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UNIVERSITY OF CALIFORNIA, SAN DIEGO

**Controlling the Dynamics of Classical and Quantum Information in  
Spin Systems**

A dissertation submitted in partial satisfaction of the  
requirements for the degree  
Doctor of Philosophy

in

Physics

by

Parin B. Dalal

Committee in charge:

Professor Lu Jeu Sham, Chair  
Professor Massimiliano Di Ventra  
Professor David Meyer  
Professor Cliff Surko  
Professor Ken Zeger

2009

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Chair

University of California, San Diego

2009

## DEDICATION

To the family, friends, and complete strangers who saw me through  
the hardest time of my life...

## EPIGRAPH

*The soul is permanent as well as impermanent. From the point of view of substance it is eternal. From the point of view of its modes it undergoes birth, decay and destruction and hence is impermanent.*

—Māhavīra, Bhagvatisūtra, 7:58-59

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The text of chapter II, in part, is a reprint of the material that is being prepared for publication as Dalal, P.; Ł. Cywiński; Sham, L. J. *The Channel Capacity of Spin-information in Semiconductors*. The co-authors in this publication directed, supervised, and co-worked on the research which forms the basis of this chapter.

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Major Field: Physics

Studies in Theoretical Condensed Matter Physics

Professor Lu Jeu Sham

## ABSTRACT OF THE DISSERTATION

### **Controlling the Dynamics of Classical and Quantum Information in Spin Systems**

by

Parin B. Dalal

Doctor of Philosophy in Physics

University of California San Diego, 2009

Professor Lu Jeu Sham, Chair

The precise, human control of quantum systems, by its definition, must combine models of the classical and the quantum world into a calculus that supports both. Open, irreversible quantum systems must interact with closed, reversible quantum systems to predict evolutions that are partially open and closed. Inevitably, the problems of quantum measurement, the assumptions of scattering, and the role of spacetime comes under scrutiny. Such considerations have extraordinary practical value: the precise control of a quantum information is the cornerstone of scalable quantum computing. Traditionally, quantum control theory as well as a formalism of redundancy and partial measurements, known as quantum error correction, attempt to remedy systematic quantum-noise and random quantum-noise respectively, but have had mixed success. This dissertation examines how the imprecision of control in quantum and classical spin systems affects the flow of select information to a receiver and how such systems may be optimized against the imprecise scattering of control fields and spins.

To this end, this dissertation intertwines the physics of state evolution with the physics of information control in classical and quantum systems. First in classical systems, a method for encoding and decoding classical spin-processing information provides an example of information flow. Then an analytic calculation of a semi-conductor spin channel's information capacity is performed. The results

limit the rate of information processing and inform the design of materials for optimal spintronic information-processing in semiconductors. Next, noisy quantum interactions are described, so that the complexities of correcting small, random phase errors using traditional control theory and quantum error correction may be explained. How these noisy processes affect the relevant information flow of a quantum algorithm (derivatives of the Quantum Fourier Transform and Grover Search) is considered, several novel methods of source-coding for these quantum channels are presented and their efficacy calculated. These methods include Unitary-Fault Tolerance, Clifford operations of locally-variant basis, and an entropic controller. Together they show classical systems in the steady state can be used to control scalable, high-precision quantum-computing machines, and ultimately may eliminate all temporal control from quantum operations.

# Chapter 1

## Introduction: Time dynamics, control, and information

In classical mechanics, entropy is understood as giving time direction. At constant entropy, all processes are reversible and there is no way to predict the temporal direction of a system's trajectory. However, irreversible processes (entropy producing processes) prescribe a definite classical direction, as entropy,  $S$ , must satisfy (1.1) by its definition as a Lyapunov function.

$$\partial_t S \geq 0 \tag{1.1}$$

If the total entropy increases, then time must evolve positive semi-definitely and vice-versa. Energy is the generator of this time evolution in both quantum mechanics and classical mechanics. However, in relativistic quantum mechanics time becomes an operator,  $T$ , rather than a parameter,  $t$ , giving rise to the famous time, energy uncertainty relationship (1.2).

$$\Delta E \Delta T \geq \frac{1}{2} \tag{1.2}$$

Fundamental questions arise as to how entropy, time, and energy are connected in quantum mechanics. This has been fertile ground for research, but remains an open question without a definitive interpretation as it defies the traditional notion of a spacetime continuum in the absence of measurements. An informal analysis

of this relationship is seen through the correspondence principle. When applied to a system's evolution, it asserts that there should be a relationship between its quantum entropy operator and its quantum time operator (1.3), and the inequality follows from (1.1). At constant energy, energy and entropy share a common eigenbasis, but when the entropy is strictly increasing, the energy uncertainty must be at least inversely proportional to the rate of entropy production, as  $\Delta S \Delta E > 0$ .

$$-i[S, E] \begin{cases} = 0 & \text{reversible} \\ > 0 & \text{irreversible} \end{cases} \quad (1.3)$$

This dissertation interprets (1.2) as applying to the precision of the interaction energy term in a Hamiltonian (or Lagrangian), and explores explicitly how to maximize the information transferred between two quantum systems (or two classical systems). In this vein, scattering between two systems may be seen as information transport and the scattering term (or interaction term) as the information channel. If a system has an exact time, then it has an unbounded amount of energy and infinite information can be encoded into every energy eigenstate. This agrees with classical thermodynamics: as the temperature of a system goes to infinity, so does its entropy. On the other hand if a system is closed, and the energy of a system is constant, then the time of the system is unknown. Only upon measuring the system, opening it to a macroscopic environment and an uncertainty in energy through a time-modulated interaction, does the system acquire a time. It appears then, that the precision in the timing of a scattering interaction exactly determines the amount to information transfer between two otherwise non-interacting systems.

## 1.1 From the viewpoint of information theory

Using the formalism of information theory, we can draw this as a sender, receiver and information channel as in Figure 1.1. Terms in the Hamiltonian or Lagrangian that interact two otherwise closed systems, allow *local information* transfer that is parametrized by the precision of the timing of the interaction. The

local information of a system can be defined by tracing over all Hilbert space dimensions external to the system. Because all quantum operations are unitary (or globally stationary), this information channel represents only a transfer of local information. In particular, quantum algorithms transfer local entropy from one set of observables to another physically convenient set of observables. If the interaction is infinitely long, then the system is simply renormalized to a new energy level and no net information is transferred. If the interaction time is infinitely precise, an unbounded amount of information may be transferred from one system to the other encoded in the phase of the system by the interaction. If the interaction time is classically modulated, quantum mechanically imprecise, it becomes an information bridge between the quantum and the classical world. This is shown in Figure 1.1 where classical information is used to encode the transfer of local information from fields to qubits and vice-versa.

For example, a qubit in the ground state that does not interact with any fields has zero entropy (only one state) and so contains no information. Upon interacting the qubit with one or more fields, a measurement of the qubit reveals information about the portion of the fields it has scattering against in the time and space of their interactions. Local entropy has been transferred to the qubit from the field, but also from the specifics of the scattering, termed as *modulation* of the scattering event. If the qubit is rotated by  $\pi$ , we know only certain combinations of field strength, frequency, and interaction time at a particular position in space could have occurred. Additionally, the acceleration of the qubit causes some degree of back-action on the field by introducing a new field over the space and time of the acceleration. It is for this reason that modern scattering theory treats the measurement of quantum degrees of freedom as information functionals over the finite portions of spacetime the fields interact with them. For the most part, incoherent objects (people and their instruments) decide the space and time of this interactions. This information, of which scattering events actually took place, represents the Kolmogorov information, or the algorithm randomness. By encoding it properly, the intended information is transferred quantum mechanically between one quantum system to the next (see caption of Figure 1.1). In Chapter 5, we

consider the control of scattering by coherent objects.

The depicted quantum channel is modulated not just by the time, but the space of the interaction, where the communication channel is the QED Green's function response,  $e\bar{\psi}\gamma^\mu\psi A_\mu$ . Additionally classical information mixes with quantum information as the permutations of field and qubit interactions is large. Classically, this is related to the free-energy and entropic-energy components of a system. Entropy increases not just with temperature, but adiabatically with expansion of volume like two gases mixing irreversibly. In the quantum case, such an entropy translates to the Kolmogorov complexity of which scatterings took place of all the possible scatterings.

## 1.2 From the viewpoint of time-evolution

S-Matrix theory idealizes the time between the in-motion of free particles and the out-motion of free particles as infinitely long. This assumption becomes weak when the scattering time must be very precise. As the observation time turns to the order of the scattering time, the notions of time and space of the scattering becomes quantum mechanically imprecise. Such timescales do practically exist today, since the observer can be another coherent object in a cascade of quantum operations as quantum information propagates. If the temporal evolution of the scattering process directly controls the propagation of (or lack of propagation of) quantum information, the choice of information representation and information evolution is paramount to its control.

The encoding of algorithmic information (process tomography) is given by the presence or absence of the control fields' *area* at any given time. This encoding of quantum information passes through the a physical channel of evolution, for example the QED Lagrangian, of a quantum system. Classical information is input through the modulation of a control field and output through it's quantum collapse, or measurement. It is then the information structure of the control field (and how its structure reacts to noise) that ultimately determines the precision of the operation and the precision of the information. Process tomography, and its

coherent control, are fertile areas of research that are often neglected to the first order considerations of quantum state tomography.

### 1.3 From the viewpoint of control

A Hilbert space,  $H$ , is considered *controllable* if there exists scattering events, or interaction terms,  $\gamma_i(t) H_i$ , and there exists some sequence of interaction pulses,  $\gamma_i(t)$ , such that any wavefunction  $|\psi_1\rangle \in H$  can be transformed into any other wave function  $|\psi_2\rangle \in H$  by some choice of pulse sequence. Such a definition assumes that the time shaping of the pulses is deterministic; this is clearly an approximation as  $t$  is really an operator and not a parameter. But, at some course-graining of the Hilbert space this approximation turns valid. So if  $[H_i, H_j]$  is complete in  $H$ , the question of controllability of a finite dimensional Hilbert space then equates to the controllability of the co-factors,  $\gamma_i(t)$ , generating the pulse areas: the controllability of the fields. The presence of noise in the control field, either by coupling to stray, unintended fields, jumps in the vector by unintended partial measurement, or imprecision in the control of  $\gamma_i(t)$ , leads to a reduced set of controllable points in the Hilbert space. By this, I mean some group of wavefunctions  $|\psi_i\rangle$  can be transformed to some other group  $|\psi_j\rangle$  with high probability, but the specific  $i, j$  cannot be controlled. This controllable subset determines the encodable information in a Hilbert space (but not necessarily measurable). The formalism of quantum error correction creates such subspaces to encode information, and uses intentional partial measurement to correct random, unintended partial measurement for any transformation in the Hilbert subspace. However when the source of these noises are physical rather than random and only a subset of transformations are required (quantum operations of useful quantum algorithms), how does one optimally source code the Hilbert space to preserve information?

This thesis explores the control, through time and space, of classical (ensemble) spin information and the coherent control of full quantum systems: control fields and qubits. To this end it will provide several new calculations, optimizations, and

techniques that together form a new paradigm for constructing a scalable, efficient quantum processing system realizable with today's physical resources. In particular, in Section 2 I describe information transport in classical spintronic systems, where electronic spin is used to encode information. In the case of electron spins in semiconductors and metals, ensemble diffusion and drift of spin polarization is used to transport information and process it. We present a direct calculation of ensemble channel capacity for an idealized semiconductor and note the implications for the future of spintronic design. Also presented is a spin-logic circuit that shows how information may be injected and extracted for a complex network of channels. In the remaining sections, several solutions to the question posed in the previous paragraph are brought to light.

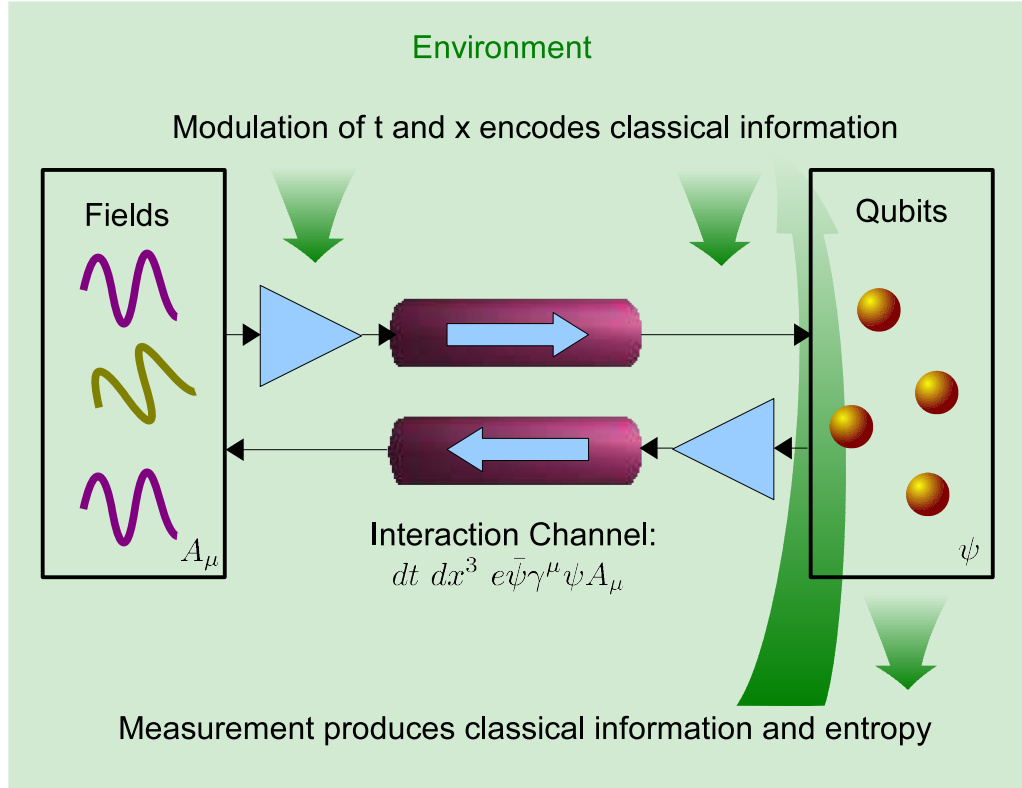


Figure 1.1: Interaction energy as information channel

Classical information from the environment (green) interacts with quantum information, by encoding (triangles) a set of scattering events between fields and qubits. In this way the scattering interaction is a channel that transfers *local quantum information* (entropy when global degrees of freedom are traced over) between fields and qubits. If time begins with all the qubits initialized to zero, then really there are just interactions between the fields mediated by entangled qubits. Because the scattering is unitary, the end result must be information neutral. Measurements of qubits (fields cannot be directly measured without extracting an infinity of information) convert quantum information to classical information by destroying coherently encoded (but not necessarily measurable) information. In this way measurable information is neither created nor destroyed but simply transferred between quantum and classical degrees of freedom that may be used to for further encoding (bottom green arrows).

## Chapter 2

# Controlling information in classical (ensemble) spins

To understand the role of modulation in spin systems, it is simplest to first consider classical systems. When a group of  $N$  spinors in the stochastic limit,  $N \rightarrow \infty$ , form an incoherent ensemble,  $\rho$ , subject to ensemble measurements of their polarization direction, they may encode any element of the  $SO(3)$  group at a given time. In the large  $N$  limit, multiple measurements can be made on the ensemble to decompose it into a Bloch vector,  $\mathbf{S}(\hat{n})$ , pointing in the direction  $\hat{n}$  with some amplitude. Then because this ideal ensemble provides a continuous support, the maximum differential entropy at any given moment in time is given by the unpolarized distribution, (2.1).

$$\begin{aligned} h(\rho) &= - \int_{\Omega} d\Omega \frac{1}{4\pi} \log \frac{1}{4\pi} \\ &= -\log 4\pi \end{aligned} \tag{2.1}$$

However, because arbitrary polarization intensity is allowed to encode arbitrary information, there must be a “power” limit,  $P_S$ , placed on the ensemble since it is at the stochastic limit. We constrain the ensemble density matrix,  $\rho$ , so that the power spiked in one direction,  $\hat{n}_0$  is constrained as  $P_S(\theta, \phi) = P_D \delta(\hat{n}_0 - \hat{n}(\theta, \phi))$ . In this, the time-independent case, the “power” is the number of polarized spins.

If instead, in the time-dependent case, the power,  $P_D$ , was the maximum, average rate at which spins could be polarized over time,  $t$ , the frequency by which the Bloch vector could evolve would be limited. In this case, as the Bloch vector evolves,  $d\mathbf{S} = (\mathbf{S}(t + dt) - \mathbf{S}(t))$ , the instantaneous power, given by  $P = d\mathbf{S}^\dagger \cdot d\mathbf{S}$ , must time-average to a bounded value,  $P_D$ . The time-dependence can be Fourier transformed,  $\tilde{\mathbf{S}}(\omega) = \mathcal{F}(\mathbf{S}(\mathbf{t}))$ , such that Parseval's Theorem (2.2) provides a band-limit to the encoding of the Bloch vector.

$$\int dt \mathbf{S}^\dagger(t) \cdot \mathbf{S}(t) = \int d\omega \tilde{\mathbf{S}}^\dagger(\omega) \cdot \tilde{\mathbf{S}}(\omega) \leq P_D \quad (2.2)$$

Currently used applications of spin-information are limited by the static power,  $P_S$ , since either the dynamic power of polarization,  $P_D$ , is small compared to other power considerations or because the time to polarize is already limited by Landau-Lifshitz-Gilbert (LLG) dynamics, as in a ferromagnet, and cannot be accelerated with extra power as per the simple model above. However, new paradigms that use spin-information to process information and spatially transport information are limited by  $P_D$ . We briefly describe the former, traditional case first. Then, a description of a spin-information processing is described followed by a detailed analysis of the information-theoretic limits associated with it.

## 2.1 Traditional information encoding and extraction: static spin-information

Below the Curie temperature, bulk ferromagnetic polarizations exist in a deep energy potential. Encoding information in such polarizations where field fluctuations minimally perturb the total energy level, create an almost stationary bulk measurable (1.2) so information can be robustly encoded into stationary degrees of freedom. However, as these bulk magnets are scaled down in size, it takes less energy to modify the polarizations, but by the same token they become susceptible to certain noise sources. The desired rate of the encoding plus the desired depth

of this will determine the power level necessary. Hard drives encode information in this way, but may read the polarization using giant magnetoresistance whose dynamics in this setting are also static (read and writes do not happen at the same time). Large ensembles provide high visibility in MRI, but after nuclear spins are polarized they are brought through free inductive decay, so no dynamical control is necessary. As these ensembles get smaller and smaller, NMR uses the coupling of nuclear spins through chemical bounds to enumerate a set of constraints to resolve molecular structure. Functional MRI (real time imaging, for invasive surgery for example) on the other hand, needs to re-polarize nuclear spins to get thirty frames per second of images. But, this rate is so exceedingly small that every spin will completely decay in time for hundreds of frames per second, again making information rate a non-issue. MRAMs use ferromagnetic precession and so are limited by LLG regardless of the power applied to fields.

## 2.2 Spin-based logic and information transport: dynamic spin-information

A semiconductor logic gate consisting of five ferromagnetic contacts is described in Figure 2.1.

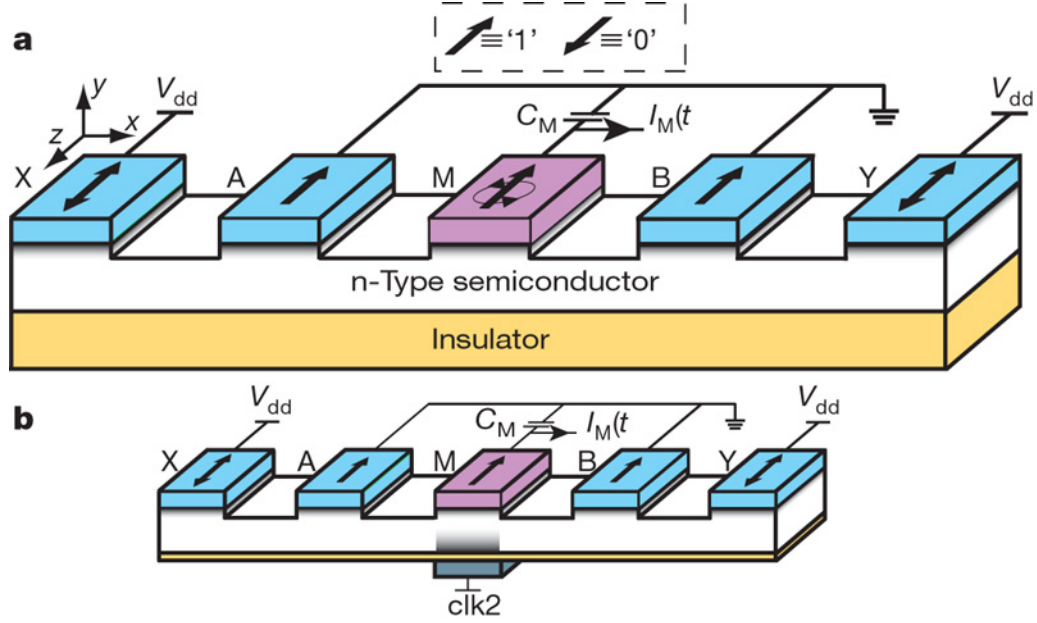


Figure 2.1: Magnetologic Gate Structure

The inputs are the magnetization directions of the contacts labeled by  $A$ ,  $B$ ,  $X$  and  $Y$ . The general Boolean expression of the gate is:  $OR(XOR(A, X), XOR(B, Y))$ . One may use the magnetization alignments of  $A$  and  $B$  to set different logic operations. For example, the figure shows a universal NAND operation between  $X$  and  $Y$  ( $A = B = '1'$ ). **a**, Steady-state currents, driven by  $V_{dd}$ , flow between  $A(B)$  and  $X(Y)$ . The output is given by a transient current response,  $I_M(t)$ , caused by an in-plane single rotation of  $M$ . The amplitude of  $I_M(t)$  is proportional to the spin accumulation in the semiconductor. In order to write the magnetization directions of the contacts in cascaded magnetologic gates, we first convert  $I_M(t)$  to a binary voltage signal (see Figure 2.4). This voltage signal will control the direction of a second, independent high-level current in wires on top of, and/or below, the contact(s) of other gate(s). **b**, A similar structure but with a pinned middle contact. The transient current,  $I_M(t)$ , is triggered by a voltage signal,  $clk2$ , applied to a (non-magnetic) back-contact beneath the mid-section. Semiconductor regions beneath the contacts are heavily doped.

By tracking the electrochemical potential when the ferromagnetic operands are rotated, one can infer one of several logical operations as described in Figure 2.2.

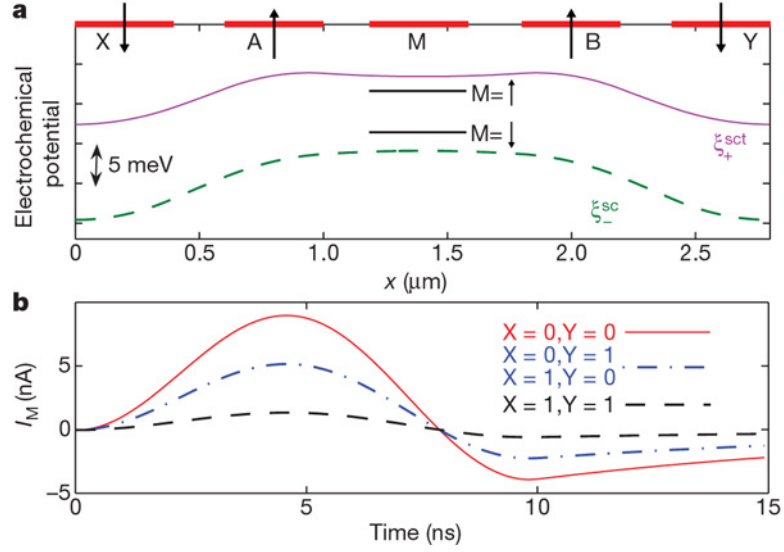


Figure 2.2: Modeled electrical behavior of a magnetologic gate set for NAND(X,Y). The diffusive transport in the planar semiconductor is modeled using a method in which the effects of traversing under the finite width of the metal contacts, and of the intrinsic capacitance across the Schottky barrier, are included in the regular spin diffusion formalism (see Supplementary Information). **a.** Typical steady state profiles of the spin dependent electrochemical potentials in the semiconductor channel for antiparallel configuration. Red sections denote the position of  $0.4\mu\text{m}$ -wide contacts, which are separated by  $0.2\mu\text{m}$  uncapped regions. In a steady state when  $I_M = 0$ , the potential level inside contact M for each magnetization alignment is represented by one of the two short black lines. **b.** Transient currents in the M terminal induced by a  $10\text{ns}$  rotation of M magnetization. The bias is  $V_{dd} = 0.5\text{V}$ , the external capacitance is  $C_M = 40\text{fF}$ , the conductive channel thickness is  $100\text{nm}$  and the contacts are  $2\mu\text{m}$  long. The semiconductor is  $10^{-16}\text{cm}^{-3}$   $n$ -type GaAs at room temperature. The mobility and spin relaxation time are, respectively,  $5,000\text{cm}^2\text{V}^{-1}\text{s}^{-1}$  and  $80\text{ps}$ . We use experimentally verified Fe/GaAs Schottky barrier parameters: the heavy doping of the interfaces results in approx  $10\text{nm}$ -thick barriers whose total conductance ( $G_+ + G_-$ ) is  $1,000\Omega^{-1}\text{cm}^{-2}$ , intrinsic capacitance per unit area is  $\sim 10^{-6}\text{Fcm}^{-2}$  and spin selectivity  $|G_+ - G_-|/(G_+ + G_-) = 1/3$ .

The decoding of a spin signal into a voltage signal is through the use of a thyristor, biased right before it's trigger point. The wire connecting the thyristor to the output is heavily shielded and the substrate must be heavily guarded, so that only Shot noise persists. This dynamical setting of the trigger point for a pipe-lined reading of spin-information is through the use of a three-phase dynamic circuit as described in Figure 2.3.

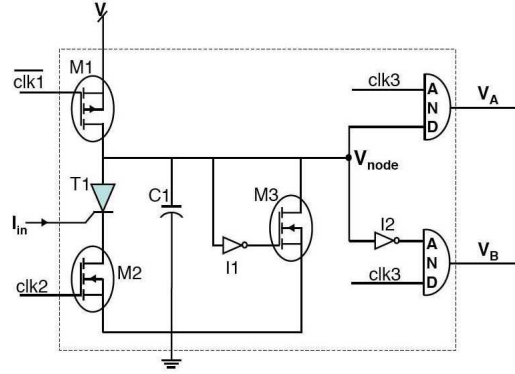


Figure 2.3: Dynamic thyristor latch to convert spin signal to voltage signal

Fig. SI(B) shows a schematic of the dynamic latch in which a transient current signal is converted into binary voltage signal. The input signals are the three clocks and the transient response of the magnetologic gate labeled by  $I_{in}$ . The outputs are the voltage signals  $V_A$  and  $V_B$ . The voltage level of  $V_A$  is high only if the voltage levels of  $clk3$  and of the node labeled by  $V_{node}$  are both high. On the other hand,  $V_B$  is high only if the voltage levels of  $clk3$  and  $V_{node}$  are, respectively, high and low. The schematic includes an intrinsic capacitor  $C1$  which is the effective parasitic capacitance between the ground and all of the devices connected to the node. This capacitor is charged when the  $clk1$  signal activates the transistor  $M1$  which acts as a voltage pull-up. This results in a high voltage in the node which is designed to be just below the forward breakdown level of the thyristor  $T1$ . When the  $clk2$  signal activates the transistor  $M2$ , the current from  $T1$  can flow easily into the ground. During 3 this time, a logic '1' output from the magnetologic gate provides just enough transient current into the inner diode of  $T1$  to push it into a self-sustaining avalanche mode which can no longer support two p-n diode drops. Consequently, the voltage across  $T1$  drops and the current across it increases. When the voltage in the node ( $V_{node}$ ) drops below the switching threshold for inverter  $I1$ , the voltage pull-down  $M3$  is activated and provides a feedback to  $I1$  which discharges  $C1$  fully. This results with zero  $V_{node}$ . On the other hand, if the logic output of the magnetologic gate is '0' then  $V_{node}$  remains high because  $T1$  is not switched from its high impedance mode and  $M3$  is not activated.

In this way, voltage signals may cascade the output using very low power. This is done by the routing of a low current bus over the calculated "next" magnetologic gates as shown in Figure 2.4.

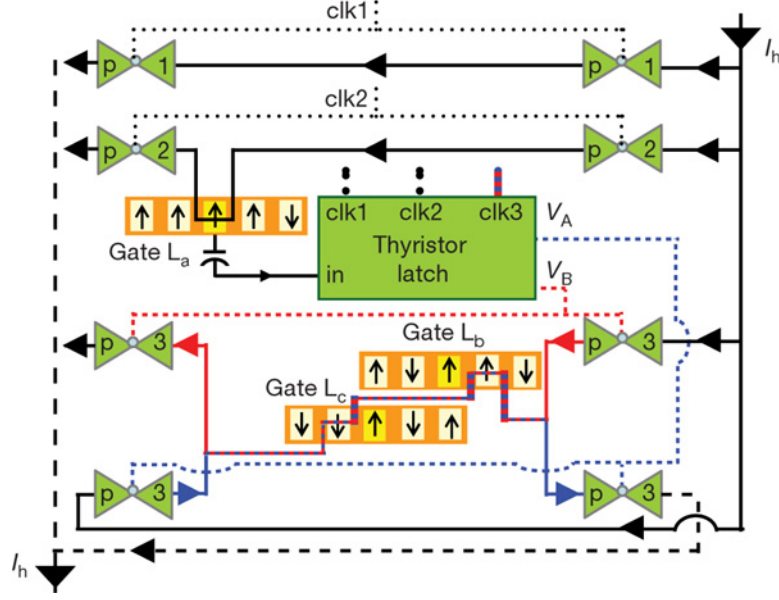


Figure 2.4: Cascading Magnetologic Gates

The output from a magnetologic gate,  $L_a$ , propagates into two similar gates,  $L_b$  and  $L_c$ . Lines with arrows denote current-carrying wires. Other lines hold binary voltage signals, which help to synchronize the flow of information and also steer the high level current  $I_h$  between the pass gates labeled by  $p1$ ,  $p2$  and  $p3$ . The synchronization process is managed in three phases by the thyristor latch (see text and Supplementary Information). In the first phase, the binary voltage signals are low except for  $clk1$  at two inputs (dotted lines). This signal synchronizes two operations: one to prepare the thyristor latch to receive a signal and one to switch the pass gates  $p1$  to their low resistance mode, thus directing the high current  $I_h$  to flow primarily in the upper horizontal black wire. During the second phase,  $clk2$  is high and  $I_h$  flows primarily in the wire over the top of the middle contact of  $L_a$ . The locally induced magnetic field rotates the magnetization of this contact and thus triggers the transient current labeled by 'in'. The thyristor latch is synchronized by  $clk2$  to capture this response. In the third phase,  $clk3$  is high and one of the voltage signals  $V_A$  or  $V_B$  at the output of the thyristor latch is also high. Whether  $V_A$  or  $V_B$  is high depends on the amplitude of the transient current. This directs  $I_h$  into one of two opposite paths (respectively, blue or red wires), resulting in magnetic encoding of '0' or '1' in the relevant operands.

A more detailed discussion of this work can be found in the publication [3], sections of which I am the principal researcher, but it mentioned largely to motivate the following quantification of spin-logic performance.

## 2.3 Computational bounds of spin-logic as an information rate

The use of electron spin to encode and process information forces consideration of non-traditional sources of noise, signal encoding and extraction. Recent experimental results suggest that the coherence time of spin in silicon is large and that both diffusion and drift may be used for information transport. Competition or cooperation for an electrochemical potential can process information using spin-physics instead of electrodynamics. However, no fundamental performance metric of this paradigm has emerged. The merits and disadvantages to using spin-based information in place of voltage-based information can be objectively quantified using information theory. The capacity of a semiconductor spin-channel is calculated under the assumptions that the momentum relaxation-time of electrons is much less than the spin relaxation-time and that using a single parabolic conduction band is sufficient. The results inform the type of materials that should be engineered for high-performance spintronics.

### 2.3.1 Introduction

Traditionally, a sender conveys information to a receiver using voltages, currents, or radiation. All of these sources are based on the transport of electromagnetic fields in a conductor or free space and the consequential, associated electron drift at the receiver. Electrical engineering's long history in control and modulation and the formalism of information theory has guided designs and materials that efficiently and maximally transport information in these mediums. Recently proposals have been put forward for spin-based logic [3] using information transport within a different type of information channel. In this paradigm, information is transported from logical operands, encoded by the polarization density of electron spins to and through an environment and space-time location where electron spin-interaction produces a logical computation. One or more logical results are encoded by spin densities or voltages, or equivalently their conjugates: spin-currents or currents. The longer the distance of information transport, the more operands

can be included in the computation and the more computationally complex an operation results.

However unlike the application of electromagnetic fields, the transport of spin is largely a diffusion-based process (in gallium-arsenide) whose level of signal leakage is orders of magnitude greater than the electronic equivalent. For this reason, researchers have sought to polarize the electron spin of materials that extend the lifetime of spin as much as possible. Recently, experiments have shown that spin-lifetimes in silicon [5] may be long enough to consider electron drift as a means of transport as well as diffusion. However, given that these modes of transport have large elements of signal drift, diffusion, and loss, the improvement, if any, in information rate remains unexamined. No information theoretic analysis informs the fundamental questions in spin transport: how quickly does one need to modulate a spin-injection source, what bulk material properties and which channel topologies are favorable to get the highest rate of information transfer, is the spin-polarizability of a material more important than the relaxation time, and in general, what are the mutually exclusive trade-offs in the design of a spin-logic gate.

Using simple assumptions, we calculate the information capacity of a spin-transport channel in relation to its fundamental material parameters of spin-relaxation time, momentum relaxation time, electric field, reduced electron mass, channel length, and channel dimension. We preserve a completely analytical result that may guide researchers in spintronics to design and develop materials that will increase the information density and computation power of spintronics. This calculation will also suggest how these paramount metrics might be affected using more accurate semiconductor properties, exotic materials, or different electronic or spin-interactions that don't follow the simple assumptions of our model.

To simplify this calculation, we reduce the constitutive differential equations of spin transport to be linear in three dimensions. We then calculate the Green's function for an impulse of spin-density injection and use the water-filling method in the information theory of time-continuous channels in the spectral domain. Spin-polarization must be bounded in the continuous spin-channel just as power

must be in a continuous-time electric channel. We reinterpret the Green's function to calculate the channel's information capacity by treating the spin-relaxation, erasure channel as an equivalent increase in noise. We extrapolate our results for two dimensions, and indicate where future work may be directed.

### 2.3.2 Model of the Channel

We create a simple model of a three-dimensional paramagnetic material in order to compute analytically its capacity as an information channel. Let  $\hat{f}(\mathbf{r}, \mathbf{k})$  be the *non-equilibrium* distribution of spin one-half density matrices in the phase space of the channel parametrized by position,  $\mathbf{r}$ , and momentum,  $\mathbf{k}$ . The density matrices may be decomposed into their polarized,  $\mathbf{s} = \text{Tr}(\hat{\sigma}\hat{f})$ , and their unpolarized,  $f = \text{Tr}\hat{f}$ , components. We assume elastic scattering processes in the spin-space and momentum-space [1], given by the connecting rates,  $W_{kk'}^{\text{sf}}$  and  $W_{kk'}$ , respectively as shown in (2.3) and (2.4). We have assumed that these scattering rates are homogeneous and isotropic in position,  $\mathbf{r}$ , but still depend on the magnitude of the momentum,  $k$ . We further assume that only ensemble spin measurements are made and that any coupled degrees of freedom relax quickly, so that otherwise coherent processes like spin-orbit interaction or the hyperfine interaction may be treated grossly as scattering with a spin relaxation-time,  $\tau_{\text{sf}}(k)$ . Likewise, the relaxation-time,  $\tau(p)$ , approximation of momentum scattering to a local equilibrium Fermi distribution due to density impurities is used. This is justified since we are using this channel only to calculate information capacity, and no further processing or coherent interaction will change that rate (data-processing inequality) when the channel is assumed to be linear. We assume the equilibrium spin-density is zero, so that there is no difference between the non-equilibrium spin-polarization and the spin-polarization.

$$\frac{d\mathbf{s}(\mathbf{r}, \mathbf{k})}{dt} = - \int d\Omega_{k'} dk' W_{kk'}^{\text{sf}} (\mathbf{s}(\mathbf{r}, \mathbf{k}) - \mathbf{s}(\mathbf{r}, \mathbf{k}')) \approx \frac{\mathbf{s}(\mathbf{r}, \mathbf{k})}{\tau_{\text{sf}}(k)} \quad (2.3)$$

$$\frac{df(\mathbf{r}, \mathbf{k})}{dt} = - \int d\Omega_{k'} dk' W_{kk'} (f(\mathbf{r}, \mathbf{k}) - f(\mathbf{r}, \mathbf{k}')) \approx \frac{f(\mathbf{r}, \mathbf{k})}{\tau(k)} \quad (2.4)$$

Implicit in this decoupled formulation is an assumption of scale. First we

assume that a point in phase-space is macroscopic enough that: (1) only an ensemble sampling of many electron spins take place and (2) we also assume that the equilibrium distribution of electrons is microscopic enough to discern all of the non-equilibrium electron populations.

$$\frac{d\hat{f}(\mathbf{r}, \mathbf{k})}{dt}\bigg|_{total} = \frac{(\text{Tr}\hat{f} - f_0) \cdot \mathbf{1}}{\tau(k)} + \frac{\mathbf{s}(\mathbf{r}, \mathbf{k}) \cdot \hat{\sigma}}{\tau_{sf}(k)} \quad (2.5)$$

We make further approximations producing a linear, analytically tractable PDE system: spin-transport takes place in one conduction band,  $\epsilon(\mathbf{k})$ , which is canonically parabolic with a reduced electron mass,  $m^*$ , elastic scattering (no photons) and no crystal anisotropy, and that thermal transport is decoupled from the transport equations of motion.

### 2.3.3 Spin-injection and spin-measurement

Currently, a non-equilibrium spin population may be created at a particular location in phase-space in one of several ways including: by tunneling electrons from energy-split spin bands through a barrier (from a ferromagnet to a normal metal or semiconductor) [8], by electrical currents using spin-orbit coupling [7], or by circularly-polarized optical excitations [6]. Each method modulates an electrical or optical source to inject some spin-current or create some spin-density, whose efficiency is related to the polarizability and current polarization of the channel. We abstract away these complexities and assume a general modulated source that can polarize electrons in a channel at a certain maximum rate at any point in time, irrespective of the state of the channel, at the injection location for any polarization encoding. This is valid when the channel is sufficiently less than fully polarized.

Likewise the measurement of spin can be accomplished in several ways, optical and electrical: the Faraday rotation of light, electrical resistance in bulk or nanostructures, or even the generation of Hall voltage. Implicit in each of these methods is a host of effects interacting with the spin signal: spin-polarization can accumulate at a tunnel junction, interfere with the dynamics of channel, or be further depolarizing. For simplicity we assume an ideal ensemble measurement of

the spin-polarization at some measurement location.

### 2.3.4 Spin-Transport Equations of Motion

Assume a Green's function response exists for a linear  $SO(3)$  transport system. This is equivalent to assuming the spin-channel is largely linear, that spin-densities don't interact in the channel, and that information transport is bosonic. These assumptions need not be true, as the spin density modulation could reveal memory effects in the channel that interacts two encoded polarizations, traditionally known as Inter-Symbol Interference (ISI). These effects need not be lossy. For example, a highly non-linear medium could give rise to spin solitons in semi-conductor (rather than in ferromagnetic material) that support an increased capacity. We assume the momentum relaxation time is orders of magnitude shorter than the spin-relaxation time. The charge of an electron is defined as  $-e$

$$\left( \partial_t + \mathbf{v} \cdot \partial_{\mathbf{r}} - \frac{e\mathbf{E}}{\hbar} \cdot \partial_{\mathbf{k}} \right) \hat{f}(\mathbf{r}, \mathbf{k}) = \frac{f \cdot \mathbf{1}}{\tau} + \frac{\mathbf{s}(\mathbf{r}, \mathbf{k}) \cdot \hat{\sigma}}{\tau_{\text{sf}}} \quad (2.6)$$

Assuming cylindrical symmetry in the channel around the direction of the electric field, we may simplify  $\hat{f}(\mathbf{r}, \mathbf{k}) = \hat{f}(x, \mathbf{k})$  and make the electric field,  $\mathbf{E}$  a scalar. Additionally we can semi-classically approximate the streaming co-factor with the shape of the conduction band,  $\mathbf{v} = \partial_{\mathbf{k}}\epsilon(\mathbf{k})/\hbar$ . Focusing on the spin-current in just the  $x$ -direction and just the polarization components of the density matrices  $\beta \in \{\sigma_x, \sigma_y, \sigma_z\}$ , we may now calculate the hydrodynamic, constitutive equations [12]. We define the spin density (2.7) and spin current (2.8) to construct the continuity equation (2.9) and Euler equation (2.10) by integrating out two modes of  $\mathbf{k}$ .

$$\hat{n}(x) \equiv \frac{1}{(2\pi)^3} \int d^3\mathbf{k} \hat{f}(x, \mathbf{k}) \quad (2.7)$$

$$\hat{j}_\alpha(x) \equiv \frac{1}{(2\pi)^3} \int d^3\mathbf{k} v_\alpha \hat{f}(x, \mathbf{k}) \quad (2.8)$$

The continuity equation simply states that any non-conservation of spin is only due to the average spin-relaxation parameter,  $\tau_{\text{sf}} = \langle \tau_{\text{sf}}(k) \rangle_k$  . and  $\tau = \langle \tau(k) \rangle_k$ .

The term  $\rho_\beta(x, t)$  relates the number of photons polarized to  $\beta$  at location  $x$  and time  $t$  and has been added to the continuity equation to represent spin-injection from the encoding source.

$$\partial_t n_\beta + \partial_x j_\beta = -\frac{n_\beta}{\tau_{\text{sf}}} + \rho_\beta(x, t) \quad (2.9)$$

$$\partial_t j_\beta + \bar{v}_x^2 \partial_x n_\beta - \frac{v_g}{\tau} n_\beta = -\frac{j_\beta}{\tau} \quad (2.10)$$

In calculating the Euler equation we have used the assumption that thermal transport is decoupled from the equations of spin motion: the second term on the left hand side of (2.10) substitutes the average,  $\bar{v}_x^2$ , for the term,  $v_x^2(k)$ , before integration. This assumption is justified when  $\tau \ll \tau_{\text{sf}}$  as shown in [2]. We have also used the fact that  $\tau e E \partial_k v_x / \hbar = v_g$ , the average conduction group velocity. Finally we have used the dominance of momentum-relaxation over spin-relaxation to simplify the relaxation of the spin-current:  $1/(\tau^{-1} + \tau_{\text{sf}}^{-1}) \approx \tau$ . Combining equations (2.9) and (2.10), produces an PDE that is second order in the time and space (2.11).

$$[\tau \tau_{\text{sf}} \partial_t^2 - \tau \partial_t + \tau \tau_{\text{sf}} \bar{v}_x^2 \partial_x^2 - v_g \tau_{\text{sf}} \partial_x + \tau_{\text{sf}} \partial_t - 1] n_\beta = \tau_{\text{sf}} \rho_\beta(x, t) \quad (2.11)$$

We reduce the temporal order by again using the fact that  $\tau \ll \tau_{\text{sf}}$  to eliminate the two leftmost terms on the left-hand side of (2.11). Then, we can solve for the Green's function response at  $x$  and  $t$ ,  $G(x, t)$ , when an impulse of spin-polarization density is injected at  $(x, t) = (0, 0)$  (2.12).

$$\left[ \tau \bar{v}_x^2 \partial_x^2 - v_g \partial_x - \partial_t - \frac{1}{\tau_{\text{sf}}} \right] G(x, t) = \delta(t) \delta(x) \quad (2.12)$$

This Green's function can be algebraically calculated by taking the Fourier transforming in the spacial domain, where  $\tilde{G}(k, t) \equiv \int dx e^{-2\pi i k x} G(x, t)$ , and the new use of the variable  $k$  is noted (2.13).

$$\tilde{G}(k, t) = e^{-\left(\frac{1}{\tau_{\text{sf}}} + k^2 \tau \bar{v}_x^2\right)t} e^{ikv_g t} \quad (2.13)$$

Hence, if the spacial modulation frequency of the spin-information is so high that equilibrating spin-currents dominate spin-flip decay, the information gain is marginal. This cross over occurs when (2.14) is satisfied.

$$k = \pm \sqrt{\frac{1}{\tau \tau_{\text{sf}} \bar{v}_x^2}} \quad (2.14)$$

According to the simple model above, virtually all information is diffusion dominated since the use of a spacial period greater than the spin-diffusion length is likely unhelpful. Even in materials such as silicon, where the spin lifetime, can exceed 500ns [5] trying to modulate a single polarization at a reasonable frequency will be difficult in this simple model, as spin-currents destroy the non-equilibrium spin populations at a rate proportional to the momentum relaxation time and the square of their spacial frequency. A more ideal material would have a low momentum relaxation-rate as well as spin-relaxation rate. Nevertheless, the exact information capacity will reveal how power should be distributed to frequencies before this threshold when the evolution is as modeled. The encoding of spin in all three dimension available is essential to maximizing information processing.

### 2.3.5 Capacity of a 3D Spin-Channel

Although the Green's is calculated in (2.13), the form necessary to compute the channel capacity is  $G^+(L, \omega)$ . In order to preserve causality, we use the field-theoretic technique of introducing the positive step function,  $\theta_t$ , when we extend the integral bounds to plus and minus infinity (2.15).

$$\tilde{G}^+(L, \omega) = \int_{-\infty}^{+\infty} d(t_2 - t_1) e^{i\omega(t_2 - t_1)} G(L, t_2 - t_1) \theta_{t_2 - t_1} \quad (2.15)$$

We assume the optimal time of measurement is the peak of the drifting pulse of polarization or,  $t_2 - t_1 = L/v_g$ . This then simplifies the integral to (2.16).

$$\tilde{G}^+(L, \omega) = \frac{-i}{2\pi} \int_{-\infty}^{\infty} dk \frac{e^{-ikL}}{\omega - v_g k + i \left( \frac{1}{\tau_{\text{sf}}} + k^2 \tau v_x^2 \right)} \quad (2.16)$$

Integrating this function via a contour integral as in Figure 2.5, we identify the poles  $k_{\pm}(\omega)$  as in , which in field theory would correspond to the energies and lifetimes of quasi-particles transporting information (2.17).

$$k_{\pm} = \frac{1}{2i\tau v_x^2} \left[ v_g \pm \sqrt{v_g^2 + 4\tau v_x^2 \left( \frac{1}{\tau_{\text{sf}}} - i\omega \right)} \right] \quad (2.17)$$

To perform this integration, the contour must be closed in the lower-half plane as  $|e^{-ikL}|$  converges when  $\text{Im}(k) < 0$ . If we identify the term under the radical as  $re^{i\theta} = x + iy$ , we can identify the real and imaginary part of the poles. We identify the diffusion constant  $D = \tau v_x^2$ .

$$\begin{aligned} \text{Im}(k_{\pm}) &= \frac{1}{2D} \left[ v_g \mp \sqrt{r} \cos \frac{\theta}{2} \right] \\ &= \frac{1}{2D} \left[ v_g \mp \left( \left[ v_g^2 + \frac{4D}{\tau_{\text{sf}}} \right]^2 + [4D\omega^2]^2 \right)^{-\frac{1}{4}} \cos \frac{1}{2}\theta \right] \end{aligned} \quad (2.18)$$

$$\text{Re}(k_{\pm}) = \frac{1}{2D} \left[ \sqrt{r} \sin \frac{\theta}{2} \right] \quad (2.19)$$

$$\begin{aligned} &= \frac{1}{2D} \left( \left[ v_g^2 + 4D \frac{1}{\tau_{\text{sf}}} \right]^2 + [4D\omega^2]^2 \right)^{-\frac{1}{4}} \sin \frac{1}{2}\theta \\ \theta &= \arg \left( v_g^2 + 4D \left( \frac{1}{\tau_{\text{sf}}} - i\omega \right) \right) \end{aligned} \quad (2.20)$$

Since all the variables in (2.18) are real and positive, the pole  $k_+$  is in the lower half plane. The integration can then be performed by separating the damping components,  $a$  (2.21), from the traveling components,  $b$  (2.23), of the Green's function.

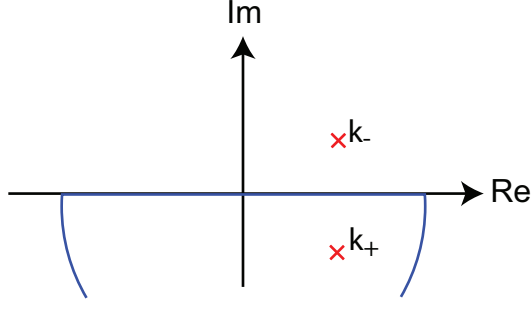


Figure 2.5: Contour integral to calculate Green's function

The poles  $k_{\pm}$  correspond to the lifetimes and energies of the information carriers in the model spin-semiconductor. By integrating the contour in the lower-half plane, the pole  $k_+$  determines how a frequency impulse will respond to the spin channel, and when increasing the frequency of modulation leads to diminishing information throughput. If the total power of encoding is constrained, these bottlenecks determine the capacity of the spin-channel.

$$a = \sqrt{\sqrt{\left(\frac{v_g^2}{2} + 2\bar{v}_x^2 \frac{\tau}{\tau_{sf}}\right)^2 + (2\tau\bar{v}_x^2\omega)^2} - \left(\frac{v_g^2}{2} + 2\bar{v}_x^2 \frac{\tau}{\tau_{sf}}\right)} \quad (2.21)$$

$$\approx \sqrt{\sqrt{\left(\frac{v_g^2}{2}\right)^2 + (2\tau\bar{v}_x^2\omega)^2} - \left(\frac{v_g^2}{2}\right)} \quad (2.22)$$

$$b = \sqrt{\sqrt{\left(\frac{v_g^2}{2} + 2\bar{v}_x^2 \frac{\tau}{\tau_{sf}}\right)^2 + (2\tau\bar{v}_x^2\omega)^2} + \left(\frac{v_g^2}{2} + 2\bar{v}_x^2 \frac{\tau}{\tau_{sf}}\right)} \quad (2.23)$$

$$\approx \sqrt{\sqrt{\left(\frac{v_g^2}{2}\right)^2 + (2\tau\bar{v}_x^2\omega)^2} + \left(\frac{v_g^2}{2}\right)} \quad (2.24)$$

In the regime that  $v_g^2 \gg 2\bar{v}_x^2\tau/\tau_{sf}$  these variables can be approximated as in (2.22) and (2.24). This form highlights the conjugacy between the information carrier's energy and its lifetime. To maximize information the group velocity should be kept as high as possible while keeping the diffusion length as low as possible:  $eE\partial_k v_x \gg \bar{v}_x^2$ . This translates to using a very high electric field and the lowest energy carriers in a regime of high positive dispersion. With these substitutions, the integration can be carried out to produce (2.25).

$$\tilde{G}(L, \omega) = -\frac{1}{2\tau v_x^2} \left[ \frac{e^{-(v_g+a)L/2\tau v_x^2} e^{-ib/2\tau v_x^2}}{\omega + a + ib} \right] \quad (2.25)$$

Because we ignored any interactions with coherent spin-torque, spin-spin interaction, or any anisotropy in general, the Green's function resides completely in the identity subspace of the density matrix. Then the time-continuous channel [Shannon] for each spin polarization component,  $\mathcal{X}_i \rightarrow \mathcal{Y}_i$ , may be written separately as the Green's function response to the input plus any stochastic noise in the channel (any systematic noise should be included in the Green's function) (2.26). Here the random variable  $\mathcal{X}_i$  encodes the information to be sent using Pauli-polarization,  $i$ , in a noisy channel whose noise power (variance) is given by the random variable  $\mathcal{N}_i(t_2)$  to produce the random variable  $\mathcal{Y}_i$ .

$$\mathcal{Y}_i(t_2) = \mathcal{N}_i(t_2) + \int_{-\infty}^{t_2} dt_2 G(L, t_2 - t_1) \cdot \mathcal{X}_i \quad (2.26)$$

To eliminate the integral, we take the Fourier transform of the channel, and consider each frequency of transport (2.27).

$$\tilde{\mathcal{Y}}_i(\omega) = \tilde{\mathcal{N}}_i + \tilde{G}^+(L, \omega) \cdot \tilde{\mathcal{X}}_i \quad (2.27)$$

We have assumed that all the noise sources (encoded, channel, and receiver) are white and Gaussian and so may be added together to produce a spectrally flat  $\tilde{\mathcal{N}}_i$ . Because this channel is continuous, a power bound must be placed on the encoding system in order for a maximum to exist. This bound is related to the susceptibility of the medium and the efficiency of the encoder in polarizing electrons. Additionally, there is power dissipation in the electric field used to drift information across the channel (2.28).

$$P_E = \frac{V^2}{R} = \frac{\sigma A (E \cdot L)^2}{L} = \frac{N (eE)^2 \tau}{m^*} \quad (2.28)$$

In this formula,  $V$  is voltage across of the length of the channel and  $R$  its resistance. The terms for cross sectional area,  $A$ , an average conductivity,  $\sigma$ , can be re-written using the total carrier number,  $N$ . In total then, we can bound the power used to

some amount,  $P$ , by converting the number of electrons polarized per second to Watts with the co-factor,  $\alpha$  (2.29), as the efficiency of the polarization process.

$$P = P_E + \alpha \int_0^\infty d\omega P_\omega \quad (2.29)$$

The parameters  $P_\omega$  bound the power used by each frequency of transport (2.30). By Parseval's theorem, an integration of the power spectrum is equivalent to a integration of the real power.

$$P_\omega \geq E \mathbf{X}_\omega \cdot \mathbf{X}_\omega^\dagger \quad (2.30)$$

We can combine the polarization channels for  $\tilde{\mathcal{X}}_i$  into the Bloch vector  $\mathbf{X}_\omega$ , and similarly for  $\mathbf{N}$  and  $\mathbf{G}_\omega = \tilde{G}^+(L, \omega) \cdot \mathbb{I}$ . Now we may calculate the mutual information of the channel as in (2.31), (2.32).

$$I(\mathbf{Y}; \mathbf{X}) = \int_0^\infty d\omega I_\omega(P_\omega) \quad (2.31)$$

$$I_\omega = H(\mathbf{G}_\omega \mathbf{X}_\omega + \mathbf{N}) - H(\mathbf{N}) \quad (2.32)$$

We define the dampening of the multi-dimensional spin channel as  $D_\omega^{-1} \equiv \mathbf{G}_\omega \mathbf{G}_\omega^\dagger | \mathbf{G}_\omega = \tilde{G}^+(L, \omega) \cdot \mathbb{I}$ . The dampening is the paramount parameter upon which the capacity turns (2.33).

$$D_\omega^{-1} = \left( \frac{1}{2\tau v_x^2} \right)^2 \frac{e^{-(v_g+a)L/\tau v_x^2}}{(\omega+a)^2 + b^2} \quad (2.33)$$

Then, the mutual information for such a linear, continuous channel [11] may be summed for the three polarization channels (2.34).

$$I = \frac{3}{2} \int_0^\infty d\omega \log \left( 1 + \frac{P_\omega D_\omega^{-1}}{N} \right) \quad (2.34)$$

To maximize the mutual information (solving the capacity), we treat the power dampening as isotropic and convert it, equivalently, to noise augmentation. This admits a known solution given noise imbalance across a continuum of channels. The optimal solution is given by “water-filling” power in frequency of encoding,

known as water filling in the spectral domain [4]. In the general case, where  $D_\omega$  is not necessarily given by (2.33), the capacity is given by (2.35).

$$C = \frac{3}{2} \int d\omega \log \left( 1 + \frac{(\nu - ND_\omega)^+}{ND_\omega} \right) \quad (2.35)$$

$$P = P_E + \int d\omega (\nu - ND_\omega)^+ \quad (2.36)$$

The “water-level” is given by  $\nu$  in the equation (2.36) derived from linear optimization. The notation  $(x)^+$  indicates the positive part of  $x$ . The capacity is different for each possible electric field value, also included power budget. However, in the case that  $D_\omega$  is monotonically increasing this expression may be simplified (2.38), since the power can be allotted to each frequency from spectral zero until some  $\omega_{max}$  is reached (2.37).

$$\frac{P - P_E}{N} = \int_0^{\omega_{max}} d\omega D_{\omega_{max}} - D_\omega \quad (2.37)$$

$$C = \frac{3}{2} [D_{\omega_{max}} - (v_g + a) L / \tau \bar{v}_x^2 - 2 \log (2\tau \bar{v}_x^2)] \omega_{max} \quad (2.38)$$

$$- \int_0^{\omega_{max}} d\omega \log [(\omega + a)^2 + b^2] \quad (2.39)$$

This informs the trade-off between power placed in charged transport, versus spin polarization. In general, given a multidimensional space of system variables, a landscape can be formed using  $D_\omega$ . Water filling then will determine the dominant channels to use for information processing. Additionally, the parameters of the semiconductor medium can now be traded off against a real information rate. For example, the spin relaxation-time strongly depends on temperature [9]. The power to cool the system can be traded off against the power to polarize and transport spin to find an optimum temperature. Finally, in order to assess the processing power of this spin information, the number of spin encoders and spin-decoders, that may be placed on this channel must be taken into account. In computer architecture the performance of a system is given by the operational complexity times the number

of operations per second. Then, a suitable metric for the processing performance,  $C_P$ , of a particular spin-based logic operation can be given by the Kolmogorov complexity of its truth table,  $K(L)$ , encoding the logical operation multiplied by the total information rate leaving the spin-channel (2.40).

$$C_P = \max_L K(L) \cdot C(L) \quad (2.40)$$

The dependence on  $L$  in (2.40) is explicit to suggest that the number of operands in the operation depend largely on  $L$  as does the capacity. At the maxima, increasing the length of the spin channel should be offset by a large increase in the number of spin-encoded logical operands to make the change in information-processing performance neutral.

### 2.3.6 Future work

Future work should explore information transport when  $\tau$  is of the same order as  $\tau_{sf}$ , as in certain two dimensional systems. In this limit, drift-based information can be modulated with a spacial period greater than the spin-diffusion length without dramatic information decay. The model used suggests that through a long spin relaxation time, as in Silicon, may extend the distance of communication, it does not necessarily increase the rate of communication. Future work should examine the effects of real semiconductor bands and anisotropic dampening. Some research predicts a heavy dependence on the quantization of sub-band structure [10]. Finally, a real consideration of information transport would include a combined capacity calculation of the information from a varying electric field plus the spin information encoded.

## 2.4 Acknowledgment

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in this publication directed, supervised, and co-worked on the research which forms the basis of this chapter.

The text of chapter II, in part, is a reprint of the material that is being prepared for publication as Dalal, P.; Ł. Cywiński; Sham, L. J. *The Channel Capacity of Spin-information in Semiconductors*. The co-authors in this publication directed, supervised, and co-worked on the research which forms the basis of this chapter.

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## Chapter 3

# The Dynamics of Quantum information

The evolution of a quantum system of fields and qubits is marked by deterministic, unitary evolution and non-deterministic, non-unitary jumps. Because the information we can extract from a quantum system critically depends on what we believe the evolution of the system was, to determine the best measurement basis to use, we must track our partial knowledge of the quantum system as well as its actual state. Two information measures then arise: the accessible information and the encodable information. The accessible information is the maximum entropy of measurable results by choosing from a set of possible measurement operators using our partial knowledge of the system. The encodable information is the maximum entropy of states in the system given the precision and completeness of operators than can act on the Hilbert space of the system. The measurable information is then necessarily less than the encodable entropy, and in general the measurable alphabet is a subset of the encodable alphabet.

This is the source of performance in quantum information processing. A quantum circuit is a transformation of the measurement basis that simply selects a subset of the encodable information that is relevant for the intended operation. Partial measurement along the way may destroy coherence information, but it can also focus the accessible information spotlight in ways that unitary evolution cannot. In this way, the factoring of a large number into its primes does not increase

the information of the original data (the large number) but simply identifies a portion of the information that the user (the controller) considers useful. It is the richness of this selection process available in the full Hilbert space that completely outpaces the classical information selection process for many large-scale problems. Furthermore partial measurements (such as in the Hidden Subgroup Problem) are not necessarily at odds with quantum circuits.

Evolving the state of a quantum system so that the information measured reflects the desired information of the original encoding involves both the dynamics of the quantum systems state and the dynamics of the encoded information. The mechanics of these evolutions, important quantum algorithms that select the information of value, and the role of partial measurement are explained in this chapter. Given this physical reality of non-deterministic encodings and evolutions, one must both encode the information into quantum states and chose an evolution of the system so that the user gains the most information of value. This task is termed “source-coding” to the quantum channel [?]. In Chapter 4, we present a new theory of error-correction for unintended unitary evolution and present new channels for quantum evolution that source-code very well against the physical noises that arise in quantum systems.

### 3.1 Open quantum system theory

Most research work on decoherence focuses on system-environment noise and has produced several well-accepted models. If information that is second order or less in the wave-function (e.g Hermitian observables) is of interest then the Lindblad master-equation [12] describes the evolution of density matrices subject to instantaneous, indirect measurements of a system through its average environmental coupling in the reduced matrix. If higher-order information degradation is to be modeled, then a stochastic Schrödinger equation may be used in conjunction with such an approach [2]. Both models rely on the short-bath-correlation-time approximation and characterize decoherence and relaxation due to system-environment interactions by two phenomenological times  $T_2$  and  $T_1$ . Though these approaches

are rather general and do not require modeling of bath dynamics, they are invalid on timescales small enough that the bath memory is again relevant. A striking example is the decoherent coupling of a spin bath [16], where a system decoheres completely within the non-Markovian regime. Local decoherence models alone also describe non-local decoherence, for instance, a sudden-death of entanglement [18]. Due to these dominating effects by local decoherence, little attention has been paid to non-local decoherence models that work specifically on higher-order coherences (e.g. entanglements).

The AFFT [6] approximation, as mentioned in Section (5.1.5), requires that we need to perform eight controlled rotations to accommodate hundreds of thousands of qubits. Therefore a viable quantum computing system should be able to perform on the order of 100 quantum operations before local-decoherence dephases the system more than  $\frac{\pi}{512}$ , or be error-correcting on single-qubit rotations to that precision, or an encoding to a decoherence-free subspace. We show that for many quantum systems, this system-environment Markovian noise is too small to decohere the system beyond the 0.9999 fidelity alone. For example, consider a system where qubits are represented as electrons in singly-doped, self-assembled quantum dots whose energy states are split by a magnetic field, whose control is afforded by optically-induced Raman transitions, and whose environmental interaction is dominated by the quantum dot's nuclear spin-bath[4]. In this system,  $0.1ns - 1.0ns$  laser pulses can perform  $\pi$ -rotations [4], while transverse spin decoherence works on the order of  $10\mu s$  at low temperature[16]. Moreover, coherence of such a qubit can be restored on much longer timescale by controlling its system-bath dynamics. In Figure 3.1, we graph the effect of various  $T_2$  and  $T_1 = 1000T_2$  on the fidelity of a single-qubit  $\pi$  rotation. It is clear that the fidelity loss due to system-environment noise in this example is negligible. Moreover, coherence of such a qubit can be restored on much longer timescale by controlling system-bath dynamics [17]. Without loss of generality, we will refer to this same quantum system to show a low-decoherence system can also impart high-precision operations.

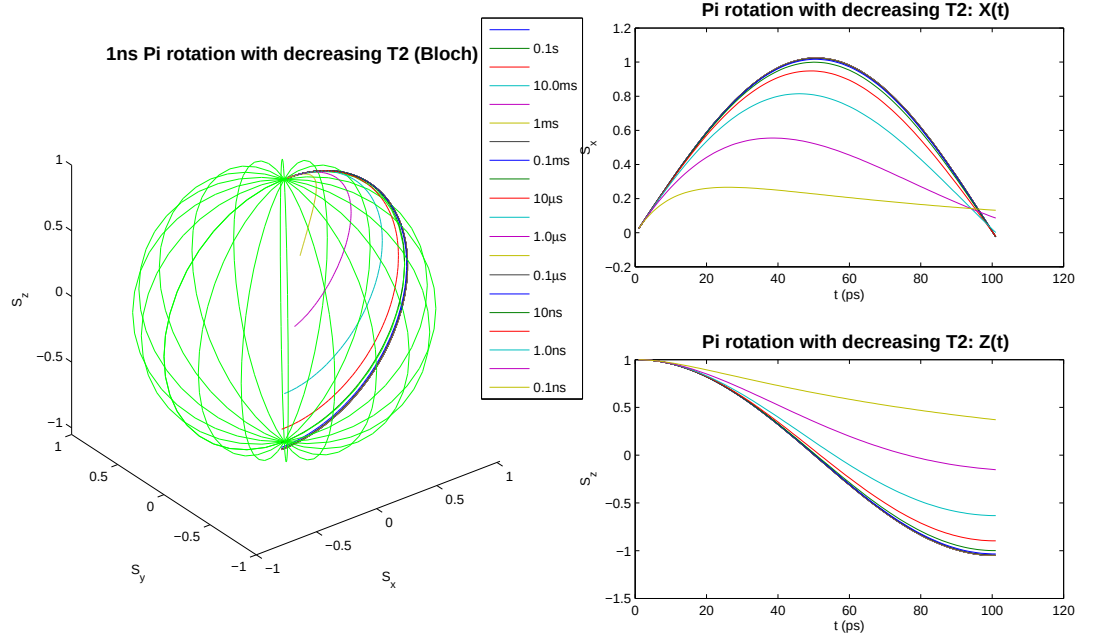


Figure 3.1: Decoherence due to  $T_2$

Given a  $1ns$   $\pi$ -rotation, in the above simulation  $T_1$  is set to  $1000T_2$  and  $T_2$  is swept between  $0.1ns$  to  $1s$  in half-decades. The decoherence due to  $T_2 > 25ns$  is negligible for a single  $1ns$  operation

## 3.2 Quantum information theory

Quantum information theory is fundamentally different than classical information theory in two major ways. First, the accessible information of a quantum system is less than or equal to the information it takes to encode a quantum state. If it were not so, it would take no more than one qubit of information to specify an arbitrary qubit state, which must consist of three real parameters. In other words, the non-unitary act of measuring a quantum system is generally not information preserving, and in fact can be very destructive locally. For example if information was only encoded in the phase of a qubit, measurements in the  $S_z$  basis would produce no information destroying the local information and irrecoverably transferring it to its global environment. Yet if information was only encoded in eigenstates of  $S_z$ , such measurements would not destroy any local information, as

the state could be re-encoded. The Holevo bound [9] proves the most information one can extract from one encoded qubit is one bit of information. However, the measurement basis may choose what subset of the information used to encode that state should comprise the extracted bit. Consider for an example, a qubit that is encoded using positions  $|S_x; \uparrow\rangle$ ,  $|S_x; \downarrow\rangle$ ,  $|S_z; \uparrow\rangle$ , and  $|S_z; \downarrow\rangle$  on the Bloch sphere that can be measured in the computational basis of  $S_z$  or  $S_x$  as in Figure 3.2.

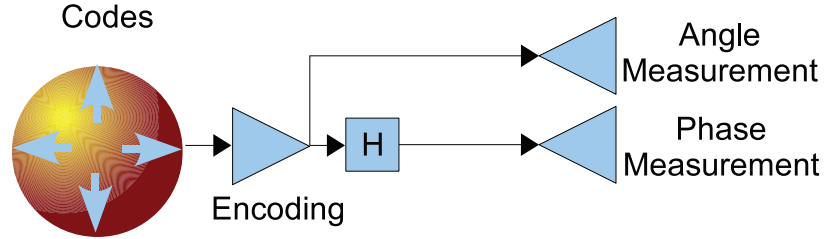


Figure 3.2: Quantum circuits choose desired information by selecting a measurement basis

If the only quantum operation available is  $S_y(\pi/4)$ , there is a dictionary of only four possible codes on a single qubit as indicated by arrows on the Bloch sphere at the left. While the encoding has two bits of entropy the accessible information has far less, if the sender acts independently of the receiver. However, the receiver by choosing the quantum circuit to implement before measurement in the  $S_z$  basis, either the Hadamard transform  $H$  or the nothing  $I$ , can select what subset of information to receive. If the receiver implements a  $H$  (bottom flow), he or she will receive phase information. If the receiver implements  $I$  (top flow) he or she will receive angle information. If the sender and receiver agree on using only the  $S_z$  or only the  $S_x$  basis to send and receive, the measurable entropy will be higher while the encodable entropy will be lower. If the goal is information transmission such an arrangement makes sense, however if the goal is information processing, “how much information” is lost is less important than “which information is lost.”

### 3.2.1 Shannon information

Shannon information is a measure of the accessible information in a quantum or classical system. If the random variable  $X$  represents a data-source with an alphabet  $\mathcal{X}$  and a probability function  $p(x) = \Pr\{X = x\}$ ,  $x \in \mathcal{X}$ , the entropy of that data source is given by the Boltzmann  $H$  function (3.1) in the discrete case and (3.2) in continuous case

$$H(X) \equiv - \sum_{x \in \mathcal{X}} p(x) \log p(x) \quad (3.1)$$

$$h(X) \equiv - \int_{\mathcal{X}} dx p(x) \log p(x) \quad (3.2)$$

The quantum analog of Shannon Entropy is the von Neumann Entropy. If the alphabet represents an orthonormal basis of the Hilbert space, then the alphabet and probability function may be combined into the positive, semi-definite density matrix  $\rho = \sum p(x) |\mathcal{X}\rangle \langle \mathcal{X}|$  to produce the discrete entropy, which supports the continuous case as well where the Hilbert space is a field.

$$S(\rho) \equiv - \sum_{x \in \mathcal{X}} \text{Tr}(|\mathcal{X}\rangle \langle \mathcal{X}| \rho) \log \text{Tr}(|\mathcal{X}\rangle \langle \mathcal{X}| \rho) \quad (3.3)$$

$$= -\text{Tr}(\rho \log \rho) \quad (3.4)$$

Because the von Neumann Entropy only evaluates over independently measurable values, the density-matrix formalism (covariance matrix) is sufficient. All observables are second order in the wavefunction. However, when one considers not “how much” information is measurable but “which information” is measurable of the encoded information, as in the case of quantum algorithms, the Shannon Entropy is inapplicable. That is a job for Fisher information.

### 3.2.2 Fisher information

Fisher information is a measure of how well a set of sample values  $X_i$  can estimate an unknown parameter  $\theta$ . It is an information measure since it provides the reduction in uncertainty of the possible  $\theta$  in bits of information. Just as Shannon information decouples the interpretation of a data source from the actual information content of the source, Fisher information decouples predictive models,  $T$ , that attempt the mapping  $T : X_i \rightarrow \theta$ , from the predictive content of data (3.5). In recent years, Fisher information has been seen as fundamental to all of

physics and indeed all of science. This is not remarkable since science is a simply a predictive model of the parameters of the universe.

$$J(\theta) = E \left[ \frac{\partial}{\partial \theta} \ln p(x|\theta) \right]^2 \quad (3.5)$$

In 1967, Helstrom proposed that quantum Fisher information should be written as in (3.6).

$$\begin{aligned} J_H(\rho_\theta) &= -\text{Tr}(\rho_\theta L_\theta^2) \\ \frac{\partial}{\partial \theta} \rho_\theta &= \frac{1}{2} (L_\theta \rho_\theta + \rho_\theta L_\theta) \end{aligned} \quad (3.6)$$

Fisher information considers a family of density matrices, instead of a single one in order to calculate non-Hermitian parameters. In this sense, Fisher information can operate upon the Stochastic Schrodinger Equation, which maintains the distribution of many  $\rho_\theta$ , and not the Lindblad Equation, which would combine them. The relationship between the classical Fisher information and the quantum Fisher information concerns how the phase of the wavefunction,  $\phi$ , changes with  $\theta$ . The difference between the two for quantum systems is stark. When information is encoded in the value of  $\theta$ , though the measurable information may still be calculated with Von Neumann entropy, the amount of information retained about the angle  $\theta$  will be different for different ensembles that compose the same density matrix,  $\rho$ . If a system is noisy, the encodable entropy (relating to  $\theta$ ) may be affected in ways that don't affect the density matrix.

### 3.2.3 Traditional quantum error correction and fault tolerant operation

A fault-tolerant quantum gate works by encoding a quantum state into a redundant set of qubits, where the encoding allows a measurement of any errors that take place during the gate's operation. The encoding is performed before the desired operation and subsequent partial measurements of the redundant bits either

herald a successful operation or identify (syndrome detection) additional, corrective fault-tolerant operations to perform. Traditionally, encodings allow parity-like checks between parallel applications of the same operation or block of operations, transversal operations, to eliminate random errors [7]. Fault-tolerance is then claimed by the statistical independence of errors: if the probability of one error in a computational block is  $\epsilon$ , then the probability of two errors is  $\epsilon^2$ , and so on with a hierarchy of measurements for fault-tolerance provided at every tier of computation [11].

For example, given a unitary operator,  $U$ , with two eigenvalues  $U_{\pm}$ , one can decompose a redundant Hilbert space into quadrants. The the unitary operator can be applied to any one of three redundant bits in the 3-qubit Shor code, so the eigenkets of  $U$  create 8 different quadrants. In only two of the quadrants do all of the eigenvalues agree, and the remaining quadrants can correct any one error projected by measurement. Such a measurement of the parity check bits is made on the the top qubits in 3.4(a).

However, this argument is problematic: the probability that any quantum operation produces an error is one. No analog operation can be regarded as perfectly precise. Due to the continuum nature of control fields and the general uncertainty principle, at some scale the operation is not exact. To overcome this hurdle, researchers often use the approximation that the chance of measuring an error in the output can provide an upper probability bound for the fact that the process is errant. This approximation affords practical solutions for certain operations (e.g. the Clifford group), but completely breaks down when applied to small quit rotations, the staple operation for almost all useful applications of quantum computing.

Consider for example a precise rotation operation applied to  $m$  redundant qubits in parallel as shown in Figure 3.4(a). Because the control field is classical modulated, it must contain an element of  $1/f$  error: as the pulse duration,  $\Delta t$ , goes down, the power of the this noise increases. If the intensity of the field is lowered to lengthen the control pulse, shot noise easily outgrows a  $\pi/128$  angular error threshold. Therefore there exists a lower bound of noise power in the coherent dimensions of the field modulator that cannot be eliminated as illustrated in Figure

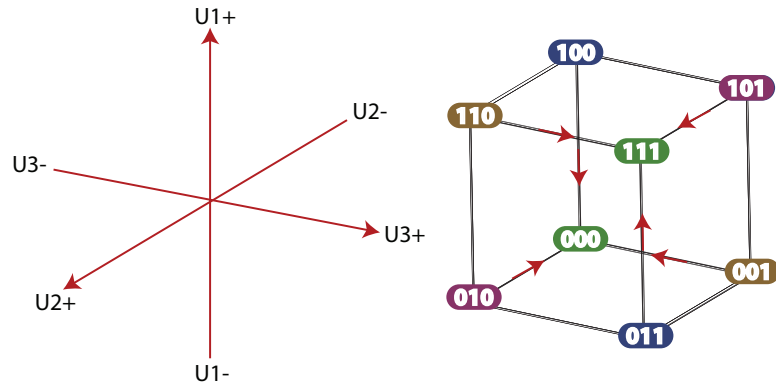


Figure 3.3: Errant subspaces of simple 3-qubit Shor code

For any error that is spanned by the eigenkets  $U_{\pm}$ , partial measurement of which quadrant it is in collapses each of three redundant qubits to subspace of the other. Labeling of the subspaces, is equivalent to labeling the vertices of a cube that identify quadrant subspaces of the total Hilbert space. By making two partial measurements, a qubit worth of information remains identifying one of the colors of the vertices. The only correct states are when all qubits have the same value (green). The arrow identify corrective operation without further knowledge of which of the two identically colored vertices the wavefunction occupies. Each errant vertex can distinguish between 000 and 111 for any errors that are error order  $\epsilon$ . Errors of order  $\epsilon^2$ , cannot distinguish the correct vertex.

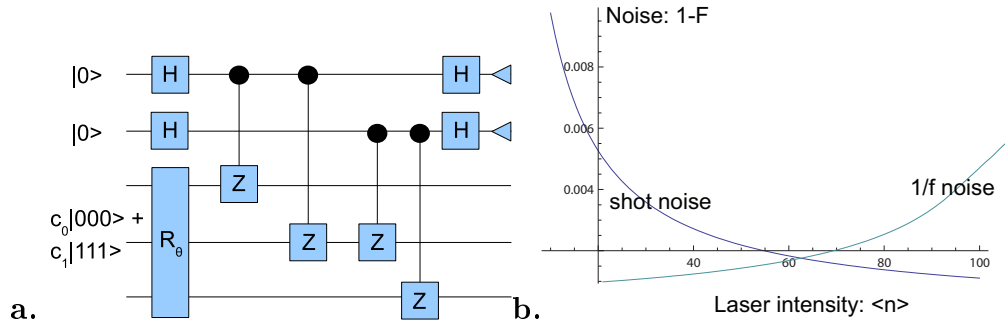


Figure 3.4: Traditional fault-tolerant approach to correct unitary under/over-rotation error in a small rotation operation

**a.** Encoding a qubit with the simple three-qubit Shor code before a precision rotation is problematic for fault-tolerance. If the rotation operation,  $R_\theta$ , is a multi-qubit operation then it introduces correlated error. If it is decomposed into several operations, then each operation must have a lower-bound noise since the control field is classically modulated as shown in **b**. If a control pulse is too long (weak intensity) it is easily dominated by shot noise, but if it is too short (high intensity) it is dominated by  $1/f$  noise in its modulation. Although, this noise is unitary in the matter-field object, it appears stochastic to a measured qubit. Given the small sampling of the mean (3 qubits of encoding) statistical independence fails for a very small pulse area. The physical system's modulation determines the actual intensity of the  $1/f$  noise, and so determines at what pulse area does common noise appears on all three qubits. Measurement, after such a process, of the top ancilla qubits cannot collapse the wavefunction to an isolated errant subspace.

3.4(b). When the control field is measured, and the modulator degrees of freedom are traced over, it will appear as stochastic noise. So, the standard deviation of the mean error for a short pulse converges at a slower rate than for a long pulse when increasing the number of redundant qubits,  $m$ , that sample the error distribution. Only in the stochastic limit,  $m \rightarrow \infty$ , can a strict Gaussian sampling be considered, and is approximated by large ensemble measurement as in liquid NMR computing [15]. But for quantum systems with scalable interactions, the number of redundant qubits is impractical when one considers the square-root convergence of the standard deviation of the mean error,  $\sigma_{\bar{\epsilon}} \propto \sigma_{\epsilon}/\sqrt{\Delta t m}$ . In Figure 3.4(a), the sampling of  $m = 3$  of this error distribution will appear to have a static error term that cannot be compensated with pulse-shaping or quantum control. This leads to a violation of the basic statistical assumptions of the quantum fault tolerance. The probability that two or more operations are errant in their measurement is not  $O(\sigma_{\epsilon}^2)$  but rather  $O(\sigma_{\epsilon}^2 + \sigma_{\bar{\epsilon}})$ . As the direct measurability of these small phase errors is similarly small, one might be tempted to think these are unimportant [?]. However, the specific quantum algorithm in question determines how measurable these errors become through a series of non-commuting operations, eventually with quantum chaos manifesting itself to prevent scalability. Instead the precise error analyses for each algorithm alluded to earlier calculate the actual sensitivity of the computations to phase errors [6, 5, 19, 3].

A parity check between two errant operations, can only collapse the wave-vector to one of two errant operations. Without a *reference structure* for the correct operation, this brand of projective correction will only improve the operational accuracy if enough parallel qubits and operations are added so that the standard deviation of the mean starts to converge. Lloyd and Slotine suggest that this operational imprecision discretizes the accessible space of quantum error correction in continuous systems [13]. The only solution was mentioned in Shor's concluding remarks in his seminal paper on quantum error correction [14]. In order to make fault-tolerant, small-rotation operations, the ancilla state  $\cos(\theta)|s_0\rangle + \sin(\theta)|s_1\rangle$  should be fault-tolerantly prepared for arbitrary  $\theta$ . Derivatives of this condition assume  $|\frac{\pi}{8}\rangle$  may be fault-tolerantly prepared and the appropriate rotation reference

structure constructed though the non-commutativity of a series of fault-tolerant operations or encodings. Shor portends the cumbersome nature of small rotations and the need to quantitatively understand their resource requirement. Fowler performs this analysis and concludes that this qubit and operation heavy construction makes the scaling of phase fault-tolerance impractical at the necessary precisions, “Given that (the simplest fault-tolerant phase gate) is both extremely complex and the simplest fault tolerant circuit sufficiently close to  $R_{128}$ , for practical computation non-fault tolerant circuits...are likely to remain the best way to implement arbitrary rotations for the foreseeable future” [5]. Other analyses comes to similar conclusions, such as in [10].

As the rotation angle is increased two orders of magnitude, the previous  $1/f$  no longer dominates the noise power, an increased control field intensity becomes optimal, and the statistical underpinnings of quantum fault tolerance are again valid. Sections 4.2.3 and 4.2.5 show that only large-angled Clifford operations are necessary with locally-variant bases. In Section (4.1), we report an alternate form of fault-tolerant operation for Clifford operations that requires no parity checks and can correct small, stochastic unitary phase errors to arbitrary order.

### 3.3 Flow of information in Quantum Algorithms

#### 3.3.1 Quantum Fourier Transform (QFT) and the Approximate Quantum Fourier transform (AQFT)

Given a vector in the Hilbert space,  $|X\rangle = \sum_{a=0}^{N-1} x_a |a\rangle$ , it may be related to a vector in its Fourier transformed Hilbert space,  $|Y\rangle = \sum_{c=0}^{N-1} y_c |c\rangle$ , by relating the coefficients of the two bases,  $\{|a\rangle\}$  and  $\{|c\rangle\}$ , using the unitary (since it is invertible) discrete Fourier transform, (3.7).

$$y_c = \frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} x_a \omega_N^{ac} \quad (3.7)$$

The  $N$ -th root of unity is defined as  $\omega_N \equiv \exp(2\pi i/N)$ . Since this transform is linear, it is popularly written as a transform of each basis vector,  $|a\rangle$ , in a bit-factored form (3.8), to suggest a particular quantum circuit implementation. Because this notation identifies the fundamental quantum circuit topology needed, we introduce the function  $f(a, l) = a_{n-l} \dots a_{n-1} a_n$ , for future reference.

$$QFT : |a\rangle \rightarrow (|0\rangle_1 + e^{2\pi i 0.a_n} |1\rangle_1) (|0\rangle_2 + e^{2\pi i 0.a_{n-1}a_n} |1\rangle_2) \dots \quad (3.8)$$

$$\dots (|0\rangle_n + e^{2\pi i 0.a_1 a_2 \dots a_n} |1\rangle_n) \quad (3.9)$$

In this notation, the index  $a$ , is written with the binary bits  $\{a_1, a_2, \dots, a_n\}$  where  $a = \sum_{i=1}^n a_i 2^{n-i}$  and the index  $c$  is written similarly. This notation hides the amplitudes of the transformation that can now be combined to write the Fourier transform more formally as in (3.10).

$$|Y\rangle \equiv \mathcal{F}(|X\rangle) = \sum_{a=0}^{N-1} x_a [QFT(|a\rangle)] \quad (3.10)$$

For all practical purposes (less than ten-thousand qubits), this exact transform is not necessary and one can simply eliminate all phase factors in the above formula when they are less than  $\pi/512$ . This phase-truncated transform (also called the AFFT) makes such a quantum algorithm practically possible as all. Research by Coppersmith shows that maximum error in the measured phase,  $\epsilon_\phi$ , when the phase operations are truncated at  $\pi/2^m$  is

$$|\epsilon_\phi| < \frac{2\pi N}{2^m}$$

However, when this transform is used to perform Shor factoring, even less precision is required as the measured information obeys:

$$|\epsilon_\phi| < \frac{2\pi N}{4^m}$$

Such a bound means that  $m = 7$  can effectively resolve the required algorithmic information with a small polynomial order of runs for a thousand qubit system.

### 3.3.2 Amplitude Amplification (AA)

Consider that of a full Hilbert space, there is a “good” subspace and a “bad” subspace. By this I mean that one would like to unitarily transform all amplitudes in the bad Hilbert space evenly into amplitudes in the good Hilbert space, to do this one may use the Amplitude Amplification algorithm [1]. This is done by using some arbitrary transformation of the initial wave function,  $|0\rangle$ , which we term  $A$ , a reflection about the bad subspace which we term,  $S_B$ , and a reflection about the original transformed wave function,  $A|0\rangle$ , which we term  $S_0$ . Here one assumes some sort of oracle that can recognize a bad subspace but cannot compute it, much like the verification of a PDE solution is a much simpler task than deriving a solution. Then by repeating the unitary transform,  $U(A, B)$ , as in (3.11) for a set number of times (depending on the relative size of the good and bad subspaces and the transform  $A$ ), one transforms the bad space to exactly or very close to the good subspace.

$$U_{AA}(A, B) = -AS_0A^\dagger S_B \quad (3.11)$$

If  $A$  is the Hadamard transform, then this is equivalent to the Grover Search [8]. For example if the two eigenvectors form the good subspace from an eight dimensional Hilbert space, it only takes two reflections, or one iteration of the algorithm, to transform the bad subspace (haystack) to the good subspace (needle) as in Figure 3.5.

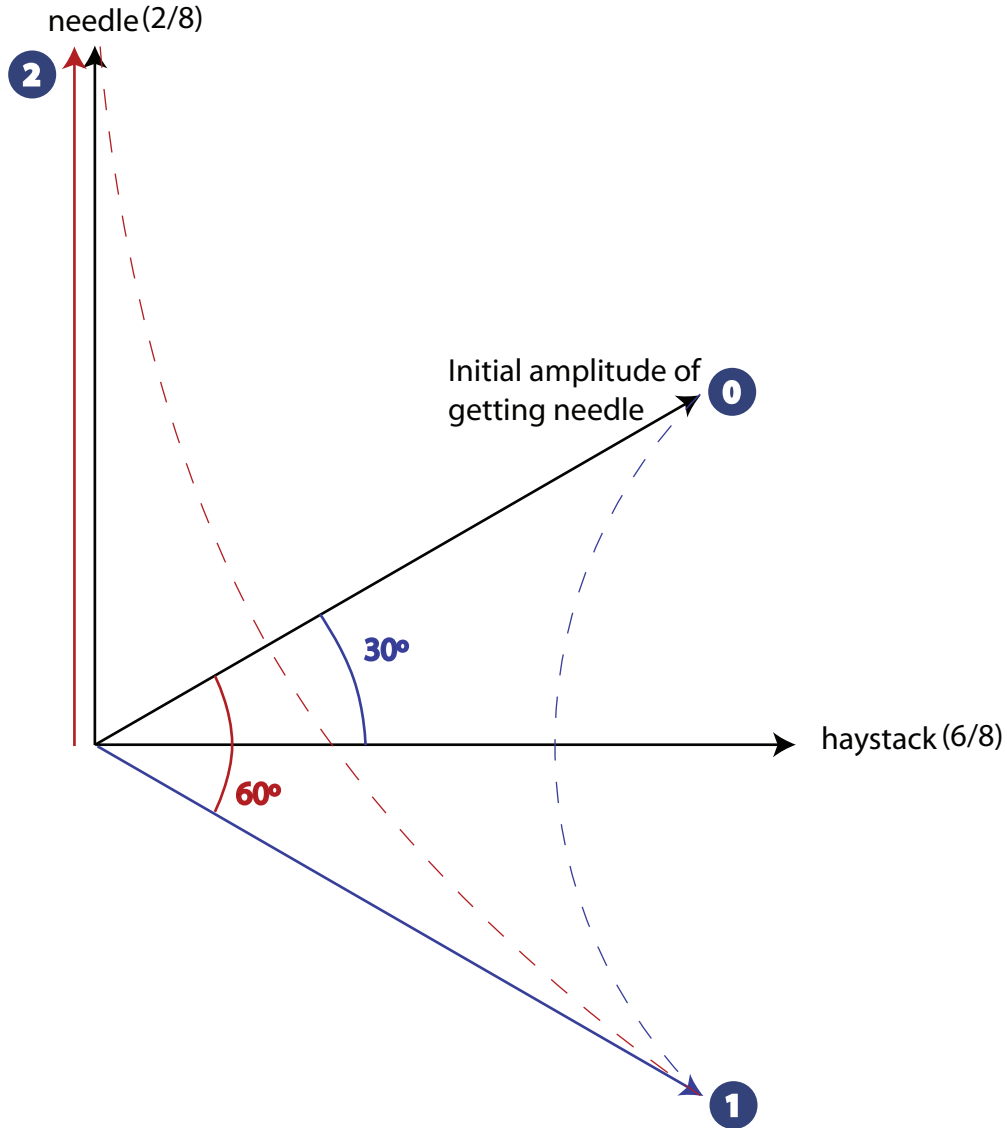


Figure 3.5: Geometric interpretation of example search of Grover's algorithm  
 If the two eigenvectors form the good subspace from an eight dimensional Hilbert space, it only takes two reflection, one interaction of the algorithm, to transform the bad subspace (haystack) to the good subspace (needle). The initial state is represented at (0), the first reflection at (1) and the second reflection at (2).

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## Chapter 4

# Controlling information dynamics in quantum spins: Unitary-fault tolerance and locally-variant bases

Given uncertainty is quantum scattering, two prescriptions to avoid classical, temporal control are detailed below. They predict a very high precision operation in systems not dominated by environmental coupling and with very coherent control fields, such as laser fields.

### 4.1 Unitary Fault-Tolerance using Quantum Coherent Feedback

We derive a new theory of noise in quantum control fields and present a new method of fault tolerance. Traditional fault-tolerant quantum operation is a theoretical formalism of partial measurement that collapses a wave-function into encoded Hilbert subspaces which identify correct or correctable operations, without further destructive measurement. However, traditional fault-tolerant operations for small, high-precision rotations are widely regarded as practically infeasible, while at the same time experimental fidelity losses are much greater for such operations than predicted by phenomenological theories of noise. We derive a new

theory of field error that purifies the control field using one level of indirection: a coherent modulation field shaping of the control field pulse. Using this theory and two coherent noise parameters, we calculate an optimal field intensity which predicts much lower fidelity than the standard treatment. Most significantly using this theory, we develop a new method of fault-tolerance using quantum-coherent feedback rather than the measurement of qubits, applicable to the unitary fluctuations controlling field precision. This unitary fault tolerance vastly simplifies the resource requirements otherwise needed by general fault-tolerant operation, and it conceptually advances useful entanglement techniques to control fields rather than just qubits. The theory can be readily tested on existing quantum systems.

#### 4.1.1 Introduction

Traditional quantum fault tolerance adds complexity in space and in time, entanglement and syndrome detection, to correct generic errors. This is done by noting every generator of error and creating a larger Hilbert space encoding that includes several errant subspaces [25, 30]. Conceptually such methods create a measurement structure that projects a continuum of operational errors to a correctable operation or the correct operation. The complexity of doing so is widely regraded as impractical and resource-taxing for small rotational errors [21]. Creating a measurement structure that distinguishes error states from correct states is difficult when only a small phase separates them [20]. It is often for this reason that traditional fault tolerance must correct all errors: the temporal and entanglement cost of implementing traditional fault-tolerance induces random errors due to qubit-decoherence times. In recent work [14], the authors have shown efficient quantum computing may be done using only “digital” quantum operations (operations from the Clifford group) that are more practical to make fault-tolerant [41]. However, when qubit-decoherence timescales are orders of magnitude greater than operational times and if operations have a fidelity greater than 0.9999, approximation algorithms allow quantum computing for thousands of qubits to be done without any fault-tolerance [21, 13]. Therefore using today’s resources, Clifford operations, and approximation algorithms, a new, simpler paradigm for produc-

ing scalable quantum computing exists if an operationally simple means of fault-tolerance for the physical errors (rather than random errors) introduced by Clifford group operations is found.

We present a new method of fault-tolerance using quantum-coherent feedback rather than the measurement of qubits, applicable to the unitary fluctuations controlling field precision. To do so: we show that control field error dominates the loss in fidelity for direct quantum operations in Section 4.1.2; we derive a new theory of field error that purifies the control field using a coherent modulation field shaping the control field pulse in Section 4.1.3; we calculate an optimal field intensity using two coherent parameters that predicts much lower fidelity than the standard treatment in Section 4.1.4; and finally we present a method for unitary fault tolerance capitalizing on the coherence of physical errors in the remaining sections.

### 4.1.2 Error Modeling in Quantum Fields

Consider a quantum system of qubits and quantum fields contained within a macroscopic environment (in which is contained a measuring apparatus) where the interaction between the fields and qubits is controlled by environmental control fields. Such a system is depicted in Figure 4.1, where environmental degrees of freedom entangle with fields and qubits, particular fields entangle with particular qubits according to a classically controlled modulator, followed by controlled coupling to macroscopic parameters that are measured. Classical information is input to the quantum system only through the modulator and classical information is output only through measurement. The literature has divided the quantum errors of this system into three categories: (I) unitary, but unintended evolution of a quantum system; (II) non-unitary evolution due to known partial measurements; and (III) decoherence by partial, unknown measurements of entangled environmental degrees of freedom. For a single qubit, the difference between these concerns our knowledge rather than any physical state. A quantum system may collapse or not in time, but it still tracks a single evolution. Our ignorance of any partial measurement of the system may create a loss of measurable information in the

qubits and fields, which for convenience is modeled as a stochastic process. For example, consider two observers, where one has measured the value of a qubit and not shared the information with the other. One observer is forced to model the qubit as entirely depolarized while the other can model it as entirely pure, though it is the same physical system. In an ensemble quantum system, with ensemble measurements, these quantum errors become physically distinct. Errors of type (I) are entirely due to errors in modulation, which cannot precisely control the interaction time, the interaction structure, or the intended field(s) and qubit(s) to interact. Errors of type (II) occur when our measuring apparatus makes an unintended measurement. While errors of type (III) occur when our measuring device measures an environmental degree of freedom that has an unknown entanglement structure within the quantum system.

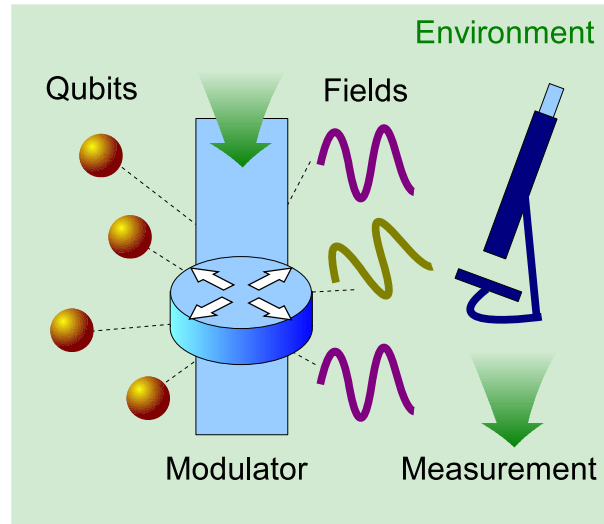


Figure 4.1: Model of Quantum System

In order to scale quantum computing, quantum operations must net a total Fidelity greater than 0.9999. Traditional treatments of quantum noise have focused on the decoherence of qubits through environmental coupling, or quantum fields with radiative reservoirs. Most quantum systems are now sufficiently isolated that this simple relaxation no longer accounts for the experimental error, and measurement Fidelities can now exceed 0.999. The precision of quantum operations, then is largely determined by the how accurately modulators interact particular quantum fields with particular qubits to encode the correct quantum operations.

Simple noise models of quantum systems immediately trace over external de-

degrees of freedom whether or not they have been measured, “creating” noise of type (III) even though the noise is physically type (I). When these external degrees of freedom are uncontrollable, this approach is justified. However, these errors are fundamentally unitary faults, as no measurement has been made. For example, it is well-known that an electron spin interacting with a nuclear spin-bath will decohere if the bath is uncontrollable. However, control through the electron [12, 48] and/or partial control of the bath will restore the coherence of the electron. We show the same restoration can be applied to coherent fields whose errant unitary interactions with the external degrees of freedom of a modulator, that were once traced over to produce type (III) noise, can be re-cohered. Operations between the qubits affect the history of control fields that interacted with it, and the right ancilla and operations can feedback their state to partially correct the fields. This formalism of fault-tolerance for unitary errors in the field external to a quantum computation basis is significant in systems that are not already dominated by non-unitary errors.

For many quantum systems, the non-unitary errors of system-environment Markovian noise are too small to decohere the system beyond the 0.9999 fidelity alone or interfere with an operation, as was shown in (3.1).

### 4.1.3 Coherent Modulation Error

The encoding of a quantum operation into a finite spacetime region of a field is limited by the quantum controllability, that is modulation, of the dynamical field. The free evolution of a field with one or more qubits is given by a precise interaction structure in the Hamiltonian, for which any deterministic errors may be compensated [35, 31, 27, 38]. However in the literature, indeterminacy in the modulation does not typically merit additional terms in the Hamiltonian due their added complexity and additional Hilbert space dimensions. For example, in a laser, two stochastic, phenomenological approximations are made. First, the interaction Hamiltonian term receives a time dependent co-factor to shape its entry and exit from the rest-frame of the qubits. Second, the atoms in the gain medium are given an exponential lifetime leading to delta-correlated white-noise [24, 40]. Other

noise sources are approximated similarly with a reservoir as in 4.1.2, thermal or otherwise, also producing delta-correlated white noise. These methods immediately trace over the the atoms in the gain medium and modulator, producing a stochastic control field before any attempt to collapse the control field into correct or correctable subspaces. This is physically misleading as the absorption and emission properties of the gain medium and modulator are highly coherent.

We propose a coherent treatment of the external dimensions of the control field by transforming them into a coherent superposition of operations parametrized by two quantum variables to retain the simplicity of stochastic procedures but allow for error-correction of the control field. Without loss of generality to all quantum operations, consider the modulation of a control field for the quantum swap operation in a two qubit system with a wave function spanned by four energy eigenstates,  $|\psi\rangle = \sum_{i=0}^3 c_i |i\rangle$ , subject to a control-field,  $f(b_k, b_k^\dagger)$ , for performing a quantum swap over time  $\Delta t$ . The non-interacting Hamiltonian is given by  $H_0$ . The interaction of maximum coherence is thought to travel along the time optimal path, geodesic, connecting the identity operation to quantum swap to produce a  $\pi$  pulse [29]. The generator of this interaction is given by (4.1), where  $\Omega_k$  is the peak interaction frequency given by a composite of physical parameters. The specific physical system (e.g. the absorption line-width of an atom) determines how “frequency-selective” the interaction actually is.

$$G = i\Omega_k \left[ b_k e^{-i\Delta t + \phi(t)} |1\rangle \langle 2| - b_k^\dagger e^{i\Delta t + \phi(t)} |2\rangle \langle 1| \right] \quad (4.1)$$

$$-i\partial_t |\alpha, \Omega(t)\rangle |\psi\rangle = (H_0 + e^{-iH_0 t} G e^{iH_0 t}) |\alpha, \Omega(t)\rangle |\psi\rangle \quad (4.2)$$

The phase factor,  $\phi(t)$ , is used to represent any phase drift in the laser. Given current experimental results [36], we now presume the effects of phase noise are minimized or may be minimized using squeezing for the period of the operation. For simplicity we keep control resonant with the laser,  $\Delta = 0$ . We introduce the value of the control field modulator as a coherent quantum state,  $|\Omega_k(t)\rangle$ , which represents its transitivity, or more physically for a laser, the quantum state of electrons in the gain medium and the modulator. In order words, we assume

the modulator's control is produced by scattering from some many-body object. The scattering of the coherent state with each individual body,  $q$ , switches from reflecting,  $|q, R\rangle$ , to transmitting,  $|q, T\rangle$ , and back again through the time evolution of the body's absorption spectrum,  $c_q(t)$ . We can write the many-body state of the modulation bodies as in (4.3)

$$|\Omega_k(t)\rangle = \prod_{q=1}^Q \left( c_q(t) |q, T\rangle + \sqrt{1 - c_q(t)} |q, R\rangle \right) \quad (4.3)$$

This approximation is compact and convenient. Many more exact, complex formulations of the time-dependent scattering such as the Schwinger Multi-Channel method are unnecessary as the error in process is already a weak perturbation. Given noise in the canonical variables of the control field, the field of optimal information transport is a coherent state,  $|\alpha\rangle$ , which retains the highest average fidelity in transport from the modulator [11]. Such a control field minimizes the product of fluctuations in the field energy, a function of  $k$ , and the time,  $\Delta t$ , which determines the precision of the swap.

Upon moving into a co-rotating frame where only the interaction term,  $G$ , remains, the operation,  $O$ , is given by (4.4) in the absence of phase-noise and decoherence of the control field. Additionally, the time dependence has been moved to the transmission operator,  $\Omega_k(t)$ , by moving into the Heisenberg picture and leaving the initial quantum state time-independent for easy integration. The transmission operator returns the transmission strength of the control field due the state of the gain medium and the modulator. The operator presents the coherent state as seen from the rest frame of the qubits, transitioning from the vacuum coherent state, to the control coherent state, and back again ( $|0\rangle \rightarrow |\alpha\rangle \rightarrow |0\rangle$ ) ideally within time  $\Delta t$ . As such the transmission operator has a pseudo-continuous, degenerate set of eigenvalues  $[0, 1]$  representing full reflection to full transmission of the coherent state across the modulation barrier.

$$O \, |\alpha, \Omega\rangle |\psi\rangle = T(\exp \left[ \int_0^\infty dt' \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b_k & 0 \\ 0 & -b_k^\dagger & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Omega_k(t') \right]) |\alpha, \Omega\rangle |\psi\rangle \quad (4.4)$$

Unless the qubit's absorption line-width is very precise, the time of the operation must be long enough to keep the control field's line-width much less than the qubit's absorption line-width to minimize off-resonant interaction. However, there are always time-dependent fluctuations in the interaction energy for any physical system (e.g. spontaneous emission, exchange energy [28], etc.) that contribute to  $\Omega(t)$ . The modulator is simply a partially coherent control field that creates a superposition of pulse shapes by coupling to a larger environment (e.g. the refractive index control of a laser consists of changing quantum mechanical absorption and emission properties of a bulk material) during the ramp-up and the ramp-down of the control field. This environmental interaction causes some level of decoherence of the field but it is assumed to be minor. Expanding the exponentiation reveals the number state operator,  $N = b_k^\dagger b_k$ , underlying the operation as shown in (4.5).

$$O = \sum_{n=\text{even}}^{\infty} \frac{1}{n!} \left( \int_0^\infty dt' \Omega_k(t') \right)^n \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & N+1 & 0 & 0 \\ 0 & 0 & N & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\frac{n}{2}} \quad (4.5)$$

$$+ \sum_{n=\text{odd}}^{\infty} \frac{1}{n!} \left( \int_0^\infty dt' \Omega_k(t') \right)^n \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (N+1)^{\frac{n-1}{2}} b_k & 0 \\ 0 & -b_k^\dagger (N+1)^{\frac{n-1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.6)$$

The formula (4.5) agrees with the Area Theorem if the control-field modula-

tion were classical: only the *area* and not the *shape* of a classical modulation  $\Omega_k$  determine the operation. But as  $\Omega_k$  is an operator, its time-dependence is given by its Heisenberg-picture evolution. It is further complicated by the fact that at small timescales  $t$  is a relativistic operator rather than a parameter. To abstract away these complexities, we denote this integrated operator as the *area operator*,  $\Omega_k \Delta t$ , written suggestively to indicate its relationship to the Area theorem. Since the area has no effect on this single frequency operation, an optimal classical pulse-shape is  $\Omega_k(t) |\alpha, \Omega\rangle = \sinh(t/\beta \Delta t) |\alpha, \Omega\rangle$  [42]. This pulse shape simultaneously minimizes the phase-noise due to frequency fluctuations and any shape distortion due to the absorption properties of the environment surrounding the qubit (the optical thickness of the qubits in the Maxwell-Bloch equations). However given the unitary-fault tolerance method described in this paper, it may be advantageous to squeeze the unitary fluctuations in the control field to minimize the frequency span further, as coherent area errors may be error-corrected as described in this paper. The parameter  $\beta$  represents the extended time needed to perform the operation (over the idealized  $\Delta t$ ) and for the particular pulse shape selected, the case of  $\sinh$ ,  $\beta \approx 2.6$ . We use the identity  $b f(N) = f(N+1) b$ , for any analytic function  $f$  of the number operator, to transform the term  $b_k N^{\frac{n-1}{2}}$  to  $(N+1)^{\frac{n-1}{2}} b_k$  in the summation of the odd terms. We identify the Susskind-Glogower operators (for purely the photonic degrees of freedom in the equations of motion, marked with a ph) in (4.6) and use the following substitutions (4.7), (4.8).

$$E_+^{\text{ph}} = b_k^\dagger (N+1)^{-\frac{1}{2}} \quad (4.7)$$

$$E_-^{\text{ph}} = (N+1)^{-\frac{1}{2}} b_k \quad (4.8)$$

These operators are “one-sided” unitary (for a non-zero number of photons, they obey  $(E_\pm^{\text{ph}})^\dagger = E_\mp^{\text{ph}}$ ) and exactly integrate (4.4) to produce the quantum operation (4.9).

$$O = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos [\sqrt{N+1}\Omega_k\Delta t] & \sin [\sqrt{N+1}\Omega_k\Delta t] E_-^{\text{ph}} & 0 \\ 0 & -E_+^{\text{ph}} \sin [\sqrt{N+1}\Omega_k\Delta t] & E_+^{\text{ph}} \cos [\sqrt{N+1}\Omega_k\Delta t] E_-^{\text{ph}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.9)$$

The actual pulse-shape is a super-position of several possibilities that are traced-over when finding line-width. The coherent errors and deviations in the control field interaction from (4.9) the desired classical secant pulse-shape are of central interest. Because we made phase-error negligible, we observe that the structure of the operation as a non-local rotation is constant: even if the terms no longer sum into a sin and cos series, a shift in the rotation parameter always preserves the phase-less structure of the rotation. Therefore, we may aggregate these errors into a coherent shift of the rotation parameter. This shift must span the full Hilbert space,  $\{|\alpha, \Omega\rangle\}$ . The difference from the ideal area operator in the integrated interaction can be represented with the operator  $\delta_\omega(\Delta t)|\Omega\rangle = \left(\int_0^{\beta\Delta t} \Omega_k(t') dt' - \Omega_k\Delta t\right)|\Omega\rangle$ . To relate this operation to the semi-classical sinh pulse approximation, one can trace over the modulator degrees of freedom as shown in (4.10), through which the exact definition of  $\delta_\omega(\Delta t)$  may be understood. This notation abstracts the particular time evolution of the modulation and any relativistic consequences of time being an operator rather than a parameter. Because there is little coupling of the qubit through the modulator, through the coherent state, to the other degrees of freedom controlling the modulator the operation is still largely coherent.

$$\text{Tr}_q(\delta_\omega(\Delta t)|\Omega_k\rangle) = \text{Tr}_q\left(\left[\frac{1}{Q}\int_0^{\beta\Delta t} dt \sum_{q=1}^Q |q, T\rangle \langle q, T| - \Omega_k\Delta t\right] |\Omega_k\rangle\right) \quad (4.10)$$

Alternatively one could add a noise operator,  $\xi(t)$ , to the area before integration, but if the form of the noise is not specified the time evolution becomes unwieldy. Because the control field is only approximated by a coherent state we represent the difference in the actual number state operator with,  $\delta_n|\alpha\rangle =$

$(N_{\text{actual}} - N) |\alpha\rangle$ . The errant operation is then given by the real rotation parameter  $\varphi = \sqrt{N + \delta_n}(\Omega\Delta t + \delta_\omega)$ . Because the new operators apply to different, independent parts of the Hilbert space (modulator v. cavity) no symmetrization is necessary. The Susskind-Glogower operators are in turn offset as in (4.11),(4.12).

$$E_+^{\text{ph}'} = b_k^\dagger (N + \delta_n + 1)^{-\frac{1}{2}} \quad (4.11)$$

$$E_-^{\text{ph}'} = (N + \delta_n + 1)^{-\frac{1}{2}} b_k \quad (4.12)$$

The modulator's intended dynamics are designed for the classical idealization that  $\langle \alpha | \sqrt{N} \Omega_k \Delta t | \alpha \rangle = \pi/2$ . Since experimentally the shift from this ideal, classical  $\pi/2$ -pulse is small, we can Taylor expand the operation in the rotation parameter to the second order to produce the operator (4.13).

$$\begin{aligned} \varphi \cong & \sqrt{N} \Omega_k \Delta t + \delta_n \sqrt{N}^{-1} \Omega_k \Delta t \left( \frac{1}{2} [1 + \delta_\omega] + \frac{1}{2} \delta_n^2 N^{-1} \right) + \sqrt{N} \delta_\omega \quad (4.13) \\ & + O(\delta_n^3) + O(\delta_n^2 \delta_\omega) \quad (4.14) \end{aligned}$$

This expansion presents two sources of error. When the intensity of the pulse is low, number fluctuations can be an appreciable source of error and the Susskind-Glogower operators become relevant. However, when the intensity of the pulse is high, pulse-shape error dominates with an error proportional to the intensity of the control field. To write this operation in terms of coherent fluctuations, we introduce an operator for the perturbations to the rotation,  $\epsilon_p(\delta_n, \delta_\omega) = \varphi - \sqrt{N} \Omega_k \Delta t$ , and quantum-uncertainty fluctuations (Shot noise) about the classical ideal,  $\epsilon_f(\delta_n, \delta_\omega) = \sqrt{N} \Omega_k \Delta t - \pi/2$ , to produce (4.15) with total errant angle operator,  $\epsilon_\theta = \epsilon_p + \epsilon_f$ .

$$O = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & E_+^{\text{ph}'} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin \epsilon_\theta & \cos \epsilon_\theta & 0 \\ 0 & -\cos \epsilon_\theta & \sin \epsilon_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & E_-^{\text{ph}'} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.15)$$

To show that measurement can collapse the value of unitary error operator,  $\epsilon_\theta$ , one must identify the Hermitian measurables associated with the phase of the system. Phase operators like,  $\epsilon_\theta$ , are infamously non-Hermitian and are only approximated as so for weakly perturbed harmonic oscillators at high excitation levels.

$$\Delta N \Delta \phi \geq \frac{\hbar}{2} \quad (\text{wrong}) \quad (4.16)$$

For example the famous uncertainty (4.16) for oscillators is a classical approximation that is only true when  $N$  is large are noted by Carruthers and Nieto [9]. Instead they introduced a formalism, using the quantum correspondence of action-angle mechanics, to show that the phase of a system is in fact only a classical variable approximated by  $\theta$  [10]. To show this, they created canonically conjugate operators of the system:  $C$  and  $S$  for a simple harmonic oscillator that are manifestly Hermitian. And while these operators are constructed to describe the cosine and sine of the phase parameter, respectively, their inverses yield two different operators that do not commute as both  $C$  and  $S$  do not commute. To derive the eigenstates of the collapse for 4.15, we use the same action-angle tools and apply them instead to the non-stationary field-matter interaction 4.1.

Except for,  $\Omega(t)$ , the interaction Hamiltonian 4.1 has no explicit time varying components. Given the frequency-limiting pulse shape of  $\Omega(t)$ , its dynamics are considered adiabatic when calculating the Rabi-frequency, keeping it the well known,  $\omega_n(n, \Omega, t) = \sqrt{(n+1)\Omega^2(t) + \Delta^2}$ . This frequency tracks the pulse shape  $\Omega(t)$ , for each energy, causing the angular frequency to increase or decrease in time, but the action (in this case the structure of the interaction) is invariant.

$$\begin{aligned}\omega &= \sum_n \omega_n |n\rangle \langle n| \\ G &= \omega J\end{aligned}$$

We have assume the initial phase of the system is zero. Because the idea that  $\dot{\theta}_n = \omega_n$  is purely classical, we seek operators of the angle,  $C_n \equiv \cos(\theta_n)$  and  $S_n \equiv \sin(\theta_n)$ , as did Carruthers and Nieto. These operators obey the time evolution

$$\begin{aligned}\dot{C}_n &= -i [C_n, G] = \omega_n S_n \\ \dot{S}_n &= -i [S_n, G] = -\omega_n C_n\end{aligned}$$

The total operators  $C$  and  $S$  may be similarly summed as  $C = \sum_n C_n |n\rangle \langle n|$  and  $S = \sum_n S_n |n\rangle \langle n|$ .

To calculate  $C$  and  $S$  we can solve for the angle evolution exponential  $E_{n,\pm} \equiv C_n \mp i S_n$ . We exploit the relationship  $[E_{n,\pm}, G] = \mp E_{n,\pm}$  to constrain the operator's properties. Since any multi-qubit operation can be decomposed into local or non-local rotations, we may now restrict our considerations to single-qubit operations. This is without loss of generality since we only consider unitary errors.

$$\begin{aligned}GE_{n,\pm} |n, \uparrow\rangle &= E_{n,\pm} \left( \sqrt{n+1} |n+1, \downarrow\rangle \pm |n, \uparrow\rangle \right) \\ GE_{n,\pm} |n, \downarrow\rangle &= E_{n,\pm} \left( \sqrt{n} |n-1, \uparrow\rangle \pm |n, \downarrow\rangle \right)\end{aligned}$$

These four equations are insufficient to solve for the eight operators that construct  $E_{n,\mp}$ . However observing that  $(E_{n,\pm})^\dagger = E_{n,\mp}$ , by its definition, provides the remaining four equations. We propose the ansatz (4.17) based on the generator of the interaction(4.1).

$$E_{\pm} = i \begin{pmatrix} \pm(N+1)^{-\frac{1}{2}} & E_-^{\text{ph}} \\ -E_+^{\text{ph}} & \mp N^{-\frac{1}{2}} \end{pmatrix} \quad (4.17)$$

It can be verified that this form (4.17) satisfies all of the required properties. The operators (4.17) may be compared to (4.5) to understand its relationship to the unitary evolution method. We may then calculate the operators of interest (4.18) and (4.19), which are manifestly Hermitian, and so measurable.

$$C = i \begin{pmatrix} 0 & E_-^{\text{ph}} \\ -E_+^{\text{ph}} & 0 \end{pmatrix} \quad (4.18)$$

$$S = \begin{pmatrix} -(N+1)^{-\frac{1}{2}} & 0 \\ 0 & N^{-\frac{1}{2}} \end{pmatrix} \quad (4.19)$$

Under classical approximations, a precise  $\theta$  would yield both a precise  $C$  and a precise  $S$  operator. However, there is a fundamental imprecision in the actual angle given by:

$$\begin{aligned} [C, S] &= 2i \begin{pmatrix} 0 & b_k N^{-1} \\ N^{-1} b_k^\dagger & 0 \end{pmatrix} \\ &= 2 \begin{pmatrix} 1 & 0 \\ 0 & N^{-1} \end{pmatrix} \left( \frac{1}{\Omega} \right) G \begin{pmatrix} 1 & 0 \\ 0 & N^{-1} \end{pmatrix} \end{aligned}$$

This is consistent with the fact that the inverses of (4.18) and of (4.19) do not produce phase operators that commute. There is in fact a co-evolution of two conjugate phase operators, but the size of their commutator is negligible when  $N$  is large. The eigenstates of  $S_n$  are given by the spin states,  $|\downarrow\rangle$  and  $|\uparrow\rangle$ , with eigenvalues  $-\sqrt{n+1}$  and  $\sqrt{n}$  for every  $N$ . On the other hand, only one of the eigenstates of  $C_n$  is measurable,  $1/\sqrt{2}(|n\rangle \otimes |\downarrow\rangle + |n-1\rangle \otimes |\uparrow\rangle)$ , with an eigenvalue of 1 for every  $N$ . The other eigenstate has an eigenvalue of  $i$  (fermionic

loop) and so does not collapse the system providing a “dark-state” measurement. The conjugate nature of these field-matter operators  $C$  and  $S$  is then clear: one operates in the spin-space to control the photon number, while the other operates in photon-space to control the spin number. These operators reveal exactly how projective measurement of the now entangled, non-local matter-field object affect both the qubit and field. Directly from the eigenstates, we see that a measurement on  $S$  collapses the spin and field into photon absorbed or not absorbed eigenstates while a measurement of the  $C$  operator is much more complex. The massive degeneracy of the the  $C$  operator implies that one may measure an eigenvalue related to the field (extremely partial measurement) and retain most of the coherence of the spin state. We exploit this fact in Section 4.1.5, to show one can separate errant field amplitudes into correct and correctable subspaces.

Finally, we can combine (4.19), (4.18), and (4.5) and (4.6). Since  $\cos$  is odd and  $\sin$  is even, it is readily verified that the total operation can now be written as (4.20).

$$O = \cos(\Omega\Delta t S) + i \sin(\Omega\Delta t C) \quad (4.20)$$

$$\approx \begin{pmatrix} 1 & 0 \\ 0 & E_+^{\text{ph}} \end{pmatrix} \left( I \otimes \cos \left[ \frac{\pi}{2} + \epsilon_\theta \right] \right) \begin{pmatrix} 1 & 0 \\ 0 & E_-^{\text{ph}} \end{pmatrix} \quad (4.21)$$

$$+ i \begin{pmatrix} 1 & 0 \\ 0 & E_+^{\text{ph}} \end{pmatrix} \left( S_y \otimes \sin \left[ \frac{\pi}{2} + \epsilon_\theta \right] \right) \begin{pmatrix} 1 & 0 \\ 0 & E_-^{\text{ph}} \end{pmatrix} \quad (4.22)$$

The approximate-equals sign is used since equation (4.20) does not include modulation error while (4.21) does. However having used an action-angle method to derive (4.20), we can now exploit the adiabatic invariant and apply error directly to the angle operators  $S' = S(N + \delta_n)$  and  $C' = C(N + \delta_n)$ . We equate the real and imaginary parts to isolate the measurable components of the operational phase to show how the field-matter operators collapse the phase angle in our result (4.23), 4.24.

$$I \otimes \cos \left[ \frac{\pi}{2} + \epsilon_\theta \right] = \begin{pmatrix} 1 & 0 \\ 0 & E_-^{\text{ph}'} \end{pmatrix} \cos ([\Omega \Delta t + \delta_\omega] S') \begin{pmatrix} 1 & 0 \\ 0 & E_+^{\text{ph}'} \end{pmatrix} \quad (4.23)$$

$$iI \otimes \sin \left[ \frac{\pi}{2} + \epsilon_\theta \right] = \begin{pmatrix} 0 & E_-^{\text{ph}'} \\ -1 & 0 \end{pmatrix} \sin ([\Omega \Delta t + \delta_\omega] C') \begin{pmatrix} 1 & 0 \\ 0 & E_+^{\text{ph}'} \end{pmatrix} \quad (4.24)$$

#### 4.1.4 Optimal Field Intensity when error is Semi-Classical

The first-order unitary error,  $\epsilon_\theta$ , provides both lower and upper bounds on the operations ideal field intensity. We can use this analysis to find the operational precisions required to meet the traditional fault-tolerance fidelity threshold of 0.9999 (or equivalently, being able distinguish phase rotation operations of  $\pi/2^9$ ). For example in the presence of Shot noise alone, this operational precision requires (4.25).

$$\pi/2^9 > \left\langle \alpha \left| \left( \sqrt{1 + 1/\sqrt{N}} - 1 \right) \pi/2 \right| \alpha \right\rangle \quad (4.25)$$

This requirement in turn requires a strong coherent state of  $|\alpha| \gtrsim 2^7$ . But, using this strong coherent state creates sensitivity to the area error operator,  $\delta_\omega(\Delta t)$ , and requires  $\langle \Omega_k \Delta t | \delta_\omega(\Delta t) | \Omega_k \Delta t \rangle < \pi/2^{16}$  to meet the threshold. Although this area precision does not violate the uncertainty principle, it is practically impossible to perform. In virtually all systems proposed for quantum computing, the separation of qubit energy states is greater than  $1\text{meV}$  and their operational times are greater than  $100\text{fs}$ , which produce an area parameter on the order of 0.1 as a strong lower-bound value. Then according to (4.15), even a 0.01% error in the shape of the pulse will violate the threshold for this lower bound. Worse still, finding a fundamentally new quantum computing system to decrease the operational time or decrease the energy separation (to decrease the area) may actually lead to greater shape distortion or greater environmental coupling given the  $1/f$  noise in the classical (bulk) apparatus modulating the pulse-shape. One solution to this impasse is presented in this paper, creating a system of fault-tolerance specific to these unitary errors in the field. As a first step, however, we may optimize the

intensity of the control field to minimize the unitary error at the start.

Given that the pulse-shape, operation, and field mode have already been optimized in our consideration, the desired coherent state is the only remaining degree of freedom. To optimize for the field intensity, any metric may be chosen but the simplest is the worst-case fidelity,  $F$ . The Fidelity measure of over a superposition of areas,  $A$ , is given in (4.26).

$$F_1(\delta_\omega, \delta_n) = \min_{|\psi\rangle} |\langle \alpha', \Omega'_k | \langle \psi | X^\dagger O | \psi \rangle | \alpha, \Omega_k \rangle| \quad (4.26)$$

$$X^\dagger O = \begin{pmatrix} 1 & 0 \\ 0 & E_+ \end{pmatrix} \begin{pmatrix} \cos[\epsilon_\theta] & -\sin[\epsilon_\theta] \\ -\sin[\epsilon_\theta] & \cos[\epsilon_\theta] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & E_- \end{pmatrix} \quad (4.27)$$

$$F_1(\delta_\omega, \delta_n) = \min_{|\psi\rangle} \left| \int_A dA' \int_A dA \sum_{n'=0}^{\infty} \sum_{n=0}^{\infty} \langle n', A' | \langle \psi | X^\dagger O | \psi \rangle | n, A \rangle \langle n, A | \alpha, \Omega_k \rangle \right|$$

The fidelity of interest is assumed to be the worst-case using the particular initial spin-state  $|\psi\rangle$  that minimizes it. It could be argued that the state of the field is irrelevant after the operation, so we should not require that the ideal and actual operations agree on the *out*-field. However, the qubit is composed solely of field operators and so we cannot rightly separate the two. The physical, fermionic phase,  $-1$ , is included in both the ideal and actual quantum operations (4.27). To actually perform the calculation, a amplitude distribution of areas is required from the trace of the modulator degrees of freedom (4.3). A reasonable approximation might be a coherent Gaussian distribution,  $\mathcal{N}$ , with a mean of  $\pi/2|\alpha|$  and a standard deviation of  $\delta_\omega$ :  $\text{Tr}_q(\delta_\omega(\Delta t) |\Omega_k\rangle) = \int_A dA \mathcal{N}(\pi/2|\alpha|, \delta_\omega) |A\rangle$ . However, lacking the operational-time dependence of  $\delta_\omega(\Delta t)$ , using such a distribution will be meaningless for field optimization. Instead we may find the optimal  $\alpha$  for relevant  $\delta_\omega$  and  $\delta_n$ . Then given a particular quantum systems, with error parameters for the purity of the gain medium and modulator, a distribution function may be chosen over these states.

Since coherent states are Poisson distributed, when  $N$  is very large it is sufficient

to optimize against the mean of the distribution, instead of  $\alpha$ . In this simple case where we assume  $n \approx \langle n \rangle$ , the minima is given by  $\langle n \rangle = (\delta_n + 2)/2\delta_\omega$ , where the mean intensity is inversely proportional to absolute shape error. However the above shot noise analysis shows a large  $N$  is not possible for any practical  $\delta_\omega$ . The full distribution of number states will have to be considered. Averaging over the phase of the coherent state  $\alpha = |\alpha|$ , we can rewrite the average fidelity as (4.28)

$$\begin{aligned} \sum_{n'=0}^{\infty} \sum_{n=0}^{\infty} \langle n | X^\dagger O | n, \rangle &= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{\langle n \rangle^{n/2} e^{-\langle n \rangle/2}}{\sqrt{n!}} \frac{\langle n \rangle^{n'/2} e^{-\langle n \rangle/2}}{\sqrt{n'!}} \cdot \\ &\cdot [\langle n' | n \rangle \cos(\epsilon_\theta) - \langle n' | n-1 \rangle \sin(\epsilon_\theta)] |\uparrow\rangle \langle \uparrow| \\ &+ \frac{1}{2} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{\langle n \rangle^{n/2} e^{-\langle n \rangle/2}}{\sqrt{n!}} \frac{\langle n \rangle^{n'/2} e^{-\langle n \rangle/2}}{\sqrt{n'!}} \cdot \end{aligned} \quad (4.28)$$

$$\begin{aligned} &\cdot [\langle n'-1 | n-1 \rangle \cos(\epsilon_\theta) - \langle n'-1 | n \rangle \sin(\epsilon_\theta)] |\downarrow\rangle \langle \downarrow| \\ F &= \sum_{n=0}^{\infty} \frac{\langle n \rangle^n e^{-\langle n \rangle}}{n!} \left( \cos(\epsilon_\theta) \left[ \frac{n}{\langle n \rangle} \right] - \sin(\epsilon_\theta) \sqrt{\frac{n}{\langle n \rangle}} \right) \end{aligned} \quad (4.29)$$

The Susskind-Glogower operators create an anisotropy of error in the spin state. The spin state of minimum fidelity is spin down (4.29) given the transition probabilities between joint Poisson distributions. The extremization  $\partial_{\langle n \rangle} F = 0$  does not have an algebraic solution. Under various semi-classical approximations of the pulse shape error, we can calculate the optimal  $\langle n \rangle$  using the constraint that  $\langle \alpha | \sqrt{N} \Omega_k \Delta t | \alpha \rangle = \frac{\pi}{2}$ . For example by setting the parameter to Shot noise alone  $\epsilon_\theta \rightarrow \pi/2 - \sqrt{n} \left[ \pi / \left( 2\sqrt{\langle n \rangle} \right) \right]$ , we can see that a very large  $N$  is needed to meet the traditional fault-tolerance threshold as shown in Figure 4.2(a). If the operators  $\delta_n$  and  $\delta_\omega$  are semi-classically replaced with numbers, we can find the optimum  $\langle n \rangle = |\alpha|^2$  trajectory by plotting likewise. Because Shot noise dominates at very low intensity, we set  $\delta_n \rightarrow 1$  and contour plot over the variables  $\delta_\omega$  and  $\langle n \rangle$  in Figures 4.2(b,c). Barring the actual operational time-dependence of the area operator, we plot 4.29 with the area deviation as both an absolute number (4.30) in Figure 4.2(b) and as a percent of the total area (4.31) in Figure 4.2(c).

$$\epsilon_\theta \rightarrow \frac{\pi}{2} - \sqrt{n-1} \left( \frac{\pi}{2\sqrt{\langle n \rangle}} - \delta_\omega \right) \quad (4.30)$$

$$\epsilon_\theta \rightarrow \frac{\pi}{2} - \sqrt{n-1} \left( \frac{\pi}{2\sqrt{\langle n \rangle}} (1 - \delta_\omega) \right) \quad (4.31)$$

These plots reveal that there must be an optimal field strength and that pulse-shape error dominates the fidelity. The modulator, because it is classically controlled bulk, must have an element of low-frequency error [18]. As the intensity of the field,  $\langle n \rangle$ , goes up, and the operational time,  $\Delta t$ , goes down, there is a lower bound constant area error. Even if this area error is 0.005 it precludes the fidelity from exceeding  $\sim 0.95$  at all. Even a constant percentage error in the area with decreasing  $\Delta t$  would impose this limit with a 5% error. A real physical system can be parametrized by both types of errors and an optimal fidelity and fidelity bound calculated.

Noise models which assume Gaussian under-over-rotation channels with a standard deviation of just the intrinsic coherent state number uncertainty,  $\sqrt{N}\pi/2N$ , may be overly simplistic for quantum computing and particularly for short pulses. Static errors in the pulse shape error,  $\delta_\omega$ , do not scale with the duration of the pulse: short pulses share a similar error in ramping up and ramping down a coherent state as long pulses. Because these static errors are off the computational basis, they cannot be corrected with control sequences. This is true of non-unitary error as well, e.g. the environmental coupling of the modulator to transmit and quench a laser pulse creates a static imperfection from environmental coupling, irrespective of the steady-state pulse duration. This may explain the large discrepancy between the precision predictions of simple models of operational decoherence and the imprecision of actual experiments.

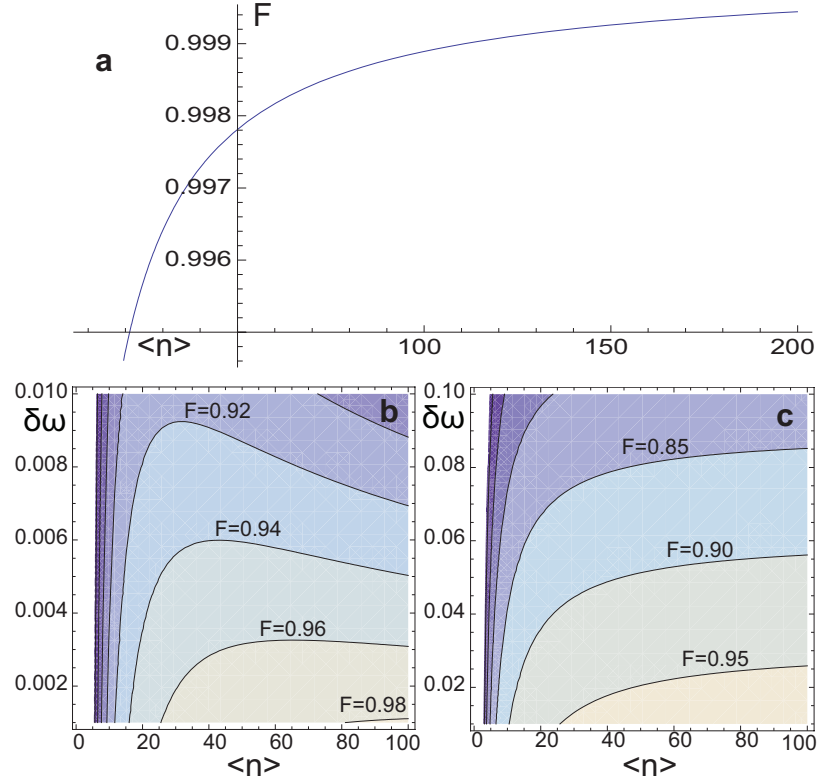


Figure 4.2: Fidelity versus field intensity for semi-classical approximations

**a.** With only Shot noise (Poisson distribution of a coherent state) as causing unitary error, the minimum field intensity,  $\langle n \rangle$ , needed to meet the the 0.9999 Fidelity threshold well passes the knee in the curve. By contrast if unitary error is corrected, a threshold of 0.99 is met with a low field strength. **b.** A plot of the Fidelity for small pulse area error that is constant reveals that an optimum field strength exists, beyond which the Fidelity will only decrease. **c.** A plot of the Fidelity for small pulse area error as a percentage reveals a very high sensitivity. Real systems will be a combination of the two error types, as classical objects like field modulators always have a component of  $1/f$  noise. See main text for details of area error.

### 4.1.5 Single Qubit Unitary Fault-Tolerant Design and Physical Realization

Traditional quantum fault-tolerance is often described as “measuring” the operation rather than the qubit, in order to correct any decoherence or unitary error in the operation. In practice, this has reduced to measuring the effect of the operation on redundant, entangled qubits. The analysis presented in Section 4.1.3 suggests that an explicit, actual measurement of the operation may be made. However in the case of unitary error, one need not use immediate measurement as there are no stochastic parameters to collapse. Rather, in this section and the following section we outline a system of unitary-fault tolerance for single and multi-qubit operations that exploits the implicit measurement at the end of any quantum circuit.

The basic postulates of quantum measurement, along with the above formalism, (4.23) (4.24), require that a collapse of a qubit to nearly (precision of the measurement) one or zero, requires that the  $C$  or  $S$  operator collapses similarly. If multiple fields control one-qubit, such a collapse creates a correlation between the two fields that is precise up to the  $[C, S]$  limit. In effect, the commutator  $[C, S]$  entangles the control fields which determine the amplitude of the projected field and hence the angle of the projected state. This distinction is significant: on timescales that turn local, qubit decoherence negligible, qubits become simple, stationary degrees of freedom, completely subservient to the quantum field. This point of view agrees with contemporary, finite-time scattering theories which “treat the states vectors as functionals of the field potential,  $\Psi[\mathcal{A}]$ , in which  $\mathcal{A}$  extends over a *finite region* of spacetime” [50].

To make a measurement that leaves one qubit of entropy after measurement, a system greater than one qubit is required. We show that two qubits is sufficient to coordinate the partial collapse of the control fields in Pauli-X operation by one order of error. This is done by isolating first-order errant amplitudes into a Hilbert subspace and then feedback the errant state of the qubits to “self-correct” the amplitude. That is to say, the error itself directs its correction as formalized by the theory of quantum coherent feedback [34]. This is possible due to the coherence of the amplitudes, if they were traced over apriori as is commonly done,

Table 4.1: Transformation of single-qubit using unitary fault tolerance

| $ r\rangle$ | $ s\rangle$ | $d_i^t$   | Input ( $t = 0$ ) | Op. 1( $t = 1$ ) | Op. 2 ( $t = 2$ ) | Op. 3 ( $t = 3$ )           |
|-------------|-------------|-----------|-------------------|------------------|-------------------|-----------------------------|
| 0           | 0           | $d_0^t =$ | $c_0$             | $c_0$            | $p_2 c_0$         | $q_3 p_1 c_1 + p_3 p_2 c_0$ |
| 0           | 1           | $d_1^t =$ | $c_1$             | $p_1 c_1$        | $p_1 c_1$         | $q_3 p_2 c_0 + p_3 p_1 c_1$ |
| 1           | 0           | $d_2^t =$ | 0                 | $q_1 c_1$        | $q_1 c_1$         | $q_1 c_1$                   |
| 1           | 1           | $d_3^t =$ | 0                 | 0                | $q_2 c_0$         | $q_2 c_0$                   |

the fields would turn stochastic and decohering. This is analogous to quantum state coherence, for example, in the tracing over nuclear degrees of freedom in a spin-bath apriori, rather than using their coherences to refocus themselves,

Consider the a four state quantum system  $|\psi\rangle = \sum_{i=0}^3 d_i^t |i\rangle$  with unique transition energies,  $\omega_{ij}$ , between every pair of states,  $|i\rangle, |j\rangle$ , e.g. the an-harmonic oscillator potential of a superconducting qubit. The superscript  $t$  denotes, discretely, the time-dependence of the coefficients before and after each quantum operation. This system is drawn in Figure 4.3(a) and represents both the ancilla qubit  $|r\rangle$  and system qubit  $|s\rangle$  states in the total Hilbert space  $|rs\rangle$ . For now, assume the ancilla is initialized to  $|0\rangle$  and the qubit is in some arbitrary state for a total initial state,  $|\psi\rangle = |0\rangle \otimes (c_0 |0\rangle + c_1 |1\rangle)$ . If the qubit is directly rotated by using a modulated pulse of frequency  $\omega_{01}$ , the fidelity of the operation would be  $|\cos(\epsilon_\theta)|^2$  due to area and intensity errors. Of course this assumes again that this error is much greater than the error due the relaxation time of the operation. Consider instead the three operations shown in Figure 4.3(b). Because the first two pulses,  $|\alpha, \Omega_1\rangle$  and  $|\alpha, \Omega_2\rangle$ , operate in separate subspaces they may applied at the same time, however quantum coherent feedback requires a temporally-spaced third pulse,  $|\alpha, \Omega_3\rangle$  to correct the first order error. We can map the transformation of the coefficients  $c_1$  and  $c_2$  over time using the amplitudes  $q_t |\alpha, \Omega_t\rangle \equiv \cos(\epsilon_\theta) |\alpha, \Omega_t\rangle$  and  $p_t |\alpha, \Omega_t\rangle \equiv \sin(\epsilon_\theta) |\alpha, \Omega_t\rangle$  as shown in Table 4.1.

It is now critical that we treat the parameters  $\epsilon_i$  in their true form as operators and not as values. At the end of the experiment, implicit measurement of all qubits occurs. If  $|r\rangle$  is measured to be one, then the amplitudes for  $q_3$  OR  $p_1$  must collapse to nearly zero as well as:  $q_3$  OR  $p_2$ ,  $p_3$  OR  $p_2$ , and  $p_3$  OR  $p_1$ . The possibility that  $q_3 p_1 = -p_3 p_2$  is negligible given that  $\epsilon_\theta$  is very small. Because these are quantum amplitudes these all must obey the condition,  $p_i = 0 \longleftrightarrow q_i = 1$ , and vice-versa.

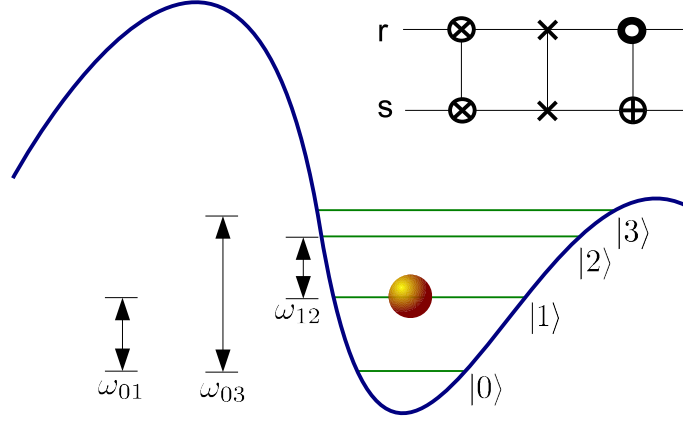


Figure 4.3: Anharmonic oscillator potential

In the local vicinity of a washboard phase-potential trapping a superconducting qubit, every transition between bound states is of a different frequency. **a.** The line-width of the control field must be able to differentiate the bound states  $|0\rangle \dots |3\rangle$  using the transitions  $\omega_{0,1}$ ,  $\omega_{0,3}$ , and  $\omega_{1,2}$  to implement **b.** a circuit to implement a unitary fault-tolerant Pauli-X gate.

Therefore the back-action of the measurements on the field due to the amplitudes,  $d_0^3$  and  $d_1^3$ , must allow satisfiability of this condition. It is not possible for both  $q_3$  and  $p_3$  to be zero, so  $p_1$  OR  $p_2$  must be equal to zero for  $d_0^3 = 1$ . However, the second condition then mandates that the other is then zero. Therefore if  $|r\rangle$  is measured as one, it implies that both  $p_1$  and  $p_2$  are both zero, leaving the remaining qubit at the correct angle, since the coefficients with  $q_1$  and  $q_2$  must be identically one. If  $|r\rangle$  is measured to be zero, which requires one of the first two operations to be measured as errant, then the amplitudes  $q_1$  and  $q_2$  must be zero. This implies that  $p_1$  and  $p_2$  must be one. Therefore the amplitudes  $q_3$  and  $p_3$  determine this next order of correctness. In total, this provides the correct answer to the second order in error probability. The new fidelity is then given by 4.32.

$$F = (|q_1 c_1|^2 + |q_2 c_0|^2) + (1 - [|q_1 c_1|^2 + |q_2 c_0|^2]) [|q_3|^2 + |p_3|^2 (c_0^* c_1 + c_1^* c_0)] \quad (4.32)$$

The fidelity improves as its loss is now  $|p_3 p_2|^2 + |p_3 p_1|^2$  instead of  $|p_3|^2$  as with a direct rotation. In general this provides the same bounds as traditional fault-tolerance: when the number of operations is less  $O(1/\langle p \rangle)$  there is a gain in fidelity.

Because the first two control fields are correlated in either qubit measurement outcome, they are by definition entangled. One interpretation of this phenomenon is that the operational information (quantum not) was encoded in two quantum fields instead of one, so an additional qubit can direct a third quantum field to correct any single error in the first two. Information-wise this is akin to quantum state correction: when if there is an measured error in two of the three fields it cannot be corrected, but any single measured error is corrected (in this case an error is in the first two fields is also corrected due to a additional qubit).

An unique set of transition energies are not necessary for this implementation. An equivalent operation can be done using virtual states and/or additional selection rules (e.g. optical polarization). For example, consider the use of bi-excitons in a quantum dot controlled using the virtual states shown in 4.4. In this case, there are only two interaction frequencies  $\Omega_1$  and  $\Omega_2$ , but every subspace is controllable given virtual states and polarization selection between the horizontal,  $H$ , and vertical,  $V$ , optical bases. Optical polarizers can be set to very high precision presenting an error of less than  $10^{-3}$  radians. However due to the new interaction frequency, operations using virtual states are slower or must use a non-optimal control field intensity. If these slower operations are still much faster than the qubit relaxation time, the same quantum circuit in 4.3(b) can be implemented.

Given that implicit measurement is used at the end of the circuit (for convenience), the question of how many ancilla qubits are necessary in a general quantum circuit arises. We have used the fact that the qubit  $|r\rangle$  is not necessarily used for any further computation so any subsequent quantum gates will operate in both the  $|r, 1\rangle$  and the  $|r, 0\rangle$  subspaces identically. However, the use of one more operation to encode the system, allows reuse of the ancilla for any other unitary-fault-tolerant operation involving the qubit. Consider the diagram shown in 4.5. In this case, we must assume that there exists errant amplitudes in the ancilla space, and it must be reflected in the initial state, where without loss of generality we have set them identically. It should be noted again that the amplitudes are mutually exclusive, meaning  $p' \simeq 1$  implies that  $q' \simeq 0$  as per  $[C, S]$ , and hence the collapse may continue in the same manner as before. First, encode the ancilla with the intended

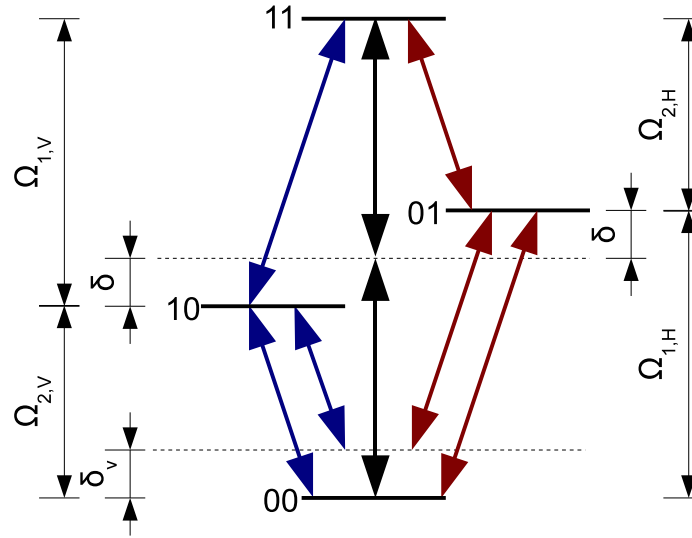


Figure 4.4: Mediating virtual energy states in a bi-exciton system

A quantum system may use virtual states to mediate an interaction in a Hilbert subspace without disturbing other states. In a bi-exciton system such as in [32], two transition energies,  $\Omega_1$  and  $\Omega_2$ , exist for four distinct transitions. Polarization control,  $H$  and  $V$ , of the laser pulse allows individual transitions. The additional transitions,  $|00\rangle \leftrightarrow |11\rangle$  and  $|01\rangle \leftrightarrow |10\rangle$  are created using virtual states offset by  $\delta_v$  and  $\delta$  in frequency. In a real bi-exciton system, the energy difference between the  $|01\rangle \leftrightarrow |10\rangle$  states is probably small and a different  $\delta$  may be used.

operation, in this case the Pauli-X gate. Next perform the operation as in 4.5. Optionally an X operation on the ancilla, restore the initial form, but subsequent operations can simply be adapted as well. The diagram below the quantum circuit in 4.5 tracks the Hilbert space amplitudes after each quantum operation. To simplify the amplitude flow, only first order errant amplitudes from operations in the errant subspace are kept. The final amplitudes verify that the net operation is correct up to second order instead of the first. In this case, the final quantum feedback operation may be interpreted as a decoding step, preparing the qubit for other operations. Analogously, the encoding operation may be interpreted as quantum coherent feedback demonstrated in the encoding function,  $f_{not}$ , written in 4.33. This encoding from one qubit to two qubits is performed using the errant amplitude of the encoding itself.

$$f_{not} : \sum_{j=0}^1 c_j |j\rangle |\alpha_1, \Omega_2\rangle \rightarrow \cos(\epsilon_\theta) (c_0 |00\rangle + c_1 |01\rangle) |\alpha_2, \Omega_2\rangle |\alpha_1, \Omega_1\rangle + (4.33)$$

$$\sin(\epsilon_\theta) (c_1 |10\rangle + c_0 |11\rangle) |\alpha_2, \Omega_2\rangle |\alpha_1, \Omega_1\rangle \quad (4.34)$$

The method presented here can be generalized to multi-qubit, unitary-fault tolerant Clifford operations and to arbitrary orders of reduction in the errant amplitudes. These generalizations come at the cost of additional qubits, additional operations, and additional operational complexity.

#### 4.1.6 Multi-Qubit Unitary Fault-Tolerant Design

The unitary fault-tolerant method presented in 4.1.5 used the specific operational example of the Pauli  $X$  gate. To make a general operation unitary fault-tolerant it should be decomposed into encodings that track the structure of operation (rather than the structure of noise as in traditional fault tolerance). To this end, one may use non-fault tolerant gates to transform the operation to the correct state, as long as each operation obeys at least one of the two following rules:

1. The operation's subspace involves terms that are all at least first order in

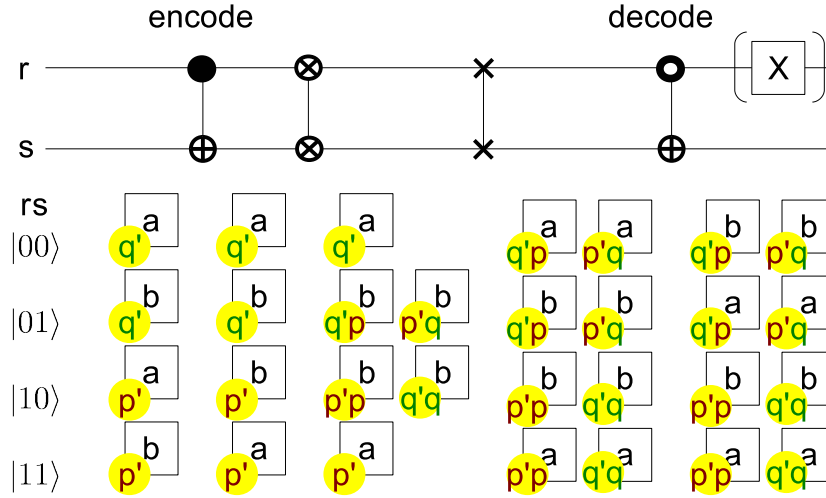


Figure 4.5: Unitary Fault-Tolerant X gate

Instead of encoding the information of qubits in multiple ancilla, as in traditional quantum fault-tolerance, unitary fault-tolerance encodes the information of the operation in multiple fields and qubits. In this way an ancilla is required per operation of a qubit. The quantum circuit shown demonstrates the ancilla may be reused for subsequent operations without any necessary measurement, due to the coherence of the fields. The variables in boxes represent the control-field independent coefficient of the eigenstate ( $c_i$  in the text) in the left column, while the variables in the attached circles are field-dependent amplitudes for their respective coefficient. Multiple boxes indicate their sum produces the respective eigenstate amplitude. After operations are made, the coefficients evolve but the first-order errant subspace always remains distinct. For simplicity, terms second order in the error amplitudes,  $p$ , have been eliminated from the encoding and decoding operations. The final coefficients verify that a unitary fault-tolerant X gate has been performed: any collapse of field amplitudes lead to the correct qubit angle in the terms shown. Only the dropped terms that were second order in the error amplitude would be incorrect.

the error.

2. The operation's subspace involves terms whose coefficient are identical to the first order in the error, but are only distinguished by error amplitudes.

For example, consider the construction of a unitary fault-tolerant quantum swap. As with the X gate we will need one more ancilla qubit to produce a correctable errant subspace. First, consider a system with individually addressable transitions as in 4.3 but with eight non-overlapping bound states. If such a system exists, one can apply  $\pi$ - pulses as depicted in 4.6. Each pair of arrows in the diagram represent a control field pulse whose frequency bridges the energy corresponding to their head and tail energy states. The green arrows show transitions that obey rule (2) while red arrows obey rule (1). The operational steps are similar to the one qubit gate: encode the operation in the errant of subspace, perform the operation using an exchange between the correct and errant subspace, and then use quantum coherent feedback to correct the errant subspace, hence decoding the operation. Additionally in this diagram, the Hilbert subspace of the correct operation that is identity is exchanged between the correct and errant eigenstates at any point during the operation. In the diagram we have arbitrarily chosen to perform this exchange during the first operation as shown with green arrows. Three simplifications are made in the diagram to simplify the labeling of amplitudes:

1. Only terms first order in error or greater are kept.
2. Because implicit measurement will correctly correlate the separate control pulses to preserve the operational angle unchanged and because the error in each pulse is roughly the same, we denote all the amplitudes with the same variable  $p^i \rightarrow p$  and  $q^i \rightarrow q$ .
3. We assume the error amplitudes are so small that we can approximate  $pq = p$ .
4. Control pulses that may applied simultaneously are done so.

The final coefficients show verify that a unitary fault tolerant swap has been performed correct in the first order of error.

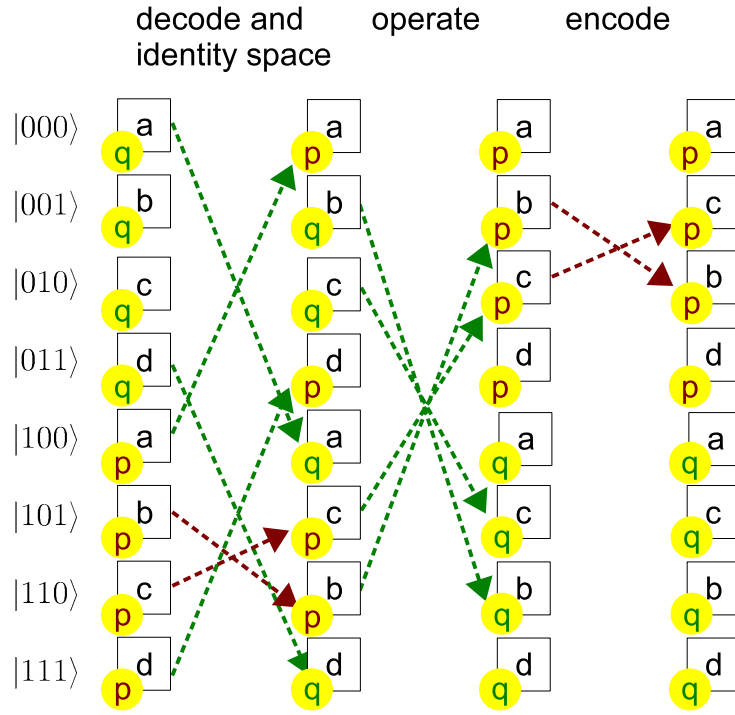


Figure 4.6: Unitary-Fault Tolerant Swap

The arrows also suggest the structure of a quantum system that would require the least number of pulses and the least time to make this quantum swap:

- $\omega_{04} = \omega_{37}$
- $\omega_{16} = \omega_{25}$
- $\omega_{56} \neq \omega_{12}$
- All other transitions energies are not equal to energies above

This two-qubit unitary fault-tolerant construct is similar to the one qubit, since it also performs an X operation, just in the non-local subspace. In general, other multiqubit operations that perform an X operation in some subspace local of non-local (e.g.  $\text{cnot} = P_0 \otimes I + P_1 \otimes X$ ) can be constructed unitary fault-tolerantly in the same way. However having shown that quantum swap works, we may now use the construction of locally-variant bases as presented in our recent work XX to prove that any Clifford operation may performed unitary-fault-tolerantly. The classical

angle at which a qubit basis is oriented may be used to as the rotation parameter in a quantum rotation of another qubit, by performing two swap operations between the qubits that respect a common spacial basis and a common energy basis in succession. Given two locally-variant qubits bases pointing in the  $|S_x, \uparrow\rangle$  and the  $|S_y, \uparrow\rangle$  direction , providing unitary fault-tolerant rotations about the Bloch x-axis,  $R_x(\pi/4)$ , and Bloch y-axis  $R_y(\pi/4)$ , any unitary fault-tolerant, single-qubit Clifford operation can be generated. Adding the above unitary fault-tolerant swap operation between any qubits of the same local basis generates all unitary fault-tolerant, multi-qubit Clifford operations.

The above methods rely on the ability of a quantum system to make particular three and two qubit operations to isolate the subspace of the operation. These operations may not be available for a given quantum system. Traditionally these systems are constructed for arbitrary qubit rotations and a single non-local operation such as iswap or cnot that make the system universal. To make a quantum system compatible with unitary fault-tolerance for general quantum computing, it must allow three-qubit operations in order to correctly apply quantum coherent feedback to errant two-qubit processes. Furthermore, the system must be operationally rich enough to isolate the operational subspace required for each of the three steps: encoding, operation, and decoding. We show that the three-qubit controlled swaps, that are physically natural in many systems, basic two qubit swaps (of both the non-local and local eigenstates) are sufficient to perform the above unitary fault-tolerant swap in Figure 4.7. In the general, of the available operations in a quantum system, only the operations that reduce to point transformations on the eigenkets are obey the two rules. It something of a transformation puzzle then to see if the analogy of a conjugacy class of these point transformations, subject to pairing only when then obey the aforementioned two rules, contains the quantum swap. If it does then the system is amenable to unitary fault-tolerance.

The use of locally-variant bases may be not necessary hidden degrees of freedom are allowed. For example, if the  $\pi$  pulses in 4.5 were replaced with  $\pi/2$  pulses, then then ancilla qubit could be used as the system qubit having undergone a Hadamard transform. Although such an example requiring hidden degrees

of freedom only increases the space complexity of an algorithm polynomially, they make them practically impossible. Likewise alternate system can be constructed using dynamical decoupling by measurement: the measurement collapses common errant amplitudes to rescale the collapsed angle. We have ignored these system of operation, since the current temporal cost of making highly precise measurements in the middle of a long quantum circuit is often catastrophic. Continuously applied traditional quantum error correction would again be required. Instead by neither requiring any measurement in the quantum circuit, nor adding more than one-qubit of ancilla per system qubit (even fewer can be used with added time-complexity), nor increasing the number of operational time steps beyond six, the requirements of unitary-fault tolerance are practically feasible.

#### 4.1.7 Multi-Ordered Reduction in Fidelity-Loss

Though several exist, the quantum circuits shown all obeyed the same algorithm for construction: perform the operation in the errant subspace first, exchange the amplitudes between the errant and correct subspaces, and then apply quantum coherent feedback to the correct subspace. This method can be extended to apply error correction to the  $n$ -th order, by using  $\log_2(n - 1)$  ancilla qubits. Again non-fault tolerant operations may be used to construct the method as long as they each obey one of the following rules:

1. The operation's subspace involves terms that are all at least  $n$ -th order in the error amplitude.
2. The operation's subspace involves terms whose coefficients are identical and whose leading error amplitudes terms are separated by one order at most.

One method is of satisfying these rules for any desired operation is to construct quantum operations that follow the inductive ordering below:

1. Perform the desired operation on terms that are  $n$ -th order errant amplitude subspace.

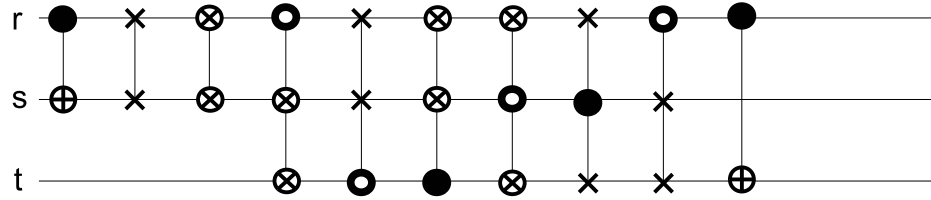


Figure 4.7: Unitary fault-tolerant quantum swap using only: controlled swap and controlled outer-swap of the three qubit operations

When  $|r\rangle$  is the ancilla qubit and  $r = 1$  is an errant amplitude subspace, the above quantum circuit perform a unitary fault-tolerant swap between the qubits  $|s\rangle$  and  $|t\rangle$ . This net operations is performed using 10 operations of a restricted pool: controlled-local-swap, controlled-swap, local-swap, swap, cnot, where each controlled operation may operate in the control 0 or 1 subspace.

2. Perform operation(s) that exchange like coefficients between the  $n$ -th order errant amplitude subspace and the  $n - 1$ -order errant amplitude subspace.
3. Perform the desired operation in the  $n$ -th errant amplitude subspace.
4. Perform the above steps for  $n - 1$  instead of  $n$

Using large  $n$  is impractical and unnecessary as  $n+2$ -qubit operations are necessary for general quantum computing and a portion of the operation's errors in non-unitary.

#### 4.1.8 Summary

Given the ubiquitous trade-off in quantum computing systems between the isolation of the system and the precision and speed of quantum operations upon it, our method suggests that systems supported by strongly coherent control fields should be strongly isolated. The resulting imprecision in the operations may be corrected to an appropriate order using unitary-quantum fault tolerance, where unitary refers to the otherwise external degrees of freedom (the modulator) controlling the error in the control field. To show this, we first presented a new formulation of noise by eliminating the traditional semi-classical approximation of field-modulation, and eliminating qubit collapse as a semi-classical approximation of partial field collapse. This theory suggests an extreme sensitivity to pulse shape

for scalable quantum computing and the existence of an optimal coherent state whose resulting operational precision that is much lower than simple under-over rotation channels hypothesize. We have shown that a system can be made unitary-fault tolerant using one reusable ancilla bit for every operation. The construction presented uses implicit measurement of qubits at the end of the circuit to collapse multiple control fields into a correlated form, and it conceptually advances re-coherence techniques to control fields rather than just qubits. When applicable, this unitary fault tolerance vastly simplifies the resource requirements otherwise needed by general fault-tolerant operation.

Systems already capitalize on isolating qubits that are not involved in the current quantum operations, the so-called “parking” of qubits. By decoupling them further from any quantum fields their “relaxation” time is shown to improve dramatically. The methods of this paper apply to the same principle to quantum states within a statistical hierarchy of error amplitudes. No states that are not second order in error are used as intermediaries to connect other states. Eigenstates that cannot be corrected with coherent feedback are not involved in the operation. In the final section, we have shown that this hierarchy can be extended to arbitrary levels at the cost of commensurately more complex qubit operations.

Finally, the experimental setups suggested allows a fundamental probe of the quantum nature of phase angles and the assumptions of S-matrix scattering theory. Conventional quantum descriptions of field-matter interaction describe field-modulation as a semi-classical perturbation to a steady-state field. However, the quantum states of matter are simply projections of the information contained in the modulation of their interacting fields. Given the non-commutation of the phase variables  $C$  and  $S$  there is a bound on how granular one may speak of interaction time and operational phase. In this projection, a steady-state field, where the interaction time is completely uncertain, contains no information. On the other hand, a modulated field may encode information commensurate with the precision of modulation. Wheeler and Feynman interpret quantum fields to be simple information-storing structures that preserve only enough information of their past dynamics to predict their future dynamics. We have shown how the information

content of a qubit is so bound by the precision of its control field and its implications for quantum computing. Because in this sense the measurement of a qubit is a partial measurement of field, one may directly correct the control field by feedback, rather than the qubits by measurement.

## 4.2 High-Precision quantum operation using quantum coherent feedback and locally-variant bases

We present a method by which general quantum operations may be performed with a precision greater than the direct noise-limit of their operating fields. We derive a theory for the transformation of macroscopic, classical precisions to microscopic, quantum precisions by generalizing quantum coherent feedback and quantum amplitude amplification to locally-variant bases. This method affords a simple construction of the quantum Fourier transform using only Clifford group operators on a quantum system in its ground state, and generalizes amplitude amplification to quantum operations rather than quantum states. We present a simulation of our theory for arbitrary single qubit rotation to illustrate the precision gain in systems that are not already dominated by environmental noise, and we explain how this discovery may directly lead to the scaling of quantum computers.

### 4.2.1 Introduction

Frequently proposals for quantum computing systems discuss the fidelity with which one may manipulate a small number of quantum bits (qubits) of information, but “pay lip-service” to the fidelity required for scalable quantum computing [2]. Scalable quantum computers are analog machines requiring quantum operational precisions that scale up, irrespective of quantum decoherence, with the number of quantum bits used in the operation [13]. Extensive research to quantify this relationship between the number of computational qubits and the analog precision required for useful quantum algorithms, find that operations with a fidelity greater than  $\cos^2\left(\frac{\pi}{512}\right) \cong 0.9999$  are needed to scale to thousands of qubits [13, 21, 20],

with quantum approximation algorithms. Years of research to engineer such precisions, have considerably improved operational fidelities. However still requiring orders of magnitude reduction in their lost fidelity to meet this bound, they are still grossly lacking [36] for the precise small rotation operations required, on otherwise scalable quantum systems. For example, previous work to create these high precision operations are limited to deterministic errors [35, 31, 38, 27], require multi-qubit encodings that still require high-precision pulse timings [17] or combat decoherence rather than imprecision [5], or use ensemble statistics that are not scalable [45]. In general, the degree by which one qubit is made to rotate another determines the desired strength of coupling between the two qubits. When enough errant coupling is introduced into the system to overshadow the required coupling, quantum chaos ensues and precludes any useful measurements [23, 22]. This errant coupling threshold drops algebraically or exponentially with the number of qubits depending on the computational regime [49]. Attempting to address this operational imprecision, many researchers have constructed a fault-tolerant quantum gate formalism [41, 30, 25]. While fault-tolerance may be useful for certain operations (particularly for a subgroup of “digital-like” quantum operations called the Clifford group [25]), these methods are unhelpful for the precise quantum rotations of arbitrary angles when a system’s operational fidelity is far from the error correction threshold (see Section 4.2.2). The most optimistic threshold calculations require greater than 0.9999 operational fidelity to produce any information gain for these precise analog quantum operations [20, 3]. And so, the scaling of quantum computers has reached an impasse due to the precision of quantum operations: fault-tolerant quantum operations do not improve quantum information unless quantum operations are already precise enough to make them almost superfluous, by way of quantum approximation algorithms. To break this impasse, a means of ultra-precise quantum operation in the midst of stochastic-noise must be found.

We present a method, generalizing the theory of quantum coherent feedback pioneered by Seth Lloyd [34] and of Amplitude Amplification by Grover, Brassard, and Hoyer [26, 6] amongst others, that enables precise, general quantum opera-

tions despite noisy fluctuations in control field interactions. In short, instead of applying a quantum field that is the generator of the desired operation, coherent feedback from the classical bases, not the specific quantum states, of qubits direct the transformation of a quantum state to its rotated state by amplifying the desired operation. Because quantum operations are linear, quantum signal-to-noise ratios usually scale linearly, tracking the error of the control field. This is in stark contrast to the exponential signal-to-noise ratios found in the non-linear transistors used for classical, digital computation. Quantum error correction introduces non-linearity through measurement, but as mentioned above the domain of its improvement is practically limited. In recent work [16] examining the actual quantum measureables of an operation, we show that these difficulties are very fundamental in nature. This contrasts with the fidelity improvements in the initialization and measurement [4] of a quantum system in which macroscopic, non-linear processes are available and essential. Because quantum bases can be aligned with non-linear, classical precisions (e.g. the angle of an optical polarizer, or the direction of ferromagnet), our method of basis-directed feedback leads to a dramatic increase in possible quantum precisions for a pseudo-continuous set of operations in the full Hilbert space.

In Section 4.2.2, we identify the dominant sources of error for most quantum systems and identify the practical shortcomings of traditional quantum fault-tolerant design. Next in Section 4.2.3, we present the means by which coherent feedback in locally-variant bases may convert classical information into quantum information with a precision that exceeds the quantum operating field used. In Section 4.2.5, we show that this feedback may be generalized into nearly arbitrary quantum operations in a practical way using only Clifford operations. We present simulation data and methods for a precise single qubit rotation and finally summarize our conclusions in 4.2.6.

## 4.2.2 Precision Errors in Quantum Circuits

Most research work on decoherence focuses on system-environment noise and has produced several well-accepted models. If information that is second order or

less in the wave-function (e.g Hermitian observables) is of interest then the Lindblad master-equation [33] describes the evolution of density matrices subject to instantaneous, indirect measurements of a system through its average environmental coupling in the reduced matrix. If higher-order information degradation is to be modeled, then a stochastic Schrödinger equation may be used in conjunction with such an approach [7]. Both models rely on the short-bath-correlation-time approximation and characterize decoherence and relaxation due to system-environment interactions by two phenomenological times  $T_2$  and  $T_1$ . However according to these models, for many quantum systems, this system-environment Markovian noise is too small to decohere the system beyond the 0.9999 threshold fidelity alone. For example, consider a system where qubits are represented as electrons in singly-doped, self-assembled quantum dots whose energy states are split by a magnetic field, whose control is afforded by optically-induced Raman transitions, and whose environmental interaction is dominated by the quantum dot's nuclear spin-bath[19]. In this system,  $0.1 - 1.0ns$  laser pulses can perform  $\pi$ -rotations [19], while transverse spin decoherence works on the order of  $10\mu s$  at low temperature[47]. Moreover, coherence of such a qubit can be restored on much longer timescale by controlling system-bath dynamics [48]. Then, a straight forward calculation reveals that operations applied to this system, with this noise alone, will not loss more than  $10^{-6}$ . Later, we will refer to this same quantum system to show the same low-decoherence system can also impart high-precision operations.

The other source of noise then, is the coherence and precision of the quantum control field. Previous work [35, 31, 38, 27] has shown that static operational errors like inhomogeneous magnetic fields can be corrected when systematic. However, when the error is in dimensions external to the total Hilbert space (e.g. the state and modulation of a laser) the interaction terms appears stochastic when these dimensions are traced over. To solve this problem, fault-tolerant gates attempt to indirectly measure the control field and collapse errant subspaces or redirect them.

### 4.2.3 Quantum Coherent Feedback with Locally-Variant Bases

Most systems considered for quantum computing have qubits that are oriented in the same direction, that is to say the total, global computational eigenbasis of these systems is simply a tensor product of identical, local basis vectors. “Locally-variant” bases refer to basis vectors that vary from one qubit to another due to a varying of the axes of asymmetry creating their local energy splittings, e.g. in the above quantum dot system, this could mean magnetic fields pointing in different directions. These locally-variant bases naturally give rise to two different, complete computational bases: a computational basis,  $\mathcal{Z}$ , containing spacial eigenkets invariant to the angles of asymmetry, and a computational basis,  $\mathcal{E}$ , containing energy eigenkets that are intrinsically parametrized by the locally-variant angles of asymmetry. A particular physical interaction between the qubits may can be represented in either basis, but the selection rules of the interaction term can only correspond to one basis.

While an external control field may initiate and terminate interactions between qubits, it need not select the operation to perform. In his original work on quantum coherent feedback, Lloyd formalized a means by which one set of quantum bits could “sense” and coherently encode the operation to be performed on other qubits. In this section, we present an alternate formulation where the locally-variant bases of a set of quantum bits can encode the desired quantum operation given their precise initializations. In the next section, we show that this operation may be coherently selected from a quasi-continuous domain of possible operations in  $U(2^n)$  where  $n$  is the number of qubits. Hence, we demonstrate quantum control need only involve operations from the Clifford group, which have a natural resilience to quantum noise, to be satisfactory for quantum computing. These “digital” operations can be made fault-tolerant at a practical level today, as is shown in [16]. Previous attempts to digitize quantum computing either are simply too rigid to be useful as suggested by the Gottesman-Knill theorem, use multi-bit encodings that continue analog precisions in timing, or require a pool of temporal and spacial resources that are not workable given the decoherence times of quantum systems. The Gottesman-Knill theorem states that with initialization and measurement in

one computational basis, quantum operations limited to the Clifford group in that basis can provide no exponential speedup over classical computation. However we now show when there are two computational bases, even though all operations may be limited to Clifford operations in a local basis, these operations can in fact can implement Shor factoring and Grover search globally.

Consider without loss of generality the above quantum dot model with two different qubit operations,  $H_1^Z$  and  $H_2^E$ , driven respectively by  $\pi$ -pulses,  $\gamma_1$  and  $\gamma_2$ . Given two qubits,  $i = \{1, 2\}$ , each within their own local two-level basis,  $|E^i, \uparrow\rangle$  and  $|E^i, \downarrow\rangle$ , we have the Hamiltonian

$$H = H_0 + \gamma_1(t) H_1^Z + \gamma_2(t) H_2^E \quad (4.35)$$

$$H_0 = \prod_{i=\{1,2\}} \sum_{j=\{\uparrow,\downarrow\}} E_j^i |E^i, j\rangle \langle E^i, j| \quad (4.36)$$

$$H_1^Z = \frac{i\Omega_1}{2} [|\hat{z}_+\rangle |\hat{z}_-\rangle \langle \hat{z}_-| \langle \hat{z}_+| - |\hat{z}_-\rangle |\hat{z}_+\rangle \langle \hat{z}_+| \langle \hat{z}_-|] \quad (4.37)$$

$$H_2^E = \frac{i\Omega_2}{2} [|E_+^1\rangle |E_-^2\rangle \langle E_-^1| \langle E_+^2| - |E_-^1\rangle |E_+^2\rangle \langle E_+^1| \langle E_-^2|] \quad (4.38)$$

where  $H_1^Z$  performs a quantum swap relative the spacial  $\hat{z}$  bases while  $H_2^E$  performs a quantum swap relative to energy levels of  $H_0$ . The  $+$  and  $-$  subscripts refer to ladder operators within the referents' bases. If the magnetic field of the first qubit is oriented in the  $+\hat{z}$  direction and the field of the second qubit is oriented at an polar angle  $\alpha$  to the  $+\hat{z}$  direction, we can rotate the quantum ground state of the first qubit by applying  $\pi$ -pulses to  $\gamma_1$  and  $\gamma_2$  consecutively in 4.39, where  $t_1 = \Omega_1 t / \pi$  and  $t_2 = \Omega_2 (t - t/t_1) / \pi$  apply sequential operations to the original wavefunction.

$$\exp \left[ -\frac{i}{\hbar} (H_0 + \gamma_1(t_1) H_1^Z + \gamma_2(t_2) H_2^E) \right] \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \frac{\theta+\alpha}{2} \\ \sin \frac{\theta+\alpha}{2} \end{bmatrix} \begin{bmatrix} \cos \frac{\alpha}{2} \\ -\sin \frac{\alpha}{2} \end{bmatrix} \quad (4.39)$$

It should be noted that the  $H_0$  term creates local precession in each qubit which do not necessarily commute during the time of operational pulses. Although these processes have been ignored in (4.39), in Section we show that these effects are

negligible to the overall fidelity under the system properties assumed in 4.2.2 or when  $\theta$  is small. Of course the square pulses,  $\square(t_1)$  and  $\square(t_2)$ , are not physically possible and generate off-resonant interactions. But the Area theorem allows for the appropriate pulse-shaping to minimize the frequency bandwidth, while other unitary error may be corrected as in [16]. We can rewrite 4.39 using a convenient short-hand where we enumerate which eigenbasis is being used in the subscript of the operation and parametrize eigenkets confined to the  $Z-Y$  great circle of a local Bloch sphere with it's azimuthal angle:  $|\theta + \alpha\rangle_z |\alpha\rangle_E = \text{Swap}_E \circ \text{Swap}_z |\theta\rangle_z |0\rangle_E$ . This process is illustrated in 4.8 where the local energy basis of one qubit is denote with the subscript  $z$  and the other with the subscript  $\alpha$ . It is important to note that unentangled initial states stay unentangled after any such operation. In this sense, each qubit acts like a coherent controller of the other qubit rather than comprising a combined Hamiltonian.

For a physical example of systems that have interactions in both the global basis,  $\mathcal{Z}$ , and local basis,  $\mathcal{E}$ , again consider the quantum dot system described in [19]. Such a scheme can use an photon-mediated, polarization-exchange interaction to perform a swap in the local basis. But it may also use a virtual exciton-mediated electronic exchange interaction to perform a swap in the global basis [37]. In the former, transitions to and from a trion-state,  $|T_+^i\rangle$ , from an excited local energy state,  $|E_+^i\rangle$ , absorb or emit a vertically-polarized photon,  $b_V^\dagger$  in each qubit,  $i$ , as shown in (4.40). In the later, a laser pulse broad enough to couple both the spin-up and the spin-down energy states of each qubit,  $S_1$  and  $S_2$  to a continuum of free exciton states in the host structure is applied. This allows these virtual, delocalized exciton states to broaden the extent the quantum dots' electronic wavefunction and mediates an interaction based on the energy of anti-symmetrization. This term appears as a Heisenberg exchange interaction as in (4.41). Physically, one may think of the selection rules governing the exchange interaction as independent of local magnetic fields, while the selection rules governing the emission or absorption of a photon are intimately tied to the construction of local energy levels. In this way, virtual particles that mediate interactions between spin are different than virtual particles that mediate interaction between energy-eigenstates. This con-

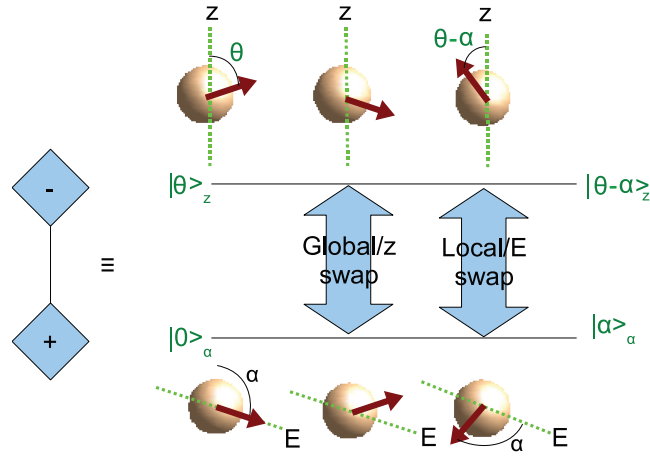


Figure 4.8: Precision rotation through feedback of locally-variant basis

To add a classical angle,  $\alpha$ , to a quantum angle,  $\theta$ , one can twice swap two quantum dots whose energy splitting asymmetries (magnetization bases) may be at very precise relative angles, illustrated by dotted lines. Through each of these idealized swap operations, the qubits are completely unentangled and their states may be represented on two separate Bloch Spheres. The first swap respects a common spacial basis, for example the optical  $z$ -axis, while the second swap respects the local energy bases of each respective quantum dot. The control qubit (in the  $\alpha$  basis) is assumed to be precisely initialized to its local ground state, as denoted by the alignment of its Bloch vector with its local asymmetry axis. The system qubit is oriented at some arbitrary, unknown angle,  $\theta$ . We have defined this “double-swap operation” with diamonds in place of direct product symbols. The text “+” and “-” are only a diagrammatic tool to denote the order of swap operations between the two qubits. If we were to reverse the operation, do an local swap followed by a global swap, the + and - would flip positions in the swap symbol as well. The non-local phase associated with each fermion loop is easily collapsed by any disentangling measurement the “control” qubit  $|\alpha\rangle_\alpha$ . Likewise, this example decouples the decoherence rates of the control qubit and the system qubit as the two qubits are completely disentangled.

ceptual framework can then be generalized to other quantum systems where two independent quantum degrees of freedom form a two-level system, but interactions can select either independently.

$$H_1^{\mathcal{E}} = \sum_{i=1,2} b_V^\dagger |E_+^i\rangle \langle T_+^i| + b_V |T_+^i\rangle \langle E_+^i| \quad (4.40)$$

$$H_2^{\mathcal{Z}} = -JS_1 \cdot S_2 \quad (4.41)$$

Using this example, and the novel error-correction scheme in recent work [16] it is a small step to theorize a scalable, fault-tolerant quantum Fourier transform circuit to an almost arbitrarily large number of  $n$  qubits using less than 21 additional qubits and three-qubit interactions. The seven controlled rotations per qubit required for the transform can implemented using the above described doubled swaps with a third qubit controlling each interaction. This type of three-qubit interaction is physically natural as one qubit may dress the interaction medium as resonant or non-resonant with the desired operation [46]. Then, each of seven control qubits, initialized to a binary span of seven angles  $\otimes_{i=1}^7 |0\rangle_{\pi/2^i}$ , may conditionally rotate by its basis angle one of the  $n$  qubits controlled by the appropriate neighbor qubit of the total  $n$ . These seven conditional rotations applied to any one of the  $n$  qubits is illustrated in 4.9. Additionally, each control qubit need contain only two qubit ancilla to isolate and correct errant amplitudes as described in [16]. The system will be detailed in later work [15].

This system possesses two properties that enable coherent feedback. The interactions  $H_1$  and  $H_2$  diagonalize the system's Hamiltonian into two different computational bases. The asymmetries admitting these two different computational bases is parametrized by an  $SO(3)$  group of operations to transform one computational basis into another. By alternating between representations, we have shown one type of  $SU(2)$  operation can be effected using only quantum swaps. It is clear that by aligning the local basis angle,  $\alpha$ , in the  $X - Y$  great circle at  $|S_x, \pm\rangle$ , the Hadamard transform may also be effected for quantum states on the  $Z - Y$  great circle. The natural question in quantum information processing then

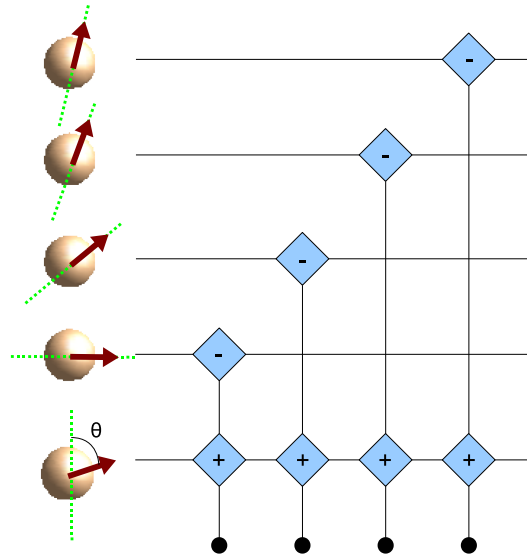


Figure 4.9: Simple Quantum Fourier Transform

The four top qubits depict four of the seven required control qubits and are oriented at binary fractions of  $\pi/4$  to encode the operation required at any time. That fact that these operations are all quantum swaps facilitates practical fault-tolerant incarnations of the operations. Additionally, given that the limited error is in the operational pulse and is largely unitary, we can use a very simple method of fault tolerance

arises: what is the density of encodable states in the Hilbert space reachable from a polynomial number of fault-tolerant operations? Traditionally, this question is answered by quantum information theory, where common, static bases are usually used to sum the encodable information of a quantum system. However in the above method, the quantum state really amounts to a intermediary to allow classical control of a quantum degree of freedom with negligible environmental coupling due to a steady-state field instead of a modulated one (fluctuations in  $B$  field can be kept to a minimum). These angles are classical variables,  $SO(3)$ , and do not form an eigenbasis for the system,  $SU(2)$ . So instead, the question is answered using quantum controllability, that is by identifying the maximal subspace of the Hilbert space that is “controllable” using only a linear number of operations from the Clifford Group with respect to these two different computational basis. In the above example the zero-information quantum state,  $|0\rangle_E$ , is simply used as a control qubit while the state  $|\theta\rangle_z$  is used as a system qubit. Considering a Hilbert space given by  $n$  system qubits and  $m$  control qubits, in the Section 4.2.5 we show at least  $\prod_{i=1}^n 2^{m_i}$  points are completely controllable in the systems Hilbert space, where  $\sum_{i=1}^n m_i = m$ . Because the number of points scale exponentially with the number of control qubits, and because quantum algorithms are generally bound by the same scaling law, this number is sufficient and practical to maintain the spacial and temporal computational bounds of any quantum algorithm.

#### 4.2.4 Operations between Qubits with Non-commutation Precession

When performing a swap operation between qubits whose bases have a local phase between them, we calculate the fidelity the system’s evolution under a common control field. In particular we calculate it’s fidelity with an operation between qubits of a common basis where the local phase is now the initial quantum state of one qubit. Because the Larmor precession of these two two-level systems do not commute, only a subset of quantum operational dynamics between the two level systems will have high fidelity. Intuitively, if the operations if fast enough then the precession should have little effect. The question is how fast does the operation

need to be with respect to the rate of Larmor precession. The other question is how does that speed depend on the phase angle between the two bases. For the quantum swap operation, we calculate greater than 0.9999 fidelity for pulse speeds 300 times faster than the Larmor precession—or for any speed, angles smaller than  $\frac{\pi}{5}$ .

If we begin with two qubits each within their own local two-level basis, we must start with a generalized Hamiltonian (4.42).

$$H = \mu \mathbf{S}^1 \cdot \mathbf{B}^1 + \mu \mathbf{S}^2 \cdot \mathbf{B}^2 + \frac{i\Omega}{2} [|\hat{z}_+\rangle |\hat{z}_-\rangle \langle \hat{z}_-| \langle \hat{z}_+| - |\hat{z}_-\rangle |\hat{z}_+\rangle \langle \hat{z}_+| \langle \hat{z}_-|] \quad (4.42)$$

where the control field (to swap the two qubits) references a common  $\hat{z}$  spacial direction. So in this common basis,  $\mathcal{Z}$ , (and not the local bases) the generator of the above swap interaction appears without any local angle as in 4.43.

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.43)$$

We orient the first qubit to have magnetic field oriented in the  $+\hat{z}$  direction, the second qubit's field is oriented at an angle  $\theta$  to that direction, to produce the Hamiltonian in 4.44.

$$H = \mu S_z^1 B + \mu S_z^2 B \cos \theta + \mu S_y^2 B \sin \theta + \frac{\Omega}{2} G \quad (4.44)$$

Going into the co-rotating frame  $\psi' = e^{-iS_\theta^2 \omega t} e^{-iS_z^1 \omega t} \psi$ , we examine the effect of the local basis on the swap operation 4.45.

$$\partial_t \psi' = i \frac{\Omega}{2} \left( e^{-i\omega S_z^1 t} e^{-i\omega S_\theta^2 t} [G] e^{i\omega S_\theta^2 t} e^{i\omega S_z^1 t} \right) \psi' \quad (4.45)$$

The equations of motion are then given by  $\partial_t \psi' = i \frac{\Omega}{2} Q G Q^\dagger \psi'$ . Using the matrix form of exponentiated spin operators, we can simplify the operator  $Q$  as:

$$Q = \begin{bmatrix} e^{-i\frac{\omega}{2}t} & 0 \\ 0 & e^{i\frac{\omega}{2}t} \end{bmatrix} \otimes \mathbf{1}. \quad (4.46)$$

$$\cdot \mathbf{1} \otimes \begin{bmatrix} \cos \frac{\omega t}{2} - i \cos \theta \sin \frac{\omega t}{2} & -\sin \theta \sin \frac{\omega t}{2} \\ \sin \theta \sin \frac{\omega t}{2} & \cos \frac{\omega t}{2} + i \cos \theta \sin \frac{\omega t}{2} \end{bmatrix} \quad (4.47)$$

We numerically solve this differential equation across a sweep of the variables  $N$ ,  $M$  given by  $\theta = \frac{N\pi}{256}$  and  $\frac{\Omega}{\omega} = \frac{1000}{M}$ . We plot the Fidelity in 4.10 at each point between an evolution by the ideal  $e^{-iG\frac{\pi}{2}}$  and the non-ideal  $e^{-iQGQ^\dagger\frac{\pi}{2}}$  (which is just the overlap because all states are pure). If the rotation angle is small then operational speed is irrelevant. If the pulse speed is 1000 times faster than the Larmor precession frequency (as in our trion-trion operations) the rotation angle is irrelevant. The contour regions are almost circular, but the elongations rotate due to the periodic influence of  $\theta$  with pulse time and Larmor frequency.

#### 4.2.5 Amplitude Amplification with Locally-Variant Bases

In this section, we show how one may create exponential parallelism in the use of base angles in much the same way phase kick-back creates parallelism in the quantum state. The seed step in this process we term a “quantum-walk on a Bloch sphere” due to it’s analogy with quantum-walks on lattices, where an exponential number of lattice steps may be taken in parallel in linear time [1]. The fundamental quantum-step in this quantum-walk on a Bloch sphere is akin to an S-matrix scattering event. We construct a qubit-operation,  $A_2$  which performs this scattering between two angles (4.48).

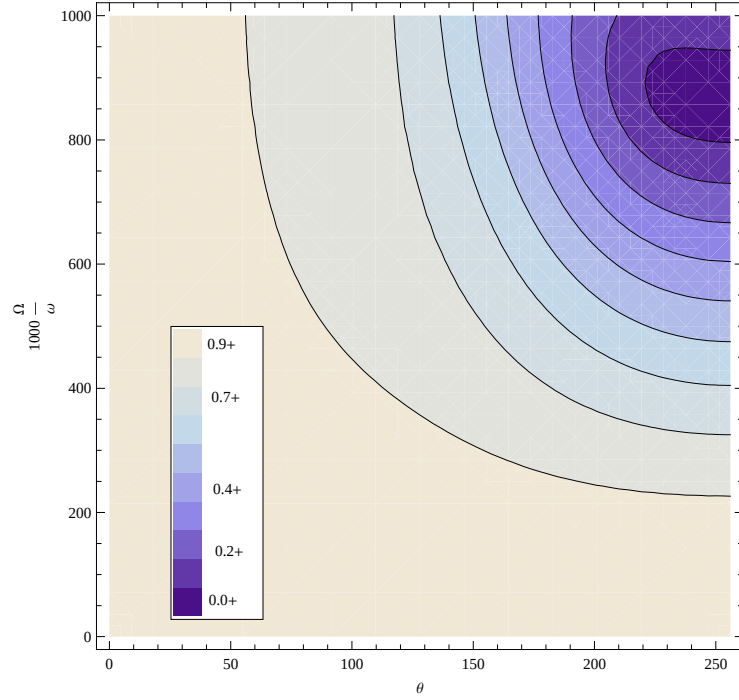


Figure 4.10: Fidelity for swap in  $\mathcal{Z}$  basis with local Larmor precession  
 For pulse speeds 300 times faster than the Larmor precession—or for any speed, angles smaller than  $\frac{\pi}{5}$ , the affect of non-commutativity of the local precession with the global operation does not cause Fidelity loss beyond 0.9999. This suggests that a fast global operate. The vertical axis plotted for the ratio of the local procession to the global operational time  $\omega/\Omega$  ranging from .001 to 1, while the horizontal axis is the angular difference between the local basis vectors

$$A_2 |\theta_0\rangle |\theta_1\rangle \equiv \exp \left( i \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \frac{\pi}{4} \right) |\theta_0\rangle |\theta_1\rangle \quad (4.48)$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \cos \frac{\theta_0 - \theta_1}{2} \\ \sin \frac{\theta_0 - \theta_1}{2} \\ \sin \frac{\theta_0 + \theta_1}{2} \\ \cos \frac{\theta_0 + \theta_1}{2} \end{bmatrix} \quad (4.49)$$

This operation is based on the  $\pi/4$  rotation of the Pauli-Y operator in both the local and non-local basis. By their definition, the Clifford operators include the Hadamard and controlled-not gates. We show the operation  $A_2$  is decomposed into these operators in (4.12) and hence it is a Clifford operation that can be easily made fault-tolerant. This does not mean that the gate need be implemented as in Figure (4.12), but rather that a simple fault-tolerant system of partial measurements is available to correct the operation. For example, the seven-qubit Calderbank-Shor-Steane (CSS) code [8, 44, 43] can fault-tolerantly implement any Clifford Gate transversally. Given a particular noise structure and/or scaling requirements more economized codes like [39] are available as well.

When performed pairwise on a consecutive set of  $m$  qubits, the scattering operation,  $A_2$ , generates all  $M = 2^m$  permutations of adding and subtracting the angles  $\{\theta_i | i \in (1 \dots m-1)\}$  to the first angle  $\theta_0$ . Or more formally, if we represent the Hilbert space vectors with the binary label  $j = j_{m-1} \dots j_1 j_0$  as commonly represented in the Quantum Fourier Transform, we can write the operation as

$$A_m \prod_{i=0}^{m-1} |\theta_i\rangle = \sum_{j_1=0}^1 \sum_{j_2=0}^1 \dots \sum_{j_{m-1}=0}^1 \cos \frac{\xi_j + \theta_0}{2} |f(j)\rangle + \sin \frac{\xi_j + \theta_0}{2} |g(j)\rangle \quad (4.50)$$

where  $\xi_j = (-1)^{j_n} \theta_m + \dots + (-1)^{j_1} \theta_1$  and the functions  $f$  and  $g$  map integers to two particular Grey Codes. When done fault-tolerantly, the amplitudes for every

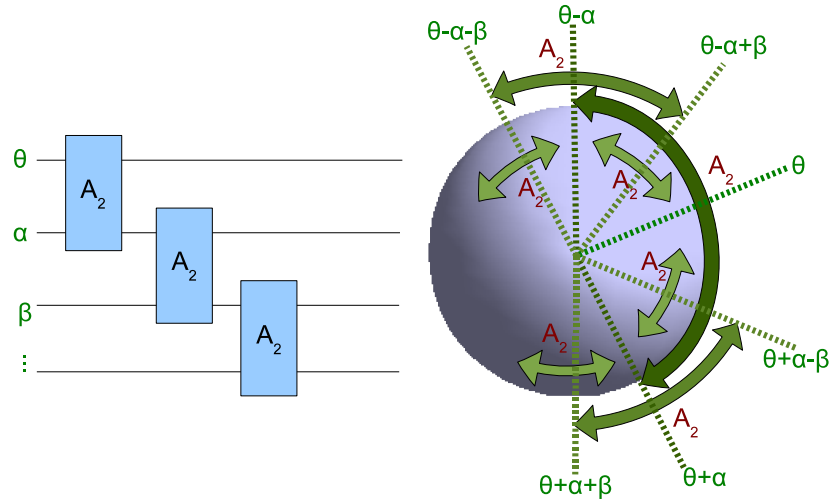


Figure 4.11: Quantum walk on Bloch Spheres

An exponential number of rotation operations are performed in the full Hilbert space of three qubits using three operations. The outcome is a traversal of four rotations on four topologically disconnected Bloch spheres that have been drawn overlaid to illustrate the quantum walk from the original angle  $\theta$  to  $|\theta - \alpha - \beta\rangle$ ,  $|\theta - \alpha + \beta\rangle$ ,  $|\theta + \alpha - \beta\rangle$ , and  $|\theta + \alpha + \beta\rangle$  in parallel.

pair of eigenstate( $|f(j)\rangle, |g(j)\rangle$ ) can be parametrized by a point on a Bloch sphere. Therefore, the resulting wavefunction is an exponential traversal of rotations on  $\frac{M}{2}$  topologically disconnected Bloch spheres. A picture of this traversal, parametrized by the angles  $\theta_i$ , is illustrated on overlaid Bloch spheres in Figure 4.11. Because the number of these angles resolved is exponential in the number of operational steps taken, it turns quasi-continuous in short order. Just as phase-kickback creates a superposition of all eigenvectors with equal amplitude suitable to algorithms that probe oracles, a quantum walk on a Bloch sphere creates a superposition of eigenvectors with amplitudes used for the Hidden Subgroup Problem (including Shor factoring) and its derivatives. From the disconnected Bloch sphere interpretation, the wavefunction is restricted to an  $SU(2^{m-1})$  subspace of the total Hilbert space,  $SU(2^m)$ . Any vector that lies outside of this Hilbert subspace is errant, so this construction suggests a secondary system of fault-tolerance may be possible for a certain set of errors.

In order to amplify completely one disconnected Bloch sphere in a linear number of steps, we now present an extension of the quantum amplitude amplification

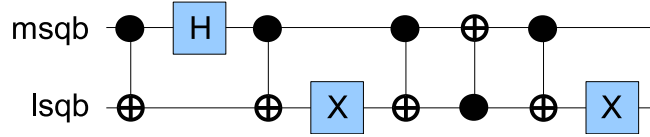


Figure 4.12: Clifford Decomposition of  $A_2$

Clifford operators may be implemented easily as fault-tolerant gates. Small-rotational gates on the other hand are infamously unwieldy in both space and time resources. This figure shows that the fundamental quantum operation in the quantum walk on a Bloch sphere, is a Clifford operation. In the diagram *msqb* and *lsqb* refers to the most and least significant qubit respectively. The operation  $A_2$  is used to create a dense superposition of precise rotation operations without performing any rotations. The gate may also be implemented as  $A_2 = (I \otimes X)^\dagger (H \otimes I) (I \otimes X)$

algorithm. This is an extension in three ways: (1) we use amplification to search for a quantum operation rather than a quantum state (2) we initialize the amplification from a Hilbert space of locally-variant bases in the zero quantum state instead of a global zero basis and (3) we implement the amplitude search “differentially” to produce a significant immunity to a type of quantum error. This method is illustrated in Figure 4.14. In this Figure two control qubits,  $m = 2$ , are used to search for and perform an exact rotation operation,  $R(\alpha + \beta)$ , applied to the quantum state,  $|\theta\rangle$ , using operations solely from the Clifford group by using new quantum operations instead of the oracle call and Grover call to amplify the angle  $\alpha + \beta$ . The number of qubits may be extended arbitrarily as with the amplitude amplification algorithm, but the depth of the circuit will increase linearly. The term “differentially” is used in the sense of traditional circuit design, where the input is a symmetric difference between two analog signals. In the quantum analog, the input to the search is two qubits with the state  $(|-\theta\rangle, |\theta\rangle)$  and output are the two qubits  $(|-(\theta + \alpha + \beta)\rangle, |\theta + \alpha + \beta\rangle)$ . We show that we may eliminate any common-mode noise between the two differential tracks including under-over rotation.

The first step is to generate a differential set of angles labeled “Seeding the table”. In one differential track all the qubits share a common local and global basis, while in the other the control qubits each have a different orientation angle. As in Figure 4.8, if  $\theta$  equals zero, a swap in the global  $\mathcal{Z}$  basis produces the differential signal for  $|\alpha\rangle_E$ . Hence, swapping each differential pair of control qubits

creates a differential set of angles in each control qubits as seen in their local bases. Next, the scattering operation,  $A_{m+1}$  is applied between the initial unknown wave function and all the control qubits in each differential track is applied in the local energy bases,  $\mathcal{E}$ , where the control qubits appear differential signals. In this fashion, we parametrize a repository of  $2^m$  angles with the ground state of  $m$  locally-variant qubits as explained in the quantum walk above. The “angle selection” step is a derivative of the Oracle call prescription used in the Grover algorithm, but in this case the sub-circuit flips the phase of the classically-known, desired operation. Although one angle,  $\theta + \alpha + \beta$ , is selected in this example by a controlled-controlled- $Z$  operation, a coherent superposition of selections could create a coherent superposition of operations with very high operational complexity. As necessary in amplitude amplification, the selection process is followed by the Hermitian conjugate of the quantum walk and seeding operations. This is labeled a “reverse walk” that inverses the initial superposition. All of the above operations take place in the energy basis, as only the seeding operations and Hermitian conjugate of the seeding operations need take place in the global, spacial basis.

In the next step, the differential nature of the algorithm may be exploited to replace the traditional Grover call. The traditional Grover call was an ingenious way to reflect a wavefunction about an equal superposition of each vector of the computational basis towards an oracle identified answer. The operation identified as angle selection in (4.14) is a means to reflect about an unknown differential wavefunction. This is done by relabeling the differential signal qubits to reverse their polarity, flipping the phase of the appropriate eigenvectors in each track, followed by the restoration of signal polarity. This is shown diagrammatically with a reordering of the qubit lines so that all operations stay exactly parallel in the tracks. This construction allows a reflection of the wave-function about the dynamic angle  $|\theta\rangle\langle\theta|$  (instead of a reflection about the static  $|0\rangle\langle 0|$  as in the Grover search algorithm). Although in this example, a controlled-controlled- $Z$  operation performs the right amplification for the desired angle, in general the Grey code used in the quantum-walk on a Bloch sphere and the desired angle selection determine the functionally correct sub-circuit. As with the traditional amplification algorithm, one repeats

all the previous steps until they have fully amplified the desired operation. In this example case, since there are only two oracle-identified answers in each track from a possible eight eigenkets, it takes only one set of reflections to completely shift all the amplitude of the wave function to  $\cos\left(\frac{\theta+\alpha+\beta}{2}\right)|000\rangle + \sin\left(\frac{\theta+\alpha+\beta}{2}\right)|001\rangle$  in the top track and  $\cos\left(\frac{\theta+\alpha+\beta}{2}\right)|000\rangle - \sin\left(\frac{\theta+\alpha+\beta}{2}\right)|001\rangle$  in the bottom track. So one final “quantum walk” fully amplifies the desired angle. This circuit may be generalized to any number of control qubits and the number of operations will scale linearly. Note that any stray phase-flips created by fermionic loops in the quantum-walk operation get reversed in the reverse-quantum-walk and so that the phases used to control reflections in Amplitude Amplification are preserved and physically correct for fermionic qubits.

To visualize the change in amplitudes as the algorithm progresses, we plot a Mathematica implementation of the amplitudes of the sixteen eigenstates before and after each step in the algorithm in Figure 4.13.

Representing the qubits differentially has a significant error correcting benefit. Because all operations between the two tracks take place concurrently, the same operating field may be used for each track simultaneously. By introducing static, common unitary imperfections in control field we can observe the response in the  $|\theta\rangle$  and  $|\neg\theta\rangle$  qubits. A results of a simulation of the circuit in Figure 4.14 using strong under/over-rotation in the angle-selection and reflection operations is shown in Figure 4.15. Static shifts in the area of the control pulse produce deviations from the correct Bloch sphere after measurements are made of the control qubits. Measurement is not necessary, but serves two purposes. First, all the control qubits should be zero at the end of the operation, and if they are measured otherwise, it herald an errant operation. Second, measuring zero’s collapses residual error and prevents it from propagating further. . Upon adding significant unitary error, we see that probability of successful measurement of the control qubits (measuring zero) is still very high and should not deter scalability as shown in Figure 4.15. Given a successful measurement, we find that any phase error in the algorithm completely limited by the differential representation of qubits as shown in Figure 4.16.

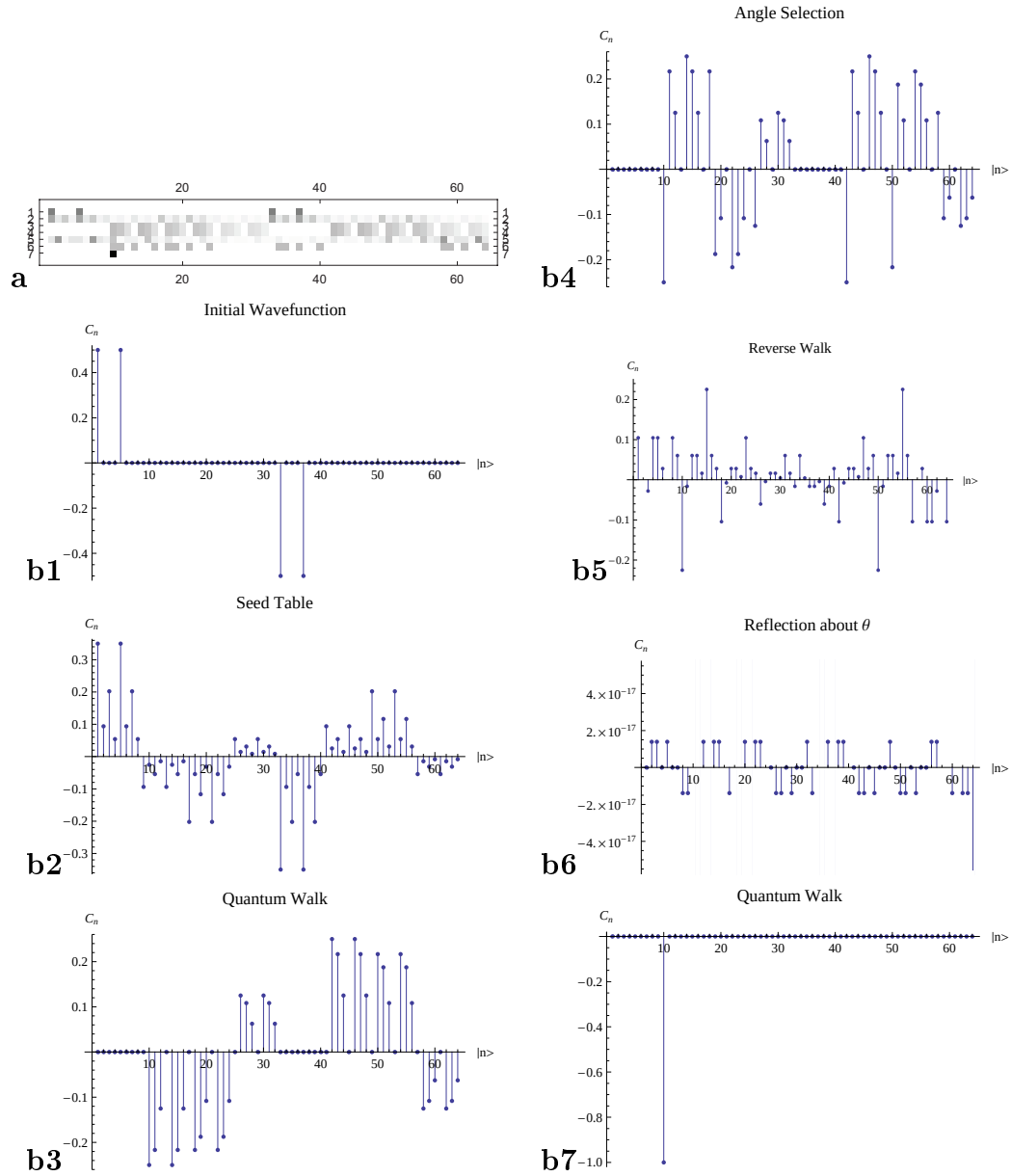


Figure 4.13: Simulation of circuit in for  $\theta = \frac{\pi}{2}$

To help understand the quantum circuit in Figure 4.14, the amplitudes of each eigenket in the local basis are plotted after each step. For simplicity we chose  $\theta = \pi/2$ ,  $\alpha = \pi/3$ , and  $\beta = \pi/6$ , though similar dynamics occur for any angle. The angle-selection applies a  $\pi/2$  rotation. **a**. A plot of amplitude intensity (phase is ignored) for each of 64 eigenkets of the total system before and after each of the seven steps. The actual amplitudes (without the phase) are plotted in **b1-b7**. The initial amplitude is  $|\pi/2, 0, 0, -\pi/2, 0, 0\rangle$  and the final amplitude is  $|\pi, 0, 0, -\pi, 0, 0\rangle = -|1, 0, 0, 1, 0, 0\rangle$

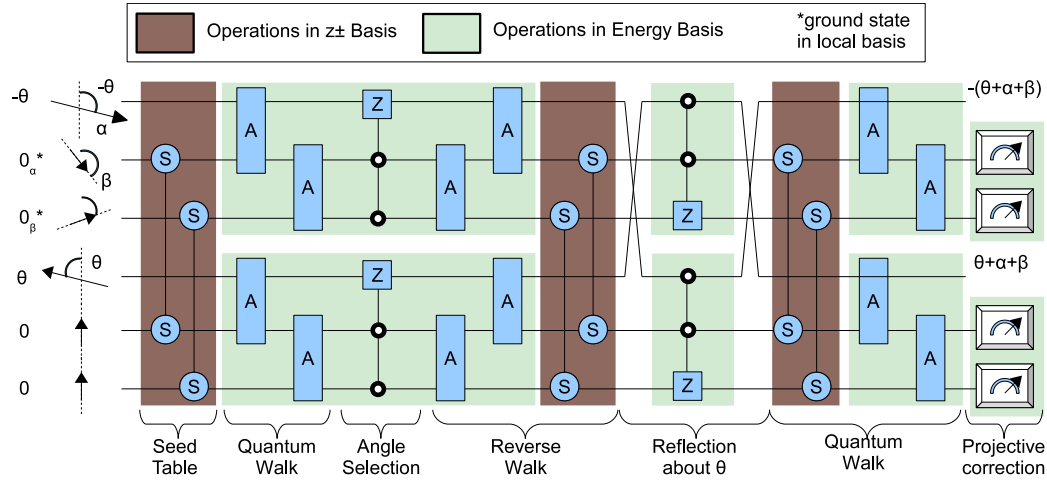


Figure 4.14: Differential Amplitude Amplification of an Operation

This circuit adds the angle  $\alpha + \beta$  to the differential qubit  $(|-\theta\rangle, |\theta\rangle)$  input to the system by using only Clifford operations and produces the differential output  $(|-(\theta + \alpha + \beta)\rangle, |\theta + \alpha + \beta\rangle)$ . Four control qubits are initialized to  $|0\rangle$  within their local basis. Operations that are represented in the global basis,  $\mathcal{Z}$ , are shaded, while operations that use the local energy basis,  $\mathcal{E}$ , are cross-hatched. A initial set of swap operations in the global  $\mathcal{Z}$  basis “seed the table.” This creates a differential set of angles,  $|-\alpha\rangle_\alpha, |-\beta\rangle_\beta$  and  $|\alpha\rangle_z, |\beta\rangle_z$  in the four control qubits as seen in their local bases. As such, a “quantum walk” on a Bloch sphere can be performed to generate a differential pair of every additive permutation of the the angles  $\alpha$  and  $\beta$ . The “angle selection” step in this example flips the phase of the desired operation followed by a “reverse walk” that inverses the initial superposition, as is prescribed by traditional amplitude amplification. Now the differential nature of the algorithm may be exploited allowing a reflection of the wave-function about the dynamic angle  $|\theta\rangle\langle\theta|$  (instead of the reflection about the static  $|0\rangle\langle 0|$ ). This is done by relabeling the differential signal qubits to reverse their polarity, flipping the phase of the appropriate eigenvectors in each track, followed by the restoration of signal polarity. The amplification algorithm repeats all the previous steps until it has fully amplified the desired operation. In this example case with only three qubits in each track and one desired angle operation, no repetitions are necessary. A final “quantum walk” fully amplifies the desired angle. This circuit may be generalize to any number of control qubits, while the number of operations will scale linearly.

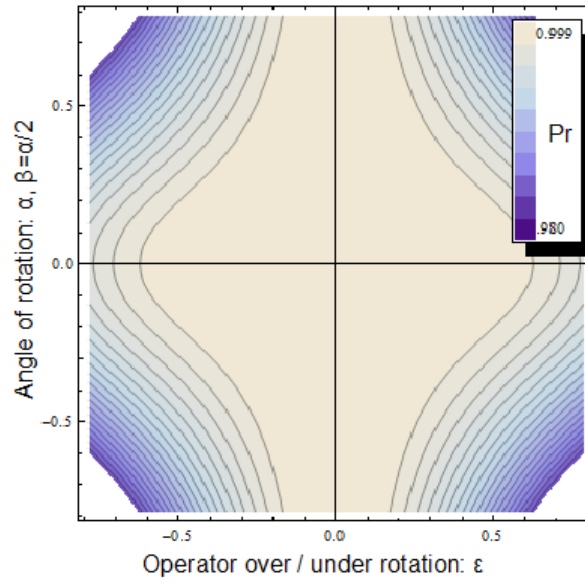


Figure 4.15: Probability of successful control qubit measurement versus initial angle and under-over-rotation angle

The horizontal axis denotes the static under-over-rotation angle while the vertical is desired angle of the rotation operation in radians. The probability that a non-zero value will be measured for any of the control bits is exceedingly low and is negligible unless several thousand qubits are used.

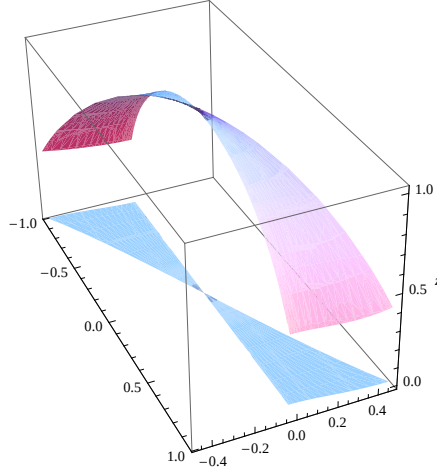


Figure 4.16: Differential error exactly tracks the phase error in reflection operations. Plotted on the top surface is the set of Bloch sphere vectors for the positive polarity output qubit  $|\theta\rangle$  when the additive angles and over/under-rotation errors are swept as in Figure (4.15), and after a successful measurement of the control qubits to be zero. The error-free points fall strictly on the  $Z - Y$  great circle, and all points with an element of phase are errant. However the bottom surface is a set of vectors denoting the phase difference the differential qubits:  $\text{phase}(|\theta\rangle) - \text{phase}|\theta\rangle$ . As expected they exactly track the unitary error in the control field, and so the differential qubits can be combined into a decoherence free ensemble, with an ensemble measurement that eliminates this error. Therefore, a successful measurement of the control qubits also heralds an elimination of any unitary under/over-rotation in the control field.

As per the traditional amplitude-amplification algorithm, the number of reflections required grows linearly with the number of control bits. However in the case of quantum Fourier transform, all seven controlled rotations needed to reach the critical bound of  $\frac{\pi}{128}$  may be done in one instantiation of the above algorithm with ten reflections, without the need for an oracle that is anymore complex. This is explained in detail in reference to a specific physical quantum dot system in future work [15]. In general, one can create a repository of precise base angles in an extended Hilbert space, and use them for quantum operations by simply amplifying the desired angles. Because the angle selection may be done coherently by using other qubits, the topology of controllable points in each local system form a product of basis vectors.

### 4.2.6 Summary

In summary, we have shown a method for encoding a pseudo-continuous set of quantum operations in classically stable degrees of freedom. Using such an encoding, both the Quantum Fourier Transform and Amplitude Amplification may be implemented efficiently using only operations from the Clifford group. Doing so, generalizes coherent quantum feedback to locally-variant bases and amplitude-amplification to differential, quantum-operation search. Traditionally, time-modulation encodes classical information into quantum degrees of freedom. In this case, quantum coherent feedback converts classical information, in the bases of one or more control qubits, into operational quantum information. Our technique breaks the traditional formalism of creating a universal set of gates to abstract physical technologies from quantum algorithm architectures. The simplicity afforded by universal quantum gates unfortunately also creates unwieldy structures for fault-tolerance, which we have demonstrated are unnecessary for quantum computing. A non-universal quantum system, where quantum information may be encoded in both spacial and temporal degrees of freedom, may be superior to a universal quantum system with only pulse-encoded operations. Finally, our work suggests that the dichotomy of quantum algorithms into derivatives of quantum Fourier transform and derivatives of Amplitude Amplification is artificial: quantum Fourier transform can be written as a specific case of Amplitude Amplification with locally-variant bases. Our theory can be tested upon today's experimental computing platforms supporting two qubits or more and whose energy splitting asymmetry is classically parametrized. Additionally, this work demonstrates that quantum operations that are dense in an operational  $n$ -qubit space can be encoded with  $O(n)$  coherent, spacial degrees of freedom.

## 4.3 Acknowledgments

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directed, supervised, and co-worked on the research which forms the basis of this chapter.

The text of chapter IV, in part, is a reprint of the material that is being prepared for publication as Dalal, P.; Saikin, S.; Sham, L. J. *Unitary-Fault Tolerance using Quantum Coherent Feedback*. The co-authors in this publication directed, supervised, and co-worked on the research which forms the basis of this chapter.

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## Chapter 5

# Eliminating time dynamics through renormalization and spacial dynamics

An operator,  $A$ , of quantum system will evolve in time,  $[H, A]$ , by two means: a time-dependent interaction energy (change of states), or an energy uncertainty (change in “phase”). In classical mechanics however, the time evolution,  $\{H, A\}$ , must separate free-energy (scattering), from entropic energy (change in microstates) . There is no role for energy uncertainty in classical mechanics, as there is no role for the free energy in quantum mechanics. As a quantum system is transitioned to a classical system at some limit, all uncertainty in energy (superpositions of states) disappears as coherence does. In the following example, we show quantum uncertainty produces classical uncertainty (entropy) in this transition, leading to irreversibility. We engineer a particular system that parametrizes this transition to perform a Quantum Fourier Transform without any time-dependent interaction energy. We show how the time scale of ergodicity grows without bound at the limit of energy.

## 5.1 Eliminating temporal control in favor of spatial control: Self-Timed Quantum Computing

In classical computation, a self-timed circuit is an analog circuit that latches a digital output without an external or reference timing signal (such as a digital clock). An external “start” signal is activated when the operands are setup and the result may be probed at some later time. All quantum systems previously realized or theorized, rely on the timing of external control fields (or measurements) to operate upon a quantum system. We present a new quantum computing paradigm that coherently operates itself from a “start” signal and then effectively latches its output through unitary evolution. This is done using an “entropic-controller” that requires an enormous time-scale to return to its initial state, while nearly all of the preceding states latch the proper output ready for measurement. We show this system can perform the quantum Fourier transform efficiently, essential to many quantum algorithms. These entropy-controlled quantum systems provide a fundamental inquiry into the meaning of entropy and information at a quantum level. Moreover given the analog nature of quantum operations, such a paradigm would increase their precisions markedly, the cornerstone of scalable, fault-tolerant quantum computation.

### 5.1.1 Introduction

Given the precision with which some quantum system can be initialized [1], measured and isolated today, the fundamental, limiting step in the scaling of quantum computers is a lack of precision in the control of highly entangled systems. Largely, this is due an inability to control the interaction between qubits and quantum fields that either couple the system to the environment or to implement the desired quantum operations. To run a quantum computer, one inputs classical information into a set of quantum states by changing their environment in a controlled way. Every quantum system proposed so far for quantum information processing, with the exception of measurement-based operation [7] such as one-way quantum computing, modulates the potential seen by qubits in time to implement

a particular quantum circuit. This is equivalent to creating a specific set of scattering events between qubits and quantum fields, where the time-evolution of their interaction is precisely controlled. For example, adiabatic quantum computing restricts itself to slow changes of the qubits' environment so that the ground state is continuously maintained [4]. However barring measurements, if this scattering is done in the absence of environmental coupling, and since quantum scattering is a unitary process, there can be no change in the total entropy of the fields and qubits, or equivalently in their net information. This creates a large non-local, field-matter object that cannot be reduced. When a measurement of a qubit takes place (or equivalently a partial measurement of a finite space-time region of fields that have interacted with it) a trace over certain degrees of freedom of the field-matter object is performed and the entropy of the now disentangled qubit system has increased. Furthermore, it is exactly the classical information of which quantum circuit is implemented, of all the possible quantum circuits implementable, that determines the new entropy of the qubit. This distribution is known as the algorithmic randomness, or the Kolmogorov complexity of the interactions [6].

As a consequence, because two different quantum circuits are related by a small change in scattering times, one major source of noise in quantum operations is the imprecise, classical control used to implement the quantum circuit. If a means of coherent control [5] can be made completely autonomous then the operation may be performed much more precisely. We present in this paper a conceptually new idea: a method of "entropic-control" of a quantum operation. By this, we mean that one quantum process having a very long time until it returns to its initial state may control another quantum process with faster dynamics, so that any measurement within a large time-scale will observe the intended quantum operation with very high fidelity. For example, a wave-packet in near vacuo will expand its spacial extent according to the energy-uncertainty principle. Because the wave-packet is not an energy eigenstate of its global environment, even in a closed system, it will experience a coherent evolution. In the strictest sense, this evolution is a coherent quantum operation. In the classical world, such an evolution pertains to diffusion: information-degradation by an increase in spacial entropy,

but a constant free energy. Such notions don't exist in the quantum world, since if a measurement is made at some later time it determines a coherent quantum operation performed by closed evolution. However in this example, the precision of the quantum measurement is proportional to its timing. A timed scattering of a measurement field is equivalent to that of a control field, and does not improve the precision of quantum operations. For example a single qubit in a closed quantum system and not in its energy eigenstate will precess, but cannot be considered a quantum operation without an autonomous end to the precession.

However, if the measurement time is very weakly coupled to the quantum operation performed, the precision of the quantum operation is no longer dependent on the spatial and temporal extent of the scattering or measurement operation. In particular, we show that quantum tunneling can coherently dress a quantum cavity so that it turns from off-resonant to resonant to off-resonant with a another quantum system, producing a coherently controlled quantum operation. Since resonant dynamics can be made to vastly outpace tunneling dynamics, the time to return the cavity to resonance is so long that any measurement that takes place in the interim will observe a quantum-controlled operation. Therefore the system can be limited by the precision of initialization and measurement alone (experimentally orders of magnitude more precise than field-controlled quantum operation). Moreover, we show that these dynamics are rich enough to implement a Quantum Fourier Transform (QFT). In a broader sense, such a paradigm demonstrates it is physically possible to increase the information of a quantum system without temporally modulating a series of control fields.

First consider the simple example of trapped bosons in a double quantum well. If bosons are initially added to the left quantum well, they coherently tunnel to the right quantum well and back again in a precession of number state, as shown in the top or bottom wells of Figure 5.1. If we assume for the moment that the photons arrive in an exponentially-peaked fashion at a much faster rate than the dynamics of dissipation, we can treat the photons as beginning their evolution with nearly the same phase. This approximation is not necessary for the physical theory presented, but rather simplifies the equations of motion. The uncertainty

in energy gives rise to a harmonic oscillation of the number state. This double well Hamiltonian,  $H_{DW}$ , can be written in terms of annihilation and creation operators,  $a_i$  and  $a_i^\dagger$  in the two harmonic wells of frequency  $\Omega_i$ ,  $i \in \{1, 2\}$ , as well as tunneling terms between them governed by the tunneling rate  $\kappa$ . The rate is a function of  $|\Omega_1 - \Omega_2|$  as well as the height of the barrier separating the two wells, as in (5.1). The the equations of motions can be calculated in the Heisenberg picture (5.2), (5.3), of the number state operators in each well,  $N_i$ . We can find the tunneling amplitude (5.4), as an intermediary term.

$$H_{DW} = \Omega_1 a_1^\dagger a_1 + \Omega_2 a_2^\dagger a_2 + \kappa (a_1^\dagger a_2 + a_2^\dagger a_1) \quad (5.1)$$

$$\dot{N}_1 = i\kappa (a_1 a_2^\dagger - a_1^\dagger a_2) \quad (5.2)$$

$$\dot{N}_2 = i\kappa (a_1^\dagger a_2 - a_1 a_2^\dagger) \quad (5.3)$$

$$\frac{d}{dt} (a_1 a_2^\dagger) = -i(\Omega_1 - \Omega_2) a_1 a_2^\dagger + i\kappa (N_1 - N_2) \quad (5.4)$$

If the wells are of different frequency only virtual tunneled states can exist due to a non-conservation of energy. If we assume that the wells are of a the same frequency, a simple second-order differential equation in the difference of number state operators is found by combining the equations (5.5), (5.6) and has the solution (5.7).

$$\frac{d}{dt} (N_1 - N_2) = i\kappa (a_1 a_2^\dagger + a_2 a_1^\dagger) \quad (5.5)$$

$$\frac{d}{dt} (a_1 a_2^\dagger + a_2 a_1^\dagger) = 2i\kappa (N_1 - N_2) \quad (5.6)$$

$$(N_1 - N_2)(t) = N_1(0) \cos(\sqrt{2}\kappa t) \quad (5.7)$$

Therefore the number of bosons in the right well is (5.8), given by the boundary conditions that  $N_2(0) = 0$  and continuity.

$$N_2(t) = \frac{N_1(0)}{2} \left[ 1 - \cos(\sqrt{2}\kappa t) \right] \quad (5.8)$$

It is significant to note that by increasing the initial number state in the left well  $N_1(0)$ , one can tune the amount of time before a fixed number of photons tunnel into the right well to compensate for a small  $\kappa$ .

The angular frequency of this procession (5.8) is not constant if the tunneling rate coherently processes as well. For example, if a similar, auxiliary procession of number state coherently dresses the tunneling interaction between the right well and the left well, a time-dependent tunneling rate,  $\kappa(t) = \kappa(0) \exp[-A\phi(t)]$ , may be induced, where  $\phi(t) = \sqrt{\frac{N_R(0)}{2}} [1 - \cos(\sqrt{2}\kappa't)]$ . In this formula,  $\kappa_0$  is the undressed, fast resonant tunneling rate through a thin barrier. It decays exponentially as the barrier is strengthened due to the renormalization of the right well's absorption spectrum, as in Figure 5.1(c), and then decreases similarly in a periodic manner. This frequency shift can be proportional to the square-root of the number of excitations,  $N_R(0)$ , dressing the cavity with proportionality coefficient  $A$ . These excitations, in turn, oscillate according to their own tunneling rate,  $\kappa'$ . This coherent dynamical dressing produces a complicated and long synchronization time between the two quantum oscillators due this non-linear interaction. The time may be approximated by the integral equation in (5.9), which integrates the accumulation of phase as the system evolves

$$2\pi = \int_0^T dt \sqrt{2}\kappa_0 \exp \left[ -A \sqrt{\frac{N_R(0)}{2}} [1 - \cos(\sqrt{2}\kappa't)] \right] \quad (5.9)$$

An entropically significant phenomenon arises in this system: in the stochastic limit as  $N_R \rightarrow \infty$ , the periodic peaks due to the argument in the exponent have diminishing width, yet the height of the peaks are unchanged and limited to one. Therefore in this limit, the integral equation has no solution, since the progression in phase at every peak is also diminishing, making the process irreversible (even though the evolution of the state is completely coherent). An unphysical unbounded energy transitions the quantum system into a classical system. If  $N_R$

is sufficiently large, any measurement of photon number in the right-well after an amount of time long compared to tunneling rate, but small compared to the synchronization time, will produce a value greater than zero with a very high fidelity. As such, by choosing a logical representation for one and zero as a number state operator in the right well, the operation appears self-timed due to a dependence on the time-scale, rather than the actual time, of the measurement. In the following sections, we show that it is physically possible to approximate such a quantum system, exploiting the non-linear interactions between photons mediated by a quantum dot. Additionally, we show such a self-timed operation can be made to implement the QFT, the underlying workhorse of many quantum algorithms. Conceptually this structure, one interaction dynamically renormalization another interaction, demonstrates that useful quantum algorithms can evolve in a closed quantum system by crafting the initial non-stationary state correctly.

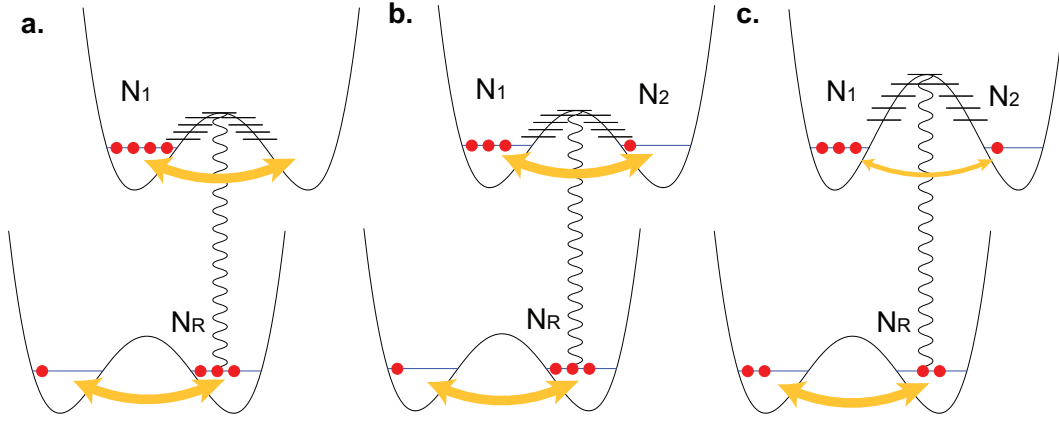


Figure 5.1: Self-timed quantum systems

The tunneling process of a boson initially in the left well of a quantum double-well potential is reversible. Because the initial state, (a, top) is not a true eigenstate of the system, the state will tunnel to the right well (b, top) and return with a frequency proportional to the tunneling rate  $\kappa$ , depicted with a bidirectional arrow (yellow). However, when the presence or absence of photons,  $N_R$ , in the right well of another oscillator (bottom) controls the potential separating the original two wells, the net system period can depend on a long synchronization time between the two oscillators. For example, the controlling oscillator (bottom) renormalizes the available energy states (the horizontal lines between wells) describing the potential of the transport oscillator (top), so that a polynomial shift in  $N_R$  exponentially decreases the rate of number state precession between  $N_1$  and  $N_2$  (c) depicted by a thinner interaction arrow. In this way, the tunneling rate is really implicitly time-dependent,  $\kappa(t)$ . In particular, by exploiting the structure of renormalization, the duration in which  $\kappa(t)$  is not negligible can be made arbitrarily small (by setting  $N_R$  very high). Such a system is still closed since all evolution is mediated by coherent tunneling, but in the stochastic limit, as the number of bosons,  $N_R$ , approach infinity (unbounded energy) in the controlling oscillator, the probability that a single photon is not in the right well after a certain amount of finite time is zero. The process appears as diffusion in this limit, yet still is coherent. If a number-state basis measurement of each well is used to logically infer the value of the quantum system, the quantum state is practically irreversible. The entropy of this system is not increased with a precisely controlled interaction Hamiltonian, but rather by choosing the time-scale of measurement commensurate with  $N_R$ .

Quantum systems that lie at the onset of chaos are associated with significant computation[3]. In the stochastic limit, this system exhibits the antithesis of the quantum-Zeno effect: if it is not measured continuously, it will absolutely change its state. While in low-energy limit, the system could be made absolutely stationary

by continuous measurement. How the system is measured and encoded completely determines the time and space of the scattering process between bosons in the different wells that will have taken place. This is an extreme example of the a long-held belief in quantum information theory: “Every it–every particle, every field of force, even the spacetime continuum itself–derives its function, its meaning, it’s very existence entirely—even if in some contexts indirectly—from the apparatus-elicited answers to yes-or-no, binary choices, bits.”

### 5.1.2 Static Cavity Renormalization

Consider two two-level systems with a common upper energy level, for example a singly-doped quantum dot as shown in Figure 5.2, where  $H$ -polarized photons directly interact with one system,  $V$ -polarized photons directly interact with the other, and  $z$ -polarized photons interact with the full three-level system. Additionally, the first system is dressed by a resonant cavity of frequency  $\Omega_1$  equal to the horizontal polarization transition frequency  $\omega_1$ , while the second is dressed by an off-resonant cavity of frequency  $\Omega_2$  equal to the vertical polarization transition frequency,  $\omega_2$  plus a detuning  $\Delta$ . A magnetic field splits the ground state into two separated by  $\omega_z$ . Because the cavities are already energy separated, polarization and frequency selection rules keep a photon in one cavity from interacting with the other, except through the quantum dot.

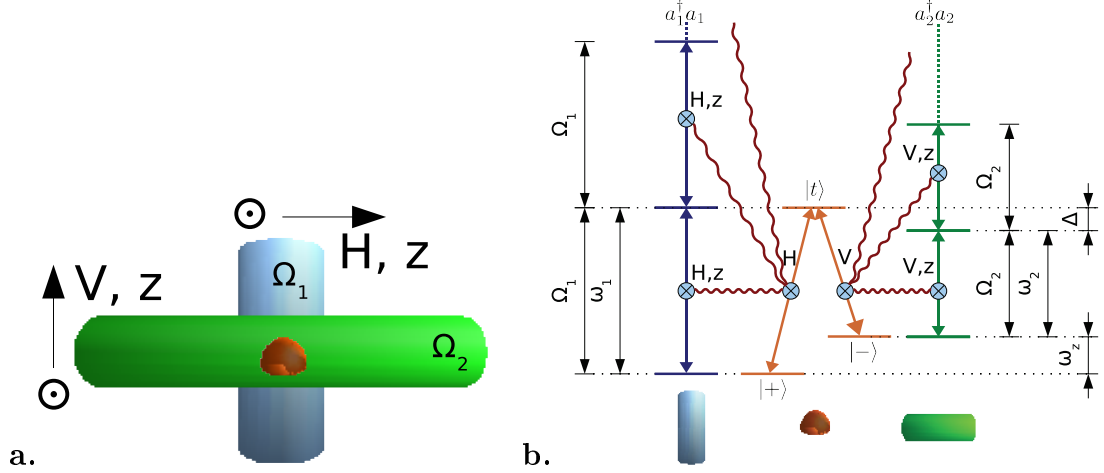


Figure 5.2: Dynamical dressing of a cavity

Two cavities of different frequencies can interact through a common singly-doped, quantum dot. The vertically-elongated cavity hosts photons,  $a_1^\dagger a_1$  of frequency  $\Omega_1$  that are polarized in the horizontal direction,  $H$ , and out of the page,  $z$ , while the horizontally-elongated cavity hosts photons,  $a_2^\dagger a_2$ , of frequency  $\Omega_2$  that are polarized vertically,  $V$ , and out of the page,  $z$ . A background DC electric field causes a Stark shift,  $\omega_z$ , between the lowest two Pauli-X levels of the quantum dot,  $|-\rangle, |+\rangle$ . An imbalance in the interaction can be effected by keeping one cavity frequency resonant with the trion transition  $|-\rangle \rightarrow |t\rangle$  while keeping the other transition  $|+\rangle \rightarrow |t\rangle$  detuned by  $\Delta$  from the cavity frequency. This imbalance in field-dot coupling is exploited to give one cavity coherent control of the other cavity's absorption spectrum. The renormalized energy levels of the system will allow the resonant interactions, the wavy, red lines on the left, to strongly perturb the off-resonant interactions, the wavy, red lines on the right—but not vice-versa.

This is a closed, static system, the total Hamiltonian,  $H_S$ , is given by (5.10) which can be decomposed into non-interacting (5.11) and interacting terms (5.12).

$$H_S = H_0 + H_{IQ} \quad (5.10)$$

$$H_0 = \omega_1 |t\rangle \langle t| + \omega_z |+\rangle \langle +| + 0 |-\rangle \langle -| + \Omega_1 a_1^\dagger a_1 + \Omega_2 a_2^\dagger a_2 \quad (5.11)$$

$$H_{IQ} = g_1 \left( |t\rangle \langle -| a_1 + |-\rangle \langle t| a_1^\dagger \right) + g_2 \left( |t\rangle \langle +| a_2 + |+\rangle \langle t| a_2^\dagger \right) \quad (5.12)$$

The eigenstates,  $|+\rangle, |-\rangle$ , and  $|t\rangle$ , reflect the ground, first excited, and first trion states of a quantum dot respectively. In the rotating wave approximation, we save the second interaction as we assume that  $\Delta = \Omega_2 - \omega_2$  is relatively small. We

assume that  $\omega_z$  is also small so that it is highly off resonant with any transition. All other interactions average quickly to zero. Let  $n$  be the number of  $H$ -polarized photons of frequency  $\omega_1$ , and let  $m$  be the number of  $V$ -polarized photons of frequency  $\omega_2$ , such that  $n \equiv a_1^\dagger a_1$  and  $m \equiv a_2^\dagger a_2$  with eigenvalues  $|n\rangle$  and  $|m\rangle$  respectively. Since the interaction only connects states  $A = |-, n+1, m\rangle$ ,  $B = |+, n, m+1\rangle$ , and  $C = |t, n, m\rangle$  we decompose the Hamiltonian,  $H = \sum_{n,m} H_{nm}$ , in order to dress each state as shown in (5.13). Photons that are  $z$ -polarized do not interact with the quantum dot.

$$H_{nm} = \begin{array}{ccc} & \begin{array}{c} A \\ B \\ C \end{array} & \begin{array}{c} A \\ B \\ C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{array}{ccc} \Omega_1(n+1) + \Omega_2 m & 0 & g_1 \sqrt{n+1} \\ 0 & \Omega_1 n + \Omega_2(m+1) + \omega_z & g_2 e^{i\Delta t} \sqrt{m+1} \\ g_1 \sqrt{n+1} & g_2 e^{-i\Delta t} \sqrt{m+1} & \Omega_1 n + \Omega_2 m + \omega_1 \end{array} \end{array} \quad (5.13)$$

Using the fact that  $\omega_1 = \Omega_1$ , and  $\omega_z + \Delta = \Omega_1 - \Omega_2$  we can simplify this Hamiltonian as:

$$H_{nm} = \left[ \Omega_1 \left( n + \frac{1}{2} \right) + \Omega_2 \left( m + \frac{1}{2} \right) \right] \cdot \mathbf{1} \quad (5.14)$$

$$+ \begin{bmatrix} \frac{1}{2}(\omega_z + \Delta) & 0 & g_1 \sqrt{n+1} \\ 0 & \frac{1}{2}(\omega_z - \Delta) & g_2 e^{i\Delta t} \sqrt{m+1} \\ g_1 \sqrt{n+1} & g_2 e^{-i\Delta t} \sqrt{m+1} & \frac{1}{2}(\omega_z + \Delta) \end{bmatrix} \quad (5.15)$$

Diagonalizing the rightmost matrix of (5.14) calculates the energy shift,  $E$ , in the absorption spectrum for the interacting versus non-interacting cavity-dot system. However, exactly diagonalizing leads to a very long and complicated expression. The exact energy levels are plotted in three different perspectives in Figure 5.3 to reveal the energy splitting structure of the renormalization. We may verify the form of the coupling with an analytical approximation by assuming  $n \gg m$ . Then a second-order approximation is calculated by first diagonalizing the  $n$ -photon interaction and then applying the new energy eigenstates to the  $m$ -photon interactions.

So first renormalizing in the subspace spanned by  $C$  and  $A$  the trion state splits into two states, separated by  $2g_1\sqrt{n+1}$ , as given in (5.16).

$$\begin{aligned} E_{n,\pm} &= \left( \frac{1}{2}(\omega_z + \Delta) \pm g_1\sqrt{n+1} \right) |t, n\rangle \\ E_0 &= 0 \end{aligned} \quad (5.16)$$

Since  $\Delta$  is positive only the lower energy of this energy splitting,  $E_{n,-}$ , approaches the resonance of the horizontally elongated cavity. Therefore we may simply use interactions with this state to continue renormalization. Considering the subspace spanned by  $C$  and  $B$ , we may diagonalize the reduced matrix (5.17) to find the energy splitting of the  $|t_+, n\rangle$  state interacting with  $m$ -photons.

$$\begin{pmatrix} \frac{1}{2}(\omega_z + \Delta) + g_1\sqrt{n+1} & 0 & 0 \\ 0 & \frac{1}{2}(\omega_z - \Delta) & g_2e^{-i\Delta+t}\sqrt{m+1} \\ 0 & g_2e^{-i\Delta+t}\sqrt{m+1} & \frac{1}{2}(\omega_z + \Delta) - g_1\sqrt{n+1} \end{pmatrix} \quad (5.17)$$

The final, renormalized energy levels for  $|t, n, m\rangle$  when both  $m > 0$  and  $n > 0$  are given in (5.18).

$$E_3 = \frac{1}{2}(\omega_z + \Delta) + g_1\sqrt{n+1} \quad (5.18)$$

$$E_2 = \frac{1}{2}(\omega_z + \Delta) - \frac{g_1}{2}\sqrt{n+1} \quad (5.19)$$

$$+ \sqrt{\left(\frac{g_1}{2}\right)^2(n+1) + \frac{g_1\Delta}{2}\sqrt{n+1} + g_2^2(m+1) + \left(\frac{\Delta}{2}\right)^2} \quad (5.20)$$

$$E_1 = \frac{1}{2}(\omega_z + \Delta) - \frac{g_1}{2}\sqrt{n+1} \quad (5.21)$$

$$- \sqrt{\left(\frac{g_1}{2}\right)^2(n+1) + \frac{g_1\Delta}{2}\sqrt{n+1} + g_2^2(m+1) + \left(\frac{\Delta}{2}\right)^2} \quad (5.22)$$

Constant energy terms can be dropped from all three levels, revealing how a change in the absorption spectrum of the combined dot-cavity system can be mod-

ulated largely by one cavity. When the number of horizontally-polarized control photons have a renormalization exceeding the detuning,  $g_1 \sqrt{a_1^\dagger a_1 + 1} > \Delta$ , there is a sharp change in the transport cavity for vertically-polarized photons if  $m > 0$ . The imbalance in control is due to an imbalance in resonance.

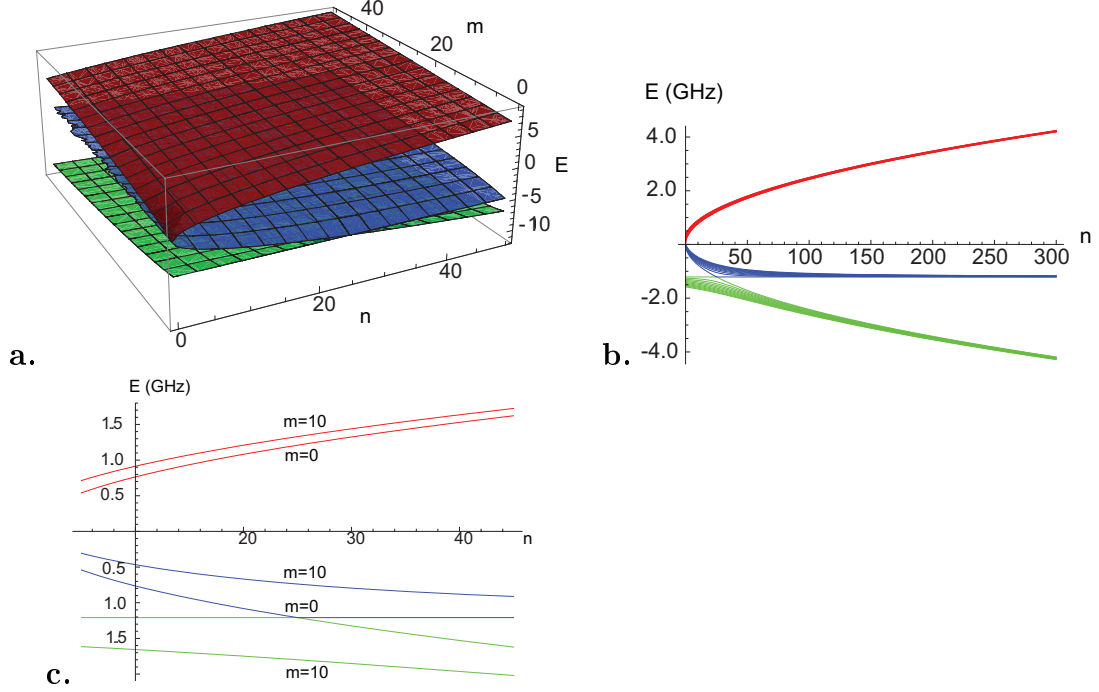


Figure 5.3: Renormalization of Lambda-system due to resonant and off-resonant absorption and emission

The shift in energy due to  $n$  photons in the vertically-elongated (interaction-control) cavity and  $m$  photons in the horizontally-elongated (interaction-transport) cavity are plotted in three different diagrams to highlight the properties of the renormalization. The parameters  $g_1 = g_2 = 0.1 \text{ meV}$  and  $\Delta = -0.5 \text{ meV}$  to reflect parameters for the example system in [8] **a.** The complete three dimensional plot reveals the marginal effect of  $m$  compared to  $n$ . **b.** A plot of energy,  $E$ , versus  $n$  is made for a series of  $m = 0 \dots 10$ . When  $n$  is large and  $m > 0$  the quantum dot effectively couples the renormalization of the interaction-control cavity to the interaction-transport cavity. The shift in the horizontally-elongated cavity appears as the bottom branch of a cavity quantum-dot renormalization after the shift is greater than the detuning of the interaction-transport cavity. **c.** The same plot zoomed in on the detuning energy. The system has an energy crossing only when  $m = 0$  and otherwise the resonant tunneling controls the renormalization of both.

The presence of even a single photon ( $m > 0$ ) effectively couples the dressing of

the vertically-elongated cavity to the dressing of the horizontally-elongated cavity. If  $m$  photons subsequently tunnel at resonance, they will have a small overall energy effect as  $n$  must necessarily be a large that time.

### 5.1.3 Number-Dynamic Cavity Renormalization: Self-Timed QED Transistor

The above static system, Figure 5.2, can mediate a quantum operation. Consider the interaction system above, where we will term the vertically-elongated cavity *interaction-control* and the horizontally-elongated cavity *interaction-transport*, with three additional polarization-independent cavities, termed: control, left, and right as in described in Figure 5.5. Each additional cavity is separated from the interaction cavities by a very thin barrier. Assume both interaction cavities contain a couple photons initially, the exact number is unimportant, while the control hosts many (hundreds) photons and the left cavity many more (millions).

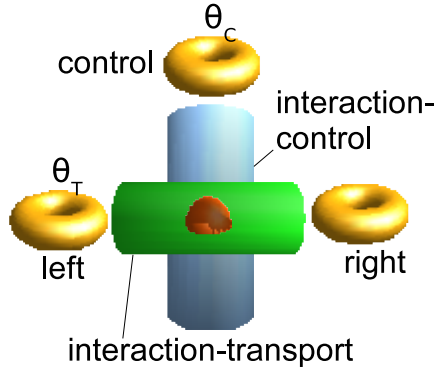


Figure 5.4: Structure for self-timed quantum operation

A structure having five cavities and a quantum dot resembles the interacting double-well structure of Figure 5.1. Photons precess between the top cavity (control) and the vertically-elongated cavity (interaction-control). The precession renormalizes the height of the energy barrier separating the left and the right cavities. The renormalization takes place through the quantum dot which couples the interaction-control cavity to the interaction-transport cavity. Once either precession process happens in resonance, the resonant tunneling can continue as there is little modulation of the absorption energy levels between the  $n + 1$  and the  $n$  energy levels for large  $n$  as explained in Figure 5.7. The barriers between the left and right cavities and the interaction-transport cavity must be exceedingly thin to enable effective tunneling, as in a quantum cascade laser.

Because the control cavity and the vertically-elongated cavity are resonant and a thin barrier exists between them, we will show that there is an efficient tunneling of the photons from one to the other as in (5.8). For example in a quantum cascade laser, tunneling through a thin barrier efficiently maintains population inversion for lasing. However for this system, the linear precession in (5.8) is an analytical simplification as the precession will slightly accelerate and decelerate in frequency given the shift from resonance at any given time. Qualitatively, the state trajectory is as follows. The horizontally-elongated cavity mediates efficient photonic tunneling between the left and right cavities for vertically-polarized photons at a very sharp resonance. The large detuning,  $\Delta$  ensures that photons cannot be strongly absorbed in one cavity and emitted in the other. If the number of photons in the control cavity is large, many photons will flow into the interaction-control cavity. When the number of photons in the control interaction cavity compensate for the detuning,  $\Delta = g_1 \sqrt{a_1^\dagger a_1 + 1}$ , an efficient coupling of the vertical to horizontal

cavity renormalization is ensured, and the absorption line for  $V$ -polarized photons drop rapidly. If the left and right cavities are initially very far from resonance in the absence of control photons, the renormalized transport interaction cavity can then become resonant with the left and right cavity invoking a mass number precession between the three cavities, left, interaction-transport, and right, during the timescale of resonance. Since the left cavity is densely populated with photons, some number of photons will appear in the right cavity with a high fidelity. This entire process will only occur with the amplitude that the control photons are horizontally-polarized and that the transport photons are vertically-polarized. Finally, the synchronization time for the intended operation is very high, and if photons continue to be pumped into the left cavity in steady state, the process is irreversible. We quantify this phenomenon in the below analysis.

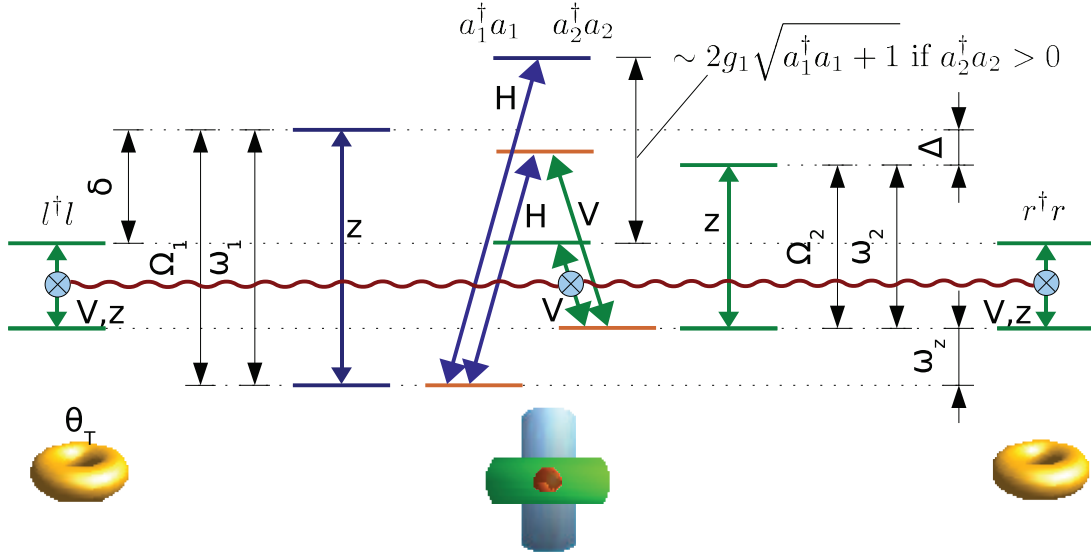


Figure 5.5: Transistor-action in QED system

A singly-doped quantum dot in both an off-resonant and a resonant cavity acts like a transistor for quantum information stored in neighboring cavities. Coherently populating photons in the cavity resonant with the quantum dot redresses the another cavity off-resonant with the quantum dot. Such an interaction changes its transport properties to be resonant or off-resonant with additional neighboring cavities,  $\Omega_T$ . Such a system can be considered a coherent, QED analog of a CMOS transistor if charged carriers are replaced with polarized photons. Initially, the collector (left cavity) has a large reservoir of photons while the emitter (right cavity) holds very few. An injection of photons into the gate (vertically-elongated cavity) changes the photonic conductivity between the left and right cavities with a certain amplification. Furthermore because the vertically oriented cavity supports polarizations orthogonal to its elongation: horizontal and out of the page, coherent control of the vertical cavity yields a complex quantum operation. In particular, the horizontally and vertically polarized photon states and the trion state from 5.2 are renormalized to two  $H$ -polarized transitions and two  $V$ -polarized transitions. One transition is common between the two and is slightly less than the frequency of the off resonant transition and the separation between the two remaining energy levels is approximately  $2g_1\sqrt{a_1^\dagger a_1 + 1}$ , when  $m$  (the number of photons in the off-resonant cavity) is low but greater than one. When the separation reaches  $\delta$ , the lowest renormalized transition becomes resonant with the horizontal component of the left and right cavities.

To begin with, consider the equations of motion for number-state precession from the control cavity into the splitting absorption band of the interaction-control cavity. We can model this state with a simple modification to the double-well

Hamiltonian (5.1). We double the Hilbert space to include each band of the absorption shift as in (5.23). This equation is only valid for the  $H$ -polarized component of photons in the control cavity. The  $z$ -polarized component tunnels like (5.1).

$$H_C = H_{DW} \cdot \mathbf{I} + \begin{pmatrix} -g_1 \sqrt{a_2^\dagger a_2 + 1} & 0 \\ 0 & g_1 \sqrt{a_2^\dagger a_2 + 1} \end{pmatrix} \quad (5.23)$$

In this analysis  $a_1$  and  $a_1^\dagger$  correspond to the control cavity and  $a_2$  and  $a_2^\dagger$  correspond to the interaction-control cavity. The time-evolution of the number states,  $N_1$  and  $N_2$  remain composite functions (5.2), (5.3) of the tunneling amplitudes  $a_1 a_2^\dagger$  and  $a_2 a_1^\dagger$ . However, the equations of motion for these tunneling amplitudes becomes more complex. They are found using the appropriate contractions to be asymmetric, due to the asymmetry of the renormalization (only one cavity is renormalized) as in (5.25)(5.25).

$$\begin{aligned} \frac{d}{dt} (a_1 a_2^\dagger) &= i \begin{pmatrix} -g_1 (\sqrt{N_2 + 1} - \sqrt{N_2 - 1}) & 0 \\ 0 & g_1 (\sqrt{N_2 + 1} - \sqrt{N_2 - 1}) \end{pmatrix} a_1 a_2^\dagger \\ &\quad + i (\Omega_2 - \Omega_1) \cdot \mathbf{I} a_1 a_2^\dagger + i \kappa (N_1 - N_2) \cdot \mathbf{I} \end{aligned} \quad (5.24)$$

$$\begin{aligned} \frac{d}{dt} (a_2 a_1^\dagger) &= i \begin{pmatrix} -g_1 (\sqrt{N_2 + 1} - \sqrt{N_2 + 3}) & 0 \\ 0 & g_1 (\sqrt{N_2 + 1} - \sqrt{N_2 - 3}) \end{pmatrix} a_2 a_1^\dagger \\ &\quad + i (\Omega_2 - \Omega_1) \cdot \mathbf{I} a_2 a_1^\dagger + i \kappa (N_1 - N_2) \cdot \mathbf{I} \end{aligned} \quad (5.25)$$

In order to preserve efficient number state precession, two similar constraints need to be met. First the co-factor to the first term on the left-hand side of (5.25) and (5.25) must be close to zero to reduce oscillations in the tunneling amplitude and second in order to avoid virtual states the line-width of the cavity must include an instantaneous absorption line of interaction-control cavity. The first condition is met if  $\Omega_2 - \Omega_1 = g_1 (\sqrt{N_2 + 1} - \sqrt{N_2 - 1})$  in the augmenta-

tion of  $N_2$  into the upper spectrum, and likewise in the depletion of  $N_2$  when  $\Omega_2 - \Omega_1 = g_1 (\sqrt{N_2 + 1} - \sqrt{N_2 + 3})$ . The second condition requires that there actually exists an occupied cavity mode in the control cavity where  $\Omega_2 - \Omega_1 = g_1 (\sqrt{N_2 + 1} - \sqrt{N_2})$ . The later is derived by the incremental change in the absorption spectrum given  $N_2$  photons have already tunneled into the interaction-control cavity. The initial  $N_2$  can be set in the system's preparation, so that the precession may start in a more active regime. As  $N$  increases the required photon mode becomes increasingly resonant with the cavity peak and the phase of the precision accelerates as shown in Figure (5.6).

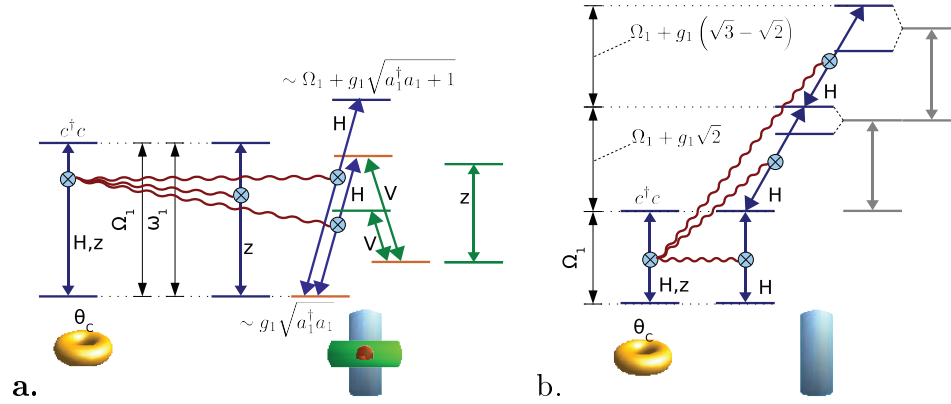


Figure 5.6: Control photons engineered to tunnel only to upper resonance

**a.** As photons from the control cavity (left) tunnel into the interaction-control cavity, their  $H$ -component interacts the  $n$ -photon absorption manifold with the  $n + 1$ -photon absorption manifold. **b.** While the first photon is resonant with the cavity, it's interaction with the quantum dot requires the next photon to have an energy increase or decrease by  $g\sqrt{2}$ . If a  $+g\sqrt{2}$  photon is available for absorption the third photon need only have a frequency increase of  $g(\sqrt{3} - \sqrt{2})$  from resonance, but will require a  $g\sqrt{3}$  decrease to be absorbed in the lower energy state. As the number of photons increase the resonance shift for an additional photon into the high energy state becomes rapidly less, while the lower energy state becomes rapidly off resonant. Therefore if the cavity's initial state includes several photons of the appropriate frequency to absorb into the upper energy level, any additional tunneled photons will overwhelmingly occupy the upper energy level, while the lower energy level will be largely avoided. This avoids any issue with  $H$ -absorption and  $V$ -emission that would leak control photons into the transport cavity.

To calculate this acceleration the density of cavity modes must be known. Assume the standard Lorentzian line-shape (5.26) normalized to have  $N_1(0)$  photons

at the peak frequency, as shown in (5.7)(a), then the effective  $N_1(0)$  as seen in simple analysis in (5.8), can be represented at any given time by the term  $N_1^{(\text{eff})}(N_2)$  defined in (5.27). The instantaneous phase of the precession is given by the inverse cosine of the number of photons,  $N_2$ , relative to the number of photons at the current resonant frequency of the cavity (the effective  $N_1$ ) as show in (5.28). Finally the instantaneous increase in  $N_2$  may be calculated as in (5.29). This differential equation assumes that the tunneling is dominated by the resonant mode and that the renormalization time is much larger than the inverse of the tunneling rate at any given time.

$$\mathcal{D}(\omega) = \frac{1}{\pi} \frac{\Omega_2/2Q}{(\omega - \Omega_2)^2 + (\Omega_2/2Q)^2} \left( \frac{N_1(0)}{\Omega_2/2Q} \right) \quad (5.26)$$

$$N_1^{(\text{eff})} \equiv \mathcal{D} \left( \Omega_1 + g_1 \left[ \sqrt{N_2 + 1} - \sqrt{N_2} \right] \right) \quad (5.27)$$

$$\phi \equiv \arccos \left( 1 - \frac{2N_2}{N_1^{(\text{eff})}} \right) \quad (5.28)$$

$$\dot{N}_2 = \frac{\sqrt{2}\kappa}{2} N_1^{(\text{eff})} \sin(\phi) \quad (5.29)$$

If we assume that the control and interaction-control cavities are exactly resonant,  $\Omega_1 = \Omega_2$ , the influx of photons into  $N_2$  is given by the non-linear differential equation (5.30) and the appropriate boundary conditions for the initial number of control photons. This equation is valid while  $N_2(t) < N_1(0)/2$  due to the definition of  $\phi$ .

$$\dot{N}_2 = \sqrt{2}\kappa \sqrt{\frac{N_1(0) N_2}{\pi (g_1 [\sqrt{N_2 + 1} - \sqrt{N_2}])^2 + \pi (\Omega_1/2Q)^2} - N_2^2} \quad (5.30)$$

Since eventually the  $N_1^{(\text{eff})}$  will converge to  $N_1$ , the above analysis suggests that with the appropriate initial photon offset, the red line in Figure 5.7(a), we can approximate the precession rate as linear,  $\kappa_C(t) \approx B\kappa_0$ , by some some co-factor,  $0.25 < B < 1$ .

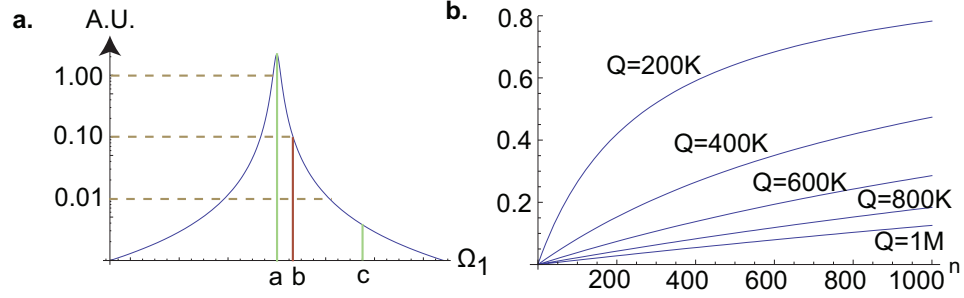


Figure 5.7: Number state precession between the control-cavity and the interaction-control cavity

**a.** A normalized Lorentzian (peak corresponds to one) is Log plotted to show the placement of  $g_1\sqrt{2}$  (offset green line) relative to the peak frequency (centered green line) for the example quantum dot system with a  $Q = 10^5$ . A small number of photons (three to five) will shrink the operating line-width (red line) of the cavity. This provides an effective DC offset into the precession sensitivity as given in **b.** The effective  $N_1$  per photon of the peak number of photons is plotted for varying values of  $Q$  for the example quantum dot system. An initial offset of thirty photons initialized in the interaction-control cavity for will provide an initial, effective  $N_1$  of 0.4 photons per resonant photon at  $Q = 10^5$ .

Finally, let us consider the effects control photons on the transport of photons from the left cavity to the right cavity. Writing down the full Hamiltonian for all five cavities and the quantum dot as shown in Figure 5.4, yields (5.31). Where  $H_0$  is the non-interacting energy levels and  $H_{IQ}$ ,  $H_{IC}$ , and  $H_{IT}$  are the interacting terms between the quantum dot and interaction cavities, the control cavity and the interaction-control cavity, and the left, right, and interaction-transport cavities respectively.

$$\begin{aligned}
 H &= H_0 + H_{IQ} + H_{IC} + H_{IT} \\
 H_0 &= \Omega_1 a_1^\dagger a_1 + \Omega_2 a_2^\dagger a_2 + \Omega_1 c^\dagger c + \Omega_T (l^\dagger l + r^\dagger r) \\
 H_{IQ} &= g_1 \left( |t\rangle \langle -| a_1 + |- \rangle \langle t| a_1^\dagger \right) + g_2 \left( |t\rangle \langle +| a_2 + |+ \rangle \langle t| a_2^\dagger \right) \\
 H_{IC} &= \kappa_C(t) \left( a_1^\dagger c + c^\dagger a_1 \right) \\
 H_{IT} &= \kappa_T(t) \left( a_2^\dagger l + l^\dagger a_2 + a_2^\dagger r + r^\dagger a_2 \right)
 \end{aligned} \tag{5.31}$$

The only unconsidered term is  $H_{IT}$ , but this interaction is a long-solved prob-

lem. It is the scattering of a particle across a well with thin, high shoulders through a resonant level. As in Baym, photons bound in the resonant level of the interaction structure resemble an unstable elementary particle with a short half-life proportional to  $1/\kappa_T$  that “decays” to the right or left well. Baym writes, “a square well with high shoulders has very sharp resonances...thus the probability amplitude for finding the particle decreases in magnitude exponentially” with a shift in energy from resonance [2]. More simply it promotes the transport of photons from the left cavity to the right cavity with amplitude  $\kappa_0^2$  during resonance an exponentially less otherwise. Therefore, we may write the transport tunneling amplitude as  $\kappa_T(t) = \kappa_0 \exp -A \left( \delta - g_1 \sqrt{a_1^\dagger a_1 + 1} \right)$ .

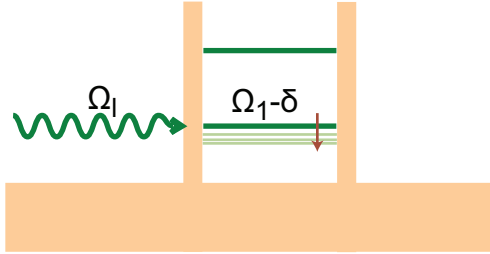


Figure 5.8: Tunneling through a square well with high shoulders

Because, the photons in the interaction-transport cavity,  $m$  of them, only begin to interact with quantum dot when  $n$  is very large, to bring the resonance level to  $\Omega_1 - \delta$ , they have little effect on the absorption spectrum. This is depicted in the cartoon above by the dense absorption lines (green lines) that are effected by the dynamical renormalization (red line). Resonant photons from the left cavity,  $\Omega_l = \Omega_1 - \delta$ , will tunnel en masse to the right cavity since the shoulders of the well are very thin. Any photons captured in the center well have a short lifetime and will decay tot he left or right cavity as no bound state exists in the real momentum space.

In total then, we may approximate the number of photons in the right well as the non-linear interaction between two oscillators as first described in Section 5.1.1. The equations of motion for the number state in the right-well (5.32) appear as a precession from the left well,  $[l^\dagger l](0)$  only during the times when an auxiliary precession of control photons,  $c^\dagger c(0)$ , renormalizes the left interaction between the left and right well to be resonant. By keep  $[c^\dagger c](0)$  and  $[l^\dagger l](0)$  very large the resonant peaks remain very sharp and the process looks almost irreversible.

$$[r^\dagger r](t) \approx \frac{[l^\dagger l](0)}{2} \left[ 1 - \cos \left( \sqrt{2} \int_0^t dt' \kappa_0^2 \exp[-\phi(t')] \right) \right] \quad (5.32)$$

$$\phi(t) = 2A \left[ \delta - g_1 \sqrt{\frac{[c^\dagger c](0)}{2} \left[ 1 - \cos \left( \sqrt{2} B \kappa_0 t \right) \right] + a_1^\dagger a_1(0) + 1} \right] \quad (5.33)$$

The significance of the above construction comes from the arbitrary polarization available to both “control” photons and “transport” photons. With a carefully designed initial state, a coherent, self-timed operation will follow that may be cascaded. Consider an encoding of a single qubit state into a condensate of  $q$  photons in the number-state and polarization basis  $\{|H, z\rangle_c\}$ , where  $H$  and  $z$  are the number photons polarized as such, as shown in (5.34). Such a condensate may be produced by repeatedly exciting a quantum dot in a superposition of states. This is not to be confused with a classical ensemble of photons  $\left( \cos\left(\frac{\theta}{2}\right) b_{H,\omega}^\dagger + \sin\left(\frac{\theta}{2}\right) e^{i\phi} b_{V,\omega}^\dagger \right)^q$  which requires control over the number-state to precisely control each amplitude.

$$\begin{aligned} |\psi\rangle &= \cos\frac{\theta}{2} |n, 0\rangle_c + \sin\frac{\theta}{2} \sum_{m=1}^{\delta n} |0, m\rangle_c \\ &\rightarrow \cos\left(\frac{\theta}{2}\right) |1\rangle \otimes \left(b_{H,\omega}^\dagger\right)^q |\emptyset\rangle + \sin\left(\frac{\theta}{2}\right) |0\rangle \otimes \left(e^{i\phi} b_{z,\omega}^\dagger\right)^q |\emptyset\rangle \end{aligned} \quad (5.34)$$

Because the dominant noise of a QED cavity is simply relaxation and not dephasing,  $q$  may decrease and be lost to the environment but  $\theta$  is largely unaffected. Experimental cavity studies show that polarization loss is negligible. If we consider the Hilbert space to include the left and right cavities and their polarization, we may write the total number state basis as the number of  $H$ -polarized,  $V$ -polarized, and  $z$ -polarized photons,  $|H, z\rangle_c |V, z\rangle_l |V, z\rangle_r$  in the control, left, and right cavities respectively. Then a coherent quantum operation may take place through the interaction cavities that resemble a quantum AND gate Table 5.1(a). In this table,  $p$  and  $q$  vary with time non-linearly. This is shown by tracing over the interaction cavities in the linear decomposition of the right and control cavity number states.

Alternatively, the same system can be used with two quantum dots, one dot

Table 5.1: Quantum operation truth table

|           |                  |                                                        |                                                    |
|-----------|------------------|--------------------------------------------------------|----------------------------------------------------|
|           | one-dot          | $ n, 0\rangle_c$                                       | $ 0, n\rangle_c$                                   |
| <b>a.</b> | $ p, 0\rangle_l$ | $ n - m, 0\rangle_c  p - q, 0\rangle_l  q, 0\rangle_r$ | $ 0, n - m\rangle_c  p, 0\rangle_l  0, 0\rangle_r$ |
|           | $ 0, p\rangle_l$ | $ n - m, 0\rangle_c  0, p\rangle_l  0, 0\rangle_r$     | $ 0, n - m\rangle_c  p, 0\rangle_l  0, 0\rangle_r$ |
|           | two-dot          | $ n, 0\rangle_c$                                       | $ 0, n\rangle_c$                                   |
| <b>b.</b> | $ p, 0\rangle_l$ | $ n - m, 0\rangle_c  p - q, 0\rangle_l  q, 0\rangle_r$ | $ 0, n - m\rangle_c  p, 0\rangle_l  0, 0\rangle_r$ |
|           | $ 0, p\rangle_l$ | $ n - m, 0\rangle_c  0, p - q\rangle_l  0, q\rangle_r$ | $ 0, n - m\rangle_c  0, p\rangle_l  0, 0\rangle_r$ |

that couples  $H$  to  $V$  but also another that couples  $H$  to  $z$  to transport the full, local polarization of the left-cavity. This produces the truth table in Table 5.1(b) and may be advantageous to keep resultant polarization decoupled from the number state.

#### 5.1.4 Time-Scale Design

There are six timescales whose relative strengths determine the degree of Fidelity in the operation: the speed of the renormalization processes, the tunneling rate of control photons, the tunneling rate of transport photons, and rate of absorption and emission of the quantum dot from the trion-state to it's resonant cavity, the time-scale of the measurement process, and the timescale of the photon leakage from cavity (or photon-phonon coupling). For rough timescales for these operations are given in table. In a sample system, a InAs self-assembled quantum dot in a  $\sim 8T$  magnetic field can produce a trion transition frequency,  $\omega_1$  of  $\sim 1.2eV$ . This is much larger than the Stark shift which is  $\sim 0.1meV$  and a reasonable detuning (given the line-width of cavities) of  $\sim 0.5meV$ . Micro-sphere cavities can be made with a  $Q = 10^5$  and a fast polarization measurement. The coupling of the cavity to a wave-guide can be made high ( $\sim 0.2meV$ ), and such coupling can approximate the resonant cavity-to-cavity coupling by a thin barrier.

The time of the operation,  $T$ , is approximately given by . This formula assumes for simplicity, with an appropriate initial number of photons  $a_1^\dagger a_1(0)$ , that  $B = 0.5$ .

$$T \approx \frac{\sqrt{2}}{\kappa_0} \arccos \left( 1 - \frac{2}{[c^\dagger c](0)} \left[ \left( \frac{\delta}{g_1} \right)^2 - a_1^\dagger a_1(0) + 1 \right] \right)$$

Table 5.2: Timescales of self-timed computation

| Parameter     | Timescale     | Function                                         |
|---------------|---------------|--------------------------------------------------|
| $c/L$         | $\sim .001fs$ | Renormalization speed                            |
| $N_T\kappa_T$ | $\sim 10ps$   | Resonant transport speed                         |
| $N_C\kappa_C$ | $\sim 100ps$  | Resonant control speed                           |
| $g_1g_2g_P$   | $\sim 10ns$   | Cavity-to-cavity leakage: phonon, $g_P$ mediated |
| $\tau_m$      | $\sim 10ns$   | Measurement speed:                               |
| $Q$           | $100ns$       | QED cavity quality                               |

A number state measurement of the photons in the right cavity,  $N_r$ , collapses a quantum state to a logical one or a logical zero. In particular, the measurement can be thresholded against a reference number,  $N_{th}$ . If  $N_r > N_{th}$  it is a one otherwise it is a zero. Calculate the optimal  $N_{th}$  for the right well, it must be run several times for all initial eigenstates which should produce a logical zero with a measurement of the right cavity. The eigenstate with the highest average photon count,  $N_L$ , provides worst case lower-bound. Conversely, when several runs and measurements of the system using eigenstates that produce a logical one are completed, the eigenstate with the lowest average photon count,  $N_H$ , provides the worst case upper-bound. Therefore a worst-case threshold is given by  $N_{th} = (N_H - N_L) / 2$ .  
T

Because the phonon-photon interaction at the cavity boundaries contains no spin-selection rules, any energy loss will subtract from both superpositions equally. In this sense, the photonic-condensate encoding is protects against decoherence: a ensemble of photons protect against relaxation, while the polarization condensate protects against dephasing. If coupling between the number state and the polarization leads to unacceptable noise through cavity leakage, then placing two orthogonally grown quantum dots in each quantum-cross point to renormalize both the  $V$  and  $z$  modes will allow full polarization transport of photons from the left and right cavities. The coupling of the quantum system to the environment is only through loss from each cavity. In a simple model, this is directly correlated to the line-width of the cavity.

### 5.1.5 Quantum Fourier Transform using Entropic Control: Self-Timed Quantum computing

The above structure may be used to construct various types of complex coherent quantum circuits by the careful design of cavity frequencies. A sample of cascades is shown in Figure 5.9, with the caveat that the net operating time must be less than the loss rate of the cavities. Such construction aside, most useful quantum circuits involve variations of the QFT. We show that using a cascadable structure like the one designed in Figure 5.10, a network like the one in Figure 5.11 can implement this transform in a scalable manner.

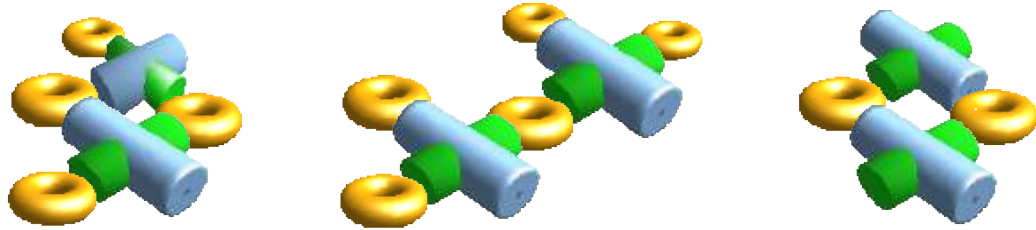


Figure 5.9: Directly cascading the self-timed quantum operation

The self-timed quantum structure may be directed cascaded to perform complex quantum operations. A careful consideration of all the cavity frequencies are necessary in order to get the desired operation. The left cascade performs differential transport with one vertically-elongated interaction-control cavity and one horizontally-elongated control cavity. With the right manufacture of electric field this structure transport either collector to a common emitter. The center cascade coherently mixes polarizations. The right cascade, performs a feedback-amplification loop. This interesting structure coherently amplifies a small number of photons that may appear in the emitter from the collectors by exploiting the all three axes  $H$ ,  $V$ , and  $z$  instead of just two.

Given this implementation, one can use the self-timed structure described in Section (5.1.3) to construct coherent multiplexer shown in Figure 5.10(a), which in turn can be cascaded to build the transform as shown in (5.11)(b). This is explained below.

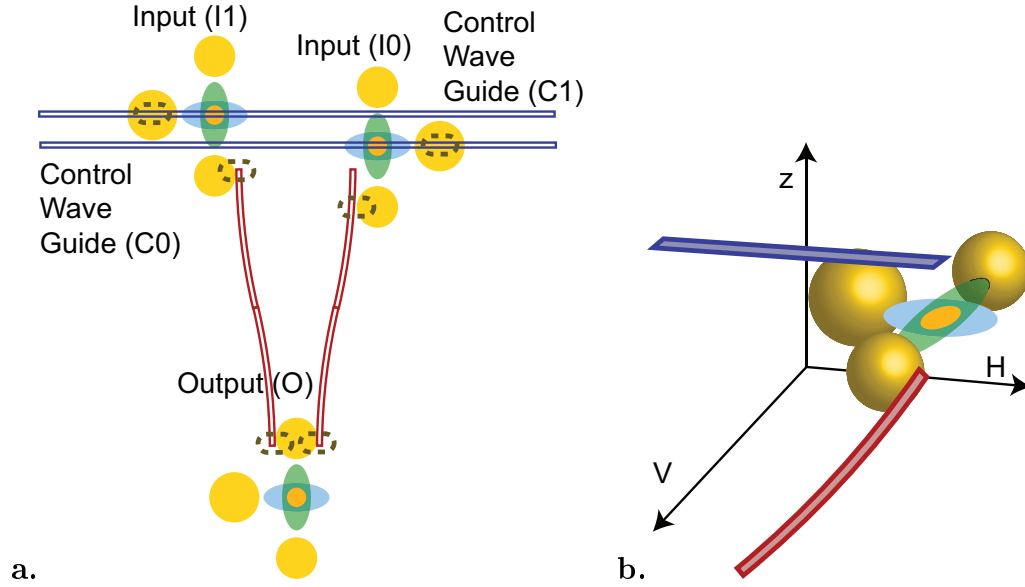


Figure 5.10: Cascading a coherent QED Multiplexer

**a.** A cascading structure uses wave guides to support the mode of propagation required by polarization encoding. Two QED inputs,  $I1$  and  $I0$ , are filled with particularly polarized photons. These photons will tunnel to one output  $O$  based on the QED control signals  $C0$  and  $C1$ . If the photons in the  $C1$  waveguide (blue) are horizontally polarized, the polarization of  $I1$  will tunnel through to  $O$  along vertically running wave guide (red) and similarly for the  $C0$  wave guide, the input  $I0$ , and the common output,  $O$ . The output waveguides (red) must bend slightly to reduce multiple input cavities to one output cavity. This requires the wave guides to be long enough that only a slight bend is necessary. The control wave guides can couple to several cavities given the number of photons in its source. Coupling between the waveguides and the cavities are indicated by the brown, dashed ellipses. If  $C0$  and  $C1$  are differentially entangled, such that when the photons in  $C0$  are  $H$ -polarized, the photons in  $C1$  are  $z$ -polarized and vice-versa, there will be a superposition of polarizations in the output  $O$ . **b.** Two layers of wave guides are needed in order to ensure horizontal propagation of control signals and vertical propagation of data signals without crossings. This is depicted with micro-sphere cavities.

Consider the structure in (5.10) which coherently multiplexes two QED inputs,  $I1$  and  $I0$ , onto one output  $O$  based on the QED control signals  $C0$  and  $C1$ . If the photons in the  $C1$  waveguide are horizontally polarized, the polarization of  $I1$  will tunnel through to  $O$  and similarly for the  $C0$  wave guide, the input  $I0$ , and the common output,  $O$ . If  $C0$  and  $C1$  are differentially entangled, such that when the photons in  $C0$  are  $H$ -polarized, the photons in  $C1$  are  $z$ -polarized and vice-versa,

there will be a superposition of polarizations in the output  $O$ .

5.11(c) uses this structure to implement a two-qubit QFT, while 5.11(b) generalizes the structure for any number of qubits. First redundantly encode the input  $X$  into an entangled set of photonic condensates in polarization as in (5.35), that are coupled to a set of control wave guides,  $C0$  and  $C1$ , as shown in 5.10(a). Such a condensate can be produced by mapping qubit states to photons with a series of  $q$  excitations. The exact value of  $q$  is not important it just needs to be large enough to transition the off-resonant to resonant interaction quickly compared to cavity loss.

$$|X\rangle = \sum_{a=0}^{N-1} x_a \prod_{j=1}^n \left( t_{H,\omega_1}^{(j)\dagger} \right)^{aq_j} \left( t_{z,\omega_1}^{(i)\dagger} \right)^{\bar{a}q_j} |\emptyset\rangle \quad (5.35)$$

$$|A_k^{(j)}\rangle = \left[ e^{2\pi i 0.k} \left( t_{V,\omega_2}^{(j)\dagger} \right)^{q_j} + \left( t_{z,\omega_2}^{(i)\dagger} \right)^{q_j} \right] |\emptyset\rangle \quad (5.36)$$

According to the calculated equations of motion, a precession of number state for the  $H$ -component of  $|\psi\rangle$  into the interaction cavity renormalizes the tunneling rate of photons from the left to the right cavity. If each multiplexer tree,  $l$ , is input with phase angles  $A_1 - A_8$  that match the power set of  $f(a, l)$  as shown in 5.11(c) and  $A_{f(a,l)}^l$  using 5.36, then each tree will select appropriate angle for that qubit. The total output is seen to exactly perform the transform 5.38.

$$Y_l = \sum_a x_a A_{f(a,l)}^{(l)} \quad (5.37)$$

$$|Y\rangle = \sum_a x_a \left[ e^{2\pi i 0.a_n} \left( t_{V,\omega_2}^{(0)\dagger} \right)^{q_j} + \left( t_{z,\omega_2}^{(0)\dagger} \right)^{q_j} \right] \quad (5.38)$$

$$\left[ e^{2\pi i 0.a_{n-1}a_n} \left( t_{V,\omega_2}^{(0)\dagger} \right)^{q_j} + \left( t_{z,\omega_2}^{(0)\dagger} \right)^{q_j} \right] \dots \quad (5.39)$$

$$\dots \left[ e^{2\pi i 0.a_1 a_2 \dots a_n} \left( t_{V,\omega_2}^{(0)\dagger} \right)^{q_j} + \left( t_{z,\omega_2}^{(0)\dagger} \right)^{q_j} \right] |\emptyset\rangle \quad (5.40)$$

Therefore, measuring the ensemble polarization of the final cavity is equivalent to the QFT. However, when implementing the  $\pi/128$  phase-truncated QFT, every

multiplexer structure is equivalent except the final seven,  $f(a, l \geq 7) = f(a, n)$ , which conform to the structure used in 5.11(c).

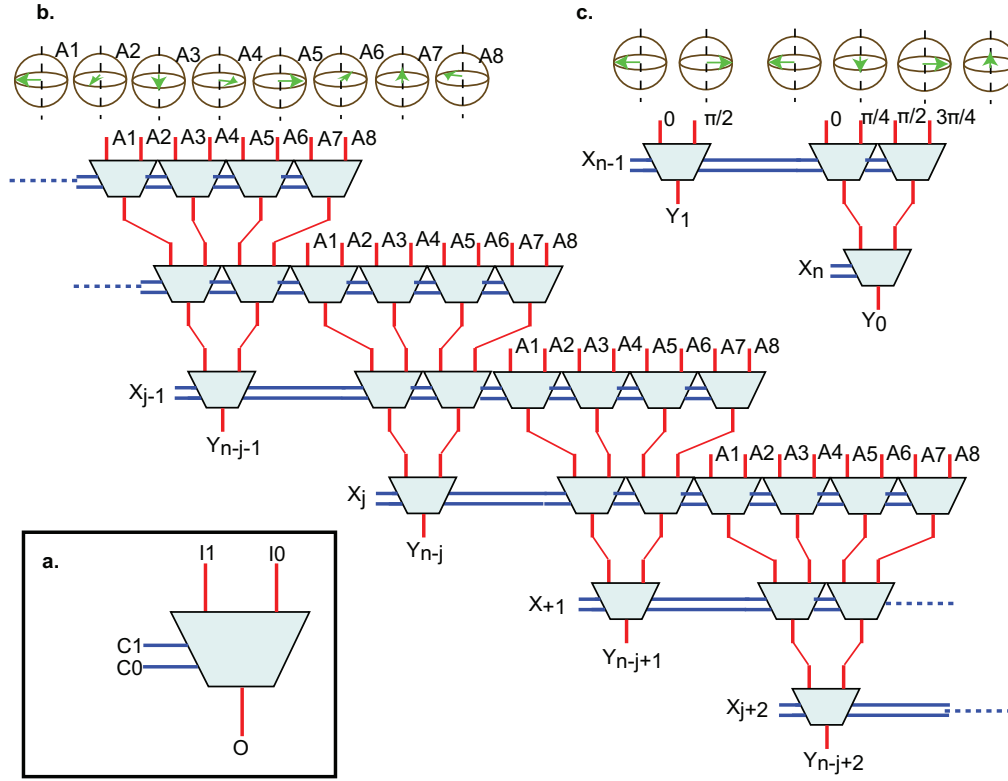


Figure 5.11: Self-Timed Quantum Fourier Transform

**a.** The structure in 5.10 is presented as a multiplexer. Each of the terminals represent a cavity that is coupled to a waveguide or a laser. Due to coherence, all terminals have some level (timescale) of bi-directionality. However, the equations of motion show that the time of the entire quantum computation will be much less than the timescale of bi-directionality between the input terminals,  $I$ , and the output terminals,  $O$ . **b.** Multiplexers are arranged so that the appropriate value is coherently “looked-up” from one of eight encoded polarizations,  $A_i | i \in \{1 \dots 8\}$ . This is the quantum analog of a traditional lookup table often used in the implementation of gate arrays. All of the differential input qubits to QFT are given by photonic condensates encoding  $n$  qubits,  $X_i | i \in \{0 \dots n-1\}$  that couple to seven control seven multiplexers. One or more lasers continually pump  $A_i$  and  $n$  photodetectors record the polarization at the output cavities  $Y_i$ . Ensemble polarization measurements encode the results of the transform or an unmeasured cavity may be used for further quantum processing.

This formulation of the transform as a look-up table has a space-complexity less than  $n \log_4 n$  connected multiplexers with a pseudo-constant time complexity, when

the operations are parallelized. Because of the phase truncation, an exponential reduction over a classical Fourier transform's complexity is still maintained.

To scale this system to thousands of qubits, simply increase the laser intensity driving the now larger lookup table  $A_i$  of 128 polarization angles. The number of photons observed at the photodetector will be commensurately more

### 5.1.6 Summary

We have shown for the first time that a closed quantum system may be constructed to precisely evaluate a Quantum Fourier Transform. This evaluation takes place by the natural evolution of an initial energy uncertainty. By “entropic control” the evaluation appears non-periodic due to a non-uniform interaction between two quantum oscillators. The dual cavity system described provides an example of such a system. With the proper representation of logical one and logical zero in a number basis, any measurement during the overwhelming majority of this system's phase will sample a high fidelity. In general, it appears that when a second quantum degree of freedom, e.g. polarization, is coupled with the controlled degree of freedom, e.g. number-state, the controlled degree of freedom will work coherently upon the uncontrolled, second quantum degree of freedom. This paradigm allows the natural evolution of a closed system to work as quantum circuit, and facilitates quantum circuit design by physical circuit topologically rather than the reversible circuits traditionally used to represent a quantum algorithm. The presented system preserves the benefits of quantum ensembles and quantum dots by using a condensate in dephasing-free environment.

## 5.2 Acknowledgements

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# Chapter 6

## Summary and conclusions

There are two overriding themes in this dissertation. First, the complexity with which one classically imparts information into a quantum system by direct modulation of the space and time of interactions between qubits and fields can be traded-off. We have shown that instead of increasing the entropy of encoding (Kolmogorov complexity) with universal operations and universal control, one can limit oneself to only Clifford operations and use quantum coherent feedback. This feedback increases the entropy of fields so that subsequent Clifford operations using said fields have greater controllability. Using such an encoding, both the Quantum Fourier Transform and Amplitude Amplification may be implemented efficiently using only operations from the Clifford group. Doing so, generalizes coherent quantum feedback to locally-variant bases and amplitude-amplification to differential, quantum-operation search. This generalization required the creation of an alternative to the Grover call where the initial wavefunction is not exactly known. We have shown such a call exists using differential tracks of qubits, such that enough information is provided about the actual initial state of the qubit to make this desired reflection possible. Given the ubiquitous trade-off in quantum computing systems between the isolation of the system and the precision and speed of quantum operations upon it, our method suggests that systems supported by strongly coherent control fields should be strongly isolated. The resulting imprecision in the operations may be corrected to an appropriate order using unitary-quantum fault tolerance, where unitary refers to the otherwise external degrees of free-

dom (the modulator) controlling the error in the control field. When applicable, this unitary fault tolerance vastly simplifies the resource requirements otherwise needed by general fault-tolerant operation. We have shown a method for encoding a pseudo-continuous set of quantum operations in classically stable degrees of freedoms. We have shown for the first time that a closed quantum system may be constructed to precisely evaluate a Quantum Fourier Transform. By “entropic control,” the evaluation appears non-periodic due to a non-uniform interaction between two quantum oscillators. When a second quantum degree of freedom, e.g. polarization, is coupled with the controlled degree of freedom, e.g. number-state, the controlled degree of freedom will work coherently upon the uncontrolled, second quantum degree of freedom. This paradigm allows the natural evolution of a closed system to work as quantum circuit, and facilitates quantum circuit design by physical circuit topologically rather than the reversible circuits traditionally used to represent a quantum algorithm. The second theme, relates the same modulation to classical ensembles of spin. We have shown the non-intuitive result that fast modulation of the spin-polarization does not necessarily lead to higher information rates in semiconductor. Because non-equilibrium populations have spin-currents proportional to the momentum relaxation time (not the spin-relaxation time), a high spacial frequency of spin polarization will quickly equilibrate and destroy spin-information. So even though spin coherence times may be long in silicon, it does not mean that a higher rate of information can be transported. The spin-logic paradigm proposed, where information is transported by currents, and computation is performed by spin-diffusion, appears optimal from an information capacity perspective. In the future, materials that have momentum-relaxation rates similar to spin-relaxation rates should be considered for fully spin-based logic paradigms.