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Lecture III

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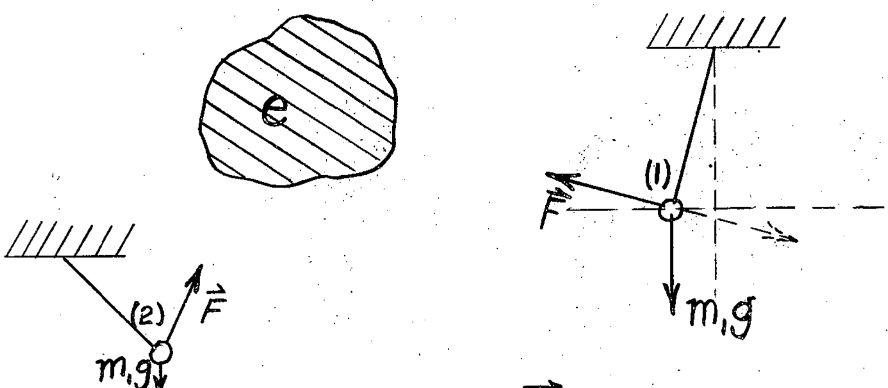
ELECTROSTATIC FIELDS IN FREE SPACE

An electrostatic field may be defined as the space surrounding a charged body.

By experiment we know that a body may be charged by rubbing, by being touched to either post of a battery, or by being touched with another charged body. We know that the body is charged because it now exerts a force on other bodies near it, which force it did not exert before it was charged. Further, we know that the magnitude of the force is independent of the method of charging, but its direction is not.

Now suppose we have a charged body whose charge is not necessarily constant, and we suspend a much smaller test body near it.



We may measure the force $\vec{F}$ by noting how far the small test body (1) is displaced. (Note $\vec{F}$ may be positive or negative) Now if $\vec{E}$ (x, y, z) is the electric field, we may say $\vec{F} = e\,\vec{E}(x, y, z)$ where e is a constant somehow related to the charge on the body. Now if we make this measurement at a given point, say (1), with e taking a value e_1 and e_2, we find:

$$|\vec{F_1}| \propto e_1 \qquad\qquad \frac{|\vec{F_1}|}{|\vec{F_2}|} = \frac{e_1}{e_2}$$
$$|\vec{F_2}| \propto e_2 \qquad \text{or}$$

DISCLAIMER

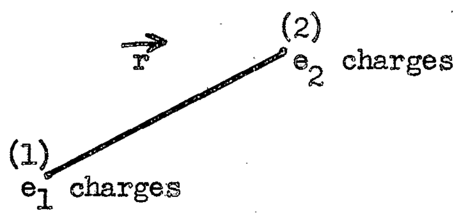
Likewise if we say $e =$ constant, i.e. charge does not change, and we move the small test body to a new position (2) where the field is $\vec{E}_2$, we find:

$$|\vec{F}_1| \propto |\vec{E}_1| \quad \text{and} \quad |\vec{F}_2| \propto |\vec{E}_2| \qquad \text{in other words} \quad \frac{|\vec{F}_1|}{|\vec{F}_2|} = \frac{|\vec{E}_1|}{|\vec{E}_2|}$$

Now these forces may be thought of in many ways. One way, due to Maxwell, is to think of the force on the small test body as due to the interaction of the body with the electric field around the charged object. Another way, due to Newton, is to think of the force on the small body as a function of the distance separating the bodies. Both ways are largely a matter of definition, and either will yield correct results.

The above equations relating the Force on a small body to the charge on the charged body and the field strength at the small body leads us to the "Law of Forces" or "Coulombs Law".

If at (1) and (2) we have respectively e_1 charges and e_2 charges separated by a distance $\vec{r}$:



we may say $|\vec{F}| = \frac{e_1 \, e_2}{|\vec{r}|^2} f$

where f is a constant.

Now if $f = 1$, $|\vec{F}|$ is in dynes and $|\vec{r}|$ in cm. Then e is in esu, or in other words, 1 dyne force at 1 cm. = 1 esu of charge. But if $|\vec{F}|$ is given in Newton's $\left(\frac{\text{Kg meter}}{\text{Sec}^2}\right) = 10^5$ dynes, $|\vec{r}|$ in meters, and e in coulombs, then

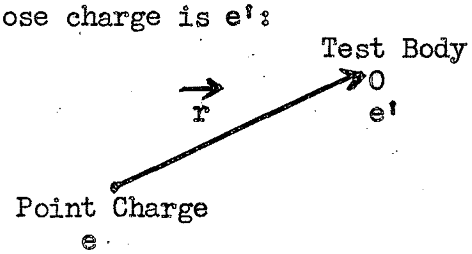
$$f = \frac{1}{4\pi K_0} \quad \text{where } K_0 = 8.85 \times 10^{-12} \quad \text{coulomb/volt meter.}$$

This latter system of units is known as the M.K.S. System and is one the most generally used in engineering problems since current has the units of amperes, potential the unit of volts, etc.

Thus in the M.K.S. System, Coulomb's Law would be written:

$$|\vec{F}| = \frac{e_1 \, e_2}{4\pi K_0 |\vec{r}|^2}$$

Now if we have a point charge e and at some distance r a small test body whose charge is e':

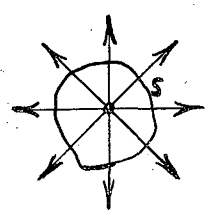


then $\vec{F} = \frac{e' \; e}{4\pi K_o \; r^2} \; \frac{\vec{r}}{|\vec{r}|}$

since $\vec{F} = e' \vec{E}$

$\vec{E} = \frac{1}{4\pi K_o} \; \frac{e}{r^2} \hat{r}$ volts/meters.

This expression is very similar to the one previously derived for the field of velocity around a point source emitting fluid.



$\vec{v} = \frac{q \; \hat{r}}{4\pi r^2}$ or $\int_s \vec{v} \cdot ds = q$

similarly: $\int_s \vec{E} \cdot ds = \frac{e}{K_o}$

which is the mathematical expression of Gauss' Law. In other words, the field leaving a closed surface is equal to the amount of charge enclosed within the surface. Now if there are a number of charges:

$$\int_s \vec{E} \cdot ds = \sum \frac{e_1}{K_o} = \int_V \frac{\rho}{K_o} dV = \int_V \operatorname{div} \vec{E} \cdot dV$$

where ρ is considered as charge density and has the units of coulombs/meter^3.

These equations indicate that a field set up by point charges is irrotational since:

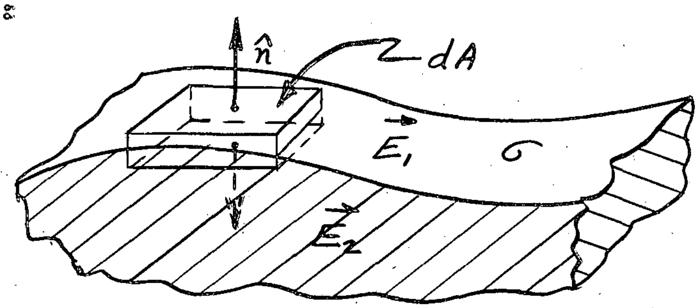
$$\int_V \frac{\rho}{K_o} dV = \int_V \operatorname{div} \vec{E} \cdot dV$$

$\operatorname{div} \vec{E} = \frac{\rho}{K_o}$ which is one of Maxwell's equations

or $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{K_o}$

DISCONTINUITY AT SURFACE OF CHARGED BODY

In considering the condition at the surface of a charged body, it may be shown that a discontinuity in the field exists at the surface. Consider a flat "pillbox" on the surface of the body such that no flux leaks out the sides of the box:



Where:

$\hat{n}$ = Unit normal vector

dA = Area of top of "pillbox"

$\vec{E}_1$ = Electrostatic field gradient outside body

$\vec{E}_2$ = Electrostatic field gradient inside body

σ = Surface charge density (coulombs/meter^2)

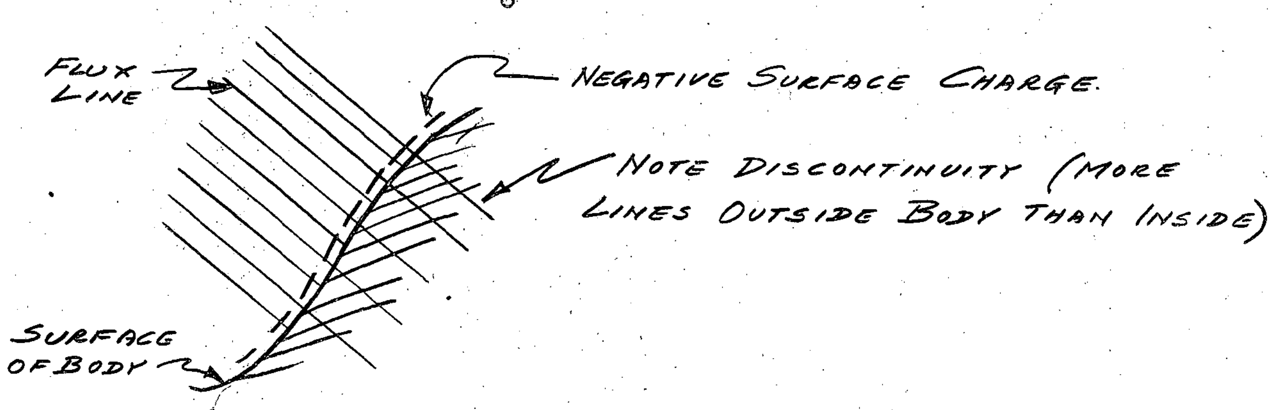
In general:

$$\int \vec{E} \cdot \hat{n} dA = \int_A \frac{\sigma}{K_o} dA$$

For this case:

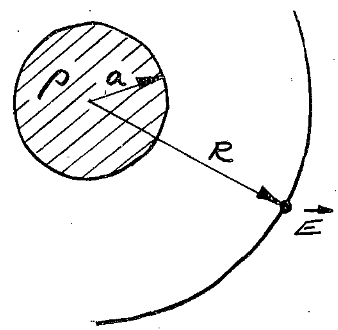
$$\int \vec{E}_1 \cdot \hat{n} dA - \vec{E}_2 \cdot \hat{n} dA = \frac{\sigma \, dA}{K_o}$$

$$\vec{E}_1 \cdot \hat{n} = \vec{E}_2 \cdot \hat{n} + \frac{\sigma}{K_o}, \text{ (i.e., } \vec{E}_1 \neq \vec{E}_2 \text{, hence a discontinuity).}$$



FIELD OUTSIDE UNIFORMLY CHARGED SPHERE

Consider a uniformly charged sphere:



Where:

ρ = charge density in sphere (coulombs/meter^3)

a = radius of sphere

R = distance of some point from center of sphere

$\vec{E}$ = Electrostatic field gradient at above point

and if V = volume of sphere, and Q = total charge on sphere, then:

$$\int \frac{\rho \, dV}{K_o} = \frac{4}{3}\pi a^3 \frac{\rho}{K_o}$$

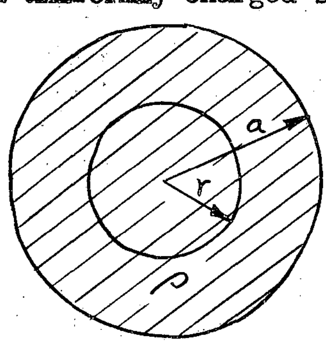
$$E \times 4\pi R^2 = \frac{Q}{K_o}$$

$$E = \frac{Q}{4\pi K_o R^2}$$

This shows that a uniformly charged sphere is equivalent to a point charge at the center of the sphere with all the charge concentrated at the point.

FIELD INSIDE UNIFORMLY CHARGED SPHERE

Consider a uniformly charged sphere:



Where:

a = radius of sphere

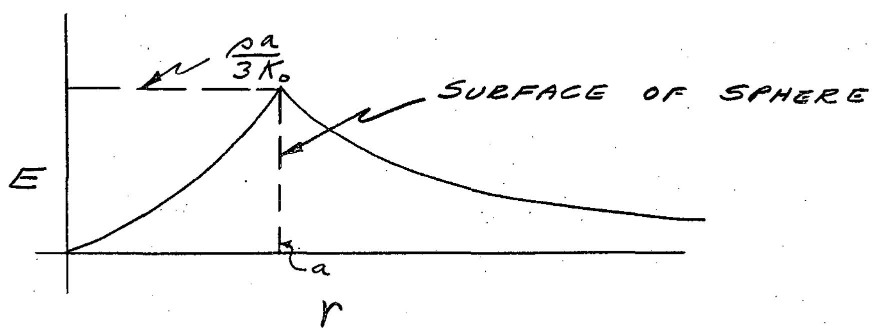
ρ = charge density in sphere

Flux leaving any sphere of radius r:

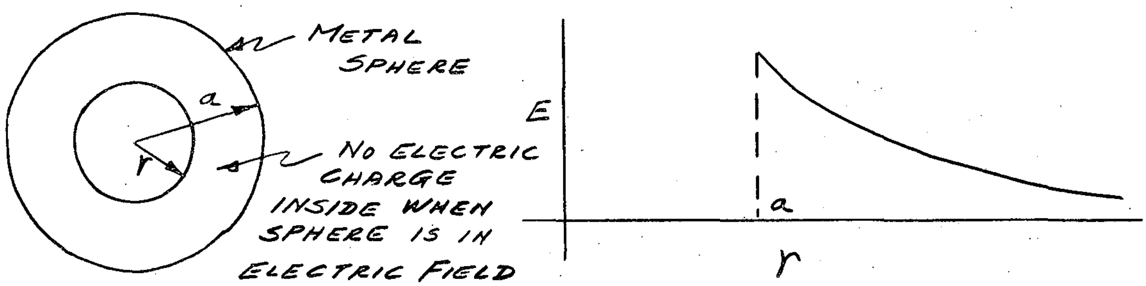
$$E\,4\pi r^2 = \frac{1}{K_o}\int_r \rho\, dV$$

$$= \frac{4}{3}\pi r^3 \frac{\rho}{K_o}$$

$$E = \frac{\rho r}{3 K_o}$$



THIN METAL SPHERE IN ELECTRIC FIELD



POISSON'S EQUATION

Consider a field around a point charge e:

$$\vec{E} = \frac{1}{4\pi K_o}\frac{e}{r^2}\hat{r} = -\text{grad}\,\frac{e}{4\pi K_o r} = \text{Volts (scalar)}$$

$$= \frac{e}{4\pi K_o r} = V$$

V = Electrostatic potential in volts (M.K.S.)

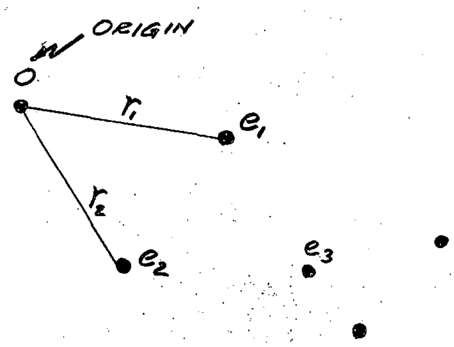
$\vec{E} = -\text{grad } V = \frac{\rho r}{3 K_o}$ (i.e. E is irrotational field)

$\text{curl } \vec{E} = 0$

$\text{div } \vec{E} = \frac{\rho}{K_o}$

$$\boxed{\text{div grad } E = -\frac{\rho}{K_o}}$$ Poisson's Equation

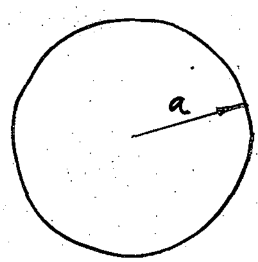
Considering a group of charges, $e_1, e_2, --- e_i$



$$V = \frac{1}{4\pi K_o} \sum \frac{e_1}{n_i}$$

$$V = \frac{1}{4\pi K_o} \int \frac{\rho}{r} \, dV$$

CAPACITY OF A METAL SPHERE



Where:

a = radius of sphere

e = charge (concentrated on surface)

potential at a, $V = \frac{e}{4\pi K_o a}$

defining $\frac{e}{V} = C$, capacity = charge/voltage

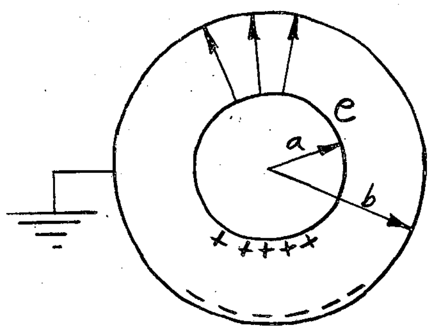
$$C \cdot V = e$$

$$C = 4\pi K_o a \text{ in farads (a in meters)}$$

For example, for a sphere one meter in diameter:

$$C = 4\pi \times 8.85 \times 1/2 \ \mu\mu F$$

CAPACITY OF CONCENTRIC SPHERES



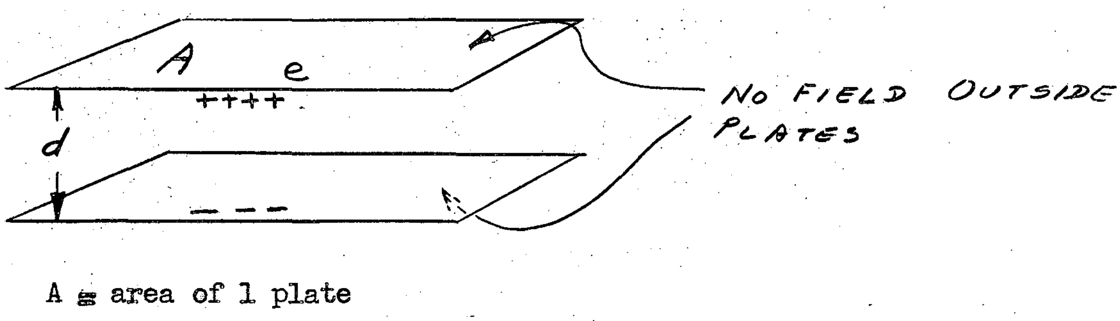
a = radius of inner sphere

b = radius of outer sphere

$$\Delta V = \frac{e}{4\pi K_o a} - \frac{e}{4\pi K_o b}$$

$$C = 4\pi K_o \frac{a\,b}{b - a} \quad \text{(where a and b are in meters)}$$

CAPACITY OF PARALLEL PLATES



A = area of 1 plate

d = distance between plates

$$\vec{E} = \frac{V}{d}$$

surface charge density $\sigma = K_o E$

$$\sigma A = K_o E A = \frac{K_o d A}{d}$$

$$C = \frac{K_o V A}{d V} = \frac{K_o A}{d}$$

$C = 8.85 \frac{A}{d} \mu\mu f$ (where A is in meters^2 and d is in meters).

plc
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