Title
UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS, LECTURE XV

Author
Culler, Glenn.

Publication Date
1953-01-27

UCRL-2291

UNCLASSIFIED

UNIVERSITY OF CALIFORNIA

Radiation Laboratory

Contract No. W-7405-eng-48

UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

Lecture XV

Glenn Culler

January 27, 1953

Berkeley, California

DISCLAIMER

University of California
Radiation Laboratory
Berkeley, California

UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

Lecture XV

January 27, 1953

Glenn Culler

THE SUMMATION OF INFINITE SERIES

1. Introduction

Various tests of convergence are given for ordinary and double infinite series, and operations on convergent series are discussed. Then, with a view to numerical applications, methods of expressiong summations in closed form and transformations of slowly convergent series are given.

2. Convergence

2.1 Definition: An infinite series is said to converge if and only if the sequence of its partial sums converges. (The i'th partial sum of the series $\sum_{n=1}^{\infty} a_n$ is $S_i = \sum_{n=1}^{i} a_n$.) In particular $\sum_{n=1}^{\infty} a_n = \lim_{i\to\infty} S_i$.

2.2 Convergence Tests. There are many convergence tests which are frequently more easily applied than this definition, but the three given below have reasonable scope and give results with minimal effort.

Suppose that, for all x greater than some fixed x_o , $f(x)$ is continuous and has a continuous non-zero derivative.

2.21 Ermakoff's Test:

If $\lim_{x\to\infty} \frac{e^x f(e^x)}{f(x)} = k$, then $\sum_{n=1}^{\infty} f(n)$

converges when $k < 1$ and diverges whenever

$$\frac{e^x f(e^x)}{f(x)} \geq 1$$

for all sufficiently large x.

2.22 Limit Test:

If $\lim_{x\to\infty} \frac{-f(x)}{xf'(x)} = k$, then $\sum_{n=1}^{\infty} f(n)$ converges when $k < 1$ and diverges if $\frac{-f(x)}{xf'(x)} \geq 1$ for all sufficiently large x.

Both of these tests are indecisive when, respectively,

$\frac{e^t f(e^t)}{f(t)} < 1$ and $\frac{-f(t)}{tf'(t)} < 1$ for sufficiently large t and $k = 1$.

2.23 Leibniz's Theorem for Alternating Series:

If $\{U_n\}$ is a non-increasing sequence and $\lim_{n\to\infty} U_n = 0$, then $\sum_{1}^{\infty} (-1)^n U_n$ is a convergent series.

2.24 Proofs for 1, 2, 3 can be constructed along the following lines:

1. Use the test conditions to show $\int_{e^{x_o}}^{e^x} f(t)dt \leq \frac{k}{1-k} \int_{x_o}^{e^{x_o}} f(t)dt$.

Hence $\int_{x_o}^{x} f(t)dt$ is bounded and Maclaurin's integral test can be applied to finish the proof.

2. If $c(t) = \frac{-f(t)}{tf'(t)}$, then

$$f(t) = e^{-\int^{t} \frac{dt}{tc(t)}} .$$

The conditions applied to this integral give a comparison series of the form $A \sum_{1}^{\infty} \frac{1}{t^{1/k}}$.

3. Every even partial sum is greater than every odd one and the limit property can be used to prove that the two sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge to the same limit.

3. The Algebra of Convergent Series

Any term by term operation on infinite series is permissible if the sum of the resulting series is related to the sum of the original series in a known way. Since every convergent subsequence of a convergent sequence must converge to the same limit, any algebraic operation is permissible if it carries the sequence of partial sums $\left\{\sum_{n=1}^{i} a_n\right\}$ onto one of its convergent subsequences.

Theorem 1: (Rearrangement of Parentheses). If $\sum_{n=1}^{\infty} a_n$ converges and $A_k = \sum_{\nu_k+1}^{\nu_{k+1}} a_n$, then $\sum_{k=1}^{\infty} A_k = \sum_{n=1}^{\infty} a_n$ if for each $S_j\left(S_j = \sum_{k=1}^{j} A_k\right)$ there exists exactly one $s_i\left(s_i = \sum_{n=1}^{i} a_i\right)$ such that $S_j = s_i$, where ν_i denotes an increasing sequence of integers.

Theorem 2: Two convergent series may be added or subtracted term by term.

Theorem 3: If $\sum_{n=1}^{\infty} a_n = S$, then $\sum_{n=1}^{\infty} (k \cdot a_n) = k \cdot S$.

Definition A: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Definition B: If $\sum_{n=1}^{\infty} |a_n|$ diverges and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Theorem 4: If $\{\nu_n\}$ is any rearrangement of the integers, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\nu_n}$ if and only if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Theorem 5: If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then for each number k, there exists some rearrangement $\{\nu_n\}$ of the integers, such that $\sum_{n=1}^{\infty} a_{\nu_n} = k$.

Theorems 1, 2, and 3 follow from the introductory statement, and proofs for 4 and 5 are given in Knopp's "Theory and Applications of Infinite Series", chapter IV.

Theorem 6: (Cauchy' Double Series Theorem). Let

$$S_i = \sum_{n=1}^{\infty} a_{ni}, \quad \sigma_i = \sum_{n=1}^{\infty} |a_{ni}|, \text{ and } Z_n = \sum_{i=1}^{\infty} a_{ni}.$$

If for every i, σ_i exists and $\sum_{i=1}^{\infty} \sigma_i$ converges, then $\sum_{i=1}^{\infty} a_{ni}$ converges absolutely for each integer n and

$$\sum_{i=1}^{\infty} S_i = \sum_{n=1}^{\infty} Z_n = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} a_{ni}.$$

This theorem is the basis for the theory of transformations of infinite series and is therefore fundamental to the study of numerical methods.

Proof: 1. Let $k = n + \frac{(n+i-1)(n+i-2)}{2}$ and consider $\sum_{k=1}^{\infty}$ where $b_k = a_{ni}$. Every a_{ni} is a term of this series exactly once.

2. $\sum_{k=1}^{\infty} b_k$ is absolutely convergent since

$$\sum_{k=1}^{\infty} |b_k| \leq \sum_{n=1}^{\infty} |a_{n1}| + \sum_{n=1}^{\infty} |a_{n2}| + \ldots + \sum_{n=1}^{\infty} |a_{nj}|$$

where j is the largest integer such that $|a_{nj}|$ is a term of the series $\sum_{k=1}^{m} |b_k|$ —

Hence $\sum_{k=1}^{m} |b_k| \leq \sum_{i=1}^{\infty} \sigma_i$ and the boundedness of the partial sums implies that $\sum_{k=1}^{\infty} |b_k|$ converges.

3. For every n, $\sum_{i=1}^{\infty} |a_{ni}| \leq \sum_{k=1}^{\infty} |b_k|$ and since $\sum_{k=1}^{\infty} b_k$ converges absolutely, all rearrangements of this series must converge to the same limit. But $\sum_{i=1}^{\infty} S_i$ and $\sum_{m=1}^{\infty} Z_n$ are just two such arrangements and hence

$$\sum_{i=1}^{\infty} S_i = \sum_{m=1}^{\infty} Z_n = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} a_{ni} .$$

4. Methods of Expressing Summations in Closed Form

4.1 If $\lim_{n\to\infty} x_n = A$, the x_n being terms of a convergent sequence, then $\sum_{n=1}^{\infty} (x_{n-1} - x_n)$ converges to the limit $x_o - A$.

Using this, series can be constructed with known sums.

Example 1: Let $x_n = \frac{1}{n+1}$, then $x_{n-1} - x_n = \frac{1}{n(n+1)}$. Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 .$$

Example 2: Choose $x_n = \frac{1}{(n+1)^2}$, then $x_{n-1} - x_n = \frac{2n+1}{n^2(n+1)^2}$.

Thus,

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = 1.$$

Conversely, whenever the difference equation $\Delta x_n = -a_n$ can be solved in closed form, the series $\sum_{m=1}^{\infty} a_n = x_o - \lim_{n\to\infty} x_n$. In his numerical calculus, Milne has listed a number of fundamental difference equations with their solutions and has given examples of this closed form technique (cf. pp. 329-331).

Example 3: Evaluate $\sum_{n=1}^{m} \sinh \frac{n}{2} \cosh \left(nS + \frac{S}{2}\right)$.

Now,

$$\tfrac{1}{2}\left\{\sinh (n+1)S - \sinh n S\right\} = \sinh \frac{n}{2} \cosh \left(n S + \frac{S}{2}\right).$$

Thus,

$$\sum_{n=1}^{m} \sinh \frac{n}{2} \cosh \left(mS + \frac{S}{2}\right) = \tfrac{1}{2}\left\{\sinh (m+1)S - \sinh S\right\}.$$

Of course, if $x_n - x_{n+q} = a_n$, then

$$\sum_{n=0}^{\infty} a_n = (x_o + x_1 + \dots + x_{q-1}) - q \lim_{n\to\infty} x_n .$$

This is evident from the partial sum

$$S_n = (x_o - x_q) + (x_1 - x_{q+1}) + \dots + (x_{q-1} - x_{2q-1}) + (x_q - x_{2q})$$

$$+ \dots + (x_n - x_{n+q})$$

$$= (x_o + x_1 + \dots + x_{q-1}) + (x_{n+1} + x_{n+2} + \dots + x_{n+q}) .$$

__4.2__ Abel's Limit Theorem.

If $\sum_{n=0}^{\infty} a_n$ converges and $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $\sum_{n=0}^{\infty} a_n x^n$ converges for all x in $-1 < x \leq +1$, and hence

$$\sum_{n=0}^{\infty} a_n = \lim_{x\to 1-0} f(x).$$

Example 4: $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \pi/4 .$$

Proof: $$\mathrm{Tan}^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$ and

$$\lim_{t \to 1-0} \tan^{-1}(x) = \pi/4 .$$

Example 5: $$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} .$$

Let $$F(x) = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots .$$

F converges uniformly for all $|x| < 1$, hence term by term differentiation is allowed, if the derived series converges uniformly.

$$F'(x) = 1 - x^3 + x^6 - x^9 + \cdots = \frac{1}{1+x^3} .$$

Thus,

$$F(x) = \int_0^x \frac{1}{1+x^3}\, dx = \frac{1}{6} \lg \frac{(x+1)^2}{(x^2 - x + 1)}$$

$$+ \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}} + \frac{\pi}{6\sqrt{3}} .$$

Thus,

$$\lim_{x \to 1-0} F(x) = \frac{1}{3} \lg 2 + \frac{\pi}{3\sqrt{3}} .$$

4.3 Application of Cauchy's Double Series Theorem.

Example 6: $$\sum_{n=1}^{\infty} \frac{1}{n^{2\ell}} = \frac{(-1)^{\ell-1} B_{2\ell} (2\pi)^{2\ell}}{2\cdot(2\ell)!}$$

where $B_{2\ell}$ is the 2ℓ'th Bernoulli number.

Proof: When x is small, the expansion of $\pi x \cot \pi x$ in Bernoulli numbers is

$$\pi x \cot \pi x = 1 + \sum_{\ell=1}^{\infty} (-1)^{\ell} 2^{2\ell} B_{2\ell} (\pi x)^{2\ell}.$$

The Bernoulli numbers are defined by the relation

$$\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots\right)\left(B_0 + \frac{B_1 x}{1!} + \frac{B_2 x^2}{2!} + \dots\right) = 1 .$$

A second expansion is given by

$$\pi x \cot \pi x = 1 + 2x^2 \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2} .$$

If this expansion is written as a power series, the coefficients of the two power series must agree.

The power series expansion of $\frac{-2x^2}{n^2 - x^2}$ is

$$-2 \sum_{\ell=1}^{\infty} \left(\frac{x^2}{n^2}\right)^{\ell} ,$$ hence

$$\pi x \cot \pi x = 1 + \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \left(-2\left(\frac{x^2}{n^2}\right)^{\ell}\right) ;$$

using the double series theorem

$$\pi x \cot \pi x = 1 + \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \left(-2 \left(\frac{x^2}{n^2} \right)^{\ell} \right)$$

$$= 1 - 2 \sum_{\ell=1}^{\infty} x^{2\ell} \sum_{n=1}^{\infty} \left(\frac{1}{n^{2\ell}} \right),$$

and equating coefficients of $x^{2\ell}$, one gets

$$\sum_{n=1}^{\infty} \frac{1}{n^{2\ell}} = \frac{(-1)^{\ell-1} \; B_{2\ell} \; (2\pi)^{2\ell}}{2 \cdot (2\ell)!} .$$

5. Transformations of Slowly Convergent Series.

5.1 Kummer's Transformation.

Let $S = \sum_{n=1}^{\infty} a_n$ be a convergent series and suppose a second series, $\sum_{n=1}^{\infty} c_n$, with a known sum, c, is such that

$$\lim_{n\to\infty} \frac{a_n}{c_n} = \gamma \neq 0 \text{, then}$$

$$S = \gamma c + \sum_{n=1}^{\infty} \left(1 - \gamma \frac{c_n}{a_n}\right) a_n .$$

Example 7: $\sum_{1}^{\infty} \frac{1}{n^2}$.

Choose

$$c_n = \frac{1}{n(n+1)}, \text{ then } c = 1 \text{ and } \gamma = 1 .$$

Therefore,

$$S = 1 + \sum_{n=1}^{\infty} \left(1 - \frac{n^2}{n(n+1)}\right) \frac{1}{n^2}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} .$$

5.2 Markoff's Transformation.

If Kummer's transformation is repeated indefinitely, the original series is transformed into a series, each term of which is the sum of an auxiliary series $\sum_{m=1}^{\infty} c_n$.

Theorem 7: If $\sum_{k=0}^{\infty} Z_k$ is a convergent series and $Z_k = \sum_{i=0}^{\infty} a_{ik}$, where $\sum_{k=0}^{\infty} a_{ik}$ converges and $\lim_{m \to \infty} \sum_{k=0}^{\infty} \left(\sum_{i=m}^{\infty} a_{ik} \right) = 0$, then

$$\sum_{k=0}^{\infty} Z_k = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik} .$$

Corollary: Euler's Transformation.

Every convergent series may be written in the form $\sum_{k=0}^{\infty} (-1)^k a_k$ and has a sum equal to $\sum_{i=0}^{\infty} \frac{\Delta^n a_o}{2^{n+1}}$. The validity of this statement follows from Markoff's transformation theorem and is established in Knopp, page 245-246.

Example 8: $\sum_{m=0}^{\infty} \frac{(-1)^n}{n+1}$.

$$\Delta(a_o) = 1 - \tfrac{1}{2} = \tfrac{1}{2} = \frac{1}{1+1}$$

$$\Delta(a_1) = \tfrac{1}{2} - 1/3 = 1/6$$

$$\Delta^2(a_o) = \tfrac{1}{2} - 1/6 = \frac{1}{1+2}$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\Delta^k(a_o) = \frac{1}{1+k} \quad .$$

Hence,
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \sum_{k=0}^{\infty} \frac{1}{(1+k)2^{k+1}} \quad .$$

Example 9: Repeated Kummer Transformation.

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + w^2} = S \quad , \qquad |w| < 1 \quad .$$

Define $A_j(n) = \dfrac{1}{n^{2j}(n^2 + w^2)}$, then $A_j(n) = \dfrac{1}{n^{2j+2}} - w^2 A_{j+1}(n)$.

Thus,
$$S = \sum_{n=1}^{\infty} A_o(n) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{w^{2(k-1)}}{n^{2k}}\right).$$

By theorem 7,
$$S = \sum_{k=1}^{\infty} (-1)^{k-1} w^{2(k-1)}\left(\sum_{n=1}^{\infty} \frac{1}{n^{2k}}\right).$$

Thus,
$$S = \sum_{k=1}^{\infty} (-1)^{k-1} w^{2(k-1)} \zeta(2k) .$$

where $\zeta(x)$ is the Riemann Zeta function defined by $\zeta(x) = \sum_{n=1}^{\infty} \dfrac{1}{n^x}$.

6. Bibliography.

6.1 Knopp, K., "Theory and Application of Infinite Series".

6.2 Bromwich, T. J. I'Anson, "Theory of Infinite Series".

6.3 Titchmarsh, E. C., "Theory of Functions".