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Some results on the qualitative behavior of solutions to the Ricci flow and other geometric evolution equations

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The dissertation of Brett L. Kotschwar is approved, and it is acceptable in quality and form for publication on microfilm:

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To my family, for their love and support.

I am not a cat man, but a dog man, and
all felines can tell this at a glance-
a sharp, vindictive glance.
—James Thurber

TABLE OF CONTENTS

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ABSTRACT OF THE DISSERTATION

Some results on the qualitative behavior of solutions to the Ricci flow and other geometric evolution equations

by

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In first part of this thesis we consider the Ricci flow, an evolution equation for Riemannian metrics introduced by Richard Hamilton. In dimensions two and three, the work of Hamilton and Perelman has effectively shown that the only gradient shrinking solitons are the round sphere, flat Euclidean space, the standard cylinder, and their quotients. Our first result is a classification of rotationally symmetric shrinking solitons in all dimensions, which we accomplish by the analysis of a certain system of ODE similar to one first considered by Bryant and Ivey for steady and expanding solitons. We also present an elementary proof of the uniqueness of a certain two-dimensional expanding soliton among those with positive curvature, which is an analog of a result of Chen, Lu, and Tian in the case of compact shrinking surface solitons. Next, we generalize an argument of Lees and Protter to prove a unique-continuation theorem for evolving tensor fields satisfying a certain parabolic differential inequality. As applications, we obtain unique-continuation theorems for solutions to the Kähler-Ricci and Ricci-DeTurck flows, as well as a proof that a solution to the Ricci flow cannot become Einstein in finite time.

In the next part, we consider differential Harnack inequalities for evolving convex hypersurfaces of the type proved by Hamilton, Chow, and Andrews. Modifying an approach of Chow, Chu, and Knopf, we exhibit a realization of the full Harnack quadratic as the second fundamental form of a certain degenerate immersion of the space-time track. By means of this realization, we provide a new geometric interpretation of Andrews's inequality in the case of isotropic flows and use the machinery to give a new proof of Hamilton's inequality for the mean curvature flow. We also show that Andrews's Gauss map technique can be used to obtain new Harnack inequalities for complete space-like surfaces in Minkowski space. Finally, via a Bernstein-type estimate and a maximum principle of Karp and Li, we extend a gradient estimate of Hamilton for the heat equation to complete manifolds. With this estimate, we obtain a sharp inequality for the heat kernel on complete manifolds of non-negative Ricci curvature.

1 Introduction

The heat flow method and the Ricci flow. The problems in this thesis are all in one way or another related to the heat flow method in geometry, wherein one attempts to improve some given geometric structure by means of a parabolic flow, with the aim, in ideal circumstances, of obtaining some canonical or otherwise desirable structure as a limit. Such methods have their genesis in the paper of Eells and Sampson [31] for harmonic maps, and have had substantial success in solving problems at the interface of Riemannian geometry and topology.

A notable example is the *Ricci flow*, the evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1.1}$$

for Riemannian metrics introduced by Richard Hamilton in [38]. In a certain sense, the Ricci tensor is the analog of the Laplacian for Riemannian metrics and, accordingly, (1.1), which induces heat-type equations on the various curvature quantities, is often informally described as a heat equation for the metric. Beginning with Hamilton's 1982 theorem for three-manifolds with positive Ricci curvature [38], and continuing through the recent and remarkable work of Grisha Perelman [62], [61] on the Geometrization Conjecture, the Ricci flow has had substantial success in the attack of many outstanding problems in Riemannian and Kähler geometry, and has been proposed as a potential approach to many others. By now there are a number of references on the Ricci flow. We recommend the interested reader to [20] for a nice introduction and to [21], [22], and [23] for more comprehensive treatments.

Ricci solitons. A basic prerequisite for the application of the Ricci flow to many geometric problems is an understanding of the nature of the singularities which develop in finite time, and the consideration of *blow-up* sequences, together with a compactness theorem of Hamilton, has been one of the most useful approaches to characterizing the local structure of these singularities. In a number of important cases, one may, with a careful choice of the points and scales with respect to which these blow-up sequences are taken, extract a limit converging to a solution which moves only under the scaling and diffeomorphism symmetries of the equation. The members of this rather privileged class are known as *self-similar* solutions or *Ricci solitons*, and when, as in many applications, the diffeomorphisms are generated by the gradient of a smooth function: *gradient Ricci solitons*. At any fixed time, these gradient solitons satisfy an equation of the form

$$R_{ij} + \nabla_i \nabla_j f - \lambda g_{ij} = 0 \tag{1.2}$$

for a constant λ whose sign identifies the scaling behavior of the solution. When λ is positive, the distances in the associated solution contract uniformly, when it is negative, they stretch uniformly, and when it is zero, the geometry remains effectively fixed. Accordingly, in these cases, one refers to the solution as a *shrinking*, *expanding*, or *steady* gradient soliton. Note that (1.2) generalizes the Einstein condition as any Einstein metric is, with the choice $f = \text{const}$, a gradient Ricci soliton.

Solitons are further distinguished in the study of the Ricci flow in their frequent roles as critical cases or cases of equality in inequalities and monotonicity formulas which hold for general solutions. Notable examples of this phenomenon occur in connection with Perelman's $\mathcal{W}$-entropy [62] – which is monotone decreasing on compact solutions to (1.1), but is constant on shrinking solitons – and Hamilton's matrix Harnack quadratic [42] – which is non-negative for solutions to (1.1) with bounded, non-negative curvature operator, but vanishes identically along certain space-time directions on expanding solitons. This phenomenon is not peculiar to the Ricci flow. The differential Harnack inequality of Li and Yau [55] for positive solutions to the heat equation is an identity for the heat kernel on $\mathbb{R}^n$ (which is a self-similar solution of that equation in the above sense) and Huisken's monotone integral quantity for solutions to the mean curvature flow [49] is constant precisely on homothetically shrinking solutions.

Shrinking solitons are of particular interest in the study of the Ricci flow because they belong to the class of *ancient* solutions – solutions which exist on half-infinite intervals of the form $(-\infty, \Omega)$ – which are local models for an important class of finite-time singularities. For applications, it is desirable to have as complete of a classification of these solitons as possible, and in low dimensions, such a classification essentially already exists. In dimension two, it follows from the work of Hamilton [45], [40] (see also [12]), that the only two complete shrinking solitons are (S^2, g_{round}), and $(\mathbb{R}^2, g_{\text{flat}})$. In dimension three, Perelman [61], building on work of Hamilton, proves that the only complete orientable shrinking solitons of nonnegative curvature are quotients of (S^3, g_{round}), or $(S^2 \times \mathbb{R}, g_{\text{cyl}})$, or $(\mathbb{R}^3, g_{\text{flat}})$. Together with an estimate of Hamilton and Ivey ([45], [52]) which implies that three-dimensional ancient solutions with bounded curvature have nonnegative curvature operator, Perelman's result effectively classifies three-dimensional shrinking solitons.

However, at present, there are no corresponding theorems in higher dimensions. One no longer has the Hamilton-Ivey estimate, and there are examples of non-Einstein shrinkers in dimensions $n \geq 4$ with curvatures of mixed sign. For compact manifolds, it is a particular consequence of recent work of Böhm and Wilking [6] that the only compact shrinking solitons with two-positive curvature operator (i.e., the sum of the two lowest eigenvalues of $\text{Rm} : \wedge^2 \to \wedge^2$ are positive) are quotients of the round sphere.

In the non-compact case, it is natural to formulate from Perelman's three-dimensional result the following potential extension.

Conjecture. There does not exist a complete non-compact shrinking soliton with bounded positive curvature operator.

In this direction, we prove in Chapter 3 the following classification theorem for rotationally invariant shrinking solitons.

Theorem. The only complete, rotationally invariant shrinking solitons on S^n, $\mathbb{R}^n$, or $S^{n-1} \times \mathbb{R}$, $n \geq 3$, are the round, flat, and standard cylindrical metrics, respectively.

Here the situation for shrinking solitons contrasts with that of expanding and steady solitons, as, by work of Bryant [8] and Ivey [22], there are known to exist complete, positively curved, rotationally symmetric examples in all dimensions. While the assumption of rotational symmetry is a considerable a priori simplification from the general condition (1.2), it is a consequence of the proof of Perelman's three-dimensional classification that (in any dimension) any potentially positively curved shrinking soliton with bounded curvature must be, in a sense, rotationally symmetric "at infinity". Thus, if any putative counterexample can be shown to be globally rotationally symmetric, our classification could be of some use.

We note that, in the Kähler category, the above conjecture has been answered. In [57], Ni classifies all non-flat gradient shrinking Kähler-Ricci solitons with non-negative bisectional curvature and, in particular, shows that any complete m-(complex) dimensional such soliton with positive bisectional curvature must be compact and isometric-biholomorphic to $\mathbb{CP}^m$. Also, Feldman, Ilmanen, and Knopf prove in [33] that the flat metric is the unique complete $\mathrm{U}(m)$-invariant shrinking soliton on $\mathbb{C}^m$, which is, in a sense, the Kähler analog of our result on $\mathbb{R}^n$. Finally, in the very recent preprint [36], H. Gu and X. P. Zhu give a different proof of our classification under the additional assumptions that the metric has bounded, non-negative curvature and obeys a certain criterion of non-collapse. They apply the classification to prove that the Ricci flow, beginning from a certain family of initial metrics on S^n, will develop a particular type of finite-time singularity (known as a Type-II singularity) which previously had been strongly conjectured, but never rigorously shown, to occur.

The first step in our argument is to show that under the constraint (1.2), the rotational symmetry of a non-flat metric will imply that of the gradient function f. (A priori, the equation involves the partial derivatives of f.) Then, representing (M^n, g) as a warped-cylinder, we may reduce the gradient Ricci soliton equation to a system of second-order ODE. This system is not integrable for dimensions $n \geq 3$, and (we find) rather unwieldy to manipulate. However, using a change of variables due to Bryant and Ivey [22] which effectively reduces the symmetries of the system, we obtain an equivalent first-order system in three unknowns that is amenable to phase-space techniques.

Surface solitons. In dimension two, gradient solitons are necessarily rotationally symmetric. Indeed, as was observed by Hamilton, if J denotes the complex structure, $J(\nabla f)$ is a Killing vector field for g and gives rise to a global circle action. In their recent note [12], Chen, Lu, and Tian use this necessary symmetry to provide a new proof of the uniqueness of the round sphere among compact surface shrinking solitons. Previous proofs of this fact made recourse to the classical uniformization theorem; by eliminating this dependence, the Ricci flow can be seen to provide a self-contained proof of the uniformization theorem. We provide in Chapter 3 a proof of the following analog of their result for expanding surface solitons.

Theorem. Suppose $(\mathcal{M}^2, g)$ is a complete, non-compact Riemannian surface with positive Gauss curvature satisfying (1.2) for some f with $\lambda = -1$. Then there exists a constant $\alpha > 0$ such that $(\mathcal{M}^2, g)$ is isometric to $(\mathbb{R}^2, \phi(r)^2 dr^2 + r^2 d\theta^2)$ with

$$\phi(r) = \frac{\alpha}{1 + W\left(\left(\frac{\alpha-\zeta}{\zeta}\right)\exp\left(\left(\frac{\alpha-\zeta}{\zeta}\right) - \frac{\alpha^2}{2}r^2\right)\right)},$$

where $W : (0,\infty) \to (0,\infty)$ is the product-log function (the inverse of xe^x) and $\zeta = \lim_{r\to 0}\phi(r) \in (0,\alpha)$.

The above soliton is described, for example, in [20], and appears in the physics literature in [37] in connection with the renormalization group flow.

Unique continuation of solutions. One interesting property of the Ricci flow is that it preserves the isometries of initial metrics in any category in which it enjoys uniqueness. This is a consequence of the diffeomorphism invariance of the Ricci tensor: If $g(t)$ is a solution to the Ricci flow on $[0,T]$ and $\hat{g}(t) := (\varphi^* g)(t)$ for some $\varphi \in \mathrm{Diff}(M^n)$, then

$$\frac{\partial}{\partial t}\hat{g}(t) = -2(\varphi^*\mathrm{Rc})[g(t)] = -2\mathrm{Rc}[\varphi^* g(t)] = -2\mathrm{Rc}[\hat{g}],$$

so $\hat{g}$ is also a solution. Thus when uniqueness holds, one has $\mathrm{Isom}(g(0)) \subset \mathrm{Isom}(g(t))$ for all t.

One of course does not expect diffusion to sponsor the creation of new isometries in finite time and so expects the reverse inclusion $\mathrm{Isom}(g(t)) \subset \mathrm{Isom}(g(0))$ to hold as well. In fact, one expects a stronger[1] result to be true .

Conjecture. (Unique continuation/backwards-uniqueness) Suppose g and $\tilde{g}$ are solutions to (1.1) on $M^n \times [0,T]$ with $g(T) = \tilde{g}(T)$. Then $g(t) = \tilde{g}(t)$ for all $t \in [0,T]$.

Theorems of this sort for linear parabolic equations are classical: see e.g., [1], [54]. However, (1.1) is a non-linear, and only weakly parabolic, system. In Chapter 5, we prove a few results in this direction including an affirmative answer to the above conjecture on compact Kähler manifolds (and hence, in particular, on compact Riemann surfaces.) Namely we prove the following theorem.

Theorem (Unique continuation for Kähler-Ricci flow)**.** Suppose $g(t)$ and $\tilde{g}(t)$ are solutions to the normalized Kähler-Ricci flow on a compact manifold defined on the interval $[0,T]$. If $g(T) = \tilde{g}(T)$, and $[\omega_g]$ is a real multiple of the first Chern class, then $g(t) = \tilde{g}(t)$ for all $t \in [0,T]$.

The key simplification in the Kähler setting is that cohomological considerations allow us to represent the evolving metric in terms of a potential function which satisfies a parabolic Monge-Ampere equation. By a device of Fan [32] (cf. also [58]) from a (forwards) uniqueness argument, we see that the difference of two potentials satisfies an equation to which a classical result of Lees and Protter [54] essentially applies. As we are ultimately interested in tensor equations, we prove a generalization of the result in [54] to tensor bundles equipped with a time-dependent family of metrics, connections and measures.

Theorem. Let $\{g(t)\}_{t\in[0,T]}$ be a smooth family of Riemannian metrics on a compact manifold M^n and suppose that $X \in C^\infty(T^k_l(M^n) \times [0,T])$ satisfies

$$\int_{M^n} |LX|^2(\cdot,t)\; d\mu \leq c_1 \int_{M^n} |X|^2(\cdot,t)\; d\mu + c_2 \int_{M^n} |\nabla X|^2(\cdot,t)\; d\mu$$

[1] Actually, the two assertions are equivalent: if the Ricci flow preserves the isometry group, then (by considering the product solution $g(t) \oplus \tilde{g}(t)$ on $M^n \times M^n$ with the map $(x,y) \mapsto (y,x)$) it must also possess a unique-continuation property.

for all $t \in [0,T]$ where $L = \Lambda^{ij}\nabla_i\nabla_j - \frac{\partial}{\partial t}$, Λ is a smooth family of symmetric, positive definite $(0,2)$-tensors, and c_1, c_2 are positive constants. (Here $|\cdot| := |\cdot|_{g(t)}$ denotes the induced metric on T^k_l, and $\nabla := \nabla_{g(t)}$ and $d\mu := d\mu_{g(t)}$, respectively, the Levi-Civita connection and volume form associated to $g(t)$). Then, if $X(\cdot,T) \equiv 0$, $X(\cdot,t) \equiv 0$ for all $t \in [0,T]$.

The original result of [54] applied to scalar functions on C^2-domains in $\mathbb{R}^n$, and while our generalization is not really necessary for the Kähler theorem, it has some useful applications. For example, with a bit of effort, it can be used to establish a unique continuation theorem for the *Ricci-DeTurck flow*, which is the equation

$$\frac{\partial}{\partial t}h_{ij} = -2(\mathrm{Rc}[h])_{ij} + (\mathcal{L}_{W_{h,\bar{g}}}h)_{ij} \tag{1.3}$$

where

$$W^k_{h,\bar{g}} = h^{pq}\left(\Gamma[h]^k_{pq} - \Gamma[\bar{g}]^k_{pq}\right).$$

Equation (1.3) was first considered by DeTurck [24] who used it to provide a much simplified proof of short-time existence for the Ricci flow. With the vector field $W_{h,\bar{g}}$ so defined, (1.3) is strictly parabolic, and, up to diffeomorphism, equivalent to the Ricci flow. Later, Hamilton [45] used (1.3) in conjunction with a harmonic map heat flow to reduce the problem of (forwards) uniqueness to the standard theory of uniqueness of ODE and strictly parabolic PDE. Though it seems the problem of backwards-uniqueness cannot be reduced to a strictly parabolic problem by a direct adaptation of this Hamilton's argument, there is nevertheless reason to believe that the following theorem may be of use.

Theorem (Unique continuation for the Ricci-DeTurck flow)**.** Suppose $h(t)$ and $\tilde{h}(t)$ are solutions to (1.3) on a compact manifold defined on $[0,T]$. If $h(T) = \tilde{h}(T)$, then $h(t) = \tilde{h}(t)$ for all $t \in [0,T]$.

Another easy application of our general theorem is the following result, which answers in a question posed in [21].

Theorem (No convergence to Einstein in finite time)**.** Suppose $g(t)$ is a solution to the Ricci flow on a compact manifold for $t \in (A,\Omega) := \mathcal{I}$ that satisfies the

Einstein condition $\mathrm{Rc}[g(t_0)] = \rho g(t_0)$ for some ρ at some $t_0 \in \mathcal{I}$. Then (extending g if $\mathcal{I}$ is not maximal), g is Einstein for all $t \in (-\infty, t_0 + 1/(2\rho))$, $(t_0 - 1/(2\rho), \infty)$, or $(-\infty, \infty)$, depending as ρ is positive, negative, or zero. Explicitly, we have

$$g(t) = (1 - 2\rho(t - t_0))g(t_0) \quad \text{and} \quad \mathrm{Rc}(t) = \frac{\rho}{1 - 2\rho(t - t_0)} g(t).$$

Harnack inequalities for evolving convex hypersurfaces. Whereas the Ricci flow is a parabolic equation for intrinsic geometry, one may also consider parabolic equations for modifying extrinsic geometric structures. The most studied of these, perhaps, is the mean curvature flow for submanifolds, introduced by Brakke [7]. From a variational standpoint, it is the (negative) gradient flow for the area functional, and it has been of most interest thus far when the evolving submanifold is of codimension one. In this case, the analysis of the attendant singular phenomena has been a particularly lively area of research with many fruitful parallels to the Ricci flow and other equations of geometric analysis (cf., e.g., the work of Huisken [48], [49], Ecker-Huisken [29], [28], Huisken-Sinestrari [51], and White [69]). Many other curvature flows of hypersurfaces have been also been studied. When the ambient manifold is Euclidean space, the basic object in the parametric approach to these flows is a family of immersions $X_t : M^n \to \mathbb{R}^{n+1}$ satisfying

$$\frac{\partial X}{\partial t} = -F(x,t)\nu(x,t) \tag{1.4}$$

where F is a smooth function and ν a smooth choice of unit normal to the hypersurface. In the cases of greatest geometric interest, $F(x,t) = F(W(x,t))$ is typically a symmetric function of the principal curvatures with with some degree of homogeneity. Examples other than the mean curvature include the Gauss curvature $F = K$, whose associated flow was introduced by Firey [34] as a model for the wear of a tumbling rock, (cf., e.g., the work of Tso [68], Chow [13], [14], and Andrews [4]) and the inverse-mean curvature ($F = 1/H$), whose flow has been studied, e.g, by Gerhardt [35] and, notably, by Huisken-Ilmanen [50]).

All differential Harnack estimates in modern geometric analysis are, in a sense, descendants of the estimate for the heat equation proved by Li and Yau in their seminal paper [55]. For the Ricci flow, such estimates, in various guises, have had substantial applications to the analysis of the long-time behavior and singularities of its solutions, most notably, perhaps, in the hands of Hamilton ([42], [41], [45]), and Perelman [62]. For curvature flows of hypersurfaces, estimates of this type were first obtained by Hamilton [46] and Chow [14] for the mean curvature and Gauss curvature flows, respectively. Both of their estimates can be stated in terms of the non-negativity of the following quadratic on the tangent bundle:

$$\frac{\partial F}{\partial t} + 2\langle \nabla F, U\rangle + h(U,U) + \frac{\kappa F}{(\kappa+1)t} \geq 0 \tag{1.5}$$

for all tangent vectors U. When $F = H$ and $\kappa = 1$ this is Hamilton's estimate; when $F = K^{\beta}$ and $\kappa = n\beta$, Chow's. Note that, in these cases, the scalar κ represents the degree of homogeneity of the speed functions in the principal curvatures. These estimates were later generalized and their proofs substantially simplified by Andrews [3]. Some noteworthy applications of these estimates and their associated entropy quantities may be found in [46], [44], and [4]. In this last reference, for example, Andrews makes use of Chow's estimate for the Gauss curvature flow to prove a long-standing conjecture of Firey [34].

A space-time approach. In [15], Chow and Chu show that the quadratic form in Hamilton's matrix Harnack inequality for the Ricci flow [42] can be identified with the curvature tensor of a degenerate metric on the space-time of the solution. On manifolds with bounded non-negative curvature operator, their result thus connects Hamilton's inequality to the non-negativity of their extension of this curvature operator. A similar construction in [16] provides a geometric interpretation of the trace-Harnack inequalities of Hamilton and Chow [18] for the linearized Ricci flow. In the later paper [19], Chow and Knopf refine and employ this sort of correspondence to prove new, generalized Harnack inequalities for the Ricci flow. Most recently, Perelman, in Section 6 of [62], exhibits a metric on $M^n \times S^N \times \mathbb{R}$ whose curvature tensor also may be (approximately) identified with Hamilton's Harnack quantity. From the perspective of his space-time geometry, the monotonicity of his reduced volume quantity has a natural interpretation in terms of the Bishop-Gromov volume comparison theorem.

In Chapter 6, we consider corresponding techniques in the setting of evolving hypersurfaces. Our construction is an adaptation of one which appears in [17], wherein the authors construct a representation of the principal part of the Harnack quadratic – that which vanishes identically on translating solitons – as a certain degenerate immersion of the space-time track. We shall combine the basics of their approach with the method in [19], and exhibit a similar representation – this time modelled on homothetically expanding solutions – to construct from the solution $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ a degenerate immersion $\widetilde{X}$ of the space-time track $\widetilde{\mathcal{M}} := M^n \times [0,T)$ into $\mathbb{R}^{n+1} \times \mathbb{R}$ whose second-fundamental form $\tilde{h}$ agrees, up to a scaling factor, with the Harnack quantity on the left-hand side of (1.5):

$$t^{\frac{1}{\kappa+1}}\tilde{h}\left(\tilde{U},\tilde{U}\right) = \frac{\partial F}{\partial t} + 2\langle \nabla F, U\rangle + h(U,U) + \frac{\kappa F}{(\kappa+1)t}. \tag{1.6}$$

Here, $\tilde{U} = \frac{\partial}{\partial t} + U^k \frac{\partial}{\partial x^k}$.

As part of the construction, we obtain a connection $\widetilde{\nabla}$ with respect to which $\widetilde{h}$ satisfies the Codazzi equations and, more importantly, an evolution-type equation along the time direction which is amenable to the application of an adapted space-time maximum principle. In the case of the mean curvature flow, for example, the form $\tilde{h}$ satisfies the particularly simple, if a bit unusual, equation

$$\widetilde{\nabla}_{t\frac{\partial}{\partial t}}\tilde{h} = \widetilde{\Box}\tilde{h} + \left(t|h_{ij}|^2 + \frac{1}{2}\right)\tilde{h}.$$

where $\widetilde{\Box} = \sum_{i,j} t\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j}$ for a local g-orthonormal frame on M^n.

The representation, in particular, provides a correspondence between the satisfaction of a Harnack inequality of the form (1.5) and the weak positivity of $\tilde{h}$ as a quadratic form on the space-time tangent bundle– that is, to the weak convexity of the immersion $\widetilde{X}$ – and suggests that such inequalities may be alternatively proved by way of the geometry on $\widetilde{\mathcal{M}}$. In the final two sections of the chapter, we explore this connection; first by demonstrating the connection between the quantities considered in the proof of Andrews's result [3], and then, in the final section of the chapter, using the machinery previously developed to provide a new proof of Hamilton's theorem for the mean curvature flow by way of a generalized tensor maximum principle argument.

Space-like hypersurfaces in Minkowski space. In Chapter 7, we consider Harnack inequalities in the related setting of evolving space-like hypersurfaces in

Minkowski space $\mathbb{L}^{n+1}$. Here the evolution is described by motion in a time-like direction at a rate determined by some function of the principal curvatures. While there is a rather well-developed theory of the geometry of space-like hypersurfaces in Lorentzian manifolds, there has been considerably less attention paid to parabolic methods in this setting, and nearly all of that exclusive to the flow by the mean curvature (see, for example, the work of Ecker [25], [26], and Ecker-Huisken [30]). The applications of the existing work include the construction of CMC hypersurfaces as limits of modified mean-curvature flows. We suspect that flows by other speeds should have applications to the construction of such prescribed-curvature space-like slices, and that Harnack inequalities for such flows should have the same applicability to questions of long time behavior and convergence as they do for their Euclidean counterparts.

One means of obtaining such estimates is through the adaption of Andrews's technique [3]. The key component in Andrews's proof in the Euclidean setting is to reparametrize the evolving hypersurfaces over the sphere by the inverse of the Gauss map. This parametrization represents the the evolution solely in terms of the support function of the hypersurface – a single scalar function – and equips the domain with a natural time-independent metric and connection.

On a space-like hypersurface, a choice of unit time-like vector gives rise to a Gauss map $M^n \to \mathbb{H}^n \subset \mathbb{L}^{n+1}$. As in the Euclidean setting, if the family of immersions is convex (say) then the flow may be reparametrized by the inverse of this map, and the local computations in Andrews's arguments carry over, essentially directly. The chief differences are that the image of the Gauss maps ν_t need not be full and will necessarily be non-compact. While the (time-dependent) open subsets on which the new parametrization is defined do not hamper our derivation of the relevant equations, our original domain (which, by our assumptions that the hypersurfaces $X_t(M^n)$ are complete and spacelike, must be diffeomorphic to $\mathbb{R}^n$) seems to be a more convenient setting to apply the maximum principle. Thus we push the results of the computations back to M^n at the sacrifice of a bit of their simplicity.

As the maximum principle does not hold in general for complete manifolds, we must impose some condition on the growth of the derivatives of the speed. At the time of this document's publication, the author is actively engaged in the study of gradient estimates for general curvature flows in this setting and we state here, primarily for illustrative purposes, only a preliminary extension of Andrews's theorem with general (and likely non-optimal) conditions on the growth and derivatives of the speed and the second fundamental form. Here F^* is the dual-function of F (see Section 2.2 of Chapter 2 for the relevant definitions).

Theorem. Suppose $X : M^n \times [0,T) \to \mathbb{L}^{n+1}$ is a family of smooth space-like immersions solving

$$\frac{\partial X}{\partial t}(x,t) = F(x,t)\nu(x,t)$$

for a function $F = F(W)$ of the principal curvatures. Assume that the induced metrics are complete and that, for all $0 < \delta < T/2$, there exists $\mathcal{O} \in M$ and positive constants $b = b(\delta)$ and $C = C(\delta)$ such that

$$|\nabla\nabla F|^2 + (h^{-1})^{ij}\nabla_i F\nabla_j F \leq F\exp(b(d(\mathcal{O},x)+1)) \tag{1.7}$$

and

$$|h|^2 + |\nabla h|^2 + F^2 + |\dot{F}|^2 \leq C(\delta) \tag{1.8}$$

on $M^n \times (\delta, T-\delta)$. Then

1. If F^* is α-concave for some $\alpha < 1$, we have

   $$\frac{\partial F}{\partial t} - (h^{-1})^{ij}\nabla_i F\nabla_j F + \frac{\alpha F}{(\alpha-1)t} \geq 0$$

   for all $(x,t) \in M^n \times (0,T)$.

2. If F^* is concave and positive, then

   $$\sup_{M^n}\left(\frac{\partial}{\partial t}\log|F| - F(h^{-1})^{ij}\nabla_i\log|F|, \nabla_j\log|F|\right) \quad \text{is decreasing.}$$

3. If F^* is α-convex for some $\alpha > 1$, we have

   $$\frac{\partial F}{\partial t} - (h^{-1})^{ij}\nabla_i F\nabla_j F + \frac{\alpha F}{(\alpha-1)t} \leq 0$$

   for all $(x,t) \in M^n \times (0,T)$.

4. If F^* is convex and positive, then

$$\inf_{M^n}\left(\frac{\partial}{\partial t}\log|F| - F(h^{-1})^{ij}\nabla_i \log|F|, \nabla_j \log|F|\right) \quad \text{is increasing.}$$

Gradient estimates for the heat kernel on complete manifolds. In [56], Lei Ni considers an entropy functional for the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = 0. \tag{1.9}$$

analogous to Perelman's $\mathcal{W}$-functional [62] for metrics evolving by the Ricci flow. The integrand of Ni's functional satisfies a pointwise inequality for manifolds of non-negative Ricci curvature, and for the positive heat kernel $H = e^{-f}/(4\pi t)^{\frac{n}{2}}$, satisfies

$$t\left(2\Delta f - |\nabla f|^2\right) + f - n \leq 0 \tag{1.10}$$

for all $t > 0$. To make rigorous some statements in the proof, which follows a clever integral argument of Perelman, one needs suitable estimates to control the derivatives of the heat kernel. (A discussion of such estimates in the context of Perelman's inequality can be found in [59] and [70]). Such considerations were the original motivation for the work in Chapter 8.

Our main result in the chapter is the following extension of a gradient estimate of Hamilton [43] to complete, non-compact manifolds with a Ricci curvature lower bound.

Theorem 1.1. *Suppose* (M^n, g) *is a complete manifold with* $R_{ij} \geq -Kg_{ij}$ *and* $0 < u(x,t) \leq M$ *is a smooth solution to* (1.9) *on* $M^n \times [0,T]$ *where* $K \geq 0$ *and* $M > 0$ *are constants. Then*

$$t|\nabla u|^2 \leq (1 + 2Kt)u^2 \log\left(\frac{M}{u}\right) \tag{1.11}$$

for all $(x,t) \in M^n \times [0,T]$.

When the Ricci curvature of g is non-negative, one has available the upper and lower bounds of Li and Yau [55] for the heat kernel. With these bounds, (1.11), and Bishop's volume comparison theorem, we then prove

Theorem 1.2. *Suppose* (M^n, g) *is a complete manifold with* $R_{ij} \geq 0$*, and let* $H(x, y, t)$ *be its positive heat kernel. Then, for all* $\delta > 0$*, there exists a constant* $C = C(n, \delta) > 0$ *such that*

$$|\nabla \log H|^2 \leq \frac{2}{t}\left(C + \frac{d^2(x, y)}{(4-\delta)t}\right) \tag{1.12}$$

for all $t > 0$ *where* $d(x, y) = \mathrm{dist}(x, y)$

By considering the heat kernel on Euclidean space, one can see that the power of t in (1.12) is sharp, and in this sense, (1.12) improves a recent estimate of Souplet and Zhang [66] at scales $d^2(x, y) >> t$. In the compact setting, estimates of the form (1.12) have already been established for the heat kernel and all of its derivatives (see, e.g., [64], [67]). To prove Theorem 1.1, we first prove a local Bernstein-type estimate for positive solutions to (1.9) to establish preliminary bounds on the gradients of solutions. For globally bounded solutions, this estimate supplies an upper bound sufficient to apply a maximum principle for complete manifolds due to Li [53] (see also [60]) to the quantity of interest in (1.11).

2 Preliminaries

2.1 Notation and Conventions

2.1.1 Metric and curvature

For a general smooth n-dimensional manifold M^n, we will denote the tangent bundle of M^n by TM^n, its dual by T^*M^n, the bundle of (k,l)-tensors by $T^k_l M^n$, and the bundle of linear maps of TM^n by $\mathrm{End}(M^n) = T^1_1 M^n$.

A *Riemannian metric* g is a smooth, positive definite section of $T^2(M^n)$, that is, a smooth choice of inner product on the tangent spaces T_pM^n, $p \in M^n$. Associated to a Riemannian metric is a torsion-free connection, the *Levi-Civita connection*, ∇, which satisfies the compatibility condition

$$Xg(Y,Z) = g\left(\nabla_X Y, Z\right) + g\left(X, \nabla_X Z\right) \tag{2.1}$$

for all vector fields X, Y, and Z. Of course, g also defines a metric on T^*M^n via the fiberwise isomorphism between T_pM^n and $(T_pM^n)^* \approx T^*_pM^n$, and thus induces a metric on all of the bundles $T^k_lM^n$ according to $g(U\otimes V, W\otimes Z) = g(U,W)g(V,Z)$. Similarly, the Leibniz rule provides a natural way to extend ∇ to T^*M^n – if ω is a section of one-forms, X a vector field, and $Y \in T_pM^n$, we define

$$\left(\nabla_Y \omega\right)(X) = Y\left(\omega(X)\right) - \omega\left(\nabla_Y X\right).$$

By the rule $\nabla(U \otimes V) = (\nabla U) \otimes V + U \otimes (\nabla V)$, ∇ then extends to the bundles $T^k_lM^n$. We will make the mild, and entirely standard, abuse of notation and also use g and ∇ to denote the induced metrics and connections on each $T^k_lM^n$. According to this convention, equation (2.1) may be expressed simply as $\nabla g = 0$.

From the connection, we define the $(3,1)$ *Riemannian curvature tensor* by

$$\mathrm{Rm}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$$

for all vector fields X, Y, Z. The *Ricci curvature* tensor is the symmetric two-tensor obtained via the following contraction of the Riemannian curvature tensor:

$$\mathrm{Rc}(U,V) = \sum_{i,j=1}^{n} g\left(\mathrm{Rm}(U,e_i)e_j, V\right)$$

where e_i is an orthonormal basis for T_pM^n, and the *scalar curvature* is the further contraction

$$\mathrm{R} = \sum_{i,j=1}^{n} \mathrm{Rc}(e_i,e_j).$$

The *sectional curvature* of a two-plane $\Pi \subset T_pM^n$ is defined to be

$$K_p(\Pi) := \frac{g(\mathrm{Rm}(X,Y)Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}$$

for any basis $\{X,Y\}$ of Π. (M^n, g) is said to have *positive sectional curvature* at p if $K_p(\Pi) > 0$ for all two-planes $\Pi \subset T_pM^n$; negative, nonpositive, and nonnegative curvature are defined analogously.

2.1.2 Conventions and identities in local coordinates

We shall often find it convenient to perform our calculations in local coordinates on a manifold. For evolution equations, this perspective has the advantage that the coordinates themselves are fixed and independent of time; were we to represent the same quantities in terms of orthonormal frames, we should typically have to evolve the frames as well so that they might stay orthonormal with respect to the changing metric.

Except where noted, we observe the following conventions.

1. *Roman and Greek indices. For local coordinates $\{x^\alpha\}$ on an $n+1$- dimensional manifold M^{n+1}, we will number our coordinates from 0 to n, reserve Roman letters i, j, k, etc., for indices strictly in the range $\{1,2,\ldots,n\}$ and use Greek letters to denote arbitrary indices in the range $\{0,1,\ldots,n\}$.*

2. *Summation convention. We will raise and lower indices by the metric g so that, e.g., for a vector V^k in TM^n we have $V_j = g_{jk}V^k$, and, except where noted, shall observe the Einstein summation convention: any character appearing twice in an expression as both an upper and lower index is to be summed over the appropriate range. Thus, for example, we should have (abiding by the convention in (1))*

$$T_\gamma^{\alpha\beta}V_\beta = \sum_{\beta=0}^{n} T_\gamma^{\alpha\beta}V_\beta \quad \textit{and} \quad U_iW^{ij} = \sum_{i=1}^{n} U_iW^{ij}.$$

3. *Tensor norms. On a Riemannian manifold (M^n, g), the metric g induces a canonical inner product on the tensor bundles $T_l^k(M^n)$. In coordinates, for example, we have*

$$\left|T_{ij}^k\right|^2 = g^{ia}g^{jb}g_{kc}T_{ij}^kT_{ab}^c$$

for an arbitrary $T \in T_1^2(M^n)$.

4. *The $(4,0)$-Riemannian curvature tensor. We shall lower the index of the $(3,1)$-Riemannian curvature tensor to the fourth position, that is*

$$R_{ijkl} := g_{lm}R_{ijk}^m.$$

In local coordinates, the Ricci curvature tensor is given by

$$R_{ij} := \operatorname{Rc}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g^{kl}R_{kijl}$$

and the scalar curvature by $R = g^{ij}R_{ij}$.

Finally, we record a local expression for the components of the Levi-Civita connection

$$\Gamma_{ij}^k = \frac{1}{2}g^{km}\left\{\frac{\partial}{\partial x^i}g_{jm} + \frac{\partial}{\partial x^j}g_{im} - \frac{\partial}{\partial x^m}g_{ij}\right\} \tag{2.2}$$

in terms of which we also may express the curvature

$$R_{ijk}^l = \frac{\partial}{\partial x^i}\Gamma_{jk}^l - \frac{\partial}{\partial x^j}\Gamma_{ik}^l + \Gamma_{jk}^p\Gamma_{ip}^l - \Gamma_{jp}^l\Gamma_{ik}^p. \tag{2.3}$$

2.2 Curvature flows of hypersurfaces

2.2.1 Definitions and conventions

For Chapters 6 and 7, we need to establish some basic notation for evolving families of immersed hypersurfaces in $\mathbb{R}^{n+1}$. To include the necessary results for both chapters in a unified setting, we will consider $\mathbb{R}^{n+1}$ to be equipped with the (pseudo-riemannian) metric

$$ds^2 = \sigma dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + \ldots + dx^n \otimes dx^n$$

for $\sigma \in \{-1, 1\}$. We shall alternatively denote the inner product by $\langle \cdot, \cdot \rangle$.

We shall use the usual notations g, ∇, and Rm, to denote the induced metric, Levi-Civita connection, and Riemannian curvature tensor on M^n, and shall use h and W to denote the second fundamental form and Weingarten maps associated to X. In most applications, W will belong to the positive cone $\Gamma_+ \subset \mathrm{End}(TM^n)$.

We define h by

$$D_U V = \nabla_U V - \sigma h(U, V)\nu \tag{2.4}$$

for all tangential vector fields U, V. Our convention for h is chosen so that we have (in both the Euclidean and Minkowski settings) the usual formula

$$h_{ij} = \left\langle \frac{\partial \nu}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle = -\left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right\rangle. \tag{2.5}$$

Also according to this convention, the Gauss equation has the local form

$$R_{ijkl} = \sigma(h_{il}h_{jk} - h_{ik}h_{jl})$$

and we have the standard commutation formula

$$(\nabla_i \nabla_j \omega - \nabla_j \nabla_i)\, \omega_k = -R_{ijkl} g^{lm} \omega_m = \sigma g^{lm} \left(h_{ik}h_{jl} - h_{il}h_{jk}\right) \omega_m$$

for any one-form ω.

For the Ricci curvature, we have

$$R_{ij} = \sigma(H h_{ij} - h_i^k h_{kj}),$$

so the Bochner formula takes the form

$$\nabla^k \Delta f \nabla_k f - \Delta \nabla_k f \nabla^k f = \sigma(\left(h^i_k h^{jk} - H h^{ij}\right) \nabla_i f \nabla_j f)$$

where $H = g^{ij} h_{ij}$ is the mean curvature.

2.2.2 Evolution equations for evolving families of hypersurfaces

In Chapters 6 and 7 we will consider smooth families of immersions $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ of a smooth manifold M^n which satisfy an equation of the form

$$\frac{\partial X}{\partial t} = -\sigma F \nu - T \tag{2.6}$$

where $F : M^n \times [0, T) \to \mathbb{R}$ is a smooth function, $\nu : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is a smooth vector field satisfying $\langle \nu, \nu \rangle \equiv \sigma$ transverse to TM^n and $T = T(x, t) \in dX_{(x,t)}(T_x M^n)$ is a smooth family of tangential vector fields. When $\sigma = 1$, ν is just a smooth choice of normal; when $\sigma = -1$, ν is a choice of time-like vector field, whose existence we ensure by imposing the additional requirement that $X_t(M^n)$ is complete and space-like for all t (cf. 7.1 for the relevant definitions).

Of course, any flow described by an equation of the form (2.6) may be adjusted by a diffeomorphism of M^n to eliminate the tangential term T; such a modification changes only the parametrization, and not the family of hypersurfaces in $\mathbb{R}^{n+1}$. Nevertheless, in some circumstances, one can simplify computations by considering an equivalent parametrization with a judicious choice of T. We shall see some example of this in Chapter 7. Here we include the term for the time being to track the Lie derivative terms it contributes to the general evolution equations.

Suppose now that (2.6) is isotropic, so that $F = F(W)$ is a smooth, symmetric function of the principle curvatures. Under equation (2.6), in local coordinates on M^n, we have

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij} &= -2\sigma F h_{ij} - g_{ik}\nabla_j T^k - g_{jk}\nabla_i T^k \\ &= -2\sigma F h_{ij} - (\mathcal{L}_T g)_{ij}\,, &(2.7)\\ \frac{\partial}{\partial t}\nu &= \left(\nabla^k F - h^k_l T^l\right)\frac{\partial X}{\partial x^k}, &(2.8)\\ \frac{\partial}{\partial t}h_{ij} &= \nabla_i\nabla_j F - \sigma F h_{ik}h^k_j - \left(\nabla_i h_{jk}T^k + h_{ik}\nabla_j T^k + h_{jk}\nabla_i T^k\right)\\ &= \nabla_i\nabla_j F - \sigma F h_{ik}h^k_j - (\mathcal{L}_T h)_{ij} &(2.9)\\ \frac{\partial F}{\partial t} &= \dot{F}^{ij}\nabla_i\nabla_j F + \sigma F\dot{F}^{ij}h_{ik}h^k_j - \nabla_k F T^k, &(2.10)\end{aligned}$$

where

$$\dot{F}_W[H] := \left.\frac{d}{ds}\right|_{s=0} F(W+sH) \quad\text{and}\quad \dot{F}^{ij} := g^{jk}\dot{F}^i_k.$$

Note that, by the Codazzi equations and the commutation formula above, we have

$$\nabla_a\nabla_b h_{ij} = \nabla_i\nabla_j h_{ab} - \sigma g^{pq}\left(h_{bi}h_{ap}h_{jq} + h_{ij}h_{ap}h_{bq} - h_{ip}h_{ab}h_{jq} - h_{ip}h_{aj}h_{bq}\right).$$

Thus we can rewrite the equation for h as

$$\begin{aligned}\frac{\partial}{\partial t}h_{ij} = \dot{F}^{ab}\nabla_a\nabla_b h_{ij} + \ddot{F}^{ab,cd}\nabla_j h_{ab}\nabla_i h_{cd} - (\mathcal{L}_T)_{ij} - \sigma F h_{ik}h^k_j\\ - \sigma\dot{F}^{ab}g^{pq}\left(h_{ip}h_{ab}h_{jq} + h_{ip}h_{aj}h_{bq} - h_{bi}h_{ap}h_{jq} - h_{ij}h_{ap}h_{bq}\right)\end{aligned}$$

where

$$\ddot{F}_W[H,K] = \left.\frac{d}{du}\frac{d}{ds}\right|_{s,u=0} F(W+sH+uK) \quad\text{and}\quad \ddot{F}^{ij,kl} := g^{jp}g^{lq}\ddot{F}^{i,k}_{j,l}.$$

In particular, we have

Lemma 2.1.

1. *(Mean curvature flow) If $F = H$ and $T^k = 0$, then $\dot{F}^{ab} = g^{ab}$,*

$\ddot{F}^{ab,cd} \equiv 0$, *and*

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2\sigma H h_{ij} \\
\frac{\partial}{\partial t} d\mu &= -\sigma H^2 d\mu \\
\frac{\partial}{\partial t} h_{ij} &= \nabla_i \nabla_j H - \sigma H h_i^k h_{kj} \\
&= \Delta h_{ij} - 2\sigma H h_{ik} h_j^k + \sigma |h_{ab}|^2 h_{ij} \\
\frac{\partial H}{\partial t} &= \Delta H + \sigma H |h_{ij}|^2.
\end{aligned}$$

2. *(Gauss curvature flow) Suppose that* h *is positive definite on all of* $M^n \times [0,T)$. *If* $F = K := \det(g^{ik} h_{kj})$, *and* $T^k = 0$, *then, setting* $A^{ab} = (h^{-1})^{ab}$, *we have* $\dot{F}^{ab} = K A^{ab}$, $\ddot{F}^{ab,cd} = K\left(A^{ab}A^{cd} - A^{ac}A^{bd}\right)$, *and*

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2\sigma K h_{ij} \\
\frac{\partial}{\partial t} d\mu &= -\sigma H K d\mu \\
\frac{\partial}{\partial t} h_{ij} &= \nabla_i \nabla_j K - \sigma K h_i^k h_{kj} \\
&= \Box h_{ij} + K\left(A^{ab}A^{cd} - A^{ac}A^{bd}\right) \nabla_j h_{ab} \nabla_i h_{cd} \\
&\qquad - \sigma(n+1) K h_{ip} h_j^p + H K h_{ij} \\
\frac{\partial K}{\partial t} &= \Box K + \sigma K^2 H \\
\frac{\partial H}{\partial t} &= \Box H + K g^{ij}\left(A^{ab}A^{cd} - A^{ac}A^{bd}\right) \nabla_i h_{ab} \nabla_j h_{cd} \\
&\qquad + K(H^2 - n|h_{ab}|^2),
\end{aligned}$$

where $\Box := K A^{ab} \nabla_a \nabla_b$.

2.2.3 Harnack estimates for evolving convex hypersurfaces

As we mentioned in the Introduction, in the setting of evolving families of convex hypersurfaces, Harnack estimates were first proved by Hamilton for the mean curvature flow [46] and by Chow [14] for positive powers of the Gauss curvature flow. Later, in [3], Ben Andrews generalized the theorems of Hamilton and Chow to a wide variety of curvature flows. His theorem will be the basic model we shall consider in Chapters 6 and 7, and it will be useful for us to recall its statement. First we need a few definitions.

Definition 2.2 (Dual function)**.** Given a function $F : \Gamma_+ \subset \mathrm{End}(TM^n) \to \mathbb{R}$, its *dual* function $F^* : \Gamma_+ \to \mathbb{R}$ is the map

$$F^*(A) = -F(A^{-1}).$$

Definition 2.3 (α-convexity/concavity)**.** A function F is α-convex (concave) if $F = \mathrm{sgn}(\alpha) B^\alpha$ for a positive convex (concave) function B.

It is not hard to see that, for $\alpha \neq 0$, the definition of α-convexity is equivalent to the following inequality on the derivatives of F.

$$\ddot{F}_A \geq \frac{(\alpha - 1)\dot{F}_A \otimes \dot{F}_A}{\alpha F(A)}. \tag{2.11}$$

Now we state Andrews's theorem.

Theorem 2.4 (B. Andrews, [3])**.** *Suppose $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is family of strictly convex immersions satisfying* (2.6) *for $F(x,t) = F(W(x,t), \nu(x,t))$ on a compact manifold M^n. Then*

1. *If F^* is α-concave for some $\alpha < 1$, we have the inequality*

$$\frac{\partial F}{\partial t} - (h^{-1})^{ij} \nabla_i F \nabla_j F + \frac{\alpha F}{(\alpha - 1)t} \geq 0.$$

2. *If F^* is concave and positive, then*

$$\sup_{M^n} \left(\frac{\partial}{\partial t} \log |F| - F (h^{-1})^{ij} \nabla_i \log |F| \nabla_j \log |F| \right)$$

is decreasing.

3. *If F^* is α-concave for some $\alpha < 1$, we have the inequality*

$$\frac{\partial F}{\partial t} - (h^{-1})^{ij} \nabla_i F \nabla_j F + \frac{\alpha F}{(\alpha - 1)t} \leq 0.$$

4. If F^* is concave and positive, then

$$\inf_{M^n}\left(\frac{\partial}{\partial t}\log|F| - F(h^{-1})^{ij}\nabla_i \log|F|\nabla_j \log|F|\right)$$

is increasing.

Remark 2.5. The α-convexity of F^* along the flow is equivalent to

$$\ddot{F}^{ab,cd} + 2\dot{F}^{ad}(h^{-1})^{bc} - \frac{\alpha-1}{\alpha F}\dot{F}^{ab}\dot{F}^{cd} \geq 0 \tag{2.12}$$

and thus is a weaker condition than that of the α-convexity of F.

In [3] and [2], Andrews provides examples of a wide variety of flows by speed functions F to which his theorem applies– note that as he permits the speed to vary by the normal direction, his theorem applies also to a number of anisotropic flows. We conclude the section by demonstrating that Andrews's theorem indeed covers the cases considered by Hamilton and Chow for compact manifolds.

Example 2.6. For the mean curvature flow, we have $\ddot{F} = 0$, so F^* is -1-concave. Then Hamilton's inequality

$$\frac{\partial H}{\partial t} + 2\langle \nabla H, V\rangle + h(V,V) + \frac{H}{2t} \geq 0, \tag{2.13}$$

which holds for all $V \in TM^n$, is a particular consequence of part (1) of Andrews's theorem, since when h is positive definite, (2.13) is minimized by $V^i = -(h^{-1})^{ij}\nabla_j F$.

For the Gauss curvature flow, F^* is $-n$-concave, since $\dot{F}^{pq} = K\left(h^{-1}\right)^{pq}$, and

$$\ddot{F}^{pq,rs} = K\left(\left(h^{-1}\right)^{pq}\left(h^{-1}\right)^{rs} - \left(h^{-1}\right)^{ps}\left(h^{-1}\right)^{qr}\right), \tag{2.14}$$

so

$$\ddot{F}^{pq,rs} + 2\dot{F}^{ps}\left(h^{-1}\right)^{qr} = K\left(\left(h^{-1}\right)^{pq}\left(h^{-1}\right)^{rs} + \left(h^{-1}\right)^{ps}\left(h^{-1}\right)^{qr}\right) \tag{2.15}$$

$$\geq \left(\frac{n+1}{n}\right) K\left(h^{-1}\right)^{pq}\left(h^{-1}\right)^{rs} \tag{2.16}$$

$$\geq 0. \tag{2.17}$$

Similarly, one may show that if $F = K^\beta$, F^* is $-n\beta$-concave, and thus Chow's estimate [14] follows from part (1) of the above theorem as well.

2.2.4 A distance-type barrier function on complete, non-compact manifolds

For some of our later analysis, we will require the following

Lemma 2.7. *Suppose $g(t)$, $t \in [0,T]$ is a smooth family of complete Riemannian metrics on M^n and that for some constant $B > 0$, $\frac{\partial}{\partial t} g_{ij} = b_{ij}$ satisfies*

$$|b|_{g(t)} \leq B \quad \text{and} \quad |\nabla b|_{g(t)} \leq C\rho(t) \tag{2.18}$$

for some $\rho \in C(0,T]$ satisfying

$$\int_0^T \rho(t)\, dt < \infty.$$

Then there exists a positive smooth function $\phi : M^n \to \mathbb{R}$ having the following properties.

1. *For any $x_0 \in M^n$, there exists a constant $C_1 = C(x_0, B, n)$ such that*

$$\frac{1}{C}(d_{g(t)}(x_0, x) + 1) \leq \phi(x) \leq C(d_{g(t)}(x_0, x) + 1). \tag{2.19}$$

2. *There exist constants $C_2 = C_2(B, n)$, $C_3 = C_3(B, n)$ such that*

$$|\nabla\phi|_{g(t)} \leq C_2 \quad \text{and} \quad (\nabla\nabla)_{g(t)}\phi \leq C_3 g(t)$$

Proof. This follows from the proof of Lemma 12.5 in [23]. The only property of the Ricci flow used is that an assumption of bounded curvature tensor implies that $|\nabla \operatorname{Rm}|$ (hence also $|\nabla \operatorname{Rc}|$) satisfies a bound of the form

$$|\nabla \operatorname{Rm}| \leq \frac{C}{\sqrt{t}},$$

the right-hand side of which is integrable over $[0, T]$.

□

3 A classification theorem for rotationally invariant shrinking gradient Ricci solitons

The principal object of our consideration in this chapter is the equation

$$\mathrm{Rc}(g) + \nabla\nabla f - \lambda g = 0 \tag{3.1}$$

for a Riemannian metric g on a manifold M, a smooth function f and a constant $\lambda \in \mathbb{R}$. A metric solving (3.1) for some smooth f and positive constant λ is called a *shrinking gradient soliton.* The analogous objects in the cases $\lambda = 0$ and $\lambda < 0$ are known as *steady* and *expanding* solitons. With the exception of the sign convention for λ, which we have reversed to favor shrinking solitons, this terminology is by now standard, though perhaps a bit misleading. It is possible for the same metric to satisfy (3.1) with different choices of f and λ, and indeed be a shrinking soliton with one gradient function and an expanding or steady soliton with another. Thus to be precise, when it is not understood that f is some canonical choice, we shall refer to the triple (g, f, λ) as a *gradient soliton structure* on the manifold M.

After the class of Einstein metrics, perhaps the most natural place to look for solitons is among the rotationally invariant metrics. In fact, in all dimensions greater than one, this class has been shown to contain complete, non-trivial examples of steady and expanding solitons (cf., e.g, [52] and [22]). However, one does not expect to find corresponding examples in the shrinking case. The goal of this chapter is to verify this expectation; our main result is the following classification.

Theorem 3.1. *Suppose $n \geq 2$, and (g, f, λ) is a complete, rotationally invariant shrinking soliton structure on a on a manifold M^{n+1} diffeomorphic to one of S^{n+1}, $\mathbb{R}^{n+1}$ or $\mathbb{R} \times S^n$. Then,*

1. *If $M^{n+1} \cong S^{n+1}$, g is isometric to a round sphere and $f \equiv \text{const}$.*
2. *If $M^{n+1} \cong \mathbb{R}^{n+1}$, g is flat.*
3. *If $M^{n+1} \cong \mathbb{R} \times S^n$, g is isometric to the standard cylinder $dr^2 + \omega_0^2 g_{S^n}$ of radius $\omega_0 = \sqrt{(n-1)/\lambda}$ and $f = f(r) = (n-1)r^2/(2\omega_0^2) + \text{linear}$.*

As we mentioned in the introduction, this theorem is known (in greater generality) in dimensions less than four. In dimension two, work of Hamilton in [40] and [45] implies that the only complete shrinking solitons are the flat metric on $\mathbb{R}^2$ and the round metric on S^2. Also, closer to the point of view of this chapter and the next, Chen, Lu, and Tian [12] have given an alternative proof of Hamilton's result on S^2 using the necessary rotational symmetry of any potential soliton. In a sense, then, our result may be considered an extension of these two-dimensional findings to higher dimensions.

In dimension three, gradient solitons need not be rotationally symmetric, however, Perelman [61], has shown that the only complete examples of shrinking solitons with non-negative sectional curvature are the flat metric on $\mathbb{R}^3$, the round metric on S^3, and the standard metric on the cylinder $\mathbb{R} \times S^2$, and their quotients. As the Hamilton-Ivey estimate (cf. [45], [52]) implies that three-dimensional ancient solutions are necessarily of non-negative curvature, Perelman's argument classifies all complete three-dimensional shrinking solitons

In higher dimensions, no complete classification exists, however, there are a number of partial results. For example, it is a consequence of the recent work of Böhm and Wilking [6] that the only compact shrinking soliton with 2-positive curvature operator is the round sphere, and, in the Kähler category, Ni [57] has classified all complete shrinking solitons of nonnegative bisectional curvature. Also, Feldman, Ilmanen, and Knopf [33] have obtained results similar to our own for Kähler-Ricci shrinking solitons under the corresponding assumption of $\mathrm{U}(n)$-invariance. In particular, their Proposition 9.2– that the flat metric is the only complete U(n)-invariant gradient shrinking soliton on $\mathbb{C}^n$–is the Kähler analog of the second case of our Theorem 3.1. Their paper also provides non-trivial examples of U(n)-invariant gradient solitons (of all types) on other spaces.

To prove Theorem 3.1, we show first that, under the constraint (3.1), the rotational symmetry of a non-flat metric implies the rotational symmetry of the gradient function. This reduces the proof of the proposition to the study of a certain second order system of non-linear ODE. By a change of variables due to Robert Bryant and Tom Ivey [22], we are able to further reduce the problem to the study of an equivalent first-order system amenable to phase-plane analysis. For the case $M^{n+1} \cong S^{n+1}$, we show that any candidate metric must have positive curvature operator, whence we may apply the result of Böhm and Wilking. It is no doubt possible to prove this case solely from the analysis of the system of ODE (without using any of the dynamic properties of the Ricci flow), however, for the present exposition, we will content ourselves with a proof with these external ingredients for convenience. For the non-compact cases, we use the criterion of completeness to eliminate all but the two standard metrics by their asymptotic behavior.

3.1 A rotationally symmetric metric as a warped cylinder.

Fix $n \geq 1$ and let $\widetilde{g}$ denote the metric on S^n of constant sectional curvature $K_{\widetilde{g}} = 1$ (or $\widetilde{g} = d\theta^2$, if $n = 1$). For $-\infty \leq A < \Omega \leq \infty$, and positive functions $\omega \in C^\infty(A, \Omega)$, consider the warped-product metric

$$g = dr^2 + \omega^2(r)\widetilde{g} \tag{3.2}$$

on the cylinder $\mathcal{C}_{A,\Omega} := (A, \Omega) \times S^n$. For this metric, we have the following standard

Lemma 3.2. *The metric $g = dr^2 + \omega(r)^2\widetilde{g}$ on $\mathcal{C}_{0,\Omega}$ extends to a smooth metric on $B_\Omega(\mathbf{0}) \subset R^{n+1}$ if and only if $\Omega > 0$, and*

$$\lim_{r\to 0} \omega(r) = 0, \ \lim_{r\to 0} \omega' = 1, \quad \textit{and} \quad \lim_{r\to 0} \frac{d^{2k}\omega}{dr^{2k}}(r) = 0 \quad \textit{for all } k.$$

The metric extends to a smooth metric on S^{n+1} if and only if in addition $\Omega < \infty$ and

$$\lim_{r\to\Omega}\omega(r) = 0, \ \lim_{r\to\Omega}\omega' = -1, \quad \textit{and} \quad \lim_{r\to\Omega}\frac{d^{2k}\omega}{dr^{2k}}(r) = 0 \quad \textit{for all } k.$$

Proof. The lemma may be proved by considering the expansion of ω about the poles $r = 0$, $r = \Omega$ in a rectangular coordinate system. See, e.g., [20], Lemma 2.10. □

Thus, for the study of rotationally symmetric gradient solitons it suffices to consider soliton structures of the form $(\mathcal{C}_{A,\Omega}, g, f, \lambda)$ for g of the form (3.2). We begin by recording the expressions of a few geometric quantities associated to g. The computations needed to derive these expressions are entirely standard. A nice reference is [63].

First, we observe that there are essentially two unique sectional curvatures of g (one when $n = 2$) , corresponding to planes that are tangent to the radial ($\frac{\partial}{\partial r}$) direction and planes spanned by directions tangent to the orbital (spherical) directions. Explicitly, we have

$$K_r = -\frac{\omega''}{\omega} \quad \text{and} \quad K_s = \frac{1-(\omega')^2}{\omega^2} \tag{3.3}$$

for the radial and orbital curvature directions, respectively (the prime denotes differentiation with respect to r). In fact, the curvature operator $\mathrm{Rm} : \Lambda^2 \to \Lambda^2$ satisfies

$$\mathrm{Rm}(dr \wedge \sigma^i) = 0$$

for a local orthonormal frame $\{\sigma^i\}_{i=1}^n$ on $\Lambda^2 S^n$, from which it follows, in particular, that the notions of positive curvature operator and positive sectional curvature coincide. By the above considerations, one may also derive that the Ricci curvature of g has the form

$$\mathrm{Rc}(g) = -n\frac{\omega''}{\omega}dr^2 + \big((n-1)(1-(\omega')^2) - \omega\omega''\big)\,\widetilde{g}. \tag{3.4}$$

A routine computation shows that for a smooth function f on $\mathcal{C}_{A,\Omega}$, one has

$$\nabla\nabla f = \begin{cases} \nabla_0\nabla_0 f &= f_{00} \\ \nabla_0\nabla_i f &= f_{i0} - (\omega'/\omega)f_i \\ \nabla_i\nabla_j f &= \widetilde{\nabla}_i\widetilde{\nabla}_j f + \omega\omega' f_0 \widetilde{g}_{ij} \end{cases}$$

in local coordinates $(\theta^0 = r, \theta^i)$. [1] In these coordinates, (3.1) has the expression

$$f_{00} - n\frac{\omega''}{\omega} - \lambda = 0 \tag{3.5}$$

$$f_{0i} - \frac{\omega'}{\omega} f_i = 0 \tag{3.6}$$

$$\widetilde{\nabla}_i \widetilde{\nabla}_j f + \left[(n-1)(1-(\omega')^2) - \omega\omega'' - \omega\omega' f_0 - \lambda\omega^2\right] \widetilde{g}_{ij} = 0. \tag{3.7}$$

The above system involves the partial derivatives of f and, despite the rotational symmetry of the metric and the Ricci tensor, there is no *a priori* reason to assume that f shares this symmetry. However, as we show next, this symmetry is implied unless g is flat, and thus for the proof of Theorem 3.1, there is no loss of generality in restricting our attention to the case $f = f(r)$.

When f is a radial function, equations (3.5)-(3.7) reduce to

$$\begin{cases} f'' - \lambda & = n\omega''/\omega \\ \omega\omega' f' - \lambda\omega^2 & = \omega\omega'' + (n-1)\left((\omega')^2 - 1\right). \end{cases} \tag{3.8}$$

Theorem 3.3. *Suppose that (g, f, λ) is a complete, rotationally symmetric gradient shrinking soliton structure on $M^{n+1} \cong S^{n+1}$, $\mathbb{R}^{n+1}$, or $\mathbb{R} \times S^n$.*

If the function f is not rotationally symmetric, then $M^{n+1} \cong \mathbb{R}^{n+1}$ and g is the flat metric.

Proof. Write $g = dr^2 + \omega^2(r)\widetilde{g}$ for $r \in (A, \Omega)$, and fix local coordinates $r = \theta^0, \theta^1, \ldots, \theta^n$ on a neighborhood $(A, \Omega) \times \mathcal{U}$ about any point. Observe that for each fixed r, equation (3.7) is a tensorial identity on S^n, and that we may therefore differentiate it using the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ to obtain

$$\widetilde{\nabla}_k \widetilde{\nabla}_i \widetilde{\nabla}_j f + \omega\omega' f_{0k} \widetilde{g}_{ij} = 0.$$

Hence

$$\widetilde{\nabla}_k \widetilde{\nabla}_i \widetilde{\nabla}_j f - \widetilde{\nabla}_i \widetilde{\nabla}_k \widetilde{\nabla}_j f = \omega\omega' \left(f_{0i}\widetilde{g}_{jk} - f_{0k}\widetilde{g}_{ij}\right).$$

[1] Here and throughout, when working in coordinates $\theta^0, \ldots, \theta^n$, we use roman letters to denote indices $1, \ldots, n$ and use a tilde to denote quantities (Levi-Civita connection, curvature, etc.) associated to the metric $\widetilde{g}$ on S^n. In particular, $\widetilde{\nabla}_i \widetilde{\nabla}_j f$ represents the hessian of $f(r, \cdot)$ considered as a function on S^n.

On the other hand, since $\widetilde{R}_{ijkl} = \widetilde{g}_{il}\widetilde{g}_{jk} - \widetilde{g}_{ik}\widetilde{g}_{jl}$, the standard commutation identities imply

$$\begin{aligned} \widetilde{\nabla}_k\widetilde{\nabla}_i\widetilde{\nabla}_j f - \widetilde{\nabla}_i\widetilde{\nabla}_k\widetilde{\nabla}_j f &= -\widetilde{R}_{kijl}\widetilde{g}^{lm} f_m \\ &= -\left(\widetilde{g}_{ij}\widetilde{g}_{kl} - \widetilde{g}_{il}\widetilde{g}_{jk}\right)\widetilde{g}^{lm} f_m \\ &= f_i\widetilde{g}_{jk} - f_k\widetilde{g}_{ij}. \end{aligned}$$

Combining the two, we find

$$\left(\omega\omega' f_{0i} - f_i\right)\widetilde{g}_{jk} = \left(\omega\omega' f_{0k} - f_k\right)\widetilde{g}_{ij},$$

and tracing yields

$$(n-1)\left(\omega\omega' f_{0i} - f_i\right) = 0$$

for all $i = 1, \ldots, n$. Together with (3.6), we conclude

$$(n-1)|\widetilde{\nabla} f|^2_{\widetilde{g}}\left[1 - (\omega')^2\right] = 0. \tag{3.9}$$

Since we assume $n > 1$, if $(Xf)(r_0, \theta_0) \neq 0$ for some $X \in T_{(r_0,\theta_0)}M^{n+1}$ tangent to the S^n factor, we must have $|\omega'| \equiv 1$ and $\omega'' \equiv 0$ on an interval $(a, b) \subset (A, \Omega)$ containing r_0. But, by equation (3.3), this means that $K_r = K_s = 0$ on (a, b). We claim that $K_r = K_s = 0$ on the entire interval (A, Ω).

Let

$$\beta = \sup\left\{r < \Omega \,|\, (\omega')^2 = 1 \text{ on } (a, r)\right\}.$$

If $\beta < \Omega$, by equations (3.7) and (3.9), we must have that $(\widetilde{\nabla}\widetilde{\nabla} f)(\beta, \cdot) = 0$, $\omega'(\beta) = \sigma \in \{\pm 1\}$, $\omega''(\beta) = 0$, $\omega(\beta) > 0$, and $f_0(\beta, \cdot) = \sigma\lambda\omega(\beta)$. Moreover, for some small ϵ, f is a function only of r on $[\beta, \beta + \epsilon)$ and on this interval, f' and ω satisfy the system (3.8), with the above initial conditions. But one may check that the functions

$$\bar{\omega}(r) = \sigma(r - \beta) + \omega(\beta) \quad \text{and } \bar{f}'(r) = \lambda((r - \beta) + \omega(\beta))$$

also satisfy (3.8) and agree with ω and f at $r = \beta$. Therefore, by uniqueness [2], these solutions must coincide and it follows that that $(\omega')^2 = (\bar{\omega}')^2 = 1$ on the

[2] Writing $x = \omega'$, and $u = f'$, we may recast (3.8) as a first-order system

$$\begin{cases} \omega' &= x := F(\omega, x, u) \\ x' &= xu - \lambda\omega + (n-1)\frac{1-x^2}{\omega} := G(\omega, x, u) \\ u' &= n\frac{xu}{\omega} + (n-1)\left(n\frac{1-x^2}{\omega^2} - \lambda\right) := H(\omega, x, u). \end{cases}$$

interval $[\beta, \beta+\epsilon)$, contradicting our choice of β. So g is flat on $(a, \Omega) \times S^n$. Using a similar argument at the other endpoint a, can show that g must be flat on the entire cylinder $(A, \Omega) \times S^n$. But this means either $\omega' \equiv 1$ or $\omega' \equiv -1$, so g cannot extend to a smooth metric on the sphere S^{n+1} or to a complete metric on the cylinder. The only possibility is $\omega' \equiv 1$ and $M^{n+1} \cong \mathbb{R}^{n+1}$. □

3.2 Classification of rotationally symmetric shrinking gradient Ricci solitons

3.2.1 An equivalent first-order system and its linearization.

In view of the result of the last section, we now assume $f = f(r)$. We are interested in solutions $(\omega(r), f(r))$ to the system (3.8) for which ω is strictly positive.

As Ivey observes in [22], (3.8) is invariant under translations of r and f. By the introduction of the variables

$$x = \omega' \quad \text{and} \quad y = n\omega' - \omega f'$$

which share this invariance, and of an independent variable t which satisfies $dt = 1/\omega dr$, one obtains the first-order system

$$\begin{cases} \frac{d\omega}{dt} & = x\omega \\ \frac{dx}{dt} & = x^2 - xy + n - 1 - \lambda\omega^2 \\ \frac{dy}{dt} & = xy - nx^2 - \lambda\omega^2. \end{cases} \tag{3.10}$$

Any solution to (3.8) gives rise to a trajectory of (3.10) and conversely, from a trajectory $(\omega(t), x(t), y(t))$ of (3.10), one may recover r, $\omega(r)$, and $f(r)$ by a succession of quadratures (see [52]). Consequently, it suffices to analyze solutions to the simpler system (3.10). We take as coordinates (ω, x, y) on the phase space $\mathbb{R}^3$ and restrict our attention to trajectories lying in the half space $\omega > 0$.

Since F, G, H are C^∞ on the region $\{\omega \neq 0\}$, the asserted uniqueness follows from standard ODE theory.

For $n > 1$, system (3.10) has two equilibrium points: $P_0 := (0, 1, n)$ and $P_1 := (0, -1, -n)$. Denoting the right hand side of (3.10) by Φ, one finds

$$d\Phi_{P_0} = -d\Phi_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2-n & -1 \\ 0 & -n & 1 \end{pmatrix}$$

which has eigenvalues 2, 1, and $1 - n$. Since we assume $n \geq 2$, both P_0 and P_1 are saddle points: P_0 (P_1) lying at the intersection of a two-dimensional unstable (stable) manifold and a one-dimensional stable (unstable) manifold. In particular, there is a one-parameter family of trajectories in the half-space $\omega > 0$ initially tangent to $(1, 0, 0)$, among which, in light of Lemma (3.2), lie the trajectories which give rise to smooth solutions on S^{n+1} and $\mathbb{R}^{n+1}$ (see Examples (3.5) and (3.6) below). Trajectories which correspond to smooth solutions on S^{n+1} must, in addition, tend to P_1 as $t \to \infty$, and hence lie in the intersection of the global unstable and stable manifolds of P_0 and P_1, respectively.

Remark 3.4. If $L : \mathbb{R}^3 \to \mathbb{R}^3$ is the map $(\omega, x, y) \mapsto (\omega, -x, -y)$, then from any solution $\gamma(t) = (\omega(t), x(t), y(t))$ of (3.10) on (S, T), one may obtain a new solution

$$\bar{\gamma}(t) := L\left(\gamma(\tau(t))\right) = (\omega(\tau(t)), -x(\tau(t)), -y(\tau(t)))$$

on an appropriate interval $(\bar{S}, \bar{T})$ where τ is chosen to satisfy $\frac{d\tau}{dt} = -1$ and $\tau(\bar{S}) = T$, $\tau(\bar{T}) = S$.

By use of this device, one immediately obtains that the global stable and unstable manifolds of P_i, S_i and U_i, $i = 1, 2$ are related by $L(S_0) = U_1$, $L(U_0) = S_1$, and moreover, if a set $V \subset \mathbb{R}^3$ is preserved by the system (3.10) for increasing t (i.e, $\gamma(t_0) \in V$ implies $\gamma(t) \in V$ for $t > t_0$, as long as the solution is defined), then $L(V)$ is preserved by the system for decreasing t.

3.2.2 The standard examples.

At this juncture, it is worthwhile to recall the standard solutions to (3.8) and locate the corresponding solution to (3.10) in ωxy-space. The content of Theorem 3.1 is that this list essentially exhausts the possibilities for complete solutions.

Example 3.5. (Round sphere) The condition $f \equiv \text{const}$ in (3.8) leads to the constant curvature soliton structure

$$\begin{cases} \omega(r) &= \sqrt{\frac{n}{\lambda}} \sin\left(\sqrt{\frac{\lambda}{n}} r\right) \\ f(r) &= \text{const} \end{cases}$$

on the sphere S^{n+1}. The corresponding trajectory in ωxy-space is the elliptical arc $\{nx^2 + \lambda\omega^2 = n\}$ lying in the plane $\{y = nx\}$ joining P_0 and P_1.

Example 3.6. (Gaussian soliton) The condition $\omega' \equiv 1$ in (3.8) leads to the flat soliton structure

$$\begin{cases} \omega(r) &= r \\ f(r) &= \frac{\lambda}{2} r^2 + \text{linear} \end{cases} \tag{3.11}$$

on $\mathbb{R}^{n+1}$ and corresponds to the trajectory $y = n - \omega^2$ in the plane $\{x = 1\}$ emanating from P_0. Applying the device of Remark (3.4), one obtains a similar trajectory in the plane $\{x = -1\}$, satisfying $y = -n + \lambda\omega^2$ and tending to P_1 as $t \to \infty$.

Example 3.7. (Standard cylinder) The condition $\omega' \equiv 0$ in (3.8) leads to the structure

$$\begin{cases} \omega(r) &= \sqrt{\frac{n-1}{\lambda}} \\ f(r) &= \frac{\lambda}{2} r^2 + \text{linear} \end{cases}$$

on $\mathbb{R} \times S^n$ corresponding to the line $\{(\sqrt{(n-1)/\lambda}, 0, y)\}$ in ωxy-space.

Remark 3.8. The flat and cylindrical solutions described in the previous two examples are the only for which $\omega' \equiv \text{const}$, and the corresponding trajectories in ωxy-space describe the intersections of the planes $\{x = 1\}$, $\{x = 0\}$, and $\{x = -1\}$ with the set $\{x^2 - xy + n - 1 - \lambda\omega^2 = 0\}$. This leads to an observation which we shall have repeated occasion to use in the sequel: the only solutions $\gamma(t)$ of (3.10) for which $\frac{dx}{dt}(t_0) = 0$ and $x(t_0) = 1$, 0, or -1 at some t_0 are, by uniqueness, those for which $x(t) \equiv 1$, 0, or -1, respectively.

Finally, we mention that by taking $\lambda = 0$ in the x and y-components of (3.10) one recovers the analogous system for rotationally symmetric solutions to the steady soliton equation. The trajectories of (3.10) which lie in the plane $\{\omega = 0\}$ are therefore naturally associated with steady soliton structures (although, of course, the warping function $\omega(r)$ of these structures no longer corresponds directly to the ω-coordinate). In particular, the (one-dimensional) intersection of the unstable manifold of P_0 with the plane $\{\omega = 0\}$ contains two candidates for a smooth steady soliton on $\mathbb{R}^{n+1}$: one with negative sectional curvature near the origin, which turns out to be incomplete, and one with positive curvature near the origin, which is the well-known *Bryant soliton*– a complete steady soliton on $\mathbb{R}^{n+1}$ of positive curvature.

3.3 Proof of Theorem 3.1

In what follows, $\gamma(t) = (\omega(t), x(t), y(t))$ will represent a trajectory of (3.10) defined for t in what we may take to be a *maximal* interval (S, T) with $-\infty \leq S < T \leq \infty$. To reduce the clutter of our expressions, we shall usually suppress the dependence of the components of the trajectory on the parameter t.

3.3.1 Some invariant sets.

We begin our analysis of trajectories of (3.10) by observing that the second and third derivatives of the x-component have the following convenient expressions.

Lemma 3.9. *The x-component of any trajectory $\gamma(t)$ of (3.10) satisfies*

$$\frac{d^2x}{dt^2} = (n-1)x(x^2-1) + (3x-y)\frac{dx}{dt} \tag{3.12}$$

and

$$\frac{d^3x}{dt^3} = 2x\left[(2n-1)x - y\right]\frac{dx}{dt} + 2\left(\frac{dx}{dt}\right)^2 + (3x-y)\frac{d^2x}{dt^2}. \tag{3.13}$$

Proof. We compute

$$\begin{aligned}\frac{d^2x}{dt^2} &= (2x-y)\frac{dx}{dt} - x^2y + nx^3 - \lambda\omega^2 x \\ &= (n-1)x(x^2-1) + (3x-y)\frac{dx}{dt},\end{aligned}$$

and

$$\frac{d^3x}{dt^3} = \left[(n-1)(3x^2-1) + \frac{dx}{dt} - \frac{dy}{dt}\right]\frac{dx}{dt} + 2\left(\frac{dx}{dt}\right)^2 + (3x-y)\frac{d^2x}{dt^2}$$
$$= 2x\left[(2n-1)x - y\right]\frac{dx}{dt} + 2\left(\frac{dx}{dt}\right)^2 + (3x-y)\frac{d^2x}{dt^2}.$$

□

With the above expressions we may easily establish the following qualitative results on the behavior of trajectories of the system.

Lemma 3.10. *The regions*

$$\left\{x \geq 1\ ,\ \frac{dx}{dt} \geq 0\right\}\ ,\ \left\{x \leq -1\ ,\ \frac{dx}{dt} \leq 0\right\},\ \textit{and}\ \{y \leq 0\}$$

are preserved under system (3.10) for increasing t, and

$$\left\{x \geq 1\ ,\ \frac{dx}{dt} \leq 0\right\}\ ,\ \left\{x \leq -1\ ,\ \frac{dx}{dt} \geq 0\right\},\ \textit{and}\ \{y \geq 0\},$$

are preserved for decreasing t.

Proof. By (3.12), if $\frac{dx}{dt}(t_0) = 0$ at some t_0 with $x(t_0) > 1$, then $\frac{d^2x}{dt^2}(t_0) > 0$, and hence both $\frac{dx}{dt}$ and x continue to increase. By Remark 3.8, we cannot have $\frac{dx}{dt} = 0$ and $x = 1$ simultaneously unless $x \equiv 1$. The argument for the case $x \leq -1$ and $\frac{dx}{dt} \leq 0$ is similar.

To see that $\{y \leq 0\}$ is preserved, observe that

$$\frac{dy}{dt} = -nx^2 - \lambda\omega^2 < 0$$

whenever $y = 0$.

The preservation of the remaining sets for decreasing t follows by applying the results already obtained to the trajectory $\bar{\gamma}(t) = L(\gamma(\tau))$ constructed as in Remark (3.4). □

Lemma 3.11.

1. *If there exists $t_0 \in (S,T)$ at which $x(t_0) = 0$, $y(t_0) \leq 0$, and $\frac{dx}{dt}(t_0) > 0$, then $x(t)$ (and $\frac{dx}{dt}$) increase until $\gamma(t)$ enters the region $\{x > 1\}$.*

2. *If there exists* $t_0 \in (S,T)$ *at which* $x(t_0) = 0$, $y(t_0) \geq 0$, *and* $\frac{dx}{dt}(t_0) > 0$, *then* $x(t) < 0$ *and* $\frac{dx}{dt}(t) > 0$ *for all* $t < t_0$ *and there exists a* $t_1 \leq t_0$ *such that* $x(t) < -1$ *for all* $t < t_1$.

In particular, in view of Lemma 3.10, if the trajectory γ *enters the region* $\{x < 0\}$, *either it remains there or eventually lies in the region* $\{x > 1\}$.

Proof. In case (1), we have $y(t) < 0$ for all $t > t_0$ by Lemma 3.10, and $\frac{d^2x}{dt^2}(t_0) \geq 0$ by equation (3.12). Since by (3.13), $\frac{d^3x}{dt^3} > 0$ on the region

$$\left\{x \geq 0,\ \frac{dx}{dt} > 0,\ \frac{d^2x}{dt^2} \geq 0\right\},$$

we have $\frac{d^2x}{dt^2}(t) > 0$ for all $t > t_0$. Consequently, $x(t) > \frac{dx}{dt}(t_0)(t - t_0)$. As bounds on x imply bounds on the derivatives of ω and y, the solution cannot expire while $0 < x < 1$. Since the interval (S,T) is maximal, $x > 1$ eventually.

Case (2) then follows from case (1) by considering again the trajectory $\bar{\gamma}(t)$ constructed according to the device in Remark 3.4: If $x(t_0) = 0$, $y(t_0) \geq 0$ and $\frac{dx}{dt}(t_0) > 0$, then $\bar{x}(t_0) = 0$, $\bar{y}(t_0) \leq 0$ and $\frac{d\bar{x}}{dt}(t_0) > 0$, and the corresponding interval of definition $(\bar{S},\bar{T})$, satisfying $\tau(\bar{T}) = S$, $\tau(\bar{S}) = T$ will also be maximal.

□

3.3.2 Proof of the case $M^{n+1} \cong S^{n+1}$

Theorem 3.12. *Suppose* (S^{n+1}, g) *is a rotationally symmetric shrinking soliton. Then* g *has positive curvature operator.*

Proof. As we observed in Section 3.1, the positivity of the curvature operator is implied by that of the sectional curvatures which have the expressions

$$K_r = -\frac{1}{\omega^2}\frac{dx}{dt}, \text{ and } K_s = \frac{(1-x^2)}{\omega^2}$$

in terms of the (ω, x, y) coordinates.

Any trajectory $\gamma(t)$ of system (3.10) which corresponds to a smooth soliton structure on the sphere must tend to $P_0 = (0, 1, n)$ as $t \to -\infty$ and to $P_1 = (0, -1, -n)$ as $t \to \infty$. Thus, by Lemma 3.10, we must have $-1 < x(t) < 1$ (hence $K_s(t) > 0$) for all t and $\frac{dx}{dt} < 0$ at least initially. We wish to show $\frac{dx}{dt} < 0$ for all t.

By equation (3.12) of Lemma 3.9, $\frac{d^2x}{dt^2} < 0$ at critical points of x in the region $\{0 < x < 1\}$, so $\frac{dx}{dt}$ remains strictly negative on this region and, by Remark 3.8, cannot vanish on $\{x = 0\}$. Since $\gamma(t)$ must tend to P_1 as $t \to \infty$, it must, in particular, enter the region $\{x < 0\}$, and, in view of Lemma 3.11, remain there for all subsequent t. Since $\frac{d^2x}{dt^2} > 0$ at critical points of x in the region $\{-1 < x < 0\}$, $\frac{dx}{dt}$ must therefore remain strictly negative if x is to approach -1. So $\frac{dx}{dt} < 0$ always, and thus for any trajectory emanating from P_0 and tending to P_1 we have $K_r > 0$ and $K_s > 0$ for all t as claimed.

Taking limits, one finds that at the "poles" $r = A$ and $r = \Omega$, the sectional curvatures agree and are at least non-negative. One may therefore apply Lemma 8.2 of [39] to conclude that the curvature operator $\mathrm{Rm}(g) : \wedge^2 \to \wedge^2$ is of constant rank, and therefore strictly positive everywhere. $\square$

That g has constant sectional curvature is then a consequence of the following general result.

Theorem. (Böhm-Wilking [6], Theorem 1) On a compact manifold, the normalized Ricci flow evolves a Riemannian metric with 2-positive curvature operator to a limit metric with constant sectional curvature.

That $f \equiv \text{const}$ in case (1) of Proposition 3.1, then follows by substituting $K_r = K_s = \text{const}$ into (3.8) or, alternatively, by considering the identity

$$R + |\nabla f|^2 - 2\lambda f = \text{const}$$

valid on any gradient Ricci shrinking soliton (see, e.g., [22]). If f attains its maximum and minimum at the points x_M and x_m, respectively, then the identity implies $f(x_M) = f(x_m)$ since R is constant.

3.3.3 The asymptotic behavior of trajectories corresponding to complete, non-compact metrics

Hereafter, we shall consider solutions $\gamma(t)$ satisfying one or both of the conditions

$$\int_{t_0}^{T} \omega(\sigma)\, d\sigma = \infty = \lim_{t \to T} r(t) \qquad (3.14)$$

and

$$\int_S^{t_0} \omega(\sigma)\, d\sigma = \infty = -\lim_{t\to S} r(t) \tag{3.15}$$

for any $t_0 \in (S,T)$. Condition (3.14) is necessarily satisfied by any trajectory corresponding to a complete metric on $\mathbb{R}^{n+1}$ and both (3.14) and (3.15) are necessarily satisfied by any trajectory corresponding to a complete metric on $\mathbb{R} \times S^n$. As we shall see, these conditions impose rather stringent conditions on the asymptotic behavior of a trajectory.

We remark that if $\gamma(t)$ satisfies (3.14) then $\bar{\gamma}(t) = L(\gamma(\tau))$ satisfies (3.15) and vice-versa. Thus, from the following results, which apply to trajectories satisfying the "forwards" condition (3.14), we may easily obtain corresponding results for trajectories satisfying the "backwards" condition (3.15). These results will be collected in Lemma 3.17, below.

We begin with a simple consequence of the forwards extendability condition by which we may obtain eventual knowledge of the sign of the y-component.

Lemma 3.13. *Along any trajectory $\gamma(t)$, the quantity $Q = y/\omega$ is strictly decreasing. If $\gamma(t)$ satisfies (3.14), $\lim_{t\to T} Q = -\infty$. In particular, y eventually becomes negative.*

Proof. We compute

$$\frac{d}{dt}Q = -n\frac{x^2}{\omega} - \lambda\omega < 0.$$

Integrating, we find that, for any $S < t_0 < t < T$,

$$\frac{y}{\omega}(t) \leq \frac{y}{\omega}(t_0) - \lambda \int_{t_0}^{t} \omega(\sigma)d\sigma.$$

□

The following observation is also immediate.

Lemma 3.14. *If $\gamma(t)$ satisfies (3.14), then* $\limsup_{t\to T} x(t) \geq 0$.

Proof. If $x(t) < -\delta$ on some $(a,T) \subset (S,T)$, then $\omega(t) \leq Ce^{-\delta t}$ on the same interval, and $\gamma(t)$ cannot satisfy (3.14). □

Thus, in view of Lemma 3.11, no trajectory satisfying (3.14) can enter the region $\{x < -1\}$. That no trajectory satisfying (3.14) can enter the region $\{x > 1\}$ is true (as we prove next), but less obvious since $\omega(t) \to \infty$. Taken together, Lemmas 3.11, 3.14, and 3.15 prove that any complete metric on $\mathbb{R}^{n+1}$ satisfies $K_s \geq 0$.

The proof follows the lines of an argument due to Bryant and Ivey (c.f. [22]) demonstrating the incompleteness of a similar trajectory of the steady soliton system. In fact, if one regards the trajectories of (3.10) in the plane $\{\omega = 0\}$ as trajectories of the the analogous system for steady solitons (cf. the remarks at the end of Section 3.2.2), then Ivey's argument pertains to the trajectory in the plane emerging from P_0 in the direction opposite the Bryant soliton. The following lemma then may be viewed as an extension of his finding to the neighboring family of trajectories in the unstable manifold U_0 which populate the sector between the flat trajectory with $x \equiv 0$ and the plane $\{\omega = 0\}$. These trajectories correspond to metrics of strictly negative sectional curvature and are all incomplete. However, Ivey's argument does not carry over directly, as, in the expression for $\frac{dx}{dt}$ in the shrinking case, one has to contend with an additional term $(-\lambda\omega^2)$ of uncooperative sign.

Lemma 3.15. *Suppose $x(t_0) > 1$ and $\frac{dx}{dt}(t_0) > 0$ at some $t_0 \in (S, T)$. Then $\int_{t_0}^{T} \omega(\sigma)\, d\sigma < \infty$.*

Proof. We proceed by contradiction. Suppose $\gamma(t)$ satisfies (3.14). Then, by (3.12), $x(t) > 1$ and $\frac{dx}{dt}(t) > 0$ for $t > t_0$, and, by Lemma 3.13, there is a $t_1 \in (t_0, T)$ such that $y(t) < 0$ for all $t \geq t_1$.

Hence, by (3.12), we have

$$\frac{d^2x}{dt^2} \geq (n-1)x(x^2-1) + 3x\frac{dx}{dt} > \frac{3}{2}\frac{d(x^2)}{dt}. \tag{3.16}$$

Now, since the interval (S, T) is assumed maximal, and since bounds on x imply bounds on the derivatives of y and ω, if $T < \infty$ we must have $\limsup_{t\to T} |x(t)| = \lim_{t\to T} x = \infty$. On the other hand, $x(t)$ is uniformly convex by (3.16), so even if $T = \infty$ we still have $\lim_{t\to T} x(t) = \infty$. Returning to (3.16) with this fact in hand, we find

$$\frac{dx}{dt} \geq \frac{5}{4}x^2 + 1$$

for all t greater than some $t_2 \geq t_1$. (The coefficient $5/4$ is chosen for convenience and could be replaced by $3/2 - \epsilon$ for any $\epsilon > 0$ – below, we merely require it to be greater than one.) From this equation it follows that $T < \infty$ and

$$\frac{d}{dt} \arctan\left(\frac{\sqrt{5}}{2} x\right) \geq \frac{\sqrt{5}}{2},$$

which implies

$$\omega(t) \leq \frac{C_2}{(T-t)^{\frac{4}{5}}}$$

for some constant C_2, contradicting (3.14). □

Together, Lemmas 3.10 and 3.15 allow us to restrict our attention to trajectories which remain in the region $\{-1 < x < 1\}$, and, consequently, to those with infinite existence time $t \in (S, \infty)$ since a trajectory with x bounded cannot satisfy condition (3.14) on a interval bounded above. Along such trajectories, y becomes negative and Lemma 3.11 implies that eventually x acquires a constant sign. As a consequence, we obtain the following refinement of Lemma 3.13:

Lemma 3.16. *If $\gamma(t)$ is a trajectory of (3.10) defined on (S, ∞) satisfying condition (3.14) and $-1 < x(t) < 1$, then $\lim_{t\to\infty} y(t) = -\infty$.*

Proof. Suppose $\limsup_{t\to\infty} y(t) \geq -M$ for some $M > 1$, and choose $t_k \in (S, \infty)$ such that $t_k \nearrow \infty$ and $y(t_k) \geq -M$. Then, by Lemma 3.13, $\lim_{k\to\infty} \omega(t_k) = 0$. Since x eventually acquires a constant sign, and $\frac{d}{dt}\omega = x\omega$, we must have $x(t) \leq 0$ eventually, and $\omega(t)$ must tend to 0 outright as $t \to \infty$.

Now, equation (3.12) shows that $x(t)$ cannot attain a local maximum on $\{-1 < x \leq 0\}$ unless $x = 0$, and we know $x = 0$ and $\frac{dx}{dt} = 0$ simultaneously only if $x \equiv 0$, in which case $dy = -(n-1)dt$. Otherwise, x is eventually monotonic in t and either increases or decreases to a limit $\bar{x} \in [-1, 0]$. By the remarks preceding this lemma, we cannot have $\bar{x} < 0$ if the trajectory is to satisfy condition (3.14). So assume $\bar{x} = 0$, which implies $\frac{dx}{dt} > 0$ eventually. Then, for any ϵ, we can choose t_ϵ such that $t > t_\epsilon$ implies both $y(t) < 0$ and $nx^2 + \lambda\omega^2 < \epsilon$. For such t, we have

$$\frac{dy}{dt} = xy - nx^2 - \lambda\omega^2 > -\epsilon.$$

Fixing $\epsilon < \frac{n-1}{2}$, we find, each for k,

$$y(t) > -M - \epsilon(t - t_k)$$

and

$$\frac{dx}{dt}(t) = x^2 - xy + n - 1 - \lambda\omega^2 > \frac{n-1}{2} + x(t_k)(M + (\epsilon(t - t_k)) ,$$

(where, in obtaining the last inequality, we used that x is monotonically increasing).

Hence,

$$x(t) - x(t_k) > \left(\frac{n-1}{2} + Mx(t_k)\right)(t - t_k) + \epsilon\frac{x(t_k)}{2}(t - t_k)^2.$$

For $k >> 0$, $Mx(t_k) > -(n-1)/4$, so that the above (with the monotonicity of $x(t)$) implies that there exists $\delta = \delta(M, \epsilon, n) > 0$ and a subsequence $t_{k_j} \to \infty$ such that $x(t_{k_{j+1}}) > x(t_{k_j}) + \delta$ for all j. This contradicts that $x \nearrow 0$, and proves $\limsup_{t\to\infty} y(t) = -\infty$. □

3.3.4 Proof of the case $M^{n+1} \cong \mathbb{R}^{n+1}$

The results of the last section are enough to assemble the

Proof of Claim (2) of Theorem 3.1. Since the underlying manifold M^{n+1} is diffeomorphic to $\mathbb{R}^{n+1}$, the smooth extension of the metric to the origin $r = 0$ requires $S = -\infty$ and our solution $\gamma(t) = (\omega(t), y(t), \omega(t))$ of (3.10) to satisfy $\lim_{t\to-\infty} \gamma(t) = P_0 = (0, 1, n)$.

We claim first that if our trajectory is to satisfy condition (3.14), then $x \leq 1$. For if ever $x > 1$, since $\lim_{t\to-\infty} x(t) = 1$, we would have to have $\frac{dx}{dt}(t_0) > 0$ and $x(t_0) > 1$ at some earlier t_0. But then, by Lemmas 3.10 and 3.15, the x-component would blow-up too fast for $\gamma(t)$ to satisfy (3.14). So we must have $x \leq 1$ for all t.

Then, if ever $x = 1$, we must also have $\frac{dx}{dt} = 0$ at the same time, which, as pointed out in Remark 3.8, happens only if $x \equiv 1$–i.e., only if $\gamma(t)$ corresponds to the flat solution of Example 3.6. We claim that this is the only trajectory emanating from P_0 which satisfies (3.14).

We may now assume that $x(t) < 1$ on our trajectory and that for some t_0 (hence all $t < t_0$), $\frac{dx}{dt}(t_0) < 0$ and $0 < x(t_0) < 1$. Since, by equation (3.12), $\frac{d^2x}{dt^2}$ is strictly negative at all critical points of x in the region $0 < x < 1$, there are two possibilities for our trajectory: either

1. x decreases monotonically to a limit $\bar{x} \in [0, 1)$ as $t \to \infty$, or

2. $\gamma(t)$ enters the region $\{x \leq 0\}$ at some time $t = t_1$.

Knowing that $y \to -\infty$ as $t \to \infty$ (in fact, just knowing that eventually $y < 0$ suffices), we can dispose of case (1) by observing that while $x \in [0, 1]$,

$$\frac{d^2x}{dt^2} \leq (n-1)x(x^2-1) + \epsilon\frac{dx}{dt} < 0 \tag{3.17}$$

once $y < -\epsilon < 0$. Hence x eventually becomes negative.

Now, we also know from Lemma 3.14 that x cannot tend to a negative limit or become strictly less than -1 if condition (3.14) is to be satisfied. Since $\frac{d^2x}{dt^2}$ is strictly positive at critical points of x in the region $\{-1 < x < 0\}$, and since the only trajectories with critical points of x on the boundary of this region are classified in Examples 3.6 and 3.7 and neither emanate from P_0, we face only two alternatives:

(2a) either γ enters the region $x > 0$ again, or

(2b) $x \nearrow 0$ as $t \to \infty$.

Alternative (2a) is immediately excluded by Lemmas 3.11 and 3.15: no trajectory which emanates from P_0 can satisfy $x(t_0) = 0$, $\frac{dx}{dt}(t_0) > 0$, $y(t_0) \geq 0$, and no trajectory which satisfies $x(t_0) = 0$, $\frac{dx}{dt}(t_0) > 0$, $y(t_0) \leq 0$ can satisfy (3.14).

For (2b), we observe that since $y \to -\infty$ as $t \to \infty$, we have $3x - y > \epsilon > 0$ eventually, and thus we may obtain the analog of equation (3.17) for t sufficiently large

$$\frac{d^2x}{dt^2} = (n-1)x(x^2-1) + (3x-y)\frac{dx}{dt} > \epsilon\frac{dx}{dt} > 0, \tag{3.18}$$

which is incompatible with $x \nearrow 0$ as $t \to \infty$.

The trajectory $x \equiv 1$ is therefore the unique trajectory emanating from P_0 satisfying (3.14), and the proof of the case $M^{n+1} \approx \mathbb{R}^{n+1}$ is complete. □

3.3.5 Proof of the case $M^{n+1} \cong \mathbb{R} \times S^n$

As remarked earlier, the results in Section 3.3.3 regarding trajectories satisfying the forwards extendability condition (3.14) have natural analogs for trajectories satisfying the backwards version (3.14).

Lemma 3.17.

1. *Suppose* $\gamma(t)$ *satisfies* (3.15).

   (a) $\lim_{t\to S} Q(t) = \infty$ *and* y *is initially positive.*

   (b) $\liminf_{t\to S} x(t) \leq 0$

   (c) If $-1 < x(t) < 1$ *for all* t *(so* $-S = T = \infty$*), then* $\lim_{t\to -\infty} y(t) = \infty$.

2. *If* $x(t_0) < -1$ *and* $\frac{dx}{dt}(t_0) > 0$ *at some* $t_0 \in (S, T)$,

$$\int_S^{t_0} \omega(\sigma)\, d\sigma < \infty \tag{3.19}$$

Proof. Let $\bar{\gamma}(t) = L(\gamma(\tau(t)))$ be as in Remark 3.4.

$$\int_S^{t_0} \omega(\sigma)\, d\sigma = \int_{t_0}^{\bar{T}} \bar{\omega}(\sigma)\, d\sigma. \tag{3.20}$$

Thus, if $\gamma(t)$ satisfies (3.15), $\bar{\gamma}(t)$ satisfies (3.14) and the claims of part 1 follow by the application of Lemmas 3.13, 3.14, and 3.16 to $\bar{\gamma}(t)$.

For part 2, note that $\bar{x}(t_0) > 1$, $\frac{d\bar{x}}{dt}(t_0) > 0$ if $x(t_0) < -1$, $\frac{dx}{dt}(t_0) > 0$, so Lemma 3.15 and equation (3.20) yield the inequality (3.19). ☐

Now we turn to the remainder of the proof of Theorem 3.1.

Proof of Claim 3 of Theorem 3.1. We shall show that the only trajectory $\gamma(t)$ satisfying both (3.14) and (3.15) is that of Example 3.7 with $x \equiv 0$ and $\omega \equiv \sqrt{(n-1)/\lambda}$.

By Lemmas 3.10 and 3.15 and case (2) of Lemma 3.17, we can assume that $-1 \leq x(t) \leq 1$ and consequently also $S = -\infty$, $T = \infty$. In fact, since neither of the trajectories with $x \equiv \pm 1$ can satisfy both conditions (3.14) and (3.15), we may assume $-1 < x(t) < 1$.

By Lemma 3.14 and part (1b) of Lemma 3.17, we know $\liminf_{t\to-\infty} x(t) \leq 0$ and $\limsup_{t\to\infty} x(t) \geq 0$. Thus since $\frac{d^2x}{dt^2}$ is strictly positive at critical points of x in the region $\{-1 < x < 0\}$ and strictly negative at these critical points in the region $\{0 < x < 1\}$, either $\gamma(t)$ crosses the plane $\{x = 0\}$ at some time or its x-component maintains a constant sign and tends monotonically to 0 as $t \to \infty$ or $t \to -\infty$. We claim that this latter option cannot occur. For, in light of the remarks in the preceding paragraph, the only scenarios of this option in which $\gamma(t)$ could potentially satisfy the extendability criteria would be $x < 0$ and $x \nearrow 0$ as $t \to \infty$, or $x > 0$ and $x \searrow 0$ as $t \to -\infty$. But the case $x < 0$, $x \nearrow 0$ was eliminated in the argument for Claim (2), and the case $x \searrow 0$ as $t \to -\infty$ reduces to the previous one by the consideration of the trajectory $\bar{\gamma}(t)$.

Thus, we conclude that there must exist a t_0 such that $x(t_0) = 0$. If $\frac{dx}{dt}(t_0) \neq 0$, then we may assume $\frac{dx}{dt}(t_0) > 0$, as the argument given in the case $M^{n+1} \approx \mathbb{R}^{n+1}$ implies that the only trajectories satisfying (3.14) and $\frac{dx}{dt}(t_0) < 0$ initially lie in the region $\{x < -1\}$ and cannot thus satisfy (3.15). However, if $\frac{dx}{dt}(t_0) > 0$, then Lemma 3.11 implies that either again $\gamma(t)$ lies initially in the region $\{x < -1\}$ or eventually in the region $\{x > 1\}$, in which case, by Lemma 3.15 and part (2) of Lemma 3.17, $\gamma(t)$ can satisfy at most one of the conditions (3.14) and (3.15).

Thus we can only have $\frac{dx}{dt}(t_0) = 0$, which implies that γ coincides with the trajectory $x \equiv 0$, $\omega \equiv \sqrt{(n-1)/\lambda}$ as claimed. □

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4 Complete, positively-curved gradient expanding solitons in two dimensions

In this chapter we will again consider the gradient soliton equation (3.1) on a Riemann surface, though it will be convenient to use the following equivalent form, in which we have reversed the sign convention to favor expanding solitons.

$$(K[g] + \lambda)g = \nabla\nabla f. \tag{4.1}$$

Here $K[g]$ denotes the Gauss curvature of g. According to this convention, the soliton is said to be *expanding*, *steady*, or *shrinking*, depending as the constant λ is positive, zero, or negative.

In dimension two, it is a simple observation of Hamilton [45] that any gradient soliton must be rotationally symmetric. Indeed, if J is the complex structure, it is not hard to verify that $J(\nabla f)$ is a Killing vector field for the metric.

On a closed M^2, Theorem 10.1 in [40] and Theorem 2 in [12] together imply that any soliton must have constant curvature. On a complete, non-compact M^2, it is known (cf. Lemma 2.7 in [22]), that a steady soliton of positive curvature must be homothetic to Hamilton's cigar soliton. The purpose of this note is to provide a simple proof of the corresponding result in the case of complete expanding solitons: in [22], and [37], the authors describe an expanding, positively curved solution to (4.1) on $\mathbb{R}^2$; here we confirm that it is essentially unique up to homothety. Our proof is elementary, and a synthesis of components either standard or extant elsewhere in the literature.

Theorem 4.1. *Suppose (M^2, g) is a complete, non-compact Riemannian surface with $K[g] > 0$, which satisfies (4.1) with $\lambda = 1$ for some f. Then there is a constant $\alpha > 0$ such that*

1. *(M^2, g) is isometric to $(\mathbb{R}^2, \phi(r)^2 dr^2 + r^2 d\theta^2)$ with*

$$\phi(r) = \frac{\alpha}{1 + W\left(\left(\frac{\alpha-\zeta}{\zeta}\right)\exp\left(\left(\frac{\alpha-\zeta}{\zeta}\right) - \frac{\alpha^2}{2}r^2\right)\right)}, \tag{4.2}$$

where $W : (0, \infty) \to (0, \infty)$ is the product-log function (the inverse of xe^x) and $\zeta = \lim_{r\to 0} \phi(r) \in (0, \alpha)$;

2. *f is a radial function satisfying*

$$\nabla f(r) = \alpha \frac{r}{\phi(r)} \frac{\partial}{\partial r}. \tag{4.3}$$

First we observe that the satisfaction of (4.1) with the assumption of positive curvature implies that f is a uniformly convex function. This is a strong condition on f, as the following lemma demonstrates.

Lemma 4.2. *Suppose f is a smooth function on a complete Riemannian manifold (M^n, g) satisfying $\nabla\nabla f \geq c > 0$. Then f has a unique critical point.*

Proof. Fix any $x \in M^n$ and for any $y \neq x$, let $\gamma_y : [0, \infty) \to M^n$ be the extension of a minimal geodesic joining x to y, parametrized by arclength. Then, by Taylor's theorem and our assumption on f,

$$f(y) \geq f(x) + \langle \nabla f(x), \dot{\gamma}_y(0) \rangle_{g(x)} d(x, y) + c\frac{d(x, y)^2}{2} \tag{4.4}$$

$$\geq f(x) - |\nabla f(x)|_{g(x)} d(x, y) + c\frac{d(x, y)^2}{2} \tag{4.5}$$

Thus there exists an $R > 0$ such that $f > f(x)$ on $B_x(R)^c$, which implies f is bounded below and attains its infimum at some $x_0 \in M^n$.

If x_1 is any other critical point of M^n, performing the above expansion about x_1 yields

$$f(x_0) \geq f(x_1) + c\frac{d(x_1, x_0)^2}{2},$$

a contradiction. Hence f has exactly one critical point. ∎

By equation (4.1), f is a conformal vector field. Therefore, by a standard argument (cf. the editors' footnote on pp. 241-2 in [10]), if J is the almost complex structure on TM^2 defined by 90° counter clockwise rotation, $J(\nabla f)$ is a Killing vector field. By the above Lemma, $J(\nabla f)$ vanishes at a unique $x_0 \in M^2$, and hence it follows from Lemma 1 of [12] that (M^2, g) is rotationally symmetric and topologically $\mathbb{R}^2$. We therefore may assume that

1. (M^2, g) is isometric to $(\mathbb{R}^2, ds^2 + \varphi^2(s)d\theta^2)$ for some positive function φ, and

2. $f = f(s)$ is a radial function.

Our assumption of positive curvature also implies the existence of the following useful coordinate system.

Lemma 4.3. *Suppose $g = ds^2 + \varphi(s)^2 d\theta^2$ is a complete, positively curved rotationally symmetric metric on $\mathbb{R}^2$. Then there exist coordinates (r, θ) on $\mathbb{R}^2 - \{\text{ray}\}$ in which g has the representation $g = \phi(r)^2 dr^2 + r^2 d\theta^2$ for some smooth positive ϕ.*

Proof. A routine calculation shows $K = -\varphi''(s)/\varphi(s)$, thus positive curvature implies $\varphi''(s) < 0$. In order for g to extend smoothly to a metric on all of $\mathbb{R}^2$, we must have $\lim_{s\to 0} \varphi'(s) = 1$. So $\varphi'(s) > 0$ for small s. We claim that, in fact, $\varphi'(s) > 0$ for all s. For if $\varphi'(s_0) = 0$ then the concavity of φ implies $\varphi'(s_0 + \epsilon) = -\delta < 0$ and hence $\varphi'(s) < -\delta$ for all $s > s_0 + \epsilon$. But then $\varphi(s) < \varphi(s_0 + \epsilon) - \delta s$ for $s > s_0 + \epsilon$ which, for large s, contradicts $\varphi(s) > 0$. Thus φ is an increasing, hence invertible, function of s. Setting $r = r(s) = \varphi(s)$, and $\phi(r) = 1/\varphi'(s(r))$, we obtain $g = \phi(r)^2 dr^2 + r^2 d\theta^2$ as desired. □

4.1 Proof of Theorem 4.1

Now we assemble the above results to verify the asserted uniqueness of the example in Theorem 4.1. The actual manipulation of the ODE presented below is essentially the same as in [22]; we include the details for the benefit of the reader.

Working in the above coordinates with the representation

$g = \phi(r)^2 dr^2 + r^2 d\theta^2$, we compute

$$K(r) = \frac{\phi'(r)}{r\phi(r)^3}, \tag{4.6}$$

and

$$\nabla\nabla f(r)\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = f''(r) - f'(r)\frac{\phi'(r)}{\phi(r)} \tag{4.7}$$

$$\nabla\nabla f(r)\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = f'(r)\frac{r}{\phi(r)^2}. \tag{4.8}$$

The soliton equation (4.1) with $c = 1$ then implies, together with equations (4.6-4.8),

$$\frac{\phi'}{r\phi} + \phi^2 = f'' - f'\frac{\phi'}{\phi}, \tag{4.9}$$

$$\frac{\phi' r}{\phi^3} + r^2 = f'\frac{r}{\phi^2}. \tag{4.10}$$

Multiplying (4.10) by ϕ^2/r^2 and combining the result with (4.9), we obtain

$$f'' = f'\left(\frac{\phi'}{\phi} + \frac{1}{r}\right),$$

or

$$f'(r) = \alpha\phi(r)r \tag{4.11}$$

for some α. Since ϕ is positive and f is convex, we must have $\alpha > 0$. This verifies equation (4.3).

Substituting (4.11) into (4.10) and simplifying, we obtain the separable equation

$$\phi' = \phi^2(\alpha - \phi)r$$

which has the solution

$$-\frac{r^2}{2} + M = \frac{1}{\alpha^2}\left(\frac{\alpha - \phi}{\phi}\log\left(\frac{\alpha - \phi}{\phi}\right)\right)$$

for some constant M, or, applying the product-log function W to both sides and solving for ϕ,

$$\phi = \frac{\alpha}{1 + W\left(\exp\left(M\alpha^2 - \frac{\alpha^2}{2}r^2\right)\right)}. \tag{4.12}$$

Setting $r = 0$ and $\zeta = \phi(0)$, we find

$$M = \frac{1}{\alpha^2} \log \left(\left(\frac{\alpha - \zeta}{\zeta} \right) \exp \left(\frac{\alpha - \zeta}{\zeta} \right) \right).$$

and equation (4.2) of Theorem 4.1 follows.

Finally observe that using (4.10) and (4.11), we obtain

$$\frac{\alpha - \phi(r)}{\phi(r)} = K(r) > 0.$$

Thus $0 < \phi(r) < \alpha$ for all $r > 0$. Sending $r \to 0$ and using that $K(0)$ is positive and well-defined, we conclude $\zeta \in (0, \alpha)$ as well.

5 A unique-continuation theorem with applications to the Ricci flow

5.1 Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^n$, it is well-known that the initial value problem

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = 0, \quad x \in \Omega, \ t \in (0,T] \\ u(x,0) = u_0, \\ u\,|_{\partial\Omega}(x,t) = 0 \end{cases} \tag{5.1}$$

admits at most one solution $u(x,t) \in C^2(\Omega \times (0,T]) \cap C(\overline{\Omega} \times [0,T])$. This, a consequence of the maximum principle, is precisely the property of uniqueness: two solutions to (5.1) which agree at time $t = 0$ must agree for as long as the solution exists. A natural question to ask is whether a similar property of uniqueness holds backwards in time, that is, if u and $\tilde{u}$ solve (5.1) on Ω and are solutions to (5.1) on Ω, and if $u(\cdot, T) = \tilde{u}(\cdot, T)$, does it follow that $u(x,t) = \tilde{u}(x,t)$ for all $(x,t) \in \overline{\Omega} \times [0,T]$? The answer to this question in this case is known classically to be yes (see, e.g., [54], [1]), and the assertion is known as a *backwards-uniqueness* or *unique continuation* property for the equation.

Note that the question is somewhat more delicate than its forwards counterpart. First, the corresponding "terminal value problem" is ill-posed and in general cannot be solved. Second, in fact one expects that the backwards uniqueness should be “approximately” false. Indeed, one expects that two temperature distributions with the same average temperature initially should approach their common average, and hence each other, as time increases. The assertion, then, is precisely that diffusion alone cannot accomplish this task in only a finite amount of time. Nevertheless, the property holds for a wide variety of linear parabolic equations and in more general contexts.

As the Ricci flow is in some sense the analog of the heat equation for the class of Riemannian metrics, one may ask if it too enjoys a unique-continuation property. At the moment this question is still open; any proof will need to overcome the obstacles posed by the non-linearity and only weak-parabolicity of the equation. However, some sort of unique-continuation is strongly conjectured to hold, and we present some results in this direction in the remainder of this chapter.

5.2 Overview and statement of results

Our objectives are to prove the following three theorems.

Theorem 5.1 (Unique continuation for the Kähler-Ricci flow). *Suppose $h_{\alpha\bar{\beta}}$ is a Kähler metric on the compact manifold M^{2n} whose Kähler form ω_h is a real multiple of the Chern class $[\rho]$, i.e., so that*

$$[\omega_h] = \frac{r_h}{n}[\rho] \quad \textit{where} \quad r_h := \frac{\int_{M^{2n}} R_h d\mu_h}{\int_{M^{2n}} d\mu_h}.$$

Then if $g_{\alpha\bar{\beta}}(x,t)$ and $\widetilde{g}_{\alpha\bar{\beta}}(x,t)$ are two solutions to the normalized Kähler-Ricci flow

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial t} = -R_{\alpha\bar{\beta}} + \frac{r_g}{n} g_{\alpha\bar{\beta}}.$$

on $M^{2n} \times [0,T]$ with

$$g_{\alpha\bar{\beta}}(\cdot,T) = h_{\alpha\bar{\beta}} = \widetilde{g}_{\alpha\bar{\beta}}(\cdot,T)$$

Then $g_{\alpha\bar{\beta}}(\cdot,t) = \widetilde{g}_{\alpha\bar{\beta}}(\cdot,t)$ agree identically on $[0,T]$.

Theorem 5.2 (No convergence to Einstein in finite time). *Suppose $g(t)$ is a solution to the Ricci flow on a compact manifold for $t \in (A,\Omega) := \mathcal{I}$ that satisfies the Einstein condition $\mathrm{Rc}[g(t_0)] = \rho g(t_0)$ for some ρ at some $t_0 \in \mathcal{I}$. Then (extending*

g if $\mathcal{I}$ is not maximal), g is Einstein for all $t \in (-\infty, t_0 + 1/(2\rho))$, $(t_0 - 1/(2\rho), \infty)$, or $(-\infty, \infty)$, depending as ρ is positive, negative, or zero. Explicitly, we have

$$g(t) = (1 - 2\rho(t - t_0))g(t_0) \quad \textit{and} \quad \mathrm{Rc}(t) = \frac{\rho}{1 - 2\rho(t - t_0)} g(t).$$

Theorem 5.3 (Unique continuation for Ricci-DeTurck flow)**.** *Suppose $\bar{g}$ is a fixed Riemannian metric and $g(t)$ and $\widetilde{g}(t)$ are two solutions to the Ricci DeTurck flow* (1.3) *with respect to $\bar{g}$ on a compact manifold M^n for t in some common interval $[0, T]$ If $g(T) = \widetilde{g}(T)$, then $g(t) = \widetilde{g}(t)$ for all $t \in [0, T]$.*

Theorems 5.1 and 5.2 answer questions posed by Chow in [21]. Given the utility of the Ricci-DeTurck flow to the questions of existence and forwards-uniqueness, it is our hope that perhaps Theorem 5.3 may have application in an eventual proof of the backwards uniqueness of the Ricci flow.

The heart of this chapter and the key to the proofs of Theorems 5.1, 5.2, and 5.3, is the following general result which may itself be of some interest. We present its proof – a modification of an argument of Lees and Protter [54]– in the next section.

Theorem 5.4. *Let $\{g(t)\}_{t\in[0,T]}$ be a smooth family of Riemannian metrics on a compact manifold M^n and suppose that $X \in C^\infty(T_l^k(M^n) \times [0, T])$ satisfies*

$$\int_{M^n} |LX|^2 (\cdot, t)\, d\mu \leq c_1 \int_{M^n} |X|^2 (\cdot, t)\, d\mu + c_2 \int_{M^n} |\nabla X|^2 (\cdot, t)\, d\mu \tag{5.2}$$

for all $t \in [0, T]$ where $L = \Lambda^{ij}\nabla_i\nabla_j - \frac{\partial}{\partial t}$, Λ is a smooth family of symmetric, positive definite $(0, 2)$-tensors, and c_1, c_2 are positive constants. (Here $|\cdot| := |\cdot|_{g(t)}$ denotes the induced metric on T_l^k, and $\nabla := \nabla_{g(t)}$ and $d\mu := d\mu_{g(t)}$, respectively, the Levi-Civita connection and volume form associated to $g(t)$). Then, if $X(\cdot, T) \equiv 0$, $X(\cdot, t) \equiv 0$ for all $t \in [0, T]$.

5.3 A unique-continuation theorem for evolving tensor fields

In [54], Lees and Protter prove a unique-continuation theorem for C^2 solutions to heat-type operators on bounded domains in $\mathbb{R}^n$. Here we adapt their proof to fit the case of tensor equations on closed manifolds with potentially evolving metrics, connections, and measures; our argument follows theirs rather closely, with some straight-forward modifications to handle the extra terms generated by the extra time-dependent quantities. We present a proof in the case that X is a $(2,0)$-tensor; the other cases are proved analogously.

5.3.1 Notation

We begin by establishing some notation. For $0 \leq \tau_1 < \tau_2 \leq T$, let $\mathcal{V}^{\tau_1,\tau_2}$ be the class of smooth families $V(t)$ of sections of $T^2(M^n)$ for $t \in [\tau_1, \tau_2]$, and define

$$\mathcal{V}_0^{\tau_1,\tau_2} = \{\, V \in \mathcal{V}^{\tau_1,\tau_2} \;:\; V(\cdot,\tau_1) \equiv 0 \equiv V(\cdot,\tau_2)\,\}.$$

Let Λ be as in Theorem 5.4 and write $\Box := \Lambda^{ij}\nabla_i\nabla_j$ and $L = \Box - \frac{\partial}{\partial t}$. Choose a_1, $a_2 > 0$ so that

$$a_1 \left|\xi\right|^2_{g(x,t)} \leq \Lambda^{ij}(x,t)\xi_i\xi_j \leq a_2 \left|\xi\right|^2_{g(x,t)} \tag{5.3}$$

for all $\xi \in T_xM^n$ and all $(x,t) \in M^n \times [0,T]$. Write $\frac{\partial}{\partial t}g_{ij} = b_{ij}$ and put $B = g^{ij}b_{ij}$ so that the volume form associated to the metric $g(t)$ satisfies

$$\frac{\partial}{\partial t}d\mu_{g(t)} = \frac{B}{2}d\mu_{g(t)}.$$

In the computations that follow, it will be convenient to make the mild abuse of notation of using $b(\cdot,\cdot)$ also to denote the time derivative of the metric on $T_2(M^n)$ induced by $g(t)$, i.e., to mean

$$b(X,Y) = -(g^{ic}g^{kd}g^{jl} + g^{ik}g^{jc}g^{ld})b_{cd}X_{ij}Y_{kl}$$

according to our original definition. For V, $W \in \mathcal{V}^{[\tau_1,\tau_2]}$ we denote the L_2-pairing by

$$\langle\langle V, W\rangle\rangle := \int_{\tau_1}^{\tau_2}\int_{M^n}\langle V, W\rangle\; d\mu_{g(t)}dt$$

where, as before,

$$\langle V, W\rangle := \langle V, W\rangle_{g(t)} = g^{ik}g^{jl}V_{ij}W_{kl},$$

and will write $\|\cdot\|$ for the norm associated to this pairing. (We use similar notation to describe the L_2-norm of smooth families of tensors of different rank as well.)

Since we assume $g(t)$ and $\Lambda(t)$ to be smooth families on $[0,T]$, and M^n is compact, there exist positive constants P, Q such that

$$\sup_{0\le t\le T}\left\{|b|^2_{g(t)}+|\nabla b|^2_{g(t)}\right\}\le P$$

and

$$\sup_{0\le t\le T}\left\{|\Lambda|^2_{g(t)}+\left|\frac{\partial\Lambda}{\partial t}\right|^2_{g(t)}+|\nabla\Lambda|^2_{g(t)}+|\nabla\nabla\Lambda|^2_{g(t)}\right\}\le Q.$$

Finally, define $\tau=\tau_2-\tau_1$ and

$$\lambda(t):=\lambda_\eta(t):=t-\tau-\eta$$

for $\eta>0$.

5.3.2 The set $\mathcal{V}_0^{[\tau_1,\tau_2]}$

We want to determine some properties implied by membership in $\mathcal{V}_0^{[\tau_1,\tau_2]}$.

Lemma 5.5. *For any $V\in\mathcal{V}_0^{[\tau_1,\tau_2]}$ and any positive integer m, there exist positive constants A_1, A_2, A_3 depending on P and Q such that*

$$\|\lambda^{-m}LV\|^2\ge mk_m(\tau,\eta)\|\lambda^{-m-1}V\|^2-A_1\|\nabla(\lambda^{-m}V)\|^2. \tag{5.4}$$

where

$$k_m(\tau,\eta):=\frac{1}{2}\left(1-\frac{A_2}{m}(\tau+\eta)-\frac{A_3}{m}(\tau+\eta)^2\right).$$

Proof. If V belongs to $\mathcal{V}_0^{\tau_1,\tau_2}$, so does $Z:=\lambda^{-m}V$, and

$$\lambda^{-m}LV=\Box Z-\frac{\partial Z}{\partial t}-m\lambda^{-1}Z.$$

So

$$\begin{aligned}\left\|\lambda^{-m}LV\right\|^2=&\left\|\frac{\partial Z}{\partial t}\right\|^2+2m\left\langle\left\langle\frac{Z}{\lambda},\frac{\partial Z}{\partial t}\right\rangle\right\rangle\\&-2\left\langle\left\langle\frac{\partial z}{\partial t},\Box Z\right\rangle\right\rangle+\left\|\Box Z-m\frac{Z}{\lambda}\right\|^2.\end{aligned} \tag{5.5}$$

We now proceed to bound the two potentially negative terms on the right-hand side of (5.5) from below. Beginning with

$$\frac{\partial}{\partial t}\left[\lambda^{-1}|Z|^2\,d\mu\right]=\left[\lambda^{-1}\left\langle\frac{\partial Z}{\partial t},Z\right\rangle+\lambda^{-1}b(Z,Z)+\lambda^{-1}\frac{B}{2}|Z|^2-\left|\lambda^{-1}Z\right|^2\right]d\mu,$$

and recalling that $Z \in \mathcal{V}_0^{\tau_1,\tau_1}$, and $|\lambda(t)| \leq \tau + \eta$, we may integrate to find

$$\begin{aligned} 2m \left\langle \left\langle \frac{Z}{\lambda}, \frac{\partial Z}{\partial t} \right\rangle \right\rangle &= m \int_{\tau_1}^{\tau_2} \int_{M^n} \left[\left|\lambda^{-1} Z\right|^2 - \lambda^{-1} \left(b(Z,Z) + \frac{B}{2} |Z|^2 \right) \right] d\mu \, dt \\ &\geq m \left(1 - A'(\tau + \eta)\right) \left\|\lambda^{-1} Z\right\|^2, \end{aligned} \tag{5.6}$$

for some positive constant $A' = A'(P, n)$.

For the third term in (5.5), we first compute

$$\begin{aligned} &\nabla_a \left(g^{ik} g^{jl} \frac{\partial Z_{ij}}{\partial t} \Lambda^{ab} \nabla_b Z_{kl} \right) \\ &\qquad = g^{ij} g^{kl} \left(\nabla_a \frac{\partial Z_{ij}}{\partial t} \Lambda^{ab} \nabla_b Z_{kl} + \frac{\partial Z_{ij}}{\partial t} \nabla_a \Lambda^{ab} \nabla_b Z_{kl} + \frac{\partial Z_{ij}}{\partial t} \Box Z_{kl} \right), \end{aligned}$$

which, after an integration over M^n, yields

$$\begin{aligned} -2 \int_{M^n} \left\langle \frac{\partial Z}{\partial t}, \Box Z \right\rangle d\mu &= \int_{M^n} 2 g^{ik} g^{jl} \left[\nabla_a \frac{\partial Z_{ij}}{\partial t} \Lambda^{ab} \nabla_b Z_{kl} + \frac{\partial Z_{ij}}{\partial t} \nabla_a \Lambda^{ab} \nabla_b Z_{kl} \right] d\mu \\ &= 2 \int_{M^n} g^{ik} g^{jl} \left[\frac{\partial}{\partial t} \left(\nabla_a Z_{ij} \right) \Lambda^{ab} \nabla_b Z_{kl} + \frac{\partial Z_{ij}}{\partial t} \nabla_a \Lambda^{ab} \nabla_b Z_{kl} \right. \\ &\qquad \left. + \left(\nabla_a \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \nabla_a \right) Z_{ij} \Lambda^{ab} \nabla_b Z_{kl} \right] d\mu \\ = \frac{d}{dt} \int_{M^n} & \Lambda^{ab} \langle \nabla_a Z, \nabla_b Z \rangle \, d\mu \\ &+ \int_{M^n} \left[2 g^{ik} g^{jl} \left(\left(\nabla_a \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \nabla_a \right) Z_{ij} \Lambda^{ab} \nabla_b Z_{kl} + \frac{\partial Z_{ij}}{\partial t} \nabla_a \Lambda^{ab} \nabla_b Z_{kl} \right) \right. \\ &\qquad \left. - \Lambda^{ab} b(\nabla_a Z, \nabla_b Z) - \left(\frac{B}{2} \Lambda^{ab} + \frac{\partial \Lambda^{ab}}{\partial t} \right) \langle \nabla_a Z, \nabla_b Z \rangle \right] d\mu. \end{aligned} \tag{5.7}$$

Now,

$$\left(\nabla_a \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \nabla_a \right) Z_{ij} = \left(\frac{\partial}{\partial t} \Gamma_{ai}^p \right) Z_{pj} + \left(\frac{\partial}{\partial t} \Gamma_{aj}^p \right) Z_{ip}$$

and

$$\left| \frac{\partial}{\partial t} \Gamma_{ij}^k \right| \leq 3 \left| \nabla_i b_{jk} \right|$$

so there is a constant $A'' = A''(P, Q, n)$ such that

$$2 g^{ik} g^{jl} \left(\nabla_a \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \nabla_a \right) Z_{ij} \Lambda^{ab} \nabla_b Z_{kl} \geq -A'' \left(|Z|^2 + |\nabla Z|^2 \right)$$

and

$$-\Lambda^{ab}b(\nabla_a Z, \nabla_b Z) - \left(\frac{B}{2}\Lambda^{ab} + \frac{\partial \Lambda^{ab}}{\partial t}\right)\langle \nabla_a Z, \nabla_b Z\rangle \geq -A'' \left|\nabla Z\right|^2.$$

Likewise, there is a constant $A''' = A'''(Q, n)$ such that for all $\delta > 0$

$$2\nabla_a \Lambda^{ab}\left\langle \frac{\partial Z}{\partial t}, \nabla_b Z\right\rangle \geq -A'''\left(\delta\left|\frac{\partial Z}{\partial t}\right|^2 + \frac{1}{\delta}\left|\nabla Z\right|^2\right).$$

With these estimates, integrating (5.7) over the interval $[\tau_1, \tau_2]$ yields

$$-2\left\langle\left\langle \frac{\partial Z}{\partial t}, \Box Z\right\rangle\right\rangle \geq -A''\|Z\|^2 - \left(2A'' + \frac{A'''}{\delta}\right)\|\nabla Z\|^2 - \delta A'''\left\|\frac{\partial Z}{\partial t}\right\|^2. \quad (5.8)$$

Hence (5.5), using (5.6) and (5.8), becomes

$$\|\lambda^{-m}LV\|^2 \geq (1-\delta_1 A''')\left\|\frac{\partial Z}{\partial t}\right\|^2 - \left(2A'' + \frac{A'''}{\delta}\right)\|\nabla Z\|^2$$
$$- A''\left\|Z\right\|^2 + m\left(1 - A'(\tau+\eta)\right)\left\|\lambda^{-1}Z\right\|^2.$$

Choosing $\delta = \delta(P, Q, n) = (A''')^{-1}$, taking $A_1 := 2A'' + (A''')^2$, $A_2 := A'$, $A_3 := A''$, and using $\left\|Z\right\|^2 \leq \left\|\lambda^{-m-1}V\right\|^2(\tau+\eta)^2$, we arrive at (5.4). □

Lemma 5.6. *There exist positive constants η_0, μ_0, m_0, depending only on n, P, Q, and a_1 such that if $0 \leq \tau_1 < \tau_2 \leq T$, $\tau := \tau_2 - \tau_1 < \mu$, $V \in \mathcal{V}_0^{\tau_1,\tau_2}$, $m \geq m_0$, $\mu \leq \mu_0$ and $0 < \eta \leq \eta_0$, then*

$$\rho_m\left\|\lambda^{-m}\right\|^2 \geq \left\|\lambda^{-m-1}V\right\|^2 + \frac{1}{2}\left\|\lambda^{-m}V\right\|^2 \quad (5.9)$$

where

$$\rho_m := \rho_m(\mu, \eta) := \frac{2}{a_1}\left[\frac{1 + a_1 + A_4(\mu+\eta)^2}{m} + A_5(\mu+\eta) + A_6(\mu+\eta)^2\right] \quad (5.10)$$

for some constants A_4, A_5, and A_6 depending on P, Q, and n.

Proof. As in [54], we begin with the identity

$$-\langle\langle \lambda^{-m-1}V, \lambda^{-m+1}LV\rangle\rangle = \left\langle\left\langle \lambda^{-2m}V, \frac{\partial V}{\partial t}\right\rangle\right\rangle - \langle\langle \lambda^{-2m}V, \Box V\rangle\rangle. \quad (5.11)$$

Now,

$$\frac{\partial}{\partial t}\left[\left\langle \lambda^{-2m}V, V\right\rangle\, d\mu\right]$$
$$= \left[2\lambda^{-2m}\left\langle \frac{\partial V}{\partial t}, V\right\rangle - 2m\lambda^{-2m-1}\left|V\right|^2 + \lambda^{-2m}b(V,V) + \lambda^{-2m}\frac{B}{2}\left|V\right|^2\right]\, d\mu,$$

which, upon integration over space and time, becomes

$$\begin{aligned}-\left\langle\left\langle \lambda^{-2m}V, \frac{\partial V}{\partial t}\right\rangle\right\rangle =& \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_{M^n}\lambda^{-2m}\left[\frac{B}{2}\left|\nabla V\right| + b(V,V)\right]\, d\mu\, dt \\ &- m\left\langle\left\langle \lambda^{-2m-1}V, V\right\rangle\right\rangle \\ &\leq \left(m(\tau+\eta) + A'(\tau+\eta)^2\right)\left\|\lambda^{-m-1}V\right\|^2 \qquad (5.12)\end{aligned}$$

for some $A' = A'(P,n) > 0$. Also,

$$\nabla_a\left[\Lambda^{ab}\left\langle \lambda^{-2m}V, \nabla_b V\right\rangle\right]$$
$$= \left\langle \lambda^{-2m}V, \Box V\right\rangle + \lambda^{-2m}\nabla_a\Lambda^{ab}\left\langle V, \nabla_b V\right\rangle + \lambda^{-2m}\Lambda^{ab}\left\langle \nabla_a V, \nabla_b V\right\rangle. \qquad (5.13)$$

There is a constant A'' depending on Q and n such that for all $\delta > 0$, we have the estimate

$$\lambda^{-2m}\nabla_a\Lambda^{ab}\left\langle V, \nabla_b V\right\rangle \leq A''\left(\frac{1}{\delta}\left|\lambda^{-m}V\right|^2 + \delta\left|\nabla(\lambda^{-m}V)\right|^2\right),$$

so, integrating and using (5.3), equation (5.13) becomes

$$\begin{aligned}-\left\langle\left\langle \lambda^{-2m}V, \Box V\right\rangle\right\rangle &\geq a_1\left\|\nabla V\right\|^2 + \int_{\tau_1}^{\tau_2}\int_{M^n}\lambda^{-2m}\nabla_a\Lambda^{ab}\left\langle V, \nabla_b V\right\rangle\, d\mu \\ &\geq -\frac{A''}{\delta}\left\|\lambda^{-1}V\right\|^2 + (a_1 - \delta A'')\left\|\nabla(\lambda^{-m}V)\right\|^2\end{aligned}$$

Now choose $\delta = a_1/(2A'')$, and put $A''' = A''/\delta$, to obtain

$$-\left\langle\left\langle \lambda^{-2m}V, \Box V\right\rangle\right\rangle \geq -A'''(\tau+\eta)^2\left\|\lambda^{-m}V\right\|^2 + \frac{a_1}{2}\left\|\nabla(\lambda^{-m}V)\right\|^2. \qquad (5.14)$$

Defining $A'''' := A' + A'''$, and returning to (5.11) with (5.12) and (5.14) in hand, we find

$$\begin{aligned}\frac{1}{2}\left\|\lambda^{-m-1}V\right\|^2 + \frac{1}{2}\left\|\nabla\lambda^{-m+1}LV\right\|^2 &\geq -\left\langle\left\langle \lambda^{-m-1}V, \lambda^{-m+1}LV\right\rangle\right\rangle \\ &\geq -\left(m(\tau+\eta) + A''''(\tau+\eta)^2\right)\left\|\lambda^{-m-1}V\right\|^2 + \frac{a_1}{2}\left\|\nabla(\lambda^{-m}V)\right\|^2,\end{aligned}$$

or

$$h_m(\tau,\eta)\left\|\lambda^{-m-1}V\right\|^2+\left\|\lambda^{-m+1}LV\right\|^2\geq a_1\left\|\nabla(\lambda^{-m}V)\right\|^2 \tag{5.15}$$

where

$$h_m(\tau,\eta):=1+2m(\tau+\eta)+2A''''(\tau+\eta)^2.$$

By Lemma 5.5, for all $m\in\mathbb{N}$, we know

$$mk_m(\tau,\eta)\left\|\lambda^{-m-1}V\right\|^2\leq\left\|\lambda^{-m}LV\right\|^2+A_4\left\|\nabla(\lambda^{-m}V)\right\|^2$$

where $k_m:=(1/2)(1-(A_1/m)(\tau+\eta)-(A_2/m)(\tau+\eta)^2)$. With (5.15), this implies

$$\frac{h_m}{mk_m}\left\|\nabla(\lambda^{-m}V)\right\|^2+\left(\frac{h_m}{mk_m}+(\tau+\eta)^2\right)\left\|\lambda^{-m}LV\right\|^2\geq a_1\left\|\nabla(\lambda^{-m}V)\right\|^2. \tag{5.16}$$

First assume μ_0, η_0 are small enough to ensure

$$1-A_1(\mu_0+\eta_0)+A_2(\mu_0+\eta_0)^2>\frac{1}{2}. \tag{5.17}$$

Then, for all $0<\tau\leq\mu\leq\mu_0$, $0<\eta\leq\eta_0$. and all m, we have

$$k_m(\mu_0,\eta_0)=\frac{1}{2}(1-\frac{A_1}{m}(\tau+\eta)-\frac{A_2}{m}(\tau+\eta)^2)>\frac{1}{4}.$$

Next choose m_0, and decrease η_0, μ_0 if necessary to ensure

$$\frac{4}{m_0}h_{m_0}(\mu_0,\eta_0)=\frac{4}{m_0}\left(1+2m_0(\mu_0+)+2A''''(\eta_0+\mu_0)^2\right)\leq\frac{a_1}{2}.$$

Then for all $m\geq m_0$, $0<\tau\leq\mu\leq\mu_0$, $\eta<\eta_0$, (5.16) gives

$$\left\|\nabla(\lambda^{-m}V)\right\|^2\leq\frac{2}{a_1}\left(\frac{4}{m}h_m(\mu,\eta)+(\mu+\eta)^2\right)\left\|\lambda^{-m}LV\right\|^2. \tag{5.18}$$

By Lemma 5.5 (recalling (5.17)), we may estimate again that (for $m\geq m_0$)

$$\left\|\lambda^{-m-1}V\right\|^2\leq\frac{2}{m}\left\|\lambda^{-m}LV\right\|^2+\frac{2A_3}{m}\left\|\nabla(\lambda^{-m}V)\right\|^2.$$

Feeding this back into (5.18), we have

$$\left\|\lambda^{-m}V\right\|^2+\left(1-\frac{2A_3}{m}\right)\left\|\nabla(\lambda^{-m}V)\right\|^2\leq$$
$$\frac{2}{a_1}\left(\frac{1}{a_1}+\frac{4}{m}h_m(\mu,\eta)(\mu+\eta)+(\mu+\eta)^2\right)\left\|\lambda^{-m}LV\right\|^2$$

Choosing m_0 larger still to ensure $1-A_3/m>1/2$, we obtain at last (5.9) for all $m\geq m_0$, $0<\tau\leq\mu\leq\mu_0\leq T$, $0<\eta\leq\eta_0$. □

5.3.3 Proof of Theorem 5.4

Armed with the estimates of the previous section, we return our attention to our primary objective.

Proof of Theorem 5.4. We may assume (increasing m_0 and decreasing μ_0 and η_0 if necessary) that

$$(\mu+\eta)^2\rho_m c_1 \le \frac{1}{2} \quad \text{and} \quad \rho_m c_2 \le \frac{1}{2}. \tag{5.19}$$

Then choose $\mu \le \mu_0$, $\eta < \eta_0$. We may also assume that $T \le \mu$, since by applying the argument successively to $[T-\mu, T]$, $[T-2\mu, T-\mu]$, etc., we can show $X \equiv 0$ on all of $[0,T]$. With this assumption in mind, choose $0 \le \tau_1 < \tau_2 < T$. Let $\xi \in C^\infty(\mathbb{R})$ be such that $\xi(t) = 0$ for $t \le \tau_1$ and $\xi(t) = 1$ for $t \ge \tau_2$. Then $V = \xi X \in \mathcal{V}_0^{0,T}$ and

$$\rho_m \left\|\lambda^{-m} LV\right\|^2 = \rho_m \int_{\tau_1}^{\tau_2}\int_{M^n} \left|\lambda^{-m} LV\right|^2 \, d\mu\, dt + \rho_m \int_{\tau_2}^{T}\int_{M^n} \left|\lambda^{-m} LX\right|^2 \, d\mu\, dt$$

Using Lemma 5.6, for $m \ge m_0$, we have

$$\begin{aligned}
\rho_m \int_{\tau_1}^{\tau_2}\int_{M^n} &\left|\lambda^{-m} LV\right|^2 \, d\mu\, dt + \rho_m \int_{\tau_2}^{T}\int_{M^n} \left|\lambda^{-m} LX\right|^2 \, d\mu\, dt \\
&\ge \left\|\lambda^{-m-1} V\right\|^2 + \frac{1}{2}\left\|\nabla(\lambda^{-m} V\right\|^2 \\
&\ge \int_{\tau_2}^{T}\int_{M^n} \left|\lambda^{-m-1} X\right|^2 \, d\mu\, dt + \frac{1}{2}\int_{\tau_2}^{T}\int_{M^n} \lambda^{-2m}\left|\nabla X\right|^2 \, d\mu\, dt,
\end{aligned} \tag{5.20}$$

and by (5.2) and (5.19), we have

$$\begin{aligned}
\rho_m &\int_{\tau_2}^{T}\int_{M^n} \left|\lambda^{-m} LV\right|^2 \, d\mu\, dt \\
&\le \rho_m c_1 (\mu+\eta)^2 \int_{\tau_2}^{T}\int_{M^n} |\lambda^{-m} X|^2 \, d\mu dt + \rho_m c_2 \int_{\tau_2}^{T}\int_{M^n} \lambda^{-2m}\left|\nabla X\right|^2 \, d\mu\, dt \\
&\qquad\qquad \le \frac{1}{2}\int_{\tau_2}^{T}\int_{M^n} |\lambda^{-m} X|^2 \, d\mu dt + \int_{\tau_2}^{T}\int_{M^n} \lambda^{-2m}\left|\nabla X\right|^2 \, d\mu\, dt.
\end{aligned}$$

Combining this with (5.20), we have

$$2\rho_m \int_{\tau_1}^{\tau_2}\int_{M^n} \left|\lambda^{-m} LV\right|^2 \, d\mu\, dt \ge \int_{\tau_2}^{T}\int_{M^n} \left|\lambda^{-m-1} X\right|^2 \, d\mu\, dt.$$

Choosing $\tau_3 \in (\tau_2, T)$, we have therefore that

$$2\rho_m(T+\eta-\tau_2)^{-m}\int_{\tau_1}^{\tau_2}\int_{M^n}|LV|^2\,d\mu\,dt \geq (T+\eta-\tau_3)^{-m-1}\int_{\tau_3}^{T}\int_{M^n}|X|^2\,d\mu\,dt.$$

Since ρ_m does not increase as m increases, by taking m sufficiently large, the above inequality leads to a contradiction unless

$$\int_{\tau_3}^{T}\int_{M^n}|X|^2\,d\mu\,dt = 0.$$

As we may take τ_3 as close as we like to τ_2, and we may take τ_2 as close as we like to 0, we conclude that $X \equiv 0$ on $M^n \times [0, T]$. □

5.4 Applications

5.4.1 Unique-continuation for the Kähler-Ricci flow

If the initial metric of a solution to the Ricci flow on M^{2n} is Kähler, both the Kähler condition and the complex structure are preserved by the equation, and the flow is known as the Kähler-Ricci flow. We will consider exclusively the case in which the Kähler class ω of the metric g is a real multiple of the first Chern class ρ on a compact manifold M^n, in which case we have

$$[\omega_g] = \frac{r}{n}[\rho] \quad \text{where} \quad r = \frac{\int_{M^n} R d\mu}{\int_{M^n} d\mu}. \tag{5.21}$$

In this case, it is most natural to consider the *normalized Kähler-Ricci flow* equation

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}} + \frac{r}{n} g_{\alpha\bar{\beta}}. \tag{5.22}$$

Under the normalized flow, the Kähler class, volume, and average scalar curvature of a solution are independent of time. Moreover, by cohomological considerations, we may reduce the study of the flow to the study of the properties of a certain parabolic Monge-Ampere equation. Our basic reference is [22], though, with an eye toward the eventual proof of Theorem 5.1, we consider an expression of the in terms of the metric at the terminal, rather than the initial, time.

We first recall the following standard result (cf., e.g., [22] or [71]).

Lemma 5.7 ($\partial\overline{\partial}$ - Lemma). *If a is a d-exact real $(1,1)$-form on a closed Kähler manifold M^{2n}, then there exists a real-valued function ϕ such that*

$$\nabla_\alpha \nabla_{\bar\beta}\phi = \frac{\partial^2 \phi}{\partial z^\alpha \partial z^{\bar\beta}} = a_{\alpha\bar\beta}.$$

Now consider a solution $g(t)$ to (5.22) on the closed manifold M^{2n} for $t \in [0,T]$ satisfying (5.21) at $t = T$ (hence all t). By Lemma 5.7, there exists a real-valued function f such that

$$\nabla_\alpha \nabla_{\bar\beta} f = R_{\alpha\bar\beta} - \frac{r}{n} g_{\alpha\bar\beta}.$$

Likewise, since the Kähler class of $\widetilde{g}$ is constant, there exists a real valued function $u = u(\cdot, t)$ such that

$$g_{\alpha\bar\beta}(t) = g_{\alpha\bar\beta}(T) + \nabla_\alpha \nabla_{\bar\beta} u(t)$$

for $t \in [0,T]$. Recalling the local formula for the Ricci curvature on a Kähler manifold

$$R_{\alpha\bar\beta} = -\frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}} \log\det(g_{\alpha\beta}),$$

we determine that the evolution of u satisfies

$$\begin{aligned}
\nabla_\alpha \nabla_{\bar\beta} \frac{\partial}{\partial t} u &= -\frac{\partial}{\partial t}(g_{\alpha\bar\beta}(t) - g_{\alpha\bar\beta}(T)) \\
&= -R_{\alpha\bar\beta}(t) + \frac{r}{n} g_{\alpha\bar\beta}(t) \\
&= \nabla_\alpha \nabla_{\bar\beta} \log\det(g_{\alpha\bar\beta}(T) + \nabla_\alpha \nabla_{\bar\beta} u(t)) + \frac{r}{n} g_{\alpha\bar\beta}(T) + \frac{r}{n} \nabla_\alpha \nabla_{\bar\beta} u(t) \\
&= \nabla_\alpha \nabla_{\bar\beta} \log\det(g_{\alpha\bar\beta}(T) + \nabla_\alpha \nabla_{\bar\beta} u(t)) + R_{\alpha\bar\beta} + \nabla_\alpha \nabla_{\bar\beta} f + \frac{r}{n} \nabla_\alpha \nabla_{\bar\beta} u(t) \\
&= \nabla_\alpha \nabla_{\bar\beta} \left[\log\left(\frac{\det(g_{\alpha\bar\beta}(T) + \nabla_\alpha \nabla_{\bar\beta} u(t))}{\det(g_{\alpha\bar\beta}(T))} \right) + \frac{r}{n} u(t) + f \right].
\end{aligned}$$

Thus, since M^{2n} is compact, we may write

$$\frac{\partial u}{\partial t} = \log\left(\frac{\det(g_{\alpha\bar\beta}(T) + \nabla_\alpha \nabla_{\bar\beta} u)}{\det(g_{\alpha\bar\beta}(T))} \right) + \frac{r}{n} u + f + c \tag{5.23}$$

for some function $c = c(t)$ of time only. Hence the potential which determines $g_{\alpha\bar\beta}(t)$ solves a parabolic Monge-Ampere type equation, and to prove the backwards uniqueness theorem for solutions to (5.22) in this category amounts to considering the equation satisfied by the difference of two such potentials.

Proof of Theorem 5.1. Suppose g, $\widetilde{g}$, and h are as in the statement of the theorem. We may choose f to satisfy

$$\nabla_\alpha \nabla_{\bar\beta} f = -R_{\alpha\bar\beta}(T) + \frac{r}{n} g_{\alpha\bar\beta}(T)$$

and consider the potentials u and $\tilde u$ solving (5.23) for g and $\widetilde{g}$, respectively. Note that f and r are the same for both, as $g(T) = \widetilde{g}(T) = h$. Thus, setting $\varphi := u - \tilde u$, we have

$$\frac{\partial}{\partial t}\varphi = \log\left(\frac{\det(h_{\alpha\bar\beta} + \nabla_\alpha\nabla_{\bar\beta} u)}{\det(h_{\alpha\bar\beta} + \nabla_\alpha\nabla_{\bar\beta}\tilde u)}\right) + \frac{r}{n}\varphi + c - \tilde c. \tag{5.24}$$

To simplify the expression we employ a trick used by Fan [32] (that we learned from [58]) in a proof of (forwards) uniqueness. Essentially, we just linearize the logarithm of the determinant:

$$\begin{aligned}\log\left(\frac{\det(h_{\alpha\bar\beta} + \nabla_\alpha\nabla_{\bar\beta} u)}{\det(h_{\alpha\bar\beta} + \nabla_\alpha\nabla_{\bar\beta}\tilde u)}\right) &= \int_0^1 \frac{d}{ds}\log\det\left(s g_{\alpha\bar\beta} + (1-s)\widetilde{g}_{\alpha\bar\beta}\right)\, ds \\ &= \left(\int_0^1 \bar g_s^{\alpha\bar\beta}\, ds\right)\nabla_\alpha\nabla_{\bar\beta}\varphi \\ &:= \Lambda^{\alpha\bar\beta}\nabla_\alpha\nabla_{\bar\beta}\varphi\end{aligned}$$

where in the penultimate line, we used $\bar g_s^{\alpha\bar\beta}$ to denote the inverse of the metric $\bar g_s = sg + (1-s)\widetilde{g}$. Note that the operator $\Lambda^{\alpha\bar\beta}\nabla_\alpha\nabla_{\bar\beta}$ is elliptic, so that the only matter left to attend to is the term $c - \tilde c$. For this we define

$$\epsilon(t) = e^{\frac{rt}{n}} \int_t^T e^{-\frac{r\tau}{n}}\left(c(\tau) - \tilde c(\tau)\right) d\tau$$

and put $\widehat{\varphi} = \phi + \epsilon$. Then $\widehat{\varphi}(\cdot, T) \equiv 0$ and

$$\frac{\partial}{\partial t}\widehat{\varphi} = \Lambda^{\alpha\bar\beta}\nabla_\alpha\nabla_{\bar\beta}\widehat{\varphi} + \frac{r}{n}\widehat{\varphi},$$

so Theorem 5.4 implies $\widetilde{\varphi}(\cdot, t) \equiv 0$ on all of $[0, T]$. Thus u and $\tilde u$ differ at most by a function of time, and g and $\widetilde{g}$ must agree identically on $[0, T]$. □

5.4.2 Solutions which become Einstein in finite time

Here we recall the Einstein equation

$$R_{ij} = \rho g_{ij} \tag{5.25}$$

for some $\rho \in \mathbb{R}$. In his 1982 paper, [38], Hamilton showed that a solution to the (normalized) Ricci flow with positive Ricci curvature converges exponentially fast to a metric of constant positive sectional curvature. By analogy with the asymptotic behavior of solutions to the heat equation, one might conjecture that in fact the rate of convergence is precisely – that is, no faster than – exponential. Theorem 5.2, whose proof we present next, in a sense provides a coarse upper bound on the rate of convergence in that it prohibits a solution of either the normalized or unnormalized flow from attaining its limit in a finite time.

The proof is simple– nearly a direct consequence of the evolution equation of the Ricci tensor,

$$\frac{\partial}{\partial t}R_{ij} = \Delta R_{ij} + 2g^{pq}g^{rs}R_{pijr}R_{qs} - 2g^{pq}R_{ip}R_{qj} \tag{5.26}$$

(cf., e.g., [20]). In fact, for the case $\rho = 0$, using that we have a smooth solution on a compact manifold, Theorem 5.4 may be applied directly to (5.26), estimating the norms of the terms of the form $\mathrm{Rc} * \mathrm{Rm}$ and $\mathrm{Rc} * \mathrm{Rc}$ above by $C|\mathrm{Rc}|$, for C depending on $\sup_{M^n\times[0,T]}|\mathrm{Rm}|$. In the general case, we argue as follows.

Proof of Theorem 5.2. Suppose g and ρ are as in the statement of the theorem. Define

$$B_{ij} := R_{ij} - \alpha(t)g_{ij}$$

where $\alpha(t)$ solves $\frac{d\alpha}{dt} = -2\alpha^2$ with $\alpha(t_0) = \rho$. So $\alpha(t) \equiv 0$ if $\rho = 0$ and

$$\alpha(t) = \frac{\rho}{1 - 2\rho(t - t_0)}$$

otherwise. It is easily checked that $\bar{g}(t) = (1 - 2\rho(t - t_0))g(t_0)$ is both Einstein and a solution to the Ricci flow for as long as $1 - 2\rho(t - t_0) > 0$. Thus by forwards-uniqueness, it follows that $g(t) = \bar{g}$ is Einstein for all $t \in (t_0, \Omega)$. If $t_0 = A$, the proof is complete. Otherwise put

$$t_k = \begin{cases} A & \text{if} \quad \rho \geq 0 \ \text{ or } \ t_0 - A > -1/2\rho \\ t_0 + 1/2\rho + \frac{1}{k} & \text{otherwise} \end{cases}$$

where $k = 0, 1, 2, \ldots$.

Then α and $g(t)$ are well-defined in each $[t_k, t_0]$. and we compute, using (5.26), that

$$\begin{aligned}\frac{\partial}{\partial t} T_{ij} &= \frac{\partial}{\partial t} R_{ij} - \alpha' g_{ij} - 2\alpha R_{ij} \\ &= \Delta R_{ij} + 2g^{pq} g^{rs} R_{pijr} R_{qs} - 2g^{pq} R_{ip} R_{qj} + 2\alpha^2 g_{ij} - 2\alpha R_{ij} \\ &= \Delta T_{ij} + 2g^{pq} g^{rs} R_{pijr} T_{qs} - 2g^{pq} R_{ip} T_{qj}.\end{aligned}$$

Choosing M so that $|\mathrm{Rm}| \le M$ on $M^n \times [A, \Omega]$, we have

$$\left| \left(\frac{\partial}{\partial t} - \Delta \right) T_{ij} \right|_g^2 \le C\, |T_{ij}|_g^2$$

for some $C = C(n, M)$, so that we may apply Theorem 5.4 to conclude that $T \equiv 0$ in $[t_k, t_0]$. If $t_k = A$, we are done. Otherwise, sending $k \to \infty$, we conclude that $g(t)$ is Einstein with negative Einstein constant α on $(t_0 + 1/2\rho, t_0]$. But $\alpha(t) \to -\infty$ as $t \to t_0 + 1/2\rho$, which contradicts that $g(t)$ was a smooth solution on $[A, \Omega]$. □

5.4.3 Unique continuation for the Ricci-DeTurck flow

In this section, let $\bar{g}$ denote an arbitrary fixed metric, and $\bar{\nabla}$ its Levi-Civita connection. Recall that a metric g evolving according to the Ricci-DeTurck flow satisfies

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i \tag{5.27}$$

where $W_j := g_{jk} g^{pq} \left(\Gamma^k_{pq} - \overline{\Gamma}^k_{pq} \right)$. In local coordinates, a computation (see e.g., [11] or [21], Lemma 7.50) shows

Lemma 5.8.

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} = g^{pq} \overline{\nabla}_p \overline{\nabla}_q g_{ij} &- g^{pq} g_{ik} \bar{g}^{kl} \overline{R}_{jplq} - g^{pq} g_{jk} \bar{g}^{kl} \overline{R}_{iplq} \\ &+ g^{pq} g^{kl} \left(\frac{1}{2} \overline{\nabla}_i g_{kp} \overline{\nabla}_p g_{lq} + \overline{\nabla}_p g_{jk} \overline{\nabla}_l g_{iq} - \overline{\nabla}_p g_{jk} \overline{\nabla}_q g_{il} \right) \\ &- g^{pq} g^{kl} \left(\overline{\nabla}_j g_{kp} \overline{\nabla}_q g_{il} + \overline{\nabla}_i g_{kp} \overline{\nabla}_q g_{jl} \right)\end{aligned} \tag{5.28}$$

From (5.28), it is clear that the Ricci-DeTurck flow is strictly parabolic. The only obstacle to applying Theorem 5.4 is the non-linearity of the equation caused by the presence of g in the highest order term. We will verify that the difference of two solutions nevertheless satisfies the appropriate parabolic inequality. To handle the highest order term, note that we have the following simple estimate

$$\begin{aligned} \left|g^{pq}\overline{\nabla}_p\overline{\nabla}_q g - \widetilde{g}^{pq}\overline{\nabla}_p\overline{\nabla}_q\widetilde{g} - g^{pq}\overline{\nabla}_p\overline{\nabla}_q(g-\widetilde{g})\right|_{\bar{g}} & \\ \leq \left|g^{-1}-\widetilde{g}^{-1}\right|_{\bar{g}}\left|\overline{\nabla}_p\overline{\nabla}_q\widetilde{g}\right|_{\bar{g}} & \\ \leq \left|g-\widetilde{g}\right|_{\bar{g}}\left|g^{-1}\right|_{\bar{g}}\left|\widetilde{g}^{-1}\right|_{\bar{g}}\left|\overline{\nabla}_p\overline{\nabla}_q\widetilde{g}\right|_{\bar{g}} & \end{aligned} \tag{5.29}$$

where in the last line, we used

$$g^{pq}-\widetilde{g}^{pq} = -g^{pk}\widetilde{g}^{ql}\left(g_{kl}-\widetilde{g}_{kl}\right). \tag{5.30}$$

Now we turn to the

Proof of Theorem 5.3. Suppose $g(t)$ and $\widetilde{g}(t)$ are solutions on $M^n\times[0,T]$ to the Ricci-DeTurck flow with background metric $\bar{g}$ that agree at $t=T$. Put $A_{ij}=g_{ij}-\widetilde{g}_{ij}$. It is straightforward, but tedious, to compute the evolution of A, and for the details, the reader is encouraged to consult [11] or [22]. Here we merely record the result, which, grouping terms and making use of (5.29) and (5.30), is

$$\begin{aligned} \frac{\partial}{\partial t}A_{ij} = g^{kl}\overline{\nabla}_k\overline{\nabla}_l A_{ij} - g^{kp}\widetilde{g}lqA_{pq}\overline{\nabla}_k\overline{\nabla}_l\widetilde{g}_{ij} & \\ - \bar{g}^{pq}\left(\widetilde{g}^{kl}A_{ip} - g_{ip}g^{kr}\widetilde{g}^{ls}A_{rs}\right)\overline{R}_{jkql} & \\ - \bar{g}^{pq}\left(\widetilde{g}^{kl}A_{jp} - g_{jp}g^{kr}\widetilde{g}^{ls}A_{rs}\right)\overline{R}_{ikql} & \\ - \frac{1}{2}\left(g^{kr}\widetilde{g}^{ls}g^{pq}A_{rs} + \widetilde{g}^{jl}g^{pr}\widetilde{g}^{qs}A_{rs}\right)B_{ipkjql} & \\ + \frac{1}{2}\widetilde{g}^{kl}\widetilde{g}^{pq}\left\{\overline{\nabla}_i A_{pk}\overline{\nabla}_j g_{ql} + \overline{\nabla}_i\widetilde{g}_{pq}\overline{\nabla}_j A_{ql}\right. & \\ + 2\left(\overline{\nabla}_k A_{jp}\overline{\nabla}_q g_{il} + \overline{\nabla}_k\widetilde{g}_{jp}\overline{\nabla}_q A_{il}\right) & \\ - 2\left(\overline{\nabla}_k A_{jp}\overline{\nabla}_l g_{iq} + \overline{\nabla}_k\widetilde{g}_{jp}\overline{\nabla} A_{iq}\right) & \\ - 2\left(\overline{\nabla}_j A_{pk}\overline{\nabla}_l g_{iq} + \overline{\nabla}_j\widetilde{g}_{pk}\overline{\nabla}_l A_{iq}\right) & \\ \left. - 2\left(\overline{\nabla}_i A_{pk}\overline{\nabla}_l g_{jq} + \overline{\nabla}_i\widetilde{g}_{pk}\overline{\nabla}_l A_{jq}\right)\right\} & \end{aligned} \tag{5.31}$$

where

$$\begin{aligned} B_{ipkjql} := \overline{\nabla}_i g_{pk}\overline{\nabla}_j g_{ql} + 2\overline{\nabla}_k g_{jp}\overline{\nabla}_q g_{il} - 2\overline{\nabla}_k\overline{\nabla}_l g_{iq} & \\ - 2\overline{\nabla}_j g_{pq}\overline{\nabla}_l g_{iq} - 2\overline{\nabla}_i g_{pk}\overline{\nabla}_l g_{jq}. & \end{aligned}$$

Since $g(t)$ and $\widetilde{g}(t)$ are smooth solutions and M^n is compact, there exist constants c_1 and c_2 depending on n, and the maximum of the $\bar{g}$-norms of $\overline{\mathrm{Rm}}$, g, $\widetilde{g}$, g^{-1}, g^{-1}, $\overline{\nabla} g$, $\overline{\nabla}\widetilde{g}$, and $\overline{\nabla}\overline{\nabla}\widetilde{g}$ on $M^n \times [0, T]$, such that

$$\left| \left(\frac{\partial}{\partial t} - g^{ij} \overline{\nabla}_i \overline{\nabla}_j \right) A \right|_{\bar{g}}^2 \leq c_1 \left| A \right|_{\bar{g}}^2 + c_2 \left| \overline{\nabla} A \right|_{\bar{g}}^2, \tag{5.32}$$

whence the result follows from Theorem 5.4. □

Of course, we expect that the results of this chapter have extensions to non-compact manifolds for solutions satisfying appropriate growth conditions. In particular, the general result for the Ricci flow should hold under at least the assumption of bounded curvature. The last result in particular also raises some related questions.

Question. What analyticity properties do solutions to the Ricci flow possess in the time variable?

(Note that it is a result of Bando [5] that solutions to the Ricci flow are analytic in geodesic coordinates [in the space variables] for each post-initial time t.)

Question. Denote by $\mathcal{O}_g$ the orbit of g under $\mathrm{Diff}(M^n)$. Given a solution $g(t)$ on $[0, T]$ of the Ricci-DeTurck flow (1.3) for a fixed background metric $\bar{g}$, can one describe the image of $\mathcal{O}_{g(0)}$ under the "time-T" Ricci-DeTurck flow operator?

It is easy to see that the time-T Ricci-DeTurck flow will map $\mathcal{O}_{g(0)}$ one-to-one onto some subset of $\mathcal{O}_{g(T)}$. If this subset were the entire orbit $\mathcal{O}_{g(T)}$, the backwards-uniqueness of the Ricci flow would follow at once from that of the Ricci-DeTurck flow.

6 Harnack inequalities for evolving hypersurfaces via a space-time approach

The basic objects of our consideration in this chapter will be the differential Harnack estimates of Andrews [3], Chow [14], and Hamilton [46], whose statements we reviewed in Section 2.2.3. Our first aim is to modify the space-time construction of Chow and Chu [17] to obtain a realization of the entire Harnack quantity as the second fundamental form $\widetilde{h}$ of an appropriate degenerate immersion $\widetilde{X}$ of the space-time track $M^n \times (0,T)$ into $\mathbb{R}^{n+1} \times \mathbb{R}$. The Harnack estimates then may be expressed in terms of the the positivity (or negativity) of $\widetilde{h}$ on $T(M^n \times (0,T))$. We also exhibit a connection on this space-time with respect to which this second-fundamental form satisfies the Codazzi equations and is compatible with the degenerate metric on $T^*(M^n \times (0,T))$ associated to the immersion. Along a certain space-time trajectory, the second fundamental form satisfies an evolution-type equation which has a convenient expression in terms of this connection, and which is amenable to a generalized tensor maximum principle. We use this latter observation to show that the Harnack inequalities may in fact be proven from within this framework, and as an application provide a new proof of Hamilton's estimate for the mean curvature flow. Our technique is in particular guided by that in [19].

6.1 Preliminaries

Our general setting will be that of Section 2.2.2: M^n will denote a smooth n-dimensional manifold and will assume that we have a one-parameter family of smooth embeddings $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ solving the isotropic equation

$$\frac{\partial}{\partial t} X(x, t) = -F(W(x, t))\nu(x, t), \tag{6.1}$$

where W is the Weingarten map $W : TM^n \times [0, T) \to TM^n$, ν is a choice of unit normal, and F is a smooth, symmetric function of the eigenvalues of W.

We will restrict our attention to solutions X_t which are strictly convex and to speed functions F for which the system (6.1) is strictly parabolic. This latter restriction is equivalent to the condition $\dot{F}_W[B] > 0$ for all $B \in \Gamma_+$ or $\dot{F}^{kl} B_{kl} > 0$ for all $B \in S^2_+(TM^n)$.

We will chiefly be interested in the case when F is homogeneous of some degree κ, that is,

$$F(\lambda W) = \lambda^\kappa F(W)$$

for all $\lambda > 0$. With such an assumption of homogeneity, equation (6.1) is invariant under dilations of Euclidean space – an invariance which is rather fundamental to the interpretation of the Harnack inequalities we are after as our construction is inspired by solutions which evolve by pure scaling. For now, we recall a standard fact about homogeneous functions.

Lemma 6.1. *Suppose F is homogeneous of degree κ. Then*

$$\dot{F}_W[W] = \kappa F(W).$$

6.2 Solitons and the modified flow

To motivate what follows, we begin by considering a special class of solutions to the flow (6.1) which evolve by pure scaling about the origin (cf., Ch. 2 of [27]). Up to tangential diffeomorphism, this phenomenon may be described by

$$\frac{\partial}{\partial t} X(x, t) = \phi(t) X(x, t_0) \tag{6.2}$$

for some $t_0 \in [0,T)$ and smooth ϕ. Let us suppose, for the time being, that our functions F are homogeneous of degree $\kappa \neq -1$ in the principal curvatures. Then if $\tilde{X} = cX$, we have $\widetilde{W} = c^{-1}W$ and $\tilde{F} = c^{-\kappa}F$. Differentiating (6.2) leads to the identity

$$\phi'(t)\,(X(x,t_0))^{\perp} = -F(W(x,t))\nu(x,t) = -\phi(t)^{-\kappa}F(W(x,t_0))\nu(x,t_0),$$

from which we deduce that $(\phi^{\kappa+1})' = \text{const}$. After we impose the condition $\phi(t_0) = 1$, we have

$$\phi(x,t) = \sqrt[\kappa+1]{(\alpha(t-t_0)+1}$$

for some constant α. The solution is called an expanding soliton if $\alpha > 0$ and a shrinking soliton if $\alpha < 0$. The Harnack inequalities of Hamilton for the Ricci and mean curvature flows ([42], [46]), and of Chow for the Gauss curvature flow [14] are modeled on expanding solitons which emanate from a cone at time $t = 0$. For such solutions we have $\alpha = 1$ and $t_0 = 1$, giving rise to equations of the form

$$X(x,t) = t^{\frac{1}{\kappa+1}}X(x,1). \tag{6.3}$$

In [19], Chow and Knopf consider the transformation

$$\begin{cases} \bar{g}(t) & = t^{-1}g(t) \\ \bar{t} & = \log t \end{cases} \tag{6.4}$$

which takes expanding solitons for the Ricci flow into steady solitons. When applied to arbitrary solutions of the Ricci flow, the transformation yields solutions to the *Ricci Flow with cosmological constant* $\mu = 1/2$, that is, to a solution of

$$\frac{\partial}{\partial \bar{t}}\bar{g}_{ij} = -2\left(\overline{R}_{ij} + \frac{1}{2}\bar{g}_{ij}\right). \tag{6.5}$$

In the hypersurface case, given an expanding soliton of the form (6.3), the analogous transformation is

$$\begin{cases} \overline{X}(x,t) & = t^{\frac{-1}{\kappa+1}}X(x,t) \\ \bar{t} & = \log t. \end{cases} \tag{6.6}$$

Under this transformation, expanding solitons evolve only by tangential diffeomorphisms, and general solutions will satisfy

$$\frac{\partial}{\partial \bar{t}}\overline{X} = -\overline{F}\overline{\nu} - \frac{1}{\kappa+1}\overline{X} \tag{6.7}$$

for all $(x, \bar{t}) \in M^n \times (-\infty, \log T)$.

These observations lead us to consider the equation

$$\frac{\partial}{\partial \bar{t}}\overline{X} = -\overline{F}\overline{\nu} + \mu\overline{X} \tag{6.8}$$

on $M^n \times \mathcal{I}$ for some interval $\mathcal{I}$, general $\overline{F}$, and smooth functions $\mu(\bar{t})$. Given its similarity to (6.5), we shall call (6.8) the *F-flow with cosmological constant μ*, or simply the *modified flow*.

6.3 Evolution equations for the modified flow

We now derive some consequences of equation (6.8). Under our assumptions on F, the operator $\overline{\Box} \doteqdot \dot{\overline{F}}^{ij}\overline{\nabla}_i\overline{\nabla}_j$ is elliptic, and routine computations yield the following generalizations of the equations in Section 2.2.

Lemma 6.2. *Under equation (6.8), the metric, normal, second fundamental form, Weingarten map, and speed function evolve according to the following equations*

$$\frac{\partial}{\partial \bar{t}}\bar{g}_{ij} = -2\overline{F}\bar{h}_{ij} + 2\mu\bar{g}_{ij} \tag{6.9}$$

$$\frac{\partial}{\partial \bar{t}}\overline{\nu} = \overline{\nabla F} \tag{6.10}$$

$$\begin{aligned}\frac{\partial}{\partial \bar{t}}\bar{h}_{ij} &= \overline{\nabla}_i\overline{\nabla}_j\overline{F} - \overline{F}\bar{h}_i^p\bar{h}_{pj} + \mu\bar{h}_{ij} \\ &= \overline{\Box}\bar{h}_{ij} + \ddot{\overline{F}}^{pq,rs}\overline{\nabla}_i\bar{h}_{pq}\overline{\nabla}_j\bar{h}_{rs} + \left(\dot{\overline{F}}\left[\overline{W}^2\right] + \mu\right)\bar{h}_{ij} \\ &\qquad - \left(\overline{F} + \dot{\overline{F}}\left[\overline{W}\right]\right)\bar{h}_i^p\bar{h}_{pj} + \overline{K}_{ij}\end{aligned} \tag{6.11}$$

$$\frac{\partial}{\partial \bar{t}}\bar{h}_j^i = \overline{\nabla}^i\overline{\nabla}_j\overline{F} + \overline{F}\bar{h}_p^i\bar{h}_j^p - \mu\bar{h}_j^i \tag{6.12}$$

$$\frac{\partial}{\partial \bar{t}}\overline{F} = \overline{\Box}\overline{F} + \dot{\overline{F}}^{pq}\left(\overline{F}\bar{h}_p^r\bar{h}_{rq} - \mu\bar{h}_{pq}\right) \tag{6.13}$$

where

$$\overline{K}_{ij} \doteqdot \dot{\overline{F}}^{pq} \bar{g}^{rs} \left(\bar{h}_{ip} \bar{h}_{qr} \bar{h}_{sj} - \bar{h}_{ir} \bar{h}_{ps} \bar{h}_{pj} \right) = \bar{h}_{ip} \left(\left[\dot{\overline{F}}, \overline{W} \right] \right)^{p}_{q} \bar{h}^{q}_{j}.$$

Example 6.3. For the mean curvature flow, $\dot{\overline{F}}^{ij} = \bar{g}^{ij}$, $\ddot{\overline{F}}^{pq,rs} = 0$, and $\overline{K}_{ij} = 0$, and so the second fundamental form and mean curvature evolve by the equations

$$\frac{\partial}{\partial \bar{t}} \bar{h}_{ij} = \overline{\Delta} \bar{h}_{ij} - 2 \overline{H} \bar{h}^{p}_{i} \bar{h}_{pj} + \left(|\bar{h}_{pq}|^2 + \mu \right) \bar{h}_{ij} \tag{6.14}$$

$$\frac{\partial}{\partial \bar{t}} \overline{H} = \overline{\Delta H} + \left(|\bar{h}_{pq}|^2 - \mu \right) \overline{H}. \tag{6.15}$$

For our purposes later, we will also need the following commutator identities.

Lemma 6.4. *Under equation (6.8), we have*

$$\begin{aligned} \left(\frac{\partial}{\partial \bar{t}} \overline{\Box} - \overline{\Box} \frac{\partial}{\partial \bar{t}} \right) \overline{F} = \ddot{\overline{F}} \left(\overline{\nabla \nabla F}, \frac{\partial}{\partial \bar{t}} \overline{W} \right) + 2 \overline{F} \dot{\overline{F}}^{ik} \bar{h}^{j}_{k} \overline{\nabla}_i \overline{\nabla}_j \overline{F} - 2 \mu \overline{F} \\ + 2 \dot{\overline{F}}^{ij} \bar{h}^{k}_{j} \overline{\nabla}_i \overline{F} \overline{\nabla}_k \overline{F} + \left(\overline{F} - \dot{\overline{F}}^{pq} \bar{h}_{pq} \right) |\overline{\nabla F}|^2 \end{aligned} \tag{6.16}$$

$$\left(\overline{\nabla}_i \overline{\Box} - \overline{\Box} \overline{\nabla}_i \right) \overline{F} = \ddot{\overline{F}} \left(\bar{g}^* \overline{\nabla \nabla F}, \overline{\nabla}_i \overline{W} \right) + \overline{F}^{pq} \left(\bar{h}_{ip} \bar{h}^{r}_{q} \overline{\nabla}_r \overline{F} - \bar{h}_{pq} \bar{h}^{r}_{i} \overline{\nabla}_r \overline{F} \right) \tag{6.17}$$

6.4 The space-time second fundamental form

6.4.1 A family of immersions of space-time

We now consider the immersion of the space-time of a solution to the modified flow into a larger space. Following [17], for any $N > 0$, equip $\mathbb{R}^{n+1} \times \mathbb{R}$ with the metric $g_{\mathcal{E},N} \doteqdot g_{\mathbb{R}^{n+1}} + N ds^2$. Define $\widetilde{\mathcal{M}} \doteqdot M^n \times \mathcal{I}$ and $\widetilde{X} : \widetilde{\mathcal{M}} \to \mathbb{R}^{n+1} \times \mathbb{R}$ by $\widetilde{X}(x, \bar{t}) \doteqdot \left(\overline{X}(x, \bar{t}), \eta(\bar{t}) \right)$ where η is a smooth function to be determined later. The image of $\widetilde{\mathcal{M}}$ under $\widetilde{X}$ is a hypersurface in $\mathbb{R}^{n+1} \times \mathbb{R}$; we will consider now the expression of a few geometric quantities on the hypersurface in local coordinates.

First the pull-back of $g_{\mathcal{E},N}$ via the map $\widetilde{X}$ defines a metric on $\widetilde{\mathcal{M}}$, which we will denote by $\widetilde{g}_N$.

$$\widetilde{g}_N = \begin{cases} \left(\widetilde{g}_N \right)_{00} &= \overline{F}^2 + 2 \mu \bar{s} \overline{F} + \mu^2 \bar{\rho} + N \left(\eta' \right)^2 \\ \left(\widetilde{g}_N \right)_{0i} &= \frac{1}{2} \mu \frac{\partial \bar{\rho}}{\partial x^i} \\ \left(\widetilde{g}_N \right)_{ij} &= \bar{g}_{ij} \end{cases} \tag{6.18}$$

where $\bar{s} = -\left\langle \overline{\nu}, \overline{X} \right\rangle_{\mathbb{R}^{n+1}}$ is the support function and $\bar{\rho} = \left\langle \overline{X}, \overline{X} \right\rangle_{\mathbb{R}^{n+1}}$ the squared norm of the position vector.

It is also easily checked that the following is a choice of unit normal to the hypersurface with respect to metric $g_{\mathcal{E},N}$.

$$\widetilde{\nu}_N \doteqdot \phi_N(x,t) \left(\overline{\nu}, \frac{\overline{F} + \mu \bar{s}}{N \eta'} \right) \tag{6.19}$$

where the normalizing factor, ϕ_N is given by

$$\phi_N \doteqdot \frac{\sqrt{N} \eta'}{\sqrt{N \left(\eta'\right)^2 + \left(\overline{F} + \mu \bar{s}\right)^2}}.$$

The second fundamental form to the hypersurface may then be computed according to the usual formula.

$$\begin{aligned} \left(\widetilde{h}_N\right)_{\alpha\beta} &= -\left\langle \frac{\partial^2 \overline{X}}{\partial x^\alpha \partial x^\beta}, \widetilde{\nu}_N \right\rangle_{g_{\mathcal{E},N}} \\ &= \begin{cases} \left(\widetilde{h}_N\right)_{00} & = \phi_N \cdot \left(\frac{\partial}{\partial \bar{t}} \overline{F} + \left(\mu - \frac{\eta''}{\eta'} \right) \overline{F} + \left(\mu' + \mu^2 - \mu \frac{\eta''}{\eta'} \right) \bar{s} \right) \\ \left(\widetilde{h}_N\right)_{0i} & = \phi_N \frac{\partial \overline{F}}{\partial x^i} \\ \left(\widetilde{h}_N\right)_{ij} & = \phi_N \bar{h}_{ij} \end{cases} \end{aligned} \tag{6.20}$$

To eliminate the term involving the support function $\bar{s}$ in $\left(\widetilde{h}_N\right)_{00}$, we now choose η to solve

$$\mu' + \mu^2 - \mu \frac{\eta''}{\eta'} = 0, \tag{6.21}$$

and write $\sigma \doteqdot \mu - (\eta''/\eta')$. so that

$$\left(\widetilde{h}_N\right)_{00} = \phi_N \cdot \left(\frac{\partial}{\partial \bar{t}} \overline{F} + \sigma \overline{F} \right).$$

Write $A = \overline{F}^2 - 2\mu s \overline{F} + \mu^2 \bar{\rho}$, so that $\left(\widetilde{g}_N\right)_{00} = A + N\left(\eta'\right)^2$. Then the expression of the inverse of the metric $\widetilde{g}_N$ in coordinates is

$$\widetilde{g}_N^{\alpha\beta} = \begin{cases} \widetilde{g}_N^{00} & = \frac{1}{A - \frac{1}{4} |\overline{\nabla} \bar{\rho}|_{\bar{g}}^2 + N(\eta')^2} \\ \widetilde{g}_N^{0i} & = -\frac{\mu}{2} \frac{\overline{\nabla}^i \bar{\rho}}{A - \frac{1}{4} |\overline{\nabla} \bar{\rho}|_{\bar{g}}^2 + N(\eta')^2} \\ \widetilde{g}_N^{ij} & = \bar{g}^{ij} + \frac{\mu^2}{4} \frac{\overline{\nabla}^i \bar{\rho} \overline{\nabla}^j \bar{\rho}}{A - \frac{1}{4} |\overline{\nabla} \bar{\rho}|_{\bar{g}}^2 + N(\eta')^2} \end{cases} \tag{6.22}$$

6.4.2 A degenerate space-time metric

Our interest is in the value of these quantities as $N \to \infty$. Clearly $\lim_{N\to\infty} \widetilde{g}_N$ is not well-defined, but

$$\widetilde{g}^{\alpha\beta} \doteqdot \lim_{N\to\infty} \widetilde{g}_N^{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha = 0 \text{ or } \beta = 0 \\ g^{\alpha\beta} & \text{if } 1 \le \alpha\ ,\ \beta \le n \end{cases} \tag{6.23}$$

defines a degenerate metric on $T^*\widetilde{\mathcal{M}}$. With an eye toward future calculations we observe that

$$(\widetilde{g}_N)^{00} = \frac{1}{N}(\eta')^{-2} + O\left(\frac{1}{N^2}\right) \tag{6.24}$$

$$(\widetilde{g}_N)^{0i} = -\frac{1}{N}\cdot\frac{\mu\overline{\nabla}^i\rho}{2} + O\left(\frac{1}{N^2}\right). \tag{6.25}$$

Similarly, noting that $\lim_{N\to\infty}\phi_N = 1$, we define the space-time normal, $\widetilde{\nu}$, and space-time second fundamental form, $\widetilde{h}$, by

$$\widetilde{\nu} \doteqdot \lim_{N\to\infty}\widetilde{\nu}_N = (\overline{\nu}, 0) \tag{6.26}$$

and

$$\widetilde{h}_{\alpha\beta} \doteqdot \lim_{N\to\infty}\left(\widetilde{h}_N\right)_{\alpha\beta} = \begin{cases} \widetilde{h}_{00} & = \frac{\partial}{\partial t}\overline{F} + \sigma\overline{F} \\ \widetilde{h}_{0i} & = \overline{\nabla}_i\overline{F} \\ \widetilde{h}_{ij} & = \bar{h}_{ij}. \end{cases} \tag{6.27}$$

6.4.3 A space-time connection

The limit of the Levi-Civita connections $\widetilde{\nabla}_N$ for the metrics $\widetilde{g}_N$ provides a compatible connection to accompany the degenerate metric $\widetilde{g}^{\alpha\beta}$.

Theorem 6.5. *The limit of the connections $\widetilde{\nabla}_N$, $\widetilde{\nabla}$, given in coordinates by*

$$\widetilde{\Gamma}^\gamma_{\alpha\beta} = \lim_{N\to\infty}\left(\widetilde{\Gamma}_N\right)^\gamma_{\alpha\beta} \tag{6.28}$$

defines a torsion-free connection that is compatible with the metric $\widetilde{g}^{\alpha\beta}$ defined by (6.23), in the sense that $\widetilde{\nabla}_\alpha\widetilde{g}^{\beta\gamma} = 0$ for all $0 \le \alpha, \beta, \gamma \le n$. Moreover, with this connection, the space-time second fundamental form satisfies the Codazzi equations:

$$\widetilde{\nabla}_\alpha\widetilde{h}_{\beta\gamma} = \widetilde{\nabla}_\beta\widetilde{h}_{\alpha\gamma} \tag{6.29}$$

Its components are

$$\widetilde{\Gamma}^0_{00} = \frac{\eta''}{\eta'} \tag{6.30}$$

$$\widetilde{\Gamma}^0_{\alpha i} = 0 \tag{6.31}$$

$$\widetilde{\Gamma}^k_{00} = -\overline{F}\overline{\nabla}^k\overline{F} \tag{6.32}$$

$$\widetilde{\Gamma}^k_{i0} = -\overline{F}\bar{h}^i_k + \mu\delta^i_k \tag{6.33}$$

$$\widetilde{\Gamma}^k_{ij} = \overline{\Gamma}^k_{ij}. \tag{6.34}$$

Proof. Since $\widetilde{\nabla}$ is the limit of the Levi-Civita connections for the metrics $\widetilde{g}_N$, it will be necessarily torsion-free and compatible with $\widetilde{g}$. Similarly, (6.29) will hold in the limit since

$$\left(\widetilde{\nabla}_N\right)_\alpha \left(\widetilde{h}_N\right)_{\beta\gamma} = \left(\widetilde{\nabla}_N\right)_\beta \left(\widetilde{h}_N\right)_{\alpha\gamma}$$

for all N.

As for the components of $\widetilde{\nabla}$, first observe that

$$\frac{\partial}{\partial x^\alpha}\left(\widetilde{g}_N\right)_{\beta\gamma} = \begin{cases} 2N\eta'\eta'' + O(1) & \text{if } \alpha=\beta=\gamma=0 \\ O(1) & \text{otherwise.} \end{cases} \tag{6.35}$$

□

Equations (6.31) and (6.34) then follow easily from this observation and the standard formula for the Christoffel symbols

$$\widetilde{\Gamma}^\gamma_{\alpha\beta} = \lim_{N\to\infty}\frac{1}{2}\left(\widetilde{g}_N\right)^{\gamma\delta}\left[\frac{\partial}{\partial x^\alpha}\left(\widetilde{g}_N\right)_{\beta\delta} + \frac{\partial}{\partial x^\beta}\left(\widetilde{g}_N\right)_{\alpha\delta} - \frac{\partial}{\partial x^\delta}\left(\widetilde{g}_N\right)_{\alpha\beta}\right]. \tag{6.36}$$

For equation (6.30), we compute

$$\begin{aligned}\left(\widetilde{\Gamma}_N\right)^0_{00} &= \frac{1}{2}\left(\widetilde{g}_N\right)^{00}\left(\frac{\partial}{\partial\bar{t}}\left(\widetilde{g}_N\right)_{00}\right) + \frac{1}{2}\left(\widetilde{g}_N\right)^{0i}\left(2\frac{\partial}{\partial\bar{t}}\left(\widetilde{g}_N\right)_{0i} - \frac{\partial}{\partial x^i}\left(\widetilde{g}_N\right)_{00}\right) \\ &= \frac{\eta''}{\eta'} + O\left(\frac{1}{N}\right).\end{aligned}$$

and for (6.33),

$$\begin{aligned}\left(\widetilde{\Gamma}_N\right)^k_{i0} &= \frac{1}{2}\left(\widetilde{g}_N\right)^{mk}\left(\frac{\partial}{\partial\bar{t}}\left(\widetilde{g}_N\right)_{im}\right) + \frac{1}{2}\left(\widetilde{g}_N\right)^{0i} + O\left(\frac{1}{N}\right) \\ &= -\overline{F}\bar{h}^k_i + \mu\delta^k_i + O\left(\frac{1}{N}\right).\end{aligned}$$

Lastly, we consider (6.32). Observe that

$$\overline{\nabla}^k \bar{s} = -\bar{g}^{pk} \left\langle \frac{\partial}{\partial x^p} \overline{\nu}, \overline{X} \right\rangle_{\mathbb{R}^{n+1}} = -\frac{1}{2} \bar{h}_p^k \overline{\nabla}^p \bar{\rho}, \tag{6.37}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \bar{t}} \overline{\nabla}_m \bar{\rho} &= 2 \frac{\partial}{\partial \bar{t}} \left\langle \frac{\partial \overline{X}}{\partial x^m}, \overline{X} \right\rangle_{\mathbb{R}^{n+1}} \\
&= 2 \left\langle -\frac{\partial \overline{F}}{\partial x^m} \overline{\nu} - \overline{F} \bar{h}_m^k \frac{\partial \overline{X}}{\partial x^k} + \mu \frac{\partial \overline{X}}{\partial x^m}, \overline{X} \right\rangle_{\mathbb{R}^{n+1}} \\
&\qquad\qquad + 2 \left\langle \frac{\partial \overline{X}}{\partial x^m}, -\overline{F} \overline{\nu} + \mu \frac{\partial \overline{X}}{\partial x^m} \right\rangle_{\mathbb{R}^{n+1}} \\
&= 2 \bar{s} \overline{\nabla}_m \overline{F} - \overline{F} \bar{h}_{mk} \overline{\nabla}^k \bar{\rho} + 2\mu \overline{\nabla}_m \bar{\rho}.
\end{aligned}$$

So

$$\begin{aligned}
\left(\widetilde{\Gamma}_N\right)_{00}^k =& \frac{1}{2} \left(\widetilde{g}_N\right)^{0k} \left(\frac{\partial}{\partial \bar{t}} \left(\widetilde{g}_N\right)_{00} \right) + \frac{1}{2} \left(\widetilde{g}_N\right)^{mk} \left(\frac{\partial}{\partial \bar{t}} \left(2\widetilde{g}_N\right)_{0m} - \frac{\partial}{\partial x^m} \left(\widetilde{g}_N\right)_{00} \right) \\
=& \frac{1}{2} \left(\mu' - \mu \frac{\eta''}{\eta'} \right) \frac{\partial}{\partial \bar{t}} \overline{\nabla}^k \bar{\rho} \\
&\qquad + \frac{\mu}{2} \bar{g}^{mk} \left(\frac{\partial}{\partial \bar{t}} \overline{\nabla}_m \bar{\rho} - \overline{\nabla}_m \left(\overline{F}^2 + 2\mu \bar{s} \overline{F} \mu^2 \bar{\rho} \right) \right) + O\left(\frac{1}{N}\right) \\
=& \frac{1}{2} \left(\mu^2 + \mu' - \mu \frac{\eta''}{\eta'} \right) \overline{\nabla}^k \bar{\rho} - \overline{F} \overline{\nabla}^k \overline{F} + O\left(\frac{1}{N}\right) \\
=& - \overline{F} \overline{\nabla}^k \overline{F} + O\left(\frac{1}{N}\right)
\end{aligned}$$

where the last line follows from our requirement on η, (6.21).

6.5 The Harnack quantity as second fundamental form

Consider again our original family of immersions $X(x,t)$ and suppose that our speed function is homogeneous of $\kappa \neq 1$. Via the transformation (6.6) we generate a solution $\overline{X}(x, \bar{t})$ to (6.8) for $\bar{t} \in (-\infty, \log T)$ with

$$\mu = -\frac{1}{\kappa + 1}.$$

For the immersion $(\overline{X}, \eta(\bar{t}))$, recall that we determined that η should satisfy

$$\mu' + \mu^2 - \mu\frac{\eta''}{\eta'} = 0,$$

and it is convenient to take $\eta(\bar{t}) = 1 + (\operatorname{sgn}\mu)e^{\mu\bar{t}}$. Then $\sigma = 0$, $\overline{F} = t^{\frac{\kappa}{\kappa+1}}F$, and

$$\widetilde{h}_{\alpha\beta} = \begin{cases} \widetilde{h}_{00} & = t^{\frac{2\kappa+1}{\kappa+1}}\left(\frac{\partial F}{\partial t} + \left(\frac{\kappa}{\kappa+1}\right)\frac{F}{t}\right) \\ \widetilde{h}_{0i} & = t^{\frac{\kappa}{\kappa+1}}\frac{\partial F}{\partial x^i} \\ \widetilde{h}_{ij} & = t^{\frac{-1}{\kappa+1}}\bar{h}_{ij}. \end{cases} \tag{6.38}$$

Given a vector $V \in T_x(M^n)$, define $\widetilde{V} \in T_{(x,\bar{t})}\widetilde{\mathcal{M}}$ by

$$\widetilde{V} \doteq \frac{\partial}{\partial t} + V^k\frac{\partial}{\partial x^k} = e^{-\bar{t}}\frac{\partial}{\partial \bar{t}} + V^k\frac{\partial}{\partial x^k}$$

and observe that

$$e^{\frac{1}{\kappa+1}\bar{t}}\widetilde{h}(\widetilde{V}, \widetilde{V}) = \frac{\partial F}{\partial t} + 2g(\nabla F, V) + h(V, V) + \left(\frac{\kappa}{\kappa+1}\right)\frac{F}{t}. \tag{6.39}$$

Thus, in light of Andrews's result,Theorem 2.4, we have the following correspondence.

Theorem 6.6. *Suppose $X : M^n \times (0,T) \to \mathbb{R}^{n+1}$ solves* (6.1) *where $F = F(W)$ is homogeneous of degree $\kappa \neq -1$, and let $\widetilde{h}$ be the associated space-time second fundamental form of* (6.38)*. Then, the satisfaction of an Andrews's-type Harnack inequality is equivalent to the positivity (negativity) of $\widetilde{h}$ as a quadratic form on $T(\widetilde{\mathcal{M}})$. In particular, when $\kappa > -1$ and F^* is $-\kappa$-concave, $\widetilde{h}$ is weakly-positive, and when $\kappa < -1$ and F^* is $-\kappa$-convex, $\widetilde{h}$ is weakly-negative.*

We will explore this correspondence further in the sequel. In Section 6.6 we will consider Andrews's theorem from the spacetime perspective and indicate the connection between his proof and the evolution equation satisfied by the minimizing vector field for $\widetilde{h}$. In Section 6.7, we will show that the machinery thus developed can be used to provide new proofs of these Harnack inequalities. For this latter application, we will need some further information about $\widetilde{h}$.

6.5.1 An evolution-type equation for the space-time second fundamental form

We now compute the derivatives of $\widetilde{h}$ with respect to the connection defined in the previous section. For the time being, we work with a space-time geometry obtained from a general solution $\overline{X}$ to the modified flow and do not assume that F has any homogeneity properties. Our goal is to prove the following

Theorem 6.7. *Suppose $\overline{X}$ evolves according to (6.8) with $\mu = (\mathrm{const})$, and $\sigma = \eta''/\eta' - \mu = 0$. Then*

$$\begin{aligned} \widetilde{\nabla}_0 \widetilde{h}_{\alpha\beta} = \widetilde{\Box}\widetilde{h}_{\alpha\beta} + \overline{\ddot{F}}^{pq,rs} \widetilde{\nabla}_\alpha \widetilde{h}_{pq} \widetilde{\nabla}_\beta \widetilde{h}_{rs} + \left(\overline{F} - \overline{\dot{F}}\left[\overline{W}\right]\right) \widetilde{g}^{pq} \widetilde{h}_{\alpha p} \widetilde{h}_{q\beta} \\ + \left(\overline{\dot{F}}\left[\overline{W}^2\right] - \mu\right) \widetilde{h}_{\alpha\beta} + L_{\alpha\beta}. \end{aligned} \tag{6.40}$$

where $\widetilde{\Box} \doteqdot \overline{\dot{F}}^{kl} \overline{\nabla}_k \overline{\nabla}_l$ and

$$L_{\alpha\beta} = \widetilde{h}_{\alpha p} \left(\overline{\dot{F}}^{pr} \bar{h}_r^q - \bar{h}_r^p \overline{\dot{F}}^{rq} \right) \widetilde{h}_{q\beta}$$

Note the resemblance between (6.40) and the evolution equation (6.11) satisfied by the second fundamental form $\bar{h}$ associated to $\overline{X}$.

Remark 6.8. Recall from Lemma 6.1 that when F is homogeneous of degree κ, $\overline{F}[\overline{W}] = \kappa \overline{F}$, and thus for flows homogeneous of degree one, the third term on the right-hand side of (6.40) vanishes identically. In particular, for the mean curvature flow, (6.40) takes the particularly simple form

$$\widetilde{\nabla}_0 \widetilde{h}_{\alpha\beta} = \widetilde{\Delta}\widetilde{h}_{\alpha\beta} + \left(\left| \widetilde{h}_{\gamma\delta} \right|^2_{\widetilde{g}} - \mu \right) \widetilde{h}_{\alpha\beta} = \widetilde{\Delta}\widetilde{h}_{\alpha\beta} + \left(\left| \bar{h}_{ij} \right|^2_{\bar{g}} - \mu \right) \widetilde{h}_{\alpha\beta}. \tag{6.41}$$

The proof of Proposition 6.7 is a lengthy, but straightforward computation. We will need to make use of the following identitites.

Lemma 6.9. *Under the flow (6.8), we have*

$$\left(\frac{\partial}{\partial\bar{t}}-\overline{\Box}\right)\frac{\partial\overline{F}}{\partial\bar{t}}=\ddot{\overline{F}}\left[\frac{\partial}{\partial\bar{t}}\overline{W},\frac{\partial}{\partial\bar{t}}\overline{W}\right]+\left(\dot{\overline{F}}\left[\overline{W}^2\right]-\mu\right)\frac{\partial\overline{F}}{\partial\bar{t}}+\left(\overline{F}-\dot{\overline{F}}\left[\overline{W}\right]\right)\left|\overline{\nabla F}\right|^2_{\bar{g}}$$
$$+2\overline{F}^2\dot{\overline{F}}\left[\overline{W}^3\right]+2\dot{\overline{F}}^{ac}\bar{h}_{bc}\left(2\overline{F}\overline{\nabla}^b\overline{\nabla}_a\overline{F}+\overline{\nabla}^b\overline{F}\overline{\nabla}_a\overline{F}\right)$$
$$-2\mu\left(\overline{\Box F}+\overline{F}\dot{\overline{F}}\left[\overline{W}^2\right]\right)-\mu'\dot{\overline{F}}\left[\overline{W}\right]. \tag{6.42}$$

$$\left(\frac{\partial}{\partial\bar{t}}-\overline{\Box}\right)\overline{\nabla}_i\overline{F}=\ddot{\overline{F}}\left[\frac{\partial}{\partial\bar{t}}\overline{W},\frac{\partial\overline{W}}{\partial x^i}\right]+\left(\dot{\overline{F}}\left[\overline{W}^2\right]-\mu\right)\overline{\nabla}_i\overline{F}-\dot{\overline{F}}\left[\overline{W}\right]\bar{h}_i^p\overline{\nabla}_p\overline{F}$$
$$+\dot{\overline{F}}^{ab}\bar{h}_{ia}\bar{h}_b^p\overline{\nabla}_p\overline{F}+2\overline{F}\dot{\overline{F}}^{ab}\bar{h}_a^p\overline{\nabla}_i\bar{h}_{bp}. \tag{6.43}$$

Proof. By equations (6.13), (6.12), and (6.16), we have

$$\frac{\partial}{\partial\bar{t}}\frac{\partial\overline{F}}{\partial\bar{t}}=\frac{\partial}{\partial\bar{t}}\left(\overline{\Box F}+\dot{\overline{F}}\left[\overline{FW}^2-\mu\overline{W}\right]\right)$$
$$=\overline{\Box}\frac{\partial\overline{F}}{\partial\bar{t}}+\ddot{\overline{F}}\left[\bar{g}^*\overline{\nabla\nabla F},\frac{\partial}{\partial\bar{t}}\overline{W}\right]+2\overline{F}\dot{\overline{F}}^{ik}\bar{h}_k^j\overline{\nabla}_i\overline{\nabla}_j\overline{F}-2\mu\overline{\Box F}$$
$$+2\dot{\overline{F}}^{ij}\bar{h}_j^k\overline{\nabla}_i\overline{F}\overline{\nabla}_k\overline{F}+\left(\overline{F}-\dot{\overline{F}}\left[\overline{W}\right]\right)\left|\overline{\nabla F}\right|^2+\frac{\partial\overline{F}}{\partial\bar{t}}\dot{\overline{F}}\left[\overline{W}^2\right]$$
$$+\overline{F}\ddot{\overline{F}}\left[\overline{W}^2-\mu\overline{W},\frac{\partial}{\partial\bar{t}}\overline{W}\right]+2\overline{F}\dot{\overline{F}}^{jk}\bar{h}_{ik}\left(\overline{\nabla}^i\overline{\nabla}_j\overline{F}+\overline{F}\bar{h}_j^k\bar{h}_k^i-\mu\bar{h}_j^i\right)$$
$$-\mu'\dot{\overline{F}}\left[\overline{W}\right]-\mu\frac{\partial\overline{F}}{\partial\bar{t}}$$
$$=\overline{\Box}\frac{\partial\overline{F}}{\partial\bar{t}}+\ddot{\overline{F}}\left[\bar{g}^*\overline{\nabla\nabla F}+\overline{FW}^2-\mu\overline{W},\frac{\partial}{\partial\bar{t}}\overline{W}\right]+\left(\dot{\overline{F}}\left[\overline{W}^2\right]-\mu\right)\frac{\partial\overline{F}}{\partial\bar{t}}$$
$$+\left(\overline{F}-\dot{\overline{F}}\left[\overline{W}\right]\right)\left|\overline{\nabla F}\right|^2+2\overline{F}\dot{\overline{F}}^{jk}\bar{h}_{ik}\left(2\overline{\nabla}^i\overline{\nabla}_j\overline{F}-\overline{\nabla}^i\overline{F}\overline{\nabla}_j\overline{F}\right)$$
$$+2\overline{F}^2\dot{\overline{F}}\left[\overline{W}^3\right]-2\mu\left(\overline{\Box F}+\overline{F}\left[\overline{W}^2\right]\right)-\mu'\dot{\overline{F}}\left[\overline{W}\right], \tag{6.44}$$

which proves (6.42). For (6.43), we apply equations (6.17) and (6.12) to (6.13), to find

$$
\begin{aligned}
\frac{\partial}{\partial t}\overline{\nabla}_i\overline{F} &= \overline{\nabla}_i\frac{\partial\overline{F}}{\partial\bar{t}} \\
&= \overline{\nabla}_i\overline{\Box}\overline{F} + \ddot{\overline{F}}\left[\overline{FW}^2 - \mu\overline{W}, \overline{\nabla}_i\overline{W}\right] \\
&\qquad + \dot{\overline{F}}^{pq}\left(\overline{\nabla}_i\overline{F}\bar{h}_p^k\bar{h}_{kq} + 2\overline{F}\overline{\nabla}_i\bar{h}_{pk}\bar{h}_q^k - \mu\overline{\nabla}_i\bar{h}_{pq}\right) \\
&= \overline{\Box}\overline{\nabla}_i\overline{F} + \ddot{\overline{F}}\left[\bar{g}^*\overline{\nabla}\overline{\nabla}\overline{F} + \overline{FW}^2 - \mu\overline{W}, \overline{\nabla}_i\overline{W}\right] + \left(\dot{\overline{F}}\left[\overline{W}^2\right] - \mu\right)\overline{\nabla}_i\overline{F} \\
&\qquad - \dot{\overline{F}}\left[\overline{W}\right]\bar{h}_i^p\overline{\nabla}_p\overline{F} + 2\overline{F}\dot{\overline{F}}^{pq}\overline{\nabla}_i\bar{h}_{pk}\bar{h}_q^k + \dot{\overline{F}}^{pq}\left(\bar{h}_{ip}\bar{h}_q^r\overline{\nabla}_r\overline{F}\right).
\end{aligned}
\tag{6.45}
$$

∎

Lemma 6.10. *The first covariant derivative of $\widetilde{h}$ with respect to $\widetilde{\nabla}$ is given in coordinates by*

$$
\widetilde{\nabla}_\gamma\widetilde{h}_{\alpha\beta} = \begin{cases}
\widetilde{\nabla}_0\widetilde{h}_{00} &= \frac{\partial}{\partial\bar{t}}\frac{\partial\overline{F}}{\partial\bar{t}} + 2\overline{F}\left|\overline{\nabla F}\right|^2 + (3\sigma - 2\mu)\frac{\partial\overline{F}}{\partial\bar{t}} + (\sigma' + 2\mu - 2\sigma\mu)\overline{F} \\
\widetilde{\nabla}_0\widetilde{h}_{i0} &= \widetilde{\nabla}_i\widetilde{h}_{00} = \overline{\nabla}_i\frac{\partial\overline{F}}{\partial\bar{t}} + 2\overline{F}\bar{h}_i^k\overline{\nabla}_k\overline{F} + (\sigma - 2\mu)\overline{\nabla}_i\overline{F} \\
\widetilde{\nabla}_0\widetilde{h}_{ij} &= \overline{\nabla}_i\widetilde{h}_{j0} = \overline{\nabla}_i\overline{\nabla}_j\overline{F} + \overline{F}\bar{h}_i^k\bar{h}_{kj} - \mu\bar{h}_{ij} \\
\widetilde{\nabla}_k\widetilde{h}_{ij} &= \overline{\nabla}_k\bar{h}_{ij}.
\end{cases}
\tag{6.46}
$$

Proof. Straightforward from equations (6.30) to (6.34) and the space-time Codazzi equations (6.29). ∎

Lemma 6.11. *In local coordinates, $\widetilde{\Box}\widetilde{h}$ is given by*

$$
\begin{aligned}
\widetilde{\Box}\widetilde{h}_{00} = \overline{\Box}\frac{\partial\overline{F}}{\partial\bar{t}} &+ 2\dot{\overline{F}}^{jk}\bar{h}_j^p\overline{\nabla}_p\overline{F}\overline{\nabla}_k\overline{F} + 2\overline{F}\left|\overline{\nabla F}\right|^2 + 4\overline{F}\dot{\overline{F}}^{jk}\bar{h}_j^p\overline{\nabla}_p\overline{\nabla}_k\overline{F} \\
&+ (\sigma - 4\mu)\overline{\Box}\overline{F} + 2\overline{F}^2\dot{\overline{F}}\left[\overline{W}^3\right] - 4\mu\overline{F}\dot{\overline{F}}\left[\overline{W}^2\right] + 2\mu^2\dot{\overline{F}}\left[\overline{W}\right]
\end{aligned}
\tag{6.47}
$$

$$
\widetilde{\Box}\widetilde{h}_{i0} = \overline{\Box}\overline{\nabla}_i\overline{F} + \dot{\overline{F}}^{kl}\bar{h}_i^p\bar{h}_{pk}\overline{\nabla}_l\overline{F} + \overline{F}\bar{h}_i^p\overline{\nabla}_p\overline{F} + 2\overline{F}\dot{\overline{F}}^{kl}\bar{h}_k^p\overline{\nabla}_i\bar{h}_{lp} - 2\mu\overline{\nabla}_i\overline{F} \tag{6.48}
$$

$$
\widetilde{\Box}\widetilde{h}_{ij} = \overline{\Box}\bar{h}_{ij} \tag{6.49}
$$

Proof. Using the previous lemma, we compute

$$\begin{aligned}\widetilde{\nabla}_k\widetilde{\nabla}_l\widetilde{h}_{00} &= \overline{\nabla}_k\left(\overline{\nabla}_l\frac{\partial\overline{F}}{\partial\bar{t}} + 2\overline{F}\bar{h}_l^p\overline{\nabla}_p\overline{F} + (\sigma-2\mu)\overline{\nabla}_l\overline{F}\right) - 2\widetilde{\Gamma}_{k0}^p\widetilde{\nabla}_l\widetilde{h}_{p0} \\ &= \overline{\nabla}_k\overline{\nabla}_l\frac{\partial\overline{F}}{\partial\bar{t}} + 2\bar{h}_l^p\overline{\nabla}_k\overline{F}\overline{\nabla}_p\overline{F} + 2\overline{F}\overline{\nabla}^p\bar{h}_{kl}\overline{\nabla}_p\overline{F} + 2\overline{F}\bar{h}_l^p\overline{\nabla}_p\overline{\nabla}_k\overline{F} \\ &\quad + (\sigma-4\mu)\overline{\nabla}_k\overline{\nabla}_l\overline{F} + 2\overline{F}\bar{h}_k^p\overline{\nabla}_p\overline{\nabla}_l\overline{F} + 2\overline{F}^2\bar{h}_k^p\bar{h}_p^q\bar{h}_{ql} \\ &\quad - 4\mu\overline{F}\bar{h}_k^p\bar{h}_{pl} + 2\mu^2\bar{h}_{kl}.\end{aligned} \tag{6.50}$$

Tracing the above with $\dot{\overline{F}}^{kl}$ and simplifying yields (6.47). Similarly, one obtains (6.48) and (6.49). □

With the above results in hand, we may return to the proof of the main result of this section.

Proof of Theorem 6.7. First, using (6.13), (6.46), (6.47), and (6.42), we compute

$$\begin{aligned}\widetilde{\nabla}_0\widetilde{h}_{00} - \widetilde{\Box}\widetilde{h}_{00} &= \left(\frac{\partial}{\partial\bar{t}} - \overline{\Box}\right)\frac{\partial\overline{F}}{\partial\bar{t}} - 2\overline{F}\dot{\overline{F}}^{kl}\bar{h}_k^p\left(2\overline{\nabla}_p\overline{\nabla}_l\overline{F} + \overline{\nabla}_p\overline{F}\overline{\nabla}_l\overline{F}\right) \\ &\quad - 2\overline{F}^2\dot{\overline{F}}\left[\overline{W}^3\right] + 2\mu\left(\overline{\Box F} + \overline{F}\dot{\overline{F}}\left[\overline{W}^2\right]\right) + 3\sigma\frac{\partial\overline{F}}{\partial\bar{t}} \\ &\quad - \sigma\overline{\Box F} + (\sigma' + 2\sigma^2 - 2\mu\sigma)\overline{F} \\ &= \ddot{\overline{F}}\left[\frac{\partial}{\partial\bar{t}}\overline{W}, \frac{\partial}{\partial\bar{t}}\overline{W}\right] + \left(\overline{F} - \dot{\overline{F}}\left[\overline{W}\right]\right)\left|\overline{\nabla F}\right|^2 \\ &\quad + \left(\dot{\overline{F}}\left[\overline{W}^2\right] - \mu + 2\sigma\right)\widetilde{h}_{00} + \left(\sigma\left(\dot{\overline{F}}\left[\overline{W}^2\right] - \mu\right) + \sigma'\right)\overline{F} \\ &\quad - (\sigma' + \sigma\mu)\,\dot{\overline{F}}\left[\overline{W}\right].\end{aligned} \tag{6.51}$$

Next, using (6.46), (6.48), and (6.43), we compute

$$
\begin{aligned}
\widetilde{\nabla}_0 \widetilde{h}_{i0} - \widetilde{\Box}\widetilde{h}_{i0} &= \left(\frac{\partial}{\partial \bar{t}} - \overline{\Box}\right) \overline{\nabla}_i \overline{F} + \overline{F}\bar{h}_i^p \overline{\nabla}_p \overline{F} + \overline{\dot{F}}^{kl} \bar{h}_k^p \bar{h}_{pi} \overline{\nabla}_l \overline{F} \\
&\quad - 2\overline{F}\overline{\dot{F}}^{kl} \bar{h}_k^p \overline{\nabla}_i \bar{h}_{lp} + \sigma \overline{\nabla}_i \overline{F} \\
&= \overline{\ddot{F}}\left[\overline{\nabla}_i \overline{W}, \frac{\partial}{\partial \bar{t}} \overline{W}\right] + \left(\overline{F} - \overline{\dot{F}}\left[\overline{W}\right]\right) \bar{h}_i^p \overline{\nabla}_p \overline{F} \\
&\quad + \overline{\dot{F}}^{kl} \left(\bar{h}_k^p \bar{h}_{il} \overline{\nabla}_p - \bar{h}_k^p \bar{h}_{pi} \overline{\nabla}_p \overline{F}\right) + \left(\overline{\dot{F}}\left[\overline{W}^2\right] - \mu + \sigma\right) \overline{\nabla}_i \overline{F} \\
&= \overline{\ddot{F}}\left[\overline{\nabla}_i \overline{W}, \frac{\partial}{\partial \bar{t}} \overline{W}\right] + \left(\overline{F} - \overline{\dot{F}}\left[\overline{W}\right]\right) \widetilde{h}_{i\gamma} \widetilde{g}^{\gamma\delta} \widetilde{h}_{\delta 0} \\
&\quad + \left(\overline{\dot{F}}\left[\overline{W}^2\right] - \mu + \sigma\right) \widetilde{h}_{i0} + \widetilde{h}_{ip} \left(\overline{\dot{F}}^{pq} \bar{h}_q^r - \bar{h}_q^p \overline{\dot{F}}^{qr}\right) \widetilde{h}_{r0}.
\end{aligned}
\tag{6.52}
$$

Finally, using (6.11), we compute

$$
\begin{aligned}
\widetilde{\nabla}_0 \widetilde{h}_{ij} - \widetilde{\Box}\widetilde{h}_{ij} &= \left(\frac{\partial}{\partial \bar{t}} - \overline{\Box}\right) \bar{h}_{ij} + 2\overline{F}\bar{h}_i^p \bar{h}_{pj} - 2\mu \bar{h}_{ij} \\
&= \overline{\ddot{F}}\left[\overline{\nabla}_i \overline{W}, \overline{\nabla}_j \overline{W}\right] + \left(\overline{\dot{F}}\left[\overline{W}^2\right] - \mu\right) \bar{h}_{ij} \\
&\quad + \left(\overline{F} - \overline{\dot{F}}\left[\overline{W}\right]\right) \bar{h}_i^p \bar{h}_{pj} + \overline{K}_{ij} \\
&= \overline{\ddot{F}}\left[\overline{\nabla}_i \overline{W}, \overline{\nabla}_j \overline{W}\right] + \left(\overline{\dot{F}}\left[\overline{W}^2\right] - \mu\right) \widetilde{h}_{ij} \\
&\quad + \left(\overline{F} - \overline{\dot{F}}\left[\overline{W}\right]\right) \widetilde{h}_{i\gamma} \widetilde{g}^{\gamma\delta} \widetilde{h}_{\delta j} + \widetilde{h}_{ip} \left(\overline{\dot{F}}^{pq} \bar{h}_q^r - \bar{h}_q^p \overline{\dot{F}}^{qr}\right) \widetilde{h}_{rj}.
\end{aligned}
\tag{6.53}
$$

Since by (6.12) and (6.46),

$$
\bar{g}_{ip} \left(\frac{\partial}{\partial \bar{t}} W\right)^p_j = \overline{\nabla}_i \overline{\nabla}_j + \overline{F}\bar{h}_i^p \bar{h}_{pj} - \mu h_{ij} = \widetilde{\nabla}_0 \widetilde{h}_{ij}
$$

and

$$
\bar{g}_{jp} \overline{\nabla}_i \bar{h}_k^p = \widetilde{\nabla}_i \widetilde{h}_{jk},
$$

it follows that

$$
\overline{\ddot{F}}\left[\frac{\partial}{\partial \bar{t}} \overline{W}, \frac{\partial}{\partial \bar{t}} \overline{W}\right] = \overline{\ddot{F}}^{pq,rs} \widetilde{\nabla}_0 \widetilde{h}_{pq} \widetilde{\nabla}_0 \widetilde{h}_{rs} \tag{6.54}
$$

$$
\overline{\ddot{F}}\left[\frac{\partial}{\partial \bar{t}} \overline{W}, \overline{\nabla}_i \overline{W}\right] = \overline{\ddot{F}}^{pq,rs} \widetilde{\nabla}_0 \widetilde{h}_{pq} \widetilde{\nabla}_i \widetilde{h}_{rs} \tag{6.55}
$$

$$
\overline{\ddot{F}}\left[\overline{\nabla}_i \overline{W}, \frac{\partial}{\partial \bar{t}} \overline{W}\right] = \overline{\ddot{F}}^{pq,rs} \widetilde{\nabla}_i \widetilde{h}_{pq} \widetilde{\nabla}_0 \widetilde{h}_{rs} \tag{6.56}
$$

$$
\overline{\ddot{F}}\left[\overline{\nabla}_i \overline{W}, \overline{\nabla}_j \overline{W}\right] = \overline{\ddot{F}}^{pq,rs} \widetilde{\nabla}_i \widetilde{h}_{pq} \widetilde{\nabla}_j \widetilde{h}_{rs}. \tag{6.57}
$$

Applying this observation and our assumption $\sigma = 0$ to the preceding identities completes the proof. □

6.6 Andrews's theorem from the space-time perspective

In the case that the speed F is homogeneous of degree κ, we have already observed (cf. Theorem 6.6), that the satisfaction of a Harnack-type inequality of the form of (1) and (3) of Theorem 2.4 is equivalent to the weak positivity of the quadratic form $\widetilde{h}$ on the space-time tangent bundle. Here we examine that correspondence a bit more closely.

In the paper [3], Andrews considers the reparametrization of the flow by the inverse of the Gauss map. Effectively this amounts to modifying the flow by a diffeomorphism φ of M^n as discussed in Section 2.2; his parametrization has the form

$$\frac{\partial Y}{\partial t}(z,t) = -F(\varphi_t(z),t)\nu(\varphi_t(z),t) - dX_{(\varphi_t(z),t)}T(z,t) \tag{6.58}$$

where $\varphi_t(z) = \nu_t^{-1}(z)$ and $T = (d\nu_{(\varphi_t(z),t)})^{-1}\nabla F(\varphi_t(z),t)$, i.e, $T^j = (h^{-1})^{ij}\nabla_i F$. Andrews demonstrates that, under this reparametrization, the entire right-hand side may be represented in terms of the support function of the hypersurface. Although here we are only interested in the relationship between his and the standard parametrization, one may consult the next chapter for the application of his technique to the very similar situation of space-like hypersurfaces in Minkowski space. For the proof of parts (1) and (3) of his theorem, he considers the evolution equation satisfied by the quantity

$$R = t\frac{\partial}{\partial t}\Phi(z,t) + \frac{\alpha}{\alpha-1}\Phi(z,t)$$

Here, $\Phi(z,t) = -F(\varphi_t(z,t)) = F^*(W^{-1}(z,t))$, so $\kappa = -\alpha$ and, in terms of the usual parametrization, using equation (2.10) together with the above description of T, we see that

$$R(\varphi_t^{-1}(x),t) = -t\left(\frac{\partial F}{\partial t} - (h^{-1})^{ij}\nabla_i F\nabla_j F + \frac{\kappa F}{\kappa+1}\right).$$

On the other hand, note that when X (hence $\overline{X}$) is convex, $\widetilde{h}$ is minimized by a vector field of the form

$$\widetilde{V} = \frac{\partial}{\partial\bar{t}} - (\bar{h}^{-1})^{ij}\overline{\nabla}_i\overline{F}\frac{\partial}{\partial x^j}$$

in which case

$$\overline{Z} := \widetilde{h}(\widetilde{V},\widetilde{V}) = \frac{\partial\overline{F}}{\partial\bar{t}} - (\bar{h}^{-1})^{ij}\overline{\nabla}_i\overline{F}\overline{\nabla}_j\overline{F}.$$

Tracing through the scaling of the various quantities involved, we have

$$\frac{\partial\overline{F}}{\partial\bar{t}} = t\frac{\partial}{\partial t}\left(t^{\frac{\kappa}{\kappa+1}}F\right) = t^{\frac{\kappa}{\kappa+1}}\left(t\frac{\partial F}{\partial t} + \frac{\kappa F}{\kappa+1}\right)$$

and

$$(\bar{h}^{-1})^{ij}\overline{\nabla}_i\overline{F}\overline{\nabla}_j\overline{F} = t^{\frac{2\kappa+1}{\kappa+1}}\left((h^{-1})^{ij}\nabla_iF\nabla_jF\right).$$

Thus

$$\begin{aligned}\overline{Z} &= t^{\frac{\kappa}{\kappa+1}}\left(t\left(\frac{\partial F}{\partial t} - (h^{-1})^{ij}\nabla_iF\nabla_jF\right) + \frac{\kappa F}{\kappa+1}\right)\\ &= -t^{\frac{\kappa}{\kappa+1}}R(\varphi_t^{-1}(x),t),\end{aligned}$$

so that the quantity R considered in Andrews's argument has a very natural interpretation from the space-time perspective. Of course, it is now clear that a proof of the weak-positivity of $\widetilde{h}$ via consideration of the minimizer reduces to Andrews's argument when X is convex. For reference, we note, however, that one may compute (either directly from the equations (6.9)-(6.13), or from the computations in [3] and translating the result back into the standard parametrization) that

$$\frac{\partial\bar{Q}}{\partial\bar{t}} = \overline{\Box}\left(\frac{(\kappa+1)\bar{Z}}{\kappa\bar{F}} + \dot{\bar{F}}(\bar{W}^2)\right)Q + S^{ab,cd}U_{ab}U_{cd}$$

where $Q = e^{\frac{\bar{t}}{\kappa+1}}\overline{Z}$,

$$S^{ab,cd} = \ddot{\overline{F}}^{ab,cd} + 2\dot{\overline{F}}^{ad}(\bar{h}^{-1})^{bc} - \frac{\kappa+1}{\kappa\overline{F}}\dot{\overline{F}}^{ab}\dot{\overline{F}}^{cd},$$

and

$$\begin{aligned}U_{ab} &= \bar{g}_{ac}\frac{\partial\bar{W}^c_b}{\partial\bar{t}} - (\bar{h}^{-1})^{pq}\overline{\nabla}_p\overline{F}\overline{\nabla}_q\bar{h}_{ab}\\ &= \widetilde{V}^\alpha\widetilde{\nabla}_\alpha\widetilde{h}_{ab}.\end{aligned}$$

When either of the conditions of parts (1) and (3) of Theorem 2.4 are met, the term $S^{ab,cd}U_{ab}U_{cd}$ will have a sign, and thus one can apply the maximum principle to the evolution of Q to obtain a proof of Andrews's Theorem (in the homogeneous, isotropic case) in terms of the space-time quantities.

6.7 Application: A space-time proof of Hamilton's Harnack inequality for the mean curvature flow

In this section, we provide a new proof of Hamilton's Harnack estimate for the mean curvature flow [46] by means of a generalized tensor maximum principle. Recall that, in this case, the evolution-type equation satisfied by the space-time second fundamental form $\widetilde{h}$ is particularly simple – the gradient terms vanish and the right hand side evidently satisfies a null-eigenvector condition, that is, it is non-negative (in fact, zero) at any null-eigenvector of $\widetilde{h}$. Thus the equation should be susceptible to an appropriate generalization of the usual tensor maximum principle [38] to tensors on $\widetilde{\mathcal{M}}$. One such generalization appears in [19], however, as $\widetilde{h}$ is only defined on $(0, T)$ and many of the quantities we are considering blow-up as $t \to 0$, it does not seem to apply directly. It is also no doubt possible to frame the following argument in somewhat more general terms, indeed into a form robust enough to apply to each (isotropic, homogeneous) case of Andrews's theorem. However, though the basic approach remains the same, the interplay of the scaling of the various quantities in conjunction with the convexity assumptions makes the statement of such a theorem rather complex. Therefore, as a simple proof of the theorem already exists, we satisfy ourselves here with an illumination of the technique in the specific situation of the mean curvature flow.

6.7.1 Preliminary estimates

When M^n is complete and non-compact, we will need some derivative estimates to control the growth of the curvature. The following theorem is due to Ecker and

Huisken [29].

Theorem 6.12 (Ecker-Huisken)**.** *Suppose $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ is a smooth solution to the mean curvature flow such that the initial immersion $X_0 := X(\cdot, 0)$ is complete and satisfies $|h_{ij}| \leq M$. Then, for all $\delta > 0$ and all $m = 0, 1, 2, \ldots$ there exists a constant $C = C(m, n, \delta, M)$ such that for all $(x,t) \in M^n \times (\delta, T-\delta)$*

$$\left|\nabla^{(m)} h_{ij}\right|(x,t) \leq C(m,n,\delta,M). \tag{6.59}$$

Of course, when M^n is non-compact, we cannot hope to have uniform derivative bounds which include $t = 0$, however, we can make effective use of the above estimates via a trick we learned from Lei Ni. Namely, observe that to prove Hamilton's theorem, it suffices to prove that

$$\frac{\partial H}{\partial t}(\cdot, t) + \langle \nabla H(\cdot, t), V\rangle + h(V,V) + \frac{H(\cdot,t)}{2(t-\delta)} \geq 0$$

for all $t \in (\delta, T-\delta)$ and then take $\delta \to 0$. In our case, this amounts to setting $\bar{t} = \log(t-\delta)$ and carrying out the rescaling and blow-up procedure as before. The resulting space-time second fundamental form is

$$\begin{aligned}\widetilde{h}_{(x,\bar{t})}\left(\widetilde{V}, \widetilde{V}\right) &= \frac{1}{\sqrt{t-\delta}}\left(\frac{\partial H}{\partial t}(x,t) + \frac{H(x,t)}{2(t-\delta)}\right)(\widetilde{V}^0)^2 \\ &\qquad + \frac{1}{\sqrt{t-\delta}}\left(2\widetilde{V}^0\langle \nabla_k H(x,t), \widetilde{V}^k\rangle_{g(x,t)} + h_{(x,t)}(\widetilde{V}^k, \widetilde{V}^k)\right) \\ &= (\widetilde{V}^0)^2\frac{\partial \overline{H}}{\partial \bar{t}} + 2\widetilde{V}^0\langle \nabla \overline{H}, \widetilde{V}^k\rangle + \bar{h}(\widetilde{V}^k, \widetilde{V}^k).\end{aligned} \tag{6.60}$$

Since the time factor in the space-time we consider is an open interval (in terms of $\bar{t}$: $(\infty, \log T - \delta)$), to use a barrier argument we need not only to ensure that $\widetilde{h}$ is bounded below and non-negative in the limit toward negative infinity, but that it actually have some form of uniform decay backward in time with respect to a reference metric. We choose the reference metric

$$\hat{g} = d\bar{t} \otimes d\bar{t} + \bar{g}_{ij}dx^i \otimes dx^j = \frac{dt \otimes dt}{(t-\delta)^2} + \frac{g_{ij}dx^i \otimes dx^j}{t-\delta} \tag{6.61}$$

and denote its Levi-Civita connection by $\widehat{\nabla}$. Via the estimates of Theorem 6.12, we can then verify next that $\widetilde{h}$ has suitable decay with respect to $\widetilde{g}$ as $\bar{t} \to -\infty$.

Lemma 6.13. *On any interval* $(-\infty, \alpha] \subset (-\infty, \log(T-\delta)]$, *there exists a constant* $C = C(\alpha)$ *such that*

$$\widetilde{h} \geq -Ce^{\bar{t}}\hat{g}. \tag{6.62}$$

Proof. By Theorem 6.12 and the evolution equations of Lemma 2.1, there is a constant C_1 so that

$$\left|\frac{\partial H}{\partial t}\right| + |\nabla H|_g \leq C_1$$

on $M^n \times [\delta, T-\delta]$. Thus, since $h_{ij} \geq 0$, we have, for any vector $\widetilde{V} = \widetilde{V}^0 \frac{\partial}{\partial \bar{t}} + \widetilde{V}^k \frac{\partial}{\partial x^k}$ in $T_{(x,\bar{t})}\widetilde{\mathcal{M}}$ for any $(x, \bar{t}) \in M^n \times (-\infty, \alpha]$,

$$\begin{aligned}
\widetilde{h}\left(\widetilde{V}, \widetilde{V}\right) &= \widetilde{h}_{00}\left(\widetilde{V}^0\right)^2 + 2\widetilde{h}_{0k}\widetilde{V}^0\widetilde{V}^k + \widetilde{h}_{ij}\widetilde{V}^i\widetilde{V}^j \\
&= e^{\frac{3\bar{t}}{2}}\left(\frac{\partial H}{\partial t} + \frac{H}{2t}\right)\left(\widetilde{V}\right)^2 + e^{\frac{\bar{t}}{2}}\nabla_k H\widetilde{V}^k + e^{-\frac{\bar{t}}{2}}h_{ij}\widetilde{V}^i\widetilde{V}^j \\
&\geq -C_1 e^{\frac{3\bar{t}}{2}}\left(\widetilde{V}^0\right)^2 - 2C_1 e^{\frac{\bar{t}}{2}}\left|\widetilde{V}^0\right|\left|\widetilde{V}^k\right|_g \\
&\geq -C_1 e^{\frac{3\bar{t}}{2}}\left(\widetilde{V}^0\right)^2 - C_1 e^{\bar{t}}\left(\left(\widetilde{V}^0\right)^2 + e^{-bt}\left|\widetilde{V}^k\right|^2\right) \\
&\geq -C(\alpha)e^{\bar{t}}\hat{g}(\widetilde{V}, \widetilde{V}).
\end{aligned}$$

□

The metric $\hat{g}$ will not in general be compatible with the connection $\widetilde{\nabla}$ defined by Lemma 6.5, but it is nearly so.

Lemma 6.14.

1. *The only non-zero components of* $\widetilde{\nabla}\hat{g}$ *are*

$$\widetilde{\nabla}_0\hat{g}_{00} = 1 \tag{6.63}$$

$$\widetilde{\nabla}_0\hat{g}_{0i} = \bar{H}\overline{\nabla}_i\bar{h} = e^{\bar{t}}H\nabla_i H \tag{6.64}$$

$$\widetilde{\nabla}_i\hat{g}_{0j} = \bar{H}\bar{h}_{ij} + \frac{1}{2}\bar{g}_{ij} = Hh_{ij} + \frac{e^{-\bar{t}}}{2}g_{ij}. \tag{6.65}$$

2. *The metric* $\hat{g}$ *satisfies the evolution-type equation*

$$\left(\widetilde{\nabla}_0 - \widetilde{\Box}\right)\hat{g}_{\alpha\beta} = \begin{cases} 1 + 2\left|\bar{P}_{ab}\right|_{\bar{g}}^2 - 2\mu & \alpha = \beta = 0 \\ -\bar{h}_i^k\overline{\nabla}_k\bar{H} & \text{either } \alpha \text{ or } \beta = i \geq 1 \\ 0 & \alpha, \beta \geq 1 \end{cases} \tag{6.66}$$

where $P_{ab} = \bar{F}\bar{h}_{ab} - \mu\bar{g}_{ab}$.

Proof. A straight-forward computation using equations (6.30) to (6.34). ∎

For later purposes, we note the following consequence

Corollary 6.15. For all $\tilde{V} = \widetilde{V}^0\frac{\partial}{\partial\bar{t}} + \widetilde{V}^k\frac{\partial}{\partial x^k}$, there exists a constant C depending on the bounds of the derivatives of h, such that

$$\left|\left(\widetilde{\nabla}_i\hat{g}\right)\left(\widetilde{V},\widetilde{V}\right)\right|_{\bar{g}} = \sqrt{\bar{g}^{pq}\widetilde{\nabla}_p\hat{g}_{0j}\widetilde{V}^0\widetilde{V}^j\widetilde{\nabla}_q\hat{g}_{0j}\widetilde{V}^0\widetilde{V}^j} \geq -C\left|\widetilde{V}^0\right|\left|\widetilde{V}^k\right|_{\bar{g}} \tag{6.67}$$

and

$$\left(\widetilde{\nabla}_0 - \widetilde{\Delta}\right)\hat{g}\left(\tilde{V},\tilde{V}\right) \geq (\widetilde{V}^0)^2 - Ce^{\frac{3\bar{t}}{2}}\hat{g}\left(\tilde{V},\tilde{V}\right). \tag{6.68}$$

Proof. Equation (6.67) follows directly from (6.65). Equation (6.68) follows from the estimates (6.59) and Cauchy's inequality:

$$\begin{aligned}
\bar{h}_i^k\overline{\nabla}_l\bar{H}\widetilde{V}^0\widetilde{V}^i &\geq -C'e^{\bar{t}}\left|\widetilde{V}^0\right|\left|\widetilde{V}^k\right|_g \\
&= -C'e^{\frac{3\bar{t}}{2}}\left(\left|\widetilde{V}^0\right|\left(e^{-\frac{\bar{t}}{2}}\left|\widetilde{V}^k\right|_g\right)\right) \\
&\geq -\frac{C'e^{3\frac{\bar{t}}{2}}}{2}\left(\left|\widetilde{V}^0\right|^2 + e^{-\bar{t}}\left|\widetilde{V}^k\right|_g^2\right) \\
&= -\frac{C'e^{\frac{3\bar{t}}{2}}}{2}\hat{g}\left(\widetilde{V},\widetilde{V}\right).
\end{aligned}$$

∎

6.7.2 A maximum principle argument

Now we turn to the main objective of this section.

Proof of Hamilton's Harnack estimate for the MCF. Given a strictly convex solution $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ to the mean curvature flow, where we assume either that M^n is compact or that the immersions $X(\cdot,t)$ are complete with bounded second fundamental form, we fix $\delta \in (0,T/2)$, set $\bar{t} = \log(t-\delta)$, $\Omega := \log(T-\delta)$ and consider the associated $\widetilde{h}$ on $\widetilde{\mathcal{M}} := M^n \times (-\infty,\Omega)$ as in Section 6.5. If M^n

is compact, let $\phi \equiv 1$, otherwise let ϕ be the function guaranteed by Lemma 2.7 (note that the derivative estimates (6.59), together with Lemma 2.1, ensure that the family of induced metrics meet the necessary requirements). Multiplying ϕ by an appropriate constant, we may assume $\phi \geq 1$.

We wish to show that $\tilde{b} := e^{-\frac{\bar{t}}{2}}\widetilde{h}$ is non-negative as a symmetric tensor on $T\widetilde{\mathcal{M}}$. For this purpose, it suffices to prove that there exists $\eta \in (0, T-\delta)$ such that for all $\epsilon > 0$, the tensor

$$\tilde{q} := \tilde{b} + \epsilon\phi(e^{\bar{t}} + \eta)\hat{g}$$

is strictly positive on $\widetilde{\mathcal{M}} \cap M^n \times (-\infty, \log \eta]$. This implies (after taking $\delta \to 0$) that the Harnack inequality holds on $(0, \eta]$, and we iterate, translating the time variable and applying the result to $(\eta, 2\eta]$, and so forth. We proceed by contradiction.

Suppose no such η exists. Fix an arbitrary η, then, and take an ϵ such that $\tilde{q}(\widetilde{V}, \widetilde{V}) < 0$ for some $(x, \bar{t}) \in M^n \times (-\infty, \log \eta)$ and some $\widetilde{V} \in T_{(x,\bar{t})}\widetilde{\mathcal{M}}$.

1. There exists a time $\bar{t}_0 > -\infty$ such that $\tilde{q}_{(x,\bar{t})}$ has a null eigenvector $\widetilde{V}$ in some tangent space $T_{(x,\bar{t}_0)}\widetilde{\mathcal{M}}$ (i.e., $\tilde{q}_{(x,\bar{t}_0)}\left(\widetilde{V}, \cdot\right) = 0$) but is strictly positive for $\bar{t} < \bar{t}_0$. To see this, note that $\tilde{b} \geq -Ce^{\frac{\bar{t}}{2}}\hat{g}$, by (6.62), and and that we also have $\phi \geq 1$, so

$$\begin{aligned}
\tilde{q}\left(\widetilde{V}, \widetilde{V}\right) &= \tilde{b}\left(\widetilde{V}, \widetilde{V}\right) + \epsilon\phi(e^{\bar{t}} + \eta) \\
&\geq \epsilon\eta\phi - Ce^{\frac{\bar{t}}{2}} \\
&\geq \epsilon\eta\left(\left(\phi - \frac{Ce^{\frac{\bar{t}}{2}}}{2\epsilon\eta}\right) + e^{\frac{\bar{t}}{2}}\left(e^{\frac{-\bar{t}}{2}} - \frac{C}{2\epsilon\eta}\right)\right).
\end{aligned}$$

at any $\widetilde{V}$ with $\left|\widetilde{V}\right|_{\hat{g}} = 1$. By our choice of ϕ, the right-hand side on the final inequality is strictly positive outside a compact set in $\widetilde{\mathcal{M}}$, and the claim follows.

2. At $\bar{t}_0$, then, choose $P_0 := (x_0, \bar{t}_0)$ and $\widetilde{V} \in T_{P_0}\widetilde{\mathcal{M}}$ such that $\left|\widetilde{V}\right|_{\hat{g}} = 1$ and $\tilde{q}_{P_0}\left(\widetilde{V}, \cdot\right) = 0$. Introduce normal coordinates $(x^i, \bar{t})$ in a neighborhood $\mathcal{U}$ of P_0, and write $e_0 = \frac{\partial}{\partial \bar{t}}$, $e_i = \frac{\partial}{\partial x^i}$ at this point.

The fact that $\tilde{W} \mapsto \tilde{q}_{P_0}(\tilde{W}, \tilde{W})$ is weakly positive as a map of tangent vectors has the following variational implications at P_0:

$$\tilde{b}_{00}\widetilde{V}^0 + \epsilon(e^{\bar{t}_0} + \eta)\widetilde{V}^0 = -\tilde{b}_{0i}\widetilde{V}^i \tag{6.69}$$

$$\tilde{b}_{0i}\widetilde{V}^i + \epsilon(e^{\bar{t}} + \eta)\bar{g}_{ij}\widetilde{V}^i = -\tilde{b}_{0j}\widetilde{V}^0. \tag{6.70}$$

3. We now extend $\widetilde{V}$ to a neighborhood $\mathcal{U} \subset \widetilde{\mathcal{M}}$ of P_0 as in the proof of Proposition 25 in [19]. First, we extend $\widetilde{V}$ by parallel translation along $\widetilde{\nabla}$-geodesics along $M^n \times \{\bar{t}\}$and then by parallel translation in the e_0-direction. Define $E : \mathcal{U} \subset \widetilde{\mathcal{M}} \to \mathbb{R}$ by

$$E(P) = q_P\left(\widetilde{V}, \widetilde{V}\right).$$

Observe that on some open $\mathcal{U}' \subset \mathcal{U}$ containing P_0, we still have $\left|\widetilde{V}\right|_{\hat{g}} > 1/2$ even with the incompatibility of $\hat{g}$ with $\widetilde{\nabla}$. Thus, by our choice of $\bar{t}_0$ it follows that E is is strictly positive on $\mathcal{U}' \cap (M^n \times (-\infty, \bar{t}_0))$ and thus P_0 is a local minimum for E on $\mathcal{U}' \cap (M^n \times (-\infty, \bar{t}_0])$. Using that $\widetilde{V}$ is a null-eigenvector and that our extension implies that the first covariant derivatives of $\widetilde{V}$ with respect to $\widetilde{\nabla}$ vanish at P_0, we have

$$0 \geq \frac{\partial}{\partial \bar{t}} E = \widetilde{\nabla}_0 \tilde{q}\left(\widetilde{V}, \widetilde{V}\right), \tag{6.71}$$

$$0 = \frac{\partial}{\partial x^i} E = \widetilde{\nabla}_i \tilde{q}\left(\widetilde{V}, \widetilde{V}\right), \tag{6.72}$$

and

$$\begin{aligned} 0 &\leq \frac{\partial^2 E}{\partial x^i \partial x^j} \\ &= \left(\widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_j} \tilde{q}\right)(\widetilde{V}, \widetilde{V}) + 2\left(\widetilde{\nabla}_{e_k} \tilde{q}\right)\left(\widetilde{\nabla}_{e_l} \widetilde{V}, \widetilde{V}\right) + 2\left(\widetilde{\nabla}_{e_l} \tilde{q}\right)\left(\widetilde{\nabla}_{e_k} \widetilde{V}, \widetilde{V}\right) \\ &\qquad + 2\tilde{q}\left(\widetilde{\nabla}_{e_k} \widetilde{\nabla}_{e_l} \widetilde{V}, \widetilde{V}\right) + 2\tilde{q}\left(\widetilde{\nabla}_{e_k} \widetilde{V}, \widetilde{\nabla}_{e_l} \widetilde{V}\right) \\ &= \left(\widetilde{\nabla}_{e_k} \widetilde{\nabla}_{e_l} \tilde{q}\right)(\widetilde{V}, \widetilde{V}). \end{aligned} \tag{6.73}$$

at P_0. Henceforth all the quantities we consider will be evaluated at P_0 and we will suppress it notation.

Upon tracing (6.73) by the degenerate metric $\widetilde{g}$, we obtain

$$0 \geq \widetilde{\Delta} E = \left(\widetilde{\Delta} \tilde{q}\right)\left(\widetilde{V}, \widetilde{V}\right)$$

where $\widetilde{\Delta} = \widetilde{g}^{ij} \widetilde{\nabla}_i \widetilde{\nabla}_j = \bar{g}^{ij} \widetilde{\nabla}_i \widetilde{\nabla}_j$. So

$$\left(\widetilde{\nabla}_0 - \widetilde{\Delta}\right) \tilde{q}\left(\widetilde{V}, \widetilde{V}\right) \leq 0 \tag{6.74}$$

at P_0.

4. On the other hand (noting that $\widetilde{\Delta} = \Delta_{\bar{g}}$ for time-independent functions on $\widetilde{\mathcal{M}}$), we compute that

$$\begin{aligned}\left(\widetilde{\nabla}_0 - \widetilde{\Delta}\right)\tilde{q} = \epsilon\phi e^{\bar{t}}\hat{g}\left(\widetilde{\nabla}_0 - \widetilde{\Delta}\right)\tilde{b} + \epsilon\phi(e^{\bar{t}} + \eta)\left(\widetilde{\nabla}_0 - \widetilde{\Delta}\right)(\hat{g}) \\ - \epsilon(e^{\bar{t}} + \eta)\left(2\bar{g}^{ij}\nabla_i\phi\widetilde{\nabla}_j\hat{g} + (\Delta\phi)\,\hat{g}\right).\end{aligned} \tag{6.75}$$

We consider the second through the last terms on the right-hand side of (6.75) in turn. First, using (6.40), we compute that

$$\begin{aligned}\left(\widetilde{\nabla}_0 - \widetilde{\Delta}\right)\tilde{b} &= \left(\widetilde{\nabla}_0 - \widetilde{\Delta}\right)e^{-\frac{\bar{t}}{2}}\widetilde{h} \\ &= e^{-\frac{\bar{t}}{2}}\left(\widetilde{\nabla}_0 - \widetilde{\Delta}\right)\widetilde{h} - \frac{1}{2}\tilde{b} \\ &= e^{-\frac{\bar{t}}{2}}\left(\left|\bar{h}_{ij}\right|_{\bar{g}} + \frac{1}{2}\right)\widetilde{h} - \frac{1}{2}\tilde{b} \\ &= \left|\bar{h}_{ij}\right|_{\bar{g}}^2\tilde{b}.\end{aligned}$$

Thus,

$$\begin{aligned}\left(\widetilde{\nabla}_0 - \widetilde{\Delta}\right)\tilde{b}\left(\widetilde{V}, \widetilde{V}\right) &= -\epsilon\phi(e^{\bar{t}} + \eta)\left|\bar{h}_{ij}\right|_{\bar{g}}^2 \\ &\geq -2e^{\bar{t}}\eta\epsilon\phi C_1,\end{aligned} \tag{6.76}$$

where we have used

$$e^{\bar{t}} \leq \eta \quad \text{and} \quad \left|\bar{h}_{ij}\right|_{\bar{g}}^2 \leq C_1 e^{\bar{t}}$$

for some $C_1 > 0$.

Then, using (6.68) and (6.67) , respectively, we find

$$\begin{aligned}\epsilon\phi(e^{\bar{t}} + \eta)\left(\widetilde{\nabla}_0 - \widetilde{\Delta}\right)\hat{g}\left(\tilde{V}, \tilde{V}\right) \\ \geq \epsilon\phi(e^{\bar{t}} + \eta)(\widetilde{V}^0)^2 - \epsilon\eta e^{\frac{3\bar{t}}{2}}C_2,\end{aligned} \tag{6.77}$$

and

$$\begin{aligned}-2\epsilon(e^{\bar{t}} + \eta)\left(\bar{g}^{ij}\nabla_i\phi\widetilde{\nabla}_j\hat{g} - (\Delta\phi)\,\hat{g}\right)\left(\widetilde{V}, \widetilde{V}\right) \\ \geq -2\epsilon\left(e^{\bar{t}} + \eta\right)\left(C_3\left|\widetilde{V}^0\right|\left|\nabla\phi\right|_{\bar{g}}\left|\widetilde{V}^k\right|_{\bar{g}} + \Delta_{\bar{g}}\phi x\right),\end{aligned}$$

and, by Lemma 2.7, we know there exists $C_4 > 0$ such that

$$\left|\nabla\phi\right|_{\bar{g}} \leq C_4 e^{\frac{\bar{t}}{2}} \quad \text{and} \quad \Delta_{\bar{g}}\phi \leq C_4 e^{\bar{t}} \leq C_4\phi e^{\bar{t}}.$$

Thus,

$$\begin{aligned} & -\epsilon(e^{\bar{t}}+\eta)\left(2\bar{g}^{ij}\nabla_i\phi\widetilde{\nabla}_j\hat{g}+(\Delta\phi)\,\hat{g}\right)\left(\widetilde{V},\widetilde{V}\right) \\ & \qquad \geq -2C_3C_4\epsilon\phi(e^{\bar{t}}+\eta)e^{\frac{\bar{t}}{2}}\left|\widetilde{V}^0\right|\left|\widetilde{V}^k\right|_{\bar{g}}-2\epsilon\eta\phi e^{\bar{t}}C_4. \end{aligned} \tag{6.78}$$

Using Cauchy's inequality again, we see that for any A,

$$e^{\frac{\bar{t}}{2}}\left|\widetilde{V}^0\right|\left|\widetilde{V}^k\right|_{\bar{g}} \leq \frac{\left(V^0\right)^2}{2A}+\frac{Ae^{\bar{t}}\left|\widetilde{V}^k\right|_{\bar{g}}^2}{2}.$$

Taking $A=2C_3C_4$ and using $\left|\widetilde{V}^k\right|_{\bar{g}}^2 \leq \left|\widetilde{V}\right|_{\hat{g}}^2=1$, (6.78) becomes

$$\begin{aligned} & -\epsilon(e^{\bar{t}}+\eta)\left(2\bar{g}^{ij}\nabla_i\phi\widetilde{\nabla}_j\hat{g}+(\Delta\phi)\,\hat{g}\right)\left(\widetilde{V},\widetilde{V}\right) \\ & \qquad \geq -\epsilon\phi(e^{\bar{t}}+\eta)\frac{\left(V^0\right)^2}{2}-C_5\epsilon\eta e^{\bar{t}}-4\epsilon\eta\phi e^{\bar{t}}C_4, \end{aligned} \tag{6.79}$$

where C_5 depends on the curvature bounds and the bounds for ϕ.

Taken together, equations (6.75), (6.76), (6.77), and (6.79) show

$$\begin{aligned} & \left(\widetilde{\nabla}_0-\widetilde{\Delta}\right)\tilde{q}\left(\widetilde{V},\widetilde{V}\right) \\ & \qquad \geq \frac{\epsilon\phi(e^{\bar{t}}+\eta)\left(\widetilde{V}^0\right)^2}{2}+\epsilon\phi e^{\bar{t}}-2e^{\bar{t}}\eta\epsilon\phi C_1-\epsilon\eta e^{\frac{3\bar{t}}{2}}C_2-C_5\epsilon\eta e^{\bar{t}}-4\epsilon\eta e^{\bar{t}}C_4 \\ & \qquad \geq \epsilon\phi e^{\bar{t}}\left(1-\eta(C_6+C_7\sqrt{\eta})\right). \end{aligned} \tag{6.80}$$

Therefore, since the constants C_6 and C_7 are independent of η we may obtain a contradiction with equation (6.74) by taking η sufficiently small, completing the proof. □

7 Harnack inequalities for space-like hypersurfaces in Minkowski space

7.1 Preliminaries

Let $\mathbb{L}^{n+1}$ denote $n+1$-dimensional Minkowski space, which which we will regard as $\mathbb{R}^{n+1}$ equipped with the Lorentzian metric

$$\langle \cdot, \cdot \rangle := -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + \ldots + dx^n \otimes dx^n.$$

Recall that vectors $v \in \mathbb{L}^{n+1}$ are said to be *space-like* if $\langle v, v \rangle > 0$, *time-like* if $\langle v, v \rangle < 0$ and *light-like* if $\langle v, v \rangle = 0$. Accordingly, a submanifold $\mathcal{N} \subset \mathbb{L}^{n+1}$ is said to be *space-like* if its tangent vectors are all space-like, or, equivalently, if the induced metric $g = \langle \cdot, \cdot \rangle|_{\mathcal{N}}$ is Riemannian.

If $\mathcal{N} = \mathcal{N}^n \subset \mathbb{L}^{n+1}$ is a space-like hypersurface, the subspace transverse to each tangent space is one-dimensional and time-like, and we may assign (a priori, only locally) a smooth choice of time-like normal, ν, satisfying $\langle \nu, \nu \rangle \equiv -1$. We may also choose this ν to be *future-directed* i.e., to satisfy $\langle \nu, e_0 \rangle < 0$. It can be shown [47] that a space-like hypersurface for which the induced Riemannian metric is complete must be complete as a subset of R^{n+1} (with respect to the standard Euclidean metric), and, by considering the projection of $\mathcal{N}$ onto the e_0 direction, the graph of a function $u : \mathbb{R}^n \to \mathbb{R}$ satisfying $|du| < 1$. Here and throughout this chapter, we will use $|\cdot|$ to denote the Euclidean length on $\mathbb{R}^n \approx \{(0, x^1, x^2, \ldots, x^n)\} \subset \mathbb{L}^{n+1}$. Conversely, the graph $\{\, (u(x), x) \,|\, x \in \mathbb{R}^n \,\}$ of any such function u gives rise to a complete space-like hypersurface. Of particular interest to us is the following

Example 7.1 (Hyperbolic space.)**.** Let $u(x) = \sqrt{1+|x|^2}$. Then

$$\frac{\partial u}{\partial x^i} = \frac{x^i}{\sqrt{1+|x|^2}},$$

so $|du| < 1$ and graph(u) is a space-like hypersurface. In fact, graph(u) is the upper sheet of the set

$$\{\, v \in \mathbb{L}^{n+1} \,|\langle v, v\rangle = -1 \,\}$$

and a standard model for n-dimensional hyperbolic space. Indeed, from equation (7.1), we see that the metric satisfies

$$g_{ij} = \delta_{ij} - \frac{x_i x_j}{1+|x|^2}.$$

In this chapter we consider a smooth family of space-like immersions $X : M^n \times [0,T) \to \mathbb{L}^{n+1}$ which evolve according to an equation of the form

$$\frac{\partial}{\partial t}X(x,t) = F(W(x,t), \nu(x,t))\nu(x,t) \tag{7.1}$$

where $\nu : M^n \times [0,T) \to \mathbb{L}^{n+1}$ is a future-directed time-like vector field on the hypersurface $X_t(M^n)$. We assume initially that the function F is a smooth function of the principal curvatures and the normal direction and satisfies $\dot{F} > 0$ – assumptions which ensure that equation (7.1) is strictly parabolic and invariant under diffeomorphisms of the domain and translations of X in $\mathbb{L}^{n+1}$. We will also assume that our immersions are strictly locally convex and that the induced Riemannian metrics $X_t^*\langle\cdot,\cdot\rangle$ are complete. By the preceding discussion, $X_t(M^n)$ must be a complete graph, and M^n must be diffeomorphic to $\mathbb{R}^n$.

In view of Example 7.1, the vector field ν_t defines a map (the *Gauss map*)

$$\nu_t : M^n \to \mathbb{H}^n \subset \mathbb{L}^{n+1}. \tag{7.2}$$

From the definition of the Weingarten map (cf. Section 2.2), the strict local convexity of the immersions X implies that $d\nu_t$ is positive-definite and, therefore, that ν_t is locally a diffeomorphism onto its image, which we shall denote by

$\mathcal{G}_t := \nu_t(M^n) \subset \mathbb{H}^n$. But since $M^n \approx \mathbb{R}^n$, it follows that for all t, ν_t is, in fact, globally a diffeomorphism onto the open subset $\mathcal{G}_t$.

For evolving hypersurfaces in Euclidean space, it was Andrews's observation in [3] that reparametrizing the hypersurfaces by the inverse of the Gauss map (in that case, the usual map $M^n \to S^n$) vastly simplified the derivation of the evolution equations for the Harnack quantities. In that situation, the compactness of M^n and the strict convexity of the immersions, implied that the manifold M^n was diffeomorphic to the sphere S^n (or some finite cover if $n = 1$) and, in particular, that the Gauss map had constant image S^n. In our case, unless the parametrization has full Gauss image $\mathcal{G}_t = \mathbb{H}^n$, the domain of ν_t^{-1} will be a time-varying open subset of $\mathbb{H}^n$. Though this will not affect our derivation of local quantities, the potential lack of completeness (and necessary non-compactness) will complicate our ultimate application of the maximum principle. In [2], Andrews also briefly considers the case where his domain M^n is non-compact and below, we follow the basic thread of his argument, which is (1) using the Gauss map parametrization to compute evolution equations and other local quantities and (2) reverting to the standard parametrization to apply the maximum principle. In this latter step Andrews considers the limit of local estimates on domains of increasing size; we will consider a general (global) maximum principle for solutions satisfying appropriate pointwise growth bounds.

7.2 The Gauss map parametrization

For $t \in [0, T)$, define $\varphi : \mathcal{G}_t \subset \mathbb{H}^n \to \mathbb{L}^{n+1}$ by

$$\varphi(z, t) = X(\nu_t^{-1}(z), t). \tag{7.3}$$

Note that the Gauss map $\bar{\nu} : \varphi(\mathcal{G}_t, t) \to \mathbb{H}^n$ is the identity map: $\bar{\nu}_t(z) = z$. As we observe next, this representation has the effect of reducing the study of the flow (7.1) to the study of an evolving scalar function on an open subset of $\mathbb{H}^n$.

Lemma 7.2. *Define $s : \mathcal{G}_t \to \mathbb{R}$ by $s(z, t) = \langle \varphi(z, t), z \rangle$, and denote by $\bar{g}$ and $\overline{\nabla}$ the canonical metric and Levi-Civita connection on $\mathbb{H}^n$. Then, for all $t \in [0, T)$ and $z \in \mathcal{G}_t$, we have the identities*

1.

$$\varphi(z,t) = -s(z,t)z + (\mathrm{grad}_{\mathbb{H}^n} s)(z,t) \tag{7.4}$$

and

2.

$$(d\varphi)^i_j = \overline{\nabla}^i \overline{\nabla}_j s - s\delta^i_j. \tag{7.5}$$

Proof. Following the argument in [3], we first write

$$\varphi(z,t) = -s(z,t)z + V(z,t)$$

for some unknown tangential vector field $V \in T_{\varphi(z,t)}(\varphi_t(\mathcal{G}_t)) \approx T_z(\mathbb{H}^n)$, and differentiate the equation along an arbitrary tangential direction U to obtain

$$d\varphi(U) = -ds(U)z - sU + D_U V.$$

Here D denotes the canonical (flat) connection on the ambient space $\mathbb{L}^{n+1}$. By the Gauss equation (2.4) (with $\sigma = -1$), and using that $\bar{g} = \bar{h}$ on $\mathbb{H}^n$, we have, according to our convention for $\bar{h}$, that

$$\begin{aligned} d\varphi(U) &= -ds(U)z - sU + \overline{\nabla}_U V + \bar{h}(U,V)z \\ &= -ds(U)z - sU + \overline{\nabla}_U V + \bar{g}(U,V)z. \end{aligned} \tag{7.6}$$

Now, by construction, $d\varphi_t(T_z(\mathbb{H}^n))$ is parallel to $T_z(\mathbb{H}^n)$. Thus the normal component of (7.6) must vanish, which, since U is arbitrary, implies the identity $V = \mathrm{grad}_{\mathbb{H}^n} s$, proving (7.4).

To prove (7.5), then, we consider the remaining (tangential) terms in (7.6) and use (7.4). □

Recalling for a moment that the Weingarten map of the hypersurface $X_t(M^n)$ at $X(\nu_t^{-1}(z),t) = \varphi(z,t)$ is given by $W = d\varphi^{-1} \circ d\bar{\nu} = d\varphi^{-1}$, we see that the map

$$A^i_j := d\varphi^i_j = \overline{\nabla}^i \overline{\nabla}_j s - s\delta^i_j \tag{7.7}$$

satisfies $A = W^{-1}$ – thus, the eigenvalues of A are the principal radii of the hypersurface.

7.2.1 Evolution equations in the Gauss map parametrization

As we noted earlier, the Gauss images $\mathcal{G}_t$ will be changing with time. Nevertheless, since each $\mathcal{G}_t$ is open and ν_t is changing smoothly with time, if $z \in \mathcal{G}_{t_0}$ for some $t_0 \in [0, T)$, then we can find an open neighborhood $\mathcal{U} \subset \mathbb{H}^n$ of z for which $\mathcal{U} \subset \mathcal{G}_t$ for all $t \in (t_0 - \epsilon, t + \epsilon) \cap [0, T)$ provided $\epsilon > 0$ is sufficiently small. Thus we may compute evolution equations for quantities associated with the flow in the Gauss map parametrization. Following [3], we first consider the evolution of s.

Differentiating (7.3) and (7.4) with respect to time and equating the normal components, we find that

$$\frac{\partial s}{\partial t}(z,t) = -F(W(v_t^{-1}(z),t),z) := \Phi_t(A(z,t),z), \tag{7.8}$$

where we define $\Phi_t : \Gamma_+ \times \mathcal{G}_t \to R$ on the subset Γ_+ of positive definite symmetric maps in $\mathrm{End}(T\mathbb{H}^n)$ by

$$\Phi_t(B,z) := -F(B^{-1}, z).$$

Note that we have

$$\dot{\Phi}_B(C) = \dot{F}(B^{-1}CB^{-1})$$

for any $B \in \Gamma$, $C \in \mathrm{End}(T\mathbb{H}^n)$, so that $\dot{\Phi} > 0$, by our assumption on F, which implies that equation (7.8) is strictly parabolic.

One of the advantages of the Gauss map parametrization is that it the various quantities are expressed in terms of the fixed metric $\bar{g}$ and connection $\overline{\nabla}$, and only depend on time through the function s. It is a simple matter, then, to compute the evolution equations for the various quantities.

Lemma 7.3. *Let $\Lambda^{ij} := \bar{g}^{jk}\Phi^i_j$ and let $\mathcal{L}$ denote the elliptic operator $\mathcal{L} := \Lambda^{kl}\overline{\nabla}_k\overline{\nabla}_l$, and define*

$$Q := \frac{\partial A}{\partial t} \text{ and } P := \frac{\partial \Phi}{\partial t}.$$

Then, in terms of the Gauss map parametrization, the quantities A, Φ, and P

satisfy the following evolution equations:

$$Q^i_j = \overline{\nabla}^i\overline{\nabla}_j\Phi - \Phi\delta^i_j \tag{7.9}$$

$$P = \mathcal{L}\Phi - \dot{\Phi}(\mathrm{Id})\Phi \tag{7.10}$$

$$\frac{\partial P}{\partial t} = \mathcal{L}P - \dot{\Phi}(\mathrm{Id})P + \ddot{\Phi}(Q, Q). \tag{7.11}$$

Proof. Equation (7.9) follows from differentiating (7.7) and recalling that the metric and connection are independent of time. Equations (7.10) and (7.11) follow then by noting that $P(z,t) = \frac{\partial}{\partial t}\Phi(A(z,t), z) = \dot{\Phi}(Q)$ and applying (7.9). □

Remark 7.4. Equations (7.9), (7.10), and (7.11) differ only from their Euclidean counterparts in the minus sign on the second term. This term will not affect the analysis that follows.

Next, we recall the definition of α-concavity (convexity) from Chapter 2 and prove the following estimates for the Harnack quantities.

Proposition 7.5. Suppose X is a solution to (7.1). Then $R = \frac{\partial}{\partial t}\log\Phi$ satisfies

$$\frac{\partial}{\partial t}R = \mathcal{L}R + \frac{2}{\Phi}\Lambda^{ab}\overline{\nabla}_a\Phi\overline{\nabla}_b R + \frac{\ddot{\Phi}(Q,Q)}{\Phi} \tag{7.12}$$

Consequently, if Φ is positive and concave (convex), we have

$$\frac{\partial}{\partial t}R - \mathcal{L}R - \frac{2}{\Phi}\Lambda^{ab}\overline{\nabla}_a\Phi\overline{\nabla}_b R \leq 0 \qquad (\geq 0), \tag{7.13}$$

if Φ is α-concave for some $0 < \alpha < 1$ we have

$$\frac{\partial}{\partial t}R - \mathcal{L}R - \frac{2}{\Phi}\Lambda^{ab}\overline{\nabla}_a\Phi\overline{\nabla}_b R - \frac{\alpha-1}{\alpha}R^2 \leq 0, \tag{7.14}$$

and the reverse inequality holds if Φ is α-concave for some $\alpha < 0$ (in which case $\Phi < 0$), or if Φ is α-convex for some $\alpha > 1$.

Proof. We compute, using (7.10) and (7.11), that

$$\begin{aligned}\frac{\partial R}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{P}{\Phi}\right)\\ &= \frac{1}{\Phi}\Big(\mathcal{L}P - P\dot{\Phi}(\mathrm{Id}) + \ddot{\Phi}(Q,Q)\Big) - R^2,\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}\left(\frac{P}{\Phi}\right) &= \frac{\mathcal{L}P}{\Phi} - \frac{2}{\Phi^2}\Lambda^{ab}\overline{\nabla}_a P\overline{\nabla}_b\Phi - \frac{P\mathcal{L}\Phi}{\Phi^2} + 2\frac{P}{\Phi^3}\Lambda^{ab}\overline{\nabla}_a\Phi\overline{\nabla}_b\Phi \\
&= \frac{\mathcal{L}P}{\Phi} - \frac{P\mathcal{L}\Phi}{\Phi^2} - \frac{2}{\Phi}\Lambda^{ab}\overline{\nabla}_a\Phi\overline{\nabla}_b R \\
&= \frac{\mathcal{L}P}{\Phi} - \frac{P\left(\mathcal{L}\Phi - \Phi\dot{\Phi}(\mathrm{Id})\right) + \Phi\dot{\Phi}(\mathrm{Id})P}{\Phi^2} - \frac{2}{\Phi}\Lambda^{ab}\overline{\nabla}_a\Phi\overline{\nabla}_b R \\
&= \frac{\mathcal{L}P}{\Phi} - R^2 + \dot{\Phi}(\mathrm{Id})R - \frac{2}{\Phi}\Lambda^{ab}\overline{\nabla}_a\Phi\overline{\nabla}_b R.
\end{aligned}$$

Thus

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)R = \frac{2}{\Phi}\Lambda^{ab}\overline{\nabla}_a\Phi\overline{\nabla}_b R + \frac{\ddot{\Phi}(Q,Q)}{\Phi},$$

and (7.13) and (7.14) follow by considering the implications of the concavity, α-concavity, etc., of Φ, and using that $P = \dot{\Phi}(Q)$.

□

7.2.2 Evolution equations in terms of the standard parametrization

Now we wish to transfer the argument back from the parametrization on $\mathcal{G}_t \subset \mathbb{H}^n$ to M^n. The Gauss map parametrization differs only from the standard parametrization by a diffeomorphism of M^n, and thus, as in [3] and [2], (cf. also the discussion in Section 6.6) we compute that P and its evolution transform as follows. We consider only the case in which the flow is isotropic, i.e., $F = F(W)$.

Lemma 7.6. *In the standard parametrization, the quantity $P = \frac{\partial\Phi}{\partial t}$ is given by*

$$P = -\frac{\partial F}{\partial t} + (h^{-1})^{ij}\nabla_i F\nabla_j P \tag{7.15}$$

and

$$\frac{\partial P}{\partial t} = \Box P - \dot{F}(W^2)P + \ddot{\Phi}(Q,Q). \tag{7.16}$$

where, as in the previous chapter, $\Box = \dot{F}^{ab}\nabla_a\nabla_b$

Proof. The first follows from $\Phi(\nu(x,t),t) = -F(x,t)$ and $\frac{d\nu}{dt} = \nabla F$. For the second identity, we note that, differentiating $P_{\mathrm{Gauss}}(\nu(x,t),t) = P_{\mathrm{Std}}(x,t)$ we obtain

$$\frac{\partial}{\partial t}P_{\mathrm{Gauss}}(\nu(x,t),t) = \frac{\partial}{\partial t}P_{\mathrm{Std}}(x,t) - (h^{-1})^{ij}\nabla_i F\nabla_j F.$$

From the definition of φ and the Codazzi equations, we may compute

$$\Gamma_{ij}^k - \overline{\Gamma}_{ij}^k = -g^{pq}(h^{-1})_p^k \nabla_i h_{jq}$$

and hence that

$$\mathcal{L}P = \Box P - \left(h^{-1}\right)^{kl} \nabla_k F \nabla_l P.$$

The last remaining term of (7.11) to reconcile with (7.16) is $-\dot{\Phi}(I)P$, and this is a consequence of the identity

$$\dot{\Phi}_{W^{-1}}(B) = \dot{F}_{W^{-1}}(WBW).$$

$\Box$

Together with Proposition 7.5, we thus have the following expressions for the Harnack quantities and their evolutions in the standard parametrization.

Corollary 7.7. In the standard parametrization,

$$R = -\frac{P}{F} = \frac{1}{F}\left(\frac{\partial F}{\partial t} - (h^{-1})^{ij}\nabla_i F \nabla_j F\right)$$

satisfies

$$\frac{\partial R}{\partial t} = \Box R + \frac{2}{F}\dot{F}^{ij}\nabla_i F \nabla_j R - \frac{\ddot{\Phi}(Q,Q)}{F}$$

and the analogs of (7.13) and (7.14) are

1. If $F^* = \Phi$ is concave and positive (convex and positive), then

$$\frac{\partial R}{\partial t} - \Box R - \frac{2}{F}\dot{F}^{ij}\nabla_i F \nabla_j R \leq 0 \quad (\geq 0), \quad \text{and} \tag{7.17}$$

2. If F^* is α-concave for some $0 < \alpha < 1$, then

$$\frac{\partial R}{\partial t} - \Box R - \frac{2}{F}\dot{F}^{ij}\nabla_i F \nabla_j R - \frac{\alpha - 1}{\alpha} R^2 \leq 0, \tag{7.18}$$

with the reverse inequality holding when F^* is α-concave with $\alpha < 0$ or α-convex with $\alpha > 1$.

7.3 Some Harnack estimates for curvature flows in Minkowski space

As we have seen in the previous section, the basic Harnack quantities satisfy some nice evolution equations. The needed ingredient now is a suitable maximum principle. Here, however, the quantities of interest satisfy nice equations with respect to a metric and connection (and elliptic operator) that are also changing with time, and whose evolution equations also depend implicitly on the very speeds and curvatures we wish to estimate. In the non-compact setting, then, to formulate a general maximum principle on the hypersurface, we need to make some assumptions about the growth of the curvature and its derivatives.

First, we recall a standard result (cf., e.g, Theorem 12.7 in [23]) that guarantees the existence of a certain super-solution to our basic parabolic equation. We describe a minor modification to the proof in the aforementioned reference to permit the elliptic operator to differ from the time-dependent Laplacian. Ultimately, the barrier function is constructed from the distance-type function described in Lemma 2.7.

Lemma 7.8 ([23], Theorem 12.7). *Suppose $g(t)$ is a family of complete metrics on the non-compact manifold M^n such that the assumptions of Lemma 2.7 are met. Further suppose $\Lambda(t) \in TM^n \otimes TM^n$ is a smooth, positive definite family satisfying $\Lambda^{ij}(t) \leq M g^{ij}(t)$ for some M and $V(t)$ is smooth family of bounded vector fields. Then, for all $\mathcal{O} \in M^n$ and constants A, $a_1 > 0$, there exists a constant b_1 and a positive function $\psi : M^n \times [0,T] \to \mathbb{R}$ such that ψ satisfies*

1. $$\left(\frac{\partial}{\partial t} - \Lambda^{ij}\nabla_i\nabla_j - V(t)\right)\psi \geq A\psi$$

   and

2. $$\exp(a_1 d_{g(0)}(\mathcal{O},x)) \leq \psi(x,t) \leq \exp(b_1 d_{g(0)}(\mathcal{O},x) + 1)$$

for $(x,t) \in M^n \times [0,T]$.

Proof. As in [23], one takes ψ in the form

$$\psi(x,t) = \exp(Bt + \gamma\phi(x))$$

and chooses B and γ appropriately, using that the inequality $\nabla_i\nabla_j\phi(x) \leq Cg_{ij}$ implies

$$\Lambda^{ij}\nabla_i\nabla_j\phi(x) \leq CMn.$$

□

Using this super-solution, one can modify the proof of Theorem 12.14 in [23] to obtain a general maximum principle for solutions to (7.1). This maximum principle will allow us to treat both of the two general cases in Andrews's theorem at once from the quantity R, rather than to consider the two quantities of the original proof in [3].

Theorem 7.9 (Theorem 12.14, [23]). *Suppose $X(t)$ is a solution to (7.1) with $F = F(W)$ on $M^n \times [0,T]$ such that the induced metrics $g(t)$ are complete and h, ∇h, F, and $\dot{F}$ are unformly bounded. Suppose that u satisfies*

$$u(x,0) \leq a \quad \textit{and} \quad |u(x,t)| \leq \exp(bd(\mathcal{O},x)+1)$$

for some $a \in \mathbb{R}$, $b > 0$, and $\mathcal{O} \in M^n$, and

$$\frac{\partial u}{\partial t} \leq \Box u + V^k\nabla_k u + Z(u,t)$$

for some smooth bounded family of vector fields $V(t)$, and some function $Z : \mathbb{R} \times [0,T] \to \mathbb{R}$ that is locally-Lipschitz in its first argument and continuous in its second.

Then, if $U_a(t)$ denotes the solution to the equation

$$\frac{dU}{dt} = Z(U,t) \quad \textit{with} \quad U(0) = a,$$

we have $u(x,t) \leq U_a(t)$ for as long as the solution exists.

Proof. Since $\frac{\partial}{\partial t}g_{ij} = 2Fh_{ij}$, the bounds on F and h imply that the metrics $g(t)$ are uniformly equivalent on the interval $[0,T]$ and, in particular, that $\delta^{-1}g(0) \leq g(t) \leq$

$\delta g(0)$. Also, the bound on $\dot{F}$ implies $\dot{F}^{ab} = g^{ac}\dot{F}_c^b \leq Cg^{ac}$. Since $\nabla_i F = \dot{F}^{ab}\nabla_i h_{ab}$, the bounds on F, h, ∇h, and $\dot{F}$ insure that $\nabla(\frac{\partial}{\partial t}g)$ is bounded. Then the super-solution of Lemma 7.8 is available to us, and the rest of the proof follows exactly as in [23]. □

Thus we may state the following analog of Andrews's theorem in Minkowski space for general flows satisfying the criterion of the above maximum principle.

Theorem 7.10. *Suppose $X(t)$ is a family of smooth, space-like immersions solving (7.1) for a function $F = F(W)$ of the principal curvatures on $M^n \times [0,T)$. Assume that the induced metrics are complete and that, for all $0 < \delta < T/2$, there exists $\mathcal{O} \in M$ and positive constants $b = b(\delta)$ and $C = C(\delta)$ such that*

$$|\nabla\nabla F|^2 + (h^{-1})^{ij}\nabla_i F \nabla_j F \leq F \exp(b(d(\mathcal{O}, x) + 1)) \tag{7.19}$$

and

$$|h|^2 + |\nabla h|^2 + F^2 + |\dot{F}|^2 \leq C(\delta) \tag{7.20}$$

on $M^n \times (\delta, T-\delta)$ Then

1. *If $F^* = \Phi$ is α-concave for some $\alpha < 1$, then*
$$\frac{\partial F}{\partial t} - (h^{-1})^{ij}\nabla_i F \nabla_j F + \frac{\alpha F}{(\alpha - 1)t} \geq 0.$$
2. *If F^* is concave and positive, then*
$$\sup_{M^n}\left(\frac{\partial}{\partial t}\log|F| - F(h^{-1})^{ij}\nabla_i \log|F|, \nabla_j \log|F|\right) \quad \textit{is decreasing.}$$
3. *If F^* is α-convex for some $\alpha > 1$, then*
$$\frac{\partial F}{\partial t} - (h^{-1})^{ij}\nabla_i F \nabla_j F + \frac{\alpha F}{(\alpha - 1)t} \leq 0.$$
4. *If F^* is convex and positive, then*
$$\inf_{M^n}\left(\frac{\partial}{\partial t}\log|F| - F(h^{-1})^{ij}\nabla_i \log|F|, \nabla_j \log|F|\right) \quad \textit{is increasing.}$$

Proof. Using again the observation of Lei Ni (cf. the remarks in Section 6.7), to prove (1), it suffices to show that for all $\delta > 0$,

$$\frac{\partial F}{\partial t} - (h^{-1})^{ij}\nabla_i F \nabla_j F + \frac{\alpha F}{(\alpha-1)(t-\delta)} \geq 0$$

on $M^n \times [\delta, T-\delta]$, where, as per our assumptions, the bounds (7.19) and (7.20) hold. These bounds imply that

$$R = \frac{1}{F}\left(\frac{\partial F}{\partial t} - (h^{-1})^{ij}\nabla_i F \nabla_j F\right) = \frac{1}{F}\left(\Box F - (h^{-1})^{ij}\nabla_i F \nabla_j F\right) - \dot{F}(W^2)$$

satisfies

$$|R(x,t)| \leq \exp(b(d(\mathcal{O},x)+1))$$

for some $b > 0$ and permit the application of the maximum principle, Theorem 7.9, to the evolution equation (7.18). When $0 < \alpha < 1$, we have

$$Z(U,t) = \frac{\alpha - 1}{\alpha} U^2.$$

A general solution $U(t)$ to the associated differential equation with $U(\delta) = M \geq 0$ will satisfy

$$U(t) = \frac{1}{M^{-1} + \frac{1-\alpha}{\alpha}(t-\delta)} \leq \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{t-\delta}\right).$$

Thus, independent of $\sup R(\cdot,\delta)$, by Theorem 7.9,

$$R(\cdot,t) \leq \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{t-\delta}\right)$$

for $t \in (\delta, T-\delta)$. Case (1) then follows since

$$R = \frac{1}{F}\left(\frac{\partial F}{\partial t} - (h^{-1})^{ij}\nabla_i F \nabla_j F\right)$$

and $F < 0$ when $\alpha > 0$.

Likewise, when $\alpha < 0$, we have, (independent of $\inf R(\cdot,\delta)$),

$$R(\cdot,t) \geq \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{t-\delta}\right)$$

for $t \in [\delta, T-\delta]$. Since F is positive when α (hence Φ) is negative, the inequality in (1) follows. An analogous argument can be used to prove Case (3).

For cases (2) and (4), we may apply Theorem 7.9 to (7.17) on any interval $[t_1, t_2] \subset (0, T)$ (using the bounds on $[\delta, T - \delta]$ for $\delta = \min\{t_1, T - t_2\}$) with $Z = 0$ and $U \equiv \sup R(\cdot, t_1)$ and $U \equiv \inf R(\cdot, t_1)$, respectively. We conclude that $\sup R(\cdot, t_1) = U(t_2) \geq \sup R(\cdot, t_2)$ and $\inf R(\cdot, t_1) = U(t_2) \leq \inf R(\cdot, t_2)$, respectively. To include $t = 0$, one may fix t_2, apply the above conclusion to the intervals $[t_1, t_2]$, and send $t_1 \to 0$. $\square$

8 Hamilton's gradient estimate for the heat kernel on complete manifolds

8.1 Introduction

In [43], Richard Hamilton established the following estimate on the gradient of the logarithm of a positive solution to the heat equation.

Theorem. (Hamilton) Suppose (M^n, g) is a closed Riemannian manifold and u a positive solution to the heat equation on M^n. If $M > 0$ and $K \geq 0$ are constants such that $Rc \geq -Kg$ and $u(x,t) \leq M$, then for all $x \in M^n$ and $t > 0$, one has

$$t\,|\nabla \log(u)|^2 \leq (1 + 2Kt)\log(M/u). \tag{8.1}$$

In this chapter, we provide a proof that Hamilton's theorem also holds on complete, non-compact manifolds with Ricci curvature bounded below. Under the additional restriction of non-negative Ricci curvature, we then obtain, via the well-known bounds of Li and Yau [55], the following estimate on the heat kernel. (Recall that on a complete, non-compact manifold, the heat kernel may be defined as the smallest positive fundamental solution to the heat equation.)

Theorem 8.1. *Suppose M^n is a complete, non-compact manifold with $Rc \geq 0$, and H its heat kernel. Then, for all $\delta > 0$, there exists a constant $C = C(n,\delta)$ such that*

$$|\nabla \log(H(x,y,t))|^2 \leq \frac{2}{t}\left(C + \frac{d(x,y)^2}{(4-\delta)t}\right) \tag{8.2}$$

for all x, y in M^n and $t > 0$.

Theorem 8.1 is sharp in the order of t for the heat kernel on $\mathbb{R}^n$ and should be compared to the recent estimate of Souplet and Zhang in [66] which applies to the heat kernel on manifolds with $Rc(g) \geq -Kg$. In the special case $K = 0$, inequality (8.2) is comparable to their estimate at scales $d^2(x, y) \leq ct$ and offers an improvement at scales $t << d^2(x, y)$.

Additionally, such an estimate is required to prove that the integrand in the entropy functional $\mathcal{W}$ for the linear heat equation (cf. [56]) is pointwise non-positive for the fundamental solution to the heat equation. The proof that the integrand in Perelman's $\mathcal{W}$-functional is non-positive for fundamental solutions to the conjugate heat equation seems also to require a non-linear analog of this result (see [59]). Perhaps the approach detailed here could serve as a model for the proof of such an estimate.

In Section 8.2, we obtain a Bernstein-type estimate for bounded solutions to the heat equation which affords pointwise control on the product of the squared norm of the gradient by the elapsed time. The estimate is similar in form to those found by W.-X. Shi [65] in the Ricci flow setting for derivatives of curvature. Such estimates have found considerable service in that setting and continue to have importance in work towards the verification of the claims of Perelman.

8.2 A Bernstein-type estimate

Henceforth we shall assume that (M^n, g) is a smooth, complete, non-compact Riemannian manifold with Ricci curvature uniformly bounded below by $-K$, and for this section, suppose u is a smooth solution to the heat equation on some open $U \subset M^n$ for $0 \leq t \leq T \leq \infty$, satisfying $|u| \leq M$. Our aim is to establish a preliminary estimate on $|\nabla u|^2$ so that the maximum principle of Karp and Li [53] may be applied to the quantity of interest in Hamilton's gradient estimate.

To do this, we employ a technique of W.-X. Shi [65] from the estimation of derivatives of curvature under the Ricci flow (see also the treatment in the forthcoming book [21]), and define $F(x, t) = (4M^2 + u^2) |\nabla u|^2$ for $t > 0$. The evolution of F then possesses an advantageous $-F^2$ term, as we see below.

Lemma 8.2. *There exist positive constants C_1 and C_2 such that*

$$\frac{\partial F}{\partial t} \leq \Delta F + C_1 K F - \frac{C_2}{M^4} F^2. \tag{8.3}$$

Proof. We have

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\nabla u|^2 = -2 |\nabla\nabla u|^2 - 2Rc(\nabla u, \nabla u) \leq -2 |\nabla\nabla u|^2 + 2K |\nabla u|^2,$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right) u^2 = -2 |\nabla u|^2.$$

So

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) F \leq (4M^2 + u^2)&(2K |\nabla u|^2 - 2 |\nabla\nabla u|^2) - 2 |\nabla u|^4 \\ &- 8u(\nabla\nabla u)(\nabla u, \nabla u) \\ \leq -10u^2 &|\nabla\nabla u|^2 + 10M^2 K |\nabla u|^2 - 2 |\nabla u|^4 \\ &- 8u(\nabla\nabla u)(\nabla u, \nabla u).\end{aligned}$$

Since

$$8|u| |\nabla\nabla u| |\nabla u|^2 \leq 10u^2 |\nabla\nabla u|^2 + \frac{8}{5} |\nabla u|^4,$$

and $4M^2 |\nabla u|^2 \leq F \leq 5M^2 |\nabla u|^2$, we find

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) F &\leq 10M^2 K |\nabla u|^2 - \frac{2}{5} |\nabla u|^4 \\ &\leq \frac{5}{2} K F - \frac{2}{625 M^4} F^2\end{aligned}$$

as claimed. ☐

Now, as in [55], for any $p \in M^n$ and $R > 0$ we may find a cut-off function $\eta(x) = \eta_{p,R}(x)$ equal to 1 on $B_p(R)$ and supported in $B_p(2R)$ satisfying the conditions

$$|\nabla \eta|^2 \leq \frac{C_3}{R^2} \eta \tag{8.4}$$

and

$$\Delta \eta \geq -\frac{C_3}{R^2} \left(1 + R\sqrt{K}\right) \tag{8.5}$$

for some $C_3 = C_3(n) > 0$. Strictly speaking, the above estimates need only hold away from the cut-locus of p, however, for the purposes of applying the maximum principle to ηF, the well-known argument of Calabi [9] allows us to assume that they hold everywhere.

The main result of this section is the following local estimate:

Theorem 8.3. *Suppose u is a smooth solution to the heat equation satisfying $|u| \leq M$ on $B_p(2R) \times [0,T]$ for some $p \in M^n$ and M, R, $T > 0$. Then there exists a constant $C_4 = C_4(K)$ such that*

$$t\,|\nabla u|^2 \leq C_4 M^2 \left(1 + T\left(1 + \frac{1}{R^2}\right)\right) \tag{8.6}$$

holds on $B_p(R) \times [0,T]$

Proof. Define F as in Lemma 8.2. On $\mathrm{supp}(\eta) \times [0,T]$, we have, by Lemma 8.2 and equations (8.4) and (8.5),

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)(t\eta F) &= \eta F + t\eta\left(\frac{\partial}{\partial t} - \Delta\right)F - tF\Delta\eta - 2t\left\langle \nabla\eta, \nabla F\right\rangle \\ &= \left(\eta + 2t\frac{|\nabla\eta|^2}{\eta} - t\Delta\eta\right)F + t\eta\left(\frac{\partial}{\partial t} - \Delta\right)F \\ &\quad - 2\left\langle \nabla(t\eta F), \frac{\nabla\eta}{\eta}\right\rangle \\ &\leq \left(\left(1 + C_1 K t\right)\eta + 3t\frac{C_3}{R^2}\left(1 + R\sqrt{K}\right)\right)F - \frac{C_2}{M^4}t\eta F^2 \\ &\quad - 2\left\langle \nabla(t\eta F), \frac{\nabla\eta}{\eta}\right\rangle.\end{aligned}$$

If ηF is not identically zero (i.e., if u is not constant on $\mathrm{supp}(\eta)$), then $t\eta F$ attains a positive maximum at $(x_0, t_0) \in M^n \times (0,T]$.

At this point,

$$\nabla(t\eta F) = 0$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right)(t\eta F) \geq 0,$$

so

$$\frac{C_2}{M^4}t_0\eta F^2 \leq \left(1 + C_1 KT + 3T\frac{C_3}{R^2}\left(1 + R\sqrt{K}\right)\right)F.$$

Consequently, for any $(x,t) \in B_p(R) \times [0,T]$,

$$\begin{aligned} tF(x,t) &= tF(x,t)\eta(x) \\ &\le t_0 F(x_0,t_0)\eta(x_0) \\ &\le \frac{M^4}{C_2}\left(1 + C_1 KT + 3T\frac{C_3}{R^2}\left(1 + R\sqrt{K}\right)\right). \end{aligned}$$

But $|\nabla u|^2 \le (1/4M^2)F$, and the claim follows. □

Remark 8.4. If $Rc \ge 0$, the above proof shows

$$t\,|\nabla u|^2 \le CM^2\left(1 + \frac{T}{R^2}\right)$$

on $B_p(R) \times [0,T]$.

Sending $R \to \infty$ in the penultimate line of the above proof, we at once obtain

Corollary 8.5. Suppose the solution u is defined on all of $M^n \times [0,T]$. Then there exists a constant C_5 such that

$$t\,|\nabla u|^2 \le C_5 M^2 (1 + KT) \tag{8.7}$$

on $M^n \times [0,T]$.

8.3 Proof of main theorem

Next, using the estimate of the previous section, we apply a maximum principle due originally to Karp and Li, whose statement we found (in more generalized form) in a paper of Ni and Tam. The statement of their theorem in our (stationary metric) case is as follows. Here $f_+(x,t) := \max\{f(x,t), 0\}$.

Theorem. (Karp-Li, [53]; Ni-Tam, [60], 1.2)

Suppose (M^n, g) is a complete Riemannian manifold and $f(x,t)$ a smooth function on $M^n \times [0,T]$ such that $\left(\frac{\partial}{\partial t} - \Delta\right) f(x,t) \le 0$ whenever $f(x,t) \le 0$. Assume that

$$\int_0^T \int_{M^n} e^{-ar^2(x)} f_+^2(x,s)\, dV ds < \infty \tag{8.8}$$

for some $a > 0$, where $r(x)$ is the distance to x from some fixed $p \in M^n$. If $f(x,0) \le 0$ for all $x \in M^n$, then $f(x,t) \le 0$ for all $(x,t) \in M^n \times [0,T]$.

Hamilton's theorem in the complete case reads as

Theorem 8.6. *Suppose* (M^n, g) *is a complete manifold with* $Rc \geq -Kg$ *for some* $K \geq 0$. *If* $0 < u(x,t) \leq M$ *is a solution to the heat equation on* $M^n \times [0,T]$ *for* $0 < T \leq \infty$, *then*

$$t \left|\nabla \log u\right|^2 \leq (1 + 2Kt) \log(M/u).$$

Proof. Defining $u_\epsilon = u + \epsilon$ for $\epsilon > 0$, we obtain a solution satisfying $\epsilon < u_\epsilon \leq M + \epsilon := M_\epsilon$. Once the estimate has been proved for u_ϵ, the theorem will follow by letting $\epsilon \to 0$.

As in [43], the function

$$P(x,t) := \varphi(t) \frac{\left|\nabla u_\epsilon\right|^2}{u_\epsilon} - u_\epsilon \log(\frac{M_\epsilon}{u_\epsilon}),$$

where $\varphi(t) := t/(1 + 2Kt)$, satisfies $\left(\frac{\partial}{\partial t} - \Delta\right) P(x,t) \leq 0$ and

$$P(x,0) = -u_\epsilon \log(M_\epsilon / u_\epsilon) \leq 0.$$

By our assumptions on u_ϵ, we also have

$$P_+(x,t) \leq \frac{1}{\epsilon} t \left|\nabla u_\epsilon\right|^2.$$

Thus using equation (8.7), for any $p \in M^n$, and $R > 0$, we have

$$\begin{aligned} \int_0^T \int_{B_p(R)} e^{-r^2(x)} P_+^2(x,t)\, dV dt &\leq \frac{1}{\epsilon^2} \int_0^T \int_{B_p(R)} e^{-r^2(x)} (t \left|\nabla u_\epsilon\right|^2)^2 \, dV dt \\ &\leq \frac{C_5^2 M_\epsilon^4}{\epsilon^2} (1 + KT)^2 \int_0^T \int_{M^n} e^{-r^2(x)} \, dV dt. \end{aligned}$$

Since we assume that $Rc \geq -Kg$, it follows from the Bishop volume comparison theorem that the rightmost integral in the above inequality is finite.

Hence,

$$\begin{aligned} \int_0^T \int_{M^n} e^{-r^2(x)} P_+^2(x,t)\, dV dt &\leq \liminf_{R \to \infty} \int_0^T \int_{B_p(R)} e^{-r^2(x)} P_+^2(x,t)\, dV dt \\ &< \infty, \end{aligned}$$

and we conclude that $P(x,t) \leq 0$ for all $t \leq T$. □

Proof. Proof of main theorem Let $H(x,y,t)$ denote the heat kernel of (M^n,g). For any $t>0$ and $y\in M^n$, set $u(x,s):=H(x,y,s+t/2)$. Then u is a smooth, positive solution to the heat equation on $[0,\infty)$. By Corollary 3.1 and Theorem 4.1 of [55], for all $\delta>0$, there is a constant $C_6=C_6(\delta)>0$ such that u satisfies

$$C_6^{-1}V\left(\sqrt{s+t/2}\right)^{-1}e^{\frac{-d^2(x,y)}{(4-\delta)(s+t/2)}}\leq u(x,s)\leq C_6V\left(\sqrt{s+t/2}\right)^{-1} \tag{8.9}$$

for all x, $y\in M^n$, and $s\geq 0$, where $V(\sqrt{s+t/2}):=\mathrm{Vol}(B_y(\sqrt{t+s/2}))$.

Defining $M=C_6V\left(\sqrt{t/2}\right)^{-1}$, the left inequality in (8.9) implies $u\leq M$ for all x and s. Moreover, since we assume $Rc\geq 0$, there exists a positive constant $C_7=C_7(n)$ such that for all $0\leq s\leq t/2$

$$V\left(\sqrt{t/2+s}\right)\leq V\left(\sqrt{t}\right)\leq C_7V\left(\sqrt{t/2}\right).$$

Thus, by the right-hand inequality in (8.9) and Theorem 8.6, we have

$$s\left|\nabla\log u\right|^2\leq\log\left(\frac{M}{u}\right)\leq\left(\log(C_6^2C_7)+\frac{d^2(x,y)}{(4-\delta)(s+t/2)}\right)$$

on $M^n\times[0,t/2]$.

Setting $C=\log(C_6^2(\delta)C_7(n))$ and evaluating at $s=t/2$, we conclude that

$$(t/2)\left|\nabla\log H\right|^2(x,y,t)=(t/2)\left|\nabla u\right|^2(x,t/2)\leq\left(C+\frac{d^2(x,y)}{(4-\delta)t}\right)$$

for all x, $y\in M^n$ and $t>0$. ☐

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