j, k) \in \mathbb{Z}_{\geq 0}^3 : 1 \leq k \leq d, j \leq \delta p^i \text{ for all } i, \text{ and } j < \delta D i \text{ if } i \geq m_D \right\}.$$

This is the set of tuples  $(i, j, k)$  such that  $c_k[a_{ij}]$  can be nonzero.

(2) The  $\pi$ -adic uniformizers are symbols  $\pi_{(i,j,k)}$  for  $(i, j, k) \in \mathfrak{X}$ .

(3) Define

$$R_p := \mathbb{Z}_p[[\pi_{(i,j,k)} : (i, j, k) \in \mathfrak{X}]], \quad R_q := R_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_q.$$

(4) Consider the monoid

$$U = \{u : \mathfrak{X} \rightarrow \mathbb{Z}_{\geq 0} : u(x) = 0 \text{ for all but finitely many } x\}.$$

Define  $\pi^u = \prod_{(i,j,k) \in \mathfrak{X}} \pi_{(i,j,k)}^{u(i,j,k)}$ , then any  $r \in R_q$  can be written as  $r = \sum_{u \in U} a_u \pi^u$ , where  $a_u \in \mathbb{Z}_q$ .

(5) The  $\pi$ -adic valuation on  $R_q$  is defined such that:

$$v(r) = \min \left\{ \sum_{(i,j,k) \in \mathfrak{X}} \frac{j}{\delta} \cdot u(i,j,k) : u \in U, a_u \neq 0 \right\} \in \frac{1}{\delta} \mathbb{Z}.$$

We set  $v(0) = +\infty$ . Note that  $v(\pi_{(i,j,k)}) = \frac{j}{\delta}$  and  $v(r) = 0$  for  $r \in \mathbb{Z}_p - \{0\}$ . Also note that the existence of the minimum is guaranteed because  $\frac{1}{\delta} \mathbb{Z}$  is discrete.

**Definition 4.7.** For a positive integer  $n$ , the  $n$ -th  $\pi$ -adic exponential sum of  $f$  over  $\mathbb{F}_{q^k}^*$  is defined by

$$S_f^*(n, \pi) = \sum_{x \in \mathbb{F}_{q^n}^*} \prod_{(i,j,k) \in \mathfrak{X}} E(\pi_{(i,j,k)})^{\text{Tr}_{\mathbb{Q}_q^n / \mathbb{Q}_p}(c_k [a_{ij}] [x]^j)} \in R_p.$$

**Definition 4.8.** The  $\pi$ -adic characteristic series associated to  $f$  is defined by

$$C_f^*(\pi, s) := \exp \left( \sum_{n=1}^{\infty} \frac{1}{1 - q^n} S_f^*(n, \pi) \frac{s^n}{n} \right) \in 1 + R_p[[s]].$$

**Definition 4.9.** For  $\underline{t} = (t_1, \dots, t_d) \in \mathcal{W}$  or  $\underline{t} = \underline{T} = (T_1, \dots, T_d)$ , the evaluation at  $\underline{t}$  map from  $R_q$  to  $C_p$  or  $\mathbb{Z}_q[[\underline{T}]]$  is the  $\mathbb{Z}_q$ -algebra homomorphism defined by

$$\text{ev}_{\underline{t}}(\pi_{(i,j,k)}) = \pi_i(t_k) = E^{-1} \left( (1 + t_k)^{p^j} \right).$$

**Proposition 4.10.** The  $\pi$ -adic,  $T$ -adic characteristic series and the character series  $C_f^*(\chi, s)$  of a continuous character  $\chi$  is related by

$$\begin{aligned} C_f^*(T, s) &= \text{ev}_{\underline{T}}(C_f^*(\pi, s)), \\ C_f^*(\chi, s) &= \text{ev}_{\chi}(C_f^*(\pi, s)) := \text{ev}_{(\chi(c_1^*)-1, \dots, \chi(c_d^*)-1)}(C_f^*(\pi, s)). \end{aligned}$$

*Proof.* After the evaluation at  $\underline{T} = (T_1, \dots, T_d)$ ,  $C_f^*(\pi, s)$  becomes identical to the generating

series form of  $C_f^*(T, s)$ . Therefore the first equation holds. Moreover, one has

$$\text{ev}_{(\chi(c_1^*)-1, \dots, \chi(c_d^*)-1)} (C_f^*(\pi, s)) = C_f^*(T, s) \Big|_{\underline{T}=(\chi(c_1^*)-1, \dots, \chi(c_d^*)-1)},$$

so by Section 3.2, the second equation also holds.  $\square$

## 4.4 Dwork Theory

Set  $E_f(X) = \prod_{(i,j,k) \in \mathfrak{X}} E(\pi_{(i,j,k)} c_k [a_{ij}] X^j)$ , and  $\sigma = \text{Frob}(\mathbb{Z}_{q^n}/\mathbb{Z}_p)$ , then

**Lemma 4.11.** *For any  $x \in \mathbb{F}_{q^n}^*$  one has*

$$\prod_{(i,j,k) \in \mathfrak{X}} E(\pi_{(i,j,k)})^{\text{Tr}_{\mathbb{Q}_{q^n}/\mathbb{Q}_p}(c_k [a_{ij}] [x]^j)} = \prod_{t=0}^{an-1} E_f^{\sigma^t}([x]^{p^t}).$$

*Proof.* To simplify the proof, we put  $\pi = \pi_{(i,j,k)}$  and  $z = c_k [a_{ij}] [x]^j$ . Recall from Assumption 4.3 that  $c_k$  is a  $(p^d - 1)$ -st root of unity, and recall that  $d|a$ , where  $q = p^a$ . Note that  $[a_{ij}]$  is a  $(q - 1)$ -st root of unity and  $[x]$  is a  $(q^n - 1)$ -st root of unity. So  $z$  is a  $(q^n - 1)$ -st root of unity. Therefore,  $\sigma(z) = z^p$ . One has  $\sum_{t=0}^{an-1} x^{p^t} = \sum_{t=0}^{an-1} x^{p^{t+s}}$ , so

$$\begin{aligned} E(\pi)^{\text{Tr}_{\mathbb{Q}_{q^n}/\mathbb{Q}_p}(z)} &= E(\pi)^{\sum_{t=0}^{an-1} z^{p^t}} \\ &= \exp \left( \left( \sum_{s=0}^{\infty} \frac{\pi^{p^s}}{p^s} \right) \cdot \left( \sum_{t=0}^{an-1} z^{p^{t+s}} \right) \right) \\ &= \exp \left( \sum_{t=0}^{an-1} \left( \frac{\pi^{p^s}}{p^s} z^{p^{t+s}} \right) \right) \\ &= \prod_{t=0}^{an-1} \exp \left( \frac{(\pi z^{p^t})^{p^s}}{p^s} \right) \\ &= \prod_{t=0}^{an-1} (E(\pi z))^{\sigma^t}. \end{aligned}$$

Substituting with  $\pi_{(i,j,k)} = \pi$  and  $c_k[a_{ij}][x]^j = z$ , we get

$$E(\pi_{(i,j,k)})^{\text{Tr}_{\mathbb{Q}_{q^n}/\mathbb{Q}_p}(c_k[a_{ij}][x]^j)} = \prod_{t=0}^{an-1} E(\pi_{(i,j,k)} c_k[a_{ij}][x]^j)^{\sigma^t}.$$

Taking product over  $\mathfrak{X}$ , we get

$$\prod_{(i,j,k) \in \mathfrak{X}} E(\pi_{(i,j,k)})^{\text{Tr}_{\mathbb{Q}_{q^n}/\mathbb{Q}_p}(c_k[a_{ij}][x]^j)} = \prod_{t=0}^{an-1} E_f^{\sigma^t}([x]^{p^t}).$$

□

**Notation 4.12.**

- (1) Define Banach  $R_q$ -module

$$B := R_q \langle X \rangle = \left\{ \sum_{i=0}^{\infty} \alpha_i X^i \in R_q[[X]] : v(\alpha_i) \rightarrow \infty \text{ as } i \rightarrow \infty \right\}.$$

The set  $\Gamma = \{X^i : i \in \mathbb{Z}_{\geq 0}\}$  is a basis for  $B$ .

- (2) Define  $\psi_p : R_q[[X]] \rightarrow R_q[[X]]$  by

$$\psi_p \left( \sum_{i=0}^{\infty} \alpha_i X^i \right) = \sum_{i=0}^{\infty} \alpha_{pi} X^i.$$

It restricts to an operator  $\psi_p : B \rightarrow B$ .

- (3) The Frobenius  $\sigma = \text{Frob}(\mathbb{Z}_{q^n}/\mathbb{Z}_p)$  acts on  $B$  by acting on  $R_q$ . Define the Dwork operator  $\psi$  by  $\psi = \sigma^{-1} \circ \psi_p \circ E_f(X)$ , where  $E_f(X)$  denotes multiplication by  $E_f(X)$ .

We will show in Lemma 4.19 that  $E_f(X) \in B$ . Therefore  $\psi$  is an operator on  $B$ .

- (4) For convenience, we write  $E_f(X) = \sum_{r=0}^{\infty} e_r X^r$ , and

$$\prod_{t=0}^{an-1} E_f^{\sigma^t}(X^{p^t}) = \sum_{r=0}^{\infty} \eta_r X^r.$$

**Lemma 4.13.** *Each  $r(X) \in B$  can be viewed as a map  $r(X) : B \rightarrow B$ , sending  $g(X)$  to  $r(X)g(X)$ . The operators  $r(X)$ ,  $\psi_p$  and  $\sigma$  satisfy:*

$$(1) \quad r(X) \circ \psi_p = \psi_p \circ r(X^p).$$

$$(2) \quad r(X) \circ \sigma^{-1} = \sigma^{-1} \circ r^\sigma(X).$$

$$(3) \quad \sigma^{-1} \circ \psi_p = \psi_p \circ \sigma^{-1}.$$

*Proof.* Put  $r(X) = \sum_{i=0}^{\infty} \alpha_i X^i$ , and let  $g(X) = \sum_{i=0}^{\infty} \beta_i X^i \in B$ . For (1), one has

$$r(X) \circ \psi_p(g(X)) = r(X) \cdot \left( \sum_{i=0}^{\infty} \beta_{pi} X^i \right).$$

On the other hand

$$\begin{aligned} \psi_p \circ r(X^p)(g(X)) &= \psi_p \left( \left( \sum_{i=0}^{\infty} \alpha_i X^{pi} \right) \cdot \left( \sum_{j=0}^{\infty} \beta_j X^j \right) \right) \\ &= \psi_p \left( \sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} \alpha_i X^{pi} \right) \beta_j X^j \right) \\ &= \left( \sum_{i=0}^{\infty} \alpha_i X^i \right) \cdot \left( \sum_{j=0}^{\infty} \beta_{pj} X^j \right). \end{aligned}$$

By comparing the two equations above, one verifies (1). Then, we directly calculate the action of left hand side of (2),

$$r(X) \cdot \sigma^{-1}(g(X)) = \sigma^{-1}(r^\sigma(X) \cdot g(X)).$$

So (2) is verified. For (3), it follows directly from the definition of  $\psi_p$  that  $\psi_p$  commutes with  $\sigma$ . □

**Corollary 4.14.** *For any integer  $n$ , one has*

$$\psi^n = \sigma^{-n} \circ \psi_p^n \circ \left( \prod_{i=0}^{n-1} E_f^{\sigma^i}(X^{p^i}) \right).$$

*Proof.* One simplifies  $(\sigma^{-1} \circ \psi_p \circ E_f(X))^n$  by applying Lemma 4.13 repeatedly.  $\square$

**REMARK 4.15.** Note that  $\psi^n$  is not in general  $R_q$ -linear, but just  $R_p$ -linear. However, it is  $R_q$ -linear if  $a : n$ . While applying Dwork's Theory, it is important to keep track of whether the operators are  $R_q$ -linear or just  $R_p$ -linear.

**Notation 4.16.** For any  $\lambda \in \mathbb{R}_{>0}$ , we define the ideal  $I^\lambda \subset R_q$  by

$$I^\lambda := \{r \in R_q : v(r) \geq \lambda\}.$$

**REMARK 4.17.** Note that if  $r_1 \in I^{\lambda_1}$  and  $r_2 \in I^{\lambda_2}$ , then  $v(r_1 r_2) \geq \lambda_1 \cdot \lambda_2$ . Therefore, for any  $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ , we have

$$I^{\lambda_1} I^{\lambda_2} \subset I^{\lambda_1 \lambda_2}.$$

**Lemma 4.18.** *Let  $f_i(X) \in 1 + XR_q[[X]]$ ,  $g(x) \in 1 + XR_q[[x]]$  be such that  $\prod_{i=1}^{\infty} f_i(X) = g(X)$ . Write  $g(X) = \sum_{n=0}^{\infty} \alpha_n X^n$ . Suppose there is a universal constant  $c \in R_{>0}$  such that the  $\pi$ -adic Newton polygon of  $f_i(X)$  lies above the line  $y = cx$ . Write*

$$f_i = \sum_{n=0}^{\infty} a_{i,n} X^n,$$

where  $a_{i,n} \in R_q$ . Also write  $g(X) = \sum_{n=0}^{\infty} \alpha_n X^n$ . Then

(1) *The  $\pi$ -adic Newton polygon of  $g(x)$  also lies above the line  $y = cx$ .*

(2)  $\alpha_n \equiv \alpha'_n \pmod{I^{cn+1/\delta}}$ , where  $\alpha'_n \in \mathbb{Z}_q[[a_{i,n} : v(a_{i,n}) = cn]]$ .

*Proof.* Write

$$f_i = \sum_{n=0}^{\infty} a_{i,n} X^n,$$

where  $a_{i,n} \in R_q$ . Then we have

$$\alpha_n = \sum_{\substack{i_1, \dots, i_s \in \mathbb{Z}_{\geq 0} \\ n_1, \dots, n_s: n_1 + \dots + n_s = n}} \prod_{k=1}^s a_{i_k, n_k}.$$

Consider each term  $\prod_{k=1}^s a_{i_k, n_k} = a_{i_1, n_1} \cdot a_{i_2, n_2} \cdots a_{i_s, n_s}$  in the sum. We have

$$\begin{aligned} v(a_{i_1, n_1} \cdot a_{i_2, n_2} \cdots a_{i_s, n_s}) &= v(a_{i_1, n_1}) \cdot v(a_{i_2, n_2}) \cdots v(a_{i_s, n_s}) \\ &\geq cn_1 + cn_2 + \cdots + cn_s \\ &= cn. \end{aligned} \tag{4.1}$$

Therefore  $v(\alpha_n) \geq cn$ . The first conclusion is proved. For the second conclusion, note that if the equality of equation 4.1 is realized, then we must have  $v(a_{i_k, n_k}) = cn_k$  for  $k = 1, 2, \dots, s$ .

Consequently,

$$a_{i_1, n_1} \cdot a_{i_2, n_2} \cdots a_{i_s, n_s} \in \mathbb{Z}_q[[a_{i,n} : v(a_{i,n}) = cn]].$$

□

**Lemma 4.19** (Kosters–Zhu [5, Lemma 5.4]). *Write  $\prod_{t=0}^{n-1} E_f^{\sigma^t}(X^{p^t}) = \sum_{r=0}^{\infty} \beta_r X^r$ , then*

$$(1) \ v(\beta_r) \geq \frac{r}{p^{n-1}\delta}.$$

(2) *For  $n \in \mathbb{Z}_{\geq 1}$ , the map  $\sigma^n \circ \psi^n$  is  $R_q$ -linear and one has*

$$\text{Mat}(\sigma^n \circ \psi^n | \Gamma) = (\beta_{p^{n_i-j}})_{i,j}.$$

*Proof.* To prove (1), recall that

$$E_f(X) = \prod_{(i,j,k) \in \mathfrak{X}} E(\pi_{(i,j,k)} c_k [a_{ij}] X^j),$$

and  $v(\pi_{(i,j,k)} c_k [a_{ij}]) = \frac{j}{\delta}$ . Since  $E(T) \in 1 + T\mathbb{Z}_p[[T]]$ , the Newton polygon of  $E(\pi_{(i,j,k)} c_k [a_{ij}] X^j)$  lies above the line  $y = \frac{x}{\delta}$ . So by Lemma 4.18, the Newton polygon of  $E_f(X)$  also lies above the line  $y = \frac{x}{\delta}$ .

This implies that the Newton polygon of  $E_f^{\sigma^t}(X^{p^t})$  lies above the line  $y = \frac{x}{p^t \delta}$  ( $0 \leq t \leq n-1$ ), all of which are above the line  $y = \frac{x}{p^{n-1} \delta}$ . So by Lemma 4.18 and that (1) is true for  $n=1$ , we have that the Newton polygon of  $\prod_{i=0}^{n-1} E_f^{\sigma^i}(X^{p^i})$  lies above the line  $y = \frac{x}{p^{n-1} \delta}$ . This is equivalent to (1).

To prove (2),  $R_q$ -linearity follows from that  $\psi_p$  and  $E_f(X)$  are  $R_q$ -linear and Cor 4.14. Then,

$$\begin{aligned} \sigma^n \circ \psi^n(X^i) &= \psi_p^k \circ \left( \sum_{r=0}^{\infty} \beta_r X^r \right) (X^i) \\ &= \psi_p^n \left( \sum_{r=i}^{\infty} \beta_{r-i} X^r \right) \\ &= \psi_p^{n-1} \left( \sum_{pr \geq i} \beta_{pr-i} X^r \right) \\ &= \cdots = \sum_{p^n r \geq i} \beta_{p^n r - i} X^r. \end{aligned}$$

□

**Theorem 4.20** (Dwork's Trace Formula). *For any positive integer  $n$ ,*

$$S^*(n, \pi) = (q^n - 1) \cdot \text{Tr}(\psi^{an} | \Gamma).$$

*Proof.* By lemma 4.11,

$$S^*(n, \pi) = \sum_{x \in \mathbb{F}_{q^n}^*} \prod_{t=0}^{an-1} E_f^{\sigma^t}([x]^{p^t}) = \sum_{x \in \mathbb{F}_{q^n}^*} \sum_{r=0}^{\infty} \eta_r[x]^r.$$

Apply lemma 4.19, and note that  $\sum_{x \in \mathbb{F}_{q^n}^*} [x]^r$  is  $(q^n - 1)$  if  $(q^n - 1) | r$ , and is 0 otherwise, then

$$\begin{aligned} S^*(n, \pi) &= (q^n - 1) \sum_{r=0}^{\infty} \eta_{(q^n-1)r} = (q^n - 1) \sum_{r=0}^{\infty} \eta_{(p^{an}-1)r} \\ &= (q^n - 1) \text{Tr}(\sigma^{an} \circ \psi^{an} | \Gamma) \\ &= (q^n - 1) \text{Tr}(\psi^{an} | \Gamma). \end{aligned}$$

because  $\sigma$  is the Frobenius of  $\text{Gal}(\mathbb{Q}_{p^{an}}/\mathbb{Q}_p)$ , so  $\sigma^{an} = \text{id}$ . □

**Corollary 4.21.**

$$C_f^*(\pi, s) = \det(I - s\psi^a | \Gamma).$$

*Proof.*

$$\begin{aligned} \det(I - s\psi^a | \Gamma) &= \exp \left( - \sum_{n=1}^{\infty} \text{Tr}(\psi^{an} | \Gamma) \frac{s^n}{n} \right) \\ &= \exp \left( - \sum_{n=1}^{\infty} \frac{1}{q^n - 1} S^*(n, \pi) \frac{s^n}{n} \right) \\ &= C_f^*(\pi, s). \end{aligned}$$

□

Therefore, we can understand  $C_f^*(\pi, s)$  by studying  $\psi^a$ . It's hard to study  $\psi^a$  directly, so we reduce to  $\sigma \circ \psi$ , which is  $R_q$ -linear by lemma 4.19 and easier to study. Let  $N$  be the matrix

$(\sigma \circ \psi|\Gamma)$ , then

$$\begin{aligned}\det(I - s\psi^a|\Gamma) &= \det(I - s(\sigma^{-1}N)^a) \\ &= \det(I - s\sigma^{-1}(N\sigma^{-1}(N\sigma^{-1}(\dots\sigma^{-1}(N))\dots))) \\ &= \det(I - s\sigma^{-1}(N)\dots\sigma^{-a}(N)).\end{aligned}$$

# Chapter 5

## The Newton polygon of the universal character series

In this chapter, we study the  $\pi$ -adic Newton polygon of the  $\pi$ -adic characteristic series  $C_f^*(\pi, s)$ , which is derived from the  $\pi$ -adic valuation  $v$  on  $R_q$ .

### 5.1 Lower bound

To treat the matrix  $N$  from previous chapter, we need to introduce the theory of twisted  $\pi$ -adic incremental matrices.

**Definition 5.1.**

- (1) Let  $M_\infty(R_q)$  be the set of matrices with entries in  $R_q$ , rows and columns indexed by  $\mathbb{Z}_{\geq 0}$ . We say a matrix  $N = (h_{mn})_{m,n \in \mathbb{Z}_{\geq 0}} \in M_\infty(R_q)$  is twisted  $\pi$ -adic incremental (in  $\delta$  steps) if  $v(h_{mn}) \geq \frac{mp-n}{\delta}$ .

(2) Let  $M \in M_\infty(R_q)$ , then

$$\begin{bmatrix} m_0 & m_D & \cdots & m_{k-1} \\ n_0 & n_1 & \cdots & n_{k-1} \end{bmatrix}_M$$

denotes the  $k \times k$  submatrix of  $M$  that consist of rows indexed by  $m_0, m_D, \dots, m_{k-1}$  and columns indexed by  $n_0, n_1, \dots, n_{k-1}$ . Note that in our notation the top row is the 0th row, and the leftmost column is the 0th column.

(3) Define  $\lambda_n := \frac{an(n-1)(p-1)}{2\delta}$ .

**Proposition 5.2.** [12, Proposition 4.6] *Let  $M_0, M_1, \dots, M_{a-1} \in M_\infty(R_q)$  be twisted  $\pi$ -adic incremental matrices, and denote*

$$\det(I - sM_{a-1} \cdots M_1 M_0) = \sum_{n=0}^{\infty} r_n s^n, \quad r_n \in R_q.$$

Then:

(1)  $v(r_n) \geq \lambda_n$ .

(2)

$$r_n \equiv \prod_{j=0}^{a-1} \det \begin{bmatrix} 0 & 1 & \cdots & n-1 \\ 0 & 1 & \cdots & n-1 \end{bmatrix}_{M_j} \pmod{I^{\lambda_n + \frac{1}{\delta}}},$$

recall that for  $\lambda \in \mathbb{R}_{>0}$ ,  $I^\lambda$  is the ideal consisting of elements in  $R_q$  whose  $\pi$ -adic valuation is greater or equal to  $\lambda$ .

*Proof.*

$$\begin{aligned}
r_n &= \sum_{0 \leq m_0 < m_D < \dots < m_{n-1}} \det \begin{bmatrix} m_0 & m_D & \dots & m_{n-1} \\ m_0 & m_D & \dots & m_{n-1} \end{bmatrix}_{M_{a-1} \dots M_1 M_0} \\
&= \sum_{\substack{0 \leq m_{0,0} < m_{0,1} < \dots < m_{0,n-1} \\ \dots \\ 0 \leq m_{a-1,0} < m_{a-1,1} < \dots < m_{a-1,n-1}}} \det \left( \prod_{j=0}^{a-1} \begin{bmatrix} m_{j,0} & m_{j,1} & \dots & m_{j,n-1} \\ m_{j+1,0} & m_{j+1,1} & \dots & m_{j+1,n-1} \end{bmatrix}_{M_j} \right) \\
&= \sum_{\substack{0 \leq m_{0,0} < m_{0,1} < \dots < m_{0,n-1} \\ \dots \\ 0 \leq m_{a-1,0} < m_{a-1,1} < \dots < m_{a-1,n-1}}} \prod_{j=0}^{a-1} \det \begin{bmatrix} m_{j,0} & m_{j,1} & \dots & m_{j,n-1} \\ m_{j+1,0} & m_{j+1,1} & \dots & m_{j+1,n-1} \end{bmatrix}_{M_j}.
\end{aligned}$$

Where we set  $m_{a,i} = m_{0,i}$  for  $0 \leq i \leq k-1$ . Then we estimate the valuation of each summand. Because each  $M_j$  is twisted  $\pi$ -adic incremental,

$$\begin{aligned}
&v \left( \det \begin{bmatrix} m_{j,0} & m_{j,1} & \dots & m_{j,n-1} \\ m_{j+1,0} & m_{j+1,1} & \dots & m_{j+1,n-1} \end{bmatrix}_{M_j} \right) \\
&\geq \min_{\sigma \in S_n} \left\{ \frac{1}{\delta} \sum_{i=0}^{n-1} p \cdot m_{j,i} - m_{j+1,\sigma(i)} \right\} \\
&= \frac{1}{\delta} \sum_{i=0}^{n-1} p \cdot m_{j,i} - m_{j+1,i}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& v \left( \prod_{j=0}^{a-1} \det \begin{bmatrix} m_{j,0} & m_{j,1} & \cdots & m_{j,n-1} \\ m_{j+1,0} & m_{j+1,1} & \cdots & m_{j+1,n-1} \end{bmatrix}_{M_j} \right) \\
&= \sum_{j=0}^{a-1} v \left( \det \begin{bmatrix} m_{j,0} & m_{j,1} & \cdots & m_{j,n-1} \\ m_{j+1,0} & m_{j+1,1} & \cdots & m_{j+1,n-1} \end{bmatrix}_{M_j} \right) \\
&\geq \frac{1}{\delta} \sum_{j=0}^{a-1} \sum_{i=0}^{n-1} p \cdot m_{j,i} - m_{j+1,i} \\
&= \frac{(p-1)}{\delta} \sum_{j=0}^{a-1} \sum_{i=0}^{n-1} m_{j,i} \\
&\geq \frac{(p-1)}{\delta} \sum_{j=0}^{a-1} (0 + 1 + \cdots + (n-1)) \\
&= \frac{a(p-1)n(n-1)}{2\delta}.
\end{aligned}$$

So each summand of  $r_n$  have valuation greater than or equal to  $\frac{a(p-1)n(n-1)}{2\delta}$ , this proves the first conclusion. And the equality can hold only if  $m_{j,i} = i$  for all  $i$ . This proves the second conclusion.  $\square$

By Cor 4.21 and Proposition 5.2, we can get a lower bound for  $C_f^*(\pi, s)$  if we can show  $N$  is a twisted  $\pi$ -adic incremental matrix. And that is a direct consequence of Lemma 4.19.

**Lemma 5.3.**  *$N$  is twisted  $\pi$ -adic incremental.*

*Proof.* By Lemma 4.19,  $v(N_{mn}) = v(e_{pm-n}) \geq \frac{pm-n}{\delta}$ .  $\square$

Recall that  $\lambda_n = \frac{an(n-1)(p-1)}{2\delta}$ .

**Theorem 5.4.** *The  $\pi$ -adic Newton polygon of  $C_f^*(\pi, s)$  lies above the polygon whose vertices are  $(n, \lambda_n)$  for  $n \in \mathbb{Z}_{\geq 0}$ .*

*Proof.* By Lemma 5.3,  $M_k = \sigma^{k-a}(N)$  are twisted  $\pi$ -adic incremental, for  $k = 0, 1, \dots, a-1$ . Then apply Proposition 5.2.  $\square$

In chapter 6, we use this  $\pi$ -adic lower bound to obtain a lower bound for  $C_f^*(\chi, s)$ , for characters  $\chi$  whose corresponding point on  $\mathcal{W}$  is close to the boundary.

## 5.2 Upper bound

As a reminder, Assumption 4.3 is still in force. The  $\pi$ -adic Newton polygon of  $C_f^*(\pi, s)$  also has an upper bound.

**Theorem 5.5.** *The  $\pi$ -adic Newton polygon of  $C_f^*(\pi, s)$  passes through the vertices  $(n, \lambda_n)$  for  $n \equiv 0$  or  $1 \pmod{d_m}$ . As a consequence, the  $\pi$ -adic Newton polygon of  $C_f^*(\pi, s)$  lies below the polygon whose vertices are  $(n, \lambda_n)$  for  $n \equiv 0$  or  $1 \pmod{d_m}$ .*

*Proof.* Recall that the integer  $m$  is such that  $\delta = \frac{d_m}{p^m}$ . Let  $\chi : \mathbb{Z}_{p^d} \rightarrow C_p^*$  be any finite continuous character of order  $p^{m_\chi} > p^m$ , then by Theorem 1.7, the  $q$ -adic Newton polygon of  $L_f^*(\chi, s)$  (recall that this is the  $q$ -adic Newton polygon of  $L_f(\chi, s)$  with one additional 0-slope edge) passes through the points

$$\left( nd_m, \frac{n(nd_m - 1)}{2p^{m_\chi - m - 1}} \right) \text{ and } \left( nd_m + 1, \frac{n(nd_m + 1)}{2p^{m_\chi - m - 1}} \right),$$

for  $0 \leq n \leq p^{m_\chi - 1}$ . Note that the  $\pi_\chi$ -adic valuation and  $q$ -adic valuation has the relation

$$v_{\pi_\chi}(x) = a(p-1)p^{m_\chi - 1}v_q(x), \quad x \in C_p^*.$$

And we verify

$$\lambda_{nd_m} = \frac{and_m(nd_m - 1)(p - 1)}{2\delta} = \frac{and_m(nd_m - 1)(p - 1)p^m}{2d_m} = a(p - 1)p^{m_\chi - 1} \cdot \frac{n(nd_m - 1)}{2p^{m_\chi - m - 1}},$$

and

$$\lambda_{nd_m + 1} = \frac{and_m(nd_m + 1)(p - 1)}{2\delta} = \frac{and_m(nd_m + 1)(p - 1)p^m}{2d_m} = a(p - 1)p^{m_\chi - 1} \cdot \frac{n(nd_m + 1)}{2p^{m_\chi - m - 1}}.$$

Therefore, the  $\pi_\chi$ -adic Newton polygon of  $L_f^*(\chi, s)$  pass through the vertices

$$(nd_m, \lambda_{nd_m}) \text{ and } (nd_m + 1, \lambda_{nd_m + 1}),$$

for  $0 \leq n \leq p^{m_\chi - 1}$ . Then, recall that

$$C_f^*(\chi, s) = L_f^*(\chi, s) \cdot L_f^*(\chi, qs) \cdot L_f^*(\chi, q^2s) \cdots$$

Therefore, the  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$  pass through the vertices

$$(nd_m, \lambda_{nd_m}) \text{ and } (nd_m + 1, \lambda_{nd_m + 1}),$$

for  $0 \leq n \leq p^{m_\chi - 1}$ . In the proof of Theorem 6.2, we proved that the  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$  lies above the  $\pi$ -adic Newton polygon of  $C_f^*(\pi, s)$ . So the  $\pi$ -adic Newton polygon of  $C_f^*(\pi, s)$  lies below the vertices  $(nd_m, \lambda_{nd_m})$  and  $(nd_m + 1, \lambda_{nd_m + 1})$  for  $0 \leq n \leq p^{m_\chi - 1}$ . By Theorem 5.4, it must pass these vertices. Letting  $m_\chi \rightarrow \infty$ , we have proved the first sentence. The second sentence is just a direct corollary.  $\square$

### 5.3 The leading term

Unlike the lower bound case, the  $\pi$ -adic upper bound does not directly imply an upper bound for the  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$ , for *all* continuous characters  $\chi$ . After specializing to  $\chi$ , the Newton polygon could possibly go up and become strictly above the  $\pi$ -adic upper bound. But as we will see later, this is not the case for *most* continuous characters  $\chi$ . To learn exactly when this happens, we study the “leading term” of the determinant of the matrix  $N$ .

**Notation 5.6.**

- (1) Denote  $\mathbb{F}_q[[\pi]] := \mathbb{F}_q[[\pi_{(i,j,k)} : (i,j,k) \in \mathfrak{X}]]$ .
- (2) For any  $r \in R_q$ , we denote its reduction modulo  $p$  in  $\mathbb{F}_q[[\pi]]$  by  $\bar{r}$ . For  $x \in \mathbb{Z}_q$ , we also use  $\bar{x}$  to denote its reduction modulo  $p$  in  $\mathbb{F}_q$ .
- (3) For  $(i,j)$  such that  $j \leq D\delta i$  if  $i \geq m_D$  and  $j < \delta p^i$  for all  $i$ , denote  $\mathfrak{T}_{(i,j)} := \sum_{k=1}^d c_k \pi_{(i,j,k)} \in R_q$ .
- (4) For the same  $(i,j)$ , denote  $\bar{\mathfrak{T}}_{(i,j)} := \sum_{k=1}^d \bar{c}_k \pi_{(i,j,k)} \in \mathbb{F}_q[[\pi]]$ .
- (5) Denote  $\mathbb{Z}_q[[\bar{\mathfrak{T}}]] = \mathbb{Z}_q[[\mathfrak{T}_{(i,j)} : (i,j,1) \in \mathfrak{X}]] \subseteq R_q$ .
- (6) Denote  $\mathbb{F}_q[[\bar{\mathfrak{T}}]] = \mathbb{F}_q[[\bar{\mathfrak{T}}_{(i,j)} : (i,j,1) \in \mathfrak{X}]] \subseteq \mathbb{F}_q[[\pi]]$ .

The following lemma helps us do row reduction on the matrix  $N \bmod p$ .

**Lemma 5.7.** *Recall  $E_f(X) = \sum_{n=0}^{\infty} e_n X^n$ , then*

$$ne_n \equiv \sum_{r=0}^{\infty} \sum_{j \leq \delta p^i} j (\bar{\mathfrak{T}}_{(i,j)}[a_{ij}])^{p^r} e_{n-jp^r} \pmod{p}$$

*Proof.* By logarithmic differentiation, the derivative of the Artin–Hasse exponential series is

$$E'(X) = \left( \sum_{r=0}^{\infty} X^{p^r-1} \right) E(X).$$

So by chain rule,

$$\begin{aligned} E'_f(X) &= \left( \sum_{r=0}^{\infty} \sum_{(i,j,k) \in \mathfrak{X}} j \left( \pi_{(i,j,k)} c_k[a_{ij}] \right)^{p^r} X^{jp^r-1} \right) \left( \sum_{n=0}^{\infty} e_n X^n \right) \\ &\equiv \left( \sum_{r=0}^{\infty} \sum_{i,j:j \leq \delta p^i} j \left( \bar{\mathfrak{T}}_{(i,j)}[a_{ij}] \right)^{p^r} X^{jp^r-1} \right) \left( \sum_{n=0}^{\infty} e_n X^n \right) \pmod{p}. \end{aligned}$$

On the other hand,

$$E'_f(X) = \left( \sum_{n=0}^{\infty} e_n X^n \right)' = \sum_{n=1}^{\infty} n e_n X^{n-1}.$$

Compare the mod  $p$  coefficients of  $X^{m-1}$  in both equations, we get

$$n \bar{e}_n \equiv \sum_{r=0}^{\infty} \sum_{i,j:j \leq \delta p^i} j \left( \bar{\mathfrak{T}}_{(i,j)}[a_{ij}] \right)^{p^r} \bar{e}_{n-jp^r} \pmod{p}.$$

□

The next lemma is the key idea of this section. Recall that by the second part of Proposition 5.2, if the  $\pi$ -adic Newton polygon of  $C_f^*(\pi, s)$  touches its lower bound at the  $n$ -th horizontal position, then the “leading term” of the  $n$ -th coefficient of  $C_f^*(\pi, s)$  depends on the upper left  $n \times n$  minor of  $N$ . Our next goal is to show that this minor mod  $p$  does not depend on the rank  $d$  of the  $\mathbb{Z}_p^d$ -tower.

**Lemma 5.8.** *The determinant*

$$\det \begin{bmatrix} 0 & 1 & \cdots & n-1 \\ 0 & 1 & \cdots & n-1 \end{bmatrix}_N \pmod p,$$

viewed as an element in  $\mathbb{F}_q[[\pi]]$ , lies in  $\mathbb{F}_q[[\bar{\mathfrak{T}}]]$ . Moreover, the coefficients of this determinant as a power series do not depend on  $d$ .

*Proof.* Consider the matrix  $\bar{N}_n$  defined by

$$\bar{N}_n^T = \begin{bmatrix} \overbrace{\bar{e}_0 \quad \bar{e}_p \quad \bar{e}_{2p} \quad \cdots \quad \bar{e}_{(n-1)p}}^n & \overbrace{0 \quad 0 \quad 0 \quad \cdots \quad 0}^{(n-1)(p-1)} \\ 0 \quad \bar{e}_{p-1} \quad \bar{e}_{2p-1} \quad \cdots \quad \bar{e}_{(n-1)p-1} & 0 \quad 0 \quad 0 \quad \cdots \quad 0 \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots & \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\ 0 \quad \bar{e}_0 \quad \bar{e}_p \quad \cdots \quad \bar{e}_{(n-2)p} & 0 \quad 0 \quad 0 \quad \cdots \quad 0 \\ 0 \quad 0 \quad \bar{e}_{p-1} \quad \cdots \quad \bar{e}_{(n-2)p-1} & 0 \quad 0 \quad 0 \quad \cdots \quad 0 \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots & \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\ 0 \quad 0 \quad * \quad \cdots \quad \bar{e}_{(n-1)(p-1)} & 0 \quad 0 \quad 0 \quad \cdots \quad 0 \\ 0 \quad 0 \quad * \quad \cdots \quad \bar{e}_{(n-1)(p-1)-1} & 1 \quad 0 \quad 0 \quad \cdots \quad 0 \\ 0 \quad 0 \quad * \quad \cdots \quad \bar{e}_{(n-1)(p-1)-2} & 0 \quad 1 \quad 0 \quad \cdots \quad 0 \\ 0 \quad 0 \quad * \quad \cdots \quad \bar{e}_{(n-1)(p-1)-3} & 0 \quad 0 \quad 1 \quad \cdots \quad 0 \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots & \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\ 0 \quad 0 \quad 0 \quad 0 \quad \bar{e}_0 & 0 \quad 0 \quad 0 \quad \cdots \quad 1 \end{bmatrix},$$

where  $\cdot^T$  means matrix transpose. Then the upper left  $k \times k$  block of  $\bar{N}_n$  is same as that of  $N \pmod p$ , so

$$\det \begin{bmatrix} 0 & 1 & \cdots & n-1 \\ 0 & 1 & \cdots & n-1 \end{bmatrix}_N \pmod p = \det(\bar{N}_n^T).$$

The next step is to study  $\det(\bar{N}_n^T)$ . In Lemma 5.7, Replace  $n$  by  $h'p - h$ , where  $h, h' \in \mathbb{Z}$ , and  $0 \leq h' \leq n - 1$ , we get

$$(h'p - h)\bar{e}_{h'p-h} \equiv -h\bar{e}_{h'p-h} \equiv \sum_{r=0}^{\infty} \sum_{i,j} j (\bar{\mathfrak{X}}_{(i,j)}[a_{ij}])^{p^r} \bar{e}_{h'p-(h+jp^r)} \pmod{p}. \quad (5.1)$$

Define the  $1 \times n$  vector

$$\bar{V}_h := (\bar{e}_{-h}, \bar{e}_{p-h}, \dots, \bar{e}_{(n-1)p-h}),$$

then the vector equation

$$-\bar{h}\bar{V}_h = \sum_{r=0}^{\infty} \sum_{i,j:j \leq \delta p^i} \bar{j} (\bar{\mathfrak{X}}_{(i,j)} a_{ij})^{p^r} \bar{V}_{h+jp^r} \quad (5.2)$$

is equivalent to equation 5.1, because comparing  $h'$ -th entry on both sides of equation 5.2 gives exactly equation 5.1. Moreover, because  $\bar{V}_h = (0, 0, \dots, 0)$  for  $h \geq np - p + 2$ , by taking the summation on  $j$  and  $r$  first, equation 5.2 can be rewritten as

$$-\bar{h}\bar{V}_h = \sum_{\{j,r:h+jp^r \leq np-p+1\}} \bar{j} \left( \sum_{i=0}^{\infty} (\bar{\mathfrak{X}}_{(i,j)} a_{ij})^{p^r} \right) \bar{V}_{h+jp^r}. \quad (5.3)$$

Note that the right hand side is a finite sum of vectors, and the coefficient  $\bar{j} \left( \sum_{i=0}^{\infty} (\bar{\mathfrak{X}}_{(i,j)} a_{ij})^{p^r} \right)$  is in  $\mathbb{F}_q[[\bar{\mathfrak{X}}]]$ . If  $p \nmid m$ , multiply both sides of equation 5.3 by the multiplicative inverse of  $\bar{h}$  in  $\mathbb{F}_q$ , we get

$$\bar{V}_h = \sum_{\{j,r:h+jp^r \leq np-p+1\}} -\bar{h}^{-1}\bar{j} \left( \sum_{i=0}^{\infty} (\bar{\mathfrak{X}}_{(i,j)} a_{ij})^{p^r} \right) \bar{V}_{h+jp^r}. \quad (5.4)$$

Now we are ready to do row reduction on the matrix  $\bar{N}_n^T$ . Fix  $n$ , view  $\bar{N}_n^T = (A_1 \ A_2)$  as a block matrix, where  $A_1$  is an  $(np - p + 1) \times k$  matrix,  $A_2$  is an  $(np - p + 1) \times (n - 1)(p - 1)$  matrix. Then  $\bar{V}_h$  is the  $h$ -th row of  $A_1$  for  $0 \leq h \leq (n - 1)p$ . So equation 5.4 implies that

standard row reduction can cancel all  $h$ -th row of the matrix  $A_1$  with  $p \nmid h$ . This means there is a  $(np - p + 1) \times (np - p + 1)$  matrix  $P$  such that

$$PA_1 = \left[ \begin{array}{cccc} \bar{e}_0 & \bar{e}_p & \cdots & \bar{e}_{(n-1)p} \\ 0 & \bar{e}_0 & \cdots & \bar{e}_{(n-2)p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{e}_0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{array} \right] \begin{array}{l} \left. \vphantom{\begin{matrix} \bar{e}_0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix}} \right\} n \\ \left. \vphantom{\begin{matrix} \bar{e}_0 \\ \bar{e}_0 \\ \vdots \\ \bar{e}_0 \\ 0 \\ \vdots \\ 0 \end{matrix}} \right\} (n-1)(p-1) \end{array} .$$

$P$  represents some replacements and  $\frac{n(n-1)(p-1)}{2}$  times of interchange, so  $\det P = 1$ . By definition of  $E_f(X)$ , we know  $e_0 = 1$ , by equation 5.4, entries of  $P$  lie in  $\mathbb{F}_q[[\bar{\mathfrak{I}}]]$ , and entries of  $A_2$  are 0 or 1, so

$$\det(\bar{N}_n^T) = \det(P\bar{N}_n^T) = \det(PA_1 PA_2) = \det \begin{bmatrix} n & n+1 & \cdots & (n-1)p \\ n & n+1 & \cdots & (n-1)p \end{bmatrix}_{PA_2} \in \mathbb{F}_q[[\bar{\mathfrak{I}}]].$$

This argument is the same for all  $d$ , so the coefficients of this power series do not depend on  $d$ . □

Recall

$$\mathfrak{I}_{(i,j)} = \sum_{k=1}^d c_k \pi_{(i,j,k)}.$$

Let  $\sigma$  be the Frobenius of  $\mathbb{Z}_p^d$  over  $\mathbb{Z}_p$ , for  $r \in \mathbb{Z}_{\geq 0}$ , we set

$$\mathfrak{I}_{(i,j)}^{\sigma^r} := \sum_{k=1}^d \sigma^r(c_k) \pi_{(i,j,k)}.$$

To reduce some writings, we denote

$$R' := \mathbb{Z}_q[[\mathfrak{X}_{(i,j)}^{\sigma^r} : (i,j,1) \in \mathfrak{X}, r \in \mathbb{Z}_{\geq 0}]] \subseteq R_q.$$

Then we have the corollary below explaining the “leading term” of the coefficients of  $C_f^*(\pi, s)$ .

**Corollary 5.9.** *Consider the  $\pi$ -adic characteristic series  $C_f^*(\pi, s) = 1 + \sum_{n=1}^{\infty} r_n s^n$ . When  $n \equiv 0$  or  $1 \pmod{d_m}$ , we have*

$$r_n \equiv \prod_{j=0}^{a-1} F^{\sigma^j} \pmod{pI^{\lambda_n} + I^{\lambda_n+1/\delta}},$$

for some  $F \in \mathbb{Z}_q[[\mathfrak{X}]]$ . In particular, the “leading term” of  $r_n$  is congruent mod  $p$  to a power series that lies in  $R'$ , and does not depend on  $d$ .

*Proof.* By Lemma 5.8, we can find a power series  $F$  lying in  $\mathbb{Z}_q[[\mathfrak{X}]]$ , does not depend on  $d$ , such that

$$F \equiv \det \begin{bmatrix} 0 & 1 & \cdots & n-1 \\ 0 & 1 & \cdots & n-1 \end{bmatrix}_N \pmod{p},$$

By Theorem 5.5, when  $n \equiv 0$  or  $1 \pmod{d_m}$ , we have  $v(r_n) = \lambda_n$ . Then by Proposition 5.2, we have

$$r_n \equiv \prod_{j=0}^{a-1} \det \begin{bmatrix} 0 & 1 & \cdots & n-1 \\ 0 & 1 & \cdots & n-1 \end{bmatrix}_{\sigma^{j-a}(N)} \equiv \prod_{j=0}^{a-1} F^{\sigma^j} \pmod{pI^{\lambda_n} + I^{\lambda_n+1/\delta}},$$

where the right hand side is a power series in terms of  $\mathfrak{X}_{(i,j)}^{\sigma^r}$  and does not depend on  $d$ .  $\square$

# Chapter 6

## Characteristic series of infinite characters

Recall that  $\{c_1, \dots, c_d\}$  is a basis of  $\mathbb{Z}_p^d$  as a free  $\mathbb{Z}_p$ -module, chosen in Section 4.1, and  $c_k^* \in \mathbb{Z}_p^d$  are such that  $\text{Tr}_{\mathbb{Q}_p^d/\mathbb{Q}_p}(c_k c_k^*) = 1$  and  $\text{Tr}_{\mathbb{Q}_p^d/\mathbb{Q}_p}(c_i c_j^*) = 0$  if  $i \neq j$ .

**Notation 6.1.** Let  $\chi : \mathbb{Z}_p^d \rightarrow \mathbb{C}_p^*$  be a continuous character, not necessarily finite. Denote  $t_k = \chi(c_k^*) - 1 \in \mathbb{C}_p^*$ , for  $k = 1, 2, \dots, d$ . Then  $v_q(t_k) > 0$  for  $k = 1, 2, \dots, d$ , since  $\chi$  is a well-defined character. Define  $\pi_\chi \in \{t_1, t_2, \dots, t_d\}$  to be any element such that

$$v_q(\pi_\chi) = \min_k \{v_q(t_k)\}.$$

This definition allows us to discuss the  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$ . As we have mentioned in Chapter 5, we can use the lower bound and the upper bound for the  $\pi$ -adic Newton polygon of  $C_f^*(\pi, s)$  to obtain bounds for the  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$ , for some characters  $\chi$ .

Suppose  $v_p(\pi_\chi) \leq 1/p^{m_D}$ , in other words, the point

$$(\chi(c_1^*) - 1, \dots, \chi(c_d^*) - 1) \in \mathcal{W}^{\geq p^{-1/p^{m_D}}},$$

where

$$\mathcal{W}^{\geq r} = \{t \in \mathcal{W} : |t| > r\} = \{t \in \mathcal{W} : 0 < v_p(t) < -\log_p r\}.$$

Then, a direct application of the  $\pi$ -adic lower bound proves the same lower bound for the  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$ . But to obtain the upper bound, by the discussion of the leading terms in coefficients of  $C_f^*(\pi, s)$ , we need to introduce one more restriction. This restriction is described by the *admissible locus*  $\mathcal{W}^{\text{adm}}$ , which is an admissible subset of  $\mathcal{W}$  which we define in Section 6.2. In this chapter, we show that if  $\chi$  is also in the *admissible locus*  $\mathcal{W}^{\text{adm}}$ , then the desired upper bound can be obtained.

## 6.1 Lower bound

By specializing Theorem 5.4 to a continuous character  $\chi$ , we can get:

**Theorem 6.2.** *If  $v_p(\pi_\chi) \leq 1/p^{m_D}$ , then the  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$  lies above the polygon with vertices  $(n, \lambda_n)$ .*

*Proof.* By Lemma 4.5, for  $i < m_D$ , by strong genus stability, we have

$$v_p(\pi_i(t_k)) \geq p^i v_p(t_k) \geq \frac{j}{\delta} v_p(\pi_\chi).$$

And for  $i < m_D$ , since  $f$  is overconvergent, we have

$$v_p(\pi_i(t_k)) > D \cdot i \cdot v_p(t_k) \geq \frac{j}{\delta} v_p(\pi_\chi).$$

So, for all  $(i, j, k) \in \mathfrak{X}$ , we have

$$v_p(\pi_i(t_k)) \geq \frac{j}{\delta} v_p(\pi_\chi) = v(\pi_{(i,j,k)}) v_p(\pi_\chi).$$

Use  $\pi_\chi$ -adic valuation instead, we get

$$v_{\pi_\chi}(\pi_i(t_k)) \geq v(\pi_{(i,j,k)}).$$

This means the  $\pi_\chi$ -adic valuation after specialization is greater than the  $\pi$ -adic valuation before specialization. In other words, let  $\underline{t} = (t_1, t_2, \dots, t_k)$ , then we have

$$v_{\pi_\chi}(\text{ev}_{\underline{t}}(r_n)) \geq v(r_n).$$

By Proposition 4.10, and the previous equation, the  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$  lies above the  $\pi$ -adic Newton polygon of  $C_f^*(\pi, s)$ . Therefore, by Theorem 5.4, the polygon with vertices  $(n, \lambda_n)$  is its lower bound.  $\square$

## 6.2 The admissible locus

In this subsection we describe the admissible locus, and prove some necessary properties. Recall that the weight space is  $\mathcal{W} = \{(t_1, \dots, t_d) \in \mathbb{C}_p^d : v_p(t_k) > 0, 1 \leq k \leq d\}$ . Each continuous character  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{Z}_p$  corresponds to a point  $\underline{t}_\chi = (\chi(c_1^*) - 1, \dots, \chi(c_d^*) - 1) \in \mathcal{W}$ .

**Definition 6.3.**

- (1) Define the polynomial

$$\mathfrak{S}(\underline{T}) := \prod_{i=1}^d \left( \sum_{k=1}^d \sigma^i(c_k) T_k \right) \in \mathbb{Z}_p[\underline{T}],$$

where  $\sigma$  denotes the Frobenius of the extension  $\mathbb{Z}_{p^d}/\mathbb{Z}_p$ , and  $\underline{T} = (T_1, T_2, \dots, T_d)$ .

(2) The *admissible locus* is defined by

$$\mathcal{W}^{\text{adm}} := \left\{ \underline{t} \in \mathcal{W}(\mathbb{C}_p) - \{0\} : v_q(\mathfrak{S}(\underline{t})) = d \cdot \min_{1 \leq k \leq d} \{v_q(t_k)\} \right\} \subseteq \mathcal{W}.$$

For a continuous character  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{Z}_p$ , the corresponding point  $\underline{t}_\chi = (\chi(c_1^*) - 1, \dots, \chi(c_d^*) - 1) \in \mathcal{W}$  depends on the choice of basis  $\{c_1, \dots, c_d\}$ , and the definition of  $\mathcal{W}^{\text{adm}}$  depends on the basis as well. We hope that whether  $\underline{t}_\chi \in \mathcal{W}^{\text{adm}}$  depends only on  $\chi$  but not on the choice of basis made in Section 4.1. Also, the name ‘‘admissible locus’’ makes sense only if  $\mathcal{W}^{\text{adm}}$  is an admissible set.

We list and prove the two desired properties of the admissible locus below.

**Proposition 6.4.**

(1) *The admissible locus is an admissible subset of the weight space  $\mathcal{W}$ .*

(2) *Let  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{Z}_p$  be a continuous character. Whether or not  $\underline{t}_\chi \in \mathcal{W}^{\text{adm}}$  does not depend on the choice of basis  $\{c_1, \dots, c_d\}$ .*

*Proof.* For (1), because  $v_q(\mathfrak{S}(\underline{t})) \leq d \cdot \min_i \{v_q(t_i)\}$ , the set

$$\begin{aligned} \mathcal{W}^{\text{adm}} \cap \mathcal{W}^{1/p^{1/n}} &= \left\{ (t_1, \dots, t_d) \in \mathcal{W}^{1/p^{1/n}} : v_q(\mathfrak{S}(\underline{t})) = d \cdot \min_k \{v_q(t_k)\} \right\} \\ &= \left\{ (t_1, \dots, t_d) \in \mathcal{W}^{1/p^{1/n}} : v_q(\mathfrak{S}(\underline{t})) \leq d \cdot \min_k \{v_q(t_k)\} \right\} \\ &= \left\{ (t_1, \dots, t_d) \in \mathcal{W}^{1/p^{1/n}} : |\mathfrak{S}(\underline{t})| \geq |t_k^d|, 1 \leq k \leq d \right\}. \end{aligned}$$

Therefore,  $\mathcal{W}^{\text{adm}} \cap \mathcal{W}^{1/p^{1/n}}$  is a rational subdomain of  $\mathcal{W}^{1/p^{1/n}}$ . By Section 2.1,  $\mathcal{W}^{\text{adm}}$  is an admissible subset of  $\mathcal{W}$ .

For (2), the group of all possible change of basis operations is generated by the following three types:

- (a) only swapping  $c_i$  with  $c_j$ ;
- (b) for a unique fixed  $i$ , scaling  $c_i$  to  $uc_i$  for  $u \in \mathbb{Z}_p^*$ ;
- (c) only changing  $c_1$  to  $c_1 + uc_2$  for  $u \in \mathbb{Z}_p$ .

It suffices to check whether  $\underline{t}_\chi \in \mathcal{W}^{\text{adm}}$  does not change under these three changes of basis operations. For simplicity, we write  $\underline{t}_\chi = (t_1, \dots, t_d)$ .

Operation (a) will result in swapping  $t_i$  with  $t_j$  in  $\underline{t}_\chi$ , and swapping  $c_i$  with  $c_j$  in  $\mathfrak{S}(\underline{t})$ . Whether  $\underline{t}_\chi \in \mathcal{W}^{\text{adm}}$  is unchanged by the definition.

Operation (b) will result in changing  $t_i$  to  $(1 + t_i)^{u-1} - 1$  in  $\underline{t}_\chi$  and changing  $c_i$  to  $uc_i$  in  $\mathfrak{S}(\underline{t})$ . One has the congruence

$$c_i t_i \equiv uc_i \left( (1 + t_i)^{u-1} - 1 \right) \pmod{(pt_i) + (t_i)^2},$$

in the ring  $\hat{\mathbb{Z}}_p = \{x \in \mathbb{C}_p : |x| \leq 1\}$ , where  $(x)$  is the ideal generated by  $x$ . So the change of  $\mathfrak{S}(\underline{t}_\chi)$  has the valuation at least  $\min\{v_q(t_i) + 1, v_q(t_i^2)\} + (d - 1) \cdot \min_k\{v_q(t_k)\}$ , which is strictly greater than  $d \cdot \min_k\{v_q(t_k)\}$ . Therefore whether  $v_q(\mathfrak{S}(\underline{t}_\chi)) = d \cdot \min_k\{v_q(t_k)\}$  is unchanged.

Operation (c) will result in changing  $t_2$  to  $(1 + t_2)(1 + t_1)^{-u} - 1$  in  $\underline{t}_\chi$  and changing  $c_1$  to  $c_1 + uc_2$  in  $\mathfrak{S}(\underline{t})$ . One has the congruence

$$c_1 t_1 + c_2 t_2 \equiv (c_1 + uc_2)t_1 + c_2 \left( (1 + t_2)(1 + t_1)^{-u} - 1 \right) \pmod{(pt_1, pt_2) + (t_1, t_2)^2}.$$

The change of  $\mathfrak{S}(\underline{t}_\chi)$  has the valuation at least  $\min_{i=1,2}\{v_q(t_i) + 1, v_q(t_i^2)\} + \sum_{k>2} v_q(t_k)$ , so

whether  $v_q(\mathfrak{S}(t_\chi)) = d \cdot \min_k \{v_q(t_k)\}$  is unchanged.  $\square$

The next lemma kills the possibility of “unexpected cancellation” for characters whose corresponding point on the weight space lie in the admissible locus.

**Lemma 6.5.** *Let  $\underline{t} = (t_1, \dots, t_d) \in \mathcal{W}^{\text{adm}}$ , and let  $r \in \mathbb{Z}_{\geq 0}$ . If  $\min_k \{v_q(t_k)\} \leq 1/ap^{m_D}$ , then*

$$v_q(\text{ev}_{\underline{t}}(\mathfrak{T}_{(i,j)}^{\sigma^r})) \geq \frac{j}{\delta} \cdot \min_i \{v_q(t_i)\},$$

with equality holds if and only if  $(i, j) = (m, d_m)$ .

*Proof.* Recall that  $\mathfrak{T}_{(i,j)} = \sum_{k=1}^d \pi_{(i,j,k)}$ , so

$$v_q(\text{ev}_{\underline{t}}(\sigma^n(\mathfrak{T}_{(i,j)}))) \geq \min_k v_q(\text{ev}_{\underline{t}}(\pi_{(i,j,k)})). \quad (6.1)$$

By Lemma 4.5, for  $i < m_D$ , since  $j \leq \delta p^i$ ,

$$v_q(\text{ev}_{\underline{t}}(\pi_{(i,j,k)})) \geq p^i v_q(t_k) \geq \frac{j}{\delta} v_q(t_k).$$

For  $i < m_D$ , since  $j < D\delta i$ ,

$$v_q(\text{ev}_{\underline{t}}(\pi_{(i,j,k)})) \geq D i v_q(t_k) > \frac{j}{\delta} v_q(t_k).$$

Therefore, for all  $i \in \mathbb{Z}_{\geq 0}$ , we have

$$v_q(\text{ev}_{\underline{t}}(\pi_{(i,j,k)})) \geq \frac{j}{\delta} v_q(t_k). \quad (6.2)$$

Combining equations 6.1 and 6.2, we get

$$v_q(\text{ev}_{\underline{t}}(\mathfrak{T}_{(i,j)}^{\sigma^r})) \geq \frac{j}{\delta} \cdot \min_k \{v_q(t_k)\}.$$

The inequality is proved.

Next, we show that if the equality holds, then we must have  $(i, j) = (m, d_m)$ . Now suppose

$$v_q(\text{ev}_{\underline{t}}(\mathfrak{T}_{(i,j)}^{\sigma^r})) = \frac{j}{\delta} \cdot \min_k \{v_q(t_k)\},$$

then the equality in equation 6.2 must hold, which means

$$v_q(\text{ev}_{\underline{t}}(\pi_{(i,j,k)})) = \frac{j}{\delta} v_q(t_k).$$

By the discussion above, this can happen only if  $i < m_D$  and  $j = \delta p^i$ . Recall that we have  $\delta > \frac{d_i}{p^i}$  for all  $i \neq m$ , and  $j \leq d_i$ , so  $j < \delta p^i$  if  $i \neq m$ . Therefore, to ensure  $j = \delta p^i$ , the only option is to set  $(i, j) = (m, d_m)$ .

The last step is to show that whenever  $(i, j) = (m, d_m)$ , the equality holds. We fix  $(i, j) = (m, d_m)$ . Recall that

$$\text{ev}_{\underline{t}}(\pi_{(m,d_m,k)}) = E^{-1}((1 + t_k)^{p^m}),$$

and as we have mentioned in the proof for Lemma 4.5, the “leading term” of  $E^{-1}((1 + t_k)^{p^m})$  comes from  $(1 + t_k)^{p^m} - 1$ . If  $v_q(t_k) \leq \frac{1}{ap^m}$ , then we have  $v_q(t_k^{p^m+1}) \leq v_q(p \cdot t)$ , so

$$(E^{-1}((1 + t_k)^{p^m})) \equiv t_k^{p^m} \pmod{t_k^{p^m+1}}.$$

Without loss of generality suppose  $t_1$  has the smallest valuation among  $t_1, \dots, t_d$ , then  $v_q(t_1) = \min_k \{v_q(t_k)\}$ , and  $v_q(t_1) \leq \frac{1}{ap^{m_D}} < \frac{1}{ap^{m+1}}$ . For all  $k$ , either  $v_q(t_k) \leq \frac{1}{ap^m}$ , then

$$(E^{-1}((1 + t_k)^{p^m})) \equiv t_k^{p^m} \pmod{t_1^{p^m+1}},$$

or  $v_q(t_k) > \frac{1}{ap^m}$ , then

$$(E^{-1}((1+t_k)^{p^m})) \equiv 0 \equiv t_k^{p^m} \pmod{p},$$

so

$$(E^{-1}((1+t_k)^{p^m})) \equiv 0 \equiv t_k^{p^m} \pmod{t_1^{p^m+1}},$$

note that the Frobenius  $\sigma$  satisfies  $\sigma(x) \equiv x^p \pmod{p}$  for  $x \in \mathbb{Z}_q$ , we have

$$\begin{aligned} \text{ev}_{\underline{t}}(\mathfrak{I}_{(m,d_m)}^{\sigma^r}) &= \sum_{k=1}^d \sigma^n(c_k) \text{ev}_{\underline{t}}(\pi_{m,d_m,k}) \\ &\equiv \sum_{k=1}^d \sigma^r(c_k) t_k^{p^m} \pmod{t_1^{p^m+1}} \\ &\equiv \sum_{k=1}^d (\sigma^{r-m}(c_k) t_k)^{p^m} \pmod{t_1^{p^m+1}} \\ &\equiv \left( \sum_{k=1}^d \sigma^{r-m}(c_k) t_k \right)^{p^m} \pmod{t_1^{p^m+1}}. \end{aligned}$$

This implies

$$\prod_{r=1}^d \text{ev}_{\underline{t}}(\mathfrak{I}_{(m,d_m)}^{\sigma^r}) \equiv \prod_{r=1}^d \left( \sum_{k=1}^d \sigma^{r-m}(c_k) t_k \right)^{p^m} = \mathfrak{S}(\underline{t})^{p^m} \pmod{t_1^{dp^m+1}}.$$

Since  $\underline{t} \in \mathcal{W}^{\text{adm}}$ , we have

$$v_q \left( \prod_{r=1}^d \text{ev}_{\underline{t}}(\mathfrak{I}_{(i,j)}^{\sigma^r}) \right) = dp^m \cdot v_q(t_1) = \frac{d \cdot d_m}{\delta} \cdot \min_k \{v_q(t_k)\}.$$

But we have proved

$$v_q(\text{ev}_{\underline{t}}(\mathfrak{I}_{(i,j)}^{\sigma^r})) \geq \frac{d_m}{\delta} \cdot \min_k \{v_q(t_k)\}.$$

So the equality must hold for all  $r = 1, 2, \dots, d$ . Therefore, when  $(i, j) = (m, d_m)$ , we have

$$v_q(\text{ev}_{\underline{t}}(\mathfrak{T}_{(i,j)}^{\sigma^r})) = \frac{d_m}{\delta} \cdot \min_k \{v_q(t_k)\}.$$

□

### 6.3 Upper bound

We now give the upper bound for the  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$ .

**Theorem 6.6.** *Let  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$  be a continuous character such that  $v_p(\pi_\chi) \leq \frac{1}{p^{m_D}}$ , and such that  $(\chi(c_1^*) - 1, \dots, \chi(c_d^*) - 1) \in \mathcal{W}^{\text{adm}}$ . The  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$  passes through the vertices  $(n, \lambda_n)$  for  $n \equiv 0$  or  $1 \pmod{d_m}$ . As a consequence, the  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$  lies below the polygon whose vertices are  $(n, \lambda_n)$  for  $n \equiv 0$  or  $1 \pmod{d_m}$ .*

*Proof.* Fix  $n$  such that  $n \equiv 0$  or  $1 \pmod{d_m}$ . Let  $\chi_0$  be a finite character of order  $p^{m_{\chi_0}}$ , such that  $p^{m_{\chi_0}} > n$  and  $m_{\chi_0} > m_D$ . To simplify notations, denote

$$\text{ev}_\chi := \text{ev}_{(\chi(c_1^*), \dots, \chi(c_d^*))}.$$

By Theorem 1.9, the  $\pi_{\chi_0}$ -adic valuation of the  $n$ -th coefficient of  $C_f^*(\chi_0, s)$  is

$$v_{\pi_{\chi_0}}(\text{ev}_{\chi_0}(r_n)) = \lambda_n.$$

Denote  $\lambda'_n := \frac{n(n-1)(p-1)}{2d_m} = \frac{\lambda_n}{ap^m}$ . By Lemma 5.9 we can find  $F \in \mathbb{Z}_q[[\mathfrak{T}]]$  such that  $F \in I^{p^m \lambda'_n}$ , and

$$r_n \equiv \prod_{j=0}^{a-1} F^{\sigma^j} \pmod{pI^{\lambda_n} + I^{\lambda_n+1/\delta}}.$$

By Lemma 4.5, we must have

$$v_{\pi_{\chi_0}} \left( \text{ev}_{\chi_0} \left( r_n - \prod_{j=0}^{a-1} F^{\sigma^j} \right) \right) > \lambda_n,$$

so

$$v_{\pi_{\chi_0}} \left( \text{ev}_{\chi_0} \left( r_n - \prod_{j=0}^{a-1} F^{\sigma^j} \right) \right) = \lambda_n. \tag{6.3}$$

In other words,  $\prod_{j=0}^{a-1} F^{\sigma^j}$  contains the “leading term” of  $r_n$ .

We take a closer look at  $F$ . Write

$$F = u \cdot \mathfrak{F}_{(m,d_m)}^{\lambda'_n} + \text{other terms} \in \mathbb{Z}_q[[\mathfrak{F}]],$$

where  $u \in \mathbb{Z}_q$ ,  $v(\text{other terms}) \geq \lambda_n$ . Then by Lemma 6.5,

$$v_{\pi_{\chi_0}} \left( \text{ev}_{\chi_0} \left( \mathfrak{F}_{(m,d_m)}^{\lambda'_n} \right) \right) = p^m \lambda'_n,$$

whereas

$$v_{\pi_{\chi_0}} (\text{ev}_{\chi_0}(\text{other terms})) > p^m \lambda'_n,$$

because any summand in “other terms” either has  $\pi$ -adic valuation greater than  $p^m \lambda'_n$ , or contains at least one factor  $\mathfrak{F}_{(i,j)}$  with  $(i,j) \neq (m,d_m)$ , therefore the valuation strictly increases after evaluation at  $\chi_0$ , according to Lemma 6.5. So, after specializing to  $\chi_0$ ,  $u$  is the coefficient of the “leading term” with  $\pi_{\chi_0}$ -adic valuation equal to  $\lambda_n$ . From this fact and equation 6.3 we know  $u \in \mathbb{Z}_q^*$ . Therefore, for a continuous character  $\chi$  such that  $v_p(\pi_\chi) \leq \frac{1}{p^m D}$

and  $(\chi(c_1^*) - 1, \dots, \chi(c_d^*) - 1) \in \mathcal{W}^{\text{adm}}$ , by Lemma 6.5 again,

$$v_{\pi_\chi}(\text{ev}_\chi(r_n)) = v_{\pi_\chi} \left( \text{ev}_\chi \left( \prod_{j=0}^{a-1} \left( u_{\mathfrak{Z}_{(m,d_m)}^{\lambda_n}} \right)^{\sigma^j} \right) \right) = \lambda_n.$$

This finishes the proof. □

# Chapter 7

## The Artin–Schreier–Witt eigenvariety

Recall that the Artin–Schreier–Witt eigenvariety associated to  $f$  is the zero locus of  $C_f^*(\underline{T}, s)$  inside  $\mathbb{G}_m^{\text{rig}} \times (\mathcal{W} - \{0\})$ , and is denoted by  $\mathcal{E}_f$ . As discussed in the introduction, it is expected that  $\mathcal{E}_f$  can be decomposed to an infinite disjoint union of nice subspaces.

In the special case where  $d = 1$  and  $f$  is a unit root polynomial, Davis–Wan–Xiao [1] obtained two different results regarding the decomposition. The first one states that  $\mathcal{E}_f$  is an infinite disjoint union  $\coprod_{i \geq 0, i \in \mathbb{Z}} X_i$ , such that each  $X_i$  is finite and flat over  $\mathcal{W}$ , and that for any point  $x \in X_i$ , the ratio

$$\frac{v_q(a_p(x))}{v_q(\text{wt}(x))}$$

is in an interval depending on  $i$ . The second result states that over the boundary annulus

$$\mathcal{W}^{\geq r} = \{t \in \mathcal{W} : |t| \geq r\} = \{t \in \mathcal{W} : 0 < v_p(t) \leq -\log_p r\},$$

where  $0 < r < 1$  is a constant,  $X_i$  can be further decomposed into a finite disjoint union, such that the ratio  $\frac{v_q(a_p(x))}{v_q(\text{wt}(x))}$  is a constant on each component. In this dissertation, we refer

to the first result as the weak decomposition, and the second as the strong decomposition.

For general  $d$  and unit root polynomial  $f$ , Ren–Wan–Xiao–Yu [12] proved a similar weak decomposition over the admissible locus. This result can be viewed as a generalization of Davis–Wan–Xiao’s weak decomposition, since when  $d = 1$ , the admissible locus becomes  $\mathcal{W} - \{0\}$ . However, they were not successful trying to generalize the strong decomposition to  $d > 1$ .

Based on our results in Chapter 6, we can prove the weak decomposition over the intersection of the admissible locus with certain boundary annulus, for general  $d$  and overconvergent  $f$ . In terms of the strong decomposition, our results in Chapter 3 suggest that the strong decomposition might not be true, but a slightly weaker form is reasonable to expect.

## 7.1 Weak decomposition

Put  $\mathcal{W}^{\text{adm}, \geq r} := \mathcal{W}^{\text{adm}} \cap \mathcal{W}^{\geq r}$ . We show that for overconvergent  $\mathbb{Z}_p^d$ -towers, there is an  $r$  such that we have a weak decomposition over  $\mathcal{W}^{\text{adm}, \geq r}$ .

**Theorem 7.1.** *Let  $\mathcal{P}$  be an overconvergent  $\mathbb{Z}_p^d$ -tower associated to  $f$ . Then there is a rational number  $r$  with  $0 < r < 1$ , such that the space*

$$\mathcal{E}_f^{\text{adm}, \geq r} := \text{wt}^{-1}(\mathcal{W}^{\text{adm}, \geq r})$$

*is an infinite disjoint union:*

$$\mathcal{E}_f^{\text{adm}, \geq r} = X_{\{0\}} \amalg X_{(0,1)} \amalg X_{\{1\}} \amalg X_{(1,2)} \amalg X_{\{2\}} \amalg \cdots,$$

*of rigid subspaces, such that:*

(1) the map  $\text{wt} : X_J \rightarrow \mathcal{W}^{\text{adm}, \geq r}$  is finite and flat. The map is degree 1 if  $J = [n, n]$ , and is degree  $d_m - 1$  if  $J = (n, n + 1)$ , and

(2) for any  $x = (\underline{t}, s) \in X_J$ , we have

$$\frac{v_q(s^{-1})}{\min_k \{v_q(t_k)\}} \in ap^m(p-1) \cdot J.$$

*Proof.* Recall  $D$  and  $m_D$  from the definition of overconvergence, and that  $m_D \geq 2$  and  $\frac{p^{m_D/2}}{m_D} \geq D$ . Let  $r = p^{-1/p^{m_D}}$ . Let  $(t_1, \dots, t_d) \in \mathcal{W}^{\text{adm}, \geq r}$ , and let  $\chi : \mathbb{Z}_{p^d} \rightarrow \mathbb{C}_p^*$  be the continuous character such that

$$\chi(c_k^*) - 1 = t_k, \quad k = 1, 2, \dots, d.$$

By Theorem 6.2 and Theorem 6.6 applied to  $\chi$ , for each  $n \in \mathbb{Z}_{\geq 0}$ , the  $\pi_\chi$ -adic Newton polygon of  $C_f^*(\chi, s)$  has:

(a) One line segment with horizontal length one, and with slope

$$\lambda_{nd_{m+1}} - \lambda_{nd_m} = ap^m(p-1) \cdot n.$$

(b) Line segments with total horizontal length  $d_m - 1$ , whose slopes are in the interval  $(ap^m(p-1) \cdot n, ap^m(p-1) \cdot (n+1))$ .

By Weierstrass Preparation Theorem, over  $\mathcal{W}^{\text{adm}, \geq r}$ , we have the factorization

$$C_f^*(T, s) = P_{\{0\}}(s) \cdot P_{(0,1)}(s) \cdot P_{\{1\}}(s) \cdot P_{(1,2)}(s) \cdots,$$

such that for each  $n \in \mathbb{Z}_{\geq 0}$ :

- (a)  $P_{\{n\}}(s) \in \mathbb{Z}_p[[\underline{T}]](s)$  is a degree one polynomial in  $s$ , with exactly one reciprocal zero of  $\pi_\chi$ -adic valuation  $ap^m(p-1) \cdot n$ , and
- (b)  $P_{(n,n+1)}(s) \in \mathbb{Z}_p[[\underline{T}]](s)$  is a degree  $d_m - 1$  polynomial in  $s$ , whose reciprocal zeroes have  $\pi_\chi$ -adic valuation in the interval  $(ap^m(p-1) \cdot n, ap^m(p-1) \cdot (n+1))$ .

For each interval  $J = [n, n]$  or  $(n, n+1)$ , let  $X_J$  be the zero locus of  $P_J$  in  $\mathcal{W}^{\text{adm}, \geq r} \times \mathbb{G}_m^{\text{rig}}$ . Then each  $X_J$  is finite and flat over  $\mathcal{W}^{\text{adm}, \geq r}$ , because it is the zero locus of a polynomial. The weight map

$$\text{wt} : X_J \rightarrow \mathcal{W}^{\text{adm}, \geq r}$$

has degree  $\deg P_J$ , which is one if  $J = [n, n]$ , and is  $d_m - 1$  if  $J = (n, n+1)$ . This proves (1).

For any  $x = (\underline{t}, s) \in X_J$ ,  $(\underline{t}, s)$  is a zero of  $P_J(s)$ , so the  $\pi_\chi$ -adic valuation of  $\frac{1}{s}$  is in the interval  $ap^m(p-1) \cdot J$ . Note that  $v_q(\pi_\chi) = \min_k \{v_q(t_k)\}$ , and  $v_q(1/s) = v_{\pi_\chi}(1/s) \cdot v_q(\pi_\chi)$ , so

$$\frac{v_q(s^{-1})}{\min_k \{v_q(t_k)\}} \in ap^m(p-1) \cdot J.$$

Therefore (2) is proved.

The intervals  $ap^m(p-1) \cdot J$  are distinct for different  $J$  in the form  $J = [n, n]$  or  $(n, n+1)$ .

As a consequence, the ratio

$$\frac{v_q(s^{-1})}{\min_k \{v_q(t_k)\}}$$

take distinct values on each  $X_J$ . Therefore,  $X_J$  are disjoint subspaces. This finishes the proof.  $\square$

REMARK 7.2. When  $f$  is a unit root polynomial, Ren–Wan–Xiao–Yu [12] proved the same weak decomposition over the entire admissible locus. In comparison, in this case, our result

only implies the decomposition over  $\mathcal{W}^{\text{adm}, \geq p^{-1/p^2}}$ . But our result covers a much larger class of  $f$ .

## 7.2 Strong decomposition

We hope to further decompose  $X_{(n, n+1)}$  such that the ratio  $\frac{v_q(s^{-1})}{\min_k \{v_q(t_k)\}}$  take constant values on each component. Up to when this dissertation is written, this question is answered only for  $d = 1$  and  $f$  being a unit root polynomial[1], and is open in other cases. It is already very hard for unit root polynomial  $f$  and  $d > 1$ .

For the rest of this section, fix a unit root polynomial  $f([X]) = b_0 + b_1[X] + \cdots + b_{\deg f}[X]^{\deg f}$ , where  $b_i$  are  $(q - 1)$ -st roots of unity in  $\mathbb{Z}_q$ ,  $p \nmid \deg f$ . Put  $r = p^{-8(\deg f)/a(p-1)(\deg f-1)^2}$ . For  $d = 1$ , the following is known.

**Theorem 7.3** (Davis–Wan–Xiao, 2016). *Let  $\mathcal{P}$  be the  $\mathbb{Z}_p$ -tower associated to a unit root polynomial  $f$ , and  $\mathcal{E}_f$  be the corresponding eigencurve. Then there exists an integer  $l \in \mathbb{N}$  and distinct rational numbers  $\alpha_1, \dots, \alpha_l \in [0, 1)$ , such that the subspace  $\text{wt}^{-1}(\mathcal{W}^{\geq r})$  of  $\mathcal{E}_f$  is an infinite disjoint union*

$$\text{wt}^{-1}(\mathcal{W}^{\geq r}) = \coprod_{i=0}^{\infty} \coprod_{j=1}^l X_{i,j}$$

of rigid subspaces, such that:

- (1) the morphism  $\text{wt} : X_{i,j} \rightarrow \mathcal{W}^{\geq r}$  is finite and flat, and
- (2) for any  $x = (T, s) \in X_{i,j}$ , we have  $v_q(s^{-1}) = a(p-1)(\alpha_j + i)v_q(T)$ .

The method used in Chapter 3 and the above theorem can help us understand the  $d > 1$  case. For a basis  $\beta = \{c_1, \dots, c_d\}$ , let  $c_1^*, \dots, c_d^*$  be such that  $\{\text{Tr}(c_k^* \cdot -)\}_{k=1}^d$  is the corresponding

dual basis. Note that

$$S_{f,\beta}^*(n, \underline{T}) \Big|_{T_k=0 \text{ for } k \geq 2} = S_{c_1 f, \{1\}}^*(n, T) \Big|_{T=T_1}.$$

Then by the generating series formula for the  $T$ -adic characteristic series, we get

$$C_{f,\beta}^*(\underline{T}, s) \Big|_{T_k=0 \text{ for } k \geq 2} = C_{c_1 f, \{1\}}^*(T, s) \Big|_{T=T_1}.$$

By [1], when  $c_1$  is a  $(q-1)$ -st root of unity, the zero locus of  $C_{c_1 f, \{1\}}^*(T, s)$  has a strong decomposition over the one-dimensional weight space. Therefore, the zero locus of  $C_{f,\beta}^*(\underline{T}, s)$ , which is the  $d$ -dimensional eigenvariety  $\mathcal{E}_f$ , has a strong decomposition over the rigid line  $\ell$  defined by the equations

$$T_k = 0 \text{ for } k \geq 2$$

inside the  $d$ -dimensional weight space  $\mathcal{W}$ . Consider another basis  $\beta' = \{c'_1, \dots, c'_d\}$ , such that  $c'_1$  is also a  $(q-1)$ -st root of unity just like  $c_1$  does. Write

$$c'_k = \sum_{j=1}^d a_{kj} c_j,$$

where the matrix  $(a_{kj}) \in \text{GL}_d(\mathbb{Z}_p)$ . By linear algebra, let  $(a_{kj}^*)$  denote the transposed inverse of the matrix  $(a_{kj})$ , then

$$c_k'^* = \sum_{j=1}^d a_{kj}^* c_j^*.$$

Therefore, if  $\chi$  is any continuous character or the universal character, then

$$\chi(c_k^*) - 1 = \chi\left(\sum_{j=1}^d a_{kj}^* c_j^*\right) - 1 = \prod_{j=1}^d (\chi(c_j^*))^{a_{kj}^*} - 1.$$

Let  $T_k = \chi(c_j^*) - 1$ ,  $T'_k = \chi(c_k^*) - 1$ ,  $k = 1, 2, \dots, d$ . If we set  $T'_k = 0$  for  $k = 2, \dots, d$ , then the universal character series becomes  $C_{c'_1, \{1\}}^*(T'_1, s)$ . Now let  $\mathcal{C}$  be the rigid curve on  $\mathcal{W}$  defined by

$$T'_k = \prod_{j=1}^d (T_k + 1)^{a_{kj}^*} - 1 = 0 \text{ for } k \geq 2,$$

such that  $c'_1 = \sum_{j=1}^d a_{1j} c_j$  is also a  $(q-1)$ -st root of unity. By the strong decomposition of  $C_{c'_1, \{1\}}^*(T, s)$ , we get a strong decomposition of  $\mathcal{E}_f$  over  $\mathcal{C}$ .

Summarizing all above, we have proved:

**Theorem 7.4.** *Fix a basis  $\beta = \{c_1, \dots, c_d\}$  of  $\mathbb{Z}_p^d$  as a free  $\mathbb{Z}_p$ -module. For each  $(a_{kj}) \in \text{GL}_d(\mathbb{Z}_p)$  with  $\left(\sum_{j=1}^d a_{1j} c_j\right)^{q-1} = 1$ , we can define a rigid curve  $\mathcal{C}$  inside  $\mathcal{W}$  by the equations*

$$\prod_{j=1}^d (T_k + 1)^{a_{kj}^*} - 1 = 0 \text{ for } k \geq 2.$$

*There is an integer  $l \in \mathbb{N}$  and distinct rational numbers  $\alpha_1, \dots, \alpha_l \in [0, 1)$ , such that  $\mathcal{E}_f \cap \mathcal{C} \cap \mathcal{W}^{\geq r}$  has the strong decomposition*

$$\mathcal{E}_f \cap \mathcal{C} \cap \mathcal{W}^{\geq r} = \prod_{i=0}^{\infty} \prod_{j=1}^l X_{i,j},$$

*such that:*

- (1) *the morphism  $\text{wt} : X_{i,j} \rightarrow \mathcal{C} \cap \mathcal{W}^{\geq r}$  is finite and flat, and*
- (2) *for any  $x = (T, s) \in X_{i,j}$ , we have  $v_q(s^{-1}) = a(p-1)(\alpha_j + i)v_q(T)$ .*

Note that the rational numbers  $\alpha_1, \dots, \alpha_l$  depend on  $\mathcal{C}$ . In fact, they are determined by the number  $c'_1$ , because they are related to the Newton polygon of  $C_{c'_1 f, \{1\}}^*(T, s)$ . If  $\frac{c'_1}{c_1} \in \mathbb{Z}_p$ , then there is an isomorphism between  $\mathbb{Z}_p$ -towers associated to  $c'_1 f$  and  $c_1 f$ , so the numbers  $\alpha_1, \dots, \alpha_l$  should be the same over two rigid lines  $\mathcal{C}, \mathcal{C}'$  corresponding to bases  $\beta, \beta'$  respectively. Therefore, in Theorem 7.4, there are at most  $\frac{\#\mathbb{F}_q^*}{\#\mathbb{F}_p^*} = \frac{q-1}{p-1}$  many distinct patterns for  $\alpha_1, \dots, \alpha_l$ . It would be interesting to know whether this number can be reduced.

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