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UNIVERSITY OF CALIFORNIA, SAN DIEGO

Groups, Nonassociative algebras, and Property (T)

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Zezhou Zhang

Committee in charge:

Professor Efim Zelmanov, Chair
Professor Vitali Nesterenko
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2014

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The dissertation of Zezhou Zhang is approved, and it is acceptable in quality and form for publication on micro-film and electronically:

Chair

University of California, San Diego

2014

EPIGRAPH

*ahuredasu namida ga utsukusi kereba
hito ha mata owaranu tabi ni
toki wo tsuiyaseru kara.*

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Chapter 2 and 4 includes a reinterpretation of, and borrows heavily from “Nonassociative algebras and groups with property (T) ”([45]). This paper has been accepted for publication in the Intern. J. Algebra and Computation, authored solely by the dissertation author.

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PUBLICATIONS

Ze Zhou Zhang, “*Nonassociative algebras and groups with property (T)*”, to appear in the Intern. J. Algebra and Computation.

Mikhail Ershov, with Andrei Jaikin-Zapirain, Martin Kassabov and Ze Zhou Zhang, “*Groups graded by root systems and property (T)*”, submitted.

ABSTRACT OF THE DISSERTATION

Groups, Nonassociative algebras, and Property (T)

by

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Doctor of Philosophy in Mathematics

University of California, San Diego, 2014

Professor Efim Zelmanov, Chair

We investigate certain groups satisfying commutator relations given by Root systems, display their connection to Nonassociative algebras, in particular Jordan and alternative algebras, and give in the process examples of new property (T) groups among this family.

Chapter 1

Overview

Notations:

In this dissertation, we adopt the following conventions:

A *ring* will always be unital, but not necessarily associative;

Φ an irreducible reduced classical root system;

When a group is concerned, $(x, y) = x^{-1}y^{-1}xy$ and $a^b = b^{-1}ab$

This dissertation focuses on groups related to nonassociative algebras.

Root graded groups (the topic of this dissertation) abound in literature in different contexts, including Chevalley groups of adjoint type and their twisted analogues. An interesting question would be when such groups possess property (T) . Recently Ershov, Jakin-Zapirain and Kassabov [13] showed that under a rather weak restriction on the grading, the union of root subgroups is a Kazhdan subset of G , where G is a group graded by Φ , an irreducible classical root system of rank ≥ 2 . As a result, for any reduced Φ and a finitely generated commutative ring R , the Steinberg group $St_{\Phi}(R)$ and the elementary Chevalley group $\mathbb{E}_{\Phi}(R)$ have property (T) . Utilizing their results, we give in chapter 4 new examples of property (T) groups, constructed from non-associative algebras.

Back in the 80's, Faulkner showed in [14] that an “elementary Chevalley group of type A_2 ” could be defined likewise over alternative rings of arbitrary characteristic. This suggested that the likes of Steinberg groups over other rings

is available, So it would be rather nice to find a unified approach to such groups. Influenced by the papers on classification of root graded Lie algebras [5][4], it is convincing that the commutator relations of the root subgroups actually will pose restrictions on the underlying base ring. We will elaborate on this in Chapter 3, and obtain classification results in this direction.

Chapter 2

Preliminaries

2.1 Property (T)

As a well known notion in representation theory, equivalent definitions of property (T) abound in literature. We refer the readers to “The (T) book” [3] for details. This paper will use a version extracted from [13], that suits our setting of discrete groups:

Definition 2.1.1. Let G be a (discrete) group and S a subset of G , let V be a unitary representation of G . A nonzero vector $v \in V$ is called (S, ϵ) -invariant if

$$\|sv - v\| \leq \epsilon\|v\| \text{ for any } s \in S.$$

If there exists a finite set $S \subset G$ and $\epsilon > 0$ such that for any unitary representation V of G , the existence of a (S, ϵ) -invariant vector implies the existence of a vector $v \in V$ fixed by G , we say that G *has property (T)* or alternatively, *is Kazhdan*. Such S is called a finite Kazhdan subset.

Remark 2.1.2. It is a rather elementary fact that $\kappa(G, S) > 0$ implies that S generates G . Conversely, if G has property (T), then $\kappa(G, S) > 0$ for any finite generating set S of G , although the Kazhdan constant $\kappa(G, S)$ depends on S .

Kazhdan subsets (not necessarily finite) always exist: although it might be quite large.

Lemma 2.1.3. *For any group G we have $\kappa(G, G) \geq \sqrt{2}$.*

Proof. See, e.g., [3, Proposition 1.1.5] □

In many circumstances, it is easier to find an infinite Kazhdan set in our group of interest G . In these situations, we shall need relative property (T) to establish that the group is Kazhdan. This notion was originally defined for pairs (G, H) where H is a normal subgroup of G (see [30, 10]):

Definition 2.1.4. Let G be a group and H a normal subgroup of G . The pair (G, H) has *relative property (T)* if there exists a finite set S and $\varepsilon > 0$ such that if V is any unitary representation of G with an (S, ε) -invariant vector, then V has a (nonzero) H -invariant vector. The supremum of the set of all such ε 's with this property (for a fixed set S) is called the *relative Kazhdan constant* of (G, H) with respect to S , denoted $\kappa(G, H; S)$.

The notion above can be generalized to pairs (G, B) , (see [10], also a remark in [12, Section 2]) where B is an arbitrary subset of a group G :

Definition 2.1.5. Let G be a group and B a subset of G . The pair (G, B) has *relative property (T)* if for any $\varepsilon > 0$ there is a finite subset S of G and $\mu > 0$ such that if V is any unitary representation of G and $v \in V$ is (S, μ) -invariant, then v is (B, ε) -invariant.

In practice, the following “bounded generation principle” provides a nice way to generate from given subsets of G with relative property (T) bigger ones with such property:

Lemma 2.1.6. *Let G be a group, $B_1, \dots, B_k$ a finite collection of subsets of G . Suppose that (G, B_i) has relative (T) for each i , then $(G, B_1 \dots B_k)$ also has relative (T). Here $B_1 \dots B_k$ is the set of all elements of G representable as $b_1 \dots b_k$ with $b_i \in B_i$.*

The Following two theorems are crucial for establishing property (T) in Chapter 4:

Theorem 2.1.7. *[13, Theorem 5.1] Let Φ be a regular root system, and let G be a group which admits a strong Φ -grading $\{X_\alpha\}$. Then $\cup X_\alpha$ is a Kazhdan subset of*

G , and moreover the Kazhdan constant $\kappa(G, \cup X_\alpha)$ is bounded below by a constant κ_Φ which depends only on the root system Φ .

Theorem 2.1.8. [20, Theorem 1.2]

Let R be any finitely generated associative ring, $R * R$ the free product of two copies of the additive group of R , and consider the semi-direct product $(R * R) \ltimes R^2$, where the first copy of R acts by upper-unitriangular matrices, that is, $a \in R$ acts as left multiplication by the matrix $e_{1,2}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, and the second copy of R acts by lower-unitriangular matrices. Then the pair $((R * R) \ltimes R^2, R^2)$ has relative property (T).

The first one will be used for establishing natural Kazhdan subsets, while the second one for relative property (T).

2.2 Case A_2 and Alternative rings

Faulkner[14] showed that the notion of a “Steinberg group of type A_2 ” could be defined over any alternative ring. We now elaborate on this example.

Definition 2.2.1. A ring $\mathcal{A}$ is said to be *alternative* if $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all x, y in $\mathcal{A}$. Such naming is due to the following fact: set $(x, y, z) := (xy)z - x(yz)$ (which we call the *associator*), then in all such $\mathcal{A}$, $(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = \text{sgn}(\sigma)(x_1, x_2, x_3)$ is true for any $x_i \in \mathcal{A}$ and any $\sigma \in S_3$.

Examples. Any associative ring; the octonions $\mathbb{O}$.

Definition 2.2.2. For R associative, we can define the following:

- (“The elementary group”): $E_n(R)$ is the group generated by the elementary matrices $I + aE_{ij}$ where $a \in R$, $i \neq j$. Denote its center by Z_0 . It has a projective version $PE_n(R) := E_n(R)/Z_0$
- (“The elementary Lie algebra”): $e_n(R)$ is the Lie ring generated by aE_{ij} , where $a \in R$, $i \neq j$. Denote its center by $\mathcal{Z}_0$. It also has a projective version $pe_n(R) := e_n(R)/\mathcal{Z}_0$.

Over a non-associative ring R , assuming $n \geq 3$, we can mimic the above construction:

- (“The Steinberg group”) $St_n(R)$ is the group generated by the symbols $x_{ij}(a)$, $1 \leq i \neq j \leq n$, $a \in R$ subject to the relations:

$$\begin{aligned} (1) \quad x_{ij}(a)x_{ij}(b) &= x_{ij}(a+b), \\ (2) \quad [x_{ij}(a), x_{ik}(b)] &= x_{ik}(ab) \quad \text{for } i, j, k \text{ distinct}, \\ (3) \quad [x_{ij}(a), x_{ik}(b)] &= 1 \quad \text{for } j \neq k \text{ and } i \neq l. \end{aligned}$$

- (“The Steinberg Lie algebra”) $st_n(R)$ is the Lie algebra generated by the symbols $l_{ij}(a)$, $1 \leq i \neq j \leq 3$, $a \in R$ subject to the relations:

$$\begin{aligned} (4) \quad l_{ij}(a) + l_{ij}(b) &= l_{ij}(a+b), \\ (5) \quad [l_{ij}(a), l_{ik}(b)] &= l_{ik}(ab) \quad \text{for } i, j, k \text{ distinct}, \\ (6) \quad [l_{ij}(a), l_{ik}(b)] &= 0 \quad \text{for } j \neq k \text{ and } i \neq l. \end{aligned}$$

Remark 2.2.3. Of course one may see “collapsing” resulting from this construction, i.e. the relations might yield groups/Lie algebras with root subgroups/subspaces smaller than R . However, this collapsing can be avoided: it was shown in [14] that if $n \geq 4$ and R is associative, or if $n = 3$ and R is alternative, then there is no collapsing.

The adjoint action of $E_n(R)$ on $e_n(R)$ is defined as:

$$(I + aE_{ij})x(I - aE_{ij}) = x + [aE_{ij}, x] - (aE_{ij})x(aE_{ij})$$

Using the fact that both Z_0 and $\mathcal{Z}_0$ lie inside $Z(R)I$, it follows that $PE_n(R)$ acts faithfully on $pe_n(R)$. By expanding the second and third term of the right hand side, Faulkner [14] gave a definition of $PE_3(R)$ for R an alternative ring :

Definition 2.2.4. If $a \in R$ is an alternative ring, define the following map on $pe_3(R)$: $e_{ij}(a) = id + d_1 + d_2$, where $d_1(x) = [aE_{ij}, x]$ and $d_2(x) = -(aba)E_{ij}$ if the j, i -component of x is bE_{ji} . Since $(ab)a = a(ba)$ in an alternative ring, we are safe to omit the parentheses for aba .

Proposition 2.2.5 ([14],(A.7)). *If R is an alternative ring, then $e_{ij}(a)$ is an automorphism of $pe_3(R)$. Moreover,*

- (a) $e_{ij}(a)e_{ij}(b) = e_{ij}(a + b)$
- (b) $[e_{ij}(a), e_{jk}(b)] = e_{ik}(a + b)$ for i, j, k distinct.
- (c) $[e_{ij}(a), e_{jk}(b)] = id$ for $j \neq k, i \neq l$.

The group generated by $\{e_{ij}(a) | 1 \leq i \neq j \leq 3, a \in \mathcal{A}\}$ will be denoted by $PE_3(\mathcal{A})$.

Proof. See [14, A.7]. □

The following is an analog of Steinberg groups over associative rings:

Definition 2.2.6. For an alternative ring $\mathcal{A}$, denote by $St_3(\mathcal{A})$ the group generated by the the set of symbols $\{e_{ij}(a) | 1 \leq i \neq j \leq 3, a \in \mathcal{A}\}$ satisfying the relations (1)-(3) of Proposition 2.2.5.

The commutator relations above endow A_2 -gradings to both $PE_3(\mathcal{A})$ and $St(\mathcal{A})$. Under the standard realization $\{w_i - w_j | 1 \leq i \neq j \leq 3\}$ of A_2 , their root subgroups are $G_{w_i - w_j} = G_{ij} := \langle e_{ij}(a), a \in \mathcal{A} \rangle$. This shall be a starting point for our later considerations.

2.3 Jordan Theory

The above construction is actually a special case of *groups arising from Jordan pairs* that we will present in this section. Nice references are [27, Section §7-§10] (or the short version [28]) and [25], from which most notations in this section are borrowed.

Definition 2.3.1. Let k be a commutative ring. A *quadratic map* between k -modules $Q : M \rightarrow N$ is one such that the following holds:

- (1) For all $a \in k, v \in M, Q(av) = a^2Q(v)$,
- (2) For $x, y \in M, Q(x + y) - Q(x) - Q(y)$ is a k -bilinear map.

Definition 2.3.2 ([28]). Let k be as in above, $\sigma \in \{\pm\}$. Then a *Jordan pair* $V = (V^+, V^-)$ is a pair of k -modules together with a pair (Q_+, Q_-) of quadratic maps $Q_\sigma : V^\sigma \rightarrow \text{Hom}_k(V^{-\sigma}, V^\sigma)$ such that the following identities hold in all base ring extensions of V :

$$D_\sigma(x, y)Q_\sigma(x) = Q_\sigma(x)D_{-\sigma}(y, x), \quad (2.1)$$

$$D_\sigma(Q_\sigma(x)y, y) = D_\sigma(x, Q_{-\sigma}(y)x), \quad (2.2)$$

$$Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x), \quad (2.3)$$

Here $D_\sigma : V^\sigma \times V^{-\sigma} \rightarrow \text{End}_k(V^\sigma)$, is the bilinear map associated to Q :

$$D_\sigma(x, y)z = Q_\sigma(x + z)y - Q_\sigma(x)y - Q_\sigma(z)y,$$

In the succeeding text, we may also use the triple product notation $\{x, y, z\} := D_\sigma(x, y)z$ where $x, z \in V^\sigma$, $y \in V^{-\sigma}$. Following [25, chapter 2.0], the subscript σ may be dropped when clearly implied by the context, and we may write Q_xy for $Q(x)y$.

Remark 2.3.3. Let k be as in above. A (*quadratic*) *Jordan algebra* J is a k -module on which a quadratic map $U_x : J \rightarrow \text{End}_k(J)$ is defined, such that the following operator identities hold in all base ring extensions of J :

$$V_{x,y}U_x = U_xV_{y,x}, \quad (2.4)$$

$$V_{U_xy,y} = V_{x,U_yx}, \quad (2.5)$$

$$U_{U_xy} = U_xU_yU_x, \quad (2.6)$$

Here $V : J \times J \rightarrow \text{End}_k(J)$ is the bilinear map defined by

$$V_{x,y}(z) = U_{x+z}(y) - U_x(y) - U_z(y),$$

denoted $\{x, y, z\}$. This operator J is called unital if there exists an element $1 \in J$ such that $U_1 = \text{Id}_J$.

Remark 2.3.4. The concept of Jordan pairs is a natural generalization of

(Quadratic) Jordan algebras: given any Jordan algebra J , the pair (J, J) with $Q_xy := U_x(y)$, where $x, y \in J$ and U_x the quadratic operator in J , is automatically a Jordan pair since Q_x and U_x satisfies exactly the same defining identities. So in this setting, we will use Q_x and U_x interchangeably. Similarly with $V_{x,y}$ and $D(x, y)$.

Examples. Among other constructions, a major source of Jordan pairs is matrices. E.g. Let k be a field. if

(α) $V^+ = V^- = \mathbb{S}_n(k)$, the symmetric $n \times n$ matrices over k , or

(β) $V^+ = M_{pq}(R), V^- = M_{qp}(R)$ where M_{ij} are $i \times j$ matrices over an k -algebra R , or

(γ) $V^+ = V^- = \mathbb{A}_n(k)$, the skew-symmetric $n \times n$ matrices over k with zero diagonal,

Then (V^+, V^-) is a Jordan pair where the quadratic map $Q_xy = xyx$ is just the matrix product.

A *homomorphism* $h : V \rightarrow W$ of Jordan pairs is a pair of linear maps (h_+, h_-) that preserves the Jordan pair structure, namely $h_\sigma : V^\sigma \rightarrow W^\sigma$ satisfies $h_\sigma(Q_xy) = Q_{h_\sigma(x)}h_{-\sigma}(y)$ for all $x \in V^\sigma, y \in V^{-\sigma}$. The definitions of endomorphism, isomorphism and automorphism of Jordan pairs clearly follow.

Remark 2.3.5. If a Jordan pair can be embedded into $V = (M_{pq}(R), M_{qp}(R))$ where R is an associative ring, we call it *special*. Elsewise we call it *exceptional*.

Examples. For a pair $(x, y) \in V^\sigma \times V^{-\sigma}$, the Bergmann operator $B(x, y) \in \text{End}V^\sigma$ is

$$B(x, y) = \text{Id}_{V^\sigma} - D(x, y) + Q_x Q_y$$

If $B(y, x) \in \text{End}V^{-\sigma}$ is invertible, then $\beta(x, y) := (B(x, y), B(y, x)^{-1})$ is an isomorphism of V . It is easily seen that the same concept is defined for a single Jordan algebra.

The notion of a derivation is defined as follows: a pair $\Delta = (\Delta_+, \Delta_-)$ of linear maps $\Delta_\sigma \in \text{End}(V^\sigma)$ is called a *derivation* if $\text{Id} + \epsilon \Delta$ is an automorphism of

the base ring extension $V \otimes_k k[\epsilon]$, where $k[\epsilon]$ is the ring of dual numbers. This is equivalent to the validity of the formula:

$$\Delta_\sigma(Q_z v) = \{\Delta_\sigma(z), v, z\} + Q_z \Delta_{-\sigma}(v)$$

for all $z \in V^\sigma$, $v \in V^{-\sigma}$. As with derivations of other algebraic structures, $\text{Der}(V)$ forms a Lie algebra under componentwise operations.

Among all derivations, we need only the following two: the inner derivations $\delta(x, y) := (D(x, y), -D(y, x))$, and the central derivation $\zeta_V = (\text{id}_{V^+}, \text{id}_{V^-})$. Invoking one of the main Jordan identities, it could be checked that $\text{Inder}(V) := \text{span}\{\delta(x, y) \mid (x, y) \in V\}$ is a Lie subalgebra of $\text{Der}(V)$.

Setting $\mathfrak{L}_0 = k \cdot \zeta_V + \text{Inder}(V)$, the celebrated procedure of Tits-Kantor-Koecher allows us to construct from any Jordan pair V the following Lie algebra

$$\mathcal{TKK}(V) = V^+ \oplus \mathfrak{L}_0(V) \oplus V^-.$$

with multiplication

$$[V^\sigma, V^{-\sigma}] = 0, \quad [D, z] = D_\sigma(z), \quad [x, y] = -\delta(x, y) \quad (2.7)$$

for $D = (D_+, D_-) \in \mathfrak{L}_0(V)$, $z \in V^\pm$ and $(x, y) \in V$. It follows immediately that if $x \in V^\sigma$, $y \in V^{-\sigma}$, then

$$2Q_x y = \{xyx\} = [x, [x, y]] \quad (2.8)$$

This suggests that $\frac{1}{2}ad_x^2$ is meaningful even when $\frac{1}{2}$ is non-existent in the base ring. Along with the fact that $ad_x^3 = 0$, we are enabled to make a quadratic definition of “exponentiation” as follows:

For any $x \in V^\sigma$ ($\sigma \in \{\pm\}$), $\exp_\sigma(x)$ is an endomorphism of $\mathcal{TKK}(V)$ given by:

$$\exp_\sigma(x)z = x, \quad \exp_\sigma(x)\Delta = \Delta + [x, \Delta], \quad \exp_\sigma(x)y = y + [x, y] + Q_x y,$$

where $z \in V^\sigma$, $\Delta \in \mathfrak{L}_0$ and $y \in V^{-\sigma}$. It could be checked that $\exp_\sigma(z)$ is an automorphism of $\mathcal{TKK}(V)$ and $\exp_\sigma : V^\sigma \rightarrow \text{Aut}(\mathcal{TKK}(V))$ is actually an injective homomorphism [26].

Putting $U^\pm := \text{Im}(\exp_\pm)$, the *projective elementary group of V* is the subgroup of $\text{Aut}(\mathcal{TKK}(V))$ generated by U^+ and U^- . Such groups will be denoted $PE(V)$.

Examples. Let $\mathcal{A}$ be an alternative ring. Then $PE_3(\mathcal{A})$, as described in §2.2, is actually a projective elementary group with respect to the exceptional Jordan pair $V = (M_{12}(\mathcal{A}), M_{21}(\mathcal{A}))$ (See [27] for definitions). We will encounter another exceptional Jordan pair in Section 4.2.

2.4 Some exceptional magic

We have now defined alternative algebras and Jordan pairs. However, we still need at our disposal some more concrete characterizations.

Unless otherwise mentioned, all the non-associative algebras in this section will be defined over some base ring containing $1/6$. The intrigued readers are encouraged to refer to the excellent expositions in [19] [31] [36] for an detailed treatment of these topics.

The non-associative algebras of interest here are quadratic alternative algebras and cubic Jordan algebras. The modifiers “quadratic” and “cubic” are natural in the following sense:

Definition 2.4.1. An algebra $\mathfrak{A}$ is of *degree n* if every element of $\mathfrak{A}$ satisfies a polynomial equation of degree n .

2.4.1 Quadratic alternative algebras

Let R be a commutative ring containing $\frac{1}{2}$. An alternative algebra $\mathfrak{a}$ over R is called a *quadratic alternative algebra* if there exists a linear function $t : \mathfrak{a} \rightarrow R$ and a quadratic form $n : \mathfrak{a} \rightarrow R$ such that

$$a^2 - t(a)a + n(a)1 = 0, \quad a \in \mathfrak{a} \tag{2.9}$$

t and n are respectively the (generic) trace and norm of $\mathfrak{a}$.

Let $\mathfrak{a}_0 = \{a \mid t(a) = 0\}$, then the direct sum decomposition $\mathfrak{a} = \mathfrak{a}_0 \oplus R1$ holds, as it is clear that $t(1) = 2$ and $n(1) = 1$. Under this decomposition, every element in $\mathfrak{a}$ can be written as $a = a_0 + \frac{1}{2}t(a)$.

One can define on $\mathfrak{a}_0$ a bilinear product $*$ as follows:

$$a * b = ab - \frac{1}{2}t(a, b)1, \quad a, b \in \mathfrak{a}_0 \quad (2.10)$$

where

$$t(a, b) := t(ab) \quad a, b \in \mathfrak{a} \quad (2.11)$$

is a symmetric bilinear form on $\mathfrak{a}$, *associative* in the sense that

$$t(ab, c) = t(a, bc) \quad a, b, c \in \mathfrak{a}. \quad (2.12)$$

This implies

$$a * b = -b * a \quad a, b \in \mathfrak{a}_0 \quad (2.13)$$

Recall that a *derivation* $\mathfrak{B}$ is a linear operator on an algebra D that satisfies the condition $D(ab) = D(a)b + aD(b)$ $a, b \in \mathfrak{B}$. For non-associative algebras, one pays special interest to the so called *inner derivations*. In particular,

Definition 2.4.2. For any alternative algebra $\mathfrak{a}$ containing $\frac{1}{2}$, $D_{a,b} = [a, b]_l - [a, b]_r - 3[a_l, b_r]$ $a, b \in \mathfrak{a}$ is a derivation. The linear span of such operators is an ideal of the Lie algebra of derivations of $\mathfrak{a}$, called the *inner derivations* of $\mathfrak{a}$. The subscripts l and r here denotes the left and right multiplication operator: e.g. $a_l(b) = ab$.

Proposition 2.4.3. Let $\mathfrak{a}$ be an alternative algebra, E and $D_{a,b}$ be respectively a derivation and an inner derivation on $\mathfrak{a}$. Then $[E, D_{a,b}] = D_{E(a),b} + D_{a,E(b)}$

Proof. It is a general fact that $[D, y_r] = (D(y))_r$ and $[D, x_l] = (D(x))_l$ $x, y \in \mathfrak{a}$. holds for any derivation D in an arbitrary ring $\mathfrak{a}$. Such equations imply the proposition immediately. $\square$

The following properties of $D_{a,b}$ (see [19, Pg91]) will be useful later:

- $D_{a,b} = -D_{b,a}$,
- $D_{\alpha 1,a} = 0$,

and

$$D_{ab,c} + D_{bc,a} + D_{ca,b} = 0 \quad (2.14)$$

Example. The most prominent class of (quadratic) alternative algebras is, arguably, the class of composition algebras.

Definition 2.4.4. ([31, Pg 136], See also [40]) An unital algebra $\mathcal{C}$ is called a *composition algebra* if it carries a non-degenerate quadratic form n (the *norm*) that permits composition, i.e.

$$n(xy) = n(x)n(y), \quad n(1) = 1 \quad \text{for all } x, y \in \mathcal{C} \quad (2.15)$$

Denote $n(x, y) := n(x + y) - n(x) - n(y)$, $t(x) = n(x, 1)$. It is known that all composition algebras $\mathcal{C}$ enjoy the following properties:

- (0) $\mathcal{C}$ is always an alternative algebra,
- (1) A standard involution is defined for all $x \in \mathcal{C}$: $\bar{x} := t(x)1 - x$,
- (2) $x\bar{x} = \bar{x}x = n(x)$.

The above immediately implies that $\mathcal{C}$ is a quadratic alternative algebra where all $x \in \mathcal{C}$ satisfies $x(t(x) - x) = n(x)$.

Hurwitz's theorem ([40, Theorem 1.6.2]) states that all composition algebras over a field F of characteristic not 2 are finite dimensional, while the only possible dimensions are 1,2,4,8. Such 8-dimensional algebras are the first examples of (non-associative) alternative algebras, called octonion or Cayley algebras.

Example. Over the real numbers, $\mathbb{R}$, $\mathbb{C}$ (the complex numbers), $\mathbb{H}$ (the quaternions), $\mathbb{O}$ (the octonions) is a well known family of composition algebras.

Remark 2.4.5. The composition algebras mentioned in Hurwitz's theorem above can actually be obtained successively by iterating the so called Cayley-Dickson process. We defer all these details regarding composition algebras to [36, 40] and Pete Clark's wonderful notes [8].

We now focus on a particular octonion algebra: the ***split octonion algebra*** that is famously involved in the construction of the simple Lie algebra $\mathfrak{g}_2$. It will be defined through the ageless *Zorn vector matrix*. The concreteness of this approach is very useful for our later purposes, and it requires no restrictions on torsion/characteristic of the base ring.

Definition 2.4.6. We fix an arbitrary base ring of scalars $\mathcal{R}$. The *Split Octonion algebra* $\mathcal{O}$ over $\mathcal{R}$ consists of all 2×2 matrices with scalar entries α, β on the diagonal and vector entries $\mathbf{v}, \mathbf{w} \in \mathcal{R}^3$ off the diagonal:

$$\mathcal{O} := \left\{ A = \begin{pmatrix} \alpha & \mathbf{v} \\ \mathbf{w} & \beta \end{pmatrix} \mid \alpha, \beta \in \mathcal{R}, \mathbf{v}, \mathbf{w} \in \mathcal{R}^3 \right\}$$

with trace, norm, involution, and product defined as:

$$\begin{aligned} t(A) &:= \alpha + \beta, \\ n(A) &:= \alpha\beta - \mathbf{v} \cdot \mathbf{w}, \\ \bar{A} &:= \begin{pmatrix} \beta & -\mathbf{v} \\ -\mathbf{w} & \alpha \end{pmatrix}, \\ A_1 A_2 &:= \begin{pmatrix} \alpha_1 \alpha_2 + \mathbf{v}_1 \cdot \mathbf{w}_2 & \alpha_1 \mathbf{v}_2 + \beta_2 \mathbf{v}_1 - \mathbf{w}_1 \times \mathbf{w}_2 \\ \alpha_2 \mathbf{w}_1 + \beta_1 \mathbf{w}_2 + \mathbf{v}_1 \times \mathbf{v}_2 & \beta_1 \beta_2 + \mathbf{w}_1 \cdot \mathbf{v}_2 \end{pmatrix}, \end{aligned}$$

here $\cdot$ and $\times$ are the usual dot and cross product of vectors.

Remark 2.4.7. The split octonion algebra $\mathcal{O}$ is the only octonion algebra over algebraically closed fields.

Let $\{c_i\}_{i=1}^3$ be an orthonormal basis for $\mathcal{R}^3$. Since $\mathcal{O}$ is an 8-dimensional free module over $\mathcal{R}$, we can fix for $\mathcal{O}$ a basis $\mathcal{B} := \{e_1, e_2, v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{23}\}$

where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, v_{1i} = \begin{pmatrix} 0 & c_i \\ 0 & 0 \end{pmatrix}, v_{2i} = \begin{pmatrix} 0 & 0 \\ c_i & 0 \end{pmatrix} \quad (i = 1, 2, 3).$$

Remark 2.4.8. A multiplication table is available at [18, Pg 105], although some terms have to be changed accordingly as the multiplication for $\mathcal{O}$ defined there is slightly different from ours.

This gives the *Pierce decomposition* for $\mathcal{O}$:

$$\mathcal{O} = \mathcal{R}e_1 \oplus \mathcal{R}e_2 \oplus \underbrace{e_1\mathcal{O}e_2}_{\oplus_{i=1}^3 \mathcal{R}v_{1i}} \oplus \underbrace{e_2\mathcal{O}e_1}_{\oplus_{i=1}^3 \mathcal{R}v_{2i}}$$

where e_1, e_2 are orthogonal idempotents satisfying $e_1 + e_2 = 1$.

When $\mathcal{R} = \mathbb{Z}$, we name $\mathcal{O}_{\mathbb{Z}} := \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus_{i,j} \mathbb{Z}v_{ij}$ the *integral split octonion algebra*.

Following [19, Pg11-16], we now review the realization of (split-) $\mathfrak{g}_2$ as $\text{Der}_k(\mathcal{O})$, where $\mathcal{O}$ is a split octonion algebra over a field k such that $\frac{1}{6} \in k$. Regardless of such restriction on the base field, we shall see that the Lie algebra structure of $\text{Der}_k(\mathcal{O})$ is totally governed by $\mathcal{O}_{\mathbb{Z}[\frac{1}{3}]}$. This allows us to obtain identical results replacing k by an arbitrary commutative ring $\mathcal{R}$ containing $\frac{1}{6}$.

Denote by $\mathfrak{D}_0$ the subalgebra of derivations that maps e_1 to 0. Since derivation kills scalars, $\mathfrak{D}_0$ kills the diagonal of $\mathcal{O}$. As a Lie algebra, $\mathfrak{D}_0$ is isomorphic to the (split-) A_2 , i.e. $\mathfrak{sl}_3$, the trace zero 3×3 matrices. Elements $x \in \mathfrak{D}_0$ act on the Pierce components $e_1\mathcal{O}e_2$ and $e_2\mathcal{O}e_1$ naturally by contragredient representations:

$$x : \begin{pmatrix} \alpha & \mathbf{v} \\ \mathbf{w} & \beta \end{pmatrix} \mapsto \begin{pmatrix} 0 & x(\mathbf{v}) \\ -x^t(\mathbf{w}) & 0 \end{pmatrix}.$$

For the remaining derivations, it turns out that the inner derivations $D_{e_1, v_{1i}}$ and $D_{e_2, v_{2i}}$ span the complement of $\mathfrak{D}_0$. Therefore

$$\text{Der}_k(\mathcal{O}) = \mathfrak{D}_0 \oplus \left(\bigoplus_{i=1}^3 kD_{e_1, v_{1i}} \right) \oplus \left(\bigoplus_{j=1}^3 kD_{e_2, v_{2j}} \right)$$

is a 14-dimensional Lie algebra over k .

Let us compute explicitly the short root subspaces. Denote by $\mathfrak{h}$ the standard Cartan subalgebra of $\mathfrak{D}_0 \cong \mathfrak{sl}_3$ spanned by the diagonal elements $h_1 := E_{11} - E_{22}$, $h_2 := E_{22} - E_{33}$. Without loss of generality, we choose an embedding of the root system A_2 into the root system G_2 such that h_1, h_2 are dual to the G_2 roots $3\alpha + \beta$ and α . In so the roots of $\mathfrak{D}_0$ are identified with the long roots of G_2 .

Consider the action of $\mathfrak{D}_0$ on $\mathcal{O}_0$. It can be shown that $\mathfrak{h}$ acts diagonally by the following weights, including all the short roots of the root system G_2 , as in the table 2.1 below.

Table 2.1: Weights on $\mathcal{O}_0$

Weight (λ)	0	α	$\alpha + \beta$	$2\alpha + \beta$
Weight space (kv_λ)	$k(e_1 - e_2)$	kv_{22}	kv_{23}	kv_{11}
	$-\alpha$	$-\alpha - \beta$	$-2\alpha - \beta$	
	kv_{12}	kv_{13}	kv_{21}	

This implies that $\mathfrak{h}$ acts diagonally on $(\oplus_{i=1}^3 kD_{e_1, v_{1i}}) \oplus (\oplus_{j=1}^3 kD_{e_2, v_{2j}})$ with root space decomposition as in Table 2.2, thus manifesting itself as a Cartan subalgebra of $\text{Der}_k(\mathcal{O})$.

Table 2.2: Root space decomposition for the complement of $\mathfrak{D}_0$

Root (γ)	α	$\alpha + \beta$	$2\alpha + \beta$
Root space ($kD_{a,b}$)	$kD_{e_2, v_{22}}$	$kD_{e_2, v_{23}}$	$kD_{e_1, v_{11}}$
	$-\alpha$	$-\alpha - \beta$	$-2\alpha - \beta$
	$kD_{e_1, v_{12}}$	$kD_{e_1, v_{13}}$	$kD_{e_2, v_{21}}$

Recall that $\mathcal{B}$ is the basis for $\mathcal{O}$ given as above. Blessed with $\frac{1}{6} \in k$, one can obtain a Chevalley basis

$$\mathfrak{b} := \{x_\gamma, h_i \mid \gamma \in G_2, i \in \{1, 2\}\} \quad (2.16)$$

of $\text{Der}_k(\mathcal{O})$ by taking Lie brackets of the elements $D_{e_i, v_{ij}}$. It could be checked that all these basis elements take the form $D_{a,b}$ where $a, b \in \mathcal{B}$ for h_i and x_γ, γ a short

root; and $a \in \mathcal{B}, b \in \frac{1}{3}\mathcal{B}$ for x_γ, γ a long root.

Remark 2.4.9. Here is an explicit description of this Chevalley basis: h_i are the diagonal elements in $\mathfrak{sl}_3$ that we specified earlier, x_γ for γ long are the matrix units. e.g. $x_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, x_γ for γ short are the $D_{a,b}$'s displayed in Table 2.2.

This completes our analysis of the $\mathfrak{g}_2$ structure of $\text{Der}_k(\mathcal{O})$.

Remark 2.4.10. Actually, $\frac{1}{6} \in k$ implies that all derivations of $\mathcal{O}$ are inner (See [36, Cor. 3.29]). This is true even if we substitute k for $\frac{1}{6} \in R$.

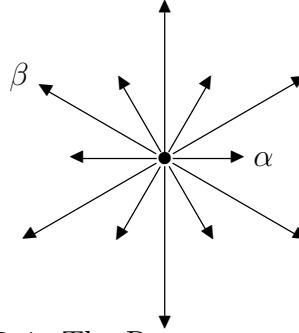


Figure 2.1: The Root system of type G_2

2.4.2 Cubic Jordan algebras

(à la [31])

Although more primitive than the concept of Jordan pairs and quadratic Jordan algebras (see [45]), the language of (linear) Jordan algebras is no less powerful when the base ring contains $\frac{1}{2}$. The modifier “linear” reflects our perspective that we mainly cope with usual Jordan multiplication instead of the quadratic operators.

Definition 2.4.11. A (linear) *Jordan algebra* $\mathfrak{J}$ over a commutative ring R of $\text{char} \neq 2$ is a R -algebra with a bilinear product $\bullet$ (the *Jordan product*) satisfying the following two axioms:

$$x \bullet y = y \bullet x; \quad x^2 \bullet (x \bullet y) = x \bullet (x^2 \bullet y) \quad \text{for } x, y \in \mathfrak{J} \quad (2.17)$$

(Denote $x^2 := x \bullet x$).

The product is commutative by definition, but $\bullet$ need not be associative. We also require $\mathfrak{J}$ to be *unital*, i.e. it contains a unit element for the $\bullet$ operation in the usual sense.

Example. Take any unital associative algebra A over a field F of characteristic $\neq 2$. Then A carries a Jordan algebra structure with product rule $a \bullet b := \frac{1}{2}(ab + ba)$. When A is non-commutative, the Jordan product is usually not associative. We denote the underlying Jordan algebra as A^+ . Jordan algebras that can be embedded as a subalgebra of some A^+ are called *special*, *exceptional* otherwise.

In literature, it seems that the terminology “Cubic Jordan algebras” is reserved for a special class of degree 3 Jordan algebras: Jordan algebras constructed from a cubic form. This process is often referred to as the *Springer construction*.

Our quick introduction the the construction is a reorganization of [31, Pg76-80 & II-4]. Although not necessary for this specific definition, we once again assume the presence of $\frac{1}{6}$.

Definition 2.4.12. A *cubic form* N on a $\mathcal{R}$ -module X is a map $N : X \rightarrow \mathcal{R}$ such that

$$N(\alpha x) = \alpha^3 N(x) \quad \text{for } \alpha \in \mathcal{R}, x \in X$$

and

$$N\left(\sum_i \omega_i x_i\right) = \sum_i \omega_i^3 N(x_i) + \sum_{i \neq j} \omega_i^2 \omega_j N(x_i, x_j) + \sum_i \omega_i \omega_j \omega_k N(x_i, x_j, x_k)$$

for $\omega_i \in \mathcal{R}, x \in X$. The *first linearization* $N(x; y)$ is quadratic in x and linear in y , while the *full linearization* $N(x, y, z)$ is trilinear and symmetric.

An explicit formula for $N(x, y, z)$ is given by

$$N(x, y, z) = \frac{1}{6} \left(N(x+y+z) - N(x+y) - N(x+z) - N(y+z) + N(x) + N(y) + N(z) \right).$$

In particular,

$$N(x, x, x) = N(x).$$

A *basepoint* for N is a point $c \in M$ with $N(c) = 1$. One then can define the following maps

- a linear map (*trace*): $T(x) := N(c; x) = 3N(c, c, x)$;
- a quadratic map (*spur*): $S(x) := N(x; c) = 3N(x, x, c)$;
- a (*spur*) bilinear form: $S(x, y) := S(x+y) - S(x) - S(y) = 6N(x, y, c)$;
- a trace bilinear form: $T(x, y) := T(x)T(y) - S(x, y)$.

Note that $T(x, y)$ is NOT the linearization of $T(x)$.

In particular, we have

$$N(c) = 1; \quad S(c) = T(c) = 3.$$

Definition 2.4.13. A cubic form with a basepoint (N, c) on a $\mathcal{R}$ -module X is said to be a *sharped cubic*, if there exists a *sharp map* $X \rightarrow X$ satisfying the three relations:

$$T(x^\#, y) = N(x; y) (= 3N(x, x, y)) \quad (2.18)$$

$$(x^\#)^\# = N(x)x. \quad (2.19)$$

$$c\#y = T(y)c - y \quad (2.20)$$

in terms of the bilinear *sharp product*

$$x\#y = (x + y)^\# - x^\# - y^\#. \quad (2.21)$$

Occasionally the term *Jordan cubic* will be used when referring to the associated Jordan algebra (see Proposition 2.4.15).

Remark 2.4.14. The following holds for all $x, y \in X$.

$$x\#x = 2x^\#. \quad (2.22)$$

$$T(x^\#) = S(x). \quad (2.23)$$

$$T(x\#y) = S(x, y) \quad (2.24)$$

Theorem 2.4.15. [31, Pg190, 4.2.2] *Any sharpened cubic form $(N, \#, c)$ gives rise to a Jordan algebra $\mathfrak{J}(N, \#, c)$ with unit $1 = c$. The Jordan product is given by*

$$x \bullet y = \frac{1}{2} \left(x\#y + T(x)y + T(y)x - S(x, y)1 \right) \quad (2.25)$$

Every element of this Jordan algebra satisfies the degree-3 identity:

$$x^3 - T(x)x^2 + S(x)x - N(x)1 = 0. \quad (2.26)$$

We also have

$$x^\# = x^2 - T(x)x + S(x)1. \quad (2.27)$$

Take the trace of the last expression, then use (2.23). This gives

$$T(x^2) = T(x)^2 - 2S(x), \quad (2.28)$$

which linearizes to

$$T(x \bullet y) = T(x, y) \quad x, y \in \mathfrak{J}. \quad (2.29)$$

This is a symmetric bilinear form on $\mathfrak{J}$, *associative* in the sense that

$$T(a \bullet b, c) = T(a, b \bullet c) \quad x, y, z \in \mathfrak{J}. \quad (2.30)$$

As in the previous section, define $\mathfrak{J}_0 = \{x \mid T(x) = 0\}$. Then the direct sum decomposition $\mathfrak{J} = \mathfrak{J}_0 \oplus \mathcal{R}1$ holds, since $T(1) = 1$ and $N(1) = 1$. Therefore every element in $\mathfrak{a}$ can be written uniquely as $x = x_0 + \frac{1}{3}T(x)$.

Define a bilinear product $\circ$ on $\mathfrak{J}_0$:

$$a \circ b = a \bullet b - \frac{1}{3}T(a, b)1, \quad x, y \in \mathfrak{J}_0 \quad (2.31)$$

equation 2.30 implies

$$x \circ y = y \circ x \quad x, y \in \mathfrak{J}_0 \quad (2.32)$$

We can also define *inner derivations* for $\mathfrak{J}$:

Definition 2.4.16. For a Jordan algebra $\mathfrak{J}$ containing $\frac{1}{2}$, $d_{a,b} = [a_r, b_l]$, $a, b \in \mathfrak{J}$ is a derivation. The linear span of such operators is an ideal of the Lie algebra of derivations of $\mathfrak{J}$, called the *inner derivations* of $\mathfrak{J}$. The subscripts l and r here still denote the left and right multiplication, respectively.

Proposition 2.4.17. Let $\mathfrak{J}$ be a Jordan algebra, e and $d_{a,b}$ be respectively a derivation and an inner derivation on $\mathfrak{J}$. Then $[e, d_{a,b}] = d_{E(a),b} + d_{a,E(b)}$

Proof. See proof of Prop.2.4.3 □

The following nice examples are taken from [24].

Example. Let V be the vector space $M_3(F)$ of 3×3 matrices over a field F . Set $N(a) = \det(a)$, $c = \text{Id}$. Then $T(a) = \text{tr}(a)$, and $x^\#$ produces the adjugate matrix of x . Equation (2.25) gives the Jordan product, which exactly is the standard Jordan product for associative algebras, given in example 2.4.2 :

$$x \bullet y = \frac{1}{2}(xy + yx).$$

Also, $T(x, y)$ is equal to $T(x \bullet y)$, the standard trace form in the matrix algebra. Summing up the facts above, the equation (2.26) simply becomes just the Cayley-Hamilton equation for 3×3 matrices.

We are now ready for something *exceptional* — a special case of the *reduced cubic factor* example of [31, I.3.9].

Example. Let $\mathcal{C}$ be a composition algebra over a commutative ring $\mathcal{R}$ with a quadratic form n and an involution $\bar{}$.

Consider $\mathcal{H}_3(\mathcal{C})$, the space of 3×3 Hermitian matrices over $\mathcal{C}$. An arbitrary element A in $\mathcal{H}_3(\mathcal{C})$ has the form

$$A = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix}, \quad \begin{array}{l} a, b, c \in \mathcal{R}, \\ x, y, z \in \mathcal{C}. \end{array}$$

It is known that $\mathcal{H}_3(\mathcal{C})$ is a Jordan algebra under the operation $A \bullet B = \frac{1}{2}(AB + BA)$, and is exceptional when $\mathcal{C}$ is an octonion algebra.

It is also a cubic Jordan algebra, defined by:

- The basepoint : the identity matrix in $\mathcal{H}_3(\mathcal{C})$
- The cubic form :

$$N(A) = abc - ax\bar{x} - by\bar{y} - cz\bar{z} + (xy)z + \bar{z}(\bar{y}\bar{x}). \quad (2.33)$$

- The trace: $T(A) = a + b + c$
- The trace bilinear form: $T(A, B) = T\left(\frac{1}{2}(AB + BA)\right)$.
- The sharp:

$$A^\# = \begin{pmatrix} bc - n(x) & \bar{y}\bar{x} - cz & zx - b\bar{y} \\ xy - c\bar{z} & ac - n(y) & \bar{z}\bar{y} - ax \\ \bar{x}\bar{z} - by & yz - ax & ab - n(z) \end{pmatrix}. \quad (2.34)$$

This example, as part of a more general construction, is attributed to Freudenthal *Remark 2.4.18*. Although $\mathcal{H}_3(\mathcal{C})$ may be exceptional, this specific cubic form is not chosen randomly: when $\mathcal{C}$ is associative, the formulas above actually coincides with that of example 2.4.2.

2.4.3 Freudenthal-Tits Magic square

In the Cartan-Killing classification of simple Lie algebras, the five *exceptional Lie algebras* E_6, E_7, E_8, F_4, G_2 are objects of mystery, garnering interest from mathematicians and physicists alike. There are many approaches to construct such objects: one can of course do this directly from the Cartan-Killing data by generators and relators (See [18]); or construct them from Spin groups and representations (See [1]); or the non-associative approach involving alternative and Jordan algebras.

Let us focus on the third approach. È.Cartan remarked in his 1908 Encyclopedia article that the real Lie group G_2 is $\text{Aut}_{\mathbb{R}}(\mathbb{O})$, where $\mathbb{O}$ is the 8-dimensional real division algebra. Since then, the non-associative constructions of Type E_6, E_7, F_4, G_2 Lie algebras have all been discovered by the 1950's, with E_8 being the sole exception.

Then came the brilliant mathematicians: Freudenthal and Tits. By 1966, it is clear that they've constructed one magic square to rule them all:

Proposition 2.4.19. (*[42]*) (See also *[19]*.) Let $A, \mathfrak{a}$ be commutative, associative R -algebras where the base ring R contains $\frac{1}{6}$. Assume $\mathfrak{J}$ is a degree 3 Jordan algebra over $\mathfrak{a}$. Let $\mathcal{C}$ denote a quadratic alternative algebra over A . Assume $\mathcal{D}(\mathcal{C})$ (resp. $\mathcal{D}(\mathfrak{J})$) is a Lie subalgebra of $\text{Der}_A^0(\mathcal{C})$ (resp. $\text{Der}_{\mathfrak{a}}^0(\mathfrak{J})$) containing the inner derivations. Then the algebra

$$\mathcal{T}(\mathcal{C}/A, \mathfrak{J}/\mathfrak{a}) := (\mathcal{D}(\mathcal{C}) \otimes \mathfrak{a}) \oplus \mathcal{C}_0 \otimes \mathfrak{J}_0 \oplus (A \otimes \mathcal{D}(\mathfrak{J}))$$

becomes a Lie algebra under suitable multiplication,

Assuming $\mathcal{C}, \mathfrak{J}$ to be composition/Jordan algebras over $\mathbb{R}$, this gives the following “magic square”:

Table 2.3: The “Magic Square”

$\mathcal{C} \backslash \mathfrak{J}$	$\mathbb{R}$	$\mathcal{H}_3(\mathbb{R})$	$\mathcal{H}_3(\mathbb{C})$	$\mathcal{H}_3(\mathbb{H})$	$\mathcal{H}_3(\mathbb{O})$
$\mathbb{R}$	0	A_1	A_2	C_3	F_4
$\mathbb{C}$	0	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	A_1	C_3	A_5	D_6	E_7
$\mathbb{O}$	G_2	F_4	E_6	E_7	E_8

One has similar results by setting $\mathcal{C}$ to be the family of composition algebras over other base field/rings, given that they contain $\frac{1}{6}$.

Before moving on, it might be necessary to explain certain notions involved in Proposition 2.4.19:

- For $\mathcal{C}$ and $\mathfrak{J}$ as above, one has

$$c^2 - t(c)c + n(c)1 = 0, \quad c \in \mathcal{C}; \quad \text{and} \quad x^3 - T(x)x^2 + S(x)x - N(x)1 = 0, \quad x \in \mathfrak{J}.$$

- The symmetric bilinear forms $t(,)$ and $T(,)$ were defined for $\mathcal{C}$ and $\mathfrak{J}$ in equations 2.11 and 2.29.
- Linear combinations of $D_{x,y} := [x, y]_l - [x, y]_r - 3[x_l, y_r]$ $a, b \in \mathcal{C}$ (resp. $d_{s,t} := [s_r, t_r]$ $s, t \in \mathfrak{J}$) form $\text{InnDer}(\mathcal{C})$, the inner derivations of $\mathcal{C}$ (resp. $\text{InnDer}(\mathfrak{J})$, the inner derivations of $\mathfrak{J}$).
- $\mathcal{C}_0$ and $\mathfrak{J}_0$ are the trace zero elements in $\mathcal{C}_0$ and $\mathfrak{J}_0$ with respect to the linear forms t and T .
- $\text{Der}_A^0(\mathcal{C})$ (resp. $\text{Der}_a^0(\mathfrak{J})$) are the $\mathcal{C}_0$ (resp. $\mathfrak{J}_0$) preserving derivations in $\text{Der}_A(\mathcal{C})$ (resp. $\text{Der}_a(\mathfrak{J})$). It is known that (1) $\text{Der}_F^0(\mathcal{A}) = \text{Der}_F(\mathcal{A})$ for any finite dimensional power-associative algebra $\mathcal{A}$ over a field F . (2) As long as quadratic alternative or cubic Jordan algebras are concerned, the existence of $\frac{1}{2}$ is enough to derive that the inner derivations always preserve the space of trace zero elements.
- Every derivation Λ in $\text{Der}_A^0(\mathcal{C})$ (resp. $\text{Der}_a^0(\mathfrak{J})$) satisfies $t(\Lambda\nu, \omega) = t(\nu, \Lambda\omega)$ (resp. $T(\Lambda\epsilon, \tau) = T(\epsilon, \Lambda\tau)$).

We can now define the “suitable” Lie bracket on $\mathcal{T}(\mathcal{C}/A, \mathfrak{J}/\mathfrak{a})$ (as mentioned in the proposition):

$$\begin{aligned} [D \otimes \alpha, E \otimes \beta] &= [D, E] \otimes \alpha\beta \quad , \quad [a \otimes \mathfrak{d}, x \otimes s] = -[x \otimes s, a \otimes \mathfrak{d}] = ax \otimes \mathfrak{d}s, \\ [a \otimes \mathfrak{d}, b \otimes \mathfrak{e}] &= ab \otimes [\mathfrak{d}, \mathfrak{e}] \quad , \quad [D \otimes \alpha, x \otimes s] = -[x \otimes s, D \otimes \alpha] = Dx \otimes \alpha s, \\ [D \otimes \alpha, b \otimes \mathfrak{e}] &= 0, \end{aligned}$$

and

$$[x \otimes s, y \otimes t] = D_{x,y} \otimes \frac{1}{12} T(s, t) + x * y \otimes s \circ t + \frac{1}{2} t(x, y) \otimes d_{s,t} \quad (2.35)$$

for $D \otimes \alpha, E \otimes \beta \in \mathcal{D}(\mathcal{C}) \otimes \mathfrak{a}$; $x \otimes s, y \otimes t \in \mathcal{C}_0 \otimes \mathfrak{J}_0$; $a \otimes \mathfrak{d}, b \otimes \mathfrak{e} \in A \otimes \mathcal{D}(\mathfrak{J})$. Here $[D, E]$ and $[d, e]$ are respectively the Lie brackets in $\mathcal{D}(\mathcal{C})$ and $\mathcal{D}(\mathfrak{J})$. Extending by linearity one obtains Lie multiplication for the whole space.

The author opt to leave out the proof and calculations, hoping the interested reader to consult [19, 36], as well as Tit’s original article [42].

Chapter 2 includes a reinterpretation of, and borrows heavily from, [45]. This paper, written by the author himself, has been submitted for publication .

Chapter 3

Root Graded Groups: Attempts at classification

3.1 Root Graded Groups 101

3.1.1 Preliminaries

The papers [4] and [5] classified all root graded Lie algebras over a ground field of characteristic zero, which are defined as:

Definition 3.1.1. Let k be a field, Φ a finite irreducible reduced root system of rank ≥ 2 , Γ the root lattice generated by Φ , $\mathfrak{g}$ a Lie algebra over k .

We say $\mathfrak{g}$ is *graded by Φ* if

- (1) $\mathfrak{g}$ has a Γ gradation

$$\mathfrak{g} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{\alpha},$$

where $\mathfrak{g}^{\alpha} \neq 0$ if and only if $\alpha \in \Phi \cup \{0\}$;

- (2) $\mathfrak{g}$ contains a split simple Lie algebra $\dot{\mathfrak{g}}$ over k (whose root system is Φ) as a subalgebra. Also, relative to some split Cartan subalgebra $\dot{\mathfrak{h}}$ of $\dot{\mathfrak{g}}$ we have $\dot{\mathfrak{g}}^{\alpha} \subset \mathfrak{g}^{\alpha}$ for all $\alpha \in \Phi \cup \{0\}$;

- (3) For all $h \in \dot{\mathfrak{h}}$, $ad_{\mathfrak{g}}(h)$ acts diagonally on $\mathfrak{g}^{\alpha}$ with eigenvalue $\langle \alpha, h \rangle = \alpha(h)$;

(4) $\mathfrak{g}$ is generated by nonzero root spaces $\mathfrak{g}^\alpha, \alpha \in \Phi$.

Remark 3.1.2. It is a simple conclusion from the above definition that

(a) $\mathfrak{g}$ is Φ -graded $\Rightarrow \mathfrak{g}$ is perfect, i.e. $\mathfrak{g} = (\mathfrak{g}, \mathfrak{g})$;

(b) $\mathfrak{g} = \sum_{\alpha \in \Phi} \mathfrak{g}^\alpha \oplus \sum_{\alpha \in \Gamma} (\mathfrak{g}^\alpha, \mathfrak{g}^\alpha)$;

As in Definition 3.1.1, we assume $\mathfrak{g}$ is a Lie algebra graded by Φ , and $\dot{\mathfrak{g}} \subset \mathfrak{g}$. It is well known (see [7]) that $\dot{\mathfrak{g}}$ possesses a (Non-unique) Chevalley basis :

$$\{x_\alpha\}_{\alpha \in \Phi} \cup \{h_{\alpha_i}\}_{1 \leq i \leq l} \quad (\alpha_i \text{ are the simple roots, and } l = \text{rank}(\Phi))$$

Such that $x_\alpha \in \dot{\mathfrak{g}}^\alpha$, $\{h_{\alpha_i}\}_{1 \leq i \leq l}$ a basis of $\dot{\mathfrak{h}}$, and

$$(x_\alpha, x_{-\alpha})_L = h_\alpha, \quad (3.1)$$

$$(x_\alpha, x_\beta)_L = \begin{cases} \pm(r+1)x_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \\ 0 & \text{if } \alpha + \beta \notin \Phi \end{cases}, \quad (3.2)$$

where $\text{length}(\beta) \geq \text{length}(\alpha)$ and $r = \max\{s \in \mathbb{Z} : \beta - s\alpha \in \Phi\}$.

We denote by $\dot{\mathfrak{g}}_{\mathbb{Z}}$ the integral Lie subring of $\dot{\mathfrak{g}}$ generated by the Chevalley basis. For any commutative ring R , if we put $\dot{\mathfrak{g}}_R = R \otimes_{\mathbb{Z}} \dot{\mathfrak{g}}_{\mathbb{Z}}$, then for a fixed root α , the elements $x_\alpha(t) = \exp(t \otimes \text{ad}(x_\alpha))$, $t \in R$ generate a subgroup of $\text{Aut}(\dot{\mathfrak{g}}_R)$ isomorphic to $(R, +)$, called the root subgroup corresponding to α . It is denoted as X_α .

Definition 3.1.3. For R a commutative ring,

(a) $\mathbb{E}_\Phi^{\text{ad}}(R) = \langle X_\alpha \rangle_{\alpha \in \Phi}$ is called the elementary Chevalley group,

(b) The Steinberg group $St_\Phi(R)$ is the graded cover of $\mathbb{E}_\Phi^{\text{ad}}(R)$. (See [41],[13]for details).

Remark 3.1.4. The following remarks are taken from [13, chapter 7]:

(1) The isomorphism class of $St_\Phi(R)$ and $\mathbb{E}_\Phi^{\text{ad}}(R)$ does not depend on the choice of Chevalley basis.

- (2) The Steinberg group $St_\Phi(R)$ can also be defined as the group generated by the elements $\{x_\alpha(r) : \alpha \in \Phi, r \in R\}$ subject to the following relations for every $\alpha \neq -\beta \in \Phi$ and $t, u \in R$:

$$x_\alpha(t)x_\alpha(u) = x_\alpha(t+u) \quad (3.3)$$

$$(x_\alpha(t), x_\beta(u)) = \prod_{i,j \in \mathbb{N}, i\alpha+j\beta \in \Phi} x_{i\alpha+j\beta}(c_{ij}(\alpha, \beta)t^i u^j). \quad (3.4)$$

The equation above is called the Chevalley commutator formula.

- (3) If we do not assume that R is commutative, then $\mathfrak{g}_R$ does not have a natural structure of a Lie algebra over R , and so the original construction of $St_\Phi(R)$ is not valid. However in the case $\Phi = A_n$, we can still define the Steinberg group as the graded cover of $EL_{n+1}(R)$. Note that in this case, The formulas in the above paragraph are the defining relations of the group (See also [32]).

3.1.2 The Setup

Given the facts presented in the last section, we define here the protagonist of the paper.

In [38], Shi defined Groups graded by simply laced root systems. His definition is essentially the same to the following

Definition 3.1.5. (ref. [15, 13]) Let Φ be a classical root system. A group G is Φ – *graded* if the following holds:

- (1) G is generated by the family of abelian subgroups $\{X_\alpha\}_{\alpha \in \Phi}$ that intersects trivially pairwise. They will be referred to as the *root subgroups*.
- (2) For any $\alpha, \beta \in \Phi$, with $\alpha \notin \mathbb{R}_{<0}\beta$, the root subgroups are subject to the relation

$$(X_\alpha, X_\beta) \subseteq \langle X_\gamma \mid \gamma = a\alpha + b\beta \in \Phi, a, b \geq 1 \rangle$$

- (3) G contains a homomorphic image of $St_\Phi(\mathbb{Z})$ where the corresponding map ϕ is defined as:

$$\phi : St_\Phi(\mathbb{Z}) \rightarrow G, \quad s_\alpha(1) \mapsto x_\alpha(1) \neq \text{id}$$

here $s_\alpha(1)$ is the image of 1 under the standard identification of $(\mathbb{Z}, +)$ to the root subgroup S_α of $St_\Phi(\mathbb{Z})$, $x_\alpha \in X_\alpha$. In other words, $\phi|_{S_\alpha}$ is always a homomorphism between root subgroups.

(4) (Preservation of the Weyl element)

$$X_\beta^{(n_\alpha(1)^{-1})} = X_{w_\alpha(\beta)}, \text{ where } w_\alpha(\beta) \text{ is the reflection of } \beta \text{ by } \alpha,$$

$$\text{and } n_\alpha(1) = x_\alpha(1)x_{-\alpha}(1)^{-1}x_\alpha(1).$$

(5) For some set P of positive roots, $X_P \cap X_{-P} = 1$

Notation We shall denote from now on by $x_\alpha(m)$ the element $\phi(e_\alpha(m))$, by n_α the element $n_\alpha(1)$. Under this notation, we have $x_\alpha(-n) = x_\alpha(n)^{-1}$, and $n_\alpha(-1) = n_\alpha^{-1}$.

Remark 3.1.6. This definition is understandable as the counterpart of root graded Lie algebras in groups, with the classical duo of split simple Lie algebra and corresponding Chevalley groups as typical examples.

Remark 3.1.7. (5) of the definition above allows us to argue that for any subset $\{\alpha_1, \dots, \alpha_t\}$ of the positive roots, if $\prod_{i=1}^t X_{\alpha_i}(a_i) = \text{id}$, then all a_i 's involved are equal to zero.

Definition 3.1.8. (ref. [13]) Fix a root system Φ , and groups G' and G Φ graded by Φ . We say the group G' *covers* G if there exists a surjective homomorphism that (1) preserves root subgroups and the image of $St_\Phi(\mathbb{Z})$ (2) restricts to isomorphisms on root subgroups.

For notation-wise convenience, we adopt the following standard root system realizations:

$$A_n = \{e_i - e_j : 1 \leq i, j \leq n+1, i \neq j\};$$

$$B_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n, \} \cup \{\pm e_i : 1 \leq i \leq n\};$$

$$C_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n, \} \cup \{\pm 2e_i : 1 \leq i \leq n\};$$

$$D_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n, \}.$$

3.2 The simply laced case

In this section, Φ will always be a simply laced irreducible reduced root system, i.e. of type A,D,E. The results in this section were obtained by Shi (see [38]), and we see it as a preparation for the more complicated proofs in later section of the chapter.

Theorem 3.2.1. *Assume Φ a simply laced irreducible reduced root system of rank ≥ 2 . G a Φ -graded group. Then there is a ring R with identity such that*

- (i) *If $\Phi = A_l, l \geq 3$, then R is associative and $St_n(R)$ covers G , where $n = l + 1$,*
- (ii) *$\Phi = D_n$ or E_n , then R is commutative and associative and $St_\Phi(R)$ covers G .*

The case $\Phi = A_2$ brings some difficulty. We will discuss this in detail in the following chapter. For all the above cases, we say G has R as its *base ring*, or G is *parametrized* (or *coordinatized*) by R .

We first list here two basic commutator identities for groups:

Remark 3.2.2. The following are well known identities valid for any group:

$$(I) \quad (x, zy) = (x, y)(x, z)^y, \text{ and } (xz, y) = (x, y)^z(z, y);$$

$$(II) \quad (Hall - Witt)$$

$$((x, y^{-1}), z)^y((y, z^{-1}), x)^z((z, x^{-1}), y)^x = 1,$$

$$((x, y), z^x)((z, x), y^z)((y, z), x^y) = 1.$$

And some facts on the Chevalley commutator formula and structure constants $N_{\alpha, \beta}$:

Lemma 3.2.3.

- (1) *For type ADE root systems, the Chevalley commutator formula for the corresponding Chevalley groups reads:*

$$(x_\alpha(u), x_\beta(t)) = x_{\alpha+\beta}(N_{\alpha, \beta}ut),$$

where $N_{\alpha, \beta}$ is an integer,

(2) If $\alpha, \beta, \alpha + \beta \in \Phi$, it follows that (i) $N_{\alpha, \beta} N_{\beta, \alpha} = -1$, (ii) $N_{\alpha, \beta} = -N_{\beta, \alpha}$, (iii) $N_{\alpha, \beta} = \pm 1$, (iv) If $\alpha_1 + \alpha_2 + \alpha_3 = 0$ for $\alpha_1, \alpha_2, \alpha_3 \in \Phi$, then $N_{\alpha_1, \alpha_2} = N_{\alpha_2, \alpha_3} = N_{\alpha_3, \alpha_1}$.

Proof. See Theorem 4.1.2 and Theorem 5.2.2 of [7]. $\square$

Our approach to the theorem is very similar to the one taken in [4, chapter 1]: One fix $\alpha \in \Phi$ to be a simple root which has only one adjacent vertex in the corresponding dynkin diagram (with respect to P , the set of positive roots), and denote $R = X_\alpha$ as an abelian group. The objective is to endow R with a ring structure. Then one can naturally define this multiplication on other root subgroups thru the standard isomorphism, which arises as a slight modification of conjugating the root subgroup by some Weyl element $n_\gamma(1), \gamma \in \Phi$. We'll show later that R is independent of the choice of α .

Consistent with previous notations, given $\forall r \in R$, we write the corresponding element in X_α as $x_\alpha(r)$. Obviously, $x_\alpha(r)^{-1} = x_\alpha(-r)$. Recall the following Lemma for elementary Chevalley groups:

Lemma 3.2.4. [7, Lemma 7.2.1] Let $r, s \in \Phi$, then

$$\begin{aligned} (1) \quad n_r \cdot x_s(t) \cdot n_r^{-1} &= x_{w_r(s)}(\eta_{r,s} t), \\ (2) \quad n_r X_s n_r^{-1} &= X_{w_r(s)}, \end{aligned}$$

Here $\eta_{r,s} = \pm 1$ is a constant dependent on r, s and the structure constants $N_{\alpha, \beta}$.

Definition 3.2.5. Let α the simple root as previously indicated. Then for any root β , the *standard isomorphism* $\lambda_{\alpha, \beta}$ between X_α and X_β is

$$\begin{aligned} \lambda_{\alpha, \beta} : X_\alpha &\rightarrow X_\beta \\ x_\alpha &\mapsto (x_\alpha^{\prod_{i=1}^m n_{\alpha_i}(-1)})^{\prod_{i=1}^m \eta_{\alpha_i, \alpha'_i}} \end{aligned} \tag{3.5}$$

Here $\alpha'_i = (\prod_{j=i}^1 w_{\alpha_j})(\alpha)$, $w := \prod_{i=m}^1 w_{\alpha_i}$ some element of the Weyl group such that $w(a) = \beta$.

Note. We adopt the convention that $\prod_{j=1}^n a_j = a_1 a_2 \dots a_n$. So in $\prod_{j=m}^1 a_j$ the product is taken in reversed order. And we do allow the m that appeared in Definition 3.2.5 to be 0: in such case the product is defined to be 1.

Definition 3.2.6. For $\forall \beta \in \Phi$, we define

$$x_\beta(r) := \lambda_{\alpha,\beta}(x_\alpha(r))$$

Where α is the simple root as previously indicated.

Proposition 3.2.7. $\lambda_{\alpha,\beta}$ is well-defined.

Proof. The proposition is equivalent to saying that the λ 's are independent of the choice and presentation of w . Let $w_1 = \prod_{i=1}^m w_{\alpha_i}$, $w_2 = \prod_{j=1}^n w_{\beta_j}$ where $w_1(\alpha) = w_2(\alpha)$. So the proposition is equivalent to :

$$(x_\alpha^{\prod_{i=1}^m n_{\alpha_i}(-1) \prod_{j=1}^n n_{\beta_j}(1)})^{\prod_{i=1}^m \eta_{\alpha_i, \alpha'_i} \cdot \prod_{j=1}^n \eta_{\beta_j, \beta'_j}} = x_\alpha, \quad \forall x_\alpha \in X_\alpha. \quad (3.6)$$

Before we continue, we need the following definitions:

Definition 3.2.8. (ref.[4])

- (1) For an arbitrary irreducible reduced root system Φ , $(\beta, \gamma) \in \Phi \times \Phi$ is called an A_2 pair if $(\beta|\gamma) = -1$ and $\|\beta\| = \|\gamma\|$.
- (2) If Φ is simply laced, we define on R multiplication laws

$$m_{(\beta,\gamma)} : R \times R \rightarrow R, \quad (x_\beta(r), x_\gamma(s)) = x_{\beta+\gamma}(N_{\beta,\gamma} m_{(\beta,\gamma)}(r, s))$$

- (3) Two A_2 pairs (β, γ) and (β', γ') are equivalent if they lie on the same orbit under the diagonal action of the Weyl group W . The corresponding equivalence class is denoted $((\beta, \gamma))$.

Remark 3.2.9. Bilinearity of the multiplication laws in (2) depends on the fact that root strings in simply laced Φ has length one. The result is then a direct consequence of Remark 3.2.2(1), along with the following:

Lemma 3.2.10. For an A_2 pair (α, β) of a simply laced Φ , $\forall x_\beta \in X_\beta$, $x_\alpha \in X_\alpha$, the following holds:

$$(1) x_\beta^{n_\alpha(-1)} = (x_\alpha(1), x_\beta)$$

$$\begin{aligned}
(2) \quad & (x_{-\alpha}(1), (x_{\alpha}(1), x_{\beta})) = x_{\beta}, \\
(3) \quad & x_{\beta}^{n_{\alpha}(1)} = (x_{\beta}, x_{\alpha}(1)).
\end{aligned}$$

Proof. We derive (1) and (2) simultaneously.

$$\begin{aligned}
x_{\beta}^{n_{\alpha}(-1)} &= x_{\alpha}(1)x_{-\alpha}(-1)x_{\alpha}(1) \cdot x_{\beta} \cdot x_{\alpha}(-1)x_{-\alpha}(1)x_{\alpha}(-1) \\
&= x_{\alpha}(1)x_{-\alpha}(-1) \cdot x_{\beta} (x_{\beta}, x_{\alpha}(-1)) \cdot x_{-\alpha}(1)x_{\alpha}(-1) \\
&= x_{\alpha}(1) \cdot x_{\beta} (x_{-\alpha}(1), (x_{\alpha}(-1), x_{\beta})) (x_{\beta}, x_{\alpha}(-1)) \cdot x_{\alpha}(-1) \\
&= x_{\beta} (x_{-\alpha}(1), (x_{\alpha}(-1), x_{\beta})) (x_{\beta}, x_{\alpha}(-1))
\end{aligned}$$

Where $(x_{\beta}, x_{\alpha}(-1)) \in X_{\alpha+\beta}$, $(x_{-\alpha}(1), (x_{\alpha}(-1), x_{\beta})) \in X_{\beta}$, $x_{\beta}^{n_{\alpha}(-1)} \in X_{\alpha+\beta}$, and $x_{\beta} \in X_{\beta}$. It follows that $(x_{-\alpha}(1), (x_{\alpha}(-1), x_{\beta})) = x_{\beta}^{-1}$ when we compare both sides of the equation and apply Lemma 3.1.5(5). Furthermore, we have $(x_{\beta}, x_{\alpha}(-1)) = x_{\beta}^{n_{\alpha}(-1)}$. Using Remark 3.2.2(I), it is clear that the last two equations we derived are equivalent to the desired result(1)(2). Result (3) is proved similarly. $\square$

Lemma. $n_{\alpha}(1)n_{-\alpha}(1) = \text{Id}$, any $\alpha \in \Phi$.

Proof. $n_{\alpha}(1)n_{-\alpha}(1)n_{\alpha}(-1) = n_{\alpha}(-1)$. $\square$

This allows us to rewrite (3.6) above as

$$(x_{\alpha}^{\prod_{i=1}^{m+n} n_{\alpha_i}(-1)})^{\prod_{i=1}^{m+n} \eta_{\alpha_i, \alpha'_i}} = x_{\alpha}, \quad \forall x_{\alpha} \in X_{\alpha}. \quad (3.7)$$

Proof of Prop'n continued.

We apply induction to $s = m + n$.

If $s = 0$, the statement is vacuously true.

If $s = 1$, we denote the only root involved in $\prod_{i=1}^s n_{\alpha_i}(-1)$ as β . since w_{β} fixes α , we have that $(\alpha|\beta) = 0$. Since in this case $\eta_{\alpha, \beta} = 1$ (see [7, Proposition 6.4.3]), the statement is valid.

If $s = 2$, the left hand side of (3.6) should essentially be either $x_{\alpha}^{n_{\alpha_1}(-1)n_{\beta_1}(-1)}$ (when $(\alpha|\alpha_1) = (\alpha|\beta_1) = 0$), $x_{\alpha}^{n_{\alpha_1}(-1)n_{-\alpha_1}(-1)}$, or $x_{\alpha}^{n_{\alpha_1}(-1)^2}$. Using the fact that Φ is a simply-laced system and Lemma 3.2.10 if necessary, we see that all these cases equal x_{α} .

Now assume the statement holds for $s < k$. When $s = k$, equation (3.7) gives $\alpha + \delta_{\alpha_1}\alpha_1 + \delta_{\alpha_2}\alpha_2 + \dots + \delta_{\alpha_{k-1}}\alpha_{k-1} + \delta_{\alpha_k}\alpha_k = \alpha$ where the sum of the first r terms is the image of α after the first $r - 1$ reflections, and the δ_* 's equal to ± 2 , ± 1 or 0 .

If any $\delta_{\alpha_i} = 0$, we can take that term and the corresponding η away (as it equals to 1) from the product, and we're left with an expression with smaller s . Therefore we can assume that this doesn't happen.

We now take care of the n_{α_i} 's where $\delta_{\alpha_i} = \pm 2$:

- If $\|\delta_{\alpha_i}\| = \|\delta_{\alpha_{i+1}}\| = 2$, then $\alpha_i = \pm\alpha_{i+1}$. Treating as in $s = 2$ above, term $n_{\alpha_i}(-1)n_{\alpha_{i+1}}(-1)$ would be eliminated from the exponent, achieving a reduction in length.

- If instead of adjacent such α_i 's, only a single one appear, we can use the following

Claim. Take an A_2 pair (α, β) in Φ . The following diagram commutes:

$$\begin{array}{ccc} X_\alpha & \xrightarrow{n_\beta} & X_{\alpha+\beta} \\ n_\alpha \downarrow & \curvearrowright & \downarrow n_{\alpha+\beta} \\ X_{-\alpha} & \xrightarrow{n_\beta} & X_{-\alpha-\beta} \end{array}$$

One should remember in these diagrams each arrow denotes conjugation by the Weyl element AND exponentiation to the relevant $\eta_{*,*}$ -th power.

Proof.

$$\begin{aligned} & (X_\alpha(a))^{\Pi \eta_{*,*}} n_\beta(-1) n_{\alpha+\beta}(-1) n_\beta(1) n_\alpha(1) \\ &= (X_\alpha(a))^{\Pi \eta_{*,*}} n_\alpha(-\eta_{\beta,\alpha}) n_\alpha(1) \end{aligned}$$

If $-\eta_{\beta,\alpha} = -1$, we invoke Lemma 3.2.10 again to finish the proof. The other case is trivial.

Thus whenever $\delta_{\alpha_i} = \pm 2$ appears, we can push it to the very beginning of the exponent. Thus we can assume that if $\delta_{\alpha_i} = \pm 2$, then i must be equal to 1. The crucial observation here is that these procedures does not increase s .

s can also be reduced in the following scenarios:

(i) Some α_j in the sum is orthogonal to α . Then the conjugation of $n_{\alpha_j}(-1)$ to $(x_\alpha^{\prod_{i=1}^{j-1} n_{\alpha_i}(-1)})$ gives us a shorter expression $(x_\alpha^{\prod_{i=1}^{j-1} n_{\beta_i}(-1)})$;

(ii) $\alpha_j = \pm\alpha_{j'}$ for some $j < j' \leq k$. Then

$$\begin{aligned} & n_{\alpha_j}(-1) \left(\prod_{i=j+1}^{j'-1} n_{\alpha_i}(-1) \right) n_{\alpha_{j'}}(1) \\ &= \begin{cases} (\prod_{i=j+1}^{j'-1} n_{\alpha_i}(-1))^{n_{\alpha_j}(1)} & , \text{ if } \alpha_j = -\alpha_{j'} \\ (\prod_{i=j+1}^{j'-1} n_{\alpha_i}(-1))^{n_{\alpha_j}(1)} \cdot n_{\alpha_j}(-1)^2 & , \text{ if } \alpha_j = \alpha_{j'} \end{cases} \end{aligned}$$

shows that we can do a reduction in length once we prove that $X_\gamma(t)^{n_{\alpha_i}(-1)^2} = X_\gamma(\pm t)$ for any $\gamma \in A_n$. Thanks to Lemma 3.2.10 again, this is certainly the case.

So we're left with the case where every root (up to sign) appears only once in $\{\alpha_i \mid 1 \leq i \leq k\}$, while $\alpha_i = \pm\alpha$ or $(\alpha_i|\alpha) = \pm 1$, all i . This implies that

$$0 = \delta_{\alpha_1}\alpha_1 + \delta_{\alpha_2}\alpha_2 + \dots + \delta_{\alpha_{k-1}}\alpha_{k-1} + \delta_{\alpha_k}\alpha_k.$$

Here all δ_* 's equal to ± 1 , with the sole exception $\delta_{\alpha_1} = \pm 2$ when $\alpha_1 = \mp\alpha$.

Since $(\alpha_j | \sum_{i=1}^k \delta_{\alpha_i} \alpha_i) = 0$, $\forall j \leq k$, there must exist for every $j \leq k$ some $j' \leq k$ such that $\alpha_j + \alpha_{j'}$ or $\alpha_j - \alpha_{j'} \in \Phi$ (i.e. $\|(\alpha_j|\alpha_{j'})\| = 1$). We only focus on those (j, j') where $\pm\alpha \neq \alpha_{\min\{j, j'\}}$ or j, j' are both larger than 1. Find among all such (j, j') the pair where $|j' - j|$ is the smallest. From now on assume $j < j'$.

Since the minimal assumption on $|j' - j|$ implies that $(\alpha_{j'}|\alpha_{j''}) = 0$ for all $j < j'' < j'$, $n_{\alpha_{j'}}(-1)$ can be moved back so it becomes adjacent to $n_{\alpha_j}(-1)$. Therefore we can assume that $j' = j + 1$. Let's consider for such j the roots $\beta := (\alpha + \sum_{i=1}^{j-1} \delta_{\alpha_i} \alpha_i)$, $\gamma := \alpha_j$ and $\tau := \alpha_{j+1}$. It is obvious that $w_\gamma(\beta) \in \Phi$, $w_\tau(\gamma) \in \Phi$, $w_\tau(w_\gamma(\beta)) \in \Phi$.

Claim. $w_\tau(\beta) = \beta$.

Proof: Clearly $\{\beta, \gamma\}$, $\{\tau, \gamma\}$ and $\{w_\gamma(\beta), \tau\}$ (up to sign) are A_2 pairs, and $\{\beta, \gamma, \tau\}$ generates either an A_2 or A_3 subsystem. In the former case, $\{\beta, \gamma\}$ (up to sign) serves as a set of simple roots, giving us a realization $\beta := \pm(e_1 - e_2)$, $\gamma := \pm(e_2 - e_3)$, $\tau :=$

$\pm(e_1 - e_3)$. This contradicts the condition that $\{w_\gamma(\beta), \tau\}$ (up to sign) is an A_2 pair. So $\{\beta, \gamma, \tau\}$ (up to sign) is a set of simple roots of an A_3 system, thus τ cannot be connected to β in the Dynkin diagram, proving the claim.

Since $n_\gamma(-1)n_\tau(-1) = n_\tau(-1)n_{w_\tau(\gamma)}(-\eta_{\tau,\gamma})$, the above claim allows us to eliminate $n_\tau(-1)$ on the right handside when it is conjugated to X_β . This gives us the last reduction, and the proposition is proved. $\square$

The following is a direct consequence of the last proposition:

Corollary 3.2.11. *Let α be the simple root as previously indicated. For $\forall \beta, \gamma \in \Phi$, The standard isomorphism $\lambda_{\beta,\gamma}$ between X_β and X_γ is $\lambda_{\alpha,\gamma} \circ \lambda_{\alpha,\beta}^{-1}$. These isomorphisms satisfy: (1) $\lambda_{\beta,\gamma} = \lambda_{\gamma,\beta}^{-1}$, (2) $\lambda_{\tau,\gamma} \circ \lambda_{\beta,\tau} = \lambda_{\beta,\gamma}$, (3) $\lambda_{\tau,\tau} = id_\tau$.*

Lemma 3.2.12.

(1) If Φ is of type D or E , then there is only one equivalence class of A_2 pairs, (2) If Φ is of type A , then there are exactly two classes of A_2 pairs:

If (β, γ) is an A_2 pair, then $(\beta, \gamma) \sim (-\gamma, -\beta)$, $(\beta, \gamma) \simeq (\gamma, \beta)$,

(3) For any simply laced Φ , if (β, γ) and (γ, δ) are A_2 pairs such that $(\beta|\delta) = 0$, then $(\beta, \gamma) \sim (\gamma, \delta) \sim (\beta, \gamma + \delta) \sim (\beta + \gamma, \delta)$.

Proof. This is [4, Lemma 1.4]. $\square$

Lemma 3.2.13. *If the A_2 -pair $(\beta', \gamma') \in ((\beta, \gamma))$, then $m_{(\beta', \gamma')} = m_{(\beta, \gamma)}$.*

Proof. Since $(\beta', \gamma') = (\beta, \gamma)$, there exists an element w of the Weyl group W of Φ such that $w\beta = \beta'$ and $w\gamma = \gamma'$. Letting $w := \prod_{i=1}^m w_{\alpha_i}$ we know that $\lambda_{\beta,\beta'}, \lambda_{\gamma,\gamma'}, \lambda_{\beta+\gamma,\beta'+\gamma'}$ all arise from conjugating by the same element $\mathbf{n} := \prod_{i=1}^m n_{\alpha_i}(-1)$, and only differ on the exponent $\prod_{i=1}^m \eta_{\alpha_i, \tau_j}$, where $\tau_j = (\prod_{j=0}^i w_{\alpha_j})(\beta)$, $(\prod_{j=0}^i w_{\alpha_j})(\gamma)$, and $(\prod_{j=0}^i w_{\alpha_j})(\beta + \gamma)$, respectively. i.e. for $r, s \in R$,

$$\begin{aligned} x_{\beta'+\gamma'}(N_{\beta,\gamma} m_{\beta,\gamma}(r, s)) &= \lambda_{\beta+\gamma,\beta'+\gamma'}((x_\beta(r), x_\gamma(s))) \\ &= ((x_\beta(r), x_\gamma(s))^{\mathbf{n}})^{\prod_{i=1}^m \eta_{\alpha_i, (\beta+\gamma)_j}} \end{aligned}$$

and

$$\begin{aligned} x_{\beta'+\gamma'}(N_{\beta',\gamma'} m_{\beta',\gamma'}(r, s)) &= (\lambda_{\beta,\beta'} x_\beta(r), \lambda_{\gamma,\gamma'} x_\gamma(s)), \\ &= ((x_\beta(r), x_\gamma(s))^{\mathbf{n}})^{\prod_{i=1}^m \eta_{\alpha_i, \beta_j} \cdot \prod_{i=1}^m \eta_{\alpha_i, \gamma_j}} \end{aligned}$$

Thus the problem reduces to proving

$$N_{\beta,\gamma} \cdot \prod_{i=1}^m \eta_{\alpha_i, (\beta+\gamma)_j} = N_{\beta',\gamma'} \cdot \prod_{i=1}^m \eta_{\alpha_i, \beta_j} \cdot \prod_{i=1}^m \eta_{\alpha_i, \gamma_j} \quad , \quad (3.8)$$

as the structure constants are always ± 1 .

But since all the previous equations we've shown actually hold in the root graded group $\mathbb{E}_\Phi(\mathbb{Z})$, the numerical equation above is also true. Lemma proved. $\square$

Lemma 3.2.14. *[4, Lemma 1.18] Let (β, γ) be an A_2 pair. Then the A_2 pair (γ, β) defines the opposite multiplication to $m_{(\beta, \gamma)}$. In particular if X is of type D or E then $m_{(\beta, \gamma)}$ is commutative. Also, if $\text{rank}(\Phi) \geq 3$, then the multiplications $m_{(\beta, \gamma)}$ are associative.*

Proof. Let $m := m_{(\beta, \gamma)}$, $m' := m_{(\gamma, \beta)}$. Then for all $r, s \in R$,

$$\begin{aligned} e_{\gamma+\beta}(N_{\beta, \gamma} \cdot m(r, s)) &= (e_\beta(r), e_\gamma(s)) = (e_\gamma(s), e_\beta(r))^{-1} \\ &= e_{\gamma+\beta}(-N_{\gamma, \beta} \cdot m'(s, r)) = e_{\gamma+\beta}(N_{\beta, \gamma} \cdot m'(s, r)). \end{aligned}$$

This proves the first two claims of the lemma.

If $\text{rank}(\Phi) \geq 3$, then we can find A_2 pairs (β, γ) and (γ, δ) such that $(\beta|\delta) = 0$. By lemma 3.2.12(3), $m_{\beta, \gamma} = m_{\gamma, \delta}$. Denote this common multiplication by m . Let $r, s, t \in R$.

Note that $((x_\beta(r), x_\gamma(s)), x_\delta(t)) = (x_\beta(r), (x_\gamma(s), x_\delta(t)))$ follows from the Hall-Witt identity. Comparing both sides and using lemma 3.2.12(3) again, we conclude that m is associative. $\square$

Recall that $x_\alpha(1) := \phi(s_\alpha(1))$. The following lemma will provide the final piece for Theorem 3.2.1.

Lemma 3.2.15. *1 is the multiplicative unit of R .*

Proof. Let (α, β) be an A_2 pair. Then $w_\alpha(\beta) = \alpha + \beta$. Moreover, since Φ is simply

laced, we have (see [7]): (1) $N_{-\alpha, \alpha+\beta} = N_{\alpha, \beta}$, (2) $\eta_{\alpha, \beta} = N_{\alpha, \beta}$. Therefore:

$$\begin{aligned}
X_{w_\alpha(\beta)}(\eta_{\alpha, \beta} t) &= x_\beta(t)^{n_\alpha(-1)} \\
&= x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1) \cdot x_\beta(t) \cdot x_\alpha(-1)x_{-\alpha}(1)x_\alpha(-1) \\
&= x_\alpha(1)x_{-\alpha}(-1)x_{\alpha+\beta}(N_{\alpha, \beta}(1 \cdot t))x_\beta(t)x_{-\alpha}(1)x_\alpha(-1) \\
&= x_\alpha(1)x_\beta(N_{-\alpha, \alpha+\beta}N_{\alpha, \beta}(1 \cdot (-t)) + t)x_{\alpha+\beta}(N_{\alpha, \beta}(1 \cdot t))x_\alpha(-1) \\
&= x_{\alpha+\beta}(N_{\alpha, \beta}(2(1 \cdot t) - 1 \cdot (1 \cdot t))x_\beta(t - 1 \cdot t))
\end{aligned}$$

Quoting article (5) of Definition 3.1.5, it follows that $x_\beta(t - 1 \cdot t) = 1$, i.e $1 \cdot t = 0$. By associating the product differently we find $t \cdot 1 = 0$ $\square$

So, it has been shown that root graded groups corresponding to simply laced classical systems of rank ≥ 3 are defined over associative rings, and they satisfy all the Chevalley commutator relations. Hence these groups are covered by $St_\Phi(R)$. By [13], This gives immediately that such objects are property (T) groups.

Of course, there is still one simply laced system being left out:

3.3 A_2 and alternative algebras

We first recall

Definition 3.3.1. A ring $\mathcal{A}$ is said to be *alternative* if $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all x, y in $\mathcal{A}$.

As shown in the preceding chapter, Φ -graded groups could be seen as defined over Associative rings when $rank(\Phi) \geq 3$. However, the argument won't work when $\Phi = A_2$ (although we still have that the group is defined over a unital ring), as was indicated by Faulkner's example ([14]; see also our "Preliminaries" chapter).

3.3.1 The source of examples

By [4, 14], A_2 -graded Lie algebras could be defined over an alternative ring $\mathfrak{A}$. A natural idea would be to mimic the exponential map and expect the output

to be a A_2 -graded group: when we choose $\mathfrak{A}$ to be $\mathbb{C}$ and the Lie algebra to be simple, one recovers the Lie correspondence.

However, it is undesirable to append such strong characteristic restrictions to $\mathfrak{A}$. To obtain the freedom where additive torsion is be allowed, we need a tool to generalize exponentiation. Since we're in the A_2 case where root strings have length at most 2, this boil downs to adjusting exponentiation so the lack of $1/2$ is legit. This is exactly how $\text{PE}_3(R)$ is defined for alternative R . We will revisit this idea in Section 4.2.

Remark 3.3.2. The phenomenon for A_n -graded groups (that the nonassociative base ring only appears when $n = 2$) is in accord with the fact that octonionic projective space $\mathbb{O}P^n$ only makes sense for $n \leq 2$ (See[2]).

3.3.2 Towards Alternativity

According to [4], we know that all A_2 -graded Lie algebras are defined over some weakly alternative ground ring R , a condition that is different from alternativity only when $1 = -1$; [14] showed that this result is independent of the characteristic of the base ring. Also, in his proof for our Proposition 2.2.5, the alternativity of R was explicitly used. This leads to the following:

Conjecture 3.3.3. *Notations as in the previous chapter. Then R is an alternative for all A_2 graded groups.*

We now elaborate on the following fact: under the restriction that the root subgroups of G contains no 2-torsion, the conjecture is indeed true if $\mathcal{A}$ satisfies a residue property.

Definition 3.3.4. We consider the class of all rings $\mathcal{A}$ that appear as the base ring of some A_2 -graded group; it will be called the class of A_2 -rings.

Let X be a set, $M = M(X)$ be the free nonassociative ring on X (which includes $1 \notin X$), and $E_{ij}(M)$ ($1 \leq i \neq j \leq 3$) abelian groups isomorphic to $(M, +)$, $\tilde{M} = *E_{ij}(M)$ a free product, and $\tilde{N}$ the normal subgroup generated by elements

$$(E_{ij}(a), E_{jk}(b)) E_{ik}(-ab) \text{ if } k \neq i, \quad (E_{ij}(a), E_{i',j'}(b)) \text{ if } j \neq i' \text{ and } i \neq j'.$$

Setting $I = \{a \in M | E_{12}(a) \in \tilde{N}\}$, we call $\mathcal{A} = M/I$ the *free A_2 ring on the set of generators X* , where $G = \tilde{M}/\tilde{N}$ is the corresponding A_2 -graded group.

Following the above setup, we let G_k be the subgroup of G generated by commutators containing $\geq k$ entries from the set $\{E_{ij}(x) | x \in X\}$, and we by $\mathcal{G}_k$ the set of these generators. Note here that we didn't require all the entries to be from the set X .

Lemma 3.3.5. $G_k \triangleleft G$, $(G_p, G_q) \subseteq G_{p+q}$.

Proof. To prove $(G_p, G_q) \subseteq G_{p+q}$, it suffices to show that $(g_p, g_q) \subseteq G_{p+q}$ where $\{g_p\}$ and $\{g_q\}$ are the commutators containing $\geq p$ or $\geq q$ entries from the set $\{E_{ij}(x) | x \in X\}$. Since they are generators of G_p, G_q , respectively, this is clearly true.

For $G_k \triangleleft G$, we use induction on k .

1. When $k = 0$, $G_0 = G$ follows from $G = (G, G)$.
2. Assuming $G_{k-1} \triangleleft G$, We know that the generators of G_k takes form $T := (g_s, g_{k-s})$ ($0 \leq s \leq k$), where $g_s \in \mathcal{G}_s, g_{k-s} \in \mathcal{G}_{k-s}$. Conjugating T by some $x \in E_{ij}$, we get

$$\begin{aligned}
 x^{-1}Tx &= (x^{-1}g_sx, x^{-1}g_{k-s}x) \\
 &= ((x, g_s^{-1})g_s, (x, g_{k-s}^{-1})g_{k-s}) \\
 &= ((x, g_s^{-1}), (x, g_{k-s}^{-1})g_{k-s})^{g_s} (g_s, (x, g_{k-s}^{-1})g_{k-s}) \\
 &= ((x, g_s^{-1}), g_{k-s})^{g_s} ((x, g_s^{-1}), (x, g_{k-s}^{-1}))^{g_{k-s}g_s} (g_s, g_{k-s}) (g_s, (x, g_{k-s}^{-1}))^{g_s}
 \end{aligned}$$

□

Lemma 3.3.6. The elements $(E_{ij}(X^{k-n}), E_{ji}(X^n))$, $(E_{ij}(X^k), E_{ji}(\pm 1))$, and $E_{ij}(X^k)$, where $n \leq k$, generates G_k/G_{k+1}

Proof. 1. For $k = 0$, the lemma is obviously true,

2. Assume our lemma is true for $k = r - 1$. By the identity $(x, zy) = (x, y)(x, z)^y$, It is not hard to see that the generators of G_r/G_{r+1} take either the form $E_{ij}(X^r)$ or $T := (\mathfrak{C}_s, \mathfrak{C}_{r-s})$ ($0 < s \leq r$), where $\mathfrak{C}_s$ and $\mathfrak{C}_{r-s}$ are commutators in G_s and G_{r-s} as specified in the lemma, respectively, with the exception when

$r = s$, where $\mathfrak{C}_0$ is a product of $E_{ij}(1)$'s, and $\mathfrak{C}_r$ a long commutator in G_r . It is clear that we should focus on the second case.

We now initiate another induction with respect of the commutator length of T , which we denote by l :

- (i) It is evident that the lemma holds for $l = 1$,
- (ii) Assume the lemma is true for $l = t - 1$, then for T with commutator length t , we have

$$T = \left(\overbrace{\mathfrak{C}_s}^{l=d}, \underbrace{\mathfrak{C}_{r-s}}_{l=t-d} \right)$$

Since we assumed that $s \neq 0$, we can rewrite $\mathfrak{C}_s$ as a product of elements of the form indicated in the lemma. If $d \geq 3$, this and the Hall-Witt identities mentioned before allows us to rewrite T as a product of commutators of shorter length in G_r/G_{r+1} , finishing the proof.

Remark 3.3.7. In the following computations, terms with degree bigger than r are always equated to the identity.

If $d \leq 2$, we have the following possible cases:

- (1) $d = 1, t = 1$:

This gives $T = (E_{ij}(X^{r-n}), E_{ji}(X^n))$ or $E_{ij}(X^r)$ (or the identity, of course).

- (2) $d = 2, t = 1$:

This is essentially reducible to the following cases (where $\{i, j, k\} = \{1, 2, 3\}$):

$$\begin{aligned}
(\mathbf{I}) : T &= ((E_{ij}(a), E_{ji}(b)), E_{jk}(c)) \\
&= E_{jk}(-b(ac))^{E_{ij}(a)E_{ji}(b)} \\
&= E_{jk}(-b(ac))E_{ik}(a(b(ac)))E_{jk}(-b(a(b(ac)))) \\
&= E_{jk}(-b(ac))E_{ik}(a(b(ac))) \quad (\text{since } \deg(a+b) \geq 1) \\
\text{or } T &= ((E_{ij}(a), E_{ji}(b)), E_{kj}(c)) \\
&= E_{kj}(c(ba))^{E_{kj}(c)E_{ji}(b)} = E_{kj}(c(ba))E_{ki}((c(ba))b)
\end{aligned}$$

and

$$\begin{aligned}
(\mathbf{II}) : T &= ((E_{ij}(a), E_{ji}(b)), E_{ji}(c)) \\
&= ((E_{ij}(a), E_{ji}(b)), (E_{jk}(c), E_{ki}(1))) \\
&= ((E_{ki}(-1), (E_{ji}(b), E_{ij}(a))), E_{jk}(c))^{(E_{ij}(a), E_{ji}(b))E_{ki}(1)} \\
&\quad \cdot (((E_{ij}(a), E_{ji}(b)), E_{jk}(-c)), E_{ki}(-1))^{E_{jk}(c)E_{ki}(1)} \\
(\text{By } \mathbf{I}) &= (E_{ki}(ab)E_{kj}((ab)a), E_{jk}(c))^{(E_{ij}(a), E_{ji}(b))E_{ki}(1)} \\
&\quad \cdot (E_{jk}(b(ac))E_{ik}(-a(b(ac))), E_{ki}(-1))^{E_{jk}(c)E_{ki}(1)} \\
&= E_{ji}(-c(ab))(E_{kj}((ab)a), E_{jk}(c))^{E_{ki}(1)} \\
&\quad \cdot E_{ji}(-b(ac))(E_{ik}(-a(b(ac))), E_{ki}(-1))^{E_{jk}(c)E_{ki}(1)}
\end{aligned}$$

It remains to display that

$$(E_{ik}(-a(b(ac))), E_{ki}(-1))^{E_{jk}(c)E_{ki}(1)} \text{ and } (E_{kj}((ab)a), E_{jk}(c))^{E_{ki}(1)}$$

can be represented as product of the desired terms. This is true since

$$\begin{aligned}
&(E_{ik}(-a(b(ac))), E_{ki}(-1))^{E_{jk}(c)E_{ki}(1)} \subset ((E_{ik}(-a(b(ac))), E_{ki}(-1))E_{jk}E_{ji})^{E_{ki}(1)} \\
&= (E_{ik}(-a(b(ac))), E_{ki}(-1))^{E_{ki}(1)}E_{jk}E_{ji} = (E_{ki}(1), E_{ik}(-a(b(ac))))E_{jk}E_{ji}
\end{aligned}$$

and

$$(E_{kj}((ab)a), E_{jk}(c))^{E_{ki}(1)} \subset (E_{kj}((ab)a), E_{jk}(c))E_{ji}E_{ki}$$

by $(\mathbf{I})$ and the Hall-Witt identity. Thus this specific case is settled.

(3) $t \geq 1$:

In this case, we rewrite the term $\mathfrak{C}_{r-s}$ as $\prod_{i=1}^n \omega_i$, where each ω_i lands in some

E_{ij} . This allows us to write T as a finite product of left-normed commutators:

$$T = (\mathfrak{C}_s, \mathfrak{C}_{r-s}) = \prod_j (\cdots ((\mathfrak{C}_s, \omega_{j_1}), \omega_{j_2}) \cdots, \omega_{j_{k_j}})$$

Using (1)(2), we can reduce the length of each factor with length ≥ 2 repetitively until it becomes a product of the terms we gave in the previous calculation, giving us the desired result. $\square$

Remark 3.3.8. For $a, b, c \in \bigcup_{i=0}^{\infty} X^i$ and $\deg(a) + \deg(b) + \deg(c) = n$, The above calculation reveals some useful relations in G_n/G_{n+1} :

$$((E_{ij}(a), E_{ji}(b)), E_{jk}(c)) \subset E_{jk}E_{ik}, \quad (3.9)$$

$$((E_{ij}(a), E_{ji}(b)), E_{kj}(c)) \subset E_{kj}E_{ki}, \quad (3.10)$$

$$((E_{ij}(a), E_{ji}(b)), E_{ji}(c)) \subset E_{ji}E_{jk}E_{ki}(E_{kj}, E_{jk})(E_{ki}, E_{ik}). \quad (3.11)$$

Note that all terms appeared above has degree no less than n .

Recall that $\langle X \rangle^k$ is the k -th power of the ideal generated by X .

Lemma 3.3.9. $E_{ij} \cap G_k \subseteq E_{ij}(\langle X \rangle^k)G_{k+1}$.

Proof. Without loss of generality, we assume $E_{ij}(a) \in G_k/G_{k+1}$, $i = 1, j = 2$, $k \geq 1$.

By the previous lemma, we can write $E_{12}(a) \in G_k$ as a finite product :

$$\begin{aligned} E_{12}(a) \in & E_{12}(\langle X \rangle^k)E_{21}(\langle X \rangle^k)E_{13}(\langle X \rangle^k)E_{31}(\langle X \rangle^k)E_{32}(\langle X \rangle^k)E_{23}(\langle X \rangle^k) \\ & \cdot \prod_{\substack{0 \leq n \leq k \\ s, t \in \{1, 2, 3\}}} (E_{st}(\langle X \rangle^{k-n}), E_{ts}(\langle X \rangle^n)) \end{aligned} \quad (3.12)$$

Note that since we work in G_k/G_{k+1} , all the terms commute.

Remark 3.3.10. In the following, 1: Lemma 3.3.8 will be used constantly; 2: “commute a and b ” is used in place of “take the commutator (a, b) ”.

Commuting Equation 3.12 with $E_{23}(1)$, we get:

$$E_{13}(a + \langle X \rangle^k) = E_{21}(\langle X \rangle^k) E_{23}(\langle X \rangle^k) \prod_{0 \leq n \leq k} (E_{st}(\langle X \rangle^{k-n}), E_{ts}(\langle X \rangle^n)) \quad (3.13)$$

Commute the above with $E_{21}(1)$:

$$E_{23}(a + \langle X \rangle^k) = E_{21}(\langle X \rangle^k) E_{31}(\langle X \rangle^k) \prod_{\substack{0 \leq n \leq k \\ \{s,t\} \neq \{1,2\}}} (E_{st}(\langle X \rangle^{k-n}), E_{ts}(\langle X \rangle^n)) \quad (3.14)$$

Commute this with $E_{12}(1)$:

$$E_{13}(a + \langle X \rangle^k) = (E_{21}(\langle X \rangle^k), E_{12}(1)) E_{32}(\langle X \rangle^k) E_{12}(\langle X \rangle^k) \quad (3.15)$$

Commute again with $E_{32}(1)$:

$$E_{12}(a + \langle X \rangle^k) = E_{32}(\langle X \rangle^k) E_{31}(\langle X \rangle^k) \quad (3.16)$$

A final commutation with $E_{31}(1)$ gives:

$$E_{32}(a + \langle X \rangle^k) = \text{Id.} \quad (3.17)$$

□

Recall that an alternative algebra is defined by the identities $x^2y = x(xy)$ and $yx^2 = (yx)x$. Letting $\{a, b, c\} \subseteq \bigcup_{i=1}^{\infty} X^i$, we consider the following element:

$$((E_{12}(\alpha), E_{21}(\beta)), E_{12}(\gamma\delta))$$

for the following cases:

- $\alpha = a, \beta = b, \gamma = 1, \delta = c$ and $\alpha = a, \beta = b, \gamma = c, \delta = 1$;
- $\alpha = a, \beta = 1, \gamma = b, \delta = c$ and $\alpha = a, \beta = 1, \gamma = bc, \delta = 1$;

It is clear that both cases in each pair correspond to the same triple commutator. Applying the Hall-Witt identity and our previous computation in Lemma 3.3.6 to both cases, we obtain different expressions for the same triple commutator, yielding the equations (weak alternativity):

$$\boxed{E_{12}((c, b, a) + (a, b, c)) \in G_s} \quad \text{and} \quad \boxed{E_{12}((a, b, c) - (b, c, a)) \in G_s} \quad (3.18)$$

Where $s = \deg(a) + \deg(b) + \deg(c) + 1$.

This further implies

$$\boxed{E_{12}(2(b, b, a)) \in G_s} \quad \text{and} \quad \boxed{E_{12}(2(a, b, b)) \in G_s} \quad (3.19)$$

Quoting Lemma 3.3.9, the above translates to $E_{12}((2a, 2b, 2b)) \in E_{12}(x_s)G_{s+1}$, where $x_s \in \langle X \rangle^s$. Denote by R the unital subring of $\mathcal{A}$ generated by elements $2a$ and $2b$, by I the ideal of R generated by associators of the form $(\alpha, \alpha, \beta) = \alpha^2\beta - \alpha(\alpha\beta)$ and $(\beta, \alpha, \alpha) = (\beta\alpha)\alpha - \beta(\alpha^2)$, any $\alpha, \beta \in R$. We consider $G|_R$, the subgroup of G generated by the “ R -points” of the root subgroups, i.e. the original root subgroups restricted to R .

Let's fix some notations. EL_n is the elementary group generated by transvections, R/I is the free associative ring on two generators (as an alternative ring generated by two elements is always associative), I_{2s} the ideal $\langle X^{2s}, I \rangle$, R/I_{2s} a homomorphic image of R where all $2s$ -th powers are zero. Denote by E_{ij} , $\widetilde{E}_{ij}$, $\overline{E}_{ij}$ the corresponding root subgroups in G , $\text{EL}(R/I)$, $\text{EL}(R/I_{2s})$, respectively. Without loss of generality, let assume $\deg(a) = \deg(b) \neq 0$.

It is clear that there exists an epimorphism f_0 , which is the composition $R \xrightarrow{f_1} R/I \xrightarrow{f_2} R/I_{2s}$. It induces homomorphisms $\tilde{f}_0 : G|_R \xrightarrow{\tilde{f}_1} \text{EL}_3(R/I) \xrightarrow{\tilde{f}_2} \text{EL}_3(R/I_{2s})$ with $\ker(\tilde{f}_1|_{E_{ij}}) = I, \ker(\tilde{f}_0|_{E_{ij}}) = I_{2s}$. Knowing that $\tilde{f}_1(E_{12}((2a, 2b, 2b))) = 0$ leads to $\tilde{f}_1(E_{12}(x_s)) \in \tilde{f}_1(G_s)$, we seek to strengthen it to $\tilde{f}_1(E_{12}(x_s)) = \text{Id}$.

Abusing notations, we use x_s to denote also its images under the maps f_1 and f_0 . Applying Lemma 3.3.9 repeatedly, we see that $\overline{E}_{12}(x_s + x_{s+1} \dots + x_{2s-1}) = \text{Id}$, where every x_i is an element of degree i in R/I_{2s} . So degree considerations tell us immediately that $\tilde{f}_1(E_{12}(x_s)) = \text{Id}$. Thus $x_s \in I$. Also, $x_s = \sum_i (a_i, a_i, b_i)c_i \in X^s$ where $a_i, b_i \in \bigcup_{i \geq 1} \langle 2a, 2b \rangle^i$, $c_i \in R$, and all summands are of degree s . As the lowest degree associators in R are (1) of degree $s - 1$ and (2) corresponds to elements in G_s , we arrive at the conclusion that $E_{12}(x_s) \in G_{s+1}$! It follows that $E_{12}((2a, 2b, 2b)) \in G_{s+1}$. Similarly, $E_{12}((2b, 2b, 2a)) \in G_{s+1}$. Since this process could be iterated, we just proved

Proposition 3.3.11. $8(x, x, y) \in \bigcap_{i=1}^{\infty} \langle X \rangle^i$ and $8(y, x, x) \in \bigcap_{i=1}^{\infty} \langle X \rangle^i$ for all $x, y \in \bigcup_{i=1}^{\infty} X^i$. Here $\langle X \rangle$ is the ideal generated by X in $\mathcal{A}$.

Since there is no 2-torsion in the root subgroups of G , this in turn gives:

Theorem 3.3.12. If $\bigcap_{i=1}^{\infty} \langle X \rangle^i = 0$, then the free A_2 ring is alternative.

3.4 The Symplectic case: C_n

In this section, we start to consider non-simply laced systems. It would be very useful to have in mind the following examples:

1. The usual Symplectic matrix groups ([7])
2. The matrix groups preserving a Sesquilinear symplectic form ([9, 17]), which we will elaborate on below.
3. The C_3 graded group defined over an alternative algebra with an involution (See section 4.2)

All the examples above are C_n graded groups. We now lay out a standard construction for such groups.

Example. Let R be an associative ring (or an alternative ring when $n = 3$) with an involution $*$, $1 \leq i < j \leq n$, $a \in R$, $s = s^* \in R$. Consider the following subgroups of $\text{GL}_{2n}(R)$:

$$\bullet \quad X_{e_i+e_j} = \left\{ X_{e_i+e_j}(a) := \left(\begin{array}{c|c} I_n & \begin{array}{cc} & \overset{i,n+j}{\underline{a}} \\ \ddots & \end{array} \\ \hline 0 & \begin{array}{cc} \underset{j,n+i}{\underline{a^*}} & \\ & \ddots \end{array} \\ \hline I_n & \end{array} \right) \right\}$$

$$\bullet \quad X_{-e_i-e_j} = \left\{ X_{-e_i-e_j}(a) := \left(\begin{array}{c|c} I_n & 0 \\ \hline \begin{array}{c} \cdot \cdot \cdot \\ \text{\tiny $\underline{a}$} \\ n+j,i \end{array} & \begin{array}{c} n+i,j \\ \text{\tiny $\underline{a^*}$} \\ \cdot \cdot \cdot \end{array} \\ \hline & I_n \end{array} \right) \right\}$$

$$\bullet \quad X_{e_i-e_j} = \left\{ X_{e_i-e_j}(a) := \left(\begin{array}{c|c} 1 & \\ \cdot \cdot \cdot & \text{\tiny $\underline{a}$} \\ i,j & \\ \hline & 1 \\ 0 & \begin{array}{c} 1 \\ \text{\tiny $\underline{-a^*}$} \\ n+i,n+j \end{array} & \cdot \cdot \cdot \\ \hline & & 1 \end{array} \right) \right\}$$

$$\bullet \quad X_{e_j-e_i} = \left\{ X_{e_j-e_i}(a) := \left(\begin{array}{c|c} 1 & \\ \text{\tiny $\underline{a}$} & \cdot \cdot \cdot \\ j,i & \\ \hline & 1 \\ 0 & 1 & \begin{array}{c} \cdot \cdot \cdot \\ \text{\tiny $\underline{-a^*}$} \\ n+j,n+i \end{array} \\ \hline & & 1 \end{array} \right) \right\}$$

$$\bullet \quad X_{2e_i} = \left\{ X_{2e_i}(s) := \left(\begin{array}{c|c} I_n & \begin{array}{c} \cdot \cdot \cdot \\ \text{\tiny $\underline{s}$} \\ i,n+i \end{array} \\ \hline 0 & I_n \end{array} \right) \right\}$$

$$\bullet \quad X_{-2e_i} = \left\{ X_{-2e_i}(s) := \begin{pmatrix} I_n & 0 \\ \hline \ddots & I_n \\ \begin{smallmatrix} s \\ n+i, i \end{smallmatrix} & \\ & \ddots \end{pmatrix} \right\}$$

As root subgroups, the above generate a C_n ($n \geq 3$) graded group $\mathfrak{G}$ defined over R .

We now start with the C_n case. Throughout our discussion we assume that our group has rank ≥ 3 . Frequent references will be made to the results in the simply laced case two sections ago.

Recall that we fixed the following standard realization for C_n :

$$C_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n, \} \cup \{\pm 2e_i : 1 \leq i \leq n\}$$

It is easy to read off from the Dynkin diagram the standard embedding $A_{n-1} \hookrightarrow C_n$. This gives immediately plenty A_{n-1} -graded subgroups contained in G . Among these we have G_1 , the one that is generated by the root subgroups $\{\pm(e_i - e_j) : 1 \leq i < j \leq n\}$ with simple roots chosen to be $\{e_i - e_{i+1} : 1 \leq i \leq n\}$. We denote its base ring R .

As in the previous chapter, we need to first establish the standard isomorphisms between subgroups corresponding to the roots of the same length. As this isomorphism is already well-defined for the A_{n-1} group $\mathcal{G}_1$, we focus on the long and $\{\pm(e_i + e_j)\}$ root subgroups.

Definition 3.4.1. Denote

$$\begin{aligned} \Psi^+ &:= \{e_i + e_j : 1 \leq i < j \leq n\}, & \Psi^- &:= \{-(e_i + e_j) : 1 \leq i < j \leq n\}, \\ S^+ &:= \{2e_i : 1 \leq i \leq n\}, & S^- &:= \{-2e_i : 1 \leq i \leq n\}. \end{aligned}$$

So it is clear that $C_{n-1} \setminus A_{n-1} = \Psi^+ \sqcup \Psi^- \sqcup S^+ \sqcup S^-$. For any root β inside one of these four disjoint subsets we define *standard isomorphisms* between α and β that

lie inside the same component:

$$\begin{aligned} \lambda_{\alpha,\beta} : X_\alpha &\rightarrow X_\beta \\ x_\alpha &\mapsto (x_\alpha^{\prod_{k=1}^m n_{\alpha_k}(-1)})^{\prod_{k=1}^m n_{\alpha_k, \alpha'_k}} \end{aligned} \quad (3.20)$$

with $\alpha'_i = (\prod_{k=i}^1 w_{\alpha_k})(\alpha)$ and $w := \prod_{i=m}^1 w_{\alpha_i}$ being some element of the Weyl group such that $w(a) = \beta$, with the restriction that $\alpha_i \in A_{n-1} = \{(e_i - e_j)\}$.

When $\alpha \in \Psi^\pm$, we have to define two different kinds standard isomorphisms $\lambda_{\alpha,\beta}$ and $\lambda_{\alpha,\beta}^*$: For a root $\alpha = \pm(e_i + e_j)$, the action of $w_{\pm(e_i - e_k)}$ has the effect of taking i to k . If $(i - j)(k - j) \leq 0$, we say this action is *crossing*. In particular, we call the action a *flip* if $k = j$. For a standard isomorphism λ sending $\alpha \rightarrow \beta$ in $\Psi^\pm$, we add the number of total crossings in each reflection. If the number is even, we denote it as $\lambda_{\alpha,\beta}$. and $\lambda_{\alpha,\beta}^*$ otherwise.

Compared with the simply laced cases, the above is a major feature unique to the C_n , due to the fact that the involution of R is encoded in these isomorphisms.

Why did we prescribe the isomorphisms this way? Let us resort to the detailed example above. Computations in $\mathfrak{G}$ yield:

Fact 3.4.2. *Assuming $i < j \leq n$, The following identities hold in a Symplectic group of size $2n \times 2n$ over a ring with an involution $*$:*

- $X_\alpha(a)^{n_\alpha(-1)} = X_{-\alpha}(-a)$, any $\alpha \in C_n$
- $X_{e_i+e_j}(a)^{n_{e_i-e_k}(-1)} = \begin{cases} X_{e_i+e_k}(-a) & \text{if } k > i \\ X_{e_i+e_k}(-a^*) & \text{if } k \leq i \end{cases}$
- $X_{e_i+e_j}(a)^{n_{e_i-e_k}(-1)} = \begin{cases} X_{e_j+e_k}(-a^*) & \text{if } k \geq j \\ X_{e_j+e_k}(-a) & \text{if } k < j \end{cases}$
- $X_{e_i-e_j}(a)^{n_{e_j+e_k}(-1)} = \begin{cases} X_{e_j+e_k}(-a) & \text{if } k > i \\ X_{e_j+e_k}(-a^*) & \text{if } k < i \end{cases}$
- $X_{e_i-e_j}(a)^{n_{e_i+e_k}(-1)} = \begin{cases} X_{-e_j-e_k}(-a) & \text{if } k > j \\ X_{-e_j-e_k}(-a^*) & \text{if } k < j \end{cases}$

- $X_{e_i+e_j}(a)^{n_{2e_i}(-1)} = X_{e_j-e_i}(a^*)$
- $X_{e_i+e_j}(a)^{n_{2e_j}(-1)} = X_{e_i-e_j}(a)$

In fact, one can use the exact η 's as in this example, which provides notational convenience.

Remark 3.4.3. Recall [7, Lemma 7.2.1] that we mentioned last chapter. The constants η there satisfies the property $\eta_{\alpha,-\alpha} = \eta_{\alpha,\alpha} = -1$ and $\eta_{\alpha,\beta} \cdot \eta_{\alpha,w_\alpha(\beta)} = (-1)^{A_{\alpha\beta}}$, where $A_{\alpha\beta}$ is the Cartan integer $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. In the A_n case, this gives $\eta_{\alpha,\beta} \cdot \eta_{\alpha,w_\alpha(\beta)} = -1$ for all $\beta \neq \pm\alpha$.

Lemma 3.4.4. $n_\alpha(1)n_{-\alpha}(1) = \text{Id}$.

Proof. $n_\alpha(1)n_{-\alpha}(1)n_\alpha(-1) = n_\alpha(-1)$. □

We will assume from after Proposition 3.4.5 that there exists no 2-torsion among the root subgroups of our C_n graded group G . We may also use $X_\alpha(*)$ or $*$ to denote elements whose specifics are irrelevant.

Proposition 3.4.5. $\lambda_{\alpha,\beta}$ and $\lambda_{\alpha,\beta}^*$ are well defined.

Proof. Let's revert to an arbitrary set of η 's in this proof.

Stated more casually, the proposition simply means that the isomorphism is independent of the “path” between the two root subgroups. We shall use (Comm S) to denote the case where the loop lies in S .

(Comm A_{n-1}):

This holds as the diagram depicts a situation that takes place in the standard A_{n-1} subgroup.

(Comm Ψ^+):

Set $\alpha = e_i + e_j$, $w_1 = \prod_{i=1}^m w_{\alpha_i}$, $w_2 = \prod_{j=1}^n w_{\beta_j}$, $w_1(\alpha) = w_2(\alpha)$, and let the crossing number of w_1 and w_2 has the same parity. As in the simply laced cases, we consider a “loop” that starts and ends at α , aiming to show that

$$(x_\alpha^{\prod_{i=1}^m n_{\alpha_i}(-1)} \prod_{j=1}^n n_{\beta_j}(1))^{\prod_{i=1}^m \eta_{\alpha_i,\alpha'_i} \cdot \prod_{j=1}^n \eta_{\beta_j,\beta'_j}} = x_\alpha, \quad \forall x_\alpha \in X_\alpha.$$

Since $n_\alpha(1)n_{-\alpha}(1) = \text{Id}$, this can be shortened to

$$(x_\alpha^{\prod_{i=1}^{m+n} n_{\alpha_i}(-1)})^{\prod_{i=1}^{m+n} \eta_{\alpha_i, \alpha'_i}} = x_\alpha, \quad \forall x_\alpha \in X_\alpha. \quad (3.21)$$

We prove three facts: (1) for $\tau = e_u + e_v \in \Psi^\pm$, $\delta \in A_n$, any “crossing” action of w_δ on τ can be written as a non-crossing action conjugated by $w_{\pm(e_u - e_v)}$; (2) If $(X_{e_s+e_t})^{n_{\pm(e_s-e_t)}(-1)}$ ever appears as we proceed along the loop, we can always push it all the way to the beginning, that is $(x_\alpha^{\prod_{i=1}^{m+n} n_{\alpha_i}(-1)}) = ((x_\alpha^{n_{\pm(e_i-e_j)}(-1)})^{\prod_{i=1}^{m+n-1} n_{\beta_i}(-1)})$; (3) $x_{e_s+e_t}^{n_{e_s-e_t}(-1)^2} = x_{e_s+e_t}$ (any $s \neq t$, $x_{e_s+e_t} \in X_{e_s+e_t}$); then we can proceed proving the proposition like in the A_n case. Treating the well-definition of $\lambda_{\alpha,\beta}$ and $\lambda_{\alpha,\beta}^*$ alike.

Proof of (3):

WLOG, assume $s = 1, t = 2$. By our A_n identification results, we have:

$$\begin{aligned} x_{e_1+e_2}^{n_{e_1-e_2}(-1)^2} &= (X_{e_1+e_3}(1), y_{e_3-e_2})^{n_{e_1-e_2}(-1)^2} \\ &= \left(X_{e_1+e_3}(1)^{(-1)^{A_{e_1-e_2, e_1+e_3}}}, y_{e_3-e_2}^{(-1)^{A_{e_1-e_2, e_3-e_2}}} \right) \\ &= (X_{e_1+e_3}(1), y_{e_3-e_2})^{(-1)^{(A_{e_1-e_2, e_1+e_3} + A_{e_1-e_2, e_3-e_2})}} \\ &= x_{e_1+e_2} \end{aligned}$$

as $\{e_1 + e_3, e_1 - e_2\}, \{e_3 - e_2, e_1 - e_2\}$ and $\{e_3 - e_2, e_1 + e_3\}$ all lie in some A_{n-1} subsystem of C_n .

Proof of (2): It suffices to show the commutativity of the following diagram. One should remember in these diagrams each arrow denotes conjugation by the Weyl element AND exponentiation to the relevant $\eta_{*,*}$ -th power.

$$\begin{array}{ccc} X_{e_i+e_j} & \xrightarrow{n_{e_i-e_j}} & X_{e_i+e_j} \\ \downarrow n_{e_p-e_j} & & \downarrow n_{e_p-e_j} \\ X_{e_i+e_p} & \xrightarrow{n_{e_i-e_p}} & X_{e_i+e_p} \end{array}$$

Denote $e_p - e_j$ as α , $e_i - e_j$ as β . Then:

$$\begin{aligned} &(X_{e_i+e_j}(a)^{\prod \eta_{*,*}})^{n_{e_p-e_j}(-1)n_{e_i-e_p}(-1)n_{e_p-e_j}(1)n_{e_i-e_j}(1)} \\ &= (X_{e_i+e_j}(a)^{\prod \eta_{*,*}})^{n_{e_i-e_j}(-\eta_{\alpha,\beta})n_{e_i-e_j}(1)}, \end{aligned}$$

is equal to $X_{e_i+e_j}(a)$ by our proof of (3) and the fact this is true when we set $a = 1$.

Proof of (1):

WLOG, it suffices for us to prove the commutativity of the following diagram:

$$\begin{array}{ccc}
 X_{e_1+e_2} & \xrightarrow{n_{e_2-e_1}} & X_{e_1+e_2} \\
 \downarrow n_{e_3-e_1} & & \downarrow n_{e_2-e_3} \\
 X_{e_2+e_3} & \xleftarrow{n_{e_2-e_1}} & X_{e_1+e_3}
 \end{array}$$

We denote $e_2 - e_1$ as α , $e_2 - e_3$ as β . Then:

$$\begin{aligned}
 & (X_{e_1+e_2}(a))^{\prod \eta_{*,*} n_{e_2-e_1} (-1)^{n_{e_2-e_3}} (-1)^{n_{e_2-e_1}} (-1)^{n_{e_3-e_1}} (1)} \\
 = & (X_{e_1+e_2}(a))^{\prod \eta_{*,*} n_{e_2-e_1} (-1)^{2n_{e_1-e_3}} (-\eta_{\alpha,\beta}) n_{e_3-e_1} (1)} \\
 = & (X_{e_1+e_2}(a))^{-\prod \eta_{*,*} n_{e_3-e_1} (\eta_{\alpha,\beta}) n_{e_3-e_1} (1)} \quad (\because (3)) \\
 = & \left(X_{e_1+e_3}(x), X_{e_2-e_3}(1)^{-\prod \eta_{*,*} n_{e_3-e_1} (\eta_{\alpha,\beta}) n_{e_3-e_1} (1)} \right) \quad (\because \text{Lemma 3.2.10}) \\
 = & \left(X_{e_1+e_3}(x), (X_{e_2-e_3}(1))^{-\prod \eta_{*,*} n_{e_3-e_1} (\eta_{\alpha,\beta}) n_{e_3-e_1} (1)} \right)
 \end{aligned}$$

By setting $a = 1$, one sees that

$$(X_{e_2-e_3}(1))^{-\prod \eta_{*,*} n_{e_3-e_1} (\eta_{\alpha,\beta}) n_{e_3-e_1} (1)} = X_{e_2-e_3}(1)$$

, which equates the above calculations to the desired $X_{e_i+e_j}(a)$. Note: this process actually increases length of the loop.

Proof of Proposition

Now we can prove the well-definition for $\lambda_{\alpha,\beta}$ and $\lambda_{\alpha,\beta}^*$, by induction of the length of the “loop”, l from expression (3.21). Note that by our choice of w_1 and w_2 , the crossing number of expression (3.21) is always even.

Set $\alpha = e_s + e_t$, $s < t \leq n$.

- When $l = 0$, this is vacuously true, and $l = 2$ is true by (3) above and the proof of identification of root subgroups in the A_n case.

- $l = 1$ is possible only when $(X_\alpha^{n_{e_s - e_t}(-1)})^{\eta_{*,*}}$ is the identity map on X_α .
- Assume the statement is true for $l = k - 1$. If the total number of flips is bigger than 1 for our loop, (2) and (3) allows us to reduce the length of the loop, and we're done.

So assume we started with a loop of length k , having at most one flip appearing at step 1. This means we have an expression $\alpha + \delta_{\alpha_1}\alpha_1 + \delta_{\alpha_2}\alpha_2 + \dots + \delta_{\alpha_{k-1}}\alpha_{k-1} + \delta_{\alpha_k}\alpha_k = \alpha$ where the sum of the first r terms is the image of α after the first $r - 1$ reflections, $\delta_* = \pm 1$ with the exception that $\delta_{\alpha_1} = \pm 1$ or 0 (when $\alpha_1 = \pm(e_s - e_t)$), and α_* 's are the exact roots involved in the reflections in expression 3.21. Another reduction of l can be done in the following cases (similar to that of A_n):

- (i) Some α_j in the sum involves neither e_s or e_t . Then the conjugation of $n_{\alpha_j}(-1)$ to $(x_\alpha^{\prod_{i=1}^{j-1} n_{\alpha_i}(-1)})$ gives us a shorter expression $(x_\alpha^{\prod_{i=1}^{j-1} n_{\beta_i}(-1)})$,
- (ii) Some $\alpha_j = \pm\alpha_{j'}$ ($j < j'$). Then

$$\begin{aligned}
 & n_{\alpha_j}(-1) \left(\prod_{i=j+1}^{j'-1} n_{\alpha_i}(-1) \right) n_{\alpha_j}(-1) \\
 &= \begin{cases} (\prod_{i=j+1}^{j'-1} n_{\alpha_i}(-1))^{n_{\alpha_j}(1)} & , \text{ if } \alpha_j = -\alpha_{j'} \\ (\prod_{i=j+1}^{j'-1} n_{\alpha_i}(-1))^{n_{\alpha_j}(1)} \cdot n_{\alpha_{j'}}(-1)^2 & , \text{ if } \alpha_j = \alpha_{j'} \end{cases}
 \end{aligned}$$

shows that we can do a reduction in length once we prove that $X_\gamma(t)^{n_{\alpha_i}(1)^2} = X_\gamma(\pm t)$ for any $\gamma \in \Psi^\pm$. If $(\gamma, \alpha_j) = \pm 1$. This is certainly the case since $\{\gamma, \alpha_j\}$ generate an A_2 subsystem, If $(\gamma, \alpha_j) = 0$, this is true by (3).

So we're left with the case where every root (up to sign) in $\{\alpha_i \mid 1 \leq i \leq k\}$ appears only once AND involve either e_s or e_t . Since

$$0 = \delta_{\alpha_1}\alpha_1 + \delta_{\alpha_2}\alpha_2 + \dots + \delta_{\alpha_{k-1}}\alpha_{k-1} + \delta_{\alpha_k}\alpha_k$$

where the δ 's are equal to ± 1 . As in A_n , Note that this implies that if some $\delta_{\alpha_c}\alpha_c = \pm(e_s - e_p)$, there must be another $d \leq k$ such that $\delta_{\alpha_d}\alpha_d = \mp(e_t - e_p)$.

Assume $\alpha_j = \pm(e_s - e_p)$, $\alpha_{j'} = \pm(e_t - e_p)$ where $j < j'$. Then part of the

exponent in (3.21) would look like

$$\begin{aligned}
& n_{\pm(e_s - e_p)}(-1) \left(\prod_{i=j+1}^{j'-1} n_{\alpha_i}(-1) \right) n_{\pm(e_t - e_p)}(-1) \\
&= n_{\pm(e_t - e_p)}(-1) n_{\pm(e_s - e_t)}(-\eta_{e_t - e_p, e_1 - e_2}) \left(\prod_{i=j+1}^{j'-1} n_{\alpha_i}(-1) \right)^{n_{\pm(e_t - e_p)}(-1)} \\
&= n_{\pm(e_t - e_p)}(-1) n_{\pm(e_s - e_t)}(-\eta_{e_t - e_p, e_1 - e_2}) \left(\prod_{i=j+1}^{j'-1} n_{\beta_i}(-1) \right).
\end{aligned}$$

One can obtain a similar result for $\alpha_j = \pm(e_t - e_p)$, $\alpha_{j'} = \pm(e_s - e_p)$: just swap s and t above. If $\prod_{i=j+1}^{j'-1} n_{\beta_i}(-1)$ gives any new flips, process as before so we still have the “sum of roots equal to zero” expression as before.

When $\alpha_1 = \pm(e_s - e_t)$, setting $j = 2$ in one of these cases allows a reduction on loop length as more than one reflection by $\pm(e_s - e_t)$ would be involved when we substitute the computation above into the original loop.

If $\alpha_1 \neq \pm(e_s - e_t)$, set $j = 1$. Since $n_{\pm(e_t - e_p)}(-1) n_{\pm(e_s - e_t)}(-\eta_{e_t - e_p, e_1 - e_2}) = n_{\pm(e_s - e_t)}(-\eta_{e_t - e_p, e_1 - e_2}) n_{\pm(e_t - e_p)}(\pm 1)$, we’re back to the situation described in the previous paragraph.

Remark 3.4.6. Note that if the $l = 1$ is possible in the above induction, then $\lambda_{\alpha, \beta} = \lambda_{\alpha, \beta}^*$. In light of the example $\mathfrak{G}$ we gave previously, this means that the involution of the base ring is trivial.

This concludes our discussion for $(\text{Comm } \Psi^+)$.

$(\text{Comm } S^+)$: The proof of this case is similar to the above, only simpler, as the commutativity of the following diagram always allows us to reduce length of the “loop”:

$$\begin{array}{ccc}
X_{2e_i} & \xrightarrow{n_{e_j - e_i}} & X_{2e_j} \\
& \searrow n_{e_k - e_i} & \nearrow n_{e_j - e_k} \\
& & X_{2e_k}
\end{array}$$

Proof. We denote $e_k - e_i$ as α , $e_j - e_k$ as β . Then:

$$\begin{aligned}
& (X_{2e_i}(a)^{\Pi^{\eta_{*,*}}})^{n_{e_k-e_i}(-1)n_{e_j-e_k}(-1)n_{e_j-e_i}(1)} \\
&= (X_{2e_i}(a)^{\Pi^{\eta_{*,*}}})^{n_{e_j-e_i}(-\eta_{\alpha,\beta})n_{e_k-e_i}(1)n_{e_j-e_i}(1)} \\
&= (X_{2e_i}(a)^{\Pi^{\eta_{*,*}}})^{n_{e_j-e_i}(-\eta_{\alpha,\beta})n_{e_j-e_i}(1)} \quad (\because (2e_j, e_k - e_i) = 0)
\end{aligned}$$

Consider conjugating $(\Pi^{\eta_{*,*}})^{n_{e_j-e_i}(-\eta_{\alpha,\beta})n_{e_j-e_i}(1)}$ to

$$(X_{e_i-e_k}(1), X_{2e_k}(x)) = X_{2e_i}(a)X_{e_i+e_k}(*).$$

(The validity of such expression, and the existence of x is ensured by lemma 3.4.9). By lemma 3.4.8 and the unique presentability of G , and comparing the above to the original expression, setting $a = 1$ when needed, we conclude that the unfinished calculation would have to be equal to $X_{2e_i}(a)$.

Also needed is $X_{2e_i}(a)^{n_{e_j-e_i}(1)^2} = X_{2e_i}(\pm a)$. We quote Lemma 3.4.9 for the proof.

(Comm S^-) and (Comm Ψ^-):

Proof. The proof for these are basically identical to the previous two cases. $\square$

We can also define similarly the isomorphisms $\lambda_{\alpha,\beta}$ between different sets of long and short root subgroups, although we may need in addition conjugation by $n_{2e_i}(\pm 1)$ and $n_{e_i+e_j}(\pm 1)$. Their well-definition depends on the following:

(Comm Ψ^+ , A_{n-1}):

$$\begin{array}{ccc}
X_{e_i-e_j} & \xrightarrow{n_{2e_j}} & X_{e_i+e_j} \\
\downarrow n_{e_j-e_k} & \searrow n_{e_j+e_k} & \downarrow n_{e_j-e_k} \\
X_{e_i-e_k} & \xrightarrow{n_{2e_k}} & X_{e_i+e_k}
\end{array}$$

Proof. We denote $2e_j$ as α , $e_k - e_j$ as β . Then:

$$\begin{aligned}
& (X_{e_i - e_j}(a))^{\Pi \eta_{*,*}} (-1)^{n_{2e_j}} (-1)^{n_{e_k - e_j}} (-1)^{n_{e_j + e_k}} (1) \\
&= (X_{e_i - e_j}(a))^{\Pi \eta_{*,*}} (-1)^{n_{e_k + e_j}} (-1)^{n_{2e_j}} (-1)^{n_{e_k + e_j}} (1) \\
&= (X_{e_i - e_j}(a))^{\Pi \eta_{*,*}} (-1)^{n_{e_k + e_j}} (-1)^{n_{e_k + e_j}} (1) \quad (\because (e_i + e_k, 2e_j) = 0) \\
&= (X_{e_i - e_j}(a))^{n_{e_k + e_j} (-1)^{n_{\alpha, \beta}} \Pi \eta_{*,*}} (-1)^{n_{e_k + e_j}} (1) \quad (\because (e_i - e_j, e_k + e_j) = -1)
\end{aligned}$$

And the rest is identical to the (Comm Ψ^+) case above. So does (Comm Ψ^- , A_{n-1}).

(Comm S^+ , S^-):

$$\begin{array}{ccc}
X_{2e_i} & \xrightarrow{n_{e_j - e_i}} & X_{2e_j} \\
n_{2e_i} \downarrow & \searrow n_{-e_i - e_j} & \downarrow n_{2e_j} \\
X_{-2e_i} & \xrightarrow{n_{e_j - e_i}} & X_{-2e_j}
\end{array}$$

Proof. We denote $2e_j$ as α , $e_j - e_i$ as β . Then:

$$\begin{aligned}
& (X_{2e_i}(a))^{\Pi \eta_{*,*}} (-1)^{n_{e_j - e_i}} (-1)^{n_{2e_j}} (-1)^{n_{-e_i - e_j}} (1) \\
&= (X_{2e_i}(a))^{\Pi \eta_{*,*}} (-1)^{n_{-e_i - e_j}} (-1)^{n_{\alpha, \beta}} (-1)^{n_{-e_i - e_j}} (1) \quad (\because (2e_j, 2e_i) = 0)
\end{aligned}$$

Treat as in case (Comm S^+).

(Comm Ψ^+ , Ψ^-):

$$\begin{array}{ccc}
X_{e_i + e_j} & \xrightarrow{n_{e_k - e_j}} & X_{e_i + e_k} \\
n_{e_i + e_j} \downarrow & & \downarrow n_{e_i + e_k} \\
X_{-e_i - e_j} & \xrightarrow{n_{e_k - e_j}} & X_{-e_i - e_k}
\end{array}$$

Proof. Denote $e_l - e_j$ as α , $e_i + e_k$ as β . Then:

$$\begin{aligned} & (X_{e_i+e_j}(a))^{\Pi^{\eta*,*}} n_{e_k-e_j} (-1)^{n_{e_i+e_k}} (-1)^{n_{e_k-e_j}} (1)^{n_{e_i+e_j}} (1) \\ &= (X_{e_i+e_j}(a))^{\Pi^{\eta*,*}} n_{e_i+e_j} (-\eta_{\alpha,\beta}) n_{e_i+e_j} (1) \end{aligned}$$

Note that this expression involves only roots of an A_{n-1} subsystem, therefore must be equal to $X_{e_i+e_j}(a)$. $\square$

As we have established the identification between the root subgroups of the same length, we specify the following commutator relations:

$$(X_{e_1-e_2}(a), X_{2e_2}(h)) = X_{2e_1}(f_{21}(a, h)) X_{e_1+e_2}(f_{11}(a, h)) \quad (\mathfrak{c}_1)$$

$$(X_{e_1+e_2}(a), X_{e_1-e_2}(b)) = X_{2e_1}(-g(a, b)) \quad (\mathfrak{c}_2)$$

Remark 3.4.7. All the other commutator relations will come from the ones above—through conjugation by Weyl elements.

Lemma 3.4.8. (See [14, Pg.37])

(1) f_{11} and g are linear in both components,

(2) f_{21} is linear in the second component.

(3) $f_{21}(a + b, h) = f_{21}(a, h) + g(f_{11}(a, h), b) + f_{21}(b, h)$

(4) $f_{21}(a, h) = f_{21}(-a, h)$.

Proof. (1)(2)(3) are direct consequences of the identity $(s_1 s_2, t) = (s_1, t)^{s_2} (s_2, t)$. For example, in item (2):

$$\begin{aligned} & (X_{e_1-e_2}(a), X_{2e_2}(h_1) X_{2e_2}(h_2)) \\ &= (X_{e_1-e_2}(a), X_{2e_2}(h_1)) (X_{e_1-e_2}(a), X_{2e_2}(h_2))^{X_{2e_2}(h_1)} \\ &= (X_{2e_1}(f_{21}(a, h_1)) X_{e_1+e_2}(f_{11}(a, h_1))) \\ &\quad \cdot (X_{2e_1}(f_{21}(a, h_2)) X_{e_1+e_2}(f_{11}(a, h_2)))^{X_{2e_2}(h_1)} \\ &= X_{2e_1}(f_{21}(a, h_1) + f_{21}(a, h_2)) X_{e_1+e_2}(f_{11}(a, h_1) + f_{11}(a, h_2)) \end{aligned}$$

Thus by the unique expression of the identity element, f_{11} and f_{21} are linear in the second component.

For (4), comparing

$$\begin{aligned}
(X_{e_1-e_2}(a), X_{2e_2}(h)) & \quad \quad \quad = X_{2e_1}(f_{21}(a, h))X_{e_1+e_2}(f_{11}(a, h)) \\
& \quad \quad \quad \text{and} \\
(X_{e_1-e_2}(a), X_{2e_2}(h))^{n_{2e_2}(\pm 1)^2} & \quad \quad \quad \xrightarrow{\text{rmk 3.4.3}} (X_{e_1-e_2}(-a), X_{2e_2}(h)) \\
= X_{2e_1}(f_{21}(a, h))X_{e_1+e_2}(\dots) & \quad \quad \quad = X_{2e_1}(f_{21}(-a, h))X_{e_1+e_2}(f_{11}(-a, h))
\end{aligned}$$

gives the desired result. $\square$

Lemma 3.4.9.

1. $X_{e_1+e_2}(a) = X_{e_1+e_2}(f_{11}(a, 1)),$
2. $X_{2e_1}(\eta_{e_1-e_2, 2e_2}h) = X_{2e_1}(-f_{21}(1, h)).$

In other words, the maps $f_{11}(, 1)$ and $f_{21}(1, h)$ are surjective maps on long and short root subgroups, respectively.*

Proof.

$$\begin{aligned}
(1) : \quad X_{e_1+e_2}(a) &= X_{e_1-e_2}(a)^{n_{2e_2}(1)} = X_{e_1-e_2}(a)^{X_{2e_2}(1)X_{-2e_2}(-1)} \\
&= (X_{e_1-e_2}(a)X_{2e_1}(f_{21}(a, 1))X_{e_1+e_2}(f_{11}(a, 1)))^{X_{-2e_2}(-1)} \\
&= X_{e_1-e_2}(a)X_{2e_1}(f_{21}(a, 1))(X_{e_1+e_2}(f_{11}(a, 1)))^{X_{-2e_2}(-1)} \\
&= X_{e_1-e_2}(*)X_{2e_1}(*)(X_{e_1+e_2}(f_{11}(a, 1))) \\
&\Rightarrow f_{11}(a, 1) = a, \text{ while the } * \text{ terms must be equal to zero.}
\end{aligned}$$

$$\begin{aligned}
(2) : X_{2e_2}(h)^{n_{e_1-e_2}(\pm 1)} &= X_{2e_2}(h)^{X_{e_1-e_2}(\pm 1)X_{e_2-e_1}(\mp 1)} \\
&= (X_{e_1-e_2}(\mp 1), X_{2e_2}(-h))^{X_{e_2-e_1}(\mp 1)} X_{2e_2}(h) \\
&= X_{2e_1}(f_{21}(\mp 1, -h))X_{e_1+e_2}(f_{11}(\mp 1, -h))^{X_{e_2-e_1}(\mp 1)} X_{2e_2}(h) \\
&= X_{2e_1}(-f_{21}(1, h))X_{e_1+e_2}(\pm f_{11}(1, h))^{X_{e_2-e_1}(\mp 1)} X_{2e_2}(h) \\
&= X_{2e_1}(-f_{21}(1, h))X_{e_1+e_2}(\pm f_{11}(1, h))^{X_{e_2-e_1}(\mp 1)} X_{2e_2}(h) \\
&= X_{2e_2}(\ast)X_{2e_1}(-f_{21}(1, h))X_{e_1+e_2}(\ast) \\
&= \underbrace{X_{2e_1}(\eta_{e_1-e_2, 2e_2}h)}_{\text{conjugation by } n_{e_1-e_2}(-1)} = \underbrace{X_{2e_1}(\eta_{e_1-e_2, 2e_1}h)}_{\text{conjugation by } n_{e_1-e_2}(-1)} \\
\Rightarrow -f_{21}(1, h) &= \eta_{e_1-e_2, 2e_2}h, \text{ while the } \ast \text{ terms must be equal to zero. } \square
\end{aligned}$$

Now we start our pursuit for an explicit description of f_{21} , f_{11} and g . Note that from now on, the 2-torsion assumption becomes indispensable.

Lemma 3.4.10. $f_{11}(1, h)$ is an injective map from long root subgroups to R .

Proof. Assume $f_{11}(a, h) = 0$, then:

$$\begin{aligned}
&(X_{e_1-e_2}(a)X_{e_1-e_2}(-a), X_{2e_2}(h)) \\
&= (X_{e_1-e_2}(a), X_{2e_2}(h))^{X_{e_1-e_2}(-a)} (X_{e_1-e_2}(a), X_{2e_2}(h)) \\
&= X_{2e_1}(f_{21}(a, h))^{X_{e_1-e_2}(-a)} X_{2e_1}(f_{21}(-a, h)) \\
&= X_{2e_1}(f_{21}(a, h) + f_{21}(-a, h)) \\
&= X_{2e_1}(2f_{21}(a, h))
\end{aligned} \tag{3.22}$$

Due to the no 2-torsion assumption on root subgroups, we have $f_{21}(a, h) = 0$, which is impossible when $a = 1$, according to Lemma 3.4.9(2). Thus the lemma is proved. $\square$

Remark 3.4.11. By the previous lemma, it is reasonable that we identify elements of the long root subgroup (h) with the element $(f_{11}(1, h))$ in short root subgroups.

Lemma 3.4.12. $f_{11}(1, h)$ is fixed by the involution $*$, i.e. symmetric with respect to $*$.

Proof. Since $(X_{e_1-e_2}(1), X_{2e_2}(h)) = X_{2e_1}(f_{21}(1, h))X_{e_1+e_2}(f_{11}(1, h))$ (see $(\mathfrak{c}_1)$), conjugating it by $n_{e_1-e_2}(-1)$ gives:

$$(X_{e_2-e_1}(-1), X_{2e_1}(\eta_{e_1-e_2, 2e_2}h)) = X_{2e_2}(\eta_{e_1-e_2, 2e_1}f_{21}(1, h))X_{e_1+e_2}(-(f_{11}(1, h))^*) \quad (3.23)$$

We can rewrite the commutator in the first equation as:

$$\begin{aligned} & X_{2e_1}(-h)X_{e_1+e_2}(f_{11}(1, h)) \\ &= X_{e_1-e_2}(-1)X_{2e_2}(-h)X_{e_1-e_2}(1)X_{2e_2}(h) \\ &= X_{e_2-e_1}(-1)X_{2e_2}(-h)^{X_{e_1-e_2}(-1)X_{e_2-e_1}(1)}X_{e_2-e_1}(1)X_{2e_2}(h) \\ &= X_{e_2-e_1}(-1)X_{2e_2}(-h)^{n_{e_1-e_2}(1)}X_{e_2-e_1}(1)X_{2e_2}(h) \\ &= X_{e_2-e_1}(-1)X_{2e_1}(-\eta_{e_1-e_2, 2e_1}h)X_{e_2-e_1}(1)X_{2e_2}(h) \end{aligned}$$

Since $-f_{21}(1, h) = \eta_{e_1-e_2, 2e_1}h$, this implies:

$$(X_{2e_1}(-\eta_{e_1-e_2, 2e_1}h), X_{e_2-e_1}(1)) = X_{2e_2}(-h)X_{e_1+e_2}(f_{11}(1, h)) \quad (3.24)$$

comparing the $X_{e_1+e_2}$ term above with that in equation 3.23, we get

$$f_{11}(1, h) = (f_{11}(1, h))^*$$

□

Lemma 3.4.13. $f_{21}(a, h) = -g(f_{11}(a, h), a)$

Proof. Under the no 2-torsion assumption, It suffices to prove $f_{21}(2a, h) = -g(f_{11}(2a, h), a)$.

Recall the commutator relations:

$$\begin{aligned}(X_{e_1-e_2}(2a), X_{2e_2}(h)) &= X_{2e_1}(f_{21}(2a, h))X_{e_1+e_2}(f_{11}(2a, h)) \\ (X_{e_1+e_2}(f_{11}(2a, h)), X_{e_1-e_2}(a)) &= X_{2e_1}(-g((f_{11}(2a, h)), a))\end{aligned}$$

This indicates that our desired result is equivalent to the following successive equations:

$$\begin{aligned}&(X_{e_1-e_2}(2a), X_{2e_2}(h)) \\ &= (X_{e_1+e_2}(f_{11}(2a, h)), X_{e_1-e_2}(a)) X_{e_1+e_2}(f_{11}(2a, h)) \\ &= X_{e_1-e_2}(-a)X_{e_1+e_2}(f_{11}(2a, h))X_{e_1-e_2}(a)\end{aligned}\tag{3.25}$$

$$\Updownarrow$$

$$\begin{aligned}&(X_{e_1-e_2}(-a)X_{2e_2}(-h)X_{e_1-e_2}(a))(X_{e_1-e_2}(a)X_{2e_2}(h)X_{e_1-e_2}(-a)) \\ &= X_{e_1+e_2}(2f_{11}(a, h))\end{aligned}\tag{3.26}$$

$$\Updownarrow$$

$$X_{e_1-e_2}(-a)X_{2e_2}(-h)X_{e_1-e_2}(a)X_{2e_2}(h) = X_{e_1+e_2}(2f_{11}(a, h))X_{e_1-e_2}(a)\tag{3.27}$$

$$\Updownarrow$$

$$\begin{aligned}&X_{2e_1}(f_{21}(a, h))X_{e_1+e_2}(f_{11}(a, h))X_{2e_1}(-f_{21}(-a, h))X_{e_1+e_2}(f_{11}(-a, -h)) \\ &= X_{e_1+e_2}(2f_{11}(a, h))\end{aligned}\tag{3.28}$$

$$\Updownarrow$$

$$\begin{aligned}&X_{2e_1}(f_{21}(a, h))X_{e_1+e_2}(f_{11}(a, h))X_{2e_1}(-f_{21}(-a, h))X_{e_1+e_2}(f_{11}(-a, -h)) \\ &= X_{e_1+e_2}(2f_{11}(a, h))\end{aligned}\tag{3.29}$$

Since $f_{21}(a, h) = f_{21}(-a, h)$, $f_{11}(a, h) = f_{11}(-a, -h)$ are consequences of Lemma 3.4.8, the evident validity of equation (3.29) proves the lemma. $\square$

Proposition 3.4.14. (1) $(a^*)^* = a$, (2) $(ab)^* = b^*a^*$.

Note: An easier proof of (1) have been obtained in the proof of lemma 3.4.5. But we opt to keep the following proof as it is useful in its own right in further analyses.

Proof.

(1): We consider the expression $(X_{e_1+e_2}(b))^{n_{e_1-e_2}(1)} = X_{e_1+e_2}(-b^*)$:

$$\begin{aligned} (A) : \quad & X_{e_1-e_2}(-1)X_{e_1+e_2}(b)X_{e_1-e_2}(1) \\ &= (X_{e_1+e_2}(b), X_{e_1-e_2}(-1)) X_{e_1+e_2}(b) = X_{2e_1}(-g(b, 1))X_{e_1+e_2}(b) \end{aligned}$$

In the following we denote $g := g(b, 1)$.

$$\begin{aligned} (B) : \quad & X_{e_2-e_1}(1)X_{2e_1}(-g)X_{e_2-e_1}(-1) \\ &= (X_{e_2-e_1}(-1), X_{2e_1}(g)) X_{2e_1}(-g) \\ &= X_{2e_2}(f_{21}(1, g))X_{e_1+e_2}(-f_{11}(1, g))X_{2e_1}(-g), \\ &\quad X_{e_2-e_1}(1)X_{e_1+e_2}(b)X_{e_2-e_1}(-1) \\ &= (X_{e_1+e_2}(b), X_{e_2-e_1}(-1)) X_{e_1+e_2}(b) \\ &= X_{2e_2}(\square)X_{e_1+e_2}(b) \\ (C) : \quad & (X_{2e_2}(f_{21}(1, g) + \square)X_{e_1+e_2}(-f_{11}(1, g) + b)X_{2e_1}(-g))^{X_{e_2-e_1}(1)} \end{aligned}$$

Since $X_{e_1+e_2}^{n_{e_1-e_2}(1)} = X_{e_1+e_2}$, the above implies:

- $f_{21}(1, g) + \square = 0$
- $-g(b - f_{11}(1, g(b, 1)), 1) = g(b, 1)$
- $-b^* = b - f_{11}(1, g(b, 1))$

Therefore $-(b^*)^* = b^* - f_{11}(1, g(b^*, 1)) = b^* - f_{11}(1, g(b, 1)) = -b$. The square denotes a term immaterial to our concerns.

$$\begin{aligned}
(2) : X_{e_1-e_3}(-(ab)^*)^{n_{2e_2}(-1)} &= X_{e_1-e_3}(-(ab)^*) \\
&= (X_{e_1-e_2}(a), X_{e_2-e_3}(b))^{n_{e_1+e_3}(-1)} \\
&= (X_{e_1-e_2}(a)^{n_{e_1+e_3}(-1)}, X_{e_2-e_3}(b)^{n_{e_1+e_3}(-1)}) \\
&= (X_{-e_2-e_3}(-a), X_{e_1+e_2}(-b^*)) \\
&= (X_{-e_2-e_3}(-a), X_{e_1+e_2}(-b^*))^{n_{2e_2}(-1)} \\
&= (X_{e_2-e_3}(-a^*), X_{e_1-e_2}(-b^*)) \\
&= X_{e_1-e_3}(-b^*a^*)
\end{aligned}$$

□

Corollary 3.4.15. $g(a, b) = ab^* + ba^*$.

Proof. By (1) of the Proposition immediately above, we have $b + b^* = f_{11}(1, g(b, 1)) = g(b, 1)$. So it suffices for us to prove $g(a, b) = g(ab^*, 1)$.

A direct application of the Hall-Witt identity tells us

$$\begin{aligned}
(X_{e_1+e_2}(a), X_{e_1-e_2}(b)) &= (X_{e_1+e_2}(a), (X_{e_1-e_3}(1), X_{e_1-e_3}(b))) \\
&= ((X_{e_3-e_2}(-b), X_{e_1+e_2}(-a)), X_{e_1-e_3}(1))^{X_{e_1+e_2}(-a)X_{e_3-e_2}(b)}
\end{aligned}$$

Since

$$\begin{aligned}
(X_{e_3-e_2}(-b), X_{e_1+e_2}(-a)) &= (X_{e_3-e_2}(-b), X_{e_1+e_2}(-a))^{n_{2e_1}(-1)n_{2e_1}(+1)} \\
&= (X_{e_3-e_2}(-b), X_{e_2-e_1}(-a^*))^{n_{2e_1}(1)} \\
&= X_{e_3-e_1}(ba^*)^{n_{2e_1}(1)} \\
&= X_{e_1+e_3}(ab^*)
\end{aligned}$$

It equals to

$$\begin{aligned}
&(X_{e_3+e_1}(ab^*), X_{e_1-e_3}(1))^{X_{e_1+e_2}(-a)X_{e_3-e_2}(b)} \\
&= (X_{e_3+e_1}(ab^*), X_{e_1-e_3}(1)) \\
&= X_{2e_1}(g(ab^*, 1)).
\end{aligned}$$

□

Lemma 3.4.16. $f_{11}(a, h) = a \cdot f_{11}(1, h)$.

Proof. Recall $(\mathbf{c}_1) : (X_{e_1-e_2}(a), X_{2e_2}(h)) = X_{2e_1}(f_{21}(a, h))X_{e_1+e_2}(f_{11}(a, h))$. Since $X_{e_1-e_2}(a) = (X_{e_1-e_3}(a), X_{e_3-e_2}(1))$, we can use the Hall-witt identity

$$\begin{aligned} \text{Id} &= ((X_{e_1-e_3}(a), X_{e_3-e_2}(1)), X_{2e_2}(h))^{X_{e_3-e_2}(-1)} \\ &\quad \cdot ((X_{e_3-e_2}(-1), X_{2e_2}(-h)), X_{e_1-e_3}(a))^{X_{2e_2}(h)} \end{aligned}$$

to obtain a different expression of $(\mathbf{c}_1)$:

$$\begin{aligned} &(X_{e_1-e_2}(a), X_{2e_2}(h)) \\ &= ((X_{e_3-e_2}(-1), X_{2e_2}(-h)), X_{e_1-e_3}(a))^{X_{2e_2}(h)X_{e_3-e_2}(1)} \\ &= (X_{2e_3}(*X_{e_3+e_2}(f_{11}(1, h))), X_{e_1-e_3}(a))^{X_{2e_2}(h)X_{e_3-e_2}(1)} \\ &= (X_{2e_3}(*X_{e_3+e_2}(f_{11}(1, h))), X_{e_1-e_3}(a))^{X_{e_3-e_2}(1)} \\ &\subset X_{2e_1}X_{e_1+e_3}(X_{e_3+e_2}(f_{11}(1, h)), X_{e_1-e_3}(a))^{X_{e_3-e_2}(1)} \\ &= X_{2e_1}X_{e_1+e_3}(X_{e_3+e_2}(f_{11}(1, h)), X_{e_1-e_3}(a))^{n_{2e_2}(-1)n_{2e_2}(1)X_{e_3-e_2}(1)} \\ &= X_{2e_1}X_{e_1+e_3}(X_{e_3-e_2}(f_{11}(1, h)), X_{e_1-e_3}(a))^{n_{2e_2}(1)} \\ &= X_{2e_1}X_{e_1+e_3}X_{e_1-e_2}(-a \cdot f_{11}(1, h))^{n_{2e_2}(1)} \\ &= X_{2e_1}X_{e_1+e_3}X_{e_1+e_2}(-a \cdot f_{11}(1, h)) \end{aligned}$$

Comparing the two, it is clear that $f_{11}(a, h) = a \cdot f_{11}(1, h)$. □

Corollary 3.4.17. $f_{21}(a, h) = aha^*$.

Proof. By the last lemma and Lemma 3.4.13, we have

$$f_{21}(2a, h) = -g(f_{11}(2a, h), a) = -2(ah)a^* - 2a(ah)^* = -4aha^*$$

the Corollary follows by Lemma 3.4.8(3) and 3.4.13, and :

Lemma 3.4.18. $h := f_{11}(1, h)$ lies inside the associative center of R , i.e. $(ah)b = a(hb)$, any $a, b \in R$.

Proof. Using the Hall-Witt identity, the equal expressions

$$\begin{aligned}
((X_{e_1-e_2}(a), X_{2e_2}(h)), X_{e_3-e_2}(b)) &= (X_{2e_1}(f_{21}(a, h))X_{e_1+e_2}(ah), X_{e_3-e_2}(b)) \\
&= (X_{e_1+e_2}(ah), X_{e_3-e_2}(b)) \\
&= (X_{e_1+e_2}(ah), X_{e_3-e_2}(b))^{n_{2e_1}(-1)n_{2e_1}(1)} \\
&= (X_{e_1+e_2}((ah)^*), X_{e_3-e_2}(b))^{n_{2e_1}(1)} \\
&= X_{e_3-e_1}(-b(ah)^*)^{n_{2e_1}(1)} = X_{e_3+e_1}(-(ah)b^*)
\end{aligned}$$

and

$$\begin{aligned}
&(X_{e_1-e_2}(a), (X_{2e_2}(-h), X_{e_3-e_2}(-b)))^{X_{e_3-3e_2}(b)X_{2e_2}(h)} \\
&= \left((X_{2e_2}(-h), X_{e_3-e_2}(-b))^{n_{e_1-e_3}(-1)n_{e_1-e_3}(1)}, X_{e_1-e_2}(a) \right) \\
&= \left((X_{2e_2}(-h), X_{e_3-e_2}(-b))^{n_{e_1-e_3}(-1)n_{e_1-e_3}(1)}, X_{e_1-e_2}(a) \right) \\
&= \left((X_{2e_2}(-h), X_{e_1-e_2}(-b))^{n_{e_1-e_3}(1)}, X_{e_1-e_2}(a) \right) \\
&= \left((X_{2e_1}(*))X_{e_1+e_2}(-bh))^{n_{e_1-e_3}(1)}, X_{e_1-e_2}(a) \right) \\
&= (X_{e_2+e_3}((bh)^*), X_{e_1-e_2}(a)) = X_{e_1-e_3}(-a(hb^*))
\end{aligned}$$

prove the lemma. The corollary follows. $\square$

Summing up all the above, we have

Theorem 3.4.19. *Given the root subgroups contain no 2-torsion, all the C_n graded groups ($n \geq 3$) are covered by:*

(1): *For $n \geq 4$, the Symplectic Steinberg group over an associative ring R with involution $*$,*

(2): *For $n = 3$, the Symplectic Steinberg group (see Section 4.2) over an alternative ring $\mathcal{A}$ with an involution $*$, given that Conjecture 3.3.3 is true.*

In both cases, the long root subgroups correspond to the symmetric elements of R (resp. $\mathcal{A}$), and the short root subgroups correspond to the R (resp. $\mathcal{A}$).

Chapter 4

Towards property (T): New Examples

4.1 Establishing Relative Property (T)

The main references for this section are [37, 20]

As mentioned in section 2.1, a typical way to prove that a group G has property (T) is to find a subset K of such that

- (a) K is a Kazhdan subset of G
- (b) the pair (G, K) has relative property (T).

Remark 4.1.1. Quoting [13]:

Clearly, (a) and (b) imply that G has property (T). Note that (a) is easy to establish when K is a large subset of G , while (b) is easy to establish when K is small, so to obtain (a) and (b) simultaneously one typically needs to pick K of intermediate size.

We're now ready to present the principal theorem of this section. Please note that square matrices over alternative rings do NOT form an associative ring.

Theorem 4.1.2. *Let $\mathcal{A}$ be a finitely generated alternative ring, $\mathcal{A} * \mathcal{A}$ the free product of two copies of the additive group of $\mathcal{A}$, and consider the semi-direct*

product $(\mathcal{A} * \mathcal{A}) \ltimes \mathcal{A}^2$, where the first copy of $\mathcal{A}$ acts by upper-unitriangular matrices, that is, $a \in \mathcal{A}$ acts as left multiplication by the matrix $e_{1,2}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, and the second copy of $\mathcal{A}$ acts by lower-unitriangular matrices. Then the pair $((\mathcal{A} * \mathcal{A}) \ltimes \mathcal{A}^2, \mathcal{A}^2)$ has relative property (T).

It is obvious that we only have to prove the theorem for $\mathcal{A}_n$, those free alternative rings generated by n elements. So we shall use from now on $\mathcal{A}_n$ and $\mathcal{A}$ interchangeably. Before proceeding, let's analyze the situation.

Denote by $L(\mathcal{A})$ the left multiplication ring of $\mathcal{A}$, defined as the ring generated by the left multiplication operators $L_a(a \in \mathcal{A})$ where $L_a(b) = ab$. From the way the action is defined, one can replace the group $\mathcal{A} * \mathcal{A}$ by a subgroup of $L(\mathcal{A}) * L(\mathcal{A})$. Formally speaking, this is given by the following map:

Lemma 4.1.3. *There exists a monomorphism of abelian groups $g : \mathcal{A} \rightarrow L(\mathcal{A})$ given by $a \mapsto L_a$.*

Proof. Since $1 \in \mathcal{A}$, it is obvious that the above map is injective. $\square$

Subtleties arise in this nonassociative situation of ours, although circumventable if we utilize following observations .

Remark 4.1.4.

(0) The free alternative ring exhibits certain “not so free” behaviors. E.g. it contains 3-torsion when $|X|$ is countable. See [33] for details.

(1) [35, pg8,13,19] The free alternative ring $\mathbb{Z}_{\mathfrak{A}}\langle X \rangle$ is a graded ring, namely

$$\mathbb{Z}_{\mathfrak{A}}\langle X \rangle = \bigoplus_{i=0}^{\infty} \langle X \rangle^i$$

where $\langle X \rangle$ is the ideal generated by the set X .

(2) [46][35, pg122 .3,.4] Let $\mathcal{A}$ be an alternative ring with a set of generators $\{a_i\}_{i \in I}$. Then the left multiplication ring $L(\mathcal{A})$ is generated by $S_{L(\mathcal{A})} := \{L_u \mid u = (a_{i_k}(a_{i_{k-1}}(\cdots(a_{i_3}(a_{i_2}a_{i_1})))) \dots), i_1 < i_2 < \dots < i_k\}$. Following

[35], we call the set of u 's involved in $S_{L(\mathcal{A})}$ the set of r_1 -words. Since finite generation of $\mathcal{A}$ implies the finiteness of the corresponding set of r_1 -words, $L(\mathcal{A})$ is finitely generated under this assumption. It is clear that $L(\mathcal{A})$ is associative.

For brevity we use the notation $\mathcal{S} := S_{L(\mathcal{A})} \cup \{Id_{\mathcal{A}}\}$.

Instead of proving the precise statement of Theorem 4.1.2, we'll handle a slightly stronger version that is a direct consequence of Theorem 2.1.8, which we rephrase below:

Theorem. [20, Theorem 1.2]

*Let R be any finitely generated associative ring, $R * R$ the free product of two copies of the additive group of R , and consider the semi-direct product $(R * R) \ltimes R^2$, where the first copy of R acts by upper-unitriangular matrices, that is, $a \in R$ acts as left multiplication by the matrix $e_{1,2}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, and the second copy of R acts by lower-unitriangular matrices. Then the pair $((R * R) \ltimes R^2, R^2)$ has relative property (T).*

Recall Theorem 2.1.8. A detailed analysis of the original argument reveals:

Theorem*. *Let S be the finite generating set of R adjoined by the unit element. Then the pair $((\mathbb{Z}S * \mathbb{Z}S) \ltimes R^2, R^2)$ has relative property (T). Here $\mathbb{Z}S$ is the additive subgroup of R generated by S .*

We are now ready to prove Theorem 4.1.2 as an implication of the following:

Proposition 4.1.5. *Let $\mathcal{A}$ be an abelian group; $\Phi := \{\varphi_1, \dots, \varphi_n\} \cup \{Id_{\mathcal{A}}\} \subset \text{End}_{\mathbb{Z}}(\mathcal{A})$. Assume that there exists an element $a \in \mathcal{A}$ such that $\mathcal{A} = \langle \Phi \rangle a$, where $\langle \Phi \rangle$ is the ring generated by Φ . If we define the action as in Theorem 4.1.2, then $((\mathbb{Z}\Phi * \mathbb{Z}\Phi) \ltimes \mathcal{A}^2, \mathcal{A}^2)$ has relative property (T).*

Proof. Set $R := \mathbb{Z}\langle x_1, \dots, x_n \rangle$, $S := \{x_1, \dots, x_n\}$. There exists a map f between the pairs $((\mathbb{Z}S * \mathbb{Z}S) \ltimes R^2, R^2)$ and $((\mathbb{Z}\Phi * \mathbb{Z}\Phi) \ltimes \mathcal{A}^2, \mathcal{A}^2)$, given by the data $f|_{\mathbb{Z}S} : x_i \mapsto \varphi_i; 1 \mapsto Id_{\mathcal{A}}$ and $f|_{R^2} : (P_1(x_1, \dots, x_n), P_2(x_1, \dots, x_n)) \rightarrow$

$(P_1(\varphi_1, \dots, \varphi_n)a, P_2(\varphi_1, \dots, \varphi_n)a)$. It is clear that this map extends to an epimorphism of groups. $\square$

Proof of Theorem 4.1.2 Define g as in Lemma 4.1.3. Recall that $\mathcal{S} := S_{L(\mathcal{A})} \cup \{Id_{\mathcal{A}}\}$. Setting Φ to be $\mathcal{S}$, $\mathcal{A}$ to be $\mathcal{A}$, $a = 1$ in the previous proposition, we get that $((\mathbb{Z}\mathcal{S} * \mathbb{Z}\mathcal{S}) \ltimes \mathcal{A}^2, \mathcal{A}^2) \cong ((g^{-1}(\mathbb{Z}\mathcal{S}) * g^{-1}(\mathbb{Z}\mathcal{S})) \ltimes \mathcal{A}^2, \mathcal{A}^2)$ has relative property (T) , which implies Theorem 4.1.2 since the latter embeds into $((\mathcal{A} * \mathcal{A}) \ltimes \mathcal{A}^2, \mathcal{A}^2)$, preserving $\mathcal{A}^2$.

Remark 4.1.6.

- (1) The above proof could be extended effortlessly to any non-associative unital ring R , as long as the left multiplication ring $L(R)$ is finitely generated and the action is well-defined.
- (2) The theorem works just as well if the action is given instead by right multiplication.

Combining the above result with [13, Theorem 5.1], We can now prove:

Theorem 4.1.7. *For $\mathcal{A}$ a finitely generated alternative ring, $\text{St}_3(\mathcal{A})$ is Kazhdan.*

Proof. It is obvious that $\text{St}_3(\mathcal{A})$ has a strong A_2 grading in the sense of [13, §4.4], which roughly means that if $\gamma \in A_2$ is a positive non simple root, then G_γ , the corresponding root subgroup in $\text{St}_3(\mathcal{A})$ is contained in the subgroup generated by the simple-root subgroups. This implies that [13, Theorem 5.1] applies, giving $\kappa(\text{St}_3(\mathcal{A}), \bigcup G_{ij}) > 0$

To finish the proof, it is necessary to establish $\kappa_r(\text{St}_3(\mathcal{A}), G_{ij}; S) > 0$ for some finite set $S \subset \text{St}_3(\mathcal{A})$. Every root subgroup G_{ij} of $\text{St}_3(\mathcal{A})$ belongs to some subgroup N_{ij} isomorphic to $\mathcal{A}^2$, which is in turn contained naturally in T_{ij} , a homomorphic image of $(\mathcal{A} * \mathcal{A}) \ltimes \mathcal{A}^2$ embedded inside $\text{St}_3(\mathcal{A})$ (See [37, Lemma 2.4])(For example, we can set $T_{12} = \langle G_{23}, G_{32} \rangle \ltimes \langle G_{12}, G_{13} \rangle$). Therefore Theorem 4.1.2 guarantees that for every G_{ij} there exists a finite subset $S_{ij} \subset T_{ij}$ such that

$$\delta_{ij} := \kappa_r(\text{St}_3(\mathcal{A}), G_{ij}; \bigcup_{i \neq j} S_{ij}) \geq \kappa_r(\text{St}_3(\mathcal{A}), N_{ij}; \bigcup_{i \neq j} S_{ij}) \geq \kappa_r(T_{ij}, N_{ij}; S_{ij}) > 0.$$

Since $\kappa_r(\text{St}_3(\mathcal{A}), \bigcup G_{ij}; \bigcup S_{ij}) \geq \inf_{i \neq j} \delta_{ij} > 0$, the Theorem is proved. $\square$

It is known that Property (T) is inherited by homomorphic images. Consequently we have

Corollary 4.1.8. *$PE_3(\mathcal{A})$ is Kazhdan.*

4.2 A Symplectic Group over Alternative Rings

In this section, $\mathcal{A}$ will always be a finitely generated alternative ring with a nuclear involution $*$, namely every $*$ -symmetric element α satisfies the identity $(\alpha, a, b) := (\alpha a)b - \alpha(ab) = 0$ for $a, b \in \mathcal{A}$. The set of $*$ -symmetric elements will be denoted $\text{Sym}(\mathcal{A})$. We will also use the standard realization of $C_3 : \{\pm w_i \pm w_j\} \cup \{\pm 2w_i\}$. For the Jordan identities involved, see [27, §7.6]

Aside from the the aforementioned A_2 case, there are more instances of low rank “root graded groups” defined over alternative rings. We’ll describe such an example below. We call it a “symplectic group” since such group comes with a C_3 -grading.

More precisely, the group we are to consider is $\text{PE}(V)$, where V is the Jordan pair $(\mathcal{H}_3(\mathcal{A}, *), \mathcal{H}_3(\mathcal{A}, *))$ (which is exceptional when $\mathcal{A}$ is not associative). Here $\mathcal{H}_3(\mathcal{A}, *)$ (usually called Hermitian matrices) is the linear subspace of $M_3(\mathcal{A})$ consisting of symmetric elements under the involution map $J_1 : (x_{ij}) \rightarrow (x_{ji}^*)$ of the full matrix space. To avoid confusion, we may use superscripts $\pm$ to distinguish elements from the two components.

As in Remark 2.3.4, the Jordan pair structure on V is dictated by the well understood Jordan algebra $\mathcal{H}_3(\mathcal{A}, *)$. We will refer to [31, Pg 1075] for a fully worked out (quadratic) Jordan multiplication table with respect to the following basis of $\mathcal{H}_3(\mathcal{A}, *)$, given in terms of matrix units $E_{ij} (1 \leq i, j \leq 3)$:

$$\begin{aligned} \alpha[ii] &:= \alpha E_{ii}, \quad \alpha = \alpha^* \in \mathcal{A} \\ d[ij] &:= dE_{ij} + d^*E_{ji}, \quad d \in \mathcal{A} \\ [ij] &:= E_{ij} + E_{ji}, \quad [ii] := E_{ii} \end{aligned}$$

Note that under this notation, $d[ij] := dE_{ij} + d^*E_{ji} = d^*[ji]$.

Remark 4.2.1. Denote by xy the usual matrix product in $\mathcal{H}_3(\mathcal{A}, *)$, and the brace product $\{x, y\} := xy + yx$. Then the expression $U_x(y) = \frac{1}{2}\{x, \{x, y\}\} - \{x^2, y\}$ is well defined, regardless of the presence (or lack thereof) of $\frac{1}{2}$ in $\mathcal{H}_3(\mathcal{A}, *)$.

Remark 4.2.2. ([31, Pg 1075]) To determine the products in $\mathcal{H}_3(\mathcal{A}, *)$, one has to go no further than computing them for the basis elements. They are as follows:

Quadratic products (U_x):

$$\begin{aligned} U_{\alpha[ii]}\beta[ii] &= \alpha\beta\alpha[ii], \\ U_{d[ij]}b[ij] &= db^*d[ij] \ (i \neq j), \\ U_{d[ij]}\beta[jj] &= dbd^*[ij] \ (i \neq j), \\ U_{a[ij]}b[kl] &= 0 \ \text{if } \{k, l\} \neq \{i, j\} \end{aligned}$$

Triple products ($\{x, y, z\}$):

$$\begin{aligned} \{d[ij], b[jk], c[kl]\} &= \frac{1}{2}(d(bc) + (db)c)[il], \\ \{d[ij], b[ji], c[ik]\} &= d(bc)[ik], \\ \{d[ij], b[jk], c[ki]\} &= (d(bc) + (bc)^*d^*)[ii], \\ \{d[ij], b[ji], c[ii]\} &= (d(bc) + (bc)^*d^*)[ii]. \end{aligned}$$

Also, a triple product on basis elements can be nonzero if and only if it can be written in the form $\{a[ij], b[kl], c[pq]\}$, where $j = k$ and $p = q$.

Let's compare $\mathfrak{g} = \mathcal{TKK}(V)$ to the usual symplectic Lie algebra $\mathfrak{c} = \mathfrak{sp}_6(F)$, where F is an algebraically closed field:

In matrix terms, the condition for $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{c}$ ($A, B, C, D \in \mathfrak{gl}(3, F)$) to be symplectic is that $B^T = B, C^T = C$, and $A^T = -D$. Under this realization, entries of B (resp. C) correspond to the root subspaces of $\{w_i + w_j\} \cup \{2w_i\}$ (resp. $\{-w_i - w_j\} \cup \{-2w_i\}$), while $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ contains the root subspaces of $\{w_i - w_j\}$ and the Cartan subalgebra.

This root decomposition actually carries over to $\mathcal{TKK}(V)$ (see [19]) when we treat $V^\pm$ like the upper left and lower right blocks of the matrices above. Explicitly, the root subspaces are:

Table 4.1: Root space decomposition for $\mathcal{TKK}(V)$

root γ	root space W_γ
$\pm 2w_i$	$\alpha[ii]^\pm$ ($\alpha = \alpha^* \in \mathcal{A}$)
$\pm(w_i + w_j)$	$d[ij]^\pm$ ($d \in \mathcal{A}$)
$(w_i - w_j)$	$\{\delta(x.y) x \in W_\mu, y \in W_\rho, \mu \text{ positive}, \rho \text{ negative}, \mu + \rho = w_i - w_j\}$

In particular, this tells us that $V^\pm$ are direct sums of the root subspaces with respect to $\pm\{\{2w_i\} \cup \{w_i + w_j\}\}$.

When a Lie algebra has root space decomposition (with respect to a root system Φ), one naturally expects its corresponding ‘‘Chevalley group of adjoint type’’ (see [7]), obtained from exponentiation, to obey Φ -commutator relations. This is indeed the case for our $\text{PE}(V)$, as suggested by

Theorem 4.2.3. [28, Theorem 2] *For the Jordan pair V defined above, $G = \text{PE}(V)$ has a C_3 -grading with the following root subgroups:*

$$G_\alpha = \begin{cases} \exp_\pm(W_\alpha), & \text{for } \alpha \in \pm\{\{2w_i\} \cup \{w_i + w_j\}\}, \\ \langle \bigcup \{\beta(W_\mu, W_\rho) | \mu \text{ positive}, \rho \text{ negative}, \mu + \rho = \beta\} \rangle, & \text{for } \alpha \in \{w_i - w_j\} \end{cases}$$

Although the theorem does reveal all the root subgroups, it alone is not sufficient for our purpose, requiring us to compute some commutator relations for $\text{PE}(V)$.

We first present some useful notations from [27]. For $(x, y) \in V$, since $\exp_+(x)$ and $\exp_-(y)$ are automorphisms of $\mathfrak{g} = \mathcal{TKK}(V) = V^+ \oplus \mathfrak{L}_0(V) \oplus V^-$, we can adopt the following matrix notation :

$$\exp_+(x) = \begin{pmatrix} 1 & ad_x & Q_x \\ 0 & 1 & ad_x \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp_-(y) = \begin{pmatrix} 1 & 0 & 0 \\ ad_y & 1 & 0 \\ Q_y & ad_y & 1 \end{pmatrix}$$

In the same spirit, if $h = (h^+, h^-)$ is an automorphism of V , then using the matrix notation one can write the induced automorphism on $\mathfrak{g}$ as:

$$\begin{pmatrix} h^+ & 0 & 0 \\ 0 & h_0 & 0 \\ 0 & 0 & h^- \end{pmatrix}$$

where $h_0(-\delta(x, y)) = h_0([x, y]) = [h_+(x), h_-(y)]$.

The goal of obtaining a reasonable form of commutator relations require us to “coordinatize” each root subgroup. This is easy for $\pm\{\{2w_i\} \cup \{w_i + w_j\}\}$ as they have canonical coordinates coming from the matrix structure of $\mathcal{H}_3(\mathcal{A}, *)$. In the succeeding paragraphs we resolve the problem for the remaining roots.

Proposition 4.2.4. *Let $x \in W_\gamma \subset V^+$, $y \in W_\tau \subset V^-$, $\gamma \neq -\tau, \gamma + \tau \in C_3$, then the Bergmann operators $B(x, y)$ and $B(y, x)$ are invertible.*

Proof. By [27, Pg 87], it suffices to prove invertibility for $B(x, y)$ where x, y are basis elements of $\mathcal{H}_3(\mathcal{A}, *)^+$ and $\mathcal{H}_3(\mathcal{A}, *)^-$, respectively. We shall need the following

Lemma 4.2.5. *As operators on V^+ , the following holds:*

1. $ad_x ad_y = -D(x, y)$
2. $ad_x ad_y Q_x Q_y = Q_x Q_y ad_x ad_y$
3. $(Q_x Q_y)^2 = 0$

Proof. Since for $z \in V^+$, $ad_x ad_y z = [x, [y, z]] = [[x, y], z] + [y, [x, z]]$, (1) is proved.

Using the result of (1) and symmetry, we see that (2) is equivalent to the first of the defining identities of Jordan pairs, showing its validity.

The general proof of (3) involves a lengthy calculation using the data in Remark 4.2.2, but we can give a shorter proof when $\frac{1}{2} \in \mathcal{A}$.

Recall that for a Jordan Pair V and the corresponding Lie algebra $\mathcal{T}\mathcal{K}\mathcal{K}(V)$, we can relate the bracket and the quadratic operator by equation (2.8):

$$2Q_x(y) = \{xyx\} = [x, [x, y]]$$

for $x \in V^\sigma, y \in V^{-\sigma}$. This tells us that in our case if $x \in W_\gamma, y \in W_\tau$, then $Q_x(y) \in W_{2\gamma+\tau}, Q_y(x) \in W_{2\tau+\gamma}$. Specifying γ and τ to be in $\pm\{\{w_i + w_j\} \cup \{2w_i\}\} \subset C_3$, it is clear that one of the two expressions $2\gamma + \tau, \gamma + 2\tau$ is not a root, implying either $Q_x(y)$ or $Q_y(x)$ is zero. Now (3) follows from the third defining identity of Jordan pairs. $\square$

Also, one general identity that holds for any Jordan pair:

$$\{\{xyu\}, y, z\} = \{x, Q_y u, z\} + \{u, Q_y x, z\}. \quad (4.1)$$

, which is a direct consequence of identity (2.2)

Claim 4.2.6. *Under the above assumptions, $B(x, y)^{-1} = \text{Id} - \text{ad}_x \text{ad}_y + Q_x Q_y$*

Proof. Using the lemma above for all admissible x, y 's, we have:

$$\begin{aligned} & B(x, y)(\text{Id} - \text{ad}_x \text{ad}_y + Q_x Q_y) \\ &= (\text{Id} + \text{ad}_x \text{ad}_y + Q_x Q_y)(\text{Id} - \text{ad}_x \text{ad}_y + Q_x Q_y) \\ &= \text{Id} - (\text{ad}_x \text{ad}_y)^2 + \cancel{\text{ad}_x \text{ad}_y Q_x Q_y} - \cancel{Q_x Q_y \text{ad}_x \text{ad}_y} + (Q_x Q_y)^2 + 2Q_x Q_y \\ &= \text{Id} - (\text{ad}_x \text{ad}_y)^2 + 4Q_x Q_y - 2Q_x Q_y \end{aligned}$$

Since

$$\begin{aligned} & (-(\text{ad}_x \text{ad}_y)^2 + 4Q_x Q_y - 2Q_x Q_y)a \\ &= -\{\{ayx\}, y, x\} + \{x, Q_y a, x\} = -\{x, Q_y a, x\} - \{x, Q_y x, a\} + \{x, Q_y a, x\} \\ &= -\{x, Q_y x, a\} \end{aligned}$$

(Note the usage of identity (4.1)), it would suffice to prove $\{x, Q_y x, a\} = 0$ for all $a \in V^+$. This essentially breaks down to 2 cases:

1. $x = c[ii], y = b[jj]$:

Here $Q_y(x) = b^*cb[jj]$, so $\{x, Q_y x, a\} = -\{a[ii], b^*cb[jj], a\} = 0$.

2. $x = c[ij], y = b[jj]$: Then $Q_y(x) = 0$, so $\{x, Q_y x, a\} = 0$.

Thus the claim is verified. The same argument works for $B(y, x)$ by symmetry. $\square$

Corollary 4.2.7. $B(x, y)^{-1} = B(-x, y) = B(x, -y)$.

Proof. This follows directly from the claim above. $\square$

Lemma 4.2.8. In $\mathcal{H}_3(\mathcal{A}, *)$, for $1 \leq i \neq j \neq k \leq 3$, we have the following identities involving Bergmann operators:

$$\{\iota\} \quad B(\alpha[ii], a[ij]) = B([ii], \alpha a[ij])$$

$$\{I\} \quad B(a[ij], b[jk]) = B([ij], ab[jk]) = B(ab[ij], [jk])$$

$$\{\bullet\} \quad B(a[ij], \alpha[jj]) = B(a\alpha[ij], [jj])$$

$$\{\iota\} \quad B(\alpha[ii], a[ij])B(\beta[ii], b[ij]) = B([ii], (\alpha a + \beta b)[ij])$$

$$\{II\} \quad B(a[ij], b[jk])B(c[ij], d[jk]) = B([ij], (ab + cd)[jk]) = B((ab + cd)[ij], [jk])$$

$$\{\bullet\bullet\} \quad B(a[ij], \alpha[jj])B(b[ij], b[jj]) = B((a\alpha + b\beta)[ij], [jj])$$

Proof. This again involves term by term checks against the multiplication table in Remark 4.2.2: The only basis elements that $B(\alpha[ii], a[ij])$ might not act on as identity and their images are: $b[ji] \mapsto b[ji] - (\alpha ab + (\alpha ab)^*[ii]; \gamma[jj] \mapsto \gamma[jj] - \alpha a \gamma[ij] + (\alpha a) \gamma(\alpha a)^*[ii]; c[jk] \mapsto c[jk] - \alpha(ac)[ik]$, which justifies $\{\iota\}$ & $\{u\}$; The only basis elements that $B(a[ij], b[jk])$ may not act on as identity and their images are: $\gamma[kk] \mapsto \gamma[kk] - \gamma(ab)^*[ki] + (ab) \gamma(ab)^*[ii]; c[ki] \mapsto c[ki] - ((ab)c + c^*(ab)^*)[ii]; d[kj] \mapsto d[kj] - d^*(ab)^*[ji]$, which is sufficient to show $\{I\}$ & $\{II\}$. Likewise for $\{\bullet\}$ & $\{\bullet\bullet\}$ $\square$

The computation above yields:

Corollary 4.2.9. For all $1 \leq i \neq j \neq k \leq 3$, $B(\alpha[ii], s[ij]) = B(\alpha[ik], s[kj])$ and $B(s[ik], \alpha[kj]) = B(s[ij], \alpha[kj])$ holds.

This allows the following “coordinatization” of root subgroups:

Definition 4.2.10. For any root $\gamma \in C_3$, one may define abelian group isomorphisms such that:

$$G_\gamma \cong \begin{cases} \mathcal{A}, & \text{If } \gamma \in \pm\{w_i + w_j\} \\ \text{Sym}(\mathcal{A}), & \text{If } \gamma \in \pm\{2w_i\} \\ \mathcal{A}, & \text{If } \gamma \in \{w_i - w_j\} \end{cases}$$

The isomorphisms in the first two cases are inherited from the matrix structure of $\mathcal{H}_3(\mathcal{A})$. In the third case, the isomorphism given by $\beta([ik]^+, s[kj]^-) \mapsto s$ is well-defined, ensured by the preceding lemma and its corollary.

We will also stick to the convention that is

$$G_{\pm(w_i+w_j)}(a) = \exp_{\pm}(a[ij]) = \exp_{\pm}(a^*[ji]) \quad \text{for } i < j.$$

Lemma 4.2.11. Denoting by $G_\gamma(s)$ the preimage of $s \in \mathcal{A}$ (resp. $\text{Sym}(\mathcal{A})$) under the preceding isomorphisms, the relation $G_\gamma(s)G_\gamma(r) = G_\gamma(s+r)$ holds for all $\gamma \in C_3$.

Proof. This is obvious for $\gamma \in \pm\{\{w_i + w_j\} \cup \{2w_k\}\}$. For the remaining roots, this is a direct consequence of Lemma 4.2.8 $\square$

We now compute $(\exp_+(x), \exp_-(y))$ for the cases (1) $x = \alpha[ii]^+, y = b[ij]^-$; (2) $x = a[ij]^+, y = b[jk]^-$, (3) $x = b[ij]^+, y = \alpha[ii]^-$. According to Theorem 4.2.3, it is clear that $(\exp_+(\alpha[ii]), \exp_-(b[ij]))$ is supposed to be equal to

$$\begin{pmatrix} h^+ & 0 & 0 \\ h_0 \circ \text{ad}_{y'} & h_0 & 0 \\ h^- \circ Q_{y'} & h^- \circ \text{ad}_{y'} & h^- \end{pmatrix} = \underbrace{\begin{pmatrix} h^+ & 0 & 0 \\ 0 & h_0 & 0 \\ 0 & 0 & h^- \end{pmatrix}}_{\in G_{w_i-w_j}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \text{ad}_{y'} & 1 & 0 \\ Q_{y'} & \text{ad}_{y'} & 1 \end{pmatrix}}_{\in G_{-2w_j}}$$

for some h 's given by a automorphism of V (see our notations below Theorem 4.2.3) and $y' \in V^-$. Direct computation of the left hand side of the above equation tells us that h^- is $B(b[ij], \alpha[ii])^{-1}$ while $Q_y = Q_{(b^*\alpha b)[ii]}$. This gives the relation:

$$(1) \quad (\exp_+(\alpha[ii]), \exp_-(b[ij])) = G_{w_i-w_j}(\alpha b) G_{-2w_j}(b^* \alpha b)$$

Similar computations yield:

$$(2) \quad (\exp_+(\alpha[ij]), \exp_-(b[jk])) = G_{w_i-w_k}(\alpha b)$$

$$(3) \quad (\exp_-(\alpha[ii]), \exp_+(b[ij])) = G_{w_j-w_i}(-b^* \alpha) G_{2w_j}(b^* \alpha b)$$

As are:

$$(4) \quad (G_{w_i-w_j}(a), G_{w_j-w_k}(b)) = G_{w_i-w_k}(ab)$$

$$(5) \quad (G_{w_i-w_j}(a), G_{w_j+w_k}(b)) = G_{w_i-w_k}(ab)$$

$$(6) \quad (G_{w_i-w_j}(a), G_{-w_i-w_k}(b)) = G_{-w_j-w_k}(-a^* b)$$

We can also compute the commutator relation involving both long and short root subgroups. Noting that $G_{w_i-w_j}(r) = \beta(r[ij]^+, [jj]^-)$, $G_{w_i+w_j}(s) = \exp_+(s[ij])$, the relation

$$(7) \quad (G_{w_i-w_j}(r), G_{w_i+w_j}(s)) = G_{2w_i}(-rs^* - sr^*)$$

follows from the fact that

$$\begin{aligned} & G_{w_i-w_j}(r)^{-1} G_{w_i+w_j}(s)^{-1} G_{w_i-w_j}(r) G_{w_i+w_j}(s) \\ &= \exp_+(\beta(-r[ij]^+, [jj]^-)(-s[ij])) G_{w_i+w_j}(s) \\ &= \exp_+(-s[ij] + \{r[ij], [jj], -s[ij]\} + 0) G_{w_i+w_j}(s) \\ &= \exp_+(-s[ij] - rs^* - sr^*) \exp_+(s[ij]) \\ &= \exp_+(-rs^* - sr^*) \end{aligned}$$

Remark 4.2.12. By its commutator relations, one sees that $\text{PE}(V)$ generalizes elementary hyperbolic unitary groups with full form ring and $\epsilon = -1$ (see [17]) to the setting of alternative rings.

We need two more definitions before stating the main theorem.

Definition 4.2.13. We denote by $\text{St}(V)$ the group generated by the the set of symbols $\{G_\gamma(a) \mid \gamma \in C_3, a \in \mathcal{A} \text{ if } \gamma \text{ is short, } a \in \text{Sym}(\mathcal{A}) \text{ if } \gamma \text{ is long}\}$ satisfy-

ing the commutator relations given by the commutators $(G_{\gamma_1}, G_{\gamma_2})$ ($\gamma_1 \neq -\gamma_2$) in $PE(V)$.

Remark 4.2.14. See [27] for a general exposition for Steinberg groups of Jordan pairs.

Definition 4.2.15. (See [13, 8.3] , [17]) We say that $Sym(\mathcal{A})$ is *finitely generated as a form ring* if there exists a finite set $\{a_i\}_{i=1}^k \subset Sym(\mathcal{A})$ such that

$$Sym(\mathcal{A}) = \left\{ \sum_{i=1}^k s_i a_i s_i^* + (r + r^*) \mid \forall s_i, r \in \mathcal{A} \right\}$$

Theorem 4.2.16. $St(V)$ is Kazhdan if $Sym(\mathcal{A})$ is finitely generated as a form ring.

Proof. By the commutator relations of $PE(V)$ it follows immediately that the relation is strong, therefore [13, Theorem 5.1] applies.

It remains to show relative property (T) for each root subgroup. The proof of [13, Prop. 8.6(a)] could be applied to our situation verbatim. We summarize the proof here: using the commutator relations we gave previously, the statement is true for any short root subgroup G_γ since it lies in a bigger subgroup that is a homomorphic image of $St_3(\mathcal{A})$. For long roots it is a bounded generation argument: If we set in the commutator relations (3) and (7) $\alpha = a_i$, $b = s_i$ and $s = -1$ (as in the definition of $Sym(\mathcal{A})$), then any element in G_{2w_i} can be written as a finite product (of bounded length) of the elements from short root subgroups and their conjugates by the finite set $\{G_\gamma(a_i) \mid a_i \text{ generates } Sym(\mathcal{A}), \gamma \in C_3 \text{ long}\}$. Since $(G, G_{\gamma_{short}})$ and $(G, G_{\gamma_{short}}^{G_{\gamma_{long}}(a_i)})$ are all relative (T) pairs, a direct application of Lemma 2.1.6 finishes the proof. $\square$

Since it is a homomorphic image of $St(V)$, we have:

Corollary 4.2.17. $PE(V)$ is Kazhdan.

4.3 An Exceptional Group

The main references for this section are [15, 5]. We retain the restriction that the base ring is commutative, associative, and always contain $\frac{1}{6}$. Notations as in the “Preliminaries” chapter.

The idea is straightforward: a suitable exponentiation of a Lie algebra of certain Cartan type gives a group of the corresponding type, i.e. we obtain groups $\mathbb{G}$ reminiscent of adjoint Chevalley groups $\mathfrak{I}(R)$ (See [7]).

Let R be a base ring containing $\frac{1}{6}$, $\mathcal{C} := \mathcal{O}_R$, the split octonion algebra over R , and $\mathfrak{J}$ any cubic Jordan algebra over R . The Freudenthal-Tits construction gives a G_2 graded Lie algebra over R :

$$\begin{aligned}\mathcal{T}(G_2) &= \mathcal{T}(\mathcal{C}, \mathfrak{J}) \\ &= \text{Der}_R(\mathcal{C}) \oplus \mathcal{C}_0 \otimes \mathfrak{J}_0 \oplus \text{InnDer}_R(\mathfrak{J}) \\ &= \mathfrak{g}_2 \oplus \mathcal{C}_0 \otimes \mathfrak{J}_0 \oplus \text{InnDer}_R(\mathfrak{J})\end{aligned}$$

Table 4.2: Root space decomposition for $\mathcal{T}(G_2)$

Root γ	Root space $\mathcal{T}(G_2)_\gamma$
0	$\text{Der}_R(\mathcal{C})_0 \oplus v_0 \otimes \mathfrak{J}_0 \oplus \text{InnDer}_R(\mathfrak{J})$
τ (long)	$\text{Der}_R(\mathcal{C})_\tau$
ι (short)	$\text{Der}_R(\mathcal{C})_\iota \oplus v_\iota \otimes \mathfrak{J}_0$

Let $\text{Der}_R(\mathcal{C}) = \bigoplus_{\gamma \in G_2 \cup \{0\}} \text{Der}_R(\mathcal{C})_\gamma$ be the root space decomposition of $\mathfrak{g}_2$. Similarly, $\mathcal{T}(G_2) = \bigoplus_{\gamma \in G_2 \cup \{0\}} \mathcal{T}(G_2)_\gamma$. Using Tables 2.1 and 2.2, the root decomposition for $\mathcal{T}(G_2)$ can be obtained: see Table 4.2.

Recall that the Chevalley basis $\mathfrak{b} := \{x_\gamma, h_i \mid \gamma \in G_2, i \in \{1, 2\}\}$ was obtained for $\text{Der}_R(\mathcal{C})$ in terms of inner derivations with entries in $\mathbb{Z}[\frac{1}{3}]\mathcal{B}$. (See expression 2.16). With this notation, we write the root elements in the following fashion:

- For a long root $\tau \in G_2$, one has $\mathcal{T}(G_2)_\tau = rx_\tau$, $r \in R$,

- For a short root $\iota \in G_2$, one has $\mathcal{T}(G_2)_\iota = rx_\iota + v_\iota \otimes j_0$, $r \in R, j_0 \in \mathfrak{J}_0$. we denote this element by $(r + \frac{1}{2}j_0)y_\iota$. Uniqueness of such expression is guaranteed by $\mathfrak{J} = \mathfrak{J}_0 \oplus R1$.

Since the root strings in G_2 has length ≤ 4 , the existence of $\frac{1}{6}$ allows us to exponentiate $\text{ad}(rx_\tau)$ and $\text{ad}(jy_\iota)$, where $r \in R$ and $j \in \mathfrak{J}$. For example, $\exp(\text{ad}(rx_\tau)) = \sum_{i=0}^3 \frac{1}{i!} \text{ad}(rx_\tau)^i$.

From now on, we write $e_\gamma(r)$ in place of $\exp(\text{ad}(rx_\gamma))$.

Definition 4.3.1. For any root $\gamma \in G_2$, the *root subgroup* E_γ is:

$$E_\gamma = \begin{cases} e_\gamma(r) & \text{if } \gamma \text{ is long,} \\ e_\gamma(j) & \text{if } \gamma \text{ is short.} \end{cases}$$

for all $r \in R, j \in \mathfrak{J}$.

Remark 4.3.2. It is clear that E_γ is an abelian group isomorphic to the underlying abelian group of R (for γ long). For γ short, E_γ is isomorphic to the underlying abelian group of $\mathfrak{J}$ since $[D_{e_i, v_{ij}}, v_{ij} \otimes j_0] = D_{e_i, v_{ij}}(v_{ij}) \otimes j_0 = 0$, all $i, j \in \mathbb{Z}$ and $j_0 \in \mathfrak{J}_0$.

Definition 4.3.3. $\mathbb{G} := \langle E_\gamma \mid \text{all } \gamma \in G_2 \rangle$. This is a subgroup of $\text{Aut}_R(\mathcal{T}(G_2))$.

Fix simple roots α and β of G_2 with α short, β long. We will be consistently quoting results from section 2.4.1.

Theorem 4.3.4. $\mathbb{G}$ satisfies the following relations:

0. $e_\alpha(j_1)e_\alpha(j_2) = e_\alpha(j_1 + j_2); \quad e_\beta(a_1)e_\alpha(a_2) = e_\alpha(a_1 + a_2)$
1. $(e_\alpha(j), e_\beta(a)) = e_{3\alpha+2\beta}(a^2N(j))e_{3\alpha+\beta}(aN(j))e_{2\alpha+\beta}(-aj^\#)e_{\alpha+\beta}(aj)$
2. $(e_\alpha(j_1), e_{\alpha+\beta}(j_2)) = e_{3\alpha+2\beta}(T(j_1, j_2^\#))e_{3\alpha+\beta}(-T(j_1^\#, j_2))e_{2\alpha+\beta}(j_1\#j_2)$
3. $(e_\alpha(j_1), e_{2\alpha+\beta}(j_2)) = e_{3\alpha+\beta}(T(j_1, j_2))$
4. $(e_\beta(a_1), e_{3\alpha+\beta}(a_2)) = e_{3\alpha+2\beta}(a_1a_2)$

Here the a 's lie in R , j 's lie in $\mathfrak{J}$.

It is obvious that these relations hold for any choice of simple roots of G_2 , for the current choice of roots (see section 2.4.1) is dependent on a random choice of simple root for the subsystem of long roots (A_2).

Proof. (0) is settled in remark 4.3.2; (4) holds automatically as the long root subgroups generate an adjoint Chevalley group of $\mathfrak{A}_{A_2}(R)$. We prove (1) and (3). The method for these two also applies to (2).

Recall the *Zassenhaus formula* (See [29]) for Lie algebras over characteristic zero fields:

$$\begin{aligned} e^{t(X+Y)} = & e^{tX} \cdot e^{tY} \cdot e^{-\frac{t^2}{2}[X,Y]} \cdot e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])} \\ & \cdot e^{\frac{-t^4}{24}([[[X,Y],X],X]+3[[X,Y],X],Y]+3[[X,Y],Y],Y))} \dots \end{aligned}$$

□

If all commutators of X, Y of length ≤ 4 vanishes, one obtains a “baby formula” that suits our purpose:

$$e^{t(X+Y)} = e^{tX} \cdot e^{tY} \cdot e^{-\frac{t^2}{2}[X,Y]} \cdot e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])} \quad (4.2)$$

Notation: From now on, we denote $tr(x) := \frac{1}{3}T(x)$, $x \in \mathfrak{J}$.

Computation of (3):

We compute $e_\alpha(j_1)e_{2\alpha+\beta}(j_2)e_\alpha(-j_1)e_{2\alpha+\beta}(-j_2)$.

(i) $e_{2\alpha+\beta}(j_2)e_\alpha(-j_1)e_{2\alpha+\beta}(-j_2)$: Recall that for a Lie algebra L ,

$$\phi(\text{ad } x)\phi^{-1} = (\text{ad } \phi(x)). \quad x \in L, \quad \phi \in \text{Aut}(L)$$

therefore

$$\begin{aligned}
& e_\beta(j_2) \operatorname{ad}(D_{e_2, v_{22}}) e_\beta(-j_2) \\
&= \operatorname{ad}(D_{e_2, v_{22}} + [tr(j_2)D_{e_1, v_{22}} + v_{11} \otimes 2(j_2)_0, D_{e_2, v_{22}}]) \\
&= \operatorname{ad}(D_{e_2, v_{22}} + tr(b)D_{v_{11}, v_{22}}), \\
& e_\beta(j_1) \operatorname{ad}(v_{22} \otimes 2(j_1)_0) e_\beta(-j_1) \\
&= \operatorname{ad}(v_{22} \otimes 2(j_1)_0 + [tr(j_2)D_{e_1, v_{11}} + v_{11} \otimes 2(j_2)_0, v_{22} \otimes 2(j_1)_0]) \\
&= \operatorname{ad}\left(v_{22} \otimes 2(j_1)_0 + \frac{1}{3}T((j_1)_0, (j_2)_0)\right) D_{v_{11}, v_{22}}
\end{aligned} \tag{4.3}$$

implies that

$$\begin{aligned}
& e_{2\alpha+\beta}(j_2) e_\alpha(-j_1) e_{2\alpha+\beta}(-j_2) \\
&= \exp\left(-tr(j_1) \operatorname{ad}(D_{e_2, v_{22}} + v_{22} \otimes 2j_0) - \right. \\
& \quad \left. \operatorname{ad}\left(tr(j_1)tr(j_2) + \frac{1}{3}T((j_1)_0, (j_2)_0)\right) D_{v_{11}, v_{22}}\right) \\
&= \exp\left(-\operatorname{ad}(j_1 y_\alpha) + \operatorname{ad}(T(j_1, j_2)x_{3\alpha+\beta})\right),
\end{aligned} \tag{4.4}$$

since $D_{v_{11}, v_{22}} = -\frac{1}{3}x_{3\alpha+\beta}$.

(ii) Setting $X = -\operatorname{ad}(j_1 y_\alpha)$, $Y = \operatorname{ad}(T(j_1, j_2)x_{3\alpha+\beta})$, formula (4.2) gives

$$e_\alpha(j_1) e_{2\alpha+\beta}(j_2) e_\alpha(-j_1) e_{2\alpha+\beta}(-j_2) = e^Y.$$

Summarizing, we have

$$(e_\alpha(-j_1), e_{2\alpha+\beta}(-j_2)) = e_{3\alpha+\beta}(T(-j_1, -j_2)),$$

which is equivalent to **(3)**

Computation of (1):

We compute $e_\alpha(j)e_\beta(a)e_\alpha(-j)e_\beta(-a)$.

(i) $e_\beta(a)e_\alpha(-j)e_\beta(-a)$:

$$\begin{aligned}
& e_\beta(a) \operatorname{ad}(D_{e_2, v_{22}}) e_\beta(-a) & e_\beta(a) \operatorname{ad}(v_{22} \otimes 2j_0) e_\beta(-a) \\
= \operatorname{ad}(D_{e_2, v_{22}} + [ax_\beta, D_{e_2, v_{22}}]) & = \operatorname{ad}(v_{22} \otimes 2j_0 + [ax_\beta, v_{22} \otimes 2j_0]) \\
= \operatorname{ad}(D_{e_2, v_{22}} - aD_{e_2, v_{23}}), & = \operatorname{ad}(v_{22} - av_{23}) \otimes 2j_0
\end{aligned} \tag{4.5}$$

implies that

$$\begin{aligned}
& e_\beta(a)e_\alpha(-j)e_\beta(-a) \\
= \exp\left(-\operatorname{tr}(j) \operatorname{ad}(D_{e_2, v_{22}} + v_{22} \otimes 2j_0) + a \operatorname{ad}(D_{e_2, v_{23}} + v_{23} \otimes 2j_0)\right) \\
= \exp\left(-\operatorname{ad}(jy_\alpha) + \operatorname{ad}(ajy_{\alpha+\beta})\right)
\end{aligned} \tag{4.6}$$

(ii) Setting $X = -\operatorname{ad}(jy_\alpha)$, $Y = \operatorname{ad}(ax_{\alpha+\beta})$, the baby formula (4.2) gives

$$e_\alpha(j)e_\beta(a)e_\alpha(-j)x_\beta(-a) = e^Y \cdot e^{-\frac{1}{2}[X, Y]} \cdot e^{\frac{1}{6}(2[Y, [X, Y]] + [X, [X, Y]])}$$

Let's compute the Lie brackets involved. Without loss of generality, we'll drop the ad from our computation, treating X and Y as $-jy_a$ and $ax_{\alpha+\beta}$.

We admit the following routine computations:

$$\begin{aligned}
[D_{e_2, v_{22}}, D_{e_2, v_{23}}] &= D_{e_1, v_{11}} \quad , \quad D_{e_2, v_{22}}(v_{23}) = D_{e_2, v_{23}}(v_{22}) = v_{11}, \\
[D_{e_2, v_{23}}, D_{e_1, v_{11}}] &= D_{v_{23}, v_{11}} \quad , \quad D_{e_2, v_{23}}(v_{11}) = D_{e_1, v_{11}}(v_{23}) = 0, \\
[D_{e_2, v_{22}}, D_{e_1, v_{11}}] &= D_{v_{22}, v_{11}} \quad , \quad D_{e_2, v_{22}}(v_{11}) = D_{e_1, v_{11}}(v_{22}) = 0, \\
D_{e_1, v_{11}} = D_{v_{11}, e_2} &= D_{v_{23}, v_{22}} \quad , \quad t(v_{23}, v_{11}) = t(v_{22}, v_{11}) = 0,
\end{aligned}$$

and

$$\begin{aligned}
j^2 - 2j \cdot \operatorname{tr}(j) - \operatorname{tr}(j^2) + 2\operatorname{tr}^2(j) &= j_0 * j_0, \\
v_{23} * v_{22} &= -v_{11},
\end{aligned} \tag{4.7}$$

With such computations available, we can write $[X, Y] = s+t$, $s \in R$, $t \in \mathfrak{J}_0$ where

$$\begin{aligned}
t &= (-2a \cdot \text{tr}^2(j) + \frac{1}{3}T(j_0, j_0))D_{e_1, v_{11}}, \\
s &= 4av_{11} \otimes (\text{tr}(j) \cdot j_0 - a_0 * a_0) \\
&= 4av_{11} \otimes (j^2 - 3j \cdot \text{tr}(j) - \text{tr}(j^2) + 3\text{tr}^2(j)).
\end{aligned} \tag{4.8}$$

Using equations 2.27, 2.28, we get $\text{tr}(j^\#) = -\frac{1}{2}\text{tr}(j^2) + \frac{3}{2}\text{tr}^2(j)$ and $j^\# - \text{tr}(j^\#) = j^2 - 3j \cdot \text{tr}(j) - \text{tr}(j^2) + 3\text{tr}^2(j)$. Along with formula 4.8, this proves that

$$-\frac{1}{2}[X, Y] = a(-j)^\# \tag{4.9}$$

Similarly, one obtains

$$\frac{2}{6}[Y, [X, Y]] = \frac{2}{3}a^2 \left(-\text{tr}^3(j) + \text{tr}(j)T(j_0, j_0) - \frac{2}{3}T(j_0, j_0 * j_0) \right) D_{v_{23}, v_{11}} \tag{4.10}$$

$$\frac{1}{6}[X, [X, Y]] = \frac{1}{3}a \cdot \left(+\text{tr}^3(j) - \text{tr}(j)T(j_0, j_0) + \frac{2}{3}T(j_0, j_0 * j_0) \right) D_{v_{22}, v_{11}} \tag{4.11}$$

By equations 2.27, 2.28, 2.24, [5, Pg24, 3.15]:

$$N(j) = \text{tr}(j^3) - \frac{9}{2}\text{tr}(j^2)\text{tr}(j) + \frac{9}{2}\text{tr}^3(j), \tag{4.12}$$

and the fact that

$$\begin{aligned}
\frac{1}{3}D_{v_{23}, v_{11}} &= -x_{3\alpha+2\beta}; \\
\frac{1}{3}D_{v_{22}, v_{11}} &= x_{3\alpha+\beta}
\end{aligned}$$

it turns out that

$$\frac{2}{6}[Y, [X, Y]] = -2a^2 N(j)x_{3\alpha+2\beta} \tag{4.13}$$

$$\frac{1}{6}[X, [X, Y]] = aN(j)x_{3\alpha+\beta} \tag{4.14}$$

Summarizing, we have

$$(e_\alpha(-j), e_\beta(-a)) = e_{\alpha+\beta}(aj)e_{2\alpha+\beta}(aj^\#)e_{3\alpha+\beta}(aN(j))e_{3\alpha+2\beta}(-2a^2N(j)).$$

This and commutator relation **(3)** implies **(1)**.

Remark 4.3.5. If we set $\sigma = \text{Id}$ in equations (E1)-(E3) of [13, Pg63], it is clear that we can obtain the same commutator relations by only allowing scalar elements of J for the short root subgroups. Actually, if we take R to be a field of characteristic not 2 or 3, $\mathfrak{J}$ to be a degree 3 galois extension of R with cubic form given by $N(x) = x\sigma(x)\sigma^2(x)$, then it can be shown that our Type G_2 groups covers the twisted type Chevalley group ${}^3D_4(R)$.

Remark 4.3.6. These commutator relations are first given (to the author's knowledge) in a related setting by Faulkner [15, Pg 28], and was involved in Tits and Weiss's classification of Moufang Hexagons.

Now, with theorem 4.3.4, Let us give a condition for $\mathbb{G}$ to have property (T).

Proposition 4.3.7. *The group $\mathbb{G}$ has property (T) provided*

1. *R is finitely generated as a ring*
2. *$\mathfrak{J}$ is a finitely generated module over R .*

Proof. From the commutator relations of Theorem 4.3.2 it follows immediately that the relation is strong, therefore [13, Theorem 5.1] applies.

It remains to show relative property (T) for each root subgroup. using the commutator relations we gave previously, the statement is true for any long root subgroup E_γ since they generate an adjoint Chevalley group of $\mathfrak{I}_{A_2}(R)$, a case proved in [13, Chap.7]

Now consider the short roots. It suffices to prove the statement for the short simple root α . Since one has $E_\alpha \cong E_\alpha(R) \oplus E_\alpha(\mathfrak{J}_0)$, it suffices to establish relative property (T) for each summand.

$E_\alpha(R)$: In this case, Theorem 4.3.4 can be rewritten as:

$$(e_\alpha(b), e_\beta(a)) = e_{3\alpha+2\beta}(a^2b^3)e_{3\alpha+\beta}(ab^3)e_{2\alpha+\beta}(-ab^2)e_{\alpha+\beta}(ab) \quad a, b \in R,$$

which implies that

$$\begin{aligned} (e_\alpha(-1), e_\beta(b))^{-1} \cdot (e_\alpha(1), e_\beta(b)) = \\ e_\beta(-b)e_\beta(b)^{e_\alpha(-1)}e_\beta(b)^{e_\alpha(1)}e_\beta(b) = e_{\alpha+\beta}(2b)e_{3\alpha+\beta}(2b^3) \end{aligned}$$

indicating

$$E_{\alpha+\beta}(2R) = E_\beta(R)E_\beta(R)^{e_\alpha(-1)}e_\beta(R)^{e_\alpha(1)}e_\beta(R)E_{3\alpha+\beta}(2R).$$

In other words, since $\frac{1}{2} \in R$, $E_{\alpha+\beta}(R)$ is a bounded product of long root subgroups and 2 elements $e_\alpha(\pm 1)$ that lie inside our desired finite Kazhdan subset, this case is solved.

$E_\alpha(\mathfrak{J}_0)$: We fix a finite set $J := \{j_i\}$ that generates $\mathfrak{J}_0$ as a R -module. Also Fix some $j \in J$, and let $a, b \in R$. In this case, we have

$$\begin{aligned} (e_\alpha(j), e_{\alpha+\beta}(a)) &= e_{3\alpha+2\beta}(T(j, a^\#))e_{3\alpha+\beta}(-T(j^\#, a))e_{2\alpha+\beta}(j^\#a) \\ &= e_{3\alpha+\beta}(-T(j^\#, a))e_{2\alpha+\beta}(-aj), \end{aligned}$$

indicating

$$E_{2\alpha+\beta}(Rj) = E_{\alpha+\beta}(R)^{e_\alpha(j)}E_{\alpha+\beta}(R)E_{3\alpha+\beta}(R).$$

Using the previous result on $E_{\alpha+\beta}(R)$, the scenario becomes nearly identical to the previous case.

Therefore we have

$$\kappa\left(\mathbb{G}, \left(\bigcup_{\substack{\gamma \in G_{2,short}, \\ j \in J'}} E_\gamma(j)\right) \cup \left(\bigcup_{\substack{\delta \in G_{2,long}, \\ t \in \Sigma}} E_\delta(t)\right)\right) > 0$$

where $J' = J \cup \{1\}$ and Σ the finite generating set of R . □

Remark 4.3.8. The above can be considered a parrallel result to [13, Prop. 8.10].

Chapter 4 includes a reinterpretation of, and borrows heavily from “Nonassociative algebras and groups with property (T) ” ([45]). This paper has been accepted for publication in the Intern. J. Algebra and Computation, with the dissertation author as author.

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