

UC Berkeley

UC Berkeley Electronic Theses and Dissertations

Title

Quantum Computing with Superconducting Kerr-cat Qubits

Permalink

<https://escholarship.org/uc/item/25z53150>

ISBN

9798189276477

Author

Qing, Bingcheng

Publication Date

2026-05-01

Peer reviewed|Thesis/dissertation

Quantum Computing with Superconducting Kerr-cat Qubits

by

Bingcheng Qing

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Physics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Irfan Siddiqi, Chair
Professor Hartmut Haeffner and
Professor Boubacar Kanté

Spring 2026

Quantum Computing with Superconducting Kerr-cat Qubits

Copyright 2026
by
Bingcheng Qing

Abstract

Quantum Computing with Superconducting Kerr-cat Qubits

by

Bingcheng Qing

Doctor of Philosophy in Physics

University of California, Berkeley

Professor Irfan Siddiqi, Chair

Quantum computing promises exponential advantages for simulating quantum systems and solving certain computational problems beyond the reach of classical computers, but the fragility of qubits and the resulting errors present a major challenge for scalable implementations. One promising approach is to engineer qubits whose physical noise processes are intrinsically biased, enabling more efficient quantum error correction. This dissertation investigates quantum computing with superconducting Kerr-cat qubits, a bosonic qubit architecture in which qubit states are encoded in coherent states of a nonlinear oscillator stabilized by a two-photon drive. This stabilization creates an effective double-well pseudo-potential that strongly suppresses bit-flip errors while enabling universal quantum control, leading to a noise-biased qubit. A superconducting device architecture based on SNAIL nonlinear oscillators is developed to realize Kerr-cat qubits with tunable parameters and high coherence, integrating readout resonators, dedicated filters, and microwave control lines for measurement and manipulation. Universal single-qubit gates are implemented using engineered single-photon and two-photon drives to deform the pseudo-potential, and a cat-quadrature readout method is employed to achieve high-fidelity quantum nondemolition measurement. The performance of the qubits is systematically characterized through lifetime measurements and benchmarking protocols, including quantum process tomography, gate set tomography, and dihedral randomized benchmarking, revealing a strongly biased noise structure with high performance. These results establish an experimental framework for the design, control, and characterization of Kerr-cat qubits and demonstrate their potential as hardware-efficient building blocks for scalable fault-tolerant quantum computing.

To My Youth and My Life

致我的青春与生命

Contents

Contents	ii
List of Figures	iv
List of Tables	xi
1 Introduction	1
1.1 Classical and Quantum Bits	4
1.2 Quantum Computing	6
1.3 Quantum Error Correction (QEC)	11
2 Quantum Electrical Circuits	16
2.1 Classical Circuits and Lagrangian Analysis	17
2.2 Nonlinearities in Superconducting Circuits	30
2.3 Quantum Circuits and Hamiltonian Analysis	37
3 Kerr-cat Qubit Design and Fabrication	74
3.1 Hamiltonian Structure and Qubit Encoding	75
3.2 Noise Structure	81
3.3 Kerr-cat Qubit Implementation	86
3.4 Fabrication	104
4 Kerr-cat Qubit Measurement	116
4.1 Hardware Setup	116
4.2 Cat Quadrature Readout	120
4.3 Universal Single-qubit Operation	122
4.4 Cat Size Calibration	124
4.5 Initialization	125
5 Kerr-cat Qubit Characterization	129
5.1 Lifetimes	130
5.2 Quantum Process Tomography	134
5.3 Gate Set Tomography	138

5.4	Dihedral Randomized Benchmarking	142
5.5	Outlook and Conclusion	149
Bibliography		151

List of Figures

1.1	Qubit encoding in superconducting circuits. (a) Basic electrical elements. (b) Transmon qubit potential profile and its qubit encoding. (c) Transmon qubit circuit. (d) Fluxonium qubit potential profile and its qubit encoding. (e) Fluxonium qubit circuit. (f) GKP qubit Wigner function and its qubit encoding. (g) GKP qubit circuit. (h) Kerr-cat qubit Wigner function and its qubit encoding. (i) Kerr-cat qubit circuit. The axes of the plots are omitted for clarity.	6
1.2	Transmon qubit readout in superconducting circuits. (a) The circuit diagram. (b) The measured spectrum. S_{11} and θ are the amplitude and phase of the scatter parameters, ω_r is the bare frequency of the resonator.	8
1.3	Bloch sphere representation of the qubit state and operations. (a) The qubit states on the three axes. (b) The $X(\gamma)$ rotation gates. (c) The $Y(\gamma)$ rotation gates. (b) The $Z(\gamma)$ rotation gates.	10
1.4	Common errors in superconducting qubits. (a) Coherent errors, corresponding to a mis-rotation on the Bloch sphere. (b) Pure dephasing, corresponding to random phase flips. (c) Longitudinal transitions, corresponding to random bit flips. (d) Depolarizing noise, which represents an isotropic reduction of the state purity. (e) Stochastic Pauli noise, which represents an anisotropic reduction of the state purity. (f) Leakage, corresponding to the state transition out of the qubit encoding space.	12
1.5	QEC codes. (a) Simple repetition code and its error syndrome check. (b) 9-qubit Bacon-Shor code and its error syndrome check. (c) Surface code[37]. (d) Color code[113].	14
2.1	Transmission line examples and circuit model. (a) Stripline. (b) Micro-strip. (c) Coplanar waveguide (CPW), (d) Coaxial line. (e) Circuit model for an infinitesimal segment of the transmission line.	19
2.2	Parallel RLC resonator. (a) Circuit diagram. (b) Measurement of an RLC resonator. (c) Reflection magnitude and (d) Reflection phase of the RLC resonator.	20

2.3	Distributed resonators under varying boundary conditions and their associated signal profiles. (a) Schematic of a resonator with both ends shorted. (b) Spatial profiles for voltage and flux, which share the same node/antinode positions. (c) Spatial profiles for current and flux derivative. (d–f) Corresponding schematic and signal profiles for a resonator with both ends open. (g–i) Corresponding schematic and signal profiles for a resonator with one end open and the other shorted.	22
2.4	Multi-mode circuit as a network. (a) The schematic of the network. (b) The constitutive relation of the network.	24
2.5	Commonly used circuit topologies and their capacitance and inductance matrix. (a) Capacitively coupled resonators. (b) Inductively coupled resonators. (c) Mutual-inductance coupled resonators.	25
2.6	Coupling to a distributed resonator. (a) The schematic of capacitive coupling. (b) The capacitive coupling with high coupling strength. (c) The mutual-inductance coupling with high coupling strength.	26
2.7	Simplification of circuits using conservative quantities. (a) A floating resonator can be simplified to a grounded resonator. (b) Y -connected inductors can be simplified to Δ -connection. (c) Y -connected capacitors can be simplified to Δ -connection.	27
2.8	Josephson junction (a) structure and (b) circuit diagram	30
2.9	SQUID. (a) SEM picture of a SQUID (colored in yellow) on a silicon chip. (b) Circuit diagram. (c) Effective critical current.	32
2.10	SNAIL element. (a) Circuit diagram. (b) An example of SNAIL inductive energy profile. (c) The third-order nonlinear coefficient and (d) the fourth-order nonlinear coefficient with different junction ratio β and external flux Φ_{ext}	34
2.11	Multi-loop structure circuit diagram.	35
2.12	Array of nonlinear elements. (a) Circuit diagram of a single element. (b) Circuit diagram of an array of elements to dilute the higher-order nonlinearities.	36
2.13	Driven resonators through (a) capacitance and (b) mutual inductance.	39
2.14	Circuit diagram, potential energy, eigenstates, and eigenenergy for (a) linear resonator, (b) transmon, (c) SNAILmon.	44
2.15	Dispersive coupling between (a) two states, and (b) a transmon and a resonator. The black solid lines indicate the bare energy levels, and the red and blue arrows indicate the dispersive shift due to the coupling terms given by the green arrows.	51
2.16	Flux-pumped Josephson parametric amplifier	57
2.17	Reflection coefficient of Klopfenstein impedance transformer.	60

2.18	Device layout. (a) False-color micrograph of the device with the nonlinear resonator shown in the inset. From left to right are the horn-like CPW taper (green), the parallel plate capacitor (blue), and the SQUID (yellow) forming the resonator, and the on-chip flux line (red). (b) Parameters of the horn-like CPW impedance transformer, showing decreasing characteristic impedance Z_0 along the CPW as its center-trace width becomes wider, transforming from $50\ \Omega$ at the left end to $18\ \Omega$ at the right end. One should notice that the characteristic impedance of the horn-like CPW is not exactly equal to $50\ \Omega$ and $18\ \Omega$ on the two ends because of the nature of the Klopfenstein taper. [152] (c) Electrical circuit depiction of the CIMPA. The nonlinear resonator on the right side is constructed by shunting a SQUID (yellow) with a capacitor (blue). The structure is galvanically coupled to a Klopfenstein taper (green) on the left and inductively coupled to a flux-line (red) on the right.	61
2.19	S-matrix simulation result of the horn-like CPW line.	63
2.20	The wiring diagram for measuring CIMPA	64
2.21	CIMPA's spectrum. Reflectometry spectroscopy performed using a VNA shows phase twists across the SQUID's resonant frequencies. Repeating the measurement at various flux biases yields the CIMPA's characteristic frequency spectrum, which fits the relation given by Eq. 2.177 with excellent agreement.	66
2.22	Gain performance with different signal power.	67
2.23	CIMPA's performance. (a) Instantaneous signal gain, where the 20 dB gain range is highlighted in cyan, and the 15 dB gain range is highlighted in blue, (b) noise temperature, and (c) input saturation power (dark green) and output saturation power (light green) of the CIMPA operated at $\Phi_e/\Phi_o = 0.3$ and $\omega_p = 2\pi \times 13.41\ \text{GHz}$. (d) Signal gain at different CIMPA frequencies tuned by varying Φ_e	69
2.24	(a) The microwave measurement chain model for simulating the input impedance Z_{in} , (b) the input impedance Z_{in} with respect to the frequency, and (c) the simulated gain profile at different operating frequencies.	70
2.25	The calibration of the measurement setup. (a) Fitting of the readout resonator. (b) Fitting of the ac Stark shift. (c) Fitting of the measurement-induced dephasing. (d) Calibration between the power going into CIMPA and the power at the top of the fridge.	71
2.26	Transmon's coherence time statistics for CIMPA on (pumped) and off (unpumped), with the mean values equal to $14.78\ \mu\text{s}$ and $16\ \mu\text{s}$, and the standard deviations equal to $1.41\ \mu\text{s}$ and $5.45\ \mu\text{s}$, respectively. The statistics consist of 201 individual measurements in each setting.	72
3.1	Classical pseudo-potential of Kerr-cat qubit Hamiltonian with different Δ and ϵ_2	76
3.2	Eigenstates and cat qubit encoding. (a) The pseudo-potential and the ground coherent states. (b) The Bloch sphere of a cat qubit.	77

3.3	Detuned-Kerr-cat qubit ground state and its energy with $\Delta = 2K$. a , The fractional deviation of the ground state energy estimated by the unperturbed Hamiltonian and perturbation theory. b , The difference between the coherent state and the ground state of the detuned-Kerr-cat qubit Hamiltonian calculated by numerical diagonalization (ND) and perturbation theory (PT).	79
3.4	Higher excited eigenstates and eigenenergies of Kerr-cat qubits. (a) The pseudo-potential and the higher excited states. (b) The energy spectrum of the Kerr-cat qubit with different detuning Δ , where the degenerate point is marked by a star, and N_{deg} is the number of degeneracies.	80
3.5	Bit-flip times of Kerr-cat qubits with different detunings without non-Markovian errors.	86
3.6	(a) False-color micrograph of the 2D Kerr-cat qubit device, which contains two uncoupled Kerr-cat devices based on the SNAIL nonlinear oscillators. The SNAIL nonlinear oscillator (yellow) is coupled to a readout resonator (dark green) with its Purcell filter (light green) for quantum nondemolition readout through the readout bus (pink). A weakly coupled control port (orange) delivers the single-photon drive, and a strongly coupled pump port with a bandblock filter (purple) delivers the two-photon stabilization drive and cat-qubit readout drive with suppressed Purcell decay. (b) The scanning electron microscope picture of the double SNAILs, with each consisting of three large JJs connected in parallel with a small JJ. (c) The zoom-in picture of the bandblock filter, where the stubs form destructive interference in the block band, leading to low transmission.	87
3.7	(a) SNAIL nonlinear oscillator frequency ω' , (b) Kerr-nonlinearity K , (c) third-order nonlinearity g_3 and (d) the ratio between g_3 and K with respect to different external flux Φ_{ext}	89
3.8	(a) The measured performance for the full range of the filter (purple). The dark grey line is the simulated performance using finite-element simulation. (b) Shows a zoomed view around the qubit frequency with the 20,30 dB attenuation ranges. (c) A zoomed view into one stub of the filter, where the length is chosen to have destructive interference around the qubit frequency. (d) The destructive interference results from the π phase that the signal picks from the two paths.	91
3.9	The SNAIL nonlinear oscillator spectrum in the rotating frame as a function of detuning with $\epsilon_2(t) = 0$	95
3.10	The continuous $Z(\theta)$ gates. (a) The shift of the pseudo-potential. (b) The rotation of the states along the Z -axis.	98
3.11	Wigner function evolution of a coherent state under $U(t)$	100
3.12	The energy barrier between the two wells and the tunneling rates with different detuning Δ	101

3.13	Discrete $X(\pi/2)$ gate engineered by frequency modulation of the two-photon stabilization drive. The state starts at $ +Z\rangle$, then tunnels through the potential barrier during the gate implementation and ends up with $ +Y\rangle$. The effect of this gate is also illustrated on the Bloch sphere, where we used a dashed line to indicate that it is not a continuous rotation on the Bloch sphere.	102
3.14	Kerr-cat qubit concepts. a , SNAIL nonlinear oscillator circuit with frequency ω_Q , nonlinear factors g_3, g_4 and Kerr nonlinearity K , and its energy levels with wave-functions. b , SNAIL oscillator pumped by a two-photon stabilization drive at $\omega_S = 2\omega_Q$, and the engineered pseudo-potential profile with its energy levels and wave-functions. c , Pseudo-potential profile and its energy levels engineered by a red-detuned two-photon stabilization drive, where the excited states become degenerate. d , Bloch sphere defined by the coherent states and their superpositions, where $ \pm Z\rangle = \pm\alpha\rangle$, $ \pm X\rangle = C_\alpha^\pm\rangle \propto \alpha\rangle \pm -\alpha\rangle$, $ \pm Y\rangle = C_\alpha^{\pm i}\rangle \propto \alpha\rangle \pm i -\alpha\rangle$. e , Pseudo-potential profile under a single-photon drive Ω at $\omega_S/2$ for $Z(\theta)$ gates. f , Pseudo-potential profile under a blue-detuned two-photon stabilization drive for the $X(\pi/2)$ gate. The detuning was chosen to be $5K$ as an example.	103
3.15	Brief fabrication procedure.	104
3.16	T_c measurement of the fabricated Nb film.	106
3.17	Endpoint monitoring in lam7.	107
3.18	Optical microscope picture of the developed ER.	108
3.19	SEM picture of the developed ER.	109
3.20	Optical microscope picture of the fabricated JJ.	110
3.21	SEM picture of JJs	112
3.22	Package box.	113
3.23	Wire-bonded device	115
4.1	VeriCold dilution refrigerator	117
4.2	Wiring Diagram. (a) The microwave components used in this experiment for control and readout involve an arbitrary waveform generator (AWG), local oscillators (LOs), and an analog-to-digital converter (ADC). The rest of the microwave components are shown. (b) The two schemes of up-conversion and down-conversion of the signals using the AWG and ADC equipment at a 1GS/s sampling rate.	118
4.3	Simplified wiring diagram of the chip and the measurement hardware, where the digital-to-analog converters (DACs) are implemented by an arbitrary waveform generator (AWG), and the analog-to-digital converters (ADCs) are implemented by an Alazar sampling card.	119

4.4	(a) The sequence to calibrate CQR drive strength. (b) The QNDness with different CQR drive strength for a cat size of 4 and a detuning of $\Delta = 2K$. (c) The zoom-in view around the peak of the QNDness. (d) The measured results of CQR, where the two blobs are separated and correspond to two Z-axis states. (e) The histogram of CQR. and (f) The collapse of wave-functions corresponding to CQR.	120
4.5	Calibration of $Z(\theta)$ gates. (a) The shifted energy profile when applying a single-photon drive. (b) The sequence for calibrating the $Z(\theta)$ gates. (c) Simulation of the measured Chevron-like plot. (d) The measured Chevron-like plot with the sequence above.	121
4.6	The introduced detuning during the discrete $X(\pi/2)$ gate. (a) The modulation of the two-photon stabilization drive. (b) The extra detuning term in the Hamiltonian.	122
4.7	Calibration of $X(\pi/2)$ gate. (a) The energy profile with low barrier when modulating the two-photon stabilization drive. (b) The sequence for calibrating the $X(\pi/2)$ gate. (c) Simulation of the measured $\langle \hat{\sigma}_z \rangle$ (d) The measured $\langle \hat{\sigma}_z \rangle$ with the sequence above.	123
4.8	Calibration of the cat size based on Fock and cat state Rabi frequencies.	124
4.9	Cat size calibration with different ϵ_2	125
4.10	Characterization of the resonant-Kerr-cat qubit initialization. (a) The pulse sequence, noticing that there is only a short relaxation time. (b) The measured $\langle \hat{\sigma}_z \rangle$. (c) The schematic of the tunneling dynamics.	126
4.11	Initialization of detuned-Kerr-cat qubits. (a) The sequence for detuned-Kerr-cat qubit initialization. (b) The evolution of the potential profiles and wave-functions during the initialization. (c) The numerical simulation of the detuned-Kerr-cat qubit state population transfer during the initialization.	127
4.12	(a) The tunneling between the excited states when the relaxation time is short. (b) The sequence for observing the excited states tunneling with $5\text{-}\mu\text{s}$ relaxation time in initialization. (c) The simulated population oscillation between excited states. (d), The measured population oscillation between excited states with the pulse above.	128
5.1	The Fock state relaxation times T_1 over 2 hours with 100 iterations.	130
5.2	(a) The bit-flip time T_z of resonant-Kerr-cat qubit and detuned-Kerr-cat qubit as a function of mean photon numbers. (b) The phase-flip time T_y of the resonant-Kerr-cat qubit and detuned-Kerr-cat qubit. The error bars are from fitting uncertainty, and the theory predictions of the bit-flip times are calculated by master-equation simulations illustrated in [183, 194]. The dashed lines correspond to an average photon number of 5.2, which will be the operational point for QPT, GST, and DRB.	132

5.3	(a) The bit-flip times T_z when $\Delta = 0$ with different mean photon number and their master equation simulation with the fitting parameters taking different incoherent processes into account. (b) The bit-flip times T_z with different detuning and their master equation simulation.	133
5.4	PTMs of resonant-Kerr-cat qubit extracted from QPT for (a) the $X(\pi/2)$ gate and (b) the $Z(\pi/2)$ gate. The errors extracted from the PTMs for (c) the $X(\pi/2)$ gate and (d) the $Z(\pi/2)$ gate.	137
5.5	GST results. SPAM-free Pauli transfer matrices, process infidelities, diamond distances, stochastic Pauli errors, and coherent Pauli errors for gates on resonant- and detuned-KCQs. a , resonant-Kerr-cat qubit $X(\pi/2)$ gate with 367-ns gate time. b , resonant-Kerr-cat qubit $Z(\pi/2)$ gate with 70-ns gate time. c , detuned-Kerr-cat qubit $X(\pi/2)$ gate with 418-ns gate time. d , detuned-Kerr-cat qubit $Z(\pi/2)$ gate with 70-ns gate time. The uncertainty represents the standard deviation of the model fitting, and each circuit in GST is repeated by 1024 shots.	139
5.6	$2\Delta_{\text{GST}} \log \mathcal{L}$ box plots for GST fits of each sequence with the resonant- and detuned-Kerr-cat qubit, where red marks indicate a detection of non-Markovianity at a confidence level of more than 95%.	142
5.7	DRB survival probability fitting. Circuits and fitting curve examples of the exponential decay in the $\mathbb{D}_8$ dihedral randomized benchmarking protocol when (a), (c) preparing and measuring the states in Z axis, and (b), (d) preparing and measuring the states in X axis. The error bars indicate the shot noise from 1024 repeated measurements.	143
5.8	Scaling factor a between the extracted and real bit-flip errors, and b between the extracted and real phase-flip errors	145
5.9	DRB results. a , The bit-flip p_{bit} and phase-flip p_{ph} errors of resonant- and detuned-Kerr-cat qubit, and corresponding estimation $\tilde{p}_{\text{ph}}$ and $\tilde{p}_{\text{bit}}$ based on the bit-flip and phase-flip times measured in Fig. 5.2. The error bars are smaller than the marker size and therefore not plotted. b , The noise bias as a function of phase-flip errors.	146
5.10	Analysis of bit-flip errors. a Bit-flip errors of resonant-Kerr-cat qubit from simulation $\tilde{p}_{\text{bit}}$ and experiments p_{bit} . b Simulated leakage of resonant-Kerr-cat qubit. c Power spectral density (PSD) of the gate pulse, showing that the local minima in leakage and bit-flip errors coincide with the zeros of the PSD at the transition frequency between the ground and first excited states. d , e , f Corresponding results for the detuned-Kerr-cat qubit with $\Delta = 2K$. g , h , i Corresponding results for the detuned-Kerr-cat qubit with $\Delta = 4K$, for which experimental data were not collected in this study.	147
5.11	Excited state population after applying the $Z(\theta)$ gates	149

List of Tables

2.1	Correspondence between flux-based circuit and mechanics	18
2.2	Allowed solutions for flux Φ in distributed resonators of length d	23
2.3	Qubit and resonator parameters	71
3.1	Device Parameters and Calibration Methods	88
5.1	Gate Set Tomography Germs and Fiducials	138

Acknowledgments

The completion of this thesis marks the end of a long journey. Along the way, I have been fortunate to receive the guidance, support, and generosity of many people, without whom this work would not have been possible. I would like to express my deepest gratitude to my advisor, Prof. Irfan Siddiqi, for his guidance, encouragement, and support throughout my PhD journey. He supports the QNL group in pursuing exciting scientific research with enough freedom and flexibility. I learned invaluable laboratory skills from him that go far beyond the textbooks or lectures. We worked on the plumbing of the compressor, where I was impressed by his memory of many different types of fittings and his strength in moving the compressor. We reinstalled the dilution unit of one of the world's earliest dry dilution refrigerators, during which I had the opportunity to witness his remarkable expertise and craftsmanship in building dilution refrigerator systems. We also polished the presentation slides, from which I learned to keep them visually focused, and most importantly, never to use Calibri. His insight into science and passion for experiments are truly inspiring, and I am deeply grateful for his mentorship.

I would like to thank William Livingston, who introduced me to the group and initiated my first project during PhD. He told me a great deal of E&M simulation and chip design, helping me get started with the experiments. The designs were translated into device fabrication with significant help from Xinyu Liu, Zahra Pedramrazi, and Trevor Chistolini, to whom I am very grateful. The fabrication training from Xinyu Liu helped me to become comfortable with device fabrication in the QNL cleanroom. Zahra and I maintained the Plassys tool together when I was assigned as its superuser, where I learned many aspects of nano- and micro-fabrication tools. She has also been very generous in sharing her experiences and offering encouragement during my early struggles in the group. Trevor, who is as warm-hearted as an angel, has been providing me with continuous support, from offering fabrication recipes and guidance to helping with ADR fridge setup and diagnosis, and his support and kindness even continue after his graduation. I also thank Larry Chen for training me in qubit fabrication and for many discussions on design and measurement topics.

I would like to thank Long B. Nguyen for his valuable guidance in device measurement and paper writing. With his help, I built my first time-domain qubit measurement setup and wrote my first scientific paper in the group, both of which would not have been possible without him. He has also enriched my life through skiing and boba tea, though the latter may result in some weight gain. He is also willing to share his life with me, take an interest in mine, and bring me to socialize, giving me a sense of warmth.

Thanks to the cat qubit team at QNL and Chapman University, whose efforts have made these remarkable qubits work. Gerwin Koolstra introduced me to the team and provided me with his comprehensive qubit simulation codes. Though his voice is very soft, his suggestions are always thoughtful. Ahmed Hajr introduced me to cat qubit experiments and provided support in design and measurement, which is critical for the project. The support from our theory partners at Chapman University, especially Irwin Huang, Bibek Bhandari and Kagan Yanik, provided valuable insights. I am impressed by their ability to work with

extremely complicated formulas. Irwin also provided important support to my thesis, even while recovering from a broken arm. Ke Wang joined the team midway, but he quickly became a core contributor to team management, fridge maintenance, and measurement setup. As a fellow Chinese with shared culture and language he has provided tremendous support to my life, which greatly brightened my life both practically and emotionally.

Thanks to Chuanhong Liu, who brought colorful experiences in my life. His generosity once extended to sharing his cat with me for a week, making it a memorable time. Thanks to Kan-Heng Lee, the master of our cleanroom, for sharing life lessons—especially on self-management—and for his support of my research. Ziqi Kang, Haodong Fei, Yuliang Xiong deserve plenty of thanks. The fantastic post-qualification party organized by Ziqi and Haodong was unforgettable. Interacting with Yuliang always makes me feel younger, happier, and more positive. These moments have made my life in the group truly enjoyable.

I thank Noah Goss for teaching me about multi-qubit measurements and scientific writing. Thank you to Noah Stevenson for random discussions about life. Thanks to Kyunghoon Lee for the help in the cleanroom, and to Kasra for giving me my first AQT user project. I thank Akel Hashim for encouraging me to attend APQC meetings and introducing me to the QCVV community. Thanks to Jordan Hines, who, together with Akel, offered important support on qubit benchmarking. Thanks to Thomas Ersevime for the group's safety management. Thanks to Linus Kim and Brian Marinelli for their support. Many thanks to the QubiC team at LBNL (Gang Huang, Yilun Xu, Neelay Fruitwala, Abhi Rajagopala, and Neel Vora), whose work on the control hardware has been necessary for the next step of the project. I also thank all other QNL members for the great discussions and experiences.

My thanks go to the Physics Department staff (Carlos Bustamante, Anthony Vitan and Espe Munoz-Riddle, etc.) who make our research environment safe and well-maintained. I thank Kerri Peyovich and David Santiago for their support at LBNL. I am grateful to Christopher Spitzer for his efforts on administrative matters to make QNL run smoothly.

I thank my undergrad advisors, Prof. Ruifeng Liu, Jun Feng, Xinyu Liu and Jacek Furdyna, for introducing me to research and supporting me even after my graduation. I thank Prof. Joseph Orenstein for the research opportunity during my exchange, and Prof. Holger Mueller for guidance during my rotation. I thank Dan Stamper-Kurn for his support in the early stages of my PhD. Thanks to Hartmut Haefner and Boubacar Kanté for being on my thesis committee, and Joel Moore for serving on my qual committee.

Finally, I would like to thank my family, partner and friends. Thank you to my family, especially my parents, Hongxia Liu and Guoqiang Qing, for their unconditional love and unwavering support, which makes me always feel safe and warm even in this country across the ocean. Thanks to my partner, Yuchen Wu, whose companionship, understanding, and encouragement have made me not lonely on this journey. I am grateful to my friends—in the US, in China, in other countries, online and in person, new and old—for all the support you have given to me, which makes this journey more memorable.

谁家玉笛暗飞声，散入春风满洛城。落花踏尽游何处，人间至味是清欢。
大梦一场，且插梅花，醉洛阳。

Chapter 1

Introduction

The physical world is governed by quantum mechanics microscopically, and quantum mechanics describes all matter and interactions in a system using Hilbert-space vectors and matrices¹, whose dimensions grow exponentially with the size of the system[3]. As a consequence, such exponentially high memory and execution time costs make the exploration of the microscopic world challenging, or even intractable, with the usual classical computer[4]. To give an intuition for how rapidly the computational cost increases and becomes intractable with increasing system size, it is instructive to examine a popular model in condensed matter physics studies - the 2D Ising model. In this model, spin- $\frac{1}{2}$ electrons reside on an $L \times L$ lattice with 2 degrees of freedom: spin-up or spin-down. The associated Hilbert-space dimension is therefore 2^{L^2} , which increases exponentially with the size of the lattice. As a result, simulating an $L = 6$ system would require a matrix with a dimension of $2^{6^2} \approx 10^{10}$, which is already intractable with normal classical computers, and would require hundreds of graphics processing units (GPUs) to handle[5].

Therefore, there has been a long-standing effort to seek more efficient computational methods for the microscopic world, or more formally, for quantum mechanics. The visionary idea of “simulating quantum physics with a quantum computer” was first articulated in 1982 by Richard P. Feynman in his famous keynote speech[6], and the pursuit of building a quantum computer with quantum bits (qubits) has continued ever since. Unlike classical bits, which are just single binary numbers, qubits are 2-dimensional vectors in the Hilbert space that exhibit both superposition and entanglement properties — two key quantum phenomena. Astute readers may have noticed that the definition of qubits mostly coincides with spin- $\frac{1}{2}$ electrons, and a qubit can faithfully simulate the dynamics of an electron in the 2D Ising example above. Thus, the computational cost will only increase linearly with the system size. Such an exponential speed-up compared to classical computers represents the power of quantum computers and significantly enlarges the size of the quantum system that can be efficiently simulated[7]. Later on, the potential of quantum computing is further

¹People may be more familiar with the wavefunction formalism of quantum mechanics introduced by Erwin Schrödinger[1], but the space-time wavefunction can be understood as a projection of vectors and matrices onto the position space[2].

revealed by various quantum algorithms across different fields[8], ranging from chemistry simulation to general optimization[9–11], and from linear equation solving to integer factoring[12, 13]. One of the most famous algorithms is Shor’s prime factorization algorithm, which offers an exponential speed-up for factorizing a large prime number with quantum computers.

Bringing these algorithms from theory into practice requires the ability to encode and manipulate qubits in experimental systems. Even though the concept of quantum computers was introduced in 1982, the first experimental realization of qubits can be traced back to the 1950s when researchers demonstrated the nuclear magnetic resonance (NMR)[14–16]. The coherent two-level system dynamics observed in NMR clearly match the qubit dynamics, but the word “qubit” was not yet invented at that time. However, NMR is a collective behavior of an ensemble of nuclei, and it is challenging to isolate a single nucleus to create a qubit[17]. Further development of experimental techniques for quantum computing has extended the qubit implementation to various physical platforms, including natural systems like optical photons[18, 19], trapped ions[20, 21], and neutral atoms[22, 23], and artificial systems like quantum dots[24, 25], topological systems[26] and superconducting circuits[27–29]. Among them, superconducting circuits have quickly evolved as one of the leading platforms for quantum computing due to their good scalability, fast gate control, and compatibility with modern micro-fabrication and electronic techniques. Some important contributions include the first observation of quantum tunneling in superconducting circuits, which laid the foundation for quantum mechanical description of the circuits[29]; the invention of charge-insensitive Cooper-pair box (or transmon), which significantly boosts the lifetime of superconducting qubits[27, 30, 31]; and the invention of Josephson parametric amplifiers (JPAs) and traveling-wave parametric amplifiers (TWPAs), which enables the single-shot readout of superconducting qubits[32, 33]. In this thesis, we will focus on quantum computing with superconducting circuits.

The successful realization of qubits marks a significant milestone toward building quantum computers. However, qubits are inherently fragile and prone to errors, and scaling to large multi-qubit systems requires quantum error correction (QEC). Inspired by classical error correction, QEC was first introduced in 1995 [34] to encode a low-error logical qubit using many physical qubits, thereby enabling fault-tolerant quantum computing (FTQC). This framework was subsequently extended to a wide range of QEC codes [35, 36], including the surface code [37], low-density parity-check codes [38], and color codes [39]. These QEC codes usually impose stringent requirements on qubit performance, as reflected in their low “error threshold”, namely the maximum physical qubit error rate below which QEC can successfully suppress logical errors[40, 41]. Moreover, most QEC codes entail significant hardware overhead, often requiring hundreds of physical qubits to encode a single logical qubit [28].

These challenges in QEC have motivated the development of alternative superconducting circuit designs that engineer noise-protected qubits, which provide resilience to some

noise sources² at the hardware level and thereby relax both the overhead and performance requirements imposed by QEC[42, 43]. The noise-protected qubits can be broadly classified into two categories. The first one keeps the traditional two-level structure of the qubit and introduces additional symmetry into its spectrum[44–47]. Some examples include the fluxonium[44], blochonium[45], $0 - \pi$ [46], and $\cos(2\theta)$ qubits[47]. The second category exploits the higher-dimensional Hilbert space of bosonic modes in superconducting circuits and is commonly referred to as bosonic qubits[48]. These qubits range from using only several levels of a bosonic mode, such as binomial qubits[49], to leveraging the full bosonic mode spectrum, such as Gottesman–Kitaev–Preskill (GKP) qubits[50, 51] and cat qubits[52–56]. Specifically, the cat qubits exhibit noise bias with strong resilience to bit-flip error, which can significantly reduce the hardware overhead and performance requirements with XZZX surface codes[57–61].

The advantages of cat qubits have driven the rapid development of their experimental implementations over the last half-decade. Superconducting cat qubits are mostly realized by engineered dissipation or engineered Hamiltonian, which are referred to as dissipative cat qubits[55] and Kerr-cat qubits[62], respectively. The dissipative cat qubits have demonstrated the long bit-flip time reaching up to 10 seconds[63], and a repetition code based on dissipative cat qubits has also been demonstrated recently[56], but the universal control of dissipative cat qubits remains under investigation. On the other hand, even though the bit-flip time of Kerr-cat qubits is limited by the associated heating[64, 65], the universal control of Kerr-cat qubits is comparatively more developed[53, 54, 62, 66]. The work in this thesis fits within the continuing development of Kerr-cat qubits toward scalable, high-performance quantum computing.

In this thesis, we will develop the high-coherence Kerr-cat qubit in a scalable superconducting circuit setup and demonstrate the universal quantum control on it. Moreover, we will optimize the Kerr-cat qubit performance with various drive detunings and benchmark its performance using systematic characterization protocols, including randomized benchmarking and tomography. The thesis is organized as follows. In Chapter 1, we review the basic concepts of quantum computing and QEC, providing the necessary background for the subsequent discussions. In Chapter 2, we present the quantum theory of electrical circuits, commonly referred to as circuit quantum electrodynamics (cQED), which lays the theoretical foundation for superconducting qubits. The understanding of cQED is crucial for engineering the Hamiltonian of Kerr-cat qubits. In Chapter 3, we proceed to the design and fabrication of Kerr-cat qubits, which will cover the engineered Kerr-cat Hamiltonian, the superconducting device layout, and the recipe for fabrication. With the understanding of the Kerr-cat qubit device, we will present the measurement of the Kerr-cat qubit device in Chapter 4, such as the calibration of the device, the universal local control, and the outlook on entanglement gates. Finally, in Chapter 5, the performance of the Kerr-cat qubit will be systematically examined, including the idling lifetimes and operational errors.

²We use the terms “noise” and “error” interchangeably, as they refer to the same physical effects.

1.1 Classical and Quantum Bits

In classical computers, the classical bit is a binary number with the value of either 0 or 1. However, as mentioned above, in quantum computers, the quantum bit, or qubit, is a vector in a 2-dimensional Hilbert space. To get an intuition of the qubit, we can make them analogous to the classical bit by defining an orthonormal basis in this Hilbert space labeled as $|0\rangle$ and $|1\rangle$, so that

$$\langle 0|0\rangle = 1, \quad \langle 1|1\rangle = 1, \quad \langle 1|0\rangle = \langle 0|1\rangle = 0, \quad (1.1)$$

where $\langle x|y\rangle$ represents the inner product between vector $|x\rangle$ and $|y\rangle$. Then, one can represent a qubit $|\psi\rangle$ as the superposition of these two basis vectors using two complex numbers α and β ,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1. \quad (1.2)$$

The vector nature of a qubit naturally leads to the most important property of a single qubit — quantum superposition. Unlike classical bits, which can exist only in one of two discrete states, 0 or 1, a qubit can be any superposition of $|0\rangle$ and $|1\rangle$. This immediately raises the question of the outcome obtained when measuring the qubit, and the answer to this question depends on the measurement type. The simplest measurement is a projective measurement to determine whether the qubit is in $|0\rangle$ or $|1\rangle$, for which Born's rule provides a complete description[2, 67]. According to Born's rule, this measurement yields the outcome $|0\rangle$ with probability $P(0) = |\alpha|^2$ and $|1\rangle$ with probability $P(1) = |\beta|^2$. In contrast to the measurement of a classical bit, measuring a qubit inevitably alters its state: the act of measurement destroys the superposition and collapses the qubit into the same state as the observed outcome, either $|0\rangle$ or $|1\rangle$. As we will see later in this chapter, this unique property of qubits poses significant challenges for QEC.

The full power of qubits is unlocked when we increase the number of qubits and engineer the entanglement between them — the most important property of multiple qubits. The entanglement can be understood through a simple example with two qubits. Considering two qubits with label a and b with basis vectors $|0\rangle_a, |1\rangle_a, |0\rangle_b, |1\rangle_b$, one can represent their state as

$$|\psi\rangle_{ab} = \alpha|00\rangle_{ab} + \beta|01\rangle_{ab} + \gamma|10\rangle_{ab} + \delta|11\rangle_{ab}, \quad |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1, \quad (1.3)$$

where $|xy\rangle_{ab} = |x\rangle_a|y\rangle_b$ is the Kronecker product of $|x\rangle_a$ and $|y\rangle_b$. In general, the two-qubit state $|\psi\rangle$ cannot always be factorized into the Kronecker product of two single-qubit states, i.e.

$$|\psi\rangle_{ab} \neq |\psi\rangle_a|\psi\rangle_b. \quad (1.4)$$

This inequality is a signature of quantum entanglement, indicating that the quantum state of the two-qubit system contains correlations that cannot be reduced to properties of the

individual qubits. One of the most famous examples of entangled states is the Bell states shown below[68, 69]. For clarity of presentation, the subscripts have been omitted.

$$\begin{aligned}
 |\Phi^+\rangle &= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \\
 |\Phi^-\rangle &= \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle), \\
 |\Psi^+\rangle &= \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle), \\
 |\Psi^-\rangle &= \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle).
 \end{aligned} \tag{1.5}$$

For any Bell state, a measurement of either qubit alone yields the outcome $|0\rangle$ or $|1\rangle$ with equal probability 50%. However, when we measure the two-qubit state together, we do not obtain all possible combinations of $|0\rangle$ and $|1\rangle$ with probability 25%. Instead, some combinations may have 0 probability, and the others have 50% probabilities. As a core resource of qubits, quantum entanglement is the foundation for many quantum information applications, including quantum teleportation[70], quantum key distribution[71, 72], and superdense coding[73].

The idea of quantum teleportation is particularly clear and pedagogical in revealing the power of entanglement. Suppose that Alice and Bob share a Bell state pair $|\Phi^+\rangle_{AB}$ where the qubit A is held by Alice and qubit B is held by Bob, and Alice wants to transmit another qubit $|\xi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob. The composite state of these three qubits can be written as (the order convention here is $|\xi\rangle|A\rangle|B\rangle$)

$$\begin{aligned}
 |\psi\rangle = (\alpha|0\rangle + \beta|1\rangle)|\Phi^+\rangle &= \frac{1}{2} \left(|\Phi^+\rangle (\alpha|0\rangle + \beta|1\rangle) + |\Phi^-\rangle (\alpha|0\rangle - \beta|1\rangle) \right. \\
 &\quad \left. + |\Psi^+\rangle (\beta|0\rangle + \alpha|1\rangle) + |\Psi^-\rangle (\beta|0\rangle - \alpha|1\rangle) \right).
 \end{aligned} \tag{1.6}$$

In the second expression, we group the qubit $|\xi\rangle$ with Alice's half of the Bell state and rewrite it in the Bell basis, which will be useful in later analysis. Instead of handing qubit $|\xi\rangle$ to Bob, she can apply a Bell basis measurement³ between her half of the entangled Bell state, qubit A, and the qubit $|\psi\rangle$. According to Born's rule, conditioned on the outcome of Alice's measurement, Bob's half of the Bell state collapses into one of four states that are related to the original state $|\xi\rangle$. The transformation required to map each of these states back to $|\xi\rangle$ is deterministic and uniquely determined by Alice's measurement outcome. Consequently, Alice can transmit her measurement result, encoded as 2-bit classical information, to Bob, who can then recover the qubit $|\xi\rangle$ by applying the corresponding correction operation. Finally, thanks to the entanglement shared among the three qubits, Alice can transmit the quantum state $|\xi\rangle$ to Bob by sending only 2-bit classical information, without physically transmitting any qubits!

³A Bell basis measurement only determines which Bell state the two-qubit states are in, and does not distinguish the individual qubit state.

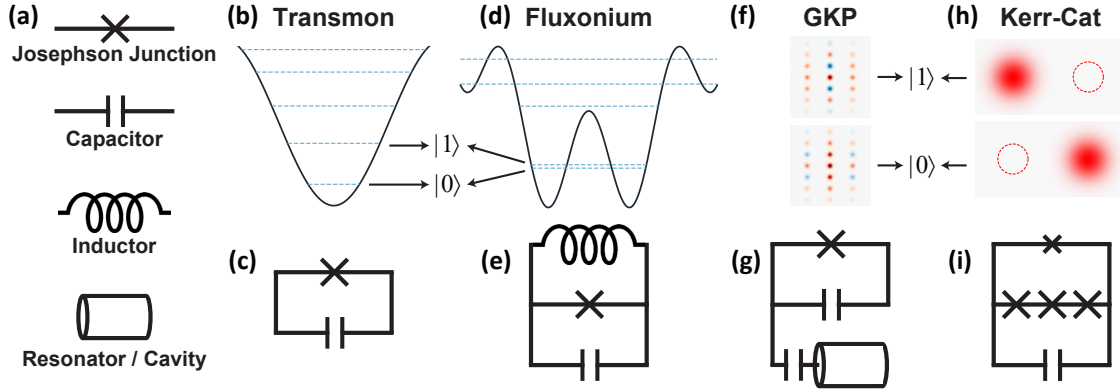


Figure 1.1: Qubit encoding in superconducting circuits. (a) Basic electrical elements. (b) Transmon qubit potential profile and its qubit encoding. (c) Transmon qubit circuit. (d) Fluxonium qubit potential profile and its qubit encoding. (e) Fluxonium qubit circuit. (f) GKP qubit Wigner function and its qubit encoding. (g) GKP qubit circuit. (h) Kerr-cat qubit Wigner function and its qubit encoding. (i) Kerr-cat qubit circuit. The axes of the plots are omitted for clarity.

1.2 Quantum Computing

Qubits present powerful resources for information processing, and quantum computing requires the programmable engineering of the transformations of and interactions between qubits. A clear checklist of requirements for a physical system to implement quantum computing is given by DiVincenzo's criteria as illustrated below[74]:

1. A scalable physical system with well-characterized qubits
2. A qubit-specific measurement capability
3. The ability to initialize the state of the qubits
4. A “universal” set of quantum gates
5. Long relevant decoherence times, much longer than the gate

As a leading platform for quantum computing, superconducting circuits meet all of the criteria above, and we will examine them in this section.

Qubit Encoding

The qubits are encoded in superconducting circuits by carefully engineering the interactions of basic electrical elements: Josephson junctions, capacitors, inductors, (distributed) resonators, and microwave cavities, as shown in Fig. 1.1(a). The combination of these elements forms various microwave oscillators with different potential profiles, leading to different qubit encodings. Some encoding isolated the lowest two levels of the oscillator to form the qubits.

The example shown in Fig. 1.1(b) and (c) is known as transmon qubits, where by connecting a Josephson junction in parallel with a capacitor, one forms a nonlinear oscillator with a weakly anharmonic potential[27]. As a result, the transition energy between the lowest two levels differs from that of higher levels, allowing the lowest two levels to be selectively addressed as a qubit. The addition of a large inductor in parallel, as shown in Fig. 1.1(e), enables the engineering of a double-well potential profile that provides improved isolation and protection of the two lowest energy levels, leading to the realization of fluxonium qubits with enhanced lifetimes[44]. On the other hand, other encoding schemes leverage the multi-level structure of a microwave oscillator to encode qubits in its bosonic modes, commonly referred to as bosonic qubits. The encoded states are usually visualized by their Wigner functions[75]. Some examples include the GKP qubit, which encodes the qubit into grid-like states as seen by their Wigner functions in Fig. 1.1(f). GKP qubits can be implemented in a resonator coupled to a transmon[51]. The focus of this thesis, Kerr-cat qubits, is also a kind of bosonic qubits, which encode the qubits into two coherent states with opposite phases as shown in Fig. 1.1(h). The Kerr-cat qubits can be engineered in a carefully designed nonlinear system consisting of three large Josephson junctions in parallel with a small Josephson junction, as shown in Fig. 1.1(i)[53, 54].

Qubit Measurement

The measurement of the qubit usually distinguishes the $|0\rangle$ and $|1\rangle$ states of the qubit, and collapses it into the corresponding measured state. In superconducting circuits, this can usually be achieved by coupling a readout resonator to the qubit dispersively (dispersive readout). As shown by the example of transmon qubit readout in Fig. 1.2, the bare frequency of the readout resonator will be shifted up or down depending on the state of the qubit, and therefore, the qubit state can be measured by probing the readout resonator frequency[76], which is known as dispersive readout.

One important evaluation of the qubit readout is quantum-nondemolition-ness (QNDness), which determines the extent to which the measurement keeps the qubit in the measured state[77, 78]. A high QNDness is crucial for repeated qubit measurement, fast qubit re-using and re-initialization, and heralding-based qubit initialization. Dispersive readout usually has high QNDness, because such a measurement drives the readout resonator rather than directly driving the qubit, thereby minimizing the perturbation on the qubit. Some research focuses on reducing the measurement-induced state transitions (MIST) or drive-induced unwanted transitions (DUST)⁴ of dispersive readout to further improve the readout fidelity and QNDness[79, 80]. In contrast, some readout schemes are inherently non-QND, such as state-selective push-out imaging methods used for trapped neutral-atom qubits[23, 81, 82]. In these methods, the atoms in $|1\rangle$ state are deliberately pushed out of the trap prior to

⁴MIST is a term historically used in the superconducting community, but some researchers believe that DUST can be a more appropriate term.

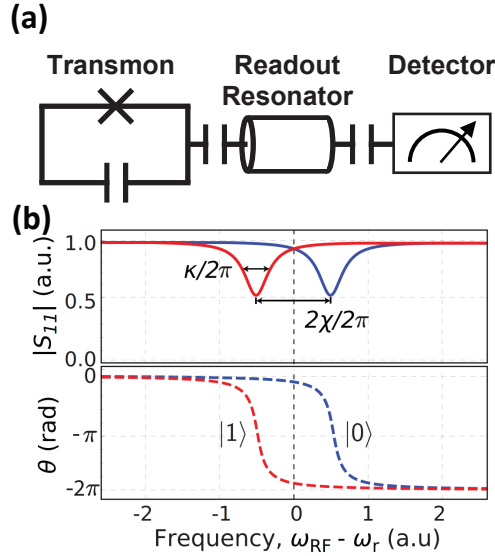


Figure 1.2: Transmon qubit readout in superconducting circuits. (a) The circuit diagram. (b) The measured spectrum. S_{11} and θ are the amplitude and phase of the scatter parameters, ω_r is the bare frequency of the resonator.

imaging, so that only atoms in the $|0\rangle$ state remain and are detected, enabling discrimination between the $|0\rangle$ and $|1\rangle$ states.

Due to the different encoding scheme of Kerr-cat qubits compared with transmon qubits, the dispersive readout mentioned above is not directly applicable to Kerr-cat qubits. Therefore, in this thesis, we introduce a cat-quadrature readout (CQR) method for reading out Kerr-cat qubits with high QNDness[53, 54, 62].

Qubit Initialization

In superconducting qubits, qubits are usually initialized through three methods: passive thermalization[83], active resetting[56, 84], and heralding[54, 85]. The passive thermalization method relies on the low environment temperature to cool down the qubit and initialize it to the ground state (usually the $|0\rangle$ state). In this method, the initialization time is set by the relaxation process of the $|1\rangle$ state and is therefore on the order of the qubit energy relaxation time T_1 . The final steady-state population of the qubit depends on the qubit transition energy $\delta E = E_1 - E_0$ and the environment temperature τ , where $E_{0/1}$ denotes the energy of the $|0\rangle/|1\rangle$ state. Superconducting qubits usually have a transition energy of around $\delta E/\hbar \approx 2\pi \times 6\text{GHz}$, and are thermalized to around 70mK ⁵. Therefore, we can

⁵Though superconducting qubits are measured in a dil-fridge with a base temperature of around 15mK , the qubit temperature is usually higher, mainly due to the radiation from the microwave lines[86–89].

estimate the final qubit state population using the Boltzmann distribution⁶[90] as

$$P(0) = \frac{e^{\frac{\delta E}{k_B \tau}}}{1 + e^{\frac{\delta E}{k_B \tau}}} \approx 98.4\%, \quad P(1) = 1 - P(0), \quad (1.7)$$

where k_B is the Boltzmann's constant, and $P(i)$ is the population in $|i\rangle$. This initialization method requires minimal effort, but has multiple limitations. The initialization time scales with the qubit T_1 , introducing a trade-off between qubit coherence and initialization speed. Moreover, the Boltzmann distribution requires the qubit transition energy to be much larger effective thermal energy of the qubit temperature, which limits its applicability to low frequency qubits with small transition energy such as fluxonium⁷[91].

The active resetting methods, rather than relying on the qubit's passive decay, employ carefully engineered drives to actively pump the qubit into a desired state [56, 84]. This approach was originally developed in cold-atom and ion-trap experiments [92, 93] and was subsequently rapidly adopted in superconducting qubit systems. For example, by resonantly driving multiphoton transitions between the qubit and the readout resonator, the qubit excitation can be swapped into the readout resonator, where it is rapidly removed via the resonator's fast decay[56]. As a result, the qubit can be initialized in its ground state $|0\rangle$ on a timescale shorter than its intrinsic T_1 .

The heralding method relies on the readout process to collapse the qubit into a specific state, thereby initializing the qubit into that state. Apparently, the heralding method requires a high-fidelity readout with high QNDness. Moreover, unlike the previous two methods, which initialize the qubit into a definite state, the initialization result of the heralding method depends on the readout outcome and is therefore probabilistic. However, this can usually be solved by post-selection and post-processing[54, 85]. In this thesis, we will initialize the Kerr-cat qubit with either the passive thermalization or the heralding methods, depending on their applicability.

Qubit Control and Gate Time

A single qubit is a unit vector in a 2D Hilbert space characterized by two complex numbers. The operations on this vector form an $SU(2)$ group, which is locally isomorphic to the 3-dimensional rotation $SO(3)$ group[94]. Therefore, we can map the qubit states and gates on a 3D sphere named the Bloch sphere, as shown in Fig. 1.3. Every point on the Bloch sphere represents a specific qubit state denoted as $|\psi\rangle = \cos(\frac{\theta}{2}) |0\rangle + e^{i\phi} \sin(\frac{\theta}{2}) |1\rangle$ with the polar angle θ and azimuthal angle ϕ . Usually, the azimuthal angle ϕ is also referred to as the phase of the qubit. Every rotation about the origin corresponds to a single-qubit gate. We can define the generators of rotations along the three axes as the Pauli matrices,

⁶Some experiments done in our group suggest that the qubit may not follow the Boltzmann distribution

⁷In quantum mechanics, the level transition energy can also be denoted by their transition frequencies, and the scaling coefficient between them is the Planck's constant h (reduced Planck's constant $\hbar$) for linear frequency (angular frequency)

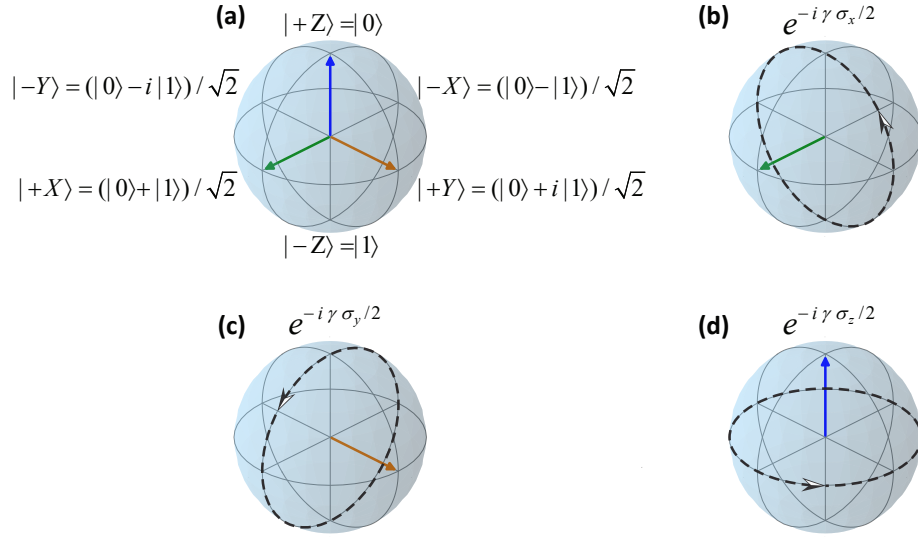


Figure 1.3: Bloch sphere representation of the qubit state and operations. (a) The qubit states on the three axes. (b) The $X(\gamma)$ rotation gates. (c) The $Y(\gamma)$ rotation gates. (d) The $Z(\gamma)$ rotation gates.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.8)$$

Therefore, the rotation of a specific state $|\psi\rangle$ along three axes can be represented as $\nu(\gamma)|\psi\rangle = e^{-i\gamma\sigma_\nu/2}|\psi\rangle$, $\nu = \{x, y, z\}$. A universal single-qubit gate set can be achieved by the combination of these rotations.

In superconducting circuits, single-qubit gates are realized by microwave drives on resonance with the qubit transition frequency[76]. This resonant drive induces the coherent population transfer between the $|0\rangle$ and $|1\rangle$ states, leading to an $X(\gamma)$ rotation when the drive is in-phase with the reference⁸. The $Z(\gamma)$ rotation is realized by a phase shift of the drive⁹, and the $Y(\gamma)$ rotation is implemented by the combination of the $X(\gamma)$ and $Z(\gamma)$ rotations[76, 95].

A multi-qubit universal gate set can be generated by the combination of a two-qubit entanglement gate and a single-qubit universal gate set[96]. The entanglement gates are usually ISWAP, Controlled-Z (CZ), and Controlled-NOT (CNOT) gates[97–99]. The ISWAP gate requires coherent population transfer between two coupled qubits, which can be achieved by either bringing the frequencies of the two qubits on resonance[97], or parametrically

⁸This reference can be chosen arbitrarily, as long as it is fixed.

⁹This is often referred to as a virtual Z gate because it does not involve any physical drive applied to the qubit and effectively takes zero time.

driving the coupler at the sideband frequency of the system[100, 101]. The CZ gate relies on the strong ZZ interactions between the two coupled qubits, which are usually engineered by off-resonantly driving the qubits[98] or bringing higher levels close to resonance[28]. The CNOT gate flips the state of the target-qubit based on the state of the control-qubit, and can be realized by a cross-resonance technique by driving the control-qubit at the target-qubit frequency[99].

Usually, single-qubit gate times are usually several tens of nanoseconds[54, 56], while two-qubit gate times range from several tens of nanoseconds to several hundreds of nanoseconds[28, 97]. Therefore, the qubit lifetimes have to be much longer than these time scales to implement quantum computing.

Qubit Lifetime

The qubit lifetime is usually evaluated by two characteristic times, T_1 and T_2 . Similar to the terminologies used in the NMR community[15], T_1 represents the decay time of state $|1\rangle$ to state $|0\rangle$, characterizing the state transition between $|+Z\rangle$ and $|-Z\rangle$ states. Therefore, T_1 is also named as bit-flip time. T_2 , on the other hand, represents the decay time of the phase of the qubit, characterizing the state transition between $|+X\rangle$ and $|-X\rangle$ states. Therefore, T_2 is also named the phase-flip time.

The lifetimes of superconducting qubits were below one microsecond before the introduction of transmon[27, 31]. By increasing the capacitance of the superconducting Cooper-pair box to reduce its sensitivity to charge noise, the transmon qubit was demonstrated to exhibit long T_1 and T_2 times on the order of hundreds of microseconds. Recent research has pushed these numbers to above one millisecond[102]. Beyond transmon qubits, other encoding regimes also achieve long lifetimes. The fluxonium qubits achieved T_1 over one millisecond due to their resilience to flux noises[103], and the cat qubits achieved bit-flip time on the order of several seconds at the cost of low phase-flip time of sub-microseconds[63].

1.3 Quantum Error Correction (QEC)

Qubits are fragile, and the average error per operation in superconducting qubits is on the order of 0.1% – 0.01%, which remains too high for practical quantum computing. Therefore, QEC is necessary for achieving fault-tolerant quantum computing (FTQC). QEC is usually much more complicated than the error correction of classical qubits, mainly because of the broader range of noise sources[36, 42], collapse of the wavefunction[2], and the no-cloning theorem[104]. Specifically, noise-biased qubits such as cat qubits, which are resilient to dominant noise sources at the expense of enhanced sensitivity to others, provide a promising pathway toward hardware-efficient quantum error correction[54, 56, 58].

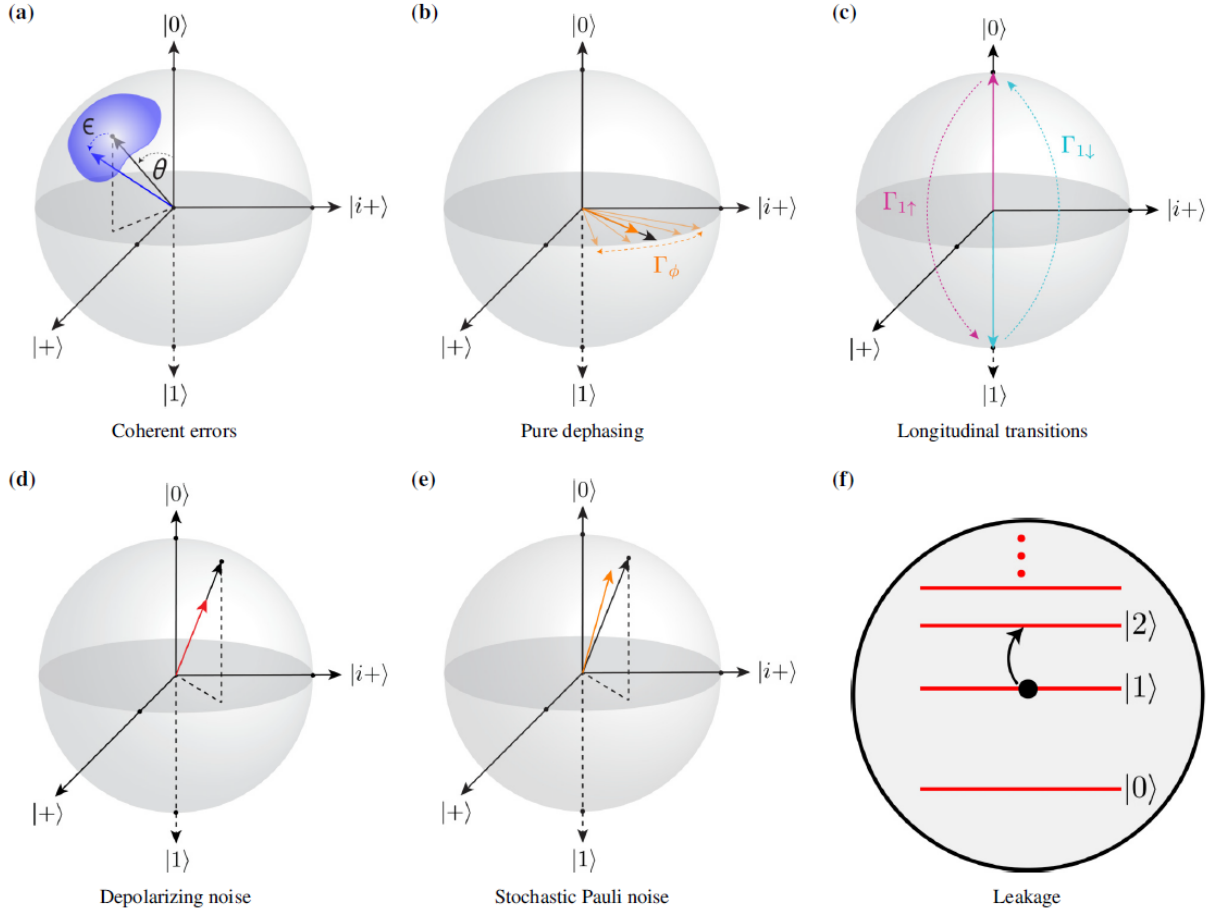


Figure 1.4: Common errors in superconducting qubits. (a) Coherent errors, corresponding to a mis-rotation on the Bloch sphere. (b) Pure dephasing, corresponding to random phase flips. (c) Longitudinal transitions, corresponding to random bit flips. (d) Depolarizing noise, which represents an isotropic reduction of the state purity. (e) Stochastic Pauli noise, which represents an anisotropic reduction of the state purity. (f) Leakage, corresponding to the state transition out of the qubit encoding space.

Noise sources

Superconducting circuits are exposed to various noise mechanisms, including the radiation loss[105], dielectric loss[106], resistive loss[107], two-level-system (TLS) -induced loss[108], control electronics noises[109], correlated errors[110, 111]. Regarding their effects on the qubit states, one can summarize some of them in Fig. 1.4[112].

When describing noisy states and processes, the density matrix representation is usually more powerful than the state vector representation, because it can represent mixed states. A mixed state is a classical mix of many pure quantum states $|\psi_i\rangle$ with probability p_i , and

its density matrix is defined as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad (1.9)$$

where $\langle\psi_i|$ is the dual (conjugate-transpose) of state vector $|\psi_i\rangle$. For a pure quantum state, only one p_i is non-zero.

The density matrix of a single qubit is a 2×2 unitary matrix, which can be decomposed into Pauli matrices as

$$\rho = \frac{1}{2} (I + \vec{r} \cdot \vec{\sigma}) \quad (1.10)$$

where $\vec{\sigma} = [\sigma_x, \sigma_y, \sigma_z]^T$, $\vec{r} = [r_x, r_y, r_z]$, $|\vec{r}| \leq 1$. In the Bloch-sphere picture, the vector $\vec{r}$ lies in the sphere. The pure states are those with $|\vec{r}| = 1$, corresponding to points on the surface of the Bloch sphere, while the mixed states are those with $|\vec{r}| < 1$, corresponding to points strictly inside the Bloch sphere.

Therefore, the effects of common errors in Fig. 1.4 can be easily interpreted by their actions on the vector $\vec{r}$. The coherent errors are usually from mis-calibrated control, which only cause a mis-rotation on the Bloch sphere without degrading the state purity. The pure dephasing introduces random phase flips, which reduce the purity of the states and shrink $\vec{r}$ towards the Z axis. The longitudinal transitions, on the other hand, flip the bit randomly, reduce the purity, and shrink $\vec{r}$ towards the equator. The depolarizing noise is a uniform purity reduction, which isotropically shrinks $\vec{r}$, while stochastic Pauli noise reduces the purity anisotropically, leading to different shrink rates along different directions. The pure dephasing, or phase-flip error, is equivalent to stochastic Pauli Z noises because it corresponds to a stochastic rotation along the Z axis. Similarly, the longitudinal transitions (with equal up- and down- transition rates, $\Gamma_{1\downarrow} = \Gamma_{1\uparrow}$)¹⁰, or bit-flip error, is equivalent to a combination of stochastic Pauli X and Y noises. Finally, the leakage represents the state transition out of the Bloch sphere space.

Error Detection and Correction

The detection and correction of qubit error is more difficult compared to classical bit error due to the state collapse and the no-cloning theorem. Measurement-induced state collapse prevents direct measurement of a qubit for error detection, while the no-cloning theorem forbids creating multiple copies of a quantum state for majority-vote correction. The state collapse has been discussed in Section 1.1, and we will briefly describe the no-cloning theorem here[104].

No cloning theorem is stated as follows. Considering a quantum state $|\psi\rangle_a = \alpha|0\rangle_a + \beta|1\rangle_a$, there is no quantum operation between this qubit a and another qubit b that can make the

¹⁰ T_1 error, which has $\Gamma_{1\downarrow} \gg \Gamma_{1\uparrow} \approx 0$ is a non-unital error and cannot be described by stochastic Pauli noises

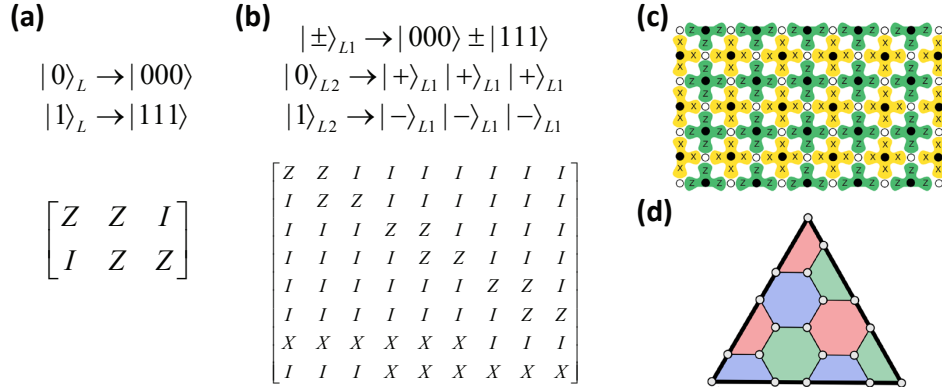


Figure 1.5: QEC codes. (a) Simple repetition code and its error syndrome check. (b) 9-qubit Bacon-Shor code and its error syndrome check. (c) Surface code[37]. (d) Color code[113].

final two-qubit state be $|\psi\rangle_a|\psi\rangle_b$. People may intuitively misinterpret the CNOT operation between two qubits as a “cloning”, which has the following effects:

$$\text{CNOT}|\psi\rangle_a|0\rangle_b = \alpha|0\rangle_a|0\rangle_b + \beta|1\rangle_a|1\rangle_b \neq |\psi\rangle_a|\psi\rangle_b. \quad (1.11)$$

But as indicated in the formula above, it generates an entangled gate rather than cloning the qubit a .

Therefore, carefully designed error correction codes and ancilla-assisted error detection are necessary for QEC. The simplest QEC codes are repetition codes, where a single logical qubit is encoded by a repetition of many $|0\rangle$ and $|1\rangle$ state qubits, i.e., $|\psi\rangle_L = \alpha|000\dots\rangle + \beta|111\dots\rangle$. Then, as illustrated by the 3-to-1 encoding example shown in Fig. 1.5(a), a single-qubit bit-flip error can be detected by measuring the two-qubit Pauli operators $ZZ = \sigma_z \otimes \sigma_z$ on the first and the last pairs of qubits. Any single-qubit bit-flip error anti-commutes with one or both of these ZZ Pauli operators, thereby flipping the sign of the corresponding measurement outcomes. Importantly, both $|0\rangle_L$ and $|1\rangle_L$ have the same value of those ZZ Pauli operators, and therefore, the encoded logical qubit will not be collapsed by such measurement.

Qubits typically suffer from both bit-flip and phase-flip errors, whereas the repetition code discussed above can correct only one type of error. As illustrated in Fig. 1.5(b), this limitation can be overcome by concatenating two repetition codes: one for correcting bit-flip errors (denoted by the subscript $L1$) and the other for correcting phase-flip errors (denoted by the subscript $L2$). The resulting codes belong to the class of CSS codes [114]. The specific construction shown here is one of the earliest examples of CSS codes, known as the Bacon–Shor code. Some QEC codes have a planar geometry and require only nearest-neighbor interactions, making them particularly well suited to superconducting qubit platforms. These codes include surface codes and color codes shown in Fig. 1.5(c) and (d).

Noise Bias

Most superconducting qubits have similar T_1 and T_2 , leading to similar phase-flip and bit-flip errors. However, with careful qubit encoding, one can achieve strongly biased noise in which the bit-flip error rate is greatly suppressed[63]. With such noise-biased qubits, some tailored surface codes, such as the XZZX surface code, are demonstrated to have low hardware overhead and high error threshold, making them a promising platform for FTQC[58–60].

The noise bias is usually defined as the ratio between the phase-flip error and the bit-flip error. As mentioned previously, the bit-flip and phase-flip errors can all be described as stochastic Pauli errors along different axes. Therefore, the noise bias of a qubit is usually given by

$$\eta = \frac{p_z}{p_x + p_y} \quad (1.12)$$

where $p_\nu, \nu = \{x, y, z\}$ is the stochastic Pauli error along the ν axis. In this thesis, we demonstrate a Kerr-cat qubit with a noise bias of 250 with $p_z < 2\%$, establishing a state-of-the-art performance benchmark for Kerr-cat qubits.

Chapter 2

Quantum Electrical Circuits

The behavior of electrical circuits is governed by the collective dynamics of charge carriers in conducting materials, most commonly free electrons in metals. These electrons generally occupy distinct quantum states and are subject to frequent scattering, rendering their collective behavior effectively classical. Therefore, the description of electrical circuits is usually classical.

However, in the superconducting phase of metals, electrons form bound pairs known as Cooper pairs, which condense into a single macroscopic quantum state described by a coherent wavefunction. This superconducting condensate is characterized by a well-defined global phase, whose dynamics determine supercurrent flow and enable quantum behavior at the macroscopic circuit level[115, 116].

As a result, in the superconducting microwave circuit regime, linear microwave resonators behave analogously to optical cavities and can be described as simple harmonic oscillators with equally spaced energy levels, supporting microwave photons[76]. The nonlinear microwave resonators, by contrast, exhibit anharmonic energy spectra with non-uniform level spacing, effectively mimicking a single atom in the context of light-matter interactions[117].

The quantum theory of electrical circuits, or circuit quantum electrodynamics (circuit QED), lies at the intersection between the circuit theory in the electrical engineering community and the cavity quantum electrodynamics (cavity QED) in the atomic, molecular, and optical (AMO) physics community. We recommend three references that assist readers in learning the theoretical approaches of, and building connections between, the two communities.

- D. M. Pozar, **Microwave Engineering: Theory and Techniques**. This book provides the classical theory for microwave circuits, which is useful before quantizing the circuits.
- P. Krantz *et al.*, **A quantum engineer's guide to superconducting qubits**. This review bridges the classical and quantum theories of microwave circuits.

- D. A. Steck, **Quantum and Atom Optics**. This book provides the quantum theory of light-matter interaction, which is useful after quantizing the circuits.

2.1 Classical Circuits and Lagrangian Analysis

We will start with the classical description of the electrical circuits and construct the Lagrangian analysis of basic circuit components. This will serve as the foundation for circuit quantization and Hamiltonian analysis.

Lumped Elements

Lumped elements are idealized circuit components whose physical dimensions are much smaller than the relevant electromagnetic wavelengths, allowing the abstraction of their properties by capacitance C , inductance L , and resistance R . Usually, the behavior of circuits is characterized by the voltage V at different nodes, and the current I through different paths. When constructing Lagrangian analysis, it is useful to integrate their time dynamics and examine the flux Φ and charge Q variables defined as

$$\phi(t) = \int_{-\infty}^t V(t')dt', \quad Q(t) = \int_{-\infty}^t I(t')dt', \quad (2.1)$$

so that $V = \dot{\Phi}$ and $I = \dot{Q}$. The Lagrangian formalism is based on the Euler-Lagrange equation for generalized coordinates and their first-order derivatives[118]. Therefore, we can choose either the variable pair $\{\Phi, \dot{\Phi}\}$ or $\{Q, \dot{Q}\}$ to build the theory. People usually choose the $\{\Phi, \dot{\Phi}\}$ pair, because the flux variable Φ is directly related to the phase of the superconducting condensate wavefunction with a constant scaling factor, leading to a more physically intuitive and simpler theoretical description. In this thesis, we will also follow this choice. For intuitive understanding, $\{\Phi, \dot{\Phi}\}$ can be treated as analogous to the {position, velocity} pair in mechanics.

Capacitance

The constitutive relation of a capacitor reads

$$Q = CV, \quad (2.2)$$

and by integrating it, we obtained the capacitive energy stored in the capacitor, given by

$$T = \frac{1}{2}CV^2 = \frac{1}{2}C\dot{\Phi}^2. \quad (2.3)$$

The capacitive energy follows a similar formula to the kinetic energy in mechanics, so we denote it as T . Applying the Euler-Lagrange equation to this kinetic energy,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \right) - \frac{\partial \mathcal{L}}{\partial \Phi} = 0, \quad \mathcal{L}(\Phi, \dot{\Phi}) = T = \frac{1}{2} C \dot{\Phi}^2, \quad (2.4)$$

we obtained the result as

$$C \ddot{\Phi} = 0, \quad \text{i.e. } \dot{Q} = 0, \quad (2.5)$$

where we recovered the charge conservation law for an isolated capacitor.

Inductance

The constitutive relation for an inductor is given by

$$\Phi = LI. \quad (2.6)$$

Similar to the capacitor, the integration of it leads to the inductive energy stored in the inductor, given by

$$U = \frac{\Phi^2}{2L}. \quad (2.7)$$

In contrast to capacitive energy, inductive energy is similar to the potential energy in mechanics, and we denote it as U . Applying the Euler-Lagrange equation to this will give the trivial result of $\Phi = 0$, meaning that there is no dynamics in an isolated undriven inductor.

Resistance

The constitutive relation of a resistor is given by

$$V = IR. \quad (2.8)$$

Resistance, unlike capacitors and inductors, does not store energy. Instead, by integrating the equation above, the resistor dissipates energy at the rate of

Table 2.1: Correspondence between flux-based circuit and mechanics

Circuit	Mechanics
Flux (Φ)	Position
Voltage ($\dot{\Phi} = V$)	Velocity
Capacitance (C)	Mass
Inverse inductance (L^{-1})	Restoration force coefficient
Capacitive energy ($\frac{1}{2} C \dot{\Phi}^2$)	Kinetic energy
Inductive energy ($\frac{1}{2L} \Phi^2$)	Potential energy
Conductance/Resistance ($G = 1/R$)	Friction coefficient

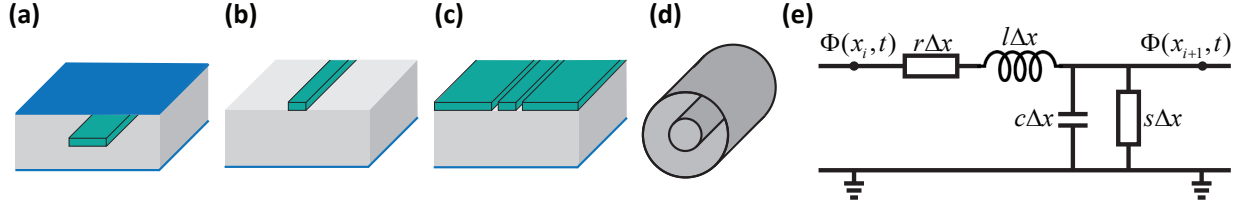


Figure 2.1: Transmission line examples and circuit model. (a) Stripline. (b) Micro-strip. (c) Coplanar waveguide (CPW), (d) Coaxial line. (e) Circuit model for an infinitesimal segment of the transmission line.

$$F = \frac{V^2}{R} = \frac{\dot{\Phi}^2}{R}. \quad (2.9)$$

To include this dissipative element into the Lagrangian analysis, one needs to generalize the Euler-Lagrange equation with the inclusion of the Rayleigh dissipation function as follows,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \right) - \frac{\partial \mathcal{L}}{\partial \Phi} + \frac{1}{2} \frac{\partial F}{\partial \dot{\Phi}} = 0. \quad (2.10)$$

Tab. 2.1 provides a comprehensive overview of the analogies between the core electrical elements and mechanical systems as derived through the Euler-Lagrange formalism.

Distributed Elements

Beyond lumped elements, the distributed elements are another important building block for microwave circuits. Distributed elements have sizes that are comparable to or larger than the relevant electromagnetic wavelength. A primary example is the transmission line, which supports the propagation of microwave electromagnetic fields over long distances. Some examples of transmission lines are shown in Fig. 2.1, including the stripline, the micro-strip, the coplanar waveguide (CPW), and the coaxial line. In superconducting circuits, the dominant transmission line being used is the CPW.

To analyze the transmission line, one can construct the Lagrangian of an infinitesimal segment of it, or the Lagrangian density. As illustrated in Fig. 2.1(e), an infinitesimal segment of the transmission line of length dx is modeled using lumped-element parameters. The signal path is characterized by an inductance l and resistance r per unit length. The coupling to the ground plane is represented by a shunt capacitance c and conductance s per unit length. For superconducting circuits, the lossy terms are approaching zero¹, i.e., $s \approx 0, r \approx 0$.

¹The lossy terms can be important for analyzing the internal quality factor of the high-performance transmission line

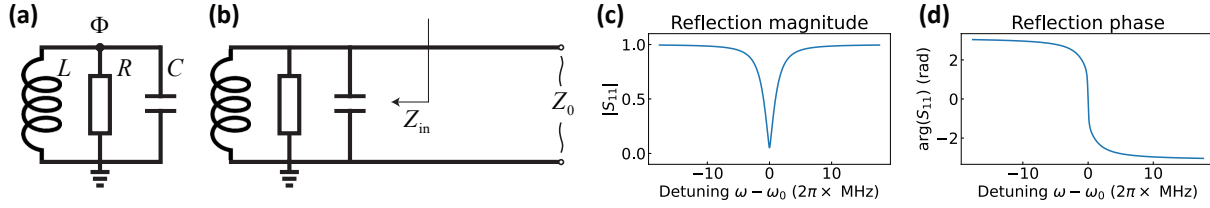


Figure 2.2: Parallel RLC resonator. (a) Circuit diagram. (b) Measurement of an RLC resonator. (c) Reflection magnitude and (d) Reflection phase of the RLC resonator.

For lossless transmission lines, the Lagrangian density $\mathfrak{L}$ and the Lagrangian $\mathcal{L}$ are given by

$$\mathfrak{L} = \frac{1}{2}c\dot{\Phi}^2 - \frac{1}{2l} \left(\frac{\partial \Phi}{\partial x} \right)^2, \quad \mathcal{L} = \int \mathfrak{L} dx \quad (2.11)$$

where $\Phi = \Phi(x, t)$, and we omit the explicit arguments hereafter for brevity. The dynamics of a transmission line can be described using the Euler-Lagrange equation for fields,

$$\frac{\partial \mathfrak{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathfrak{L}}{\partial (\partial_\mu \Phi)} \right) = 0 \quad \rightarrow \quad \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \right) + \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \Phi'} \right) - \frac{\partial \mathcal{L}}{\partial \Phi} = 0, \quad (2.12)$$

where μ iterates over all dependencies of Φ , and $\Phi' = \partial \Phi / \partial x$. Plugging in the Lagrangian density of the transmission line, we end up with the lossless wave equation with the propagation speed $1/\sqrt{lc}$,

$$\frac{\partial^2 \Phi}{\partial x^2} - lc \frac{\partial^2 \Phi}{\partial t^2} = 0. \quad (2.13)$$

Resonators

Lumped Element Resonators

By combining the capacitance, inductance, and resistance in series or parallel, one can construct a microwave resonator. The resonators in the superconducting qubit community are usually configured in parallel, making the flux-based Lagrangian analysis particularly well-suited for describing their dynamics.

Considering a circuit shown in Fig. 2.2 (a), the Lagrangian and the energy dissipation rate are given by

$$\mathcal{L} = \frac{1}{2}C\dot{\Phi}^2 - \frac{1}{2L}\Phi^2, \quad F = \frac{\dot{\Phi}^2}{R}. \quad (2.14)$$

Plugging this into the Euler-Lagrange equation, we obtained

$$\ddot{\Phi} + \frac{1}{RC}\dot{\Phi} + \frac{1}{LC}\Phi = 0, \quad (2.15)$$

which is exactly the equation of motion for a damped simple harmonic oscillator, with the resonant frequency $\omega_0 = 1/\sqrt{LC}$ and intrinsic damping rate $\kappa_i = (RC)^{-1}$.

The properties of the resonator can be measured by connecting this resonator to a transmission line with characteristic impedance Z_0 and measuring the reflection coefficient S_{11} . The input impedance of the resonator is

$$Z_{\text{in}} = \left(\frac{1}{i\omega L} + i\omega C + \frac{1}{R} \right)^{-1}, \quad (2.16)$$

where ω is the signal frequency. Therefore, the reflection coefficient is

$$S_{11} = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \frac{(Z_0 C)^{-1} - (RC)^{-1} - i\frac{\omega^2 - \omega_0^2}{\omega}}{(Z_0 C)^{-1} + (RC)^{-1} + i\frac{\omega^2 - \omega_0^2}{\omega}}. \quad (2.17)$$

For a near-resonance signal, $\omega^2 - \omega_0^2 \approx 2\omega(\omega - \omega_0)$. Using this approximation, we obtained a clean expression

$$S_{11} = \frac{(\kappa_e - \kappa_i)/2 - i(\omega - \omega_0)}{(\kappa_e + \kappa_i)/2 + i(\omega - \omega_0)} \quad (2.18)$$

with $\kappa_e = (Z_0 C)^{-1}$ as the external damping rate of the resonator. By fitting the reflection spectrum to this equation, the intrinsic damping rate κ_i , external damping rate κ_e and resonant frequency ω_0 can all be extracted, and a calculated example of S_{11} is shown in Fig. 2.2(c) and (d).

However, when κ_e differs significantly from κ_i , this equation fails to accurately extract κ_i . In superconducting circuits, κ_i is typically small due to low dielectric and resistive losses, while κ_e is often too large for a directly connected transmission line with the commonly used characteristic impedance $Z_0 = 50\Omega$. Therefore, in practice, people usually introduce a coupling capacitance between the transmission line and the resonator to reduce the κ_e . But one should also notice that in certain applications, including Josephson parametric amplifiers, a larger κ_e is favorable. In these cases, directly connecting the circuit to a low-impedance transmission line can further increase κ_e and lead to better performance[119–122].

Distributed Resonators

Beyond explicitly combining the lumped element resistance, inductance, and capacitance together to form a resonator, carefully designed distributed elements, or transmission line segments, could also form a resonator. In fact, due to the simplicity in design and fabrication, the distributed resonators are more common in superconducting circuits.

Recalling that the transmission line dynamics is described by the wave equation in Eq. 2.13, which has the general solution given by

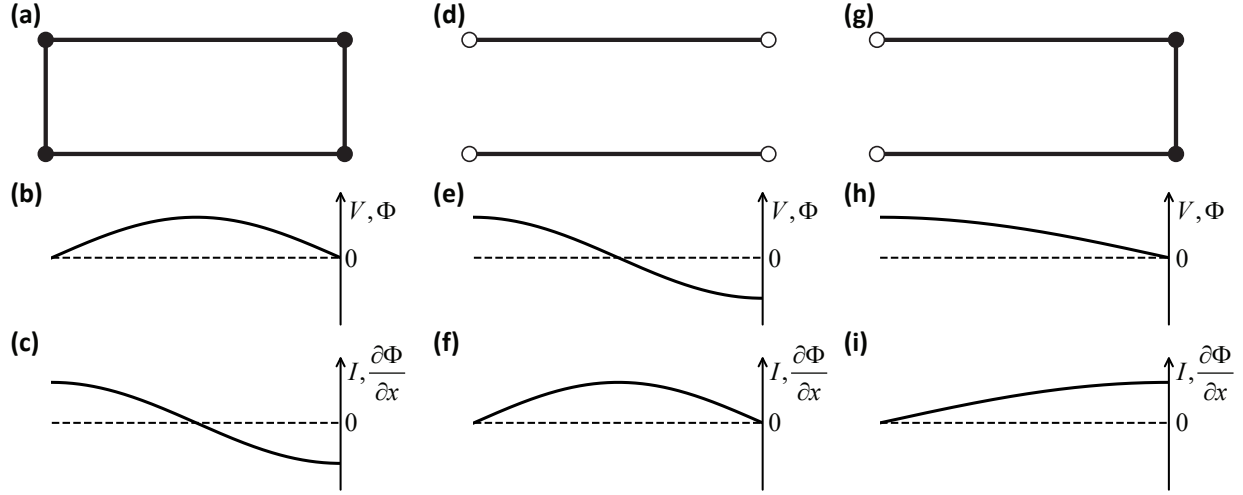


Figure 2.3: Distributed resonators under varying boundary conditions and their associated signal profiles. (a) Schematic of a resonator with both ends shorted. (b) Spatial profiles for voltage and flux, which share the same node/antinode positions. (c) Spatial profiles for current and flux derivative. (d–f) Corresponding schematic and signal profiles for a resonator with both ends open. (g–i) Corresponding schematic and signal profiles for a resonator with one end open and the other shorted.

$$\Phi(x, t) = \sum_k (A_k e^{ikx} + B_k e^{-ikx}) e^{-i\omega_k t}, \quad \omega_k = \frac{|k|}{\sqrt{lc}}. \quad (2.19)$$

Taking a segment of transmission line and terminating two ends with either open or short circuit boundary conditions, one can impose different restrictions on the allowed modes on the transmission line, which effectively form a multi-mode resonator. The boundary condition for open ends is

$$I(x_{\text{open}}, t) = \frac{1}{l} \frac{\partial \Phi}{\partial x}(x_{\text{open}}, t) = 0 \quad \rightarrow \quad \frac{\partial \Phi}{\partial x}(x_{\text{open}}, t) = 0, \quad (2.20)$$

and the boundary condition for short ends is

$$V(x_{\text{short}}, t) = \frac{\partial \Phi}{\partial t}(x_{\text{short}}, t) = 0 \quad \rightarrow \quad \Phi(x_{\text{short}}, t) = 0. \quad (2.21)$$

We analyze resonators formed by the three possible terminations: both ends shorted (short-short), both ends open (open-open), and one end open while the other is shorted (open-short), as shown in Fig. 2.3. Plugging the general solutions to the corresponding boundary conditions, we obtained the allowed solutions for a segment with length d shown in Tab. 2.2, and the signal profile in the fundamental mode ($n = 1$) is sketched in Fig. 2.3.

Boundary Condition	$\Phi(x)$	k_n	Mode Frequency f_n
Short-Short, $\Phi(0) = \Phi(d) = 0$	$\sin\left(\frac{n\pi x}{d}\right)$	$\frac{n\pi}{d}$	$\frac{n}{2d\sqrt{lc}}$
Open-Open, $\Phi'(0) = \Phi'(d) = 0$	$\cos\left(\frac{n\pi x}{d}\right)$	$\frac{n\pi}{d}$	$\frac{n}{2d\sqrt{lc}}$
Short-Open, $\Phi(0) = \Phi'(d) = 0$	$\sin\left(\frac{(2n-1)\pi x}{2d}\right)$	$\frac{(2n-1)\pi}{2d}$	$\frac{2n-1}{4d\sqrt{lc}}$

Table 2.2: Allowed solutions for flux Φ in distributed resonators of length d .

The distributed resonator is, in principle, a multi-mode resonator supporting an infinite set of resonant frequencies indexed by the integer n . However, in most superconducting qubit practice, people only focus on the fundamental mode with $n = 1$, and keep all relevant frequencies far detuned from higher order modes with $n > 1$, effectively treating the distributed resonator as a single-mode resonator. For the resonators with the same boundary condition for both ends (open-open and short-short), the fundamental resonant mode wavelength is $2d$ and is therefore referred to as $\lambda/2$ resonators, while those with different boundary conditions for two ends (open-short) have a fundamental resonant mode wavelength of $4d$ and are therefore referred to as $\lambda/4$ resonators.

A handy strategy for analyzing the fundamental mode of a distributed resonator is to integrate its Lagrangian density and approximate it to a lumped element resonator. The integration of fundamental mode profiles gives

$$\Phi(x, t) = \Phi(t) \sin\left(\frac{\pi x}{d}\right) \rightarrow \int_0^d \dot{\Phi}(x, t)^2 dx = \frac{d}{2} \dot{\Phi}(t)^2, \quad \int_0^d \left(\frac{\partial \Phi(x, t)}{\partial x}\right)^2 dx = \frac{\pi^2}{2d} \Phi(t)^2, \quad (2.22)$$

$$\Phi(x, t) = \Phi(t) \cos\left(\frac{\pi x}{d}\right) \rightarrow \int_0^d \dot{\Phi}(x, t)^2 dx = \frac{d}{2} \dot{\Phi}(t)^2, \quad \int_0^d \left(\frac{\partial \Phi(x, t)}{\partial x}\right)^2 dx = \frac{\pi^2}{2d} \Phi(t)^2, \quad (2.23)$$

$$\Phi(x, t) = \Phi(t) \sin\left(\frac{\pi x}{2d}\right) \rightarrow \int_0^d \dot{\Phi}(x, t)^2 dx = \frac{d}{2} \dot{\Phi}(t)^2, \quad \int_0^d \left(\frac{\partial \Phi(x, t)}{\partial x}\right)^2 dx = \frac{\pi^2}{8d} \Phi(t)^2. \quad (2.24)$$

Therefore, integrating the Lagrangian density of a distributed resonator gives the Lagrangian

$$\mathcal{L} = \int_0^d \mathfrak{L} dx = \frac{1}{2} C_{\text{eff}} \dot{\Phi}^2 - \frac{1}{2L_{\text{eff}}} \Phi^2, \quad (2.25)$$

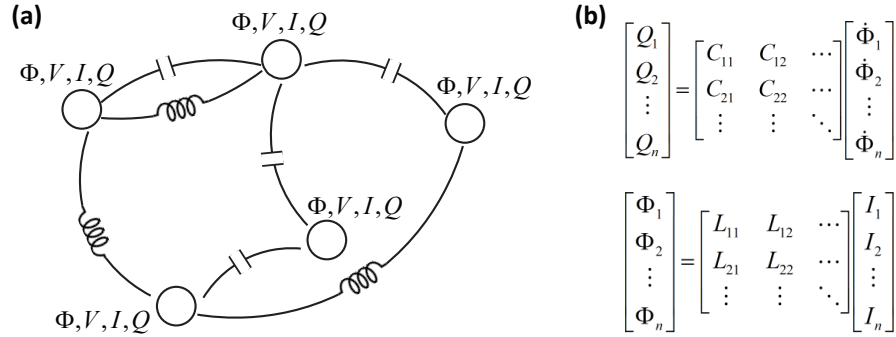


Figure 2.4: Multi-mode circuit as a network. (a) The schematic of the network. (b) The constitutive relation of the network.

where we omit the time argument of $\Phi(t)$ for clarity, and $C_{\text{eff}} = \frac{1}{2}cd$, $L_{\text{eff}} = \frac{2ld}{\pi^2}$ for $\lambda/2$ resonator and $L_{\text{eff}} = \frac{8ld}{\pi^2}$ for $\lambda/4$ resonator. The resultant distributed resonator Lagrangian follows the same pattern as the lumped element resonator as long as the relevant frequency is close to the fundamental resonant frequency.

Multiple Modes

Up to now, the analysis of microwave resonators has been restricted to a single mode. However, the qubit readout and entanglement usually require the interaction between different modes. In this section, we will extend the Lagrangian analysis to multi-mode circuits. For simplicity and clarity, we assume circuits to be lossless in this section (i.e., $R = \infty$).

We can view the general multi-mode circuit as a network. In this network shown in Fig. 2.4(a), each node is characterized by its flux Φ , voltage $V = \dot{\Phi}$, charge Q , and total current leaving the node I , and each edge is characterized by capacitance C or inductance L . By arranging all the relevant node parameters as a vector, we can write the constitutive relation of the network as

$$\mathbf{Q} = \mathbf{C}\mathbf{V} = \mathbf{C}\dot{\mathbf{\Phi}}, \quad \mathbf{\Phi} = \mathbf{L}\mathbf{I}, \quad (2.26)$$

where the node parameter vector is correlated by the Maxwell capacitance and inductance matrix². The capacitance and inductance matrices are always symmetric[124]. One can then write down the Lagrangian of the network as

$$\mathcal{L} = \frac{1}{2}\dot{\mathbf{\Phi}}^T \mathbf{C} \dot{\mathbf{\Phi}} - \frac{1}{2}\mathbf{\Phi}^T \mathbf{L}^{-1} \mathbf{\Phi}, \quad (2.27)$$

²Another convention of the matrix construction is given by the Spice matrix[123].

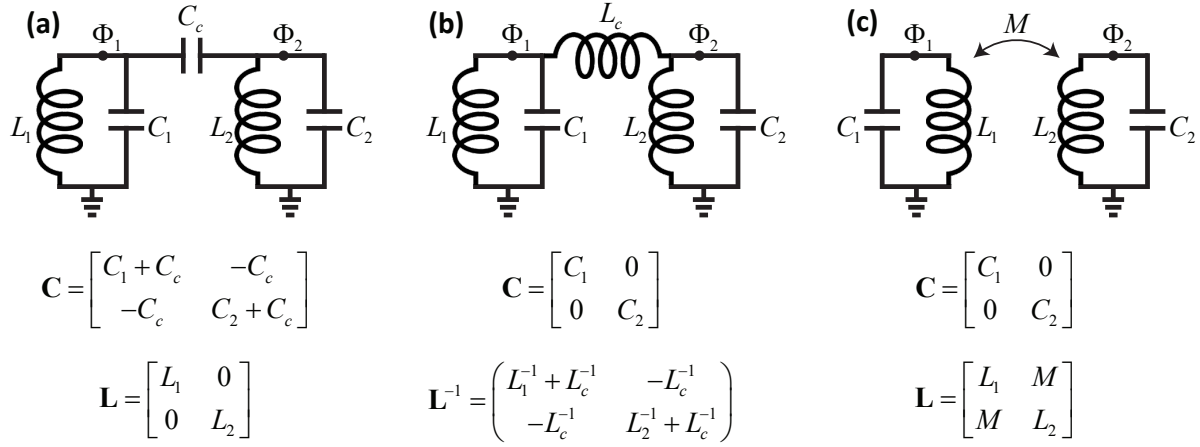


Figure 2.5: Commonly used circuit topologies and their capacitance and inductance matrix. (a) Capacitively coupled resonators. (b) Inductively coupled resonators. (c) Mutual-inductance coupled resonators.

with $\Phi = [\Phi_1, \Phi_2, \dots, \Phi_n]^T$, and construct the equations of motion by the Euler-Lagrange equation for each node

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i} \right) - \frac{\partial \mathcal{L}}{\partial \Phi_i} = 0. \quad (2.28)$$

In practice, the capacitance and inductance matrix of a realistic superconducting circuit layout can be extracted by finite element method (FEM) simulation. The capacitance and inductance matrices for several common topologies are presented in Fig. 2.5. Note that for each topology, either the matrix or its inverse is provided, depending on which form yields a more concise expression.

Transmission Line Coupling: position dependence

The above analysis is based on the assumption that nodes and edges are described by lumped-element parameters, but the properties of distributed elements, or transmission lines, can not always be idealized as lumped elements. As an example, the capacitive coupling between a distributed resonator and a lumped-element resonator is shown in Fig. 2.6 (a). In contrast to lumped elements, the coupling is distributed along the transmission line and characterized as the coupling capacitance density $c(x)$, and the electrostatic energy due to the coupling is given by

$$T_{\text{coup}} = \frac{1}{2} \int_0^d c(x) (\dot{\Phi}_1 - \dot{\Phi}_t(x))^2 dx. \quad (2.29)$$

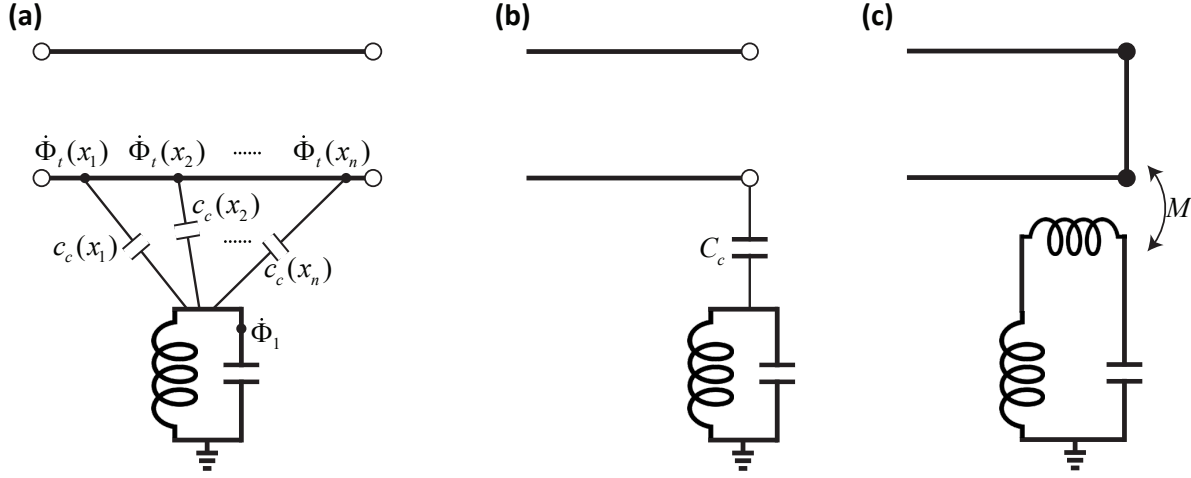


Figure 2.6: Coupling to a distributed resonator. (a) The schematic of capacitive coupling. (b) The capacitive coupling with high coupling strength. (c) The mutual-inductance coupling with high coupling strength.

Therefore, the coupling strength is dependent on the relative position between the lumped-element resonator and the distributed resonator. Generally speaking, the capacitive coupling strength is positively associated with the voltage amplitude and therefore is maximized at the open end of a distributed resonator, where the voltage forms an antinode, as shown in Fig. 2.6 (b). Correspondingly, the mutual-inductance coupling strength is positively associated with the current amplitude and therefore is maximized at the short end of a distributed resonator, as shown in Fig. 2.6 (c). On realistic superconducting chips, such couplings will always be a combination of capacitive and mutual inductance coupling.

Conserved Quantities

Similar to classical mechanics, identifying and exploiting conserved quantities can significantly simplify the analysis of electrical circuits and reduce the number of nodes. We will show some examples in this section.

Floating Resonator

Considering the circuit shown in Fig. 2.7 (a), the parallel LC resonator is floating from the ground with the capacitance C_g on each side. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}C_g(\dot{\Phi}_1^2 + \dot{\Phi}_2^2) + \frac{1}{2}C(\dot{\Phi}_1 - \dot{\Phi}_2)^2 - \frac{1}{2L}(\Phi_1 - \Phi_2)^2. \quad (2.30)$$

The inductive energy is only related to the flux difference between the two nodes. Recalling that the inductive energy is analogous to the potential energy in mechanics and the flux

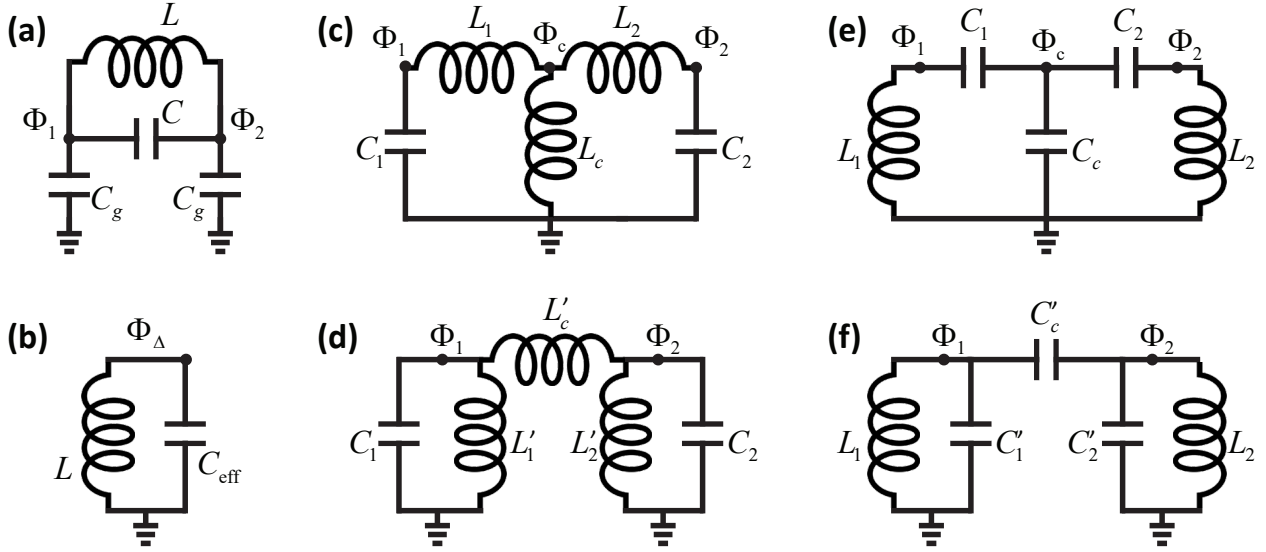


Figure 2.7: Simplification of circuits using conservative quantities. (a) A floating resonator can be simplified to a grounded resonator. (b) Y-connected inductors can be simplified to Δ -connection. (c) Y-connected capacitors can be simplified to Δ -connection.

coordinate is analogous to the position, this scheme is reminiscent of the two-body central-force problem. By defining new coordinates $\Phi_{\Delta} = \Phi_1 - \Phi_2$ and $\Phi_{\Sigma} = \frac{1}{2}(\Phi_1 + \Phi_2)$, we can rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{2}C_{\Sigma}\dot{\Phi}_{\Sigma}^2 + \frac{1}{2}C_{\text{eff}}\dot{\Phi}_{\Delta}^2 - \frac{1}{2L}\Phi_{\Delta}^2, \quad (2.31)$$

with $C_{\Sigma} = 2C_g$, $C_{\text{eff}} = C + C_g/2$. The two coordinates Φ_{Σ} and Φ_{Δ} are completely decoupled. If we make an analogous to mechanics, C_{Σ} and Φ_{Σ} corresponds to the total mass and center-of-mass coordinate, C_{eff} and Φ_{Δ} corresponds to the reduced mass and relative coordinate, and $1/L$ associates with the restoring force between the two bodies³. If we apply the Euler-Lagrange equation to this Lagrangian, the coordinate Φ_{Σ} has no related potential term, meaning that this coordinate corresponds to a free particle (analogous to mechanics) with

$$\ddot{\Phi}_{\Sigma} = 0, \quad (2.32)$$

and considering that the sum of all charges on the two nodes is $Q_{\Sigma} = C_{\Sigma}\Phi_{\Sigma}$, this implies the charge on the floating resonator is conserved:

$$\dot{Q}_{\Sigma} = 0. \quad (2.33)$$

³Notice that in this analogous, the reduced mass here is not necessary “reduced” because the capacitance between two nodes provide an intrinsic mass term C for the relative coordinate.

Using this conserved quantity, we can effectively remove the Φ_Σ coordinate and consider the effective Lagrangian

$$\mathcal{L} = \frac{1}{2}C_{\text{eff}}\dot{\Phi}_\Delta^2 - \frac{1}{2L}\Phi_\Delta^2, \quad (2.34)$$

which corresponds to a grounded resonator shown in Fig. 2.7 (b). Therefore, the dynamics of the floating resonator can be effectively described by a single-node grounded resonator.

Y- Δ Transformation

Considering the circuit with Y-connected inductors shown in Fig. 2.7 (c), the circuit has three nodes involved. However, using some conserved quantities, one can transform it into a Δ -connected inductor system with only two nodes involved. If we arrange the coordinate vector as $\mathbf{\Phi} = [\Phi_1, \Phi_c, \Phi_2]$, the capacitance and inductance matrix will be

$$\mathbf{C} = \begin{pmatrix} C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_2 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} L_1 + L_c & L_c & L_c \\ L_c & L_c & L_c \\ L_c & L_c & L_2 + L_c \end{pmatrix} \quad (2.35)$$

The capacitance matrix has all zero values in the second row and second column, because there is no capacitor connected to the node associated with Φ_c . As a result, the Euler-Lagrange equation for Φ_c leads to

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_c} \right) = \frac{\partial \mathcal{L}}{\partial \Phi_c} = \frac{1}{2} [(\mathbf{L}^{-T} + \mathbf{L}^{-1})\mathbf{\Phi}]_c = 0, \quad (2.36)$$

where $[\cdot]_c$ denotes the vector element corresponding to Φ_c . Plugging into the inductance matrix, we obtained

$$L_\Sigma^{-1}\Phi_c = L_1^{-1}\Phi_1 + L_2^{-1}\Phi_2, \quad (2.37)$$

where the node flux Φ_c is not independent and completely determined by the other two mode fluxes, and $L_\Sigma^{-1} = L_1^{-1} + L_c^{-1} + L_2^{-1}$. As the result, we can eliminate node Φ_c to arrange the effective node flux as $\mathbf{\Phi}_{\text{eff}} = [\Phi_1, \Phi_2]$, and therefore obtain the effective capacitance and inductance matrices:

$$\mathbf{C}' = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \quad \mathbf{L}'^{-1} = \begin{pmatrix} L_1'^{-1} + L_c'^{-1} & -L_c'^{-1} \\ -L_c'^{-1} & L_2'^{-1} + L_c'^{-1} \end{pmatrix}, \quad (2.38)$$

where the effective inductance is given by

$$L_1' = \frac{L_1 L_2 + L_2 L_c + L_1 L_c}{L_2}, \quad L_2' = \frac{L_1 L_2 + L_2 L_c + L_1 L_c}{L_1}, \quad L_c' = \frac{L_1 L_2 + L_2 L_c + L_1 L_c}{L_c}. \quad (2.39)$$

These two-node inductance and capacitance matrices correspond to an effective circuit shown in Fig. 2.7 (d), where two resonators are connected through a shunt inductance with only two nodes being involved. In fact, the process above has transformed the Y -connected (Fig. 2.7 (c)) inductor to Δ -connected inductors (Fig. 2.7 (d)), known as the Y - Δ transformation in electrical engineering[124].

The conserved quantity being applied here is not apparent from the first glance. However, Eq. 2.37 is equivalent to the equation below:

$$L_1^{-1}(\Phi_1 - \Phi_c) + L_1^{-1}(\Phi_1 - \Phi_c) = L_c^{-1}\Phi_c, \quad (2.40)$$

meaning that the total current flowing into and out of node c should be the same. This equation also has a perfect analogy to classical mechanics. Recalling that in Tab. 2.1 we made the analogy that the capacitance can be mapped to mass, flux can be mapped to position, and inverse inductance can be mapped to restoration force coefficient. Therefore, the node c with no capacitor connected corresponds to a massless joint in a constraint system[125], and the forces on this massless joint have to be balanced⁴. The Eq. 2.40 above reflects this force balance constraint in its analogous mechanical system.

Similar treatment can also be applied to a Y -connected capacitor system shown in Fig. 2.7(e), which has the capacitance and inductance matrices as:

$$\mathbf{C} = \begin{pmatrix} C_1 & -C_1 & 0 \\ -C_1 & C_1 + C_c + C_2 & -C_2 \\ 0 & -C_2 & C_2 \end{pmatrix}, \quad \mathbf{L}^{-1} = \begin{pmatrix} L_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L_2^{-1} \end{pmatrix}. \quad (2.41)$$

Noticing the zero-valued second row and second column of the inductance matrix, meaning that the potential energy for node Φ_c is zero. The Euler-Lagrange equation for this node will lead to

$$\left[\frac{d}{dt} \left((\mathbf{C}^T + \mathbf{C}) \dot{\Phi} \right) \right]_c = 0 \quad \rightarrow \quad \frac{d}{dt} \left(C_c \dot{\Phi}_c + C_1(\dot{\Phi}_c - \dot{\Phi}_1) + C_2(\dot{\Phi}_c - \dot{\Phi}_2) \right) = 0. \quad (2.42)$$

This equation reflects exactly the charge conservation at node c . Usually, the total charge on a DC-isolated node is zero unless some specific process intentionally charges that node. Therefore, we can force the total charge on node c to be zero, leading to

$$C_c \dot{\Phi}_c + C_1(\dot{\Phi}_c - \dot{\Phi}_1) + C_2(\dot{\Phi}_c - \dot{\Phi}_2) = 0. \quad (2.43)$$

Using this to eliminate the node c , one obtained the effective capacitance and inductance matrices as

$$\mathbf{C}' = \begin{pmatrix} C'_1 + C'_c & -C'_c \\ -C'_c & C'_2 + C'_c \end{pmatrix}, \quad \mathbf{L}'^{-1} = \begin{pmatrix} L_1^{-1} & 0 \\ 0 & L_2^{-1} \end{pmatrix}, \quad (2.44)$$

⁴For a massless joint, if the forces are not balanced, the acceleration of this joint will be infinite and therefore, nonphysical.

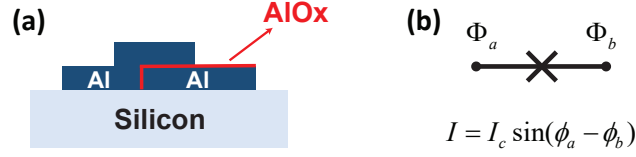


Figure 2.8: Josephson junction (a) structure and (b) circuit diagram

with the effective capacitance given by

$$C'_1 = \frac{C_c C_1}{C_1 + C_c + C_2}, \quad C'_2 = \frac{C_c C_2}{C_1 + C_c + C_2}, \quad C'_c = \frac{C_1 C_2}{C_1 + C_c + C_2}. \quad (2.45)$$

Our process transforms the Y -connected capacitance system in Fig. 2.7 (e) into the Δ -connected capacitance system in Fig. 2.7 (f) with the elimination of node c .

2.2 Nonlinearities in Superconducting Circuits

The AMO physics and cavity QED are mostly exploring the interaction between nonlinear atoms and linear cavities. In circuit QED, the linear microwave resonators we discussed before correspond to the linear cavities, while the nonlinear “atoms” are provided by the various applications of Josephson junctions (JJ), which can be viewed as a nonlinear inductor in superconducting circuits.

Josephson Junctions

A Josephson junction (JJ) is formed by sandwiching a thin insulator between two superconductors. In superconducting qubits, this structure is usually fabricated by an Al/AlOx/Al stack, as shown in Fig. 2.8. When the two Al metal pads are used as conductors, the current will be blocked by the AlOx insulator. However, when the two Al metal pads are superconductors, the current can tunnel through the thin AlOx insulator.

To extract the dynamics of the tunneling current, it is necessary to connect the circuit parameter, node flux Φ , to the superconducting electron (or Cooper pair) wavefunction parameter, phase ϕ . According to the Schrödinger equation and WKB approximation theory[2], the Cooper pair wavefunction is given by

$$\psi(x, t) = \psi(x) e^{-\frac{i}{\hbar} \int_{-\infty}^t H(t') dt'}, \quad (2.46)$$

where H is the Hamiltonian of the Cooper pair. In the absence of a magnetic field, the Hamiltonian of the Cooper pair of a circuit node is usually the same as its energy, given by $H = -2eV$, where V is the voltage of the node, e is the elementary charge, and the factor 2

is because the Cooper pair is formed by two electrons. Plugging this into the wavefunction of the Cooper pair, we obtained

$$\psi(x, t) = \psi(x) e^{\frac{2ie}{\hbar} \int_{-\infty}^t V(t') dt'} = \psi(x) e^{\frac{2ie}{\hbar} \Phi} = \psi(x) e^{i\phi}, \quad (2.47)$$

where we found the relation between the node flux Φ and the cooper pair wavefunction phase ϕ

$$\phi = \frac{2\pi\Phi}{\Phi_0}, \quad (2.48)$$

where Φ_0 is the magnetic flux quantum given by $\Phi_0 = \frac{h}{2e}$.

Then, we can model the JJ as two condensation of cooper pairs with wavefunctions $\psi_a(x, t) = \psi_a(x) e^{i\phi_a}$ and $\psi_b(x, t) = \psi_b(x) e^{i\phi_b}$ with have their intrinsic site energy U_a and U_b , and the coupling between them through the thin insulator is then modeled as the tunneling energy K between them. The Schrödinger equation of this system is given by

$$i\hbar \frac{\partial \psi_a}{\partial t} = U_a \psi_a + K \psi_b, \quad i\hbar \frac{\partial \psi_b}{\partial t} = U_b \psi_b + K \psi_a. \quad (2.49)$$

Solving these equations, one obtained

$$\frac{\partial |\psi_a(x)|}{\partial t} = \frac{2K}{\hbar} \sqrt{\psi_a \psi_b} \sin(\phi_b - \phi_a). \quad (2.50)$$

Noticing that $|\psi_a(x)|$ is proportional to the Cooper pair density, so the time derivative of it is proportional to the tunneling current (up to a negative sign). Therefore, by defining a proportionality coefficient I_c named as critical current, one obtains the current-flux relation of a JJ as

$$I = I_c \sin(\phi_a - \phi_b) = I_c \sin\left(\frac{2\pi}{\Phi_0}(\Phi_a - \Phi_b)\right). \quad (2.51)$$

To the first order, the current I is proportional to the flux difference between the two nodes, leading to the same constitutive relation as the linear inductor shown as

$$I = \frac{\Phi_a - \Phi_b}{L_j} \quad (2.52)$$

where $L_j = \frac{\Phi_0}{2\pi I_c}$. However, as the flux difference between the two nodes increases, the nonlinearity appears to be more and more important, leading to the behavior of a nonlinear inductor.

The energy associated with the JJ is given by

$$U = \int V I dt = I_c \int \frac{d\Phi}{dt} \sin\left(\frac{2\pi\Phi}{\Phi_0}\right) dt = I_c \int \sin\left(\frac{2\pi\Phi}{\Phi_0}\right) d\Phi = -E_j \cos\left(\frac{2\pi\Phi}{\Phi_0}\right) = -E_j \cos(\phi), \quad (2.53)$$

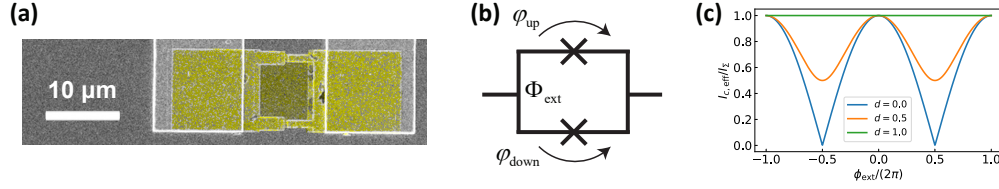


Figure 2.9: SQUID. (a) SEM picture of a SQUID (colored in yellow) on a silicon chip. (b) Circuit diagram. (c) Effective critical current.

where, for simplicity, we have replaced the flux (phase) difference $\Phi_a - \Phi_b$ ($\phi_a - \phi_b$) by a single symbol Φ (ϕ), and $E_j = \frac{\Phi_0 I_c}{2\pi}$ is known as the Josephson energy of the JJ. JJ is modeled as a nonlinear inductor, and the energy associated with JJ is only dependent on the node flux. Therefore, we use the U for its energy to emphasize that it should be a “potential” like term in Lagrangian analysis.

Superconducting Quantum Interference Device

For a single Josephson Junction (JJ), the critical current I_c is determined by its physical parameters, such as the barrier thickness and the superconducting energy gap. However, by configuring two junctions in parallel to form a superconducting loop, known as a Superconducting QUantum Interference Device (SQUID), the tunneling currents through the two paths undergo quantum interference. By applying an external magnetic flux Φ_{ext} through the loop, this interference can be modulated, providing a practical mechanism to tune the system’s effective critical current.

As shown in Fig. 2.9, the current flowing through the upper and lower branches is

$$I_{\text{up}} = I_{c, \text{up}} \sin(\varphi_{\text{up}}), \quad I_{\text{down}} = I_{c, \text{down}} \sin(\varphi_{\text{down}}). \quad (2.54)$$

In the presence of magnetic fields, the cooper pair phase difference across the JJ is not only defined by the node flux/voltage difference, but also has an additional term from the magnetic vector potential $\vec{A}$ [126]. Therefore, we reserved the label ϕ for the phase purely from the node flux, and use the label φ to denote the Cooper pair phase across the junction here. The relation between them is

$$\varphi_\mu = \phi_\mu + \frac{2e}{\hbar} \int_\mu \vec{A} \cdot d\vec{l} \quad (2.55)$$

, where $\mu = \{\text{up}, \text{down}\}$, and the integral is taken from one edge of the junction to the other edge. The phase due to the node flux should be the same for the parallel connected “up” and “down” branches $\phi_{\text{up}} = \phi_{\text{down}} = \phi$. For the phase due to the magnetic field, by making a loop integral of the SQUID loop, we obtained

$$\oint_{\text{SQUID}} \vec{A} \cdot d\vec{l} = \int_{\text{down}} \vec{A} \cdot d\vec{l} - \int_{\text{up}} \vec{A} \cdot d\vec{l} = \iint_{\text{SQUID}} \vec{B} \cdot d\vec{S} = \Phi_{\text{ext}}, \quad (2.56)$$

where the loop integral around the SQUID loop is in a counterclockwise direction, so the integral along the "upper" branch carries a negative sign. This derivation implicitly assumes the Meissner effect, such that the magnetic vector potential is negligible within the bulk of the superconductor[127]. Under this assumption, the contribution to the loop integral is localized almost entirely at the Josephson junctions. As a result, we can obtain the relation between the two Cooper pair phases as

$$\varphi_{\text{down}} = \varphi_{\text{up}} + \frac{2\pi}{\Phi_0} \Phi_{\text{ext}} = \varphi_{\text{up}} + \phi_{\text{ext}}. \quad (2.57)$$

With this relation, we can calculate the total current flowing through the SQUID as

$$I = I_{\text{up}} + I_{\text{down}} = I_{\text{c,eff}} \sin(\phi_{\text{eff}}), \quad (2.58)$$

with

$$I_{\text{c,eff}} = 2I_{\Sigma} \sqrt{\cos^2\left(\frac{\phi_{\text{ext}}}{2}\right) + d^2 \sin^2\left(\frac{\phi_{\text{ext}}}{2}\right)} \quad (2.59)$$

$$\phi_{\text{eff}} = \phi + \frac{e}{\hbar} \int_{\text{up}} \vec{A} \cdot d\vec{l} + \frac{e}{\hbar} \int_{\text{down}} \vec{A} \cdot d\vec{l} - \arctan\left(d \tan\left(\frac{\phi_{\text{ext}}}{2}\right)\right)$$

where $I_{\Sigma} = \frac{I_{\text{up}} + I_{\text{down}}}{2}$ and $d = \frac{I_{\text{up}} - I_{\text{down}}}{I_{\text{up}} + I_{\text{down}}}$ characterizes the asymmetry of the SQUID. The expression of ϕ_{eff} seems complicated, but one should notice that $\frac{e}{\hbar} \int_{\text{up}} \vec{A} \cdot d\vec{l} + \frac{e}{\hbar} \int_{\text{down}} \vec{A} \cdot d\vec{l} - \arctan(\tan(\phi_{\text{ext}}/2)d)$ is a constant for a specific magnetic field configuration. Recalling the analogy to classical mechanics, where flux and phase variables are analogous to position, and the inductive energies are analogous to potential, the flux variable should always stay at the minimum location of the inductive energy when idling, and any excitation to the circuit will perturb the flux variable away from the minimum location. The inductive energy is written as

$$U = -E_{j,\text{eff}} \cos(\phi_{\text{eff}}), \quad (2.60)$$

with $E_{j,\text{eff}} = \frac{\Phi_0 I_{\text{c,eff}}}{2\pi}$, which minimize at $\phi_{\text{eff}} = 0$. Therefore, in realistic analysis, we can safely write the current flowing through a SQUID as

$$I = I_{\text{c,eff}} \sin(\phi), \quad (2.61)$$

where we have used the variable ϕ to denote the phase/flux deviation from the minimum location of the inductive energy. The expression above indicates that the SQUID loop is effectively a single JJ with a tunable critical current. By modulating the external magnetic flux threading the SQUID loop, one can tune its effective critical current in situ, as shown in Fig. 2.9 (c).

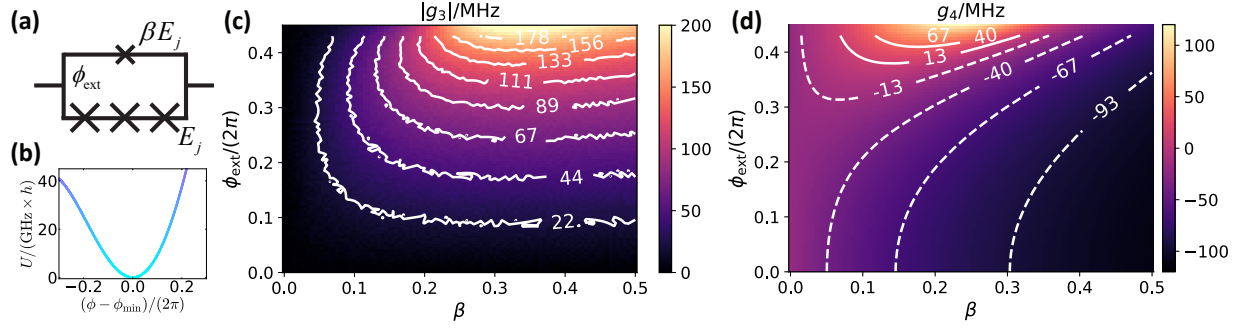


Figure 2.10: SNAIL element. (a) Circuit diagram. (b) An example of SNAIL inductive energy profile. (c) The third-order nonlinear coefficient and (d) the fourth-order nonlinear coefficient with different junction ratio β and external flux Φ_{ext}

Superconducting Nonlinear Asymmetric Inductive Element

The inductive energy profiles for a single JJ and SQUID are all in the form of a cosine function, which is an even function and restricts the available nonlinearities to even orders. However, many important applications, including Josephson parametric amplifier[128], dissipative cat[63] and Kerr cat[54] engineering, would require odd-order nonlinearities.

By connecting three large JJs in parallel with a single small JJ, as shown in Fig. 2.10 (a), one can engineer a nonlinear inductive energy profile with both even and odd orders of nonlinearities, as shown by the non-even function profile in Fig. 2.10 (b). The lower branch of SNAIL has three large JJs with Josephson energy E_J for each, and the upper branch of SNAIL has a single small JJ with Josephson energy βE_J ($0 < \beta < 1$). The inductive energy (potential energy) of the SNAIL is therefore given by

$$U_{\text{SNAIL}}(\phi) = -\beta E_J \cos \phi - 3E_J \cos \left(\frac{\phi - \phi_{\text{ext}}}{3} \right), \quad (2.62)$$

where the flux is equally distributed among the three identical junctions on the lower branch, as each sustains an equivalent voltage drop.

For a fixed external flux, one can Taylor expand this potential energy around its minimum point,

$$U_{\text{SNAIL}}(\phi) \approx U_{\text{SNAIL}}(\phi_{\min}) + \sum_{k=2}^{+\infty} r_k (\phi - \phi_{\min})^k, \quad (2.63)$$

where the term with $k = 2$ corresponds to the linear part of this element with the quadratic dependence on the “position-like” variable ϕ , and terms with $k > 2$ correspond to higher order nonlinearities. The analytical formula for r_k is complicated, and we provided the

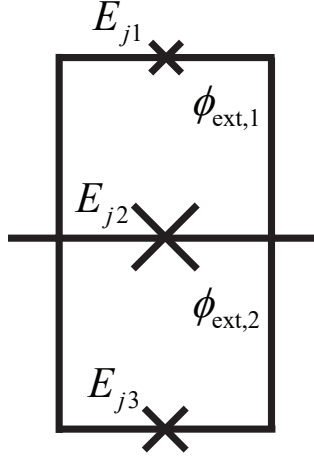


Figure 2.11: Multi-loop structure circuit diagram.

numerical results of r_3 and r_4 with $E_j = 2\pi \times 1\text{GHz}^5$ in Fig. 2.10 (c) and (d). Importantly, the third-order nonlinear coefficient r_3 can be significantly higher than the fourth-order nonlinear coefficient r_4 , which is necessary for engineering Kerr-cat qubits[53, 62].

Multiple Loops

For both SQUIDs and SNAILS, modulating the external flux not only changes the nonlinearities but also changes the linear part of the inductive potential (second-order coefficient r_2), i.e., the effective E_j of the element. However, for many applications of parametric processes, a pure modulation of the nonlinearities without influencing the linear behavior is favorable[55, 63, 129], leading to cleaner multi-wave mixing. Such pure modulation can be achieved with a multi-loop structure shown in Fig. 2.11.

The derivation of the inductive energy of this multi-loop structure is out of the scope of this thesis, so we will directly provide the expression of its inductive energy here,

$$U(\phi) = -E_{j2} \cos(\phi) - 2E_{j,\Sigma} \cos(\phi_{\text{ext},\Sigma}) \cos(\phi + \phi_{\text{ext},\Delta}) + 2E_{j,\Delta} \sin(\phi_{\text{ext},\Sigma}) \sin(\phi + \phi_{\text{ext},\Delta}), \quad (2.64)$$

where $E_{j,\Sigma} = \frac{E_{j1} + E_{j2}}{2}$, $E_{j,\Delta} = \frac{E_{j1} - E_{j2}}{2}$, $\phi_{\text{ext},\Sigma} = \frac{\phi_{\text{ext},1} + \phi_{\text{ext},2}}{2}$, $\phi_{\text{ext},\Delta} = \frac{\phi_{\text{ext},1} - \phi_{\text{ext},2}}{2}$. We can see an un-modulated Josephson energy term associated with E_{j2} , which will be the foundation for separating the external-flux modulation of linear and nonlinear terms.

By choosing different circuit parameters and biasing the circuit with different external flux, one can engineer either an Asymmetrically Threaded SQUID (ATS)[55] or a Symmetrically Threaded SQUID (STS)[129]. For both ATS and STS, people design the multi-loop with

⁵The exact values of E_j do not matter too much here as everything is proportional to E_j .

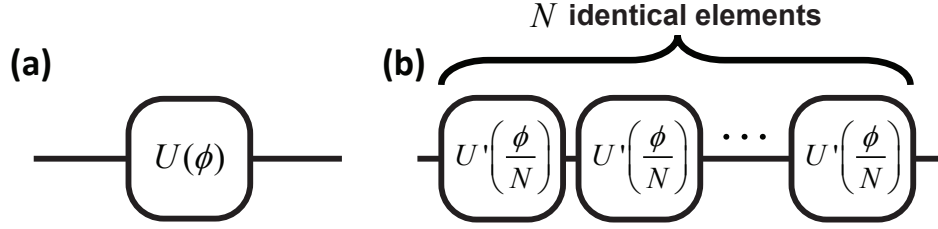


Figure 2.12: Array of nonlinear elements. (a) Circuit diagram of a single element. (b) Circuit diagram of an array of elements to dilute the higher-order nonlinearities.

$$E_{j1} = E_{j3} \rightarrow E_{j,\Delta} = 0 \quad (2.65)$$

and modulate the circuit by the sum of the flux threading the two loops following

$$\phi_\Sigma = \pi/2 + \epsilon_0 \cos(\omega_p t). \quad (2.66)$$

However, they bias the differential flux at different points, with ATS biasing at $\phi_{\text{ext},\Delta} = \pi/2$ and STS biasing at $\phi_{\text{ext},\Delta} = 0$. As a result, ATS is able to engineer a clean parametric four-wave mixing process, leading to efficient two-photon dissipation for high-performance dissipative cat qubits[55, 56, 63], while STS is able to engineer a clean parametric three-wave mixing process, leading to efficient two-photon driving for high-performance Kerr-cat qubits and parametric amplification[129].

Nonlinear Element Arrays

The nonlinearities of superconducting circuit elements can also be adjusted by connecting them in series, forming an array of nonlinear elements, as shown in Fig. 2.12. The inductive energy of the single element in Fig. 2.12 (a) can be represented as

$$U(\phi) \approx r_2 \phi^2 + r_3 \phi^3 + r_4 \phi^4 + \dots, \quad (2.67)$$

and the inductive energy of the element array in Fig. 2.12 is given by

$$NU'(\frac{\phi}{N}) \approx r'_2 \frac{\phi^2}{N} + r'_3 \frac{\phi^3}{N^2} + r'_4 \frac{\phi^4}{N^3} + \dots, \quad (2.68)$$

Typically, people do not want to modify the quadratic term because that term determines the base oscillation frequency of resonators formed with those nonlinear elements. Therefore, one can increase the effective E_j of all elements in Fig. 2.12 (b) by a factor of N^6 , and make $r'_i = Nr_i, i = \{2, 3, 4, \dots\}$. As a result, the inductive energy of the N nonlinear element array is

⁶Usually, the inductive energy of any JJ-based nonlinear elements is proportional to its effective E_j

$$NU'(\frac{\phi}{N}) \approx r_2\phi^2 + r_3\frac{\phi^3}{N} + r_4\frac{\phi^4}{N^2} + \dots, \quad (2.69)$$

where the quadratic term is the same as a single nonlinear element. Notably, the nonlinear terms are “diluted” by the number of nonlinear elements N in the array. With the increasing nonlinear orders, these dilution effects are exponentially enhanced.

Therefore, replacing a single nonlinear element with an array of identical nonlinear elements can dilute the higher-order nonlinearities, providing an easy way to adjust the nonlinearities of JJ-based elements. One popular application is to use an array of a large number of JJs to form a so-called super-inductor, which exhibits a large linear inductance with negligible nonlinearities. This structure is widely applied in fluxonium[44], STS[129] and ATS[55].

2.3 Quantum Circuits and Hamiltonian Analysis

The condensate of Cooper pairs allowed us to connect circuit variables to Cooper pair wavefunctions, facilitating the quantization of electrical circuits. In this section, we will provide the canonical quantization of electrical circuits, introduce several useful transformations, and derive some widely used effective Hamiltonians for superconducting qubits.

Hamiltonian of Electrical Circuits

While Hamiltonian analysis and Lagrangian analysis are mathematically equivalent, Hamiltonian is usually more convenient for quantum mechanical applications, as it facilitates the direct transition to the Schrödinger equation.

The Hamiltonian is the Legendre transformation[130] of the Lagrangian. Suppose we have a Lagrangian given by

$$\mathcal{L} = \mathcal{L} \left(\{ \dot{\Phi}_i, i = 1, 2, \dots \}, \{ \Phi_i, i = 1, 2, \dots \} \right), \quad (2.70)$$

we can define the conjugate variables of Φ_i as $Q_i = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i}$, which is usually named “canonical charge” in reminiscence of the “canonical momentum” in mechanics. It should be noticed that even though Q_i has the unit of charge, it may not necessarily be equal to any charges on the electrodes in the circuits.

Using the canonical charge and Legendre transformation, we can obtain the Hamiltonian by

$$H = \sum_i Q_i \dot{\Phi}_i - \mathcal{L}, \quad (2.71)$$

and finally, we replace any remaining variables $\{\dot{\Phi}_i\}$ with $\{Q_i\}$ to obtain the final expression of Hamiltonian

$$H = H(\{Q_i, i = 1, 2, \dots\}, \{\Phi_i, i = 1, 2, \dots\}). \quad (2.72)$$

Circuits with Multiple Modes

In previous sections, we derived that the Lagrangian of a circuit with multiple modes is

$$\mathcal{L} = \frac{1}{2} \dot{\Phi}^T \mathbf{C} \dot{\Phi} - U(\Phi), \quad (2.73)$$

where we wrote the inductive energy as $U(\Phi)$ rather than $\frac{1}{2} \Phi^T \mathbf{L}^{-1} \Phi$ to include the possible nonlinearities. The canonical charge $\mathbf{Q}$ is given by

$$\mathbf{Q} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \frac{1}{2} (\mathbf{C} + \mathbf{C}^T) \dot{\Phi} = \mathbf{C} \dot{\Phi}. \quad (2.74)$$

Then, the Hamiltonian is given by

$$H = \mathbf{Q}^T \dot{\Phi} - \mathcal{L} = \frac{1}{2} \mathbf{Q}^T \mathbf{C}^{-1} \mathbf{Q} + U(\Phi). \quad (2.75)$$

For the situations with capacitance matrix $\mathbf{C}$ being singular (like the case for floating resonators), we should replace the normal inverse $\mathbf{C}^{-1}$ with the Moore-Penrose pseudo-inverse $\mathbf{C}^+$ [131].

Single Resonator

For a single linear resonator, the Hamiltonian is given by

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}, \quad (2.76)$$

which clearly mimics the Hamiltonian for a simple harmonic oscillator. When we replace the inductor with a JJ, one obtained the nonlinear Hamiltonian

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L_j} - \frac{\Phi^2}{24L_j} \left(\frac{2\pi\Phi}{\Phi_0} \right)^2 + \dots, \quad (2.77)$$

where we have omitted the terms that are not related to Q or Φ without losing generality. The fourth-order term characterizes the anharmonicity of the nonlinear resonator, which is critical for engineering superconducting qubits.

For a general nonlinear element with inductive energy $U(\Phi) = \sum_k r_k \phi^k$, the Hamiltonian is given by

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L_{\text{eff}}} + \sum_{k=3}^{+\infty} r_k \left(\frac{2\pi\Phi}{\Phi_0} \right)^k, \quad (2.78)$$

with $L_{\text{eff}} = \frac{1}{2r_2} \left(\frac{\Phi_0}{2\pi} \right)^2$.

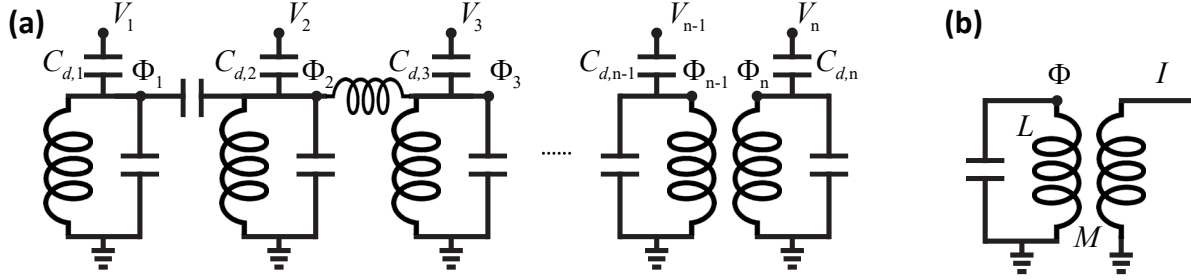


Figure 2.13: Driven resonators through (a) capacitance and (b) mutual inductance.

Coupled Resonators

For multiple uncoupled resonators, the inductance and capacitance matrices should be diagonal. Therefore, the coupling between different resonators manifests in the off-diagonal terms of the capacitance and inductance matrices.

In the expression of Hamiltonian, the diagonal terms in $\mathbf{C}$ and $\mathbf{L}$ lead to quadratic terms, and the off-diagonal terms lead to crossing terms. Therefore, the capacitive coupling and inductive coupling between two modes i and j will appear as

$$H_{c,ij} = \frac{Q_i Q_j}{C_{c,\text{eff}}}, \quad H_{l,ij} = \frac{\Phi_i \Phi_j}{L_{c,\text{eff}}}, \quad (2.79)$$

where the values of effective coupling capacitance $C_{c,\text{eff}}$ and inductance $L_{c,\text{eff}}$ can be derived from the capacitance and inductance matrices.

Driven Resonators

In superconducting qubit systems, direct coupling between resonators and drive lines typically results in excessive coupling strengths, except for some specific applications like parametric amplifiers[32, 119, 120]. Therefore, people usually capacitively or inductively drive the resonators, as shown in Fig. 2.13.

Considering the capacitively driven system in Fig. 2.13 (a), the Lagrangian now is given by

$$\mathcal{L} = \frac{1}{2} \dot{\Phi}^T \mathbf{C} \dot{\Phi} + \frac{1}{2} (\dot{\Phi} - \mathbf{V})^T \mathbf{C}_d (\dot{\Phi} - \mathbf{V}) - \frac{1}{2} \Phi^T \mathbf{L}^{-1} \Phi, \quad (2.80)$$

where $\mathbf{V} = [V_1, V_2, \dots, V_n]^T$ and $\mathbf{C}_d = \text{diag}(C_{d,1}, C_{d,2}, \dots, C_{d,n})$. The canonical charge now is given by

$$\mathbf{Q} = (\mathbf{C} + \mathbf{C}_d) \dot{\Phi} - \mathbf{C}_d \mathbf{V}, \quad (2.81)$$

where we can see that the canonical charge now does not correspond to any real charge on the nodes. Therefore, to construct a physically consistent Hamiltonian, one must derive

the kinetic energy via a formal Legendre transformation of the system's Lagrangian kinetic energy rather than assuming a standard quadratic form in terms of node charges⁷.

Then, using Legendre transformation and replace all $\dot{\Phi}$ with $\mathbf{Q}$, the Hamiltonian of the system is given by

$$H = \frac{1}{2} \mathbf{Q}^T (\mathbf{C} + \mathbf{C}_d)^{-1} \mathbf{Q} + \frac{1}{2} \Phi^T \mathbf{L}^{-1} \Phi + \mathbf{Q}^T (\mathbf{C} + \mathbf{C}_d)^{-1} \mathbf{C}_d \mathbf{V}, \quad (2.82)$$

where we have omitted all terms that is independent of $\mathbf{Q}$ and Φ . The capacitive drives have two effects, including a renormalization of the circuit capacitance matrix and the addition of driven terms in the Hamiltonian that are linearly related to $\mathbf{Q}$. For a single resonator, the matrix equation above is simplified to

$$H = \frac{Q^2}{2(C + C_d)} + \frac{\Phi^2}{2L} + \frac{C_d}{C + C_d} QV, \quad (2.83)$$

which is one of the most basic Hamiltonians for superconducting qubit control.

The analysis of inductively driven circuits is slightly different because the drives through a mutual inductance introduce an extra term in the potential energy, rather than kinetic energy. Potential energy is usually not influenced by the Legendre transformation, and therefore, one can directly add the driven energy into the Hamiltonian. Considering a driven node shown in Fig. 2.13 (b) through a mutual inductance M , the driven energy is given by

$$U_{\text{driven}} = M I_{\text{node}} I = \frac{M}{L} I \Phi, \quad (2.84)$$

where I_{node} is the current flowing from the node to the ground, and L is the node-to-ground inductance. Then, the Hamiltonian of the system is given by

$$H_{\text{total}} = H_{\text{undriven}} + U_{\text{driven}}. \quad (2.85)$$

Capacitive and inductive drives introduce terms in Hamiltonian that are linear to canonical charge $\mathbf{Q}$ or flux Φ , respectively.

Canonical Quantization

For simple harmonic oscillators, the canonical quantization replaces the position and momentum variables with their corresponding operators, and imposes the canonical commutation relation between the two operators, i.e., the commutator of two operators is $i\hbar$.

In electrical circuits, we follow a similar procedure because the flux and canonical charge variable are analogous to the position and momentum variable. By replacing them with the corresponding operators,

⁷In fact, the Hamiltonian is not necessarily the total energy of the system. The fact that it matches the total energy is only a coincidence for quadratic kinetic energy terms.

$$\Phi_i \rightarrow \hat{\Phi}_i, \quad Q_i \rightarrow \hat{Q}_i \quad (2.86)$$

we can impose the canonical commutation relation

$$[\Phi_i, Q_j] = i\hbar\delta_{ij}. \quad (2.87)$$

For a specific resonator mode, the quadratic part (corresponding to the linear resonator part) is written as

$$\hat{H}_{\text{lin}} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}, \quad (2.88)$$

We can then define the creation and annihilation operators as

$$\hat{a} = \frac{1}{2} \left(\frac{\hat{\Phi}}{\Phi_{\text{zpf}}} + i \frac{\hat{Q}}{Q_{\text{zpf}}} \right), \quad \hat{a}^\dagger = \frac{1}{2} \left(\frac{\hat{\Phi}}{\Phi_{\text{zpf}}} - i \frac{\hat{Q}}{Q_{\text{zpf}}} \right), \quad (2.89)$$

where the zero-point fluctuations of flux and canonical charge are given by

$$\Phi_{\text{zpf}} = \sqrt{\frac{\hbar Z}{2}} = \sqrt{\frac{\hbar}{2} \sqrt{\frac{L}{C}}}, \quad Q_{\text{zpf}} = \sqrt{\frac{\hbar}{2Z}} = \sqrt{\frac{\hbar}{2} \sqrt{\frac{C}{L}}}, \quad (2.90)$$

with $Z = \sqrt{L/C}$ as the characteristic impedance of the resonator. Then, the flux and charge operators can now be written as

$$\hat{\Phi} = \Phi_{\text{zpf}}(\hat{a} + \hat{a}^\dagger), \quad \hat{Q} = -iQ_{\text{zpf}}(\hat{a} - \hat{a}^\dagger), \quad (2.91)$$

and the linear part of the Hamiltonian can be written as

$$H_{\text{lin}} = \hbar\omega\hat{a}^\dagger\hat{a}, \quad (2.92)$$

where $\omega = 1/\sqrt{LC}$ is the resonant frequency and the half-photon vacuum energy has been omitted for clarity.

The remaining higher-order terms (nonlinear part) of the Hamiltonian usually have the form of $\sum_k r_k \phi^k$. Therefore, one can express the remaining higher-order terms as

$$H_{\text{nl}} = \sum_k r_k \phi_{\text{zpf}}^k (\hat{a} + \hat{a}^\dagger)^k = \sum_k \hbar g_k (\hat{a} + \hat{a}^\dagger)^k, \quad (2.93)$$

with the zero-point-fluctuation for phase as $\phi_{\text{zpf}} = \frac{2\pi\Phi_{\text{zpf}}}{\Phi_0}$.

Useful Transformations and Approximations

The analysis of Hamiltonians in quantum mechanics usually involves many versatile approximations and transformations. Most of those approximations and transformations aim to reduce the time-dependence of the Hamiltonian and try to (block) diagonalize the Hamiltonian. We will introduce three of them, which will serve as a basis for further analysis.

Rotating Wave Approximation (RWA)

One class of commonly used Hamiltonians in circuit QED is shown below

$$\hat{H}_d/\hbar = \epsilon_d \hat{a}^{\dagger m} \hat{a}^n e^{i\omega' t}, \quad (2.94)$$

where m, n are any non-negative integers. This Hamiltonian can describe the single- and multi-photon drives on a resonator mode. Similarly, the Hamiltonian above can also have the creation and annihilation operators from different resonator modes, which will thereby describe the multi-photon couplings and drives on multiple resonator modes.

When the oscillating frequency ω' in this term is much larger than any relevant energy scale in the Hamiltonian⁸, e.g. $\omega' \gg \epsilon$, this term will quickly average to zero with minimal effect on the secular long-term dynamics. Therefore, when people are only interested in the dynamics at the time scale much longer than $\frac{2\pi}{\omega'}$, this term can be approximately ignored. Since the logic to ignore this term is based on the fast rotating waves, this approximation is usually named “rotation wave approximation (RWA)”.

Unitary Transformation

Unitary transformation is a method to transform the analysis of a Hamiltonian from one frame to another. Considering a Hamiltonian $\hat{H}$ and quantum state $|\psi\rangle$, the Schrödinger equation is given as

$$i\hbar|\dot{\psi}\rangle = \hat{H}|\psi\rangle. \quad (2.95)$$

Considering a unitary operator U ⁹, which transforms the quantum state to a new frame with $|\psi'\rangle = U|\psi\rangle$, the Schrödinger equation in the new frame for $|\psi'\rangle$ is

$$i\hbar|\dot{\psi}'\rangle = \hat{H}'|\psi'\rangle. \quad (2.96)$$

The change of frame with a unitary operator should not change the true physics being observed. Therefore, the above two Schrödinger equations should be equivalent. Therefore, we have

⁸In quantum mechanics, energy is equivalent to frequency up to a scaling factor of $\hbar$. Therefore, in this thesis, we will mix the language of frequency and energy, and the readers should assume the scaling factor when necessary.

⁹For visual clarity, we omit the “ $\hat{}$ ” symbol where the context clearly distinguishes them from classical variables.

$$i\hbar|\dot{\psi}'\rangle = i\hbar\left(\dot{U}|\psi\rangle + U|\dot{\psi}\rangle\right) = \hat{H}'U|\psi\rangle \rightarrow i\hbar|\dot{\psi}\rangle = U^\dagger\hat{H}'U - i\hbar U^\dagger\dot{U} = \hat{H}|\psi\rangle. \quad (2.97)$$

The equation above implies that the Hamiltonian in the new frame should be

$$\hat{H}' = U\hat{H}U^\dagger + i\hbar\dot{U}U^\dagger. \quad (2.98)$$

The unitary transformation is extremely useful in the analysis of driven and coupled systems, which can significantly reduce the time dependence of the Hamiltonian.

Schrieffer-Wolff Transformation

Schrieffer-Wolff (SW) transformation is a useful tool for getting the effective Hamiltonian by removing the off-diagonal terms in the Hamiltonian to the first order. SW transformation can be viewed as a special case of unitary transformation with $U = e^S$, where S is a time-independent anti-Hermitian operator.

Considering a Hamiltonian with its diagonal part as $\hat{H}_0$ and off-diagonal part as $\hat{H}_1$,

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \quad (2.99)$$

the effective Hamiltonian under the SW transformation will be

$$\hat{H}_{\text{eff}} = e^S \hat{H} e^{-S} = \hat{H} + [S, \hat{H}] + \frac{1}{2}[S, [S, \hat{H}]] + \dots, \quad (2.100)$$

where we have expanded the operator exponential using the Baker-Campbell-Hausdorff expansion[132]. By imposing the balanced condition

$$[S, \hat{H}_0] = -\hat{H}_1, \quad (2.101)$$

one can rewrite the effective Hamiltonian as

$$H_{\text{eff}} = \hat{H}_0 + \frac{1}{2}[S, \hat{H}_1] + \frac{1}{3}[S, [S, \hat{H}_1]] + \frac{1}{8}[S, [S, [S, \hat{H}_1]]] + \dots + \frac{n-1}{n!}C_S^{n-1}(\hat{H}_1) \quad (2.102)$$

where $C_S^n(\hat{O}) = \underbrace{[S, [S, \dots [S, \hat{O}] \dots]]}_{n \text{ times}}$. Usually, the scale of off-diagonal terms $\hat{H}_1$ is much

smaller than the scale of diagonal terms $\hat{H}_0$, so the scale of S is usually much smaller than 1. The SW transformation above removed the off-diagonal terms to the first order, and expanded the remaining off-diagonal contributions with higher orders of S . In most applications, people only keep the first two terms, leading to an effective Hamiltonian given by

$$H_{\text{eff}} = \hat{H}_0 + \frac{1}{2}[S, \hat{H}_1]. \quad (2.103)$$

The SW transformation is very useful in deriving the effective Hamiltonian for dispersive readout of superconducting qubits.

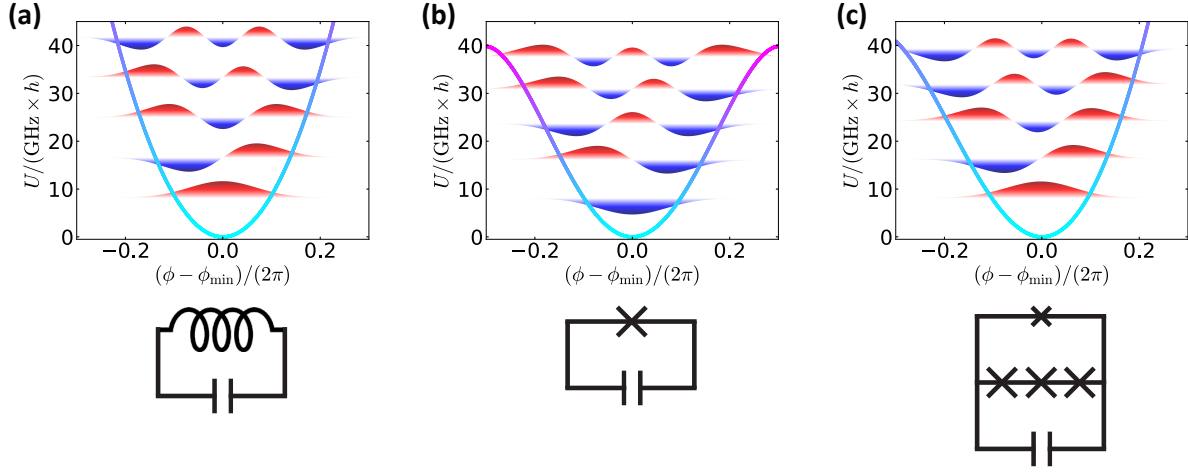


Figure 2.14: Circuit diagram, potential energy, eigenstates, and eigenenergy for (a) linear resonator, (b) transmon, (c) SNAILmon.

Qubit

As mentioned before, the Hamiltonian of a linear resonator is $\hbar\omega\hat{a}^\dagger\hat{a}$. The eigenstates and eigenenergy of this linear resonator are sketched in Fig. 2.14 (a), where the location of each eigenstate indicates its eigen-energy, and the color-filled profile indicates the scaled eigenstate. We usually denote the eigenstate as $|n\rangle$ and the eigenenergy as $E_n = n\hbar\omega$. The eigenenergy of a linear resonator is evenly distributed, preventing the isolation of the lowest two levels for qubit encoding and other multi-photon mixing processes.

Therefore, by replacing the linear inductor with a JJ, one can introduce the nonlinearity given by (to the lowest order)

$$\hat{H}_{\text{nl}} = -\frac{\Phi^2}{24L_j} \left(\frac{2\pi\Phi}{\Phi_0} \right)^2 = r_4 \phi_{\text{zpf}}^4 (\hat{a} + \hat{a}^\dagger)^4, \quad (2.104)$$

where the coefficient

$$r_4 = -\frac{\Phi_0^2}{4\pi^2} \frac{1}{24L_j}, \quad \phi_{\text{zpf}} = \frac{2\pi}{\Phi_0} \sqrt{\frac{\hbar}{2}} \sqrt{\frac{L_j}{C}} \quad (2.105)$$

Usually, people define the capacitive energy unit as $E_c = \frac{e^2}{2C}$. Thus, using E_c and E_j , one can simplify the above nonlinear Hamiltonian to be

$$\hat{H}_{\text{nl}} = -\frac{E_c}{12} (\hat{a} + \hat{a}^\dagger)^4 \quad (2.106)$$

Combining with the linear part, the total Hamiltonian is given by

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} - \frac{E_c}{12}(\hat{a} + \hat{a}^\dagger)^4 \quad (2.107)$$

with $\omega = \sqrt{8E_j E_c}/\hbar$.

To further simplify the nonlinear terms, we can move to the frame rotating at the resonator frequency by a unitary transformation with U and its effect on the creation and annihilation operators shown below

$$U = e^{i\omega t \hat{a}^\dagger \hat{a}}, \quad U\hat{a}U^\dagger = \hat{a}e^{-i\omega t}, \quad U\hat{a}^\dagger U^\dagger = \hat{a}^\dagger e^{i\omega t}. \quad (2.108)$$

Using the unitary transformation, the Hamiltonian in the rotating frame is given by

$$\hat{H}' = -\frac{E_c}{12}(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t})^4 \quad (2.109)$$

Among the terms in the expansion of the above bracket, only the terms with equal powers of $\hat{a}$ and $\hat{a}^\dagger$ will have no time dependence, while the other terms will oscillate with a high frequency on the order of ω . Since in real applications, the oscillator frequency $\omega \approx 5$ GHz, which is much higher than other interaction and drive strengths, and we are also only interested in the dynamics at the time scale larger than several nano-seconds, we can safely apply RWA and ignore all terms with unequal powers of $\hat{a}$ and $\hat{a}^\dagger$. One may also notice that RWA actually ignored the terms that do not conserve the total excitations of the system, which is a more physical implication of RWA. As a result, we can obtain the simplified Hamiltonian under RWA

$$\hat{H}'_{\text{RWA}} = -\frac{E_c}{2}\hat{a}^{\dagger 2}\hat{a}^2 - E_c\hat{a}^\dagger\hat{a}, \quad (2.110)$$

where we have omitted the operator-independent terms. This Hamiltonian under RWA provides a renormalization of the resonator frequency and an anharmonic term (nonlinear term), which leads to unevenly distributed energy levels.

Finally, we can transform this Hamiltonian under RWA back to the original frame and obtain the widely used Hamiltonian for a charge qubit[76].

$$\hat{H}_{\text{RWA}}/\hbar = \omega_q\hat{a}^\dagger\hat{a} + \frac{\alpha}{2}\hat{a}^{\dagger 2}\hat{a}^2, \quad (2.111)$$

where the qubit frequency is $\omega_q = (\sqrt{8E_j E_c} - E_c)/\hbar$ and the anharmonicity is $\alpha = -E_c/\hbar$. When the capacitive energy is much lower than the inductive energy, i.e., $E_j/E_c \approx 50$, the nonlinear resonator will be much more immune to the charge noise on the node, leading to the realization of a charge-insensitive qubit, or transmon[27]. One example of the potential energy profile and eigenstates of a transmon is shown in Fig. 2.14 (b).

Transmon is using a single JJ as the nonlinear element, which only provides even orders of nonlinearity. One can also replace the JJ in transmon with a SNAIL element to form a “SNAILmon”, which can further provide odd orders of nonlinearity, as shown in Fig. 2.14 (c). One may notice that the expansion of third-order nonlinearity terms cannot survive under

RWA because odd combinations of $\hat{a}$ and $\hat{a}^\dagger$ will never form a term with equal powers for each. However, when the SNAILmon is under some off-resonant drives or coupled to other modes, the third-order nonlinearity can provide time-independent terms with significant effects under RWA.

Driven Qubit

The qubit can be driven inductively or capacitively. Due to the large charge matrix element and the simplicity in design and fabrication, people usually choose to capacitively drive the transmons and SNAILmons¹⁰. Moreover, for flux-tunable devices like SQUIDs and SNAILs, one can also modulate the external flux threading the device, and therefore modulate the frequency and nonlinearity coefficient of the qubits, which is known as parametric drives. In superconducting qubits, the effect and treatment of near-resonant drives and off-resonant drives are usually different. Therefore, we will introduce them separately.

Off-resonant Drive

The Hamiltonian of a capacitively or inductively driven qubit is given by

$$\hat{H}/\hbar = \omega \hat{a}^\dagger \hat{a} + 2\epsilon_0 \cos(\omega_d t) f(\hat{a}, \hat{a}^\dagger) + g_3(\hat{a} + \hat{a}^\dagger)^3 + g_4(\hat{a} + \hat{a}^\dagger)^4, \quad (2.112)$$

where we have assumed that the drive follows a cosine function with frequency ω_d , $f(\hat{a}, \hat{a}^\dagger) = \hat{a} + \hat{a}^\dagger$ for inductive drives and $f(\hat{a}, \hat{a}^\dagger) = -i(\hat{a} - \hat{a}^\dagger)$ for capacitive drives. To simplify this Hamiltonian, we can apply a unitary transformation to move to a displaced frame with

$$U = e^{\xi \hat{a}^\dagger - \xi^* \hat{a}}, \quad U \hat{a} U^\dagger = \hat{a} + \xi, \quad U \hat{a}^\dagger U^\dagger = \hat{a}^\dagger + \xi^*. \quad (2.113)$$

and the resulting Hamiltonian will be

$$\begin{aligned} \hat{H}'/\hbar = & \omega \hat{a}^\dagger \hat{a} + \left[\omega \xi + 2\epsilon_0 \cos(\omega_d t) w - i \dot{\xi} \right] \hat{a}^\dagger + \left[\omega \xi^* + 2\epsilon_0 \cos(\omega_d t) w^* + i \dot{\xi}^* \right] \hat{a} \\ & + g_3(\hat{a} + \xi + \hat{a}^\dagger + \xi^*)^3 \\ & + g_4(\hat{a} + \xi + \hat{a}^\dagger + \xi^*)^4. \end{aligned} \quad (2.114)$$

where $w = 1$ for inductive drives and $w = i$ for capacitive drives. We would like to require the linear drive terms to be zero,

$$\omega \xi + 2\epsilon_0 \cos(\omega_d t) w - i \dot{\xi} = 0, \quad (2.115)$$

which leads to the solutions given by

¹⁰Inductive coupling and driving may be more popular in the context of fluxonium qubit, which has a large phase matrix element and small charge matrix element.

$$\xi(t) = -w\epsilon_0 \left(\frac{e^{-i\omega_d t}}{\omega - \omega_d} + \frac{e^{i\omega_d t}}{\omega + \omega_d} \right). \quad (2.116)$$

Noticing that ξ will diverge for near resonant drive with $\omega_d \approx \omega$, indicating that the simplification above becomes invalid when the drive is near resonance. In fact, the above treatment of the driven Hamiltonian is usually only used for off-resonant drives. Plugging the above formula into the Hamiltonian, we obtain

$$\hat{H}'/\hbar = \omega \hat{a}^\dagger \hat{a} + g_3(\hat{a} + \xi_0 e^{-i\omega_d t} + \hat{a}^\dagger + \xi_0^* e^{i\omega_d t})^3 + g_4(\hat{a} + \xi_0 e^{-i\omega_d t} + \hat{a}^\dagger + \xi_0^* e^{i\omega_d t})^4. \quad (2.117)$$

with

$$\xi_0 = -\epsilon_0 w \left(\frac{1}{\omega - \omega_d} \pm \frac{1}{\omega + \omega_d} \right). \quad (2.118)$$

where the plus sign is for inductive drives and the minus sign is for capacitive drives. One may notice that for low-frequency drives with $\omega_d \ll \omega$, ξ_0 approaches zero for capacitive drives, while remaining non-zero for the inductive drives. Therefore, for low-frequency drives, the inductive drives can be much more efficient than capacitive drives.

Up to now, we have finished the first step of Hamiltonian simplification, which removed the drive terms to the first order. Next, we will move to the rotating frame with frequency, which will enable us to identify the fast oscillating terms and apply RWA. The rotating frequency of the rotating frame needs to be chosen carefully, depending on the driven process of interest. If the driven process being considered is a multi-wave mixing process involving n photons from the drive and m photons from the qubit, then the frequency match condition should be $n\omega_d = m\omega$, and the rotating frame frequency should be chosen as $\omega_r = n\omega_d/m$.

For example, if we are interested in the three-wave mixing process involving 1 photon from the drive and 2 photons from the qubit, we will have the frequency match condition given by $\omega_d = 2\omega$, and we should move to the frame rotating at $\omega_r = \omega_d/2$. The Hamiltonian in this rotating frame is

$$\frac{\hat{H}''}{\hbar} = \Delta \hat{a}^\dagger \hat{a} + g_3(\hat{a} e^{-i\frac{\omega_d t}{2}} + \xi_0 e^{-i\omega_d t} + \hat{a}^\dagger e^{i\frac{\omega_d t}{2}} + \xi_0^* e^{i\omega_d t})^3 + g_4(\hat{a} e^{-i\frac{\omega_d t}{2}} + \xi_0 e^{-i\omega_d t} + \hat{a}^\dagger e^{i\frac{\omega_d t}{2}} + \xi_0^* e^{i\omega_d t})^4, \quad (2.119)$$

where $\Delta = \omega - \frac{\omega_d}{2}$ is the detuning between the qubit and the rotating frequency of the frame. By expanding the higher-order nonlinear terms, we can identify the time-independent terms that survive under RWA. In third-order terms, the desired three-wave mixing process leads to two-photon drive terms

$$\hat{H}_{\text{drive}}/\hbar = \epsilon_2 \hat{a}^{\dagger 2} + \epsilon_2^* \hat{a}^2 \quad (2.120)$$

with $\epsilon_2 = 3g_3\xi_0$. In fourth-order terms, beyond the anharmonicity term and frequency renormalization terms that were already analyzed for the undriven qubit, we obtained an extra term associated with the AC-Stark shift

$$\hat{H}_{\text{Stark}}/\hbar = 24g_4 |\xi_0|^2 \hat{a}^\dagger \hat{a}, \quad (2.121)$$

which shifts the qubit frequency depending on the strength of the drive.

Combining all effects together, one obtains the final effective Hamiltonian in the rotating frame as

$$\hat{H}''/\hbar = (\Delta + 12g_4)\hat{a}^\dagger \hat{a} + 6g_4 \hat{a}^{\dagger 2} \hat{a} + \epsilon_2 \hat{a}^{\dagger 2} + \epsilon_2^* \hat{a}^2 + 24g_4 |\xi_0|^2 \hat{a}^\dagger \hat{a}. \quad (2.122)$$

This Hamiltonian is actually the core for engineering Kerr-cat qubits. By redefining the detuning as $\Delta + 12g_4 \rightarrow \Delta$, the Kerr nonlinearity as $6g_4 = K$, and ignoring the AC-Stark shift term, we end up with the Kerr-cat Hamiltonian commonly quoted in the literature.

$$\hat{H}_{\text{KCQ}}/\hbar = \Delta \hat{a}^\dagger \hat{a} + K \hat{a}^{\dagger 2} \hat{a}^2 + \epsilon_2 \hat{a}^{\dagger 2} + \epsilon_2^* \hat{a}^2 \quad (2.123)$$

Typically, for capacitive or inductive drives, a k -wave mixing process can only be activated by a k th-order nonlinearity, since the interaction requires the absorption or emission of k photons supplied directly by the drive. In contrast, for parametric drives that modulate the coupling coefficient g_k itself, a k -wave mixing process can be enabled by a $(k-1)$ th-order nonlinearity. This is because the parametric modulation of g_k effectively supplies one additional photon to the interaction.

A well-known example is the flux-pumped Josephson parametric amplifier based on two-photon driving. In this case, a three-wave mixing process is activated using quadratic terms in the Hamiltonian¹¹ by parametrically modulating the SQUID inductance with an external flux drive[119, 120].

Near-resonant Drive

For near resonant drives, the treatment above becomes invalid as the denominator for ξ_0 approaches 0. Considering the driven Hamiltonian with $\omega_d \approx \omega$ in Eq. 2.112, moving directly to the rotating frame with frequency ω_d leads to

$$\hat{H}'_{\text{near-res}}/\hbar = \Delta \hat{a}^\dagger \hat{a} + 2\epsilon_0 \cos(\omega_d t) f(\hat{a} e^{-i\omega_d t}, \hat{a}^\dagger e^{i\omega_d t}) + g_3 (\hat{a} e^{-i\omega_d t} + \hat{a}^\dagger e^{i\omega_d t})^3 + g_4 (\hat{a} e^{-i\omega_d t} + \hat{a}^\dagger e^{i\omega_d t})^4 \quad (2.124)$$

By expanding the cosine function into co-rotating and counter-rotating terms with coefficient $e^{i\omega_d t}$ and $e^{-i\omega_d t}$, one can apply RWA to simplify the driven terms as

$$\hat{H}'_{\text{drive}}/\hbar = \epsilon_0 f(\hat{a}, \hat{a}^\dagger). \quad (2.125)$$

¹¹Recalling that the quadratic term is in the form of $E_j(\hat{a} + \hat{a}^\dagger)^2$ and the parametric drive can modulate E_j .

If we allow the drives to have a phase, i.e. $\epsilon_0 = |\epsilon_0| e^{-i\gamma}$, then the driven terms will become

$$\begin{aligned}\hat{H}'_{\text{drive}}/\hbar &= \epsilon_0^* \hat{a} + \epsilon_0 \hat{a}^\dagger &\rightarrow &\text{inductive drive} \\ \hat{H}'_{\text{drive}}/\hbar &= -i(\epsilon_0^* \hat{a} - \epsilon_0 \hat{a}^\dagger) &\rightarrow &\text{capacitive drive}\end{aligned}\tag{2.126}$$

It is noticed that for near-resonant drives, the inductive and capacitive drives have equivalent effects up to a 90-degree phase shift. The remaining part of the Hamiltonian is similar to the undriven qubit discussed in the previous section. By applying RWA to the remaining part of the Hamiltonian and combining it with the driven Hamiltonian, we obtained the effective Hamiltonian for near-resonant drives

$$\hat{H}''_{\text{near-res}}/\hbar = \Delta \hat{a}^\dagger \hat{a} + \frac{\alpha}{2} \hat{a}^{\dagger 2} \hat{a}^2 + \epsilon_0^* \hat{a} + \epsilon_0 \hat{a}^\dagger,\tag{2.127}$$

where we have redefined the detuning to absorb the frequency renormalization from fourth-order nonlinearity.

This driven Hamiltonian includes all the basic control operations for a single transmon qubit. The transmon qubit uses the lowest two levels $|0\rangle$ and $|1\rangle$ to encode the qubit, and relies on the anharmonicity to isolate the qubit encoding space. For ideal cases, we can ignore all higher levels and use the Pauli matrices to analyze the dynamics in just this two-level system (TLS). In the subspace of these two levels, we can map the creation and annihilation operators to Pauli operators as

$$\hat{a}^\dagger = \frac{1}{2}(\hat{\sigma}_x - i\hat{\sigma}_y), \quad \hat{a} = \frac{1}{2}(\hat{\sigma}_x + i\hat{\sigma}_y)\tag{2.128}$$

and map the Hamiltonian to

$$\hat{H}''_{\text{TLS}}/\hbar = -\frac{\Delta}{2}\hat{\sigma}_z + \text{Re}[\epsilon_0]\hat{\sigma}_x + \text{Im}[\epsilon_0]\hat{\sigma}_y.\tag{2.129}$$

Therefore, this Hamiltonian enables coherent rotations of the qubit state about all three Bloch-sphere axes. A finite detuning between the qubit and the drive produces an effective Z rotation, while a resonant drive with nonzero amplitude generates rotations about the X or Y axis, depending on the phase of the drive.

Importantly, a Z rotation does not require an actual microwave pulse. The effect of detuning is simply to accumulate a relative phase between the qubit and the reference frame of the drive. Therefore, in practice, a Z rotation is typically implemented by updating the phase of subsequent control pulses. This technique is known as a “virtual Z gate”, since it does not involve any additional physical interaction with the qubit and ideally incurs zero duration and no additional error.

Coupling

For two resonator modes with their creation and annihilation operators given by $\{\hat{a}_1, \hat{a}_1^\dagger\}$ and $\{\hat{a}_2, \hat{a}_2^\dagger\}$, the coupled Hamiltonian is usually given by

$$\hat{H}/\hbar = \hat{H}_1/\hbar + \hat{H}_2/\hbar + g(\hat{a}_1 \pm \hat{a}_1^\dagger)(\hat{a}_2 \pm \hat{a}_2^\dagger), \quad (2.130)$$

where H_1 and H_2 are the uncoupled Hamiltonians of the two modes, g is the coupling strength, and the plus sign corresponds to inductive coupling while the minus sign corresponds to capacitive coupling. Usually, the detuning between two coupled modes and the coupling strength are much smaller than the mode frequencies, $\Delta, g \ll \omega_1, \omega_2$. Therefore, similar to the analysis of the transmon qubit Hamiltonian in section 2.3, we can apply RWA to the coupling terms to get the so-called Jaynes-Cummings (JC) Hamiltonian

$$\hat{H}_{\text{JC}}/\hbar = \hat{H}_1/\hbar + \hat{H}_2/\hbar + g(\hat{a}_1\hat{a}_2^\dagger + \hat{a}_1^\dagger\hat{a}_2), \quad (2.131)$$

which only keeps the excitation-conserved terms.

We will discuss several main applications of JC coupling in the superconducting qubit community, including excitation swapping, dispersive coupling, and multi-wave mixing.

Excitation Swapping

JC Hamiltonian describes the excitation swapping between the two resonators. Therefore, when the two resonators are on resonance, such coupling will be time-independent in the rotating frame with each resonator, and a non-zero coupling strength g will induce coherent excitation swapping. This effect has been used in engineering the ISWAP gate between two qubits[133].

Suppose we have two qubits with frequency ω_1 and ω_2 , and anharmonicity α_1 and α_2 , the coupled Hamiltonian is given by

$$\hat{H} = \omega_1\hat{a}_1^\dagger\hat{a}_1 + \omega_2\hat{a}_2^\dagger\hat{a}_2 + \frac{\alpha_1}{2}\hat{a}_1^{\dagger 2}\hat{a}_1^2 + \frac{\alpha_2}{2}\hat{a}_2^{\dagger 2}\hat{a}_2^2 + g(\hat{a}_1\hat{a}_2^\dagger + \hat{a}_1^\dagger\hat{a}_2), \quad (2.132)$$

By moving to the rotation frame at the frequencies of two qubits with $U = e^{i\omega_1 t \hat{a}_1^\dagger \hat{a}_1} e^{i\omega_2 t \hat{a}_2^\dagger \hat{a}_2}$, we obtained

$$\hat{H}' = \frac{\alpha_1}{2}\hat{a}_1^{\dagger 2}\hat{a}_1^2 + \frac{\alpha_2}{2}\hat{a}_2^{\dagger 2}\hat{a}_2^2 + g(\hat{a}_1\hat{a}_2^\dagger e^{i\Delta t} + \hat{a}_1^\dagger\hat{a}_2 e^{-i\Delta t}), \quad (2.133)$$

with $\Delta = \omega_2 - \omega_1$ being the detuning between the two qubits. When the two qubits are resonant with $\Delta = 0$, the above Hamiltonian will be time-independent. For the ideal case where we treat the qubits as a TLS using Pauli operators, the above Hamiltonian will reduce to

$$\hat{H}'_{\text{TLS}} = \frac{g}{4}(\hat{\sigma}_{1,x}\hat{\sigma}_{2,x} + \hat{\sigma}_{1,y}\hat{\sigma}_{2,y}) \quad (2.134)$$

which is exactly the generator for the ISWAP gate[97].

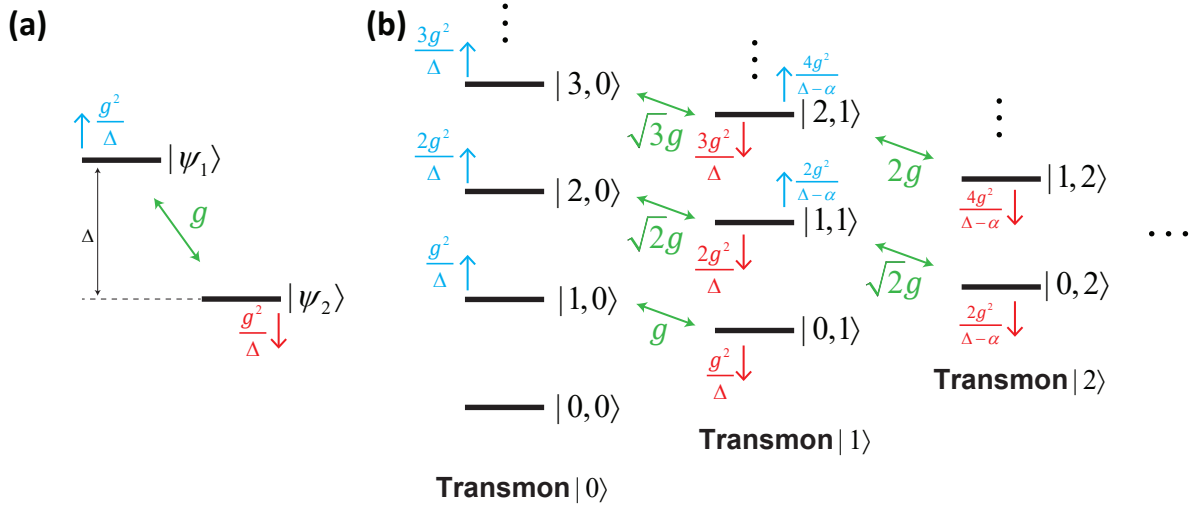


Figure 2.15: Dispersive coupling between (a) two states, and (b) a transmon and a resonator. The black solid lines indicate the bare energy levels, and the red and blue arrows indicate the dispersive shift due to the coupling terms given by the green arrows.

Dispersive Coupling

In qubit community, dispersive couplings usually refer to the ZZ type couplings, which perturb the frequencies, level energies and phases of the coupled system, and their Hamiltonian is usually given by $\hat{\sigma}_{1,z}\hat{\sigma}_{2,z}$ or $\hat{a}_1^\dagger\hat{a}_1\hat{a}_2^\dagger\hat{a}_2$. Considering the toy model shown in Fig. 2.15 (a). When two energy levels separated by an energy gap Δ are coupled with strength g , we can write down their Hamiltonian as

$$\hat{H} = \Delta|\psi_1\rangle\langle\psi_1| + g|\psi_2\rangle\langle\psi_1| + g|\psi_1\rangle\langle\psi_2| \quad (2.135)$$

When $g \ll \Delta$, i.e, the coupling strength is much smaller than the energy gap, we can treat the coupling term $\hat{H}_1 = g|\psi_2\rangle\langle\psi_1| + g|\psi_1\rangle\langle\psi_2|$ as a perturbation, and use the perturbation theory[2] to extract the energy of the two levels influenced by the coupling term.

Using the perturbation theory up to the second order, we obtained

$$\begin{aligned} E_{|\psi_1\rangle} &= \Delta + \langle\psi_1|\hat{H}_1|\psi_1\rangle + \frac{\langle\psi_1|\hat{H}_1|\psi_2\rangle\langle\psi_2|\hat{H}_1|\psi_1\rangle}{\Delta} = \Delta + \frac{g^2}{\Delta} \\ E_{|\psi_2\rangle} &= \Delta + \langle\psi_2|\hat{H}_1|\psi_2\rangle + \frac{\langle\psi_2|\hat{H}_1|\psi_1\rangle\langle\psi_1|\hat{H}_1|\psi_2\rangle}{-\Delta} = \Delta - \frac{g^2}{\Delta} \end{aligned} \quad (2.136)$$

As shown in Fig. 2.15 (a), the effective energy of $|\psi_1\rangle$ and $|\psi_2\rangle$ is both shifted by g/Δ^2 away from each other. Therefore, when $g \ll \Delta$, the interaction reduces to an effective longitudinal coupling between the two levels.

Now we can switch from the toy model to a realistic model for a linear resonator coupled to a qubit (transmon) with anharmonicity α . The bare energy levels of this system are shown in Fig. 2.15 (b). The labels of the energy levels are given by $|m, n\rangle$ where m is the excitation number of the resonator, and n is the excitation number of the qubit. Therefore, the bare energy of $|m, n\rangle$ is given by $E_{mn} = m\omega_r + n\omega_q + \frac{n(n-1)}{2}\alpha$, where ω_r is the resonator frequency and ω_q is the qubit frequency.

The Jaynes-Cummings interaction will couple the states with the same total excitation with the Hamiltonian

$$H_{\text{int}} = g(\hat{a}_r \hat{a}_q^\dagger + \hat{a}_r^\dagger \hat{a}_q), \quad (2.137)$$

where $\hat{a}_{q/r}^\dagger$ and $\hat{a}_{q/r}$ are the creation and annihilation operators of the qubit/resonator. The effect of the creation and annihilation operators will be enhanced by a factor of $\sqrt{n}$ with the excitation n . As a result, the effective coupling between each coupled level is given in Fig. 2.15 (b). The calculated energy shift of all relevant levels is also shown in Fig. 2.15, with $\Delta = \omega_r - \omega_q$ being the detuning between the resonator and the qubit.

For the column with the transmon state being in $|0\rangle$, the transition frequency of the resonator is

$$\omega_r^{(0)} = \omega_r + \frac{g^2}{\Delta} \quad (2.138)$$

For the column with the transmon state being in $|1\rangle$, the transition frequency of the resonator is

$$\omega_r^{(1)} = \omega_r + \frac{2g^2}{\Delta - \alpha} - \frac{g^2}{\Delta} \quad (2.139)$$

Therefore, the frequency of the linear resonator is shifted by χ when the transmon is excited to the $|1\rangle$ state, and χ is given by

$$\chi = \omega_r^{(1)} - \omega_r^{(0)} = 2g^2 \left(\frac{1}{\Delta - \alpha} - \frac{1}{\Delta} \right). \quad (2.140)$$

The effective Hamiltonian of this coupled system can thus be written as

$$\hat{H}_{\text{dispersive}}/\hbar = \left(\omega_r + \frac{g^2}{\Delta} \right) \hat{a}_r^\dagger \hat{a}_r + \left(\omega_q - \frac{g^2}{\Delta} \right) \hat{a}_q^\dagger \hat{a}_q + \frac{\alpha}{2} \hat{a}_q^{\dagger 2} \hat{a}_q^2 + \chi \hat{a}_q^\dagger \hat{a}_q \hat{a}_r^\dagger \hat{a}_r \quad (2.141)$$

An important application of this dispersive coupling is the dispersive readout. Since the frequency of the linear resonator is different when the qubit is in $|0\rangle$ or $|1\rangle$ state, one can read out the qubit state by monitoring the frequency of the linear resonator[78], which avoids measuring the qubit directly and leads to high QNDness.

When we replace the linear resonator with another qubit (as shown in Eq. 2.132, a similar Hamiltonian will result in a dispersive coupling between the two qubits given by

$$\hat{H}_{\text{dispersive}}/\hbar = \left(\omega_1 + \frac{g^2}{\Delta}\right) \hat{a}_1^\dagger \hat{a}_1 + \left(\omega_2 - \frac{g^2}{\Delta}\right) \hat{a}_2^\dagger \hat{a}_2 + \frac{\alpha_1}{2} \hat{a}_1^{\dagger 2} \hat{a}_1^2 + \frac{\alpha_2}{2} \hat{a}_2^{\dagger 2} \hat{a}_2^2 + \zeta \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \quad (2.142)$$

where $\zeta = 2g^2 \left(\frac{1}{\Delta - \alpha_2} + \frac{1}{\Delta + \alpha_1} \right)$ is the ZZ interaction rate between the two qubits, and $\Delta = \omega_1 - \omega_2$ is the detuning. The reason why ζ is called ZZ is because in the TLS approximation, we have

$$\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 = \frac{1}{4} \hat{\sigma}_{1,z} \hat{\sigma}_{2,z} - \frac{1}{2} \hat{\sigma}_{1,z} - \frac{1}{2} \hat{\sigma}_{2,z} \quad (2.143)$$

which involves a ZZ interaction term with some renormalization of the qubit frequency. In practice, this Hamiltonian is usually used to engineer CZ gates between two qubits.

Multi-wave mixing

Another important application of the JC coupling Hamiltonian is the multi-wave mixing process under off-resonant drives. Considering the coupled Hamiltonian between two qubits given by

$$\begin{aligned} \hat{H}/\hbar &= \hat{H}_1/\hbar + \hat{H}_2/\hbar + g(\hat{a}_1 \pm \hat{a}_1^\dagger)(\hat{a}_2 \pm \hat{a}_2^\dagger) \\ \hat{H}_1/\hbar &= \omega_1 \hat{a}_1^\dagger \hat{a}_1 + g_3^{(1)}(\hat{a}_1 + \hat{a}_1^\dagger)^3 + g_4^{(1)}(\hat{a}_1 + \hat{a}_1^\dagger)^4 \\ \hat{H}_2/\hbar &= \omega_2 \hat{a}_2^\dagger \hat{a}_2 + g_3^{(2)}(\hat{a}_2 + \hat{a}_2^\dagger)^3 + g_4^{(2)}(\hat{a}_2 + \hat{a}_2^\dagger)^4 \end{aligned} \quad (2.144)$$

we can apply the SW transformation with

$$S = -\frac{g}{\Delta}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1 \hat{a}_2^\dagger) \quad (2.145)$$

where $\Delta = \omega_1 - \omega_2$ is the detuning. The effect of this SW transformation on the creation and annihilation operators to the first order is given by

$$\begin{aligned} e^S \hat{a}_1 e^{-S} &= \hat{a}_1 + [S, \hat{a}_1] = \hat{a}_1 + \frac{g}{\Delta} \hat{a}_2, & e^S \hat{a}_1^\dagger e^{-S} &= \hat{a}_1^\dagger + [S, \hat{a}_1^\dagger] = \hat{a}_1^\dagger + \frac{g}{\Delta} \hat{a}_2^\dagger \\ e^S \hat{a}_2 e^{-S} &= \hat{a}_2 + [S, \hat{a}_2] = \hat{a}_2 - \frac{g}{\Delta} \hat{a}_1, & e^S \hat{a}_2^\dagger e^{-S} &= \hat{a}_2^\dagger + [S, \hat{a}_2^\dagger] = \hat{a}_2^\dagger - \frac{g}{\Delta} \hat{a}_1^\dagger \end{aligned} \quad (2.146)$$

Plugging this into the original Hamiltonian, we obtained the near Hamiltonian under the SW transformation given by

$$\begin{aligned} \hat{H}_{\text{SW}}/\hbar &= \hat{H}_{1,\text{SW}}/\hbar + \hat{H}_{2,\text{SW}}/\hbar \\ \hat{H}_{1,\text{SW}}/\hbar &= \left(\omega_1 + \frac{g^2}{\Delta}\right) \hat{a}_1^\dagger \hat{a}_1 + g_3^{(1)}(\hat{a}_1 + \frac{g}{\Delta} \hat{a}_2 + \hat{a}_1^\dagger + \frac{g}{\Delta} \hat{a}_2^\dagger)^3 + g_4^{(1)}(\hat{a}_1 + \frac{g}{\Delta} \hat{a}_2 + \hat{a}_1^\dagger + \frac{g}{\Delta} \hat{a}_2^\dagger)^4 \\ \hat{H}_{2,\text{SW}}/\hbar &= \left(\omega_2 - \frac{g^2}{\Delta}\right) \hat{a}_2^\dagger \hat{a}_2 + g_3^{(2)}(\hat{a}_2 - \frac{g}{\Delta} \hat{a}_1 + \hat{a}_2^\dagger - \frac{g}{\Delta} \hat{a}_1^\dagger)^3 + g_4^{(2)}(\hat{a}_2 - \frac{g}{\Delta} \hat{a}_1 + \hat{a}_2^\dagger - \frac{g}{\Delta} \hat{a}_1^\dagger)^4 \end{aligned} \quad (2.147)$$

Then, we can add an off-resonant drive ϵ_0 with frequency ω_d , and follow the similar treatment in section 2.3 to transform the Hamiltonian to the displaced. The k -th order nonlinear part of the resulting Hamiltonian will have the form given by

$$(A\hat{a}_1 + B\hat{a}_2 + C\xi + A^*\hat{a}_1^\dagger + B^*\hat{a}_2^\dagger + C^*\xi^*)^k \quad (2.148)$$

where the coefficients A, B, C depend on which qubit is being driven. One can then engineer the multi-wave mixing process with the frequency matching condition $n\omega_1 \pm \omega_2 \pm v\omega_d = 0$, where one can choose the photon numbers $\{n, m, v\}$ and the plus and minus signs at their will as long as the corresponding process is allowed by the available nonlinearity.

For example, if the system has third-order nonlinearity with $g_3 \neq 0$, one can drive one or both of the qubits at the frequency of $\omega_d = \omega_1 - \omega_2$, and engineer the coherent excitation swapping with the Hamiltonian

$$\xi\hat{a}_1\hat{a}_2^\dagger + \xi^*\hat{a}_1^\dagger\hat{a}_2. \quad (2.149)$$

One can also drive them at the sum frequency $\omega_d = \omega_1 + \omega_2$ to engineer the pair-excitation process with the Hamiltonian

$$\xi\hat{a}_1^\dagger\hat{a}_2^\dagger + \xi^*\hat{a}_1\hat{a}_2. \quad (2.150)$$

Using the fourth-order nonlinearity, one can also engineer the cross-resonance[99] process by driving one qubit at the other qubit's frequency to engineer the following Hamiltonian

$$\xi\hat{a}_1^\dagger\hat{a}_1\hat{a}_2^\dagger + \xi^*\hat{a}_1^\dagger\hat{a}_1\hat{a}_2. \quad (2.151)$$

which corresponds to an ZX interaction, and has been used to engineer the CNOT gates between two qubits.

Dissipation

Hamiltonian and Schrödinger equations usually can only describe the coherent processes that conserve the total energy of the system. However, superconducting qubit systems are open systems, which are always coupled to uncontrolled environmental degrees of freedom and random fluctuations. The dynamics of these couplings usually involve the heating or dissipation of the qubit systems, leading to an incoherent process. The description of such incoherent qubit-environment coupling is, in general, complicated. However, such processes can be approximately described by the Lindblad master equations under the following assumptions[134]:

- The system Hamiltonian $\hat{H}_S$ and environment Hamiltonian $\hat{H}_B$ is separable, and their coupling $\hat{H}_{SB}$ is weak.

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{SB} \quad (2.152)$$

- Born approximation: the hybridization between the system and environment is weak, so that the total density matrix ρ_{tot} can be approximately factorized using the system density matrix ρ_S and environment density matrix ρ_B

$$\rho_{\text{tot}} = \rho_S \otimes \rho_B \quad (2.153)$$

- Markov approximation: the environment has no memory, so that the environment will not be influenced by past interactions. Or equivalently, the environment is assumed to be an infinite bath which is not influenced by the system-environment interaction.
- RWA: the system-environment interaction conserves the total excitation and energy of the system-environment coupled system.

The Lindblad master equation for a system density matrix ρ is given by

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H}, \rho] + \sum_j \mathcal{D}\left(\sqrt{\gamma_j} \hat{L}_j\right) \quad (2.154)$$

where $\hat{H}$ is the Hamiltonian of the system, and $\hat{L}_j$ and γ_j are the jump operator and dissipation rate of each dissipation process. The Lindbladian dissipator is

$$\mathcal{D}(\sqrt{\gamma_j} \hat{L}_j) = \gamma_j \left(\hat{L}_j \rho \hat{L}_j^\dagger - \frac{1}{2} \hat{L}_j^\dagger \hat{L}_j \rho - \frac{1}{2} \rho \hat{L}_j^\dagger \hat{L}_j \right). \quad (2.155)$$

Most of the dissipation processes satisfy the assumptions above and can be described by the Lindblad master equation. The single-photon decay due to the spontaneous and stimulated emission is modeled with

$$\gamma = (n_{\text{th}} + 1) \gamma_{\text{vac}}^\downarrow, \quad \hat{L} = \hat{a} \quad (2.156)$$

where we have omitted the subscription j for clarity. The jump operator of the single-photon decay process is given by the annihilation operator, and the decay rate is enhanced by the thermal photon population n_{th} based on their vacuum decay rate $\gamma_{\text{vac}}^\downarrow$. Even though the natural decay processes are dominantly single-photon decay, one can engineer multi-photon decay processes using multi-wave mixing. The two-photon decay process with the jump operator $\hat{a}^2$, which is engineered by four-wave mixing through ATS, is the core technical process for dissipative cat qubits[55, 56].

Similarly, the heating process due to the thermal photon population can be given by

$$\gamma = n_{\text{th}} \gamma_{\text{vac}}^\uparrow, \quad \hat{L} = \hat{a}^\dagger \quad (2.157)$$

where $\gamma_{\text{vac}}^\uparrow$ is the heating rate per thermal photon. Moreover, some other processes, such as the dispersive couplings between the qubit system and the environment, can lead to fluctuations of the qubit frequency. Such processes, also known as pure dephasing, are intrinsically coherent and do not dissipate or excite energies in the system. However, because

such fluctuations are usually random and not controlled, the average effect of them will reduce the coherence of the system. In the Markovian white-noise approximation limit, the jump operator of this process is usually given by

$$\hat{L} = \hat{a}^\dagger \hat{a}. \quad (2.158)$$

Noticing that due to the intrinsically coherent nature, the pure dephasing noise can usually be mitigated by carefully qubit engineering and control. Some examples include the continuously-driven qubit and dynamical decoupling[28, 135, 136].

Beyond using bosonic creation and annihilation operators to model the dissipation process, one can also approximate the qubit dynamics as a TLS model and use Pauli matrices to describe the noise mechanism. For example, the stochastic Pauli error along the ν axis is modeled by the jump operator

$$\hat{L} = \hat{\sigma}_\nu \quad (2.159)$$

Finally, we would like to offer an intuition for the Lindbladian dissipator as a conclusion. The effect of the jump operator and dissipation rate can be intuitively understood as follows: The system density matrix will be perturbed by the jump operator randomly, with the probability per unit time given by the dissipation rate.

Example: Josephson Parametric Amplifier

In this section, we will provide an important application of the superconducting circuit theory discussed above: Josephson parametric amplifier (JPA). The contents of this section are based on the author's publication[119], which developed a novel broadband Josephson parametric amplifier by enhancing the amplifier-signal coupling using a CPW-taper.

The parametric process facilitates coherent energy transfer among electromagnetic waves by nonlinearities, which enables quantum-limited parametric amplification [137]. Microwave parametric amplification not only improves the readout of solid-state [77, 138–141] and nanomechanical [142] quantum devices, but also becomes increasingly important in emerging technologies such as hybrid quantum systems [143] and axion dark matter detection [144, 145]. It is thus important to develop amplifiers with outstanding performance: low noise, broad bandwidth, and high saturation power. Moreover, for easy adaptation of parametric amplifiers, their design and fabrication should be robust and simple.

In superconducting circuits, parametric amplification is achieved by JPAs utilizing multi-wave mixing processes in nonlinear superconducting resonators, where Josephson junctions provide the required nonlinearity without introducing dissipation [146, 147]. The nonlinear resonator of the JPA is typically coupled to a fixed 50- Ω embedding environment [148–151], which prevents tuning its linewidth and saturation power independently, resulting in a trade-off between saturation power and bandwidth[120]. Hence, its bandwidth is inherently limited to 10-50 MHz, and its input saturation power is typically -120 dBm for a nominal gain of 20 dB.

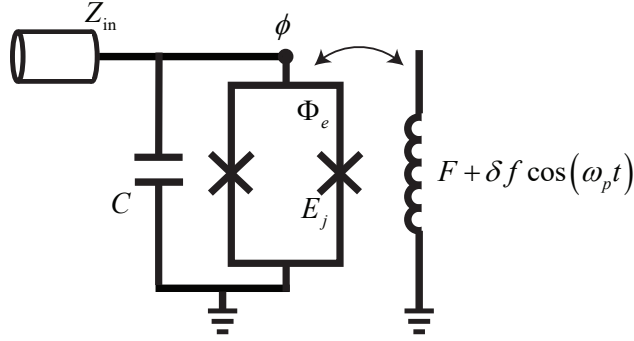


Figure 2.16: Flux-pumped Josephson parametric amplifier

In this section, we present a broadband CPW-based impedance-transformed parametric amplifier (CIMPA) design and showcase its performance. By engineering an impedance transformer based on the Klopfenstein taper[152] with reduced technical requirements, we can implement it using a single horn-like CPW, which achieves the broadband behavior while maintaining the simplicity of the design and fabrication. The horn-like CPW structure allows us to leverage streamlined RF design tools [153] and minimize the complexity of simulating and fabricating the impedance transformer, and the feature size of the horn-like CPW is well within the scope of the optical lithography technique. The amplifier displays an instantaneous bandwidth of 700(200) MHz for 15(20) dB gain, a 1.4 GHz flux-tunable bandwidth, a saturation input power of approximately -110 dBm, and no significant back action on the qubit.

Theory

Considering a symmetric SQUID¹² directly coupled to the external signal through a transmission line, as shown in Fig. 2.16, and the input impedance of the transmission line is given by Z_{in} . Then, the external damping rate of the SQUID resonator is given by $\kappa_{\text{ext}} = \text{Re}[\frac{1}{CZ_{\text{in}}}]$. Notably, JPAs are usually designed to couple strongly with the signal and have a large external damping rate for optimized bandwidth and dynamical range; therefore, the transmission line is directly coupled to the SQUID resonator without any inductance or capacitance. Typically, the κ_{ext} is on the order of hundreds of MHz or several GHz, which is significantly higher than the internal damping rate κ_{int} . Therefore, we can approximate the total damping rate of the SQUID resonator as

$$\kappa = \kappa_{\text{ext}} + \kappa_{\text{int}} \approx \kappa_{\text{ext}} \quad (2.160)$$

On the other side of the SQUID resonator, it is also coupled to a flux pump line, which DC biases and AC modulates the external flux Φ_e threading the SQUID loop, following

¹²A symmetric SQUID has two junctions in each loop with the same E_j

$$\Phi_e = \frac{\Phi_0}{\pi}(F + \delta f \cos(\omega_p t)) \quad (2.161)$$

We can write down the Hamiltonian of the SQUID resonator as

$$\hat{H} = \frac{\hat{Q}^2}{2C} - 2E_J |\cos(F + \delta f \cos(\omega_p t))| \cos(\hat{\phi}), \quad (2.162)$$

Assuming that the AC flux modulation is small, we can expand the cosine term as

$$\cos(F + \delta f \cos(\omega_p t)) = \cos(F) - \sin(F) \delta f \cos(\omega_p t) = \cos(F) - \frac{1}{2} \sin(F) \delta f (e^{i\omega_d t} + e^{-i\omega_d t}), \quad (2.163)$$

and assuming that the phase operator fluctuation is also small, we can further expand the cosine term with the phase as

$$\cos(\hat{\phi}) = 1 - \frac{1}{2} \hat{\phi}^2. \quad (2.164)$$

Plugging all of these expansions into the Hamiltonian, we obtained

$$\hat{H} = \frac{\hat{Q}^2}{2C} + E_J \cos(F) \hat{\phi}^2 + \frac{1}{2} \sin(F) \delta f (e^{i\omega_d t} + e^{-i\omega_d t}) \hat{\phi}^2 + \hat{O}(\hat{\phi}^4) \quad (2.165)$$

Translating the charge and phase operators $\hat{Q}$ and $\hat{\phi}$ into creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, we obtained the Hamiltonian

$$\hat{H}/\hbar = \omega_q \hat{a}^\dagger \hat{a} + \frac{1}{2} \sin(F) \delta f (e^{i\omega_d t} + e^{-i\omega_d t}) \phi_{\text{zpf}}^2 (\hat{a} + \hat{a}^\dagger)^2 + \hat{O}(\hat{a}^4) \quad (2.166)$$

where $\omega = 4\sqrt{E_J E_c \cos(F)} - E_c$ with $E_c = \frac{e^2}{2C}$, and $\phi_{\text{zpf}} = \frac{2\pi}{\Phi_0} \sqrt{\frac{\hbar}{2} \sqrt{\frac{L_J}{2C \cos(F)}}}$. We can then move to the rotating frame with frequency $\omega_p/2$ and extract the three-wave mixing process engineered by the parametric drive. The resulting Hamiltonian is given by

$$\hat{H}/\hbar = \Delta \hat{a}^\dagger \hat{a} + \lambda (\hat{a}^{\dagger 2} + \hat{a}^2) + \hat{O}(\hat{a}^4), \quad (2.167)$$

where $\Delta = \omega_q - \omega_p/2$ is the frequency mismatch of the three-wave mixing, and $\lambda = \delta f E_{J,\text{eff}} \sqrt{\sin(F) \tan(F) E_c / (8 E_{J,\text{eff}})}$. The flux modulation effectively realizes a two-photon drive of the SQUID resonator through the parametrically activated three-wave mixing process. This two-photon drive is the core for parametric amplification. Notably, the Kerr-cat qubit has a similar Hamiltonian, but with a relatively large fourth-order nonlinearity term. These two devices both rely on the two-photon drives to operate. However, the fourth-order terms are harmful and reduce the dynamical range of a flux-pumped SQUID, while the fourth-order terms are necessary for Kerr-cat qubits to engineer stable cat states and fast gate operations.

Assuming that the three-wave mixing frequency mismatch is zero, we can use the input-output theory to write down the equation of motion¹³ for the CIMPA coupled to a transmission line with decay rate κ ,

$$\dot{\hat{a}} = \frac{i}{\hbar}[\hat{H}, \hat{a}] - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\text{in}} = -2i\lambda\hat{a}^\dagger - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\text{in}}. \quad (2.168)$$

where $\hat{a}_{\text{in}}$ is the input field from the transmission line. We can solve this equation of motion in the frequency domain by defining $\bar{a}[\omega] = \int_{-\infty}^{\infty} dt e^{i\omega t} \hat{a}(t)$ as the frequency-domain bosonic annihilation operator, $\bar{a}_{\text{in}}[\omega] = \int_{-\infty}^{\infty} dt e^{i\omega t} \hat{a}_{\text{in}}(t)$ as the frequency-domain input field, and $\kappa[\omega]$ as a frequency dependent decay rate. The equation in the frequency domain is therefore given by

$$\begin{aligned} i\omega\bar{a}[\omega] &= -2i\lambda\bar{a}^\dagger[-\omega] - \frac{\kappa}{2}\bar{a}[\omega] + \sqrt{\kappa}\bar{a}_{\text{in}}[\omega] \\ -i\omega\bar{a}^\dagger[-\omega] &= 2i\lambda\bar{a}[\omega] - \frac{\kappa}{2}\bar{a}^\dagger[-\omega] + \sqrt{\kappa}\bar{a}_{\text{in}}^\dagger[-\omega]. \end{aligned} \quad (2.169)$$

Solving the equation above, we can obtain the solution of the fields inside the SQUID resonator as

$$\bar{a}[\omega] = \sqrt{\kappa[\omega]} \left(\frac{(\kappa[\omega]/2 - i\omega)\bar{a}_{\text{in}}[\omega] - 2i\lambda\bar{a}_{\text{in}}^\dagger[-\omega]}{|\kappa[\omega]/2 - i\omega|^2 - 4|\lambda|^2} \right). \quad (2.170)$$

One should notice here that ω is the detuning from the SQUID resonance frequency because we work in the rotating frame. Then, with the input-output boundary condition $\sqrt{\kappa[\omega]}\bar{a}[\omega] = \bar{a}_{\text{in}}[\omega] + \bar{a}_{\text{out}}[\omega]$, we can obtain the relation between the input signal tone $\bar{a}_{\text{in}}[\omega]$ and output signal tone $\bar{a}_{\text{out}}[\omega]$,

$$\begin{aligned} \bar{a}_{\text{out}}[\omega] &= \left(\kappa[\omega] \frac{\kappa[\omega]/2 - i\omega}{|\kappa[\omega]/2 - i\omega|^2 - 4|\lambda|^2} - 1 \right) \bar{a}_{\text{in}}[\omega] \\ &\quad + \left(\frac{-2i\kappa[\omega]\lambda}{|\kappa[\omega]/2 - i\omega|^2 - 4|\lambda|^2} \right) \bar{a}_{\text{in}}^\dagger[-\omega], \end{aligned} \quad (2.171)$$

from which we can identify the gain $G(\omega)$ as the ratio between the power of the output and input signals with the same frequency,

$$G(\omega) = \left| \kappa[\omega] \frac{\kappa[\omega]/2 - i\omega}{|\kappa[\omega]/2 - i\omega|^2 - 4|\lambda|^2} - 1 \right|^2. \quad (2.172)$$

From the expression of gain, we found that the maximum gain appears when the signal frequency is resonant with the SQUID resonator frequency, $\omega = 0$, and the bandwidth of the gain is determined by $\kappa = \text{Re}[\frac{1}{CZ_{\text{in}}}]$. Therefore, by decreasing the input impedance Z_{in}

¹³Also known as quantum Langevin equation[154]

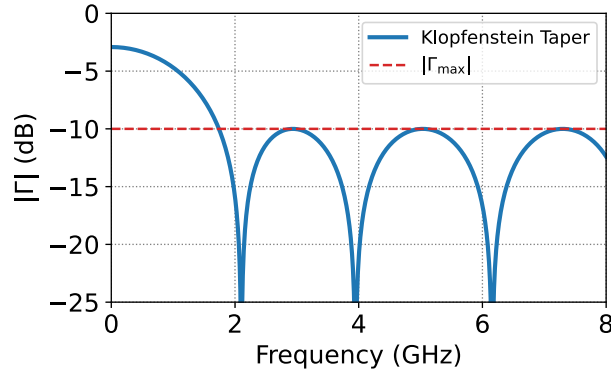


Figure 2.17: Reflection coefficient of Klopfenstein impedance transformer.

of the coupled transmission line, one can increase the bandwidth of the JPA. As discussed in the next section, the core technical approach to increasing the bandwidth of CIMPA is to decrease Z_{in} through a dedicated impedance transformer.

One may also notice that for specific values of λ , the gain will diverge and become nonphysical. This is because the derivation above assumed that the field in the SQUID resonator $\langle \hat{a}^\dagger \hat{a} \rangle$ is small, or equivalently, $\langle \hat{\phi} \rangle$ is small, and the large gain will lead to a large SQUID resonator field and break this assumption, making the equations above invalid. In fact, when the gain or the input signal is large enough, the fourth-order term will introduce a significant AC-stark shift, which distorts the output field and decreases the gain[155], leading to the limited dynamical range and saturation power of the JPA.

Broadband JPA Design

Typically, the JPA is directly coupled to an environment with $Z_{\text{in}} = 50\Omega$ because most of the microwave electronics are designed with a characteristic impedance of 50Ω . However, in order to increase the bandwidth of the JPA, we designed a CPW-based Klopfenstein taper[152] to transform the 50Ω environment impedance to 18Ω for an enhanced JPA coupling coefficient κ and large bandwidth, which is named CIMPA (CPW-based IMpedance-transformed Parametric Amplifier).

The Klopfenstein impedance transformer has many nice properties. Firstly, as an impedance transformer, we would like to maximize the transmission coefficient and minimize the reflection coefficient to have the most efficient signal delivery. For a Klopfenstein impedance transformer that transforms the impedance from Z_1 to Z_2 , the reflection coefficient is shown in Fig. 2.17 and is given by

$$\Gamma(\beta L) = \Gamma_0 e^{-i\beta L} \frac{1}{\cosh A} \begin{cases} \cos \sqrt{(\beta L)^2 - A^2} & \beta L > A \\ \cosh \sqrt{A^2 - (\beta L)^2} & \beta L < A \end{cases} \quad (2.173)$$

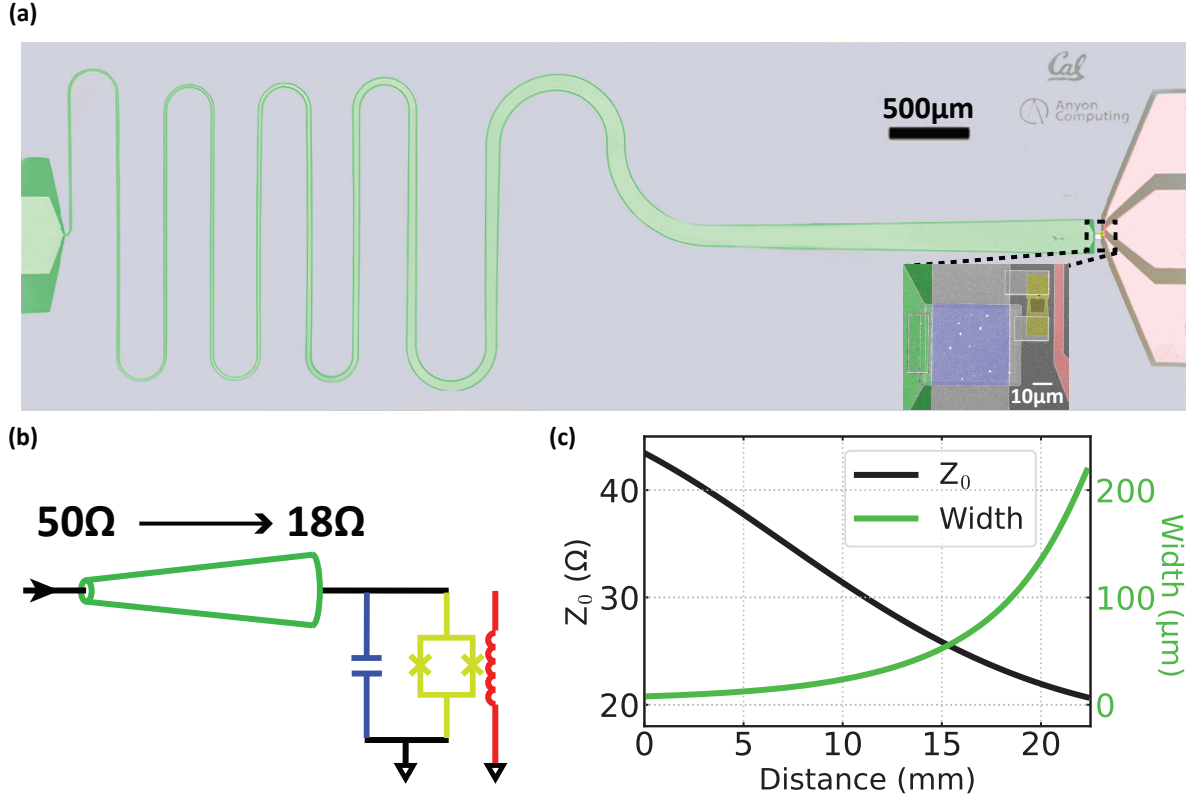


Figure 2.18: Device layout. (a) False-color micrograph of the device with the nonlinear resonator shown in the inset. From left to right are the horn-like CPW taper (green), the parallel plate capacitor (blue), and the SQUID (yellow) forming the resonator, and the on-chip flux line (red). (b) Parameters of the horn-like CPW impedance transformer, showing decreasing characteristic impedance Z_0 along the CPW as its center-trace width becomes wider, transforming from $50\ \Omega$ at the left end to $18\ \Omega$ at the right end. One should notice that the characteristic impedance of the horn-like CPW is not exactly equal to $50\ \Omega$ and $18\ \Omega$ on the two ends because of the nature of the Klopfenstein taper. [152] (c) Electrical circuit depiction of the CIMPA. The nonlinear resonator on the right side is constructed by shunting a SQUID (yellow) with a capacitor (blue). The structure is galvanically coupled to a Klopfenstein taper (green) on the left and inductively coupled to a flux-line (red) on the right.

where $\beta = \frac{\omega}{v}$ is the wave vector of the signal with frequency ω and speed v , L is the length of the impedance transformer, A is a design constant and Γ_0 is the bare-reflection coefficient without the transformer, defined as

$$\Gamma_0 = \frac{Z_1 - Z_2}{Z_1 + Z_2}. \quad (2.174)$$

Therefore, an ideal Klopfenstein impedance transformer can ensure a broadband high transmission behavior that for any signals with frequency above $\omega_{\min}$, the reflection coefficient is smaller than $\Gamma_{\max}$, where

$$\omega_{\min} = \frac{Av}{L}, \quad \Gamma_{\max} = \frac{\Gamma_0}{\cosh A}. \quad (2.175)$$

Secondly, to optimize the device layout compactness, the impedance transformer must be kept as small as possible. The Klopfenstein impedance transformer is demonstrated to be the optimal impedance transformer, which means that for a maximum reflection coefficient specification in the passband, the length of the Klopfenstein taper is the shortest.

The CIMPA device layout is shown in Fig. 2.18 (a). The device is constructed from three basic building blocks, as depicted by the circuit diagram in Fig. 2.18(b). The core component of the device is the SQUID resonator that facilitates the necessary parametric process through which amplification occurs, consisting of a SQUID (yellow) shunted by a capacitor (blue). On the right-hand side, it is inductively coupled to a flux line (red) that allows in-situ tuning of the resonator frequency and RF flux parametric pumping [149, 156]. On the left-hand side, it is galvanically connected to a Klopfenstein impedance transformer (green).

The Klopfenstein impedance transformer transforms the input impedance Z_{in} seen by CIMPA from the nominal 50Ω to 18Ω to improve the bandwidth of the CIMPA. The maximum reflection coefficient $|\Gamma_{\max}|$ and the cutoff frequency $\omega_{\min}$ is designed to be

$$|\Gamma_{\max}| = -10 \text{ dB}, \quad \omega_{\min} = 2\pi \times 2 \text{ GHz} \quad (2.176)$$

Typically, the frequencies of the relevant signals are around 6 GHz, but we choose a much lower cutoff frequency here to ensure the low signal loss under possible experimental imperfections.

The design parameters chosen above satisfy the requirement for both efficient signal delivery with small transmission loss and easy fabrication by photo lithography tools in many research facilities with high yield. On the one hand, such a design of the Klopfenstein tapers requires a $3\text{-}\mu\text{m}$ gap width for CPW, which is readily achievable with commonly used optical lithography tools. On the other hand, the -10 dB maximum reflection coefficient of the Klopfenstein taper leads to a minimum transmission loss of -0.45 dB. As the signal passes through the Klopfenstein taper twice, such an insertion loss will decrease the signal gain by -0.9 dB, which is negligible compared to the nominal 15 to 20 dB gain of the amplifier.

To construct this Klopfenstein impedance transformer, we tailor a CPW with its gap fixed at $3 \mu\text{m}$ and its center trace width tapered from $7.8 \mu\text{m}$ to $218 \mu\text{m}$, forming a horn-like shape. The characteristic impedance along this horn-like CPW line follows the Klopfenstein taper profile [152] as shown in Fig. 2.18 (c). The Klopfenstein profile is calculated using MATLAB, and the CPW characteristic impedance is calculated using TX-Line, a convenient industrial-level transmission line calculator [153]. To verify the property of the impedance transformer, we simulate the transmission and reflection coefficients (S_{21} and S_{11} , respectively) of the

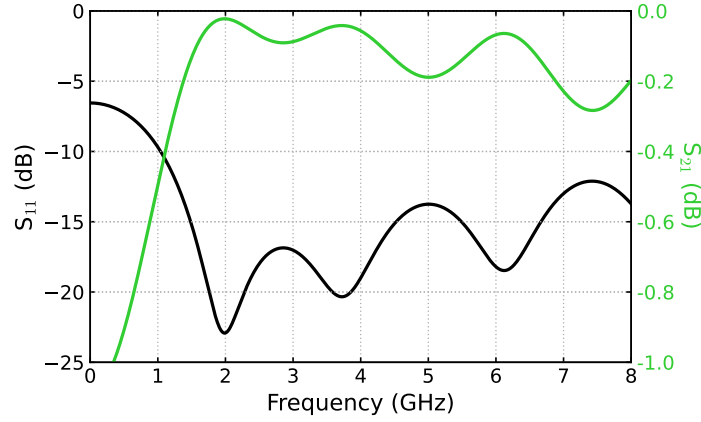


Figure 2.19: S-matrix simulation result of the horn-like CPW line.

horn-like CPW line with ANSYS HFSS. The result in Fig. 2.19 shows low reflection (below -10 dB) and high transmission (insertion loss below 0.3 dB) above the intended 2-GHz cutoff frequency.

The design geometry of this single horn-like CPW structure can be computed using simple RF design suites such as TX-Line [153], and its dimensions are compatible with single-layer photo-lithography and reactive-ion etching techniques. As a result, this approach significantly reduces the complexity overhead in both design and fabrication, which is reflected by the simple construction of our device shown in Fig. 2.18 (a).

Fabrication and Measurement Setup

The CIMPA device is fabricated on an intrinsic high-resistivity Si wafer with resistivity $\rho > 10 \text{ k-}\Omega\cdot\text{cm}$. After sputtering a 100-nm-Al film, the horn-like CPW's features are defined using photo-lithography and reactive-ion etching. Notably, the fabrication process is compatible with the photo-lithography technique thanks to the $3\text{-}\mu\text{m}$ gap. Then, the capacitor is fabricated using the lift-off e-beam lithography technique, followed by e-beam evaporation of aluminum oxide and aluminum electrodes. We note that the size of the capacitor is approximately $39 \times 39 \text{ }\mu\text{m}^2$, which can also be made using photo-lithography. The SQUID structure is defined using e-beam lithography and double-angle e-beam evaporation of aluminum films following the standard Dolan bridge technique [157]. The galvanic contact between the SQUID and the capacitor electrodes is formed using the Argon ion-milling band-aid process [158]. Finally, a layer of methyl methacrylate (MMA) resist is used to protect the wafer before dicing. The chips are diced and then cleaned using N-methylpyrrolidone (NMP) before being packaged into a copper box.

The CIMPA is wire-bonded to a printed circuit board (PCB) and packaged using a copper box. The device is then attached to the mixing chamber stage of a dilution fridge

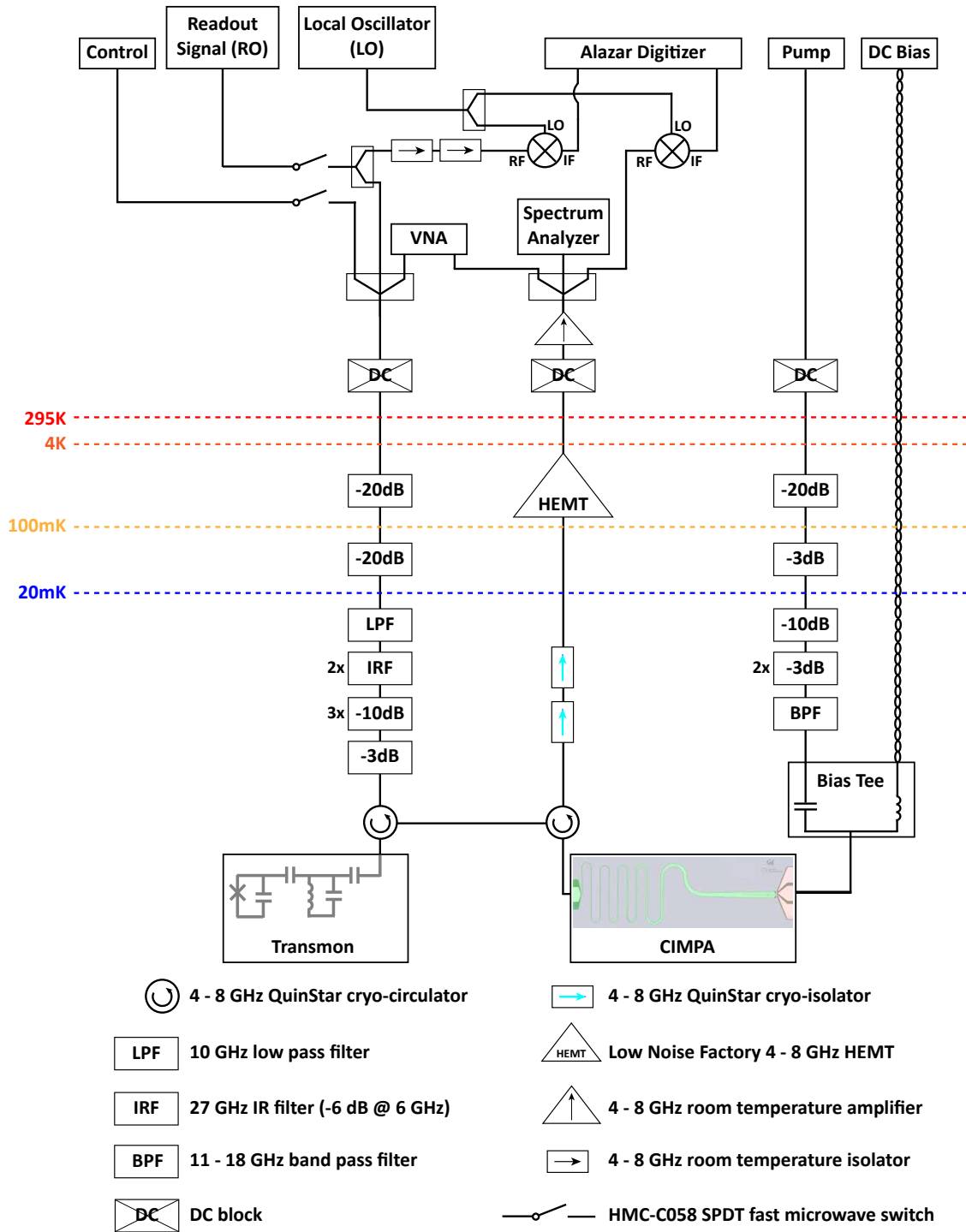


Figure 2.20: The wiring diagram for measuring CIMPA

operating at 20 mK. The wiring diagram of both the cryogenic and room-temperature setups is shown in Fig. 2.20(d). A transmon qubit is integrated into the measurement setup to calibrate the signal power and detect possible back action from the parametric process. Two QuinStar 4 - 8 GHz cryogenic circulators are used to route the signals, and the signals leaving the transmon are amplified by the CIMPA and a high-electron-mobility-transistor amplifier (HEMT).

A Rohde & Schwarz SGT100A vector RF source (Control) is used to generate qubit control pulses. An Agilent Technologies E8257D signal generator (RO) generates the readout signal at the readout resonator frequency. By mixing the readout signal with a Hewlett-Packard 8671B local oscillator (LO), the readout signal information is converted to an intermediate frequency of 50 MHz for the AlazarTech digitizer (Alazar) to record the signal. The readout signal and control signal are pulse-controlled using HMC-C058 SPDT switches. A Tektronix 5014C arbitrary waveform generator generates the pulse sequences to control the qubit, read out the resonator signals, and trigger the Alazar. The DC bias current to the CIMPA is provided by a Yokogawa GS200 DC voltage/current source, and the AC pump is supplied by a Hewlett-Packard 83732B signal generator. The CIMPA gain and noise performance are measured using an Agilent Technologies N9010A signal analyzer (SA) and an Agilent Technologies N5230C network analyzer (VNA).

Device Parameter Characterization

We extract the parameters of the SQUID nonlinear resonator by measuring the small-signal linear response of the device using a vector network analyzer (VNA). As the probe signal goes through the resonant frequency $\omega_0/(2\pi)$, the reflected signal phase is twisted, indicating a resonant condition, and the resonant frequency is extracted by fitting the reflection spectrum. [159] When we adjust the external flux bias Φ_e of the SQUID, the effective inductance L_j of the SQUID will be changed, and thereby the resonant frequency changes according to

$$\omega_0 = [C(L_{j,\text{eff}} |\cos(\pi\Phi_e/\Phi_0)|^{-1} + L_{\text{geo}})]^{-1/2}, \quad (2.177)$$

where L_{geo} is the parasitic geometric inductance originating from the leads, $L_{j,\text{eff}}$ is the effective inductance of the SQUID at zero flux, and Φ_0 is the magnetic flux quantum. Measuring the phase response at each flux point gives us the spectrum shown in Fig. 2.21. By fitting the measured spectrum, we extract the following parameters: $L_{j,\text{eff}} = 69$ pH, $C = 4$ pF, and $L_{\text{geo}} < 4$ pH. The spectrum also allows us to estimate the asymmetry of the SQUID to be less than 1%.

Gain Characterization

We then proceed to measure the three main performance metrics for an amplifier: gain, noise performance, and dynamic range. The gain profile is defined as the increase in signal power when the CIMPA is tuned and flux-pumped to provide amplification, relative to the signal measured when the CIMPA is bypassed.

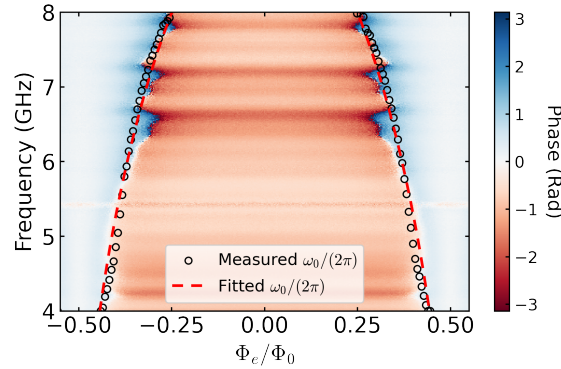


Figure 2.21: CIMPA's spectrum. Reflectometry spectroscopy performed using a VNA shows phase twists across the SQUID's resonant frequencies. Repeating the measurement at various flux biases yields the CIMPA's characteristic frequency spectrum, which fits the relation given by Eq. 2.177 with excellent agreement.

The noise performance refers to the noise temperature of the device, which is defined as

$$T = \frac{n\hbar\omega}{k_B} \quad (2.178)$$

where k_B is the Boltzmann's constant, ω is the signal frequency and n noise photon number. Typically, the microwave signal at frequency ω from an environment at temperature T_{env} has the noise photon number given by

$$n_{\text{env}} = \frac{1}{e^{\frac{\hbar\omega}{k_B T_{\text{env}}}} - 1} + \frac{1}{2}, \quad (2.179)$$

For the relevant signals at around 6 GHz to 7 GHz and the mixing chamber temperature of dilution refrigerators at 20 mK, $k_B T_{\text{env}} \ll \hbar\omega$, and therefore the signal noise is approximately half a photon. For an ideal parametric amplifier which offers quantum-limited amplification, the added noise from the amplifier is also half a photon[119, 120, 155]. As a result, for an ideal parametric amplifier, the noise temperature should correspond to one photon, i.e.

$$T_{\text{ideal}} = \frac{\hbar\omega}{k_B} \quad (2.180)$$

The dynamical range, also known as the saturation power, is usually defined as the lowest input power when the gain is reduced by 1 dB. As shown in Fig. 2.22, due to the higher-order effects, when the input power increases, the gain of the amplifier will decrease. People usually define the saturation power as the 1-dB compression point, i.e., the input power when the gain is reduced by 1 dB, indicated by the dashed line in Fig. 2.22.

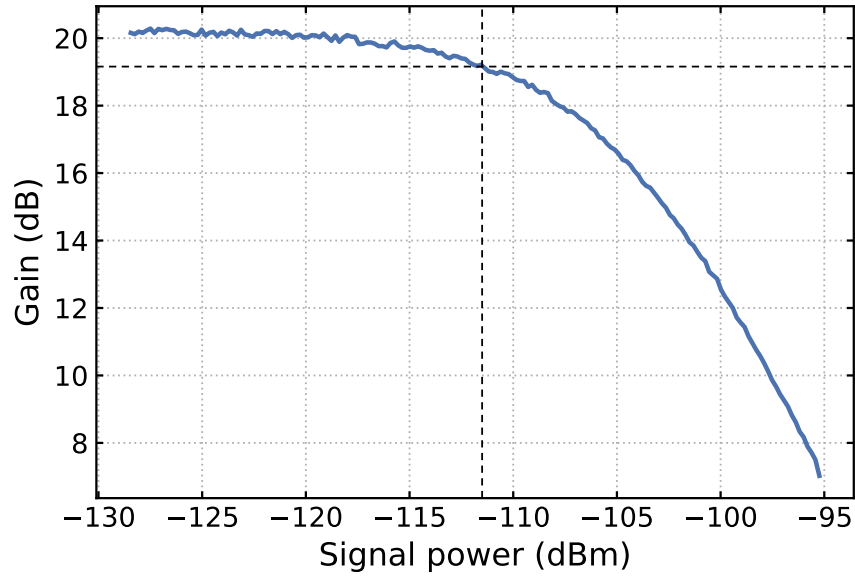


Figure 2.22: Gain performance with different signal power.

To extract the gain profile, we set the external flux bias to $\Phi_e = 0.3\Phi_0$, and apply a microwave pump tone at frequency $\omega_p/(2\pi) = 13.41$ GHz with 3.41 dBm power outside the fridge. we measure the microwave response using the VNA, observing over 20(15) dB gain over a 200(700)-MHz bandwidth (Fig. 2.23(a)). Such a high increase in the bandwidth cannot be explained solely by the increased resonator linewidth [120]. As discussed in Ref. [120], the weak frequency dependence of Z_{in} due to possible impedance mismatch strongly assists the enhancement of the bandwidth, but it hinders the quantitative comparison between the theory and experiments without the careful calibration of the impedance of the measurement chain. A qualitative comparison with the calculated gain performance will be discussed later.

The noise performance of the device can be characterized via the signal-to-noise ratio (SNR) improvement $\text{SNR}_{\text{imprv}}$ measured by a spectrum analyzer (SA) [32, 33, 120, 160]. In an amplification chain consisting of a CIMPA and a HEMT, the SNR improvement due to the CIMPA is given as

$$\text{SNR}_{\text{imprv}} = \left(\frac{T_{\text{CIMPA}}}{T_{\text{HEMT}}} + \frac{1}{G} \right)^{-1}, \quad (2.181)$$

where T_{CIMPA} and T_{HEMT} indicate the noise temperatures of the CIMPA and the HEMT, respectively. The extracted noise temperature with $T_{\text{HEMT}} = 2.3$ K - 2.9 K is shown in Fig. 2.23(b), which approaches the quantum noise limit set by the noise temperature of one photon. The uncertainty here originates from the estimated T_{HEMT} range. The device approaches the quantum noise limit over a 700 MHz range, and the noise increase beyond this range is mainly because the CIMPA's gain is not large enough to compensate for the

noise at the HEMT.

Presently, we characterize the saturation power of the device as follows. First, the relation between the actual signal power going into the CIMPA and the power set on the VNA must be established. By measuring the ac Stark shift and measurement-induced dephasing of the transmon qubit at various VNA powers, we can calibrate the cable attenuation [32]. The detailed calibration process will be discussed in the next section. Then, we gradually increase the power until the gain falls off by 1 dB, and convert this using the established attenuation. The result in Fig. 2.23(c) shows an average input 1-dB compression point of approximately -110 dBm. Compared with the input saturation power of simple JPAs coupled to a 50 Ω environment (with $Q \approx 15$), which is less than -120 dBm [120, 122, 149], our saturation power is increased by around 10 dB. This can be explained by the reduction in the environment's input impedance in our design (with $Q \approx 4$), as $P_{\text{sat}} \sim C/Z_{\text{in}}$ [120].

In addition to the instantaneous bandwidth, we can flux-tune the device and operate it at other frequencies. To showcase this capability, we calibrate the CIMPA's parameters at five different external flux points corresponding to a frequency range of 1.4 GHz, and we then characterize the gain performance of the device across this broad range. Figure 2.23 (d) shows that the combination of the engineered broadband impedance transformer and the inherent flux-tunability allows the device to amplify signals with a frequency range from 6 GHz to 7.4 GHz and at least 15 dB of maximum gain.

As mentioned before, the exact quantitative calculation of the gain profile is challenging without careful calibration of the impedance mismatch along the measurement chain. However, we can assume reasonable parameters of the measurement chain and provide a qualitative analysis of the gain profile and compare it with the experiments.

The gain performance of the CIMPA is given in Eq. 2.172, which depends on the decay rate of the nonlinear resonator, $\kappa = \text{Re}[(CZ_{\text{in}})^{-1}]$. As mentioned in Ref. [120], the impedance mismatch from the wire bonding, SMA cables and circulators will lead to complicated frequency dependence of Z_{in} and strongly influence the gain performance.

To simulate the gain performance with possible impedance mismatch, we model the measurement chain as shown in Fig. 2.24(a), where the CIMPA is coupled to an SMA cable with characteristic impedance $Z_{\text{SMA}} = 50 \Omega$ and length $L_{\text{SMA}} = 25$ cm through wire bonds with characteristic impedance $Z_{\text{wb}} = 55 \Omega$ and length $L_{\text{wb}} = 2$ mm. The SMA cable is connected to the cryo-circulator with impedance $Z_c = 55 \Omega$.

We analyze this microwave network using the ABCD matrix or the transmission matrix $\mathbf{T}$ [152]. The ABCD matrix of the horn-like CPW line $\mathbf{T}_{\text{CPW}}$ is obtained using the S-parameters from ANSYS HFSS results in Fig. 2.19, while the ABCD matrix of the wire bonds and the SMA cable is given by

$$\mathbf{T}_{\text{SMA(wb)}} = \begin{bmatrix} \cos(\beta L_{\text{SMA(wb)}}) & iZ_{\text{SMA(wb)}} \sin(\beta L_{\text{SMA(wb)}}) \\ i \cos(\beta L_{\text{SMA(wb)}})/Z_{\text{SMA(wb)}} & \cos(\beta L_{\text{SMA(wb)}}) \end{bmatrix} \quad (2.182)$$

The overall ABCD matrix of this microwave network is then given by $\mathbf{T} = \mathbf{T}_{\text{SMA}} \mathbf{T}_{\text{wb}} \mathbf{T}_{\text{CPW}}$. Together with the impedance of the cryo-circulator, one can obtain the reflection coefficient

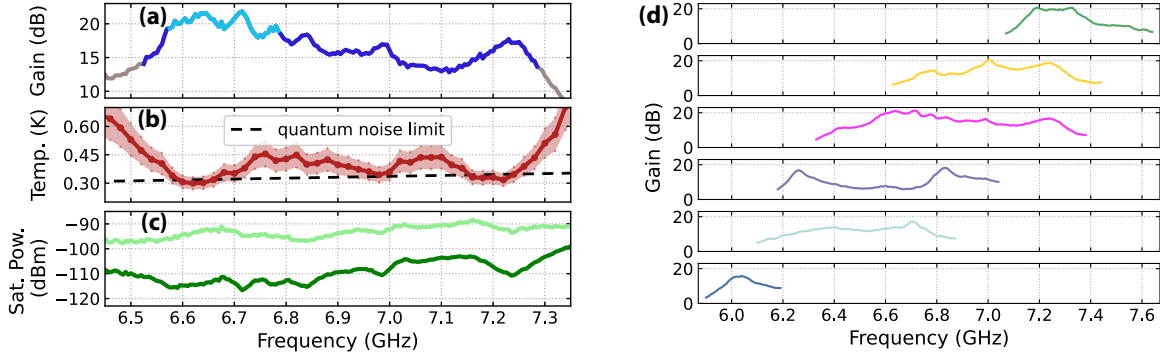


Figure 2.23: CIMPA's performance. (a) Instantaneous signal gain, where the 20 dB gain range is highlighted in cyan, and the 15 dB gain range is highlighted in blue, (b) noise temperature, and (c) input saturation power (dark green) and output saturation power (light green) of the CIMPA operated at $\Phi_e/\Phi_o = 0.3$ and $\omega_p = 2\pi \times 13.41$ GHz. (d) Signal gain at different CIMPA frequencies tuned by varying Φ_e .

S'_{11} as seen from the CIMPA's nonlinear resonator. The input impedance Z_{in} is calculated by

$$Z_{\text{in}} = 18\Omega \times \frac{1 + S'_{11}}{1 - S'_{11}} \quad (2.183)$$

The results are shown in Fig. 2.24(b). Plugging this input impedance Z_{in} into Eq. 2.172, we obtained gain profiles at different flux bias and operating frequencies shown in Fig. 2.24(c), where the gain profile shows non-Lorentzian line-shape and multi-peak behaviors due to the frequency dependence of Z_{in} . The gain profiles qualitatively agree with the experiment results shown in the main text. We can only obtain a quantitative comparison with precise calibration of the impedance along the microwave chain.

Input Power Calibration

To measure the saturation power of the CIMPA, we need to precisely calibrate the input power at the amplifier (not at the top of the fridge). Therefore, we need to calibrate the attenuation along the input line and convert the power sent out of the VNA to the power fed into the CIMPA. According to the method mentioned in Ref. [32], we can connect the CIMPA with a transmon qubit, as shown in Fig. 2.20, and the signal at the CIMPA will be equal to the signal reflected from the readout resonator of the transmon. we can then calibrate the input signal power by extracting the intra-cavity photon number occupation $\bar{n}$ inside the readout resonator. The power of the signal leaving the qubit and feeding into

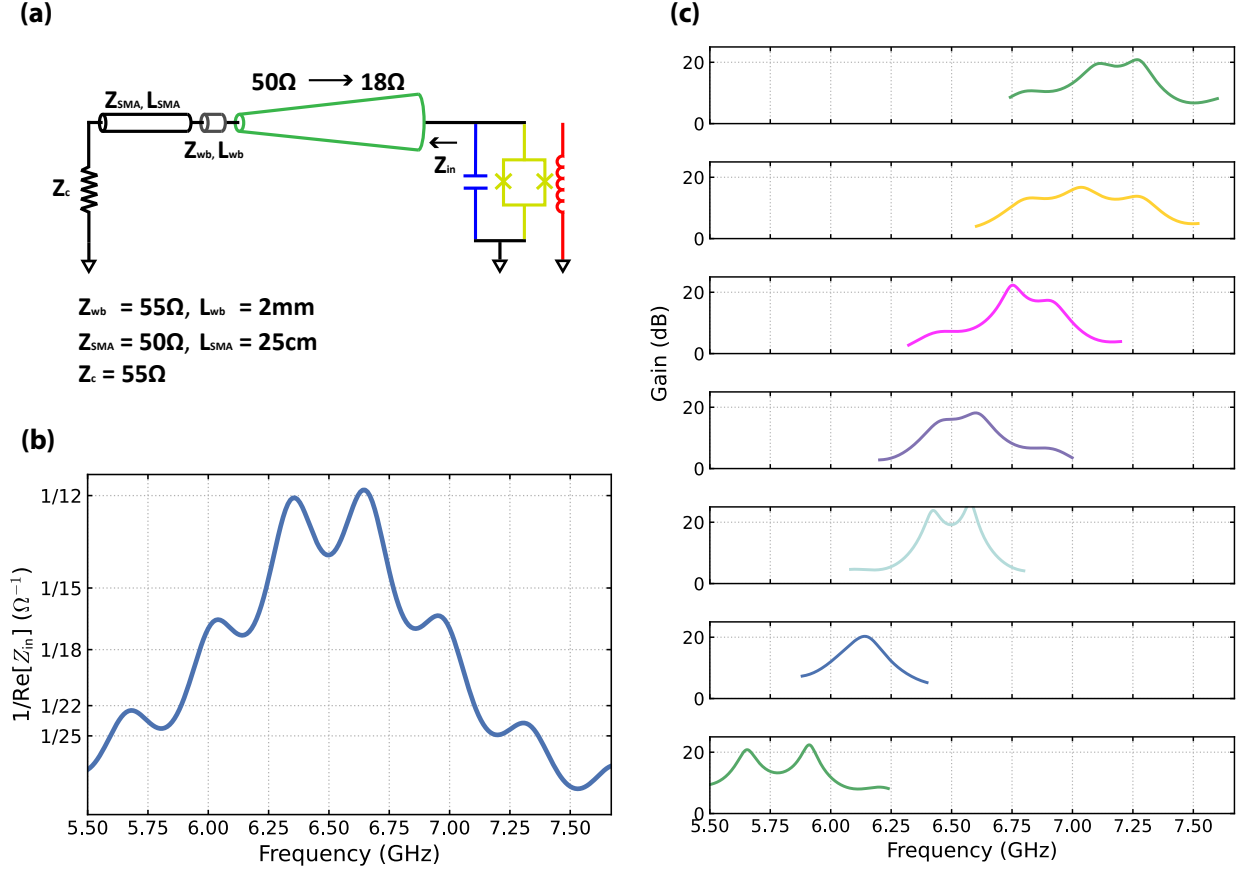


Figure 2.24: (a) The microwave measurement chain model for simulating the input impedance Z_{in} , (b) the input impedance Z_{in} with respect to the frequency, and (c) the simulated gain profile at different operating frequencies.

CIMPA is given by

$$P = \hbar\omega_r\kappa_r\bar{n}, \quad (2.184)$$

where ω_r and κ_r are the angular frequency and linewidth of the readout resonator, respectively. The resonator linewidth and frequencies are extracted by directly fitting the resonator reflection spectrum, as shown in Fig. 2.25(a).

The intra-cavity photon number is measured using the ac Stark shift and measurement-induced dephasing of the qubit, which can be extracted by fitting the Ramsey measurement. Considering a qubit dispersively coupled to a readout resonator with a dispersive shift χ , the ac Stark shift caused by the photon population in the readout resonator is $\Delta_{ac} = 2\chi\bar{n}$, and the measurement-induced dephasing is given by $\Gamma_\phi = 8\chi^2\bar{n}/\kappa_r$ [161]. Utilizing the quadratic dependence of the dephasing rate and the linear dependence of the frequency shift on χ , we can extract χ and $\bar{n}$. As we vary the VNA power outside the fridge, we can extract the

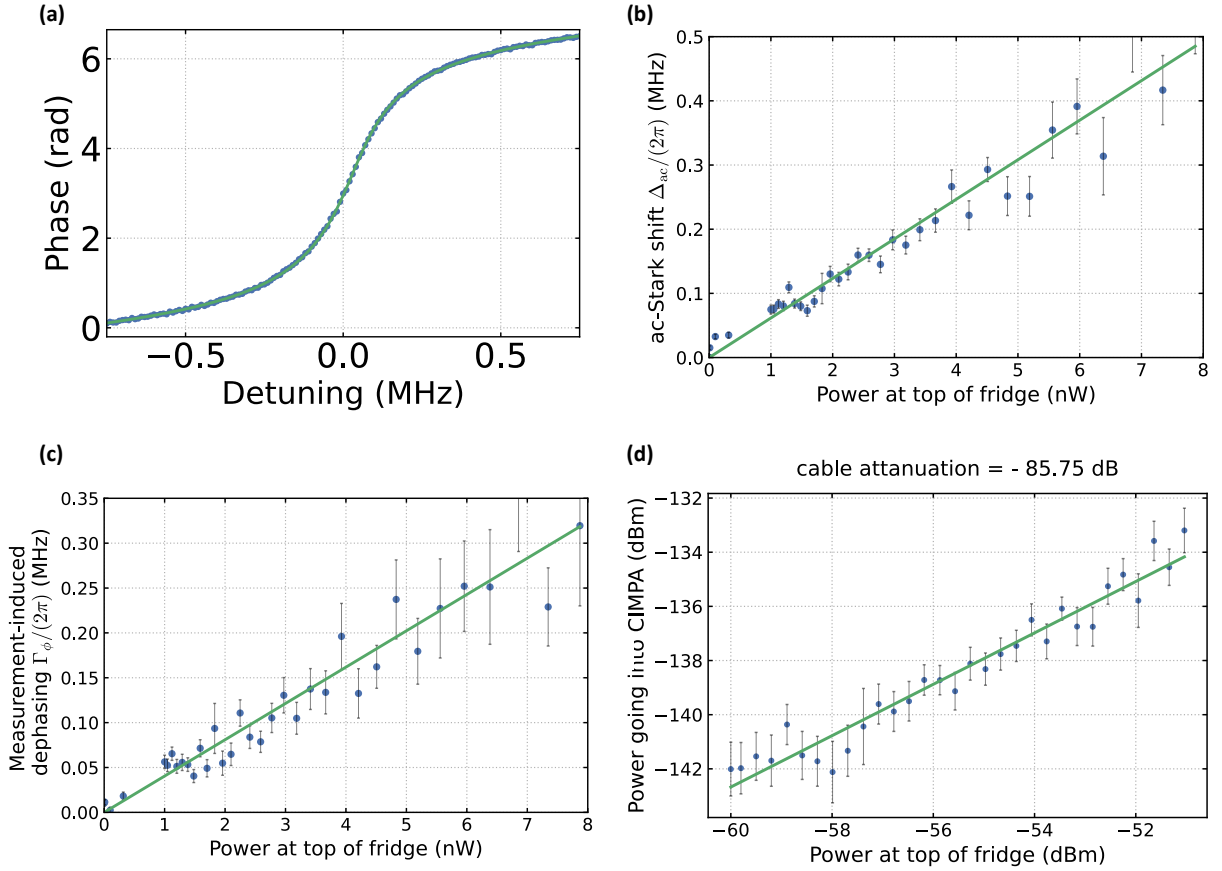


Figure 2.25: The calibration of the measurement setup. (a) Fitting of the readout resonator. (b) Fitting of the ac Stark shift. (c) Fitting of the measurement-induced dephasing. (d) Calibration between the power going into CIMPA and the power at the top of the fridge

intra-cavity photon number of the readout resonator by the Ramsey spectrum of the qubit, and then calibrate the corresponding signal power at the CIMPA. The ac Stark shift and measurement-induced dephasing at various VNA powers are shown in Fig. 2.25(b, c). The extracted correspondence between the signal sent into CIMPA and the VNA power outside the fridge is shown in Fig. 2.25 (d). All extracted parameters are listed in Table 2.3.

Parameter	$\omega_q/(2\pi)$ (GHz)	$\omega_r/(2\pi)$ (GHz)	$\kappa_r/(2\pi)$ (kHz)	T_1 (μ s)	$T_{2,\text{echo}}$ (μ s)
Value	5.350	6.633	309	45	16

Table 2.3: Qubit and resonator parameters

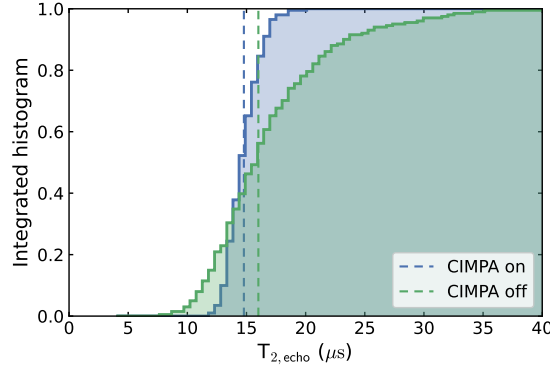


Figure 2.26: Transmon’s coherence time statistics for CIMPA on (pumped) and off (unpumped), with the mean values equal to $14.78 \mu\text{s}$ and $16 \mu\text{s}$, and the standard deviations equal to $1.41 \mu\text{s}$ and $5.45 \mu\text{s}$, respectively. The statistics consist of 201 individual measurements in each setting.

Back-action Measurement

Importantly, possible back action from the parametric process can undermine the integration of an amplifier into a qubit measurement setup. For example, spurious microwave radiation from the CIMPA can populate the readout resonator connected to the qubit, leading to photon-noise dephasing [161, 162]. To investigate the possible back action of the CIMPA device, we measure the echo time $T_{2,\text{echo}}$ of the qubit with and without amplification. To turn off the CIMPA, we remove the pump tone and flux-tune its frequency away from the qubit’s readout resonator, such that the amplifier acts as a passive mirror. When the CIMPA is turned on, it provides a gain of 21 dB for the readout signal frequency at 6.633 GHz. The $T_{2,\text{echo}}$ measurement is repeated continuously over 201 iterations in each setting to provide reliable statistics despite temporal fluctuations. We attribute the wider fluctuation range associated with the measurement when the CIMPA is off to the approximately ten times longer data acquisition time. As shown in Fig. 2.26, there is no significant degradation of the qubit’s coherence time beyond random fluctuation when the CIMPA is operated.

Summary and Discussion

In conclusion, we show that the CIMPA’s design and fabrication is as simple as the JPA’s, yet its performance is comparable to earlier IMPA implementations. Therefore, it fills the technological gap between JPAs and TWPAs, with potential applications ranging from qubit readout to axion dark matter detection. Compared to typical TWPAs, the CIMPA device reported here has a narrower bandwidth and lower dynamic range. However, we note that, on one hand, a majority of quantum experiments do not require a 3-GHz bandwidth, and

on the other hand, we can integrate SQUID arrays [163, 164] or superconducting nonlinear asymmetric inductive elements (SNAIL) [165] into the CIMPA design to increase its power handling capability.

Chapter 3

Kerr-cat Qubit Design and Fabrication

Quantum systems are fragile, subject to various errors such as decoherence [166], dephasing [23, 167], stochastic and coherent errors [168]. As a result, quantum error correction (QEC) is essential for practical and fault-tolerant quantum computing (FTQC) [37, 169]. Most QEC protocols impose stringent performance requirements on the physical device level due to significant hardware overhead and low error thresholds [28, 169]. The suppression of qubit errors via the surface code highlights the progress in this direction [28]. Alternatively, tailored QEC codes explore the noise structures of quantum systems to achieve higher efficiency and error thresholds, relaxing the requirements for the quantum device performance [170–173].

QEC codes tailored for noise-biased qubits [58–60], such as the XZZX codes [57, 174, 175], have gained extensive attention. Recent work on enhancing the phase-flip time of dissipative cat qubits using repetition codes marks an important step toward the extreme limit where physical noises are so strongly biased that only one error channel requires correction [176]. Nevertheless, such a method is challenging to scale up because the uncorrected error will still increase exponentially with the system size [176]. On the other hand, the exploration of Kerr-cat qubits (KCQs) aims to address stochastic Pauli errors based on the XZZX surface code and relies on noise bias to achieve high error thresholds [177]. Importantly, KCQs are shown to be promising through demonstrations of their universal gate control and quantum nondemolition readout [53, 62].

Kerr-cat qubits [53, 55, 62, 64, 83, 177–183] are usually realized by engineering Schrödinger cat states in quantum oscillators, and have attracted considerable attention due to their comparatively high noise bias and low hardware complexity. Leveraging strong non-linear interactions in superconducting circuits, many theoretical proposals examined cat qubit implementations [174, 184], bias-preserving gates [185, 186], and applications in XZZX surface code for FTQC [177]. Furthermore, the successful implementation of cat qubits in superconducting circuits has driven experimental advances [55, 62]. The universal quantum operations of a KCQ enclosed in a 3D cavity have been demonstrated [52, 65]. However,

Prior implementation of Kerr-cat qubits was restricted to 3D architectures, and the feasibility for realizing high-performance Kerr-cat qubits in the scalable 2D superconducting circuits remains an open question, which limits their further progress in quantum error correction. Therefore, the development of the Kerr-cat qubit in a 2D architecture is crucial for advancing Kerr-cat qubits toward further applications in quantum computing.

In this chapter, we will discuss the design and fabrication of Kerr-cat qubits in a scalable 2D architecture, which exhibit inherent resilience to bit-flip errors[53, 54]. The biased noise with suppressed bit-flip errors of Kerr-cat qubits offered the possibility for hardware-efficient quantum error corrections, which have a high error threshold and reduced hardware overhead. The contents in this chapter are based on the author's publication[53, 54].

3.1 Hamiltonian Structure and Qubit Encoding

The Kerr-cat qubits are engineered in a bosonic oscillator mode through the following effective Hamiltonian

$$\hat{H}_{\text{KCQ}}/\hbar = \Delta \hat{a}^\dagger \hat{a} + \epsilon_2 \hat{a}^{\dagger 2} + \epsilon_2^* \hat{a}^2 - K \hat{a}^{\dagger 2} \hat{a}^2, \quad (3.1)$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators of the bosonic mode. This Hamiltonian is reminiscent of the CIMPA Hamiltonian introduced in Eq. 2.167, which likewise contains a detuning term characterized by Δ and a two-photon drive term characterized by the drive strength ϵ_2 . However, the present Hamiltonian additionally includes a non-negligible fourth-order contribution, namely the Kerr nonlinearity term characterized by the Kerr coefficient K . This Kerr term plays a crucial role in engineering Kerr-cat qubits.

To obtain an intuition for the structure of this Hamiltonian, one can make a plot of the classical energy pseudo-potential of this Hamiltonian by mapping the bosonic operators to complex numbers α following

$$\hat{a} \rightarrow \alpha, \quad \hat{a}^\dagger \rightarrow \alpha^*, \quad (3.2)$$

and plot the values of the corresponding classical Hamiltonian. Unlike the CIMPA Hamiltonian Eq. 2.167, which has negligible fourth-order term and therefore maintain an approximately parabola profile, the Kerr-cat qubit Hamiltonian Eq. 3.1 forms a double-well structure for suitable choices of Δ and ϵ_2 , as shown in Fig. 3.1. The barrier energy of the pseudo-potential double-well, defined as the energy difference between the bottom of the potential wells and the top of the potential barrier between them, is given by

$$E_{\text{barrier}} = \frac{(2|\epsilon_2| + \Delta)^2}{4K}. \quad (3.3)$$

Therefore, the potential well will become deeper with a higher ϵ_2 and positive Δ .

A double-well potential structure usually supports pairs of eigenstates with even and odd parities, and the tunneling between the two potential wells is exponentially suppressed,

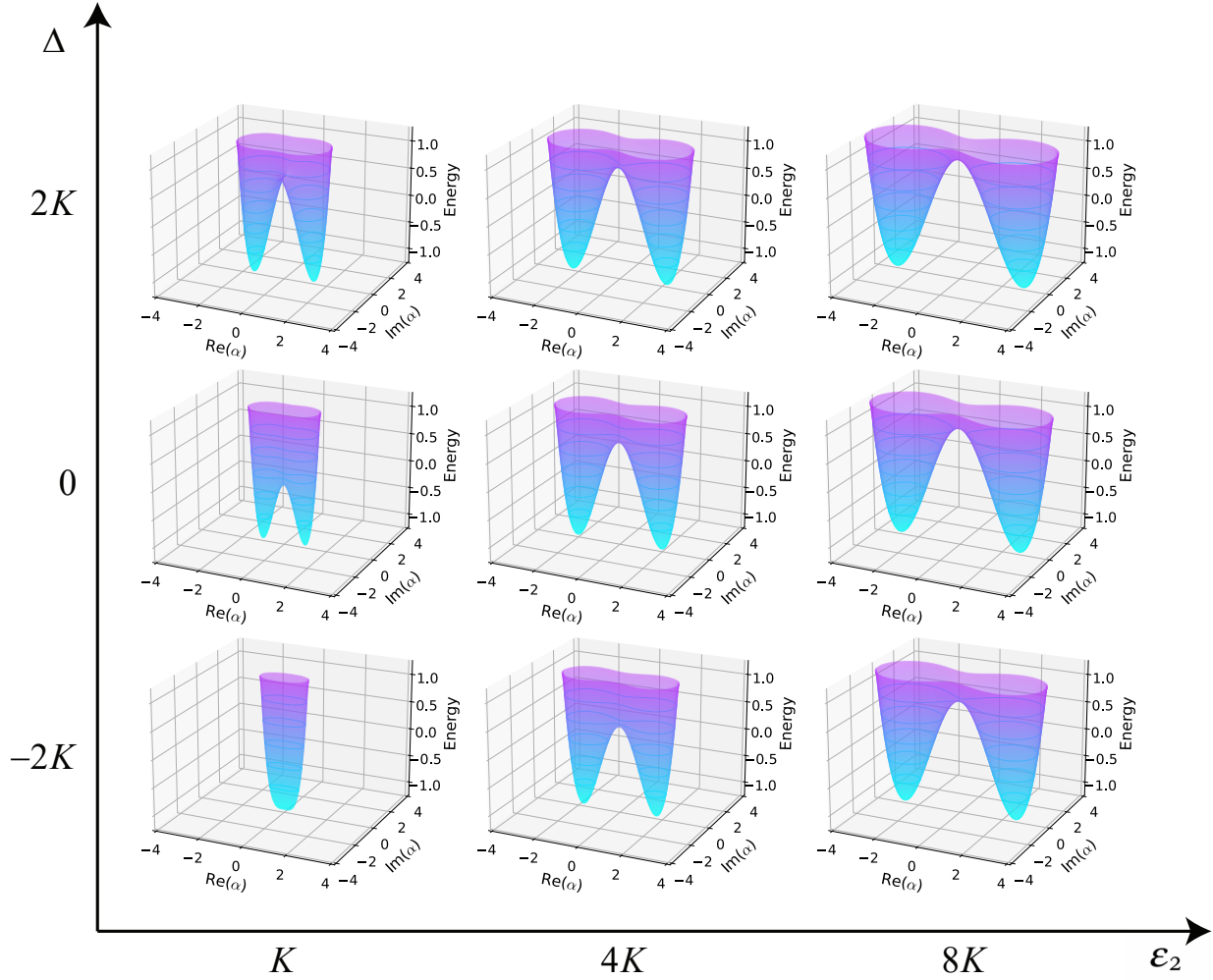


Figure 3.1: Classical pseudo-potential of Kerr-cat qubit Hamiltonian with different Δ and ϵ_2 .

enabling the protection of the states against noise. Similar properties are also observed in the Kerr-cat qubit Hamiltonian. In the following of this section, we will discuss the structure of the eigenstates and the eigenenergies of the Kerr-cat qubit Hamiltonian and their interactions with various noise mechanisms.

Resonant-Kerr-cat qubit

We first analyze the resonant-Kerr-cat qubit case with $\Delta = 0$, which has a clear analytical formula of the eigenstates and eigenenergies. When $\Delta = 0$, one can factorize the Hamiltonian to be

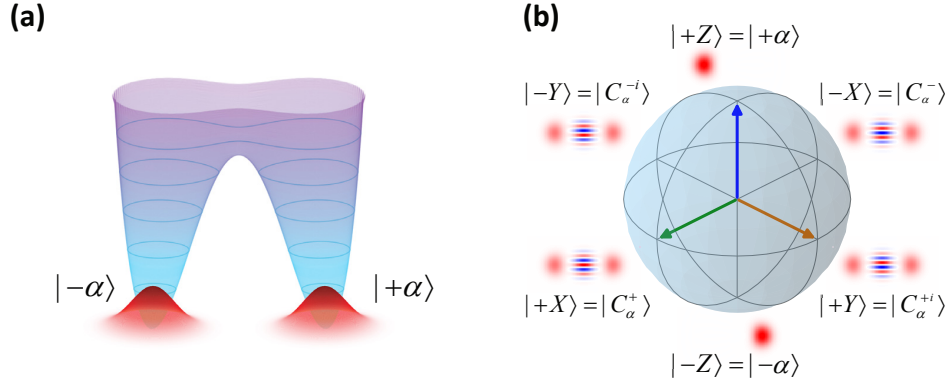


Figure 3.2: Eigenstates and cat qubit encoding. (a) The pseudo-potential and the ground coherent states. (b) The Bloch sphere of a cat qubit.

$$\hat{H}_{\text{res}}/\hbar = \epsilon_2 \hat{a}^{\dagger 2} + \epsilon_2^* \hat{a}^2 - K \hat{a}^{\dagger 2} \hat{a}^2 = -K(\hat{a}^{\dagger 2} - \alpha^{*2})(\hat{a}^2 - \alpha^2) + K|\alpha|^4 \quad (3.4)$$

where $\alpha = \sqrt{\epsilon_2/K}$. This Hamiltonian clearly supports two degenerate ground states $|\pm\alpha\rangle$, which are two coherent states with opposite phases lying at the bottom of each potential well as shown in Fig. 3.2 (a). The eigenenergies of these two ground states are $E_0 = K|\alpha|^4 = \epsilon_2^2/K$. The superposition of them then forms a pair of eigenstates with even and odd parities

$$|C_\alpha^\pm\rangle = \mathcal{N}_\pm(|\alpha\rangle \pm |-\alpha\rangle), \quad (3.5)$$

where $\mathcal{N}_\pm$ is the normalization factor, which approaches $1/\sqrt{2}$ when $|\alpha|^2 \gg 1$. The eigenstates above, consisting of the superposition of two coherent states with opposite phases, are named Schrödinger cat states¹. Therefore, using the superpositions of those cat states or coherent states, one can encode a qubit and define the Bloch sphere of the qubit as shown in Fig. 3.2 (b), where the Z axis states are coherent states, the X axis states are even and odd cat states, and the Y axis states are the imaginary even and odd cat states defined as

$$|C_\alpha^{\pm i}\rangle = \mathcal{N}_{\pm i}(|\alpha\rangle \pm i|-\alpha\rangle), \quad (3.6)$$

and $|\alpha|^2$ is usually referred to as the size of the cat qubit. One may find that ϵ_2 and α are in general complex with a non-zero phase. However, such a phase only causes a rotation in the reference phase space. Therefore, without the loss of generality, we can always assume ϵ_2 and α are real, and reference the phases of all other drives and corresponding dynamics with respect to the phase of ϵ_2 .

¹In reminiscence of Schrödinger's cat thought experiment, “quantum” superposition of “classical” states are usually referred to as cat states, and coherent states are considered a very “classical” state of the bosonic mode.

Detuned-Kerr-cat qubit

We then move to the discussion of the Hamiltonian with a non-zero detuning $\Delta \neq 0$. Unlike the resonant-Kerr-cat qubit case, where the Hamiltonian can be factorized to find the eigenstates, the detuned-Kerr-cat qubit Hamiltonian is not factorizable, and it is challenging to obtain an accurate analytical formula. Therefore, we will analyze the detuned-Kerr-cat qubit Hamiltonian using perturbation theory.

We apply a displacement transformation to the detuned-Kerr-cat qubit Hamiltonian with displacement operator $U = \hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$, and get the Hamiltonian in the displaced frame as

$$\hat{H}' = E + \Lambda a^\dagger + \Lambda^* a + \tilde{\Delta} a^\dagger a + \tilde{\epsilon}_2 (a^{\dagger 2} + a^2) + \Gamma a^{\dagger 2} a + \Gamma^* a^\dagger a^2 - K a^{\dagger 2} a^2, \quad (3.7)$$

where we have assumed ϵ_2 is real, and the coefficients are given by

$$E = \Delta |\alpha|^2 - K |\alpha|^4 + \epsilon_2 (\alpha^2 + \alpha^{*2}), \quad (3.8a)$$

$$\Lambda = -\Delta \alpha + 2K |\alpha|^2 \alpha^* - 2\epsilon_2 \alpha^*, \quad (3.8b)$$

$$\tilde{\Delta} = \Delta - 4K |\alpha|^2, \quad (3.8c)$$

$$\tilde{\epsilon}_2 = -K \alpha^2 + \epsilon_2, \quad (3.8d)$$

$$\Gamma = -2K \alpha. \quad (3.8e)$$

With a suitable choice of $\alpha = \sqrt{(\epsilon_2 + \Delta/2)/K}$, the single-photon drive terms are canceled out (i.e. $\Lambda = 0$), and the displaced Hamiltonian can be written as

$$\hat{H}' = H_0 + H_1, \quad (3.9a)$$

$$H_0 = \frac{(\Delta/2 + \epsilon_2)^2}{K} - (4\epsilon_2 + \Delta) a^\dagger a, \quad (3.9b)$$

$$H_1 = -\frac{1}{2} \Delta (a^{\dagger 2} + a^2) \quad (3.9c)$$

$$- (2\epsilon_2 + \Delta) (a^{\dagger 2} a + a^\dagger a^2) - K a^{\dagger 2} a^2, \quad (3.9d)$$

where the eigenstates of H_0 are Fock states $|n\rangle$ and their eigenenergies are given by $E_n^{(0)} = (\Delta/2 + \epsilon_2)^2 / K - (4\epsilon_2 + \Delta) n$. H_1 is treated as a perturbation as long as its energy scale is much smaller than the energy gap of the unperturbed system ($|\epsilon_2| \gg |K|, |\Delta/2|$).

Firstly, the perturbed ground-state energy is estimated as

$$\tilde{E}_0 = E_0^{(0)} + E_0^{(1)} + \dots = \frac{(\Delta/2 + \epsilon_2)^2}{K} + \frac{\Delta^2 K}{(4\epsilon_2 + \Delta)^2} + \dots, \quad (3.10)$$

In Fig. 3.3 (a), we plot the fractional deviation of the estimated ground state energy ($\tilde{E}_0$) from the exact ground state energy (E_0) given by the numerical diagonalization with various two-photon stabilization drive strengths. In the regime of interest where $\Delta/K \sim 2$ and

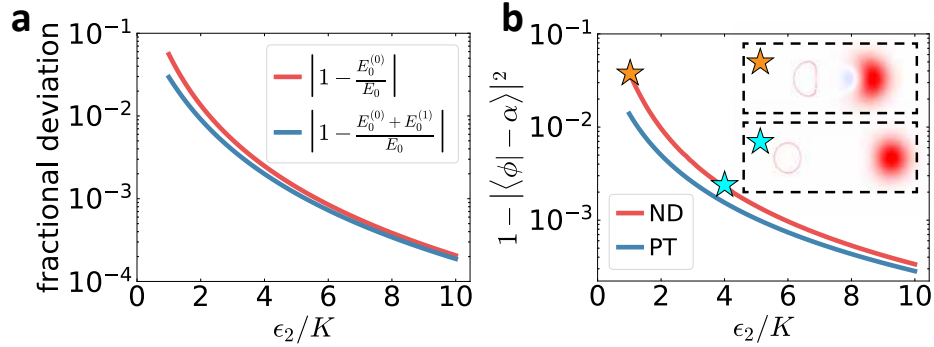


Figure 3.3: Detuned-Kerr-cat qubit ground state and its energy with $\Delta = 2K$. **a**, The fractional deviation of the ground state energy estimated by the unperturbed Hamiltonian and perturbation theory. **b**, The difference between the coherent state and the ground state of the detuned-Kerr-cat qubit Hamiltonian calculated by numerical diagonalization (ND) and perturbation theory (PT).

$\epsilon_2/K \sim 1 - 10$, the unperturbed ground state energy $E_0^{(0)}$ matches the exact ground state energy with less than 5% deviation, which is further improved slightly by the perturbation theory. Notably, the unperturbed ground-state energy can also be written as

$$E_0^{(0)} = \frac{(\Delta/2 + \epsilon_2)^2}{K} = K|\alpha|^4, \quad (3.11)$$

which is in the same form as the ground state energy of a resonant-Kerr-cat qubit. In fact, when $\Delta \rightarrow 0$, the perturbative corrections of the eigenenergies approach zero, and the unperturbed ground state energy will approach the ground state energy of the resonant-Kerr-cat qubit.

Secondly, the perturbed ground state (up to the first order) is given by

$$|\psi_0\rangle = \mathcal{N} \left(|0\rangle - \frac{\sqrt{2}\Delta}{4(4\epsilon_2 + \Delta)} |2\rangle \right), \quad (3.12)$$

where the perturbation introduces the occupation beyond the unperturbed ground states, and $\mathcal{N}$ is the normalization factor. After returning to the undisplaced frame, the ground state ends up approximately as the coherent state,

$$|\phi\rangle = \hat{D}^\dagger(\alpha)|\psi_0\rangle = \mathcal{N} \left(|-\alpha\rangle - \frac{\sqrt{2}\Delta}{4(4\epsilon_2 + \Delta)} \hat{D}^\dagger(\alpha)|2\rangle \right) \approx |-\alpha\rangle. \quad (3.13)$$

The difference between the coherent state $|-\alpha\rangle$ and the ground state $|\phi\rangle$ of the detuned-Kerr-cat qubit when $\Delta = 2K$ is shown in Fig. 3.3b. The Wigner functions of the ground

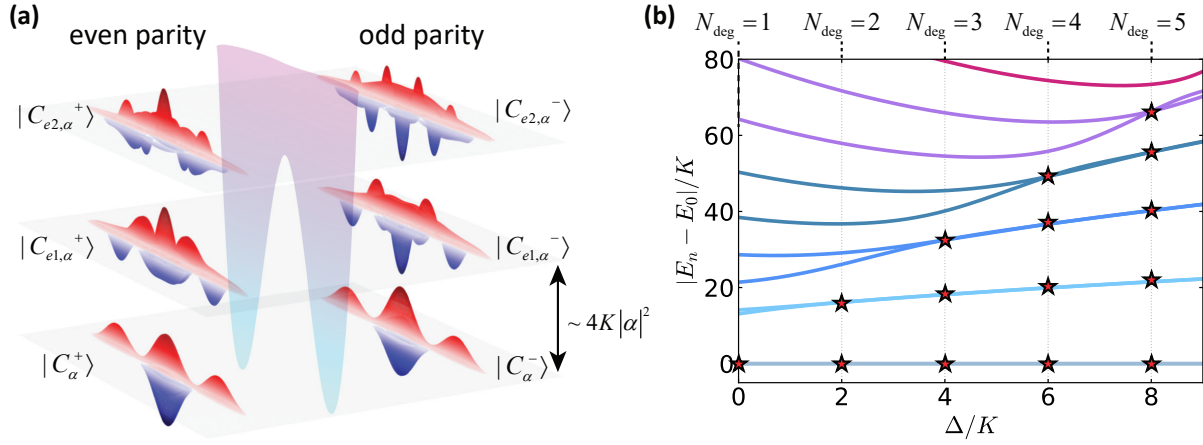


Figure 3.4: Higher excited eigenstates and eigenenergies of Kerr-cat qubits. (a) The pseudo-potential and the higher excited states. (b) The energy spectrum of the Kerr-cat qubit with different detuning Δ , where the degenerate point is marked by a star, and N_{deg} is the number of degeneracies.

states when $\epsilon_2/K = 1$ and $\epsilon_2/K = 4$ are also plotted in the inset indicated by the orange and blue stars. For non-zero positive detunings, the Wigner functions of the ground states are distorted from the coherent blobs to kidney-shaped structures. Nevertheless, both results from perturbation theory and numerical diagonalization indicate the rapidly decreasing difference between the coherent state and the detuned-Kerr-cat qubit ground state. Therefore, as long as the ϵ_2 is large enough, we can approximate the ground states of detuned-Kerr-cat qubits as coherent states and encode the cat qubit in a similar manner to resonant-Kerr-cat qubits. As an example, when $\epsilon_2 = 4K$ (blue star), the difference between the coherent state and the detuned-Kerr-cat qubit ground state is only 0.3%. Therefore, the analysis of coherent states and cat states for resonant-Kerr-cat qubits is still compatible with detuned-Kerr-cat qubits. While the above derivation employs the displacement operator $\hat{D}(\alpha)$ to obtain the eigenstate associated with $|\alpha\rangle$, the corresponding eigenstate for $|\alpha\rangle$ can be derived analogously by applying $\hat{D}(-\alpha)$.

As a summary, the encoding of Kerr-cat qubits is shown in Fig. 3.2 (b) with the states on three axes given by

$$|\pm Z\rangle = |\pm\alpha\rangle, \quad |\pm X\rangle = |C_\alpha^\pm\rangle, \quad |\pm Y\rangle = |\pm C_\alpha^{\pm i}\rangle \quad (3.14)$$

The Pauli matrices in the Kerr-cat qubit encoding are therefore defined as

$$\begin{aligned}
\hat{\sigma}_z &= |Z\rangle\langle Z| - |-Z\rangle\langle -Z| \\
\hat{\sigma}_x &= |X\rangle\langle X| - |-X\rangle\langle -X| \\
\hat{\sigma}_y &= |Y\rangle\langle Y| - |-Y\rangle\langle -Y|
\end{aligned} \tag{3.15}$$

Higher Excited States

The Kerr-cat qubits are encoded in the ground state manifold of the Kerr-cat Hamiltonian. However, the structure of the higher excited states is important for analyzing the noise mechanism of the qubit. Similarly to the ground states, the higher excited states also appear in pairs with even and odd parities, as shown in Fig. 3.4 (a). According to the perturbative analysis in the previous section, to the leading order when $\Delta \rightarrow 0$, we can write down the j -th pair of higher excited states as

$$|C_{ej,\alpha}^\pm\rangle = \mathcal{N}_{ej,\alpha}^\pm \left(\hat{D}(\alpha)|j\rangle \pm \hat{D}(-\alpha)|j\rangle \right) \tag{3.16}$$

where $|j\rangle$ is the j -th excited Fock state, and $\mathcal{N}_{ej,\alpha}$ is the associated normalization factor.

The eigenenergies of the j -th pair of higher excited states, to the leading order, can be given by $E_j = 4K|\alpha|^2 j$. Importantly, such eigenenergies indicate that the excited states are separated from the ground state manifold by an energy gap of around $E_{\text{gap}}/\hbar = 4K|\alpha|^2$, as shown in Fig.3.4 (a). This gap ensures the isolation of the ground state manifold for qubit encoding. Therefore, analogous to the role of anharmonicity in transmon qubits, this energy gap sets an upper bound on the gate speed within the ground-state manifold, beyond which significant leakage to higher excited states occurs. A rough estimation of the fastest gate in the ground-state manifold is given by $T_g \approx 2\pi\hbar/E_{\text{gap}}$.

Unlike the ground-state manifold, the higher excited-state pairs, although degenerate to leading order, are not exactly degenerate. Through numerical diagonalization of the Kerr-cat qubit Hamiltonian, one obtained the eigen energy spectrum with various detuning values Δ shown in Fig. 3.4 (b), where the two-photon drive strength $\epsilon_2 = 4K$. While the ground state manifold is always degenerate, the j -th excited state pairs are only degenerate when $\Delta = 2nK, n \geq j$, and the total number of degeneracies when $\Delta = 2nK$ is given by $N_{\text{deg}} = n + 1$. The energy differences between the even and odd states in a higher excited state pair are related to the tunneling rate across the energy barrier of the double-well pseudo-potential, and therefore influence the bit-flip time of the Kerr-cat qubits. As discussed in detail later, the bit-flip time of the Kerr-cat qubit can be significantly increased when $\Delta = 2nK, n = \{1, 2, \dots\}$.

3.2 Noise Structure

In this section, we firstly discuss the effect of an intrinsic non-ideality in the qubit encoding: $|\alpha\rangle$ and $|- \alpha\rangle$ are not orthogonal. Then, we will discuss the influence of four dominant noise

mechanisms on Kerr-cat qubits, including the single-photon decay, single-photon heating, pure dephasing, and non-Markovian low-frequency noises.

Non-orthogonality of coherent states

The conventional qubit uses the orthogonal Fock state $|0\rangle$ and $|1\rangle$ as its Z-axis state with $\langle 0|1\rangle = 0$. In contrast, the Kerr-cat qubit uses the non-orthogonal coherent states $|\pm\alpha\rangle$ as the Z-axis state, with the inner product of them given by

$$|\langle\alpha|-\alpha\rangle|^2 = e^{-4|\alpha|^2}, \quad (3.17)$$

As a result, even for a process $\hat{A}$ that does not perturb the Z-axis state, i.e.

$$\hat{A}|\pm\alpha\rangle = A_{\pm}|\pm\alpha\rangle \quad (3.18)$$

with $A_{\pm}$ being just a complex number rather than an operator, the Kerr-cat qubit will still develop an intrinsic bit-flip with the probability of $p_{\text{intrinsic}}$ given by

$$p_{\text{intrinsic}} = |\langle\mp\alpha|\hat{A}|\pm\alpha\rangle|^2 = |A_{\pm}|^2 e^{-4|\alpha|^2}. \quad (3.19)$$

This property is, in theory, harmful for Kerr-cat qubits. However, the overlap between two coherent states decreases rapidly with increasing cat size $|\alpha|^2$, leading to an exponentially small influence on the system. In fact, for a moderate cat size of $|\alpha|^2 = 2$, this overlap is $e^{-4|\alpha|^2} \approx 0.03\%$.

Single-photon Decay

The effect of single-photon decay is characterized by the jump operator $\hat{L} = \hat{a}$ with the decay rate of $\gamma^{\downarrow}$, and the relaxation time T_1 of the mode is given by $T_1 = 1/\gamma^{\downarrow}$. The effect of single-photon decay on the three axis-states are given by

$$\hat{a}|\pm Z\rangle = \pm\alpha|\pm Z\rangle, \quad \hat{a}|\pm X\rangle = \alpha|\mp X\rangle, \quad \hat{a}|\pm Y\rangle = \alpha|\mp Y\rangle \quad (3.20)$$

As shown, the single-photon decay does not influence the Z-axis state because the coherent states are eigenstates of the annihilation operator². Therefore, it will only introduce a bit-flip rate due to the intrinsic non-orthogonality of the coherent states, which is given by $\gamma^{\downarrow}|\alpha|^2 e^{-4|\alpha|^2}$. In contrast, the single-photon decay flips the X-axis and Y-axis states at an enhanced rate. Therefore, for a single-photon decay process with rate $\gamma^{\downarrow}$ and jump operator $\hat{L} = \hat{a}$, its effect can be described by a stochastic Pauli error process with the jump operator $\hat{L}_z = \hat{\sigma}_z$ with an error rate of $2\gamma^{\downarrow}|\alpha|^2$. On the Bloch sphere, this corresponds to a phase-flip process.

²For detuned-Kerr-cat qubits, the Z-axis states are not exact coherent states, but their deviations from coherent states are usually on the order of 1%, and therefore, the perturbation of the annihilation operator on them is minimal.

As a summary, the single-photon decay process will introduce a bit-flip rate of $\gamma^\downarrow |\alpha|^2 e^{-4|\alpha|^2}$ and a phase-flip rate of $2\gamma^\downarrow |\alpha|^2$.

Single-photon Heating

The single-photon heating process is described by a jump operator $\hat{L} = \hat{a}^\dagger$ and the error rate of $\gamma^\uparrow$. Usually, the error rate is proportional to the thermal photons n_{th} in the system and written as $\gamma^\uparrow = n_{\text{th}} \gamma_{\text{vac}}^\uparrow$. Noticing that the coherent states are not the eigenstates of the creation operator, the creation operator will excite the coherent states following

$$\hat{a}^\dagger |\alpha\rangle = \alpha^* |\alpha\rangle + \hat{D}(\alpha) |1\rangle = \alpha |\alpha\rangle + \hat{D}(\alpha) |1\rangle \quad (3.21)$$

where we have implicitly assume α is real. The creation operator will perturb the coherent state by introducing a leakage state $\hat{D}(\alpha) |1\rangle$ which is out of the computational space. This leakage state will have non-zero overlap with the higher excited state pairs. For non-degenerate excited state pairs, the non-zero tunneling rate across the energy barrier will introduce bit-flip errors. The contribution of the bit-flip errors given by this leakage mechanism is given by [187]

$$\gamma^\uparrow \sum_j \frac{\sum_{\pm} \left| \langle C_{ej,\alpha}^\pm | \hat{D}(\alpha) | 1 \rangle \right|^2}{2} \left(1 - \frac{\sin(\delta_j/\gamma_\downarrow)}{\delta_j/\gamma_\downarrow} \right) \quad (3.22)$$

where δ_j is the energy non-degeneracy of the j -th excited state pair. Therefore, when the j -th excited state pair is degenerate with δ_j , their contribution to the bit-flip error will be removed, and the bit-flip time will be increased. However, one should also notice that the contribution also scales as the overlap between the j -th excited state pair and the leakage state $\hat{D}(\alpha) |1\rangle$, which decreases rapidly as j increases. Therefore, people usually find that a detuning of $\Delta = 2K$ or $\Delta = 4K$ is significantly beneficial, while the effects of higher detunings become marginal.

Pure Mode Dephasing

As discussed above, pure dephasing arises from the averaged effect of random fluctuations in the mode frequency. Although each individual frequency fluctuation corresponds to a coherent evolution, ensemble averaging over these fluctuations leads to an effectively incoherent process that manifests as dephasing of the mode³. The Hamiltonian of the random fluctuation of the mode frequency is given by

$$\delta \hat{H} / \hbar = \delta \omega(t) \hat{a}^\dagger \hat{a}, \quad (3.23)$$

³The notion of pure mode dephasing here should not be confused with the phase-flip error of the Kerr-cat qubit discussed above.

where $\delta\omega(t)$ indicates the frequency fluctuation.

For conventional qubits encoded in the $|0\rangle$ and $|1\rangle$ Fock states, the evolution of the $|1\rangle$ state will be perturbed by this frequency fluctuation following

$$e^{-i \int_t \delta \hat{H} dt / \hbar} |1\rangle = e^{-i \int_t \delta \omega(t) dt} |1\rangle. \quad (3.24)$$

Therefore, any frequency fluctuation, no matter how strong and how fast the fluctuation is, will perturb the phase of the $|1\rangle$ state. By taking the average of the phase of the $|1\rangle$ state and assuming that $\delta\omega$ is a Gaussian noise, we can obtain the phase decay formula given by

$$\mathbb{E} \left[e^{-i \int_t \delta \omega(t) dt} \right] = e^{-\gamma_\phi t} \quad (3.25)$$

and

$$\gamma_\phi = \frac{1}{\pi t} \int_0^\infty d\omega S_\omega(\omega) \frac{\sin^2(\omega t/2)}{\omega^2} \quad (3.26)$$

where $S_\omega(\omega) = \int_{-\infty}^\infty d\tau \mathbb{E}[\delta\omega(t)\delta\omega(t+\tau)] e^{i\omega\tau}$ is the noise spectrum of $\delta\omega$, and $\mathbb{E}[\delta\omega(t)\delta\omega(t+\tau)]$ is the correlation function of $\delta\omega$. For the Markovian white noise limit, where the correlation function is a delta-function, $\mathbb{E}[\delta\omega(t)\delta\omega(t+\tau)] = \delta(\tau)$, the phase decay factor γ_ϕ becomes a constant $\gamma_\phi = \frac{1}{2}S_\omega(0)$ and we can therefore end up with the Lindbladian for dephasing noise with a jump operator of $\hat{L} = \hat{a}^\dagger \hat{a}$ and the dephasing rate $\gamma_\phi = \frac{1}{2}S_\omega(0)$.

However, the effect of frequency fluctuation on cat qubit states is fundamentally different from that on Fock qubit states. In a Fock-state encoding, frequency fluctuations translate directly into phase errors by altering the relative phases of the basis states. By contrast, cat-qubit states can exhibit robustness against small and slowly varying frequency fluctuations, since the encoded states adiabatically track the instantaneous eigenstates. Considering a Kerr-cat qubit Hamiltonian with a random frequency fluctuation, shown below

$$\hat{H}(t) = (\delta\omega(t) + \Delta)\hat{a}^\dagger \hat{a} + \epsilon_2 \hat{a}^{\dagger 2} + \epsilon_2^* \hat{a}^2 - K \hat{a}^{\dagger 2} \hat{a}^2, \quad (3.27)$$

the encoded states are the cat states $|C_\alpha^\pm\rangle$, imaginary cat states $|C_\alpha^{\pm i}\rangle$ and coherent states $|\pm\alpha\rangle$, and $\alpha = \sqrt{(\epsilon_2 + \Delta/2 + \delta(t)\omega/2)/K}$ is dependent on the frequency fluctuation. Therefore, as long as the encoded states can track the instantaneous eigenstates of the Hamiltonian in the presence of the fluctuation, the qubit evolution proceeds as expected without accumulating errors.

According to the adiabatic theorem, for a time-dependent Hamiltonian with instantaneous eigenstates $\psi_1(t)$ and $\psi_2(t)$ and instantaneous eigenenergies E_1 and E_2 , the transition probability $\Gamma_{1 \rightarrow 2}$ from state $\psi_1(t)$ to $\psi_2(t)$ with this time-varying Hamiltonian is [2]

$$\Gamma_{1 \rightarrow 2} = \left| \frac{\hbar \langle \psi_1 | \frac{\partial \hat{H}(t)}{\partial t} | \psi_2 \rangle}{(E_1 - E_2)^2} \right|^2, \quad (3.28)$$

and in order to allow the states to follow their instantaneous eigenstates, this transition rate has to be much smaller than 1.

For Kerr-cat qubits, the higher excited states are separated from the ground manifold by an energy gap of $E_{\text{gap}}/\hbar \approx 4K|\alpha|^2$. Therefore, we would require

$$\frac{\partial\omega(t)}{\partial t} \ll \frac{E_{\text{gap}}^2}{\hbar^2}, \quad (3.29)$$

so that the Kerr-cat qubit states will follow their instantaneous eigenstates. As a result, only the fluctuation that varies mode frequency fast enough, i.e., the high-frequency pure dephasing noises, will perturb the encoded cat qubit states, and the low-frequency pure dephasing noises will have minimal effects. Such protection against low-frequency pure dephasing noise is reminiscent of the dynamical decoupling technique, which uses “echo” pulses to correct the low-frequency dephasing noises[136].

Therefore, to describe the effect of pure dephasing on Kerr-cat qubits, we should use the same jump operator $\hat{L} = \hat{a}^\dagger \hat{a}$ but with a different dephasing rate modified by the protection against low-frequency pure dephasing noises γ'_ϕ , and typically, we have

$$\gamma'_\phi < \gamma_\phi \quad (3.30)$$

Non-Markovian Low-frequency Noises

Even though the encoded cat qubit states can follow the instantaneous eigenstates under low-frequency pure dephasing noises and avoid the associated higher excited state populations, such low-frequency noises still influence the bit-flip time of the cat qubits. As mentioned before, the bit-flip error contribution from single-photon heating is related to the degeneracy of the higher excited states, depending on the detuning value Δ . Therefore, under finite temperature with non-zero heating rate $\gamma^\uparrow$, the bit-flip time T_z is a function of the detuning Δ . $T_z(\Delta)$, and has peaks when $\Delta = 2nK, n = \{1, 2, \dots\}$, as shown in Fig. 3.5.

When the system suffers from a low-frequency noise that varies the mode frequency by a random variable $\delta\omega$, it will also vary the detuning Δ by the same random variable, which is shown as

$$\tilde{\Delta} = \Delta + \delta\omega. \quad (3.31)$$

As a result, when the encoded cat qubit states follow the instantaneous eigenstates with $\tilde{\Delta}$, the effective bit-flip time $\bar{T}_z$ being observed will be the average of the lifetime with the varying detuning,

$$\bar{T}_z = \mathbb{E} \left[T_z(\tilde{\Delta}) \right]. \quad (3.32)$$

The effect of this low-frequency noise cannot be adequately captured by a Lindblad master equation formulated in terms of jump operators and constant error rates. We therefore classify it as non-Markovian noise. Its dominant impact is the suppression and broadening

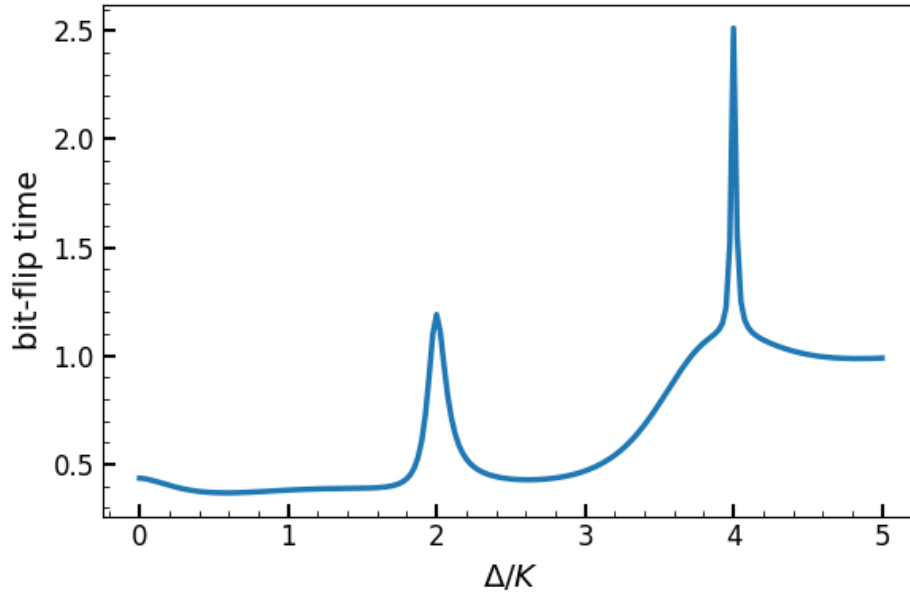
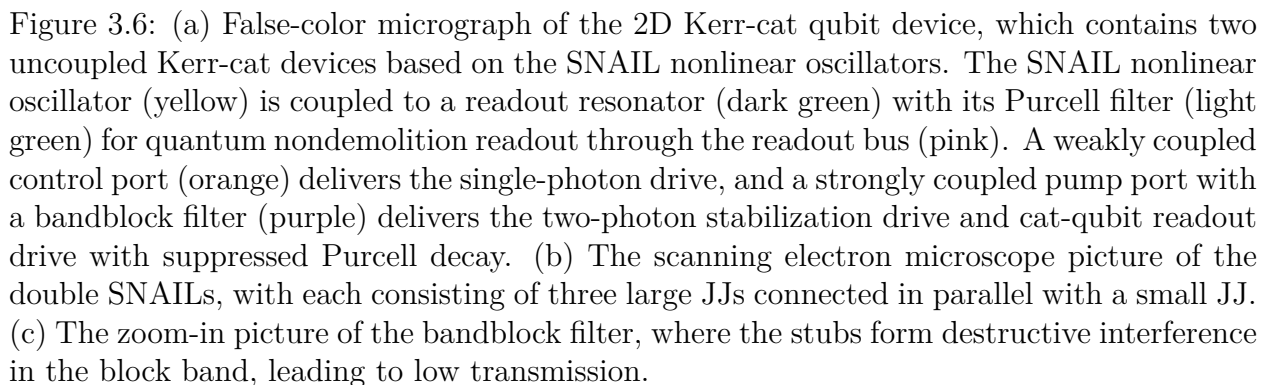


Figure 3.5: Bit-flip times of Kerr-cat qubits with different detunings without non-Markovian errors.

of the bit-flip time peaks at $\Delta = 2nK$. In the experiments discussed later, incorporating this low-frequency noise is essential for accurately predicting the bit-flip time and achieving good agreement between the theoretical model and experimental results.

3.3 Kerr-cat Qubit Implementation

With the understanding of the Kerr-cat qubit Hamiltonian, encoding, and noise structure, we proceed to the implementation of the Kerr-cat qubit Hamiltonian in a 2D superconducting circuit. We begin with a general overview of the Kerr-cat qubit device layout, followed by a description of its key components and their respective functions, including the SNAIL nonlinear oscillator, the readout resonator, the strongly coupled microwave bus, and the weakly coupled microwave bus. The SNAIL nonlinear oscillator provides the engineered nonlinearities required to encode the cat-qubit states. The readout resonator enables high-fidelity, quantum-nondemolition (QND) measurement of the cat-qubit states. The strongly and weakly coupled microwave buses deliver the various drive tones necessary for stabilization and coherent control of the cat qubits. Finally, we discuss the initialization of the cat qubits via adiabatic mapping and heralded preparation.



The Kerr-cat qubit is implemented on a planar superconducting circuit device. The false-color micrograph of the device chip is shown in Fig. 3.6 (a) and the equivalent circuit

diagram of the device is shown in Fig. 3.6 (d). The key component of the Kerr-cat qubit device, SNAIL nonlinear oscillator (yellow), consists of two SNAILs shunted by a coplanar capacitor, where each SNAIL is formed by three large Josephson junctions connected in parallel with a small Josephson junction, as shown in Fig. 3.6 (b). A readout resonator (dark green) is capacitively coupled to the SNAIL nonlinear oscillator with a dispersive shift χ to enable the quantum nondemolition readout of the Kerr-cat qubit, and a Purcell filter (light green) is introduced between the readout resonator and the readout port (pink) to engineer the desired readout resonator decay rate κ_r with suppressed Purcell decay. We engineered two microwave ports to deliver the microwave efficiently without introducing too much Purcell decay. On the one hand, a control port (orange) is weakly coupled to the SNAIL nonlinear oscillator to deliver the single-photon drive on resonance with the SNAIL nonlinear oscillator frequency. On the other hand, a pump port (purple) with a bandblock filter is strongly coupled to the SNAIL nonlinear oscillator to deliver the two-photon stabilization drive and the cat-qubit readout (CQR) drive, where the bandblock filter strongly suppresses the Purcell decay from this port by blocking the signal around the qubit frequency. A zoom-in picture of the bandblock filter structure is shown in Fig. 3.6 (c), where the stubs form destructive interference for the signals in the block band and reduce the transmission. We will introduce the functions of the components on this chip in the following sections. Finally, an external flux coil is anchored on the sample package to provide a DC flux bias to the SNAIL.

Table 3.1: Device Parameters and Calibration Methods

Parameter	Value	Methods
SNAIL oscillator frequency $\omega_Q/(2\pi)$	5.9 GHz	two-tone spectroscopy
SNAIL oscillator Kerr $K/(2\pi)$	1.2 MHz	two-tone spectroscopy
SNAIL large junction $E_J/(2\pi)$	263.2 GHz	room-temp. resistance[188]
SNAIL small-to-large junction ratio β	0.1	room-temp. resistance[188]
SNAIL oscillator charge energy $E_c/(2\pi)$	118 MHz	finite-element-method simu.
SNAIL oscillator relaxation time T_1	40 μs	time-domain coherence
SNAIL oscillator dephasing time T_2	3.2 μs	time-domain Ramsey[189]
SNAIL oscillator thermal population n_{th}	5%	T_z fitting
Readout resonator frequency $\omega_R/(2\pi)$	7.1 GHz	microwave reflectometry
Readout resonator relaxation rate $\kappa_R/(2\pi)$	0.4 MHz	microwave reflectometry
Readout resonator dispersive shift $\chi/(2\pi)$	40 kHz	time-domain spectroscopy

The parameters of the device and the calibration/estimation methods are described in Tab. 3.1. The calibration procedure will be illustrated later.

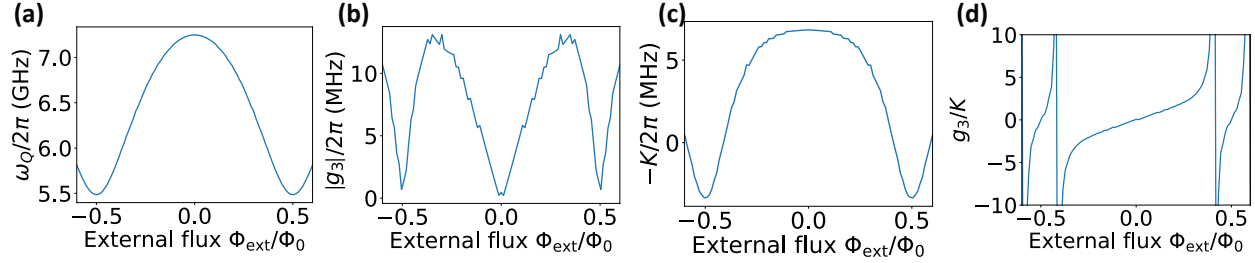


Figure 3.7: (a) SNAIL nonlinear oscillator frequency ω' , (b) Kerr-nonlinearity K , (c) third-order nonlinearity g_3 and (d) the ratio between g_3 and K with respect to different external flux Φ_{ext}

SNAIL Nonlinear Oscillator

As we mentioned in previous sections, the SNAIL nonlinear oscillator has a Hamiltonian with both fourth-order and third-order nonlinearities, given by

$$\hat{H}_{\text{SNAIL}}/\hbar = \omega a^\dagger a + g_3 (a + a^\dagger)^3 + g_4 (a + a^\dagger)^4. \quad (3.33)$$

Therefore, one can engineer two-photon drives by capacitively driving it off-resonantly at a frequency ω_S close to twice its resonant frequency, as described by the Hamiltonian below

$$\hat{H}_d = -2i\epsilon_0 \cos(\omega_S t)(\hat{a} - \hat{a}^\dagger). \quad (3.34)$$

Following the procedure mentioned in 2.3, we can apply the unitary transformation with $U = e^{\xi \hat{a}^\dagger - \xi^* \hat{a}}$ and $\xi = -i\epsilon_0 \left(\frac{e^{-i\omega_S t}}{\omega - \omega_S} + \frac{e^{i\omega_S t}}{\omega + \omega_S} \right)$ to move to the displaced frame, and then apply the unitary transformation with $U = e^{i\omega_S t/2}$ to move to the rotating frame and apply RWA, and finally, we will end up with the Kerr-cat qubit Hamiltonian shown below

$$\hat{H}_{\text{KCQ}} = \Delta \hat{a}^\dagger \hat{a} + \epsilon_2 \hat{a}^{\dagger 2} + \epsilon_2^* \hat{a}^2 - K \hat{a}^{\dagger 2} \hat{a}^2, \quad (3.35)$$

where $\Delta = \omega_Q - \frac{\omega_S}{2}$ is the detuning between the re-normalized SNAIL nonlinear oscillator frequency and half of the two-photon drive frequency, and $\epsilon_2 = 3g_3\xi_0$ is the two-photon drive strength. $K = -6g_4$ is the Kerr-nonlinearity, $\omega_Q = \omega + 24g_4|\xi_0|^2 + 12g_4$ is the re-normalized SNAIL nonlinear oscillator frequency shifted by the static fourth-order terms and AC-Stark shift due to the two-photon drive, and $\xi_0 = i\epsilon_0 \left(\frac{1}{\omega - \omega_S} - \frac{1}{\omega + \omega_S} \right)$. Noticing that the Kerr-cat qubit can only be engineered with $K > 0$, we must require that the SNAIL nonlinear oscillator has $g_4 < 0$. Using the parameters from Tab. 3.1, we can extract the SNAIL nonlinear oscillator frequency ω_Q , Kerr-nonlinearity K , third-order nonlinearity g_3 , as well as the ratio between g_3 and K with respect to different external flux Φ_{ext} , as shown in Fig. 3.7. Importantly, for suitable choices of external flux, one can obtain $g_3 \gg |K|$, leading to efficient three-wave mixing and two-photon drive. As a result, a large cat size

$|\alpha|^2 = \epsilon_2/K = 3g_3\xi_0/K$ can be achieved without unreasonably large ξ_0 . However, the microwave drive strength ϵ_0 may still be large due to the small factor $\frac{1}{\omega-\omega_S}$. Therefore, we engineered the strongly coupled port to deliver such a microwave drive efficiently.

Following the convention of the Kerr-cat qubit community, we will hereafter refer to the two-photon drive ϵ_2 as the “stabilization drive” because such a two-photon drive “stabilized” the cat qubit states.

Microwave Buses

The operation of the Kerr-cat qubit involves the microwave delivery at various frequencies, including the near-resonant frequencies at $\omega_S/2$ and ω_Q , and off-resonant frequencies at $\omega_S \approx 2\omega_Q$ and $\omega_{CQR} = \omega_R - \omega_S/2$. Therefore, we engineered two different microwave buses, one strongly coupled bus with a bandblock filter to deliver off-resonant drives while suppressing the Purcell decay near the frequency of ω_Q , and the other weakly coupled bus without any filter to deliver near-resonant drives.

Strongly Coupled Bus and Bandblock Filter

The two-photon stabilization drive and the latter discussed cat-qubit readout (CQR) drive are far off-resonant with the SNAIL nonlinear oscillator, and their effects are attenuated by the factor $\frac{1}{\omega-\omega_d}$, $\omega_d = \{\omega_S, \omega_{CQR}\}$ in the displacement transformation. Therefore, to achieve enough microwave delivery required for Kerr-cat qubit stabilization, we engineered a strongly coupled port (purple) to deliver such a signal efficiently.

The key challenge of a strongly coupled port is the Purcell decay of the SNAIL nonlinear oscillator mode through this port. To mitigate the Purcell decay while keeping efficient signal delivery, we engineered a bandblock filter to prevent the signal transmission at the SNAIL nonlinear oscillator frequency ω_Q while allowing the signals at the other frequencies to propagate, as shown in Fig. 3.6 (c).

A zoom-in view into one of the stubs of the filter is shown in Fig. 3.8 (c). The open boundary condition requires this filter design to be $\lambda/4$ to have destructive interference inside the stop band of the filter [152]. Fig. 3.8 (d) shows a visualization of this process.

Ideally, the filter should have high isolation at the qubit frequency ($\omega_Q/2\pi = 5.9$ GHz) with low attenuation at other frequencies. In the experiment, the two-photon stabilization drive at ω_S and the cat quadrature readout drive at ω_{CQR} ⁴ will be delivered through this strongly coupled bus. Therefore, specifically, the filter should have minimal attenuation at the frequency of the two-photon stabilization drive ($\omega_S/2\pi = 11.8$ GHz) and the frequency of the cat quadrature readout ($\omega_{CQR}/2\pi = 1.2$ GHz). In Fig. 3.8, we present a measurement of the transmission coefficient S_{21} of the filter in a dedicated chip. In Fig. 3.8 (a), we show the behavior of the normalized scattering parameter S_{21} for the full range of interest. The 30 dB of isolation at the qubit frequency corresponds to a three-order of magnitude suppression

⁴This is an important off-resonant drive for cat quadrature readout being discussed later.

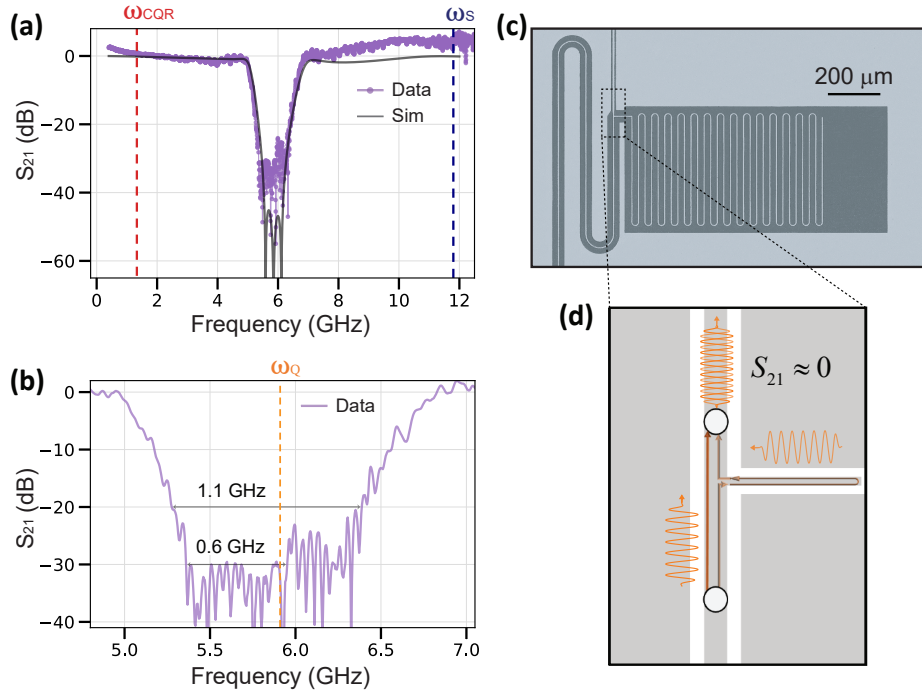


Figure 3.8: (a) The measured performance for the full range of the filter (purple). The dark grey line is the simulated performance using finite-element simulation. (b) Shows a zoomed view around the qubit frequency with the 20,30 dB attenuation ranges. (c) A zoomed view into one stub of the filter, where the length is chosen to have destructive interference around the qubit frequency. (d) The destructive interference results from the π phase that the signal picks from the two paths.

of the Purcell decay, pushing the Purcell limit energy relaxation time through the strongly coupled port to 12 ms. In Fig. 3.8 (b), we zoom in around the qubit frequency to examine the range of 20 dB and 30 dB of isolation. The large bandwidth of this filter makes it suitable for multi-qubit chips with strong qubit-qubit coupling while maintaining a negligible Purcell effect and efficient pumping for all the qubits. This filter can also be incorporated in any circuit QED experiment in which a strong off-resonance drive is required to implement the desired interaction.

Weakly Coupled Bus

In the operation and initial characterization of Kerr-cat qubits, we also need to apply near-resonant drives at ω_Q and $\omega_S/2$. Such drives will be blocked by the bandblock filter mentioned above and therefore cannot be delivered through the strongly coupled bus. Moreover, since these near-resonant drives have similar frequencies to the SNAIL nonlinear oscillator

frequencies, it is challenging to engineer dedicated filters to suppress the Purcell decay.

However, unlike off-resonant drives, the effect of near-resonant drives does not rely on multi-wave mixing processes or displacement transformations that introduce the small suppression factor $\frac{1}{\omega_Q - \omega_d}$. Consequently, the near-resonant drive amplitudes required in experiments are typically weak. This allows the use of a weakly coupled port, such as the yellow-colored bus shown in Fig. 3.6(a), to deliver the near-resonant drives without introducing significant Purcell decay.

Readout Resonator and Cat Quadrature Readout

The coupling between the readout resonator and the SNAIL nonlinear oscillator can be described by the JC coupling model given by

$$\hat{H}_{\text{KR}} = g(\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}) \quad (3.36)$$

where $\hat{a}$ ($\hat{a}^\dagger$) and $\hat{b}$ ($\hat{b}^\dagger$) denote the annihilation (creation) operators of the SNAIL nonlinear oscillator and the readout resonator, respectively. A Purcell filter is inserted between the readout resonator and the readout bus. This dedicated Purcell filter is engineered to optimize transmission at the readout resonator frequency while strongly attenuating signals at the SNAIL oscillator frequency. As a result, the energy relaxation time T_1 of the SNAIL nonlinear oscillator remains mostly unaffected by the readout chain.

The readout resonator has two functions for Kerr-cat qubits corresponding to the two coupling regimes discussed previously: dispersive coupling and multi-wave mixing. The dispersive coupling provided the readout of Fock states, which is essential for the initial characterization of the sample parameters, and the multi-wave mixing offered high-fidelity QND readout of the cat qubit states, which is necessary for high-performance quantum computing with Kerr-cat qubits.

Dispersive Readout

As mentioned in section 2.3, the coupling between the readout resonator and the SNAIL nonlinear oscillator can be described by an effective dispersive coupling Hamiltonian,

$$\hat{H}_{\text{dispersive}} = \chi \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \quad (3.37)$$

where $\chi = 2g^2 \left(\frac{1}{\omega_R - \omega_Q + 2K} - \frac{1}{\omega_R - \omega_Q} \right)$ and ω_R is the readout resonator frequency. The anharmonicity of the SNAIL nonlinear oscillator is given by twice the Kerr nonlinearity. For Kerr-cat qubits, the anharmonicity $2K = 2 \times 1.2$ MHz and the dispersive shift $\chi = 40$ kHz are much smaller than those of transmon qubits because engineering cat qubits requires a large g_3/K ratio and therefore small K and χ .

The dispersive readout of the SNAIL nonlinear oscillator is inefficient with such a small dispersive shift, and therefore, one needs to take many shots and calculate the average of them to obtain a reliable readout result. Thus, unlike transmon qubits, one cannot achieve

single-shot high-fidelity dispersive readout for Kerr-cat qubits. Such dispersive readout is only used for the initial characterization of the sample parameters, such as the SNAIL oscillator frequency, Kerr-nonlinearity, T_1 , and T_2 , rather than quantum computing with Kerr-cat qubits.

Cat Quadrature Readout

On the other hand, as mentioned in section 2.3, one can apply the cat quadrature readout (CQR) drive to the SNAIL nonlinear oscillator at the frequency difference between the rotating frame reference and the readout resonator frequency, $\omega_{CQR} = \omega_R - \omega_S/2$, and engineer the beam-splitter interaction through the three-wave mixing process to enable the Z-axis measurement of the Kerr-cat qubit, as shown by the Hamiltonian below.

$$\hat{H}_{CQR} = 6g_3 \frac{g}{\omega_Q - \omega_R} \left(\xi_{CQR}^* \hat{a}^\dagger \hat{b} + \xi_{CQR} \hat{a} \hat{b}^\dagger \right) = g_{CQR}^* \hat{a}^\dagger \hat{b} + g_{CQR} \hat{a} \hat{b}^\dagger \quad (3.38)$$

The readout resonator mode will be populated by this Hamiltonian. To solve the dynamics of the readout resonator under such a Hamiltonian, one can apply the quantum Langevin equation. Assuming the energy relaxation rate of the readout resonator mode is κ_b , we obtained

$$\begin{aligned} \partial_t \hat{b} &= i \left[\hat{H}_{CQR}, \hat{b} \right] - \frac{\kappa_b}{2} \hat{b} + \sqrt{\kappa_b} \hat{b}_{\text{in}} \\ &\approx g_{CQR} \hat{a} - \frac{\kappa_b}{2} \hat{b} + \sqrt{\kappa_b} \hat{b}_{\text{in}} \end{aligned} \quad (3.39)$$

If we set $\hat{b}_{\text{in}} = 0$ since we do not apply any input signal at the resonator frequency, we can solve for the evolution of the resonator mode

$$\hat{b} = \frac{2g_{CQR}}{\kappa_b} \hat{a} \left[1 - e^{-\kappa_b t/2} \right] = \frac{2g_{CQR}}{\kappa_b} \hat{a} \Big|_{t \rightarrow \infty} \quad (3.40)$$

Clearly, the resonator mode will be populated in different states depending on the SNAIL nonlinear oscillator mode. Therefore, by measuring the readout resonator field $\langle \hat{b} \rangle$, one can obtain the average value of the SNAIL nonlinear oscillator mode $\langle \hat{a} \rangle$. In Kerr-cat qubit encoding, the SNAIL nonlinear oscillator mode for the Z-axis states $|\pm Z\rangle = |\pm \alpha\rangle$ is given by

$$\langle \hat{a} \rangle = \langle \pm \alpha | \hat{a} | \pm \alpha \rangle = \alpha \langle \hat{\sigma}_z \rangle \quad (3.41)$$

where $\langle \hat{\sigma}_z \rangle$ is the average value of the Pauli Z operator, with $\langle \hat{\sigma}_z \rangle = \pm 1$ for $|\pm \alpha\rangle$. As a result, the resonator state will approach two points in the phase plane depending on the Z-axis state of the Kerr-cat qubits, and by detecting the resonator state, one can easily measure the qubit along its Z axis. We hereafter refer to this readout as the cat quadrature readout (CQR).

Unlike dispersive readout, where the readout fidelity is intrinsically limited by the low dispersive shift χ , one can increase the readout fidelity of CQR by increasing CQR drive to achieve a large separation of the readout resonator states for $|+\alpha\rangle$ and $|-\alpha\rangle$, because the beam-splitter interaction strength g_{CQR} is controlled by the CQR drive strength. Therefore, one can usually achieve high-fidelity Z-axis readout of the Kerr-cat qubit through CQR. However, increasing the CQR drive too much will decrease the QNDness. The CQR process depletes the photons stored in the SNAIL nonlinear oscillator by swapping them into the readout resonator for detection. Consequently, the readout can be considered quantum nondemolition only if the photons removed by the CQR are sufficiently replenished by the two-photon stabilization drive ϵ_2 , leading to the condition $g_{CQR} \ll \epsilon_2$. In Kerr-cat qubit experiments, there is usually a wide range of g_{CQR} where one can obtain both large enough readout resonator state separation and QNDness. Since the CQR drive frequency is far off-resonant with ω_Q and relies on higher order nonlinearities, it is delivered through the strongly coupled bus, and the signal transmission at ω_{CQR} for the filter is shown in Fig. 3.8 (a).

Although CQR can read out the Z-axis states of the Kerr-cat qubit, it cannot distinguish the higher excited states from the ground states. For example, the SNAIL nonlinear oscillator mode for the ground state $|\alpha\rangle$ and higher excited state $\hat{D}(\alpha)|1\rangle$ is the same as shown below

$$\begin{aligned}\langle\alpha|\hat{a}|\alpha\rangle &= \alpha \\ \langle 1|\hat{D}^\dagger(\alpha)\hat{a}D(\alpha)|1\rangle &= \langle 1|\hat{a}|1\rangle + \alpha\langle 1|1\rangle = \alpha\end{aligned}\tag{3.42}$$

Therefore, CQR can only determine in which well of the double-well pseudo-potential the Kerr-cat qubit state resides, but cannot distinguish the ground and excited states in the same pseudo-potential well. One has to be cautious when using CQR when excited states are involved.

Cat-qubit Initialization

The Kerr-cat qubit can be initialized either through adiabatic mapping or heralding. The adiabatic mapping can initialize the Kerr-cat qubit into any state, but suffers from the control errors and limited lifetimes of the Fock states. The heralding-based initialization can prepare the Kerr-cat qubit into Z-axis states ($|\pm\alpha\rangle$) using CQR, which is only limited by the QNDness of CQR.

Adiabatic Mapping

Considering the Kerr-cat qubit Hamiltonian with time-varying two-photon stabilization drive $\epsilon_2(t)$,

$$\hat{H}_{KCQ}/\hbar = \Delta\hat{a}^\dagger\hat{a} + \epsilon_2(t)\hat{a}^{\dagger 2} + \epsilon_2(t)^*\hat{a}^2 - K\hat{a}^{\dagger 2}\hat{a}^2\tag{3.43}$$

when $\epsilon_2(t) = 0$, the Hamiltonian reduced to a nonlinear oscillator Hamiltonian

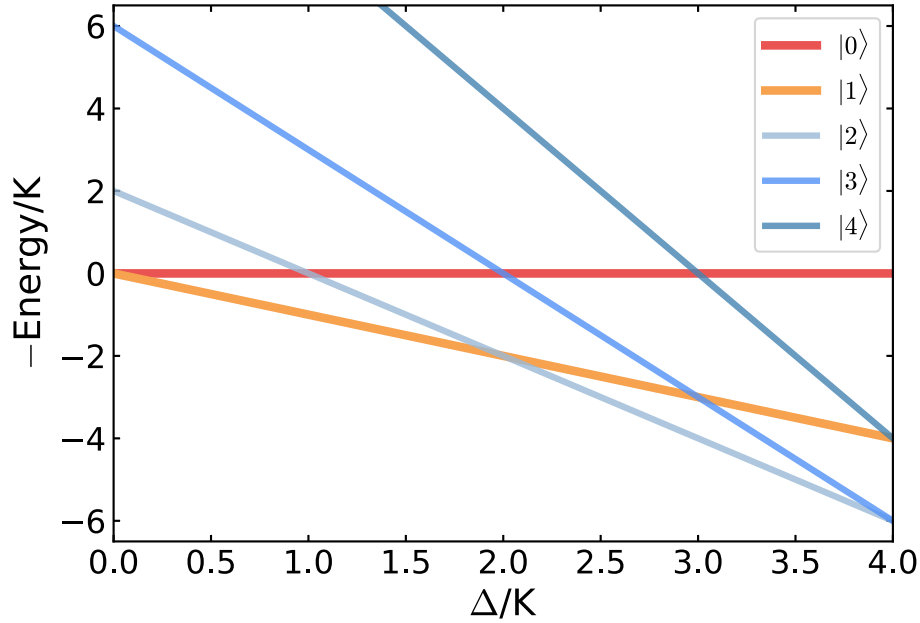


Figure 3.9: The SNAIL nonlinear oscillator spectrum in the rotating frame as a function of detuning with $\epsilon_2(t) = 0$

$$\hat{H}_{\text{KCQ}} \Big|_{\epsilon_2(t)=0} = \Delta \hat{a}^\dagger \hat{a} - K \hat{a}^{\dagger 2} \hat{a}^2 \quad (3.44)$$

and the Fock states will be the eigenstates of this Hamiltonian. Therefore, by adiabatically ramping up the $\epsilon_2(t)$, one can adiabatically map the Fock states into the corresponding eigenstates of the Kerr-cat qubit Hamiltonian and initialize the Kerr-cat qubit into a desired state.

The mapping between the Fock states and the Kerr-cat qubit states can be analyzed as follows. The X-axis states of the Kerr-cat qubit, $|\pm X\rangle = |C_\alpha^\pm\rangle$, have definite parity. Moreover, they correspond to the ground states of $\hat{H}_{\text{KCQ}}$, yet possess the highest energy in the rotating frame. It is important to note that the notion of a ground state in the rotating frame differs from that in the laboratory frame, which explains why the ground states here correspond to the highest, rather than the lowest, energy eigenvalues. Therefore, as $\epsilon_2(t)$ is adiabatically ramped from zero to a finite value, the two highest-energy eigenstates of $\hat{H}_{\text{KCQ}}|_{\epsilon_2(t)=0}$ will be mapped into $|\pm X\rangle = |C_\alpha^\pm\rangle$ with the same parity, since the adiabatic evolution preserves both the parity symmetry and the ordering of the energy levels.

The energies of several Fock states of Hamiltonian $\hat{H}_{\text{KCQ}}|_{\epsilon_2(t)=0}$ with respect to different detuning values Δ is shown in Fig. 3.9. Therefore, when $\Delta \in [0, K)$, $|0\rangle$ and $|1\rangle$ states have the highest energy and will be mapped to $|C_\alpha^+\rangle$ and $|C_\alpha^-\rangle$. People have used this method to

initialize a resonant-Kerr-cat qubit into $|C_\alpha^+\rangle$ by adiabatically ramping up ϵ_2 with the SNAIL nonlinear oscillator in vacuum $|0\rangle$ [62]. However, when $\Delta \in (K, 3K)$, $|2\rangle$ and $|1\rangle$ states have the highest energy, and one needs to excite the SNAIL nonlinear oscillator into the $|1\rangle$ state to adiabatically map it to $|C_\alpha^+\rangle$, leading to significant errors from the T_1 decay and control errors of the Fock states. Therefore, adiabatic mapping is usually only used for resonant-Kerr-cat qubits rather than detuned-Kerr-cat qubits. The author noticed that an initialization method based on a chirp pulse has been reported recently [52, 190], which initializes the detuned-Kerr-cat qubits by firstly mapping the vacuum state to a resonant-Kerr-cat qubit state, and then adiabatically modulating the two-photon stabilization drive frequency to map it to the detuned-Kerr-cat qubit state. This method provides a fast initialization method for detuned-Kerr-cat qubits, but the initialization fidelity is still limited by the coherence and adiabatic operation errors.

Heralding

The CQR provides a high-fidelity quantum nondemolition (QND) readout for Kerr-cat qubits and can be used to initialize the qubit along the Z-axis via heralding. However, the CQR can only distinguish Kerr-cat states residing in different potential wells; it cannot resolve the ground and excited states within the same well. Therefore, prior to applying the CQR to project the state, it is necessary to ensure that the Kerr-cat states remain within the ground-state manifold.

Typically, the SNAIL nonlinear oscillator will stay in the vacuum state $|0\rangle$ before turning on the two-photon stabilization drive ϵ_2 . Therefore, if $\Delta \in [0, K)$, after ramping up the two-photon stabilization drive, the Kerr-cat state will remain in the ground manifold, and one can apply CQR immediately to project the state into either $|+Z\rangle = |\alpha\rangle$ or $|-Z\rangle = |-\alpha\rangle$. However, for the detuned-Kerr-cat qubits with $\Delta > K$, the detuning changes the energies of SNAIL nonlinear oscillator states in the rotating frame, as shown in Fig. 3.9, making $|0\rangle$ no longer the ground state. Consequently, when the two-photon stabilization drive is ramped up, the $|0\rangle$ state is mapped to the higher excited states instead of the ground states. The subsequent CQR drive will then collapse the qubit into excited states in one of the double wells of the pseudo-potential, resulting in a failure to initialize the qubit into $|\pm Z\rangle$ states. To address this, it is crucial to introduce sufficient relaxation time before applying the CQR drive, allowing the system to decay into the degenerate ground states. This ensures the validity of preparing the $|\pm Z\rangle = |\pm \alpha\rangle$ state through heralding.

Universal Single Qubit Gates

A universal gate set for the Kerr cat qubit consists of continuous $Z(\theta)$ gates and a discrete $X(\pi/2)$ gate on the Bloch sphere. Here, we describe the implementation of these gates.

Continuous $Z(\theta)$ Gates

The continuous $Z(\theta)$ gates are engineered by a single-photon drive at half of the stabilization drive frequency $\omega_S/2$ according to⁵

$$\hat{H}_d = \frac{\Omega}{2}\hat{a}^\dagger + \frac{\Omega^*}{2}\hat{a} \quad (3.45)$$

This single-photon drive shifts the energy of the two pseudo-potential wells, as shown in Fig. 3.10 (a), which introduces an energy difference between the two Z -axis states $|\pm\alpha\rangle$ as shown below

$$\begin{aligned} \langle\alpha|\hat{H}_d|\alpha\rangle &= \text{Re}[\Omega]\alpha \\ \langle-\alpha|\hat{H}_d|-\alpha\rangle &= -\text{Re}[\Omega]\alpha \end{aligned} \quad (3.46)$$

This energy shift is analogous to the generator of $\{Z(\theta), \theta = \Omega_c t\}$ rotation gates, which corresponds to a rotation of the qubit states along the Z -axis at a Rabi rate of $\Omega_c = 2\text{Re}[\Omega]\alpha$, as shown in Fig. 3.10 (b). As a result, such a single-photon drive at frequency $\omega_S/2$ will realize continuous $Z(\theta)$ gates with any desired θ by adjusting the drive amplitude and width.

Importantly, the analysis above is only valid when the energy shift stays much smaller than the barrier energy of the pseudo-potential to preserve its double-well structure,

$$\text{Re}[\Omega]\alpha \ll E_{\text{barrier}} \quad \rightarrow \quad \text{Re}[\Omega] \ll \epsilon_2 \quad (3.47)$$

where in the scale estimation above, we have ignored the effect of Δ and α because their values typically do not influence the scale estimation result.

One can also understand the effect of this single-photon drive through the quantum Zeno effect. When applying the single-photon drive Ω , the strong two-photon stabilization drive ϵ_2 of the Kerr cat qubit continuously projects its dynamics into the cat space due to the quantum Zeno effects. Therefore, one can project the effect of bosonic mode operators into cat space to get:

$$\hat{P}_C \hat{a} \hat{P}_C = \alpha \hat{\sigma}_z - i\alpha e^{-2\alpha^2} \hat{\sigma}_y \quad (3.48)$$

$$\hat{P}_C \hat{a}^\dagger \hat{P}_C = \alpha^* \hat{\sigma}_z + i\alpha^* e^{-2\alpha^2} \hat{\sigma}_y \quad (3.49)$$

where $\hat{P}_C = |C_\alpha^+\rangle\langle C_\alpha^+| + |C_\alpha^-\rangle\langle C_\alpha^-|$ is the projector in to the cat-qubit space. We notice that both $\hat{a}$ and $\hat{a}^\dagger$ exponentially approach $\alpha \hat{\sigma}_z$ with increasing cat size. When Ω is in phase with α , i.e. Ω is real, the single-photon transition effectively applies continuous $Z(\theta)$ gates with a Rabi rate of $\Omega_c = 2\alpha\Omega$ according to the Hamiltonian:

$$\hat{P}_C \hat{H}_d \hat{P}_C^\dagger = \alpha\Omega \hat{\sigma}_z \quad (3.50)$$

⁵Notice that we work in the rotation frame at the frequency $\omega_S/2$. Consequently, all drive frequencies are referenced to this frame, and a drive at $\omega_S/2$ appears as a time-independent term in the Hamiltonian.

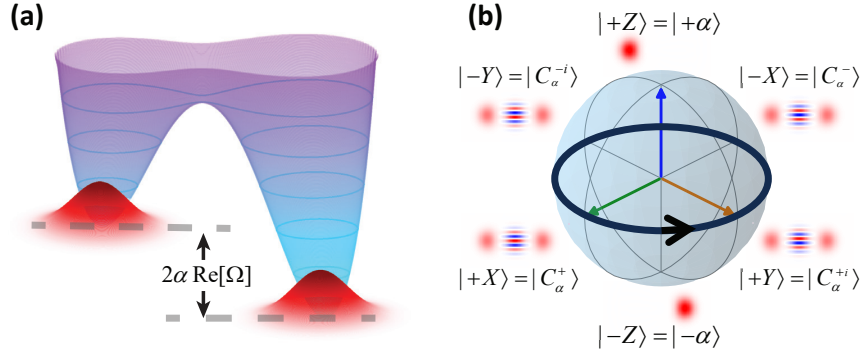


Figure 3.10: The continuous $Z(\theta)$ gates. (a) The shift of the pseudo-potential. (b) The rotation of the states along the Z-axis.

The analysis through the quantum Zeno effect then leads to the same result as the analysis through the pseudo-potential shift.

To ensure that the dynamics of Kerr-cat qubits under this single-photon drive stay within the ground state manifold, we would require that the single-photon drive strength Ω be much smaller than the energy gap between the ground states and the higher excited states E_{gap} ,

$$\Omega \ll E_{\text{gap}} \quad \rightarrow \quad \Omega \ll \epsilon_2 \quad (3.51)$$

where we have ignored some constants that do not influence the scale estimation. The analysis of scale estimation also agrees with the analysis with pseudo-potential shift, indicating the equivalence of the two understandings of the continuous $Z(\theta)$ gates.

One of the important applications of this $Z(\theta)$ gates is the calibration of cat size $|\alpha|^2$. When the single-photon drive is in phase with the two-photon stabilization drive, the cat-qubit state Rabi rate $\Omega_c = 2\alpha\Omega$ is enhanced by a factor of 2α comparing to the Fock state Rabi rate Ω . Therefore, by applying the same single-photon drive to both the Fock states and the cat-qubit states, one can extract α , and hence the cat size, from the ratio of their measured Rabi rates.

Discrete $X(\pi/2)$ gate

The discrete $X(\pi/2)$ gate change the coherent state $|\pm\alpha\rangle$ to the imaginary cat states $|C_\alpha^{\pm i}\rangle$. However, due to the insensitivity of the coherent states against single photon drive⁶, it is difficult to implement the $X(\pi/2)$ gate with photon drive. The noise-resilience in qubit design is always a double-edged sword, which not only protects the qubits against external perturbations but also prevents the manipulation of their states.

In Kerr-cat qubits, the coherent states are stabilized at the bottom of the pseudo-potential wells by the two-photon stabilization drive. Therefore, to perturb the coherent states and engineer the $X(\pi/2)$ gate, it is necessary to modify the two-photon stabilization drive and deform the pseudo-potential. There are mainly two methods to modify the two-photon stabilization drive. The first one interrupts the two-photon by turning it off abruptly[62], and the free evolution of coherent states in the undriven Kerr-nonlinear oscillator naturally implements an $X(\pi/2)$ gate. However, in the absence of the two-photon stabilization drive, this method is susceptible to T_1 decay, and the bosonic enhancement will make the high-photon-number Fock states decay even faster, leading to degraded performance for cat qubits with large cat sizes[65]. The second method, instead, keeps the two-photon stabilization drive always-on during the gate implementation, and modulates the frequency of the two-photon stabilization drive to deform the pseudo-potential and engineer the coherent state transitions between $|\alpha\rangle$ and $|\alpha\rangle$. We will explain these two methods in this section.

For the $X(\pi/2)$ gate implemented through two-photon stabilization drive interruption, we can expand the coherent state into the Fock basis as

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (3.52)$$

Then, when the two-photon stabilization drive is turned off, the coherent state will evolve under the undriven Kerr-nonlinear oscillator Hamiltonian with

$$U(t) = e^{-i\hat{H}_K t}, \quad \hat{H}_K = -K\hat{a}^{\dagger 2}\hat{a}^2 = -K(\hat{a}^\dagger\hat{a})^2 + K\hat{a}^\dagger\hat{a} \quad (3.53)$$

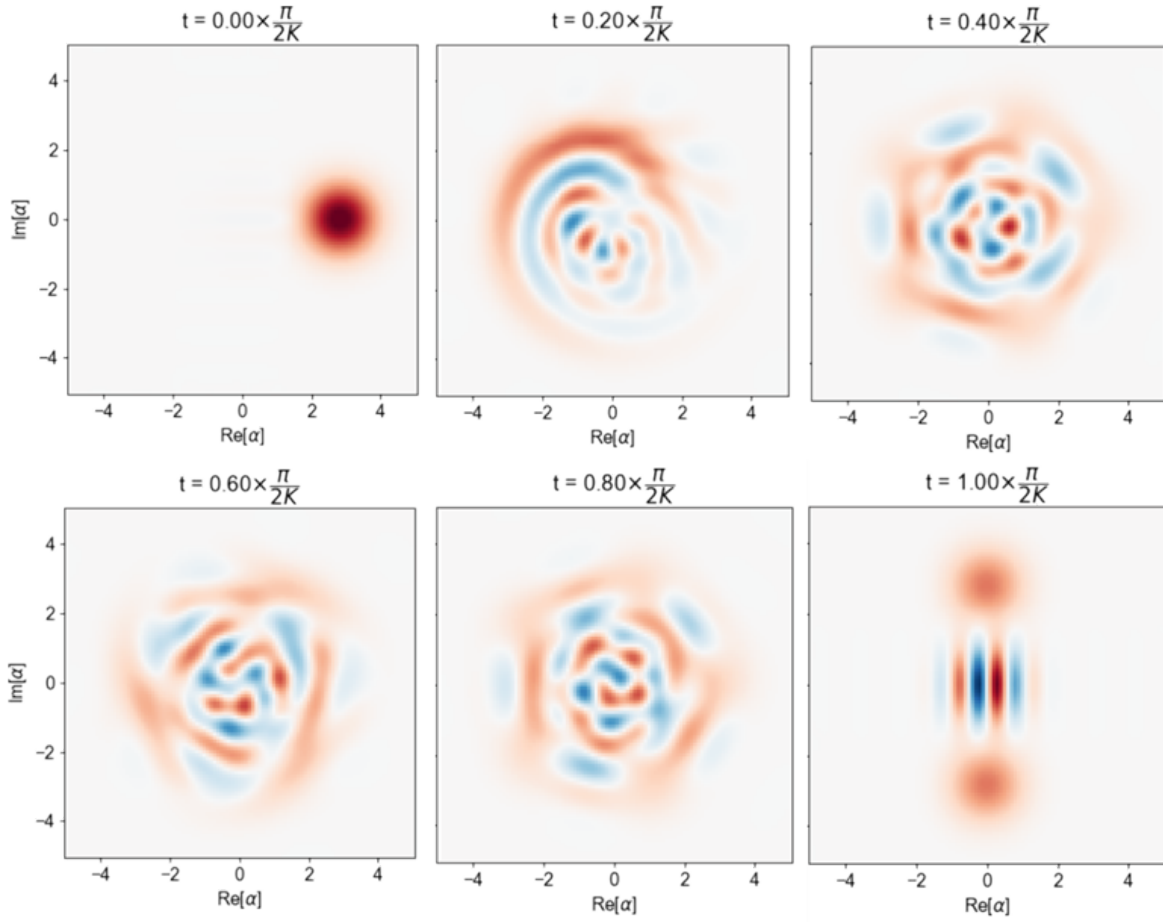
The state will be given by

$$U(t)|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{iKn^2t} e^{-iKnt} |n\rangle \quad (3.54)$$

Noticing that for some specific values of t , one can rewrite the equation above with coherent states. When $t = \frac{\pi}{qK}$, $q \in \mathbb{N}^+$, we can rewrite the equation to be

$$U(t = \frac{\pi}{qK})|\alpha\rangle = \frac{1}{2q} \sum_{p=0}^{2q-1} \sum_{k=0}^{2q-1} e^{ik(k-p)\frac{\pi}{q}} \left| \alpha e^{ip\frac{\pi}{q} - i\frac{\pi}{q}} \right\rangle \quad (3.55)$$

⁶This is also the origin of the bit-flip error protection of the Kerr-cat qubits

Figure 3.11: Wigner function evolution of a coherent state under $U(t)$

When we set $q = 2$, we will end up with the state

$$U(t = \frac{\pi}{2K})|\alpha\rangle = \frac{1}{\sqrt{2}}e^{\frac{i\pi}{4}}(|\alpha e^{-i\frac{\pi}{2}}\rangle - i|-\alpha e^{-i\frac{\pi}{2}}\rangle) \quad (3.56)$$

This state is just the imaginary cat state with a $\pi/2$ reference phase shift. The evolution of the coherent state without the two-photon stabilization drive is visualized in Fig. 3.11 by its Wigner functions.

Next, we move to the $X(\pi/2)$ gate engineered through the frequency modulations of the two-photon stabilization drive. An $X(\pi/2)$ gate requires inter-well tunneling to achieve the desired population transfer. Recalling that in Fig. 3.1, modifying detuning Δ by adjusting the two-photon stabilization drive frequency can adjust the potential barrier height. Therefore, by increasing the two-photon stabilization drive frequency to introduce a nega-

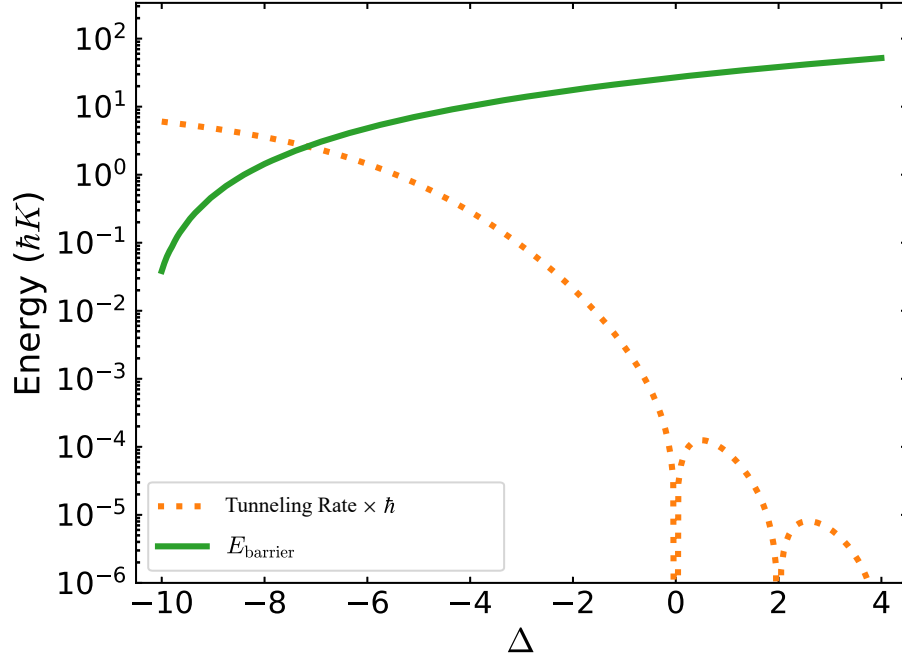


Figure 3.12: The energy barrier between the two wells and the tunneling rates with different detuning Δ

tive detuning, one can lower the energy barrier and allow the two coherent states to tunnel through it.

We calculate the barrier energy and the tunneling rate between two coherent states with different values of detuning Δ , and the result is shown in Fig. 3.12, and the tunneling rates are defined as

$$\frac{2}{\hbar} \left| \langle -\alpha | \hat{H}_{\text{KCQ}} | \alpha \rangle \right| \quad (3.57)$$

Therefore, by introducing a large enough negative detuning, one can achieve large enough tunneling rates and implement coherent population transfer between the two coherent states. Because $\Delta = \omega_Q - \omega_S/2$ is the detuning between the qubit frequency and half of the stabilization frequency, there are two ways to introduce such detuning. First, one can modulate the external flux threading the SNAILs to change the qubit frequency ω_Q . However, this method also changes the Kerr nonlinearity and the SNAIL's third-order nonlinearity, making the analysis more complicated. Therefore, we use the second approach, which involves modulating the stabilization frequency ω_S directly. Practically, we modulate the phase of the stabilization drive according to:

$$\epsilon_2(t) = \epsilon_2(0) e^{-it\delta_d(t)} \quad (3.58)$$

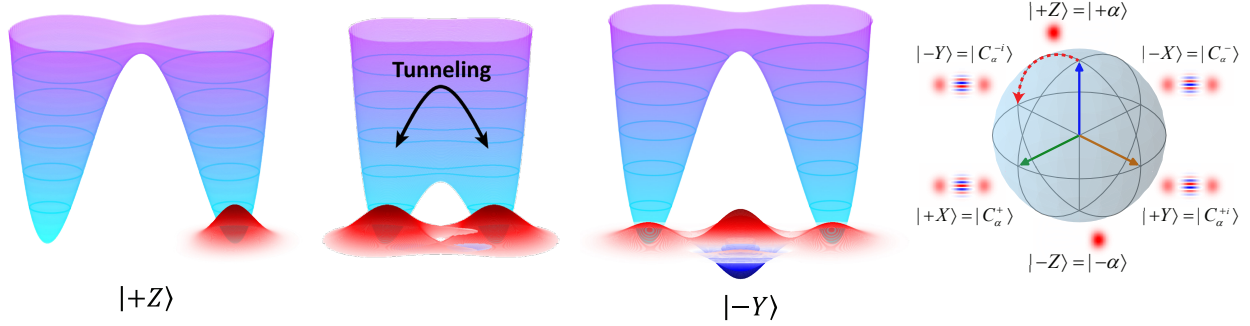


Figure 3.13: Discrete $X(\pi/2)$ gate engineered by frequency modulation of the two-photon stabilization drive. The state starts at $|+Z\rangle$, then tunnels through the potential barrier during the gate implementation and ends up with $|+Y\rangle$. The effect of this gate is also illustrated on the Bloch sphere, where we used a dashed line to indicate that it is not a continuous rotation on the Bloch sphere.

which effectively modulate the frequency by $\delta_d(t)$. Because the rotating frame is defined by half of the two-photon stabilization drive frequency $\omega_S/2$, such frequency modulation will also modify the rotating frame frequency reference and lead to the Hamiltonian in the rotating frame given by

$$\begin{aligned}\hat{H}_{\text{KCQ,rot}} &= \hat{U}_{\text{rot}} \hat{H}_{\text{KCQ}} \hat{U}_{\text{rot}}^\dagger + i\hbar \dot{\hat{U}}_{\text{rot}} \hat{U}_{\text{rot}}^\dagger \\ &= H_{\text{KCQ}} + \tilde{\Delta}(t) \hat{a}^\dagger \hat{a},\end{aligned}\tag{3.59}$$

where the unitary transformation is given as

$$\hat{U}_{\text{rot}} = e^{i\delta_d(t)ta^\dagger a/2}\tag{3.60}$$

and the effective detuning $\tilde{\Delta}(t) = -\frac{1}{2}(\delta_d(t) + t\dot{\delta}_d(t))$ as the effective detuning. It is important to notice that the effective detuning not only contains the frequency modulation, but also includes the time derivative of the frequency modulation due to the fact that we work in the rotating frame defined by $\omega_S/2$.

Therefore, by choosing a suitable modulation pulse, one can lower the energy barrier adiabatically, allow the two coherent states to tunnel and transfer their population coherently, and then increase the energy barrier height back to engineer an $X(\pi/2)$ gate, as shown in Fig. 3.13.

Up to now, we have discussed the encoding, control, and readout of the Kerr cat qubit. The encoding and control of the Kerr cat qubit can be summarized in Fig. 3.14 through the understanding of the pseudo-potential profile.

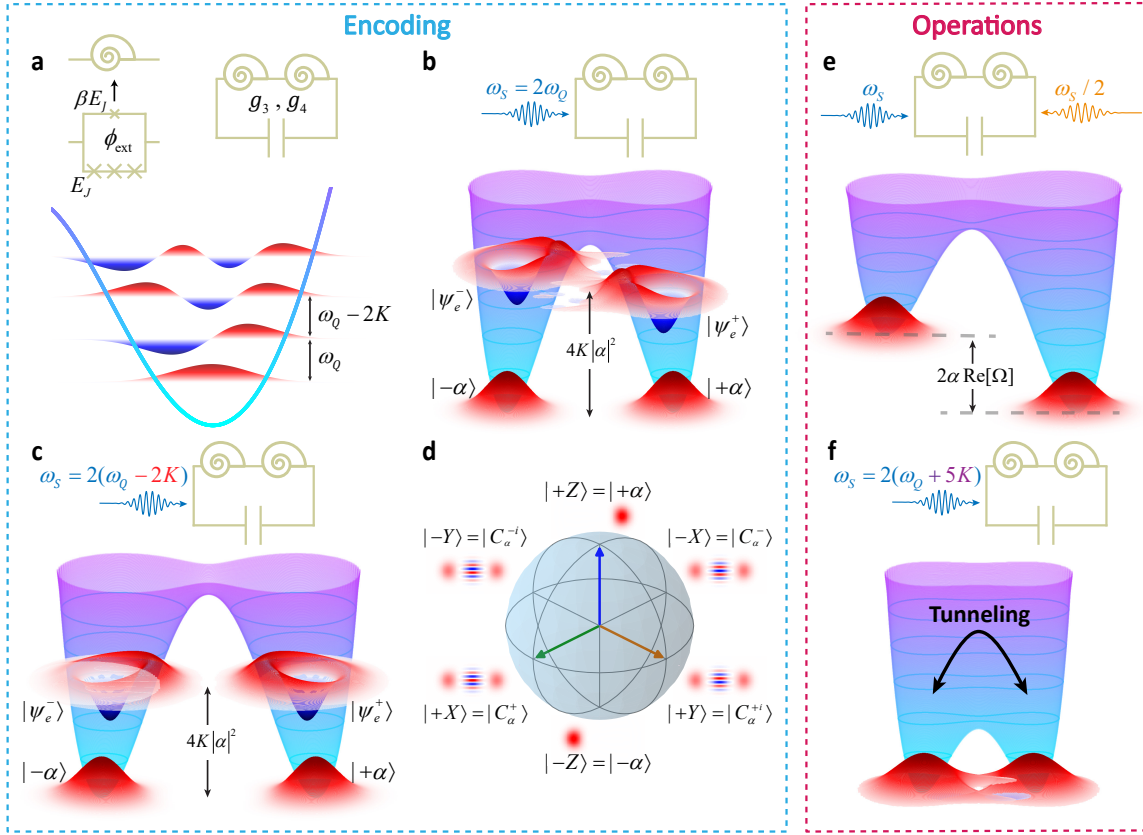


Figure 3.14: Kerr-cat qubit concepts. **a**, SNAIL nonlinear oscillator circuit with frequency ω_Q , nonlinear factors g_3, g_4 and Kerr nonlinearity K , and its energy levels with wavefunctions. **b**, SNAIL oscillator pumped by a two-photon stabilization drive at $\omega_S = 2\omega_Q$, and the engineered pseudo-potential profile with its energy levels and wavefunctions. **c**, Pseudo-potential profile and its energy levels engineered by a red-detuned two-photon stabilization drive, where the excited states become degenerate. **d**, Bloch sphere defined by the coherent states and their superpositions, where $|\pm Z\rangle = |\pm\alpha\rangle$, $|\pm X\rangle = |C_\alpha^\pm\rangle \propto |\alpha\rangle \pm |-\alpha\rangle$, $|\pm Y\rangle = |C_\alpha^{\pm i}\rangle \propto |\alpha\rangle \pm i|-\alpha\rangle$. **e**, Pseudo-potential profile under a single-photon drive Ω at $\omega_S/2$ for $Z(\theta)$ gates. **f**, Pseudo-potential profile under a blue-detuned two-photon stabilization drive for the $X(\pi/2)$ gate. The detuning was chosen to be $5K$ as an example.

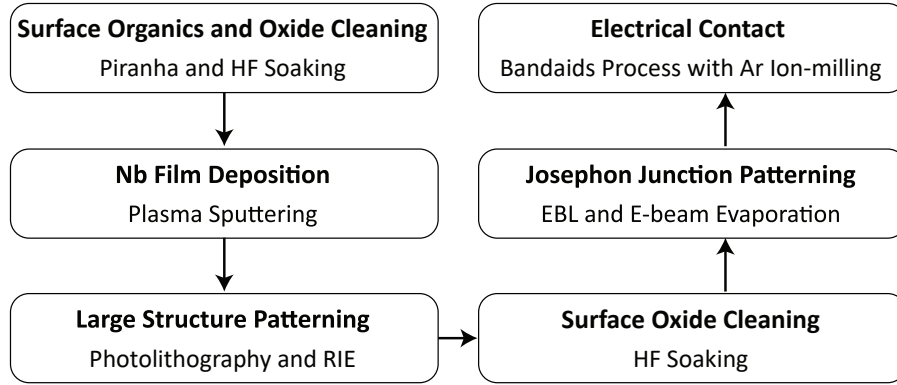


Figure 3.15: Brief fabrication procedure.

3.4 Fabrication

The device in this work is based on superconducting circuits patterned on an intrinsic silicon (Si) wafer with a resistivity $\rho \geq 10\text{k} \cdot \text{cm}$ using both the equipment in the Berkeley Nanolab (Nanolab), a shared fabrication foundry, and the Quantum Nanoelectronic Lab (QNL), a dedicated cleanroom for superconducting circuit development. The fabrication process is briefly depicted in the flow chart shown in Fig. 3.15. We will discuss them in detail in this section.

Substrate Cleaning

The substrate of our device is a double-side polished intrinsic 6-inch silicon $\langle 100 \rangle$ wafer from WaferPro with a thickness of $675 \mu\text{m} \pm 15 \mu\text{m}$. Before the metalization for large structure patterning, we perform both the organics cleaning and oxide cleaning.

We use the MOS-Clean VLSI sink (msink6) at Nanolab to perform the substrate cleaning. We soak the wafer in Piranha solutions, which consist of 19-L H_2SO_4 spiked by 250-mL H_2O_2 , for 10 minutes to clean the organics on the wafer. The solution is heated to 120°C to increase the chemical reaction rate.

After cleaning with the Piranha solutions, we load the wafer into a quick-dump-rinse (QDR) machine to clean the residual Piranha on the wafer. QDR rinses the wafer with deionized water (DI) until the container is full and then dumps the DI four times, which in total takes around 5 to 10 minutes.

After QDR, the wafer is soaked in a room-temperature 25:1 dilute hydrofluoric acid (HF) tank for 1 minute to remove the oxides (mainly SiO_2). After cleaning, the wafer is loaded into QDR again to remove HF residuals. After cleaning with QDR, the wafer is further loaded into the spin-rinse-dry (SRD) machine for a final cleaning and drying. The machine will rinse the wafer with DI and dry it by spinning it at 3000 r/min.

The wafer is then ready for metalization with Niobium. Importantly, we carry the wafer to the metal deposition chamber with a vacuum food container from Zwilling⁷ to reduce the oxidation rate and control the time interval between removing the wafer from the SRD and loading it into the metal deposition chamber to within 15 minutes.

Large Structures

The large structures, including everything on the device but the Josephson junctions and bandaids, are patterned on a 200-nm niobium layer with a high superconducting transition temperature T_c of around 9 K. We will discuss the fabrication of large structures in this section.

Niobium Sputtering

The niobium (Nb) layer is deposited in a Kurt J. Lesker sputterer in QNL. Even though the sputterer has a loadlock, we still pump the process chamber overnight after loading the sample from the loadlock to the process chamber, and reach a base pressure of 3×10^{-8} Torr before the Nb sputtering.

After reaching the base pressure, we first apply an RF Argon (Ar) plasma cleaning of the substrate with a power of 17 W. Then, we strike the Ar plasma with a DC power of 300 W to sputter the Nb target. Once the pressure of the Ar environment reaches 1.95 mTorr, we open the shutter of the substrate and deposit the film for 12 minutes to achieve the targeted 200-nm thickness. The wafer is water-cooled and spun during the cleaning and deposition process to ensure the film uniformity and quality. The sputtering pressure is carefully chosen to ensure the high T_c of the film[191] and the stability of the Ar plasma.

The Nb film fabricated using the recipe above is measured to have a T_c of around 9 K, as shown in Fig. 3.16.

Optical Lithography

The large structures are defined by optical lithography on the Nb film. We apply the photoresist (PR) using the Picotrack 6”/8” Spin Coat Track (picotrac1) in Nanolab with the default recipe for spin-coating a layer of 1-um MiR701 PR.

After spin coating, the optical lithography is done by loading the wafer in a Heidelberg MLA150 Maskless Aligner (MLA150) in Nanolab to write the structures with a 375-nm laser. The pattern is written with “high quality” mode and a dose of 140 mJ/cm². Typically, it takes around 30 minutes to 1 hour to write the patterns on the 6-inch wafer.

After lithography, the wafer is loaded into Picotrack 6”/8” Spin Develop Track (picotrac2) in Nanolab to develop the structures with the default recipe for MiR701 PR. Finally, we will apply a final hard-bake of the wafer at 110 °C for 1 minute.

⁷Of course, we will never use this box for food, it is just handy to buy this one.

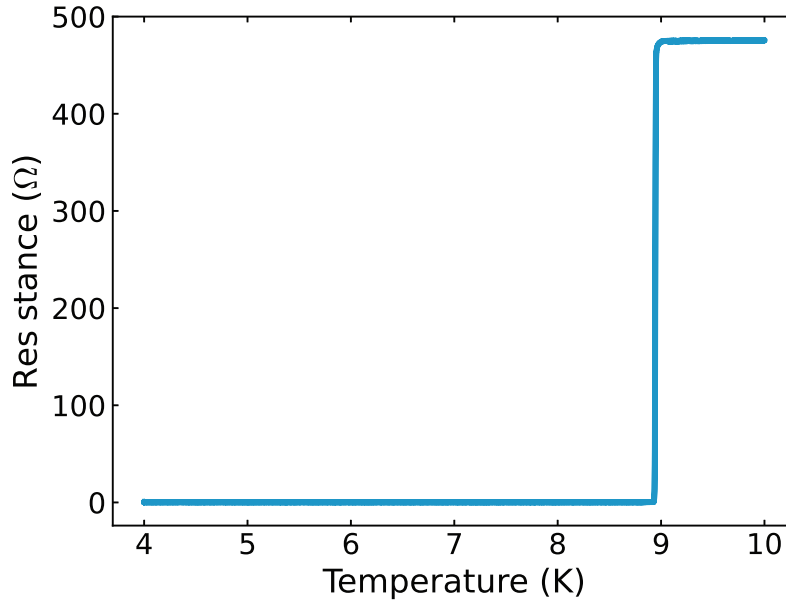


Figure 3.16: T_c measurement of the fabricated Nb film.

Reactive Ion Etching and Resist Stripping

To transfer the structure of PR onto the Nb film, we use the reaction ion etching with Metal (Al) TCP Etcher⁸(lam7) in Nanolab to etch the structures on Nb.

The etching process uses BCl_3 and Cl_2 as the reaction gas with a feeding flow of 15 sccm and 130 sccm, respectively. The TCP RF power is set to 800 W to achieve the target etching rate and quality. We monitor the byproduct of the etching process to determine the endpoint when Nb is completely etched, and the byproduct stops being generated. After the endpoint, we intentionally over-etch the structure for 7 to 10 seconds to ensure that the etched part is completely separated from the adjacent Nb layers. One example of the endpoint monitoring is shown as the purple curve in Fig. 3.17.

After finishing etching, we flow in SF_6 with no TCP RF power to flush out the etching gas and byproducts. Then, we unload the wafer from the chamber, rinse it with DI, and dry it by spinning with N_2 flow.

The wafer is then soaked in a bath of 80 °C NMP (PR remover 1165) in PR Strip & Pre-Furnace Metal Clean Sink (msink1) in Nanolab for 2 hours to strip out the PR.

⁸It is just named as Al, but it can actually etch many different metals.

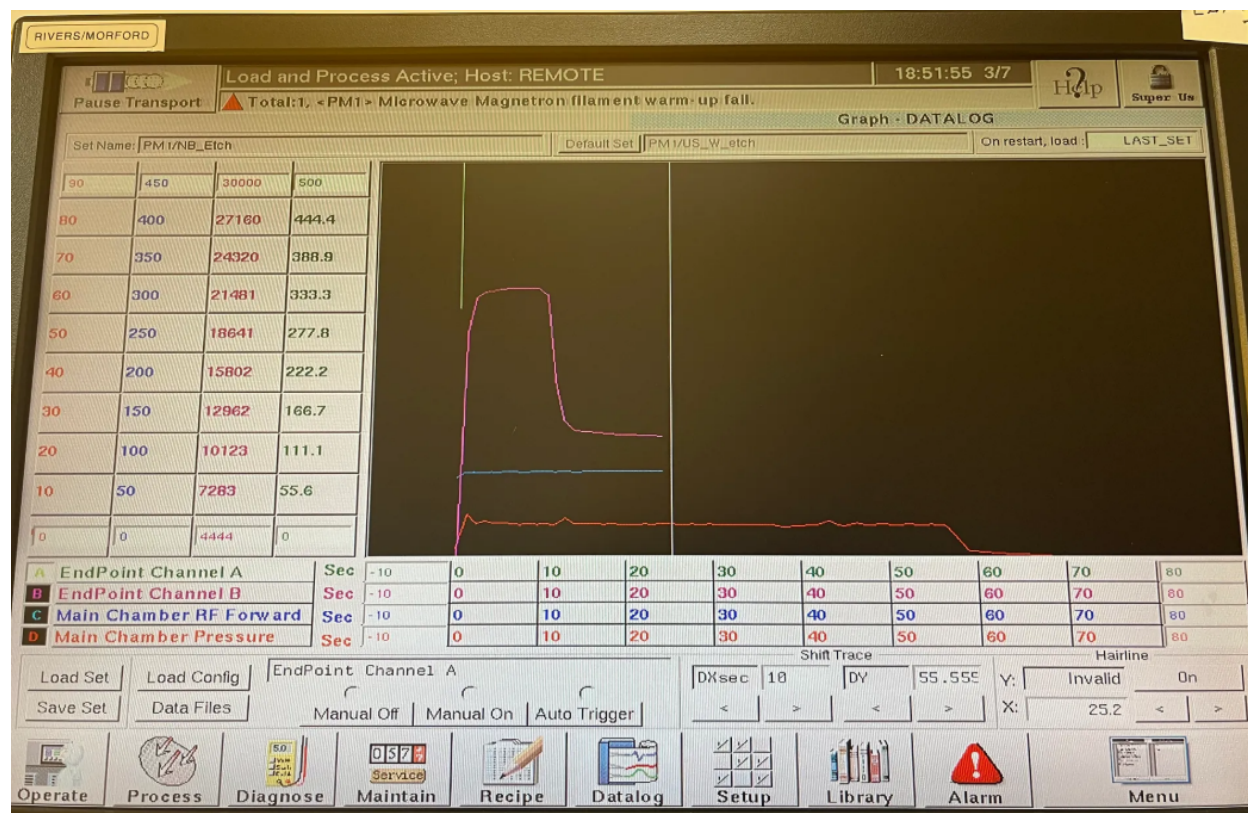


Figure 3.17: Endpoint monitoring in lam7.

Josephson Junctions

The JJs on the device are fabricated in QNL with E-beam lithography and E-beam evaporation following the lift-off technique.

Pre-lithography Cleaning

Before JJ patterning, we first clean the substrate and the Nb film with Buffered oxide etchant (BOE) 5:1 for 30s with strong agitation⁹, and clean the BOE residual by soaking it in three DI baths for 30s each. We then spray the wafer with DI to further remove the BOE residuals and load it into the spinner to coat the E-beam resist (ER) for E-beam lithography.

E-beam Lithography

We use a double-layer structure of ER to pattern JJ through E-beam lithography following the bridge-free method shown in the reference[192]. E-beam lithography is necessary here

⁹Basically, we agitate the solution and wafer a lot to ensure uniform and thorough reaction.

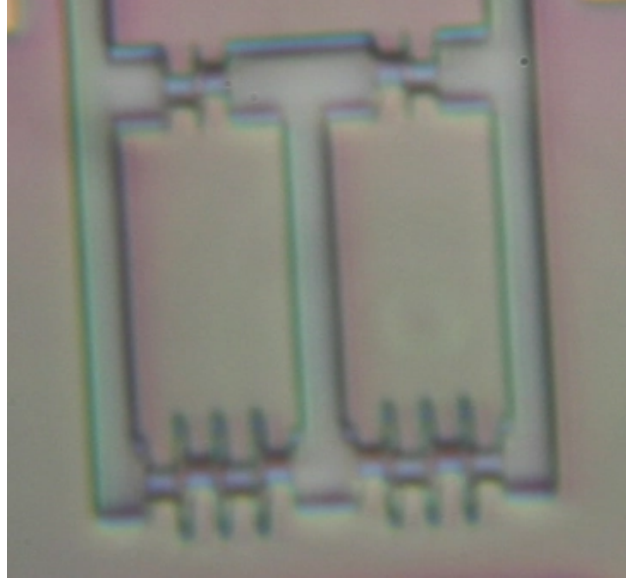


Figure 3.18: Optical microscope picture of the developed ER.

because of the small scale of the JJ and the double-layer structure of the ER. We first spray DI and Isopropyl Alcohol (IPA) on the wafer and spin it at a low speed of 1000 r/min to clean it. Then we bake it at 200 °C for 1 min to dehydrate the surface. Then, we apply MMA-EL11 ER on the wafer and spin it at 1500 r/min for 1.5 min to coat the first layer of ER with a thickness of around 300 nm. The MMA-EL11 is then softbaked at 150 °C to solidify. The wafer is then applied with AR-P 6200.9 (CSAR) ER and spun at 3000 r/min for 1 min to coat the second layer of ER with a thickness of around 100 nm. CSAR is softbaked at 150 °C for 1 min to solidify. The double-layer ER structure is essential for JJ design, and therefore the thicknesses of them should be accurately calibrated for precise fabrication.

The E-beam lithography is done in a Raith E-beam lithography tool in QNL. To ensure the quality of the lithography, we thermalize the chamber for 1 hour after loading the wafer. The relatively large patterns, such as the junction leads, are written with a 150-nA beam with 300 μm aperture to increase the writing speed, while the small patterns like the junctions themselves are written with a 1-nA beam with 300 μm aperture to make sure the accuracy. The beam voltage is 100 kV for the tool.

After E-beam writing, we develop the structure using two processes, one for CSAR and the other for MMA-EL11. We first soak the wafer in AR600-546 (N-Amyl Acetate, NAA) at 0 °C for 1 min with strong agitation to develop the structures on CSAR. We then soak it in IPA to rinse the NAA residual. Then, we soak the wafer in a 3:1 IPA:DI solution for 1min10sec to develop the structures on MMA-EL11 to clear the written structures and form the undercut, which are essential for junction patterning. After MMA-EL11 development, the wafer is soaked again in IPA to rinse the development residual, and then cleaned by

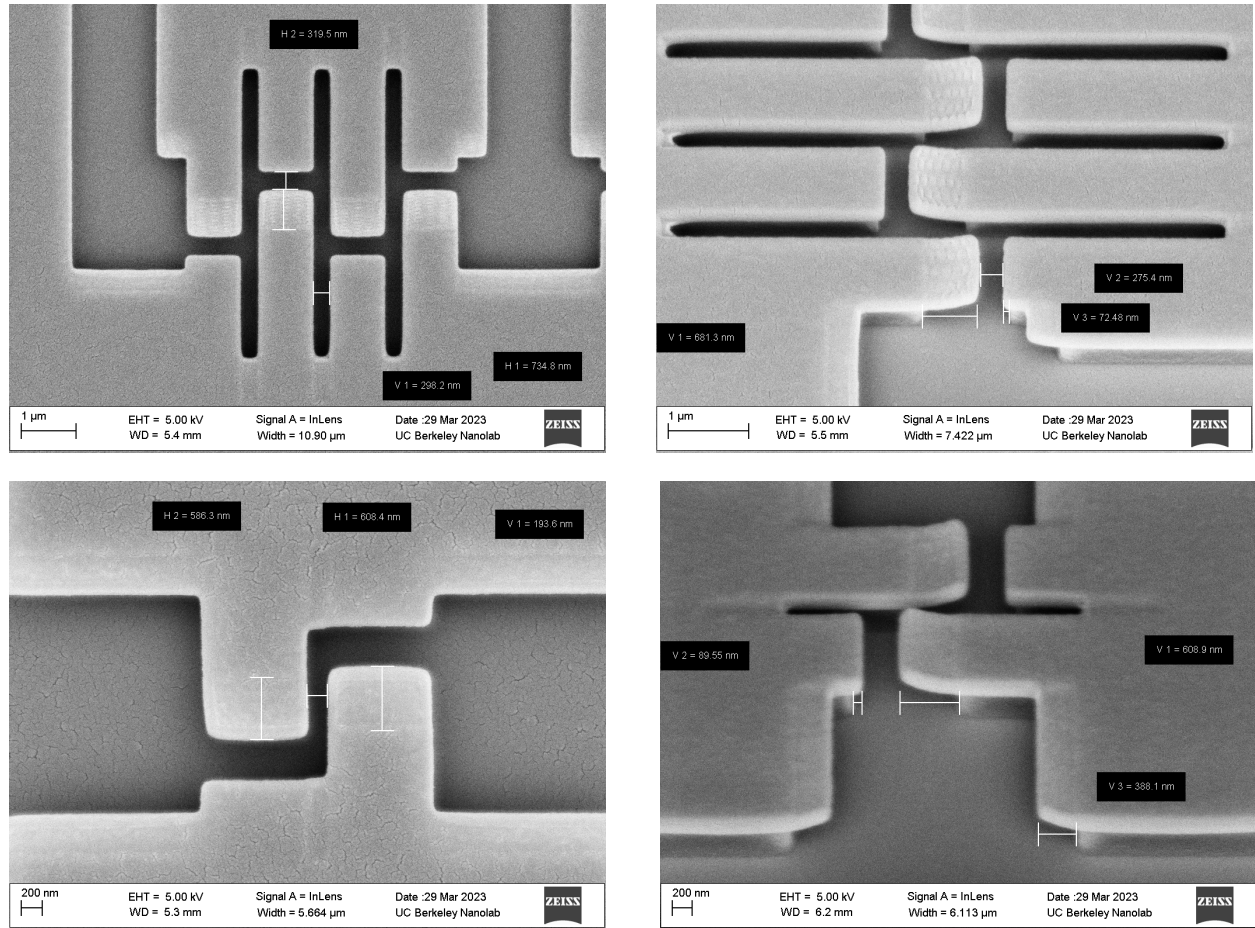


Figure 3.19: SEM picture of the developed ER.

IPA spray and spun at 1000r/min to dry out. The undercut structures are usually visible through optical microscopes. One example of the optical microscope image of the developed ER is shown in Fig. 3.18. One can also sputter the ER with some metals, such as gold, to visualize the ER in a scanning electron microscope (SEM). The developed ER structure under SEM is shown in Fig. 3.19, where the undercut is clearly visible with a side-view from SEM.

After development, the wafer is loaded in a Yield Engineering Systems G-1000 plasma asher to remove the ER residual by an oxygen plasma at a pressure of 160 mTorr and a power of 150W.

After ashing, the wafer is now ready for the Aluminum deposition to form the JJs. We usually try to minimize the time between finishing development and JJ deposition to avoid possible oxide accumulation on the wafer.

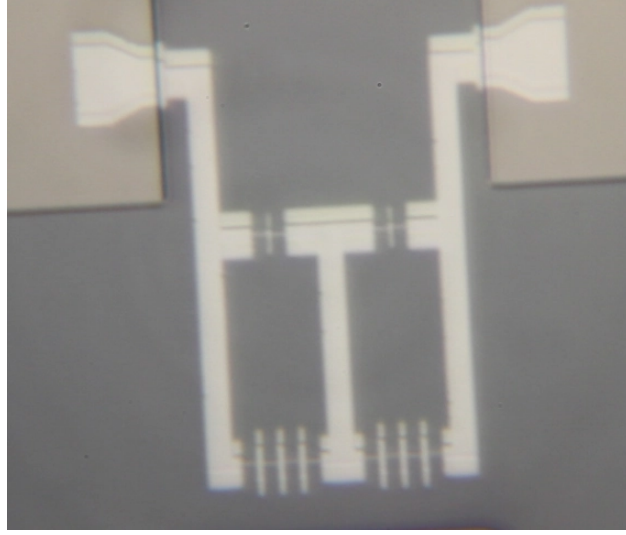


Figure 3.20: Optical microscope picture of the fabricated JJ.

E-beam Evaporation and Liftoff

The JJs are deposited by aluminum (Al) using a PLASSYS MEB550S E-beam evaporation tool. PLASSYS E-beam evaporation tools are commonly used in superconducting-circuit fabrication, particularly for Josephson junctions, because they support precise and stable fixed-angle evaporation.

After loading the wafer in the loadlock, we will pump the loadlock to below 10^{-3} mbar and open the gate between the loadlock and process chamber to pump them together with both the turbo pump of the loadlock and the cryopump of the process chamber. We require the chambers to be pumped down overnight to remove any possible contaminations introduced when loading the wafer, and the base pressure of the chambers is usually below 5×10^{-8} mbar.

Before evaporating the aluminum, we first release Titanium (Ti) into the chamber by evaporating the Ti pocket for 3 minutes with the substrate shutter closed, which prevents the Ti from being deposited on the wafer. The Ti can chemically bind residual gas molecules, such as O_2 , N_2 , CO_2 , CO , H_2 , which allows them to be pumped out more efficiently and further reduces the base pressure to around 2×10^{-8} mbar. This step is usually referred to as “Ti getter”.

After the Ti getter, we wait for 5 minutes to allow the pocket to cool down and the chamber to be pumped. We then rotate the wafer to the desired angle and start to heat up the Al pocket with the E-beam. A crystal monitor is used to monitor the deposition rate of Al, and a PID loop is utilized to stabilize the deposition rate to 0.3 nm/s. Before evaporation, the Al pocket is heated up for 1 minute to remove any possible contamination on the material surface. Then, we open the substrate shutter and deposit a 30-nm layer of

Al at a speed of 0.3 nm/s. After that, we wait for 10 minutes to allow the substrate and pocket to cool down and the chamber environment to stabilize.

We then introduce the Argon:Oxygen 95:5 mixed gas to the chamber and stabilize the chamber pressure at 20 mbar for 22 minutes to oxidize the surface of the deposited Al film, which forms the insulation layer of the JJs and leads to a critical current density of around $600 \text{ nA}/\mu\text{m}^2$ to $700 \text{ nA}/\mu\text{m}^2$. The substrate is constantly rotating during the oxidation to enhance the uniformity. We found that the critical current density is usually very unstable and can be influenced by a variety of factors, including the development method, geometric design, surface preparation method, etc.

After oxidation, we implement another round of Ti getter following the same recipe as the first deposition, and also wait for 5 minutes to allow everything in the chamber to cool down and stabilize. After that, we deposit the second layer of Al with the same speed but a thickness of 40 nm, which is intentionally made thicker than the first layer to ensure the film edge coverage. Notice that even though the Al pocket has been cleaned during the first evaporation, we still clean it for 1 minute before the evaporation here to ensure the quality of the Al film.

Finally, after the second evaporation, we introduce 10mbar Argon:Oxygen 95:5 mixed gas to form a stable oxidation layer on the second layer of Al. We implement this oxidation because the oxide surface of the second layer Al formed in Argon:Oxygen 95:5 is cleaner compared with that formed in the atmosphere. The chamber pressure during the evaporation of both layers of Al is below 10^{-7} mbar.

We then take the wafer out of the evaporation chamber, and soak it into 67 °C acetone bath for 36 hours for liftoff. To enhance the quality of liftoff, we sonicate the wafer for 5 minutes when removing unwanted Al films on the wafer.

The JJs after liftoff can be inspected under the optical microscope, and an example figure is shown in Fig. 3.20. A zoom-in SEM picture of the JJs is shown in Fig. 3.21. One should note that the SEM process will damage JJs and change their critical current. Therefore, one should never SEM a qubit device before measuring it cryogenically.

Bandbonds and Ion-Milling

The newly deposited Al JJs are separated from Nb large structures, such as the qubit capacitor pads, by a layer of Nb oxides. Therefore, we need to deposit another layer of Al to make the galvanic connection between them. This process is called bandbonds[119].

The bandbonds are defined using E-beam lithography. We use E-beam lithography here because the E-beam lithography tools are all located in QNL and dedicated to qubit development, while the optical lithography tools are in Nanolab and shared by many parties. We would like to make all processes as clean as possible, especially after the JJ is fabricated, because the Al/AlOx/Al structure of JJ is susceptible to most acids like HF and H_2SO_4 , and prevents any intense substrate cleaning methods.

To ensure the best film edge coverage, the bandbonds being deposited have a thickness of 400 nm. Therefore, we replace the first layer of ER with MMA-EL13 and spin it at 2000

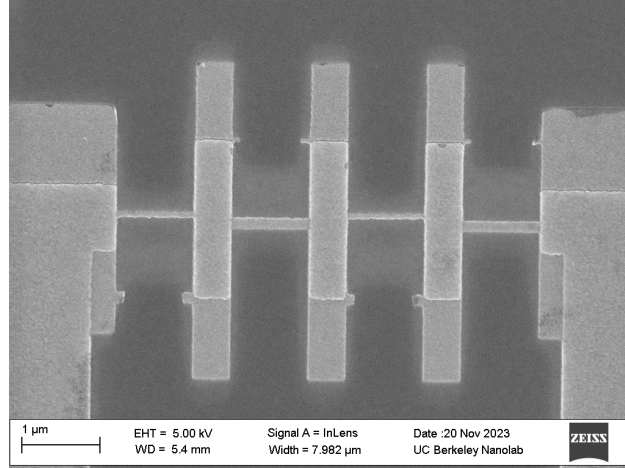


Figure 3.21: SEM picture of JJs

r/min to form a 600-nm-thick ER layer. We then apply CSAR with the same recipe as JJ fabrication to form the second layer of ER. The second layer here is purely for easier liftoff and does not relate to the design of the bandaids. Therefore, the thickness of the ER here is not critical.

The E-beam lithography, development process, and ashing process of bandaids are similar to JJ fabrication, and therefore, we do not illustrate them again here. The lithography precision requirement for bandaids is very low compared to JJ fabrication.

After E-beam lithography, we load the sample in PLASSYS MEB550S for ion-milling and bandaid evaporation. To clean the surface oxide of Nb and Al to form a galvanic connection, we apply Argon ion-milling with 400V voltage and 10mA current for a total time of 3 minutes. However, we do not apply ion-milling constantly. Instead, we apply 3-second ion-millings separated by an off-time of 20 seconds to avoid heating up the substrate. The ion-milling process physically breaks the surface oxide. After ion milling, we implement the Ti getter and wait for 5 minutes to stabilize the chamber. We then evaporate a 400-nm Al film at 1 nm/s to form the galvanic connection between the Nb layer and Al layer. After evaporation, we apply an oxidation with 10-mbar Argon:Oxygen 95:5 mixed gas, and soak it into 67 °C acetone for liftoff.

The reason for not doing ion-milling during the JJ evaporation is that the silicon substrate is fragile and can be damaged by ion-milling, which will increase the Si-Al interface loss. The author is aware that pre-JJ evaporation ion-milling is common for qubits with the sapphire substrate.

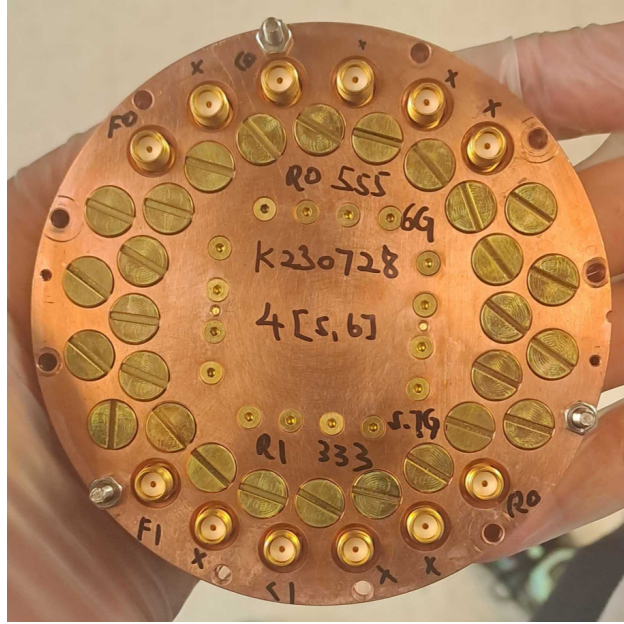


Figure 3.22: Package box.

Dicing and Package

Dicing

We usually fabricate 64 chips on a 6-in wafer, and therefore, we need to dice the wafer after patterning the device. We usually cover the wafer with a layer of MMA-EL13 ER as a protective layer before dicing.

The dicing is done with the DAD3240 Automatic Dicing Saw (disco) in Nanolab. The cutting line of this tool is $60\text{ }\mu\text{m}$, which should be taken into account when designing the device.

The author noted that the Si substrate has two advantages compared to the sapphire substrate in dicing. The first one is that Si is soft compared to sapphire, allowing easier, faster, and stabler dicing¹⁰. The second one is that Si has higher conductivity than sapphire, preventing the charge accumulation on the capacitor pads of the qubit during dicing, which may damage the JJs by electrostatic discharge. However, for the second advantage, some other projects in the group, especially the 3D-cavity based transmon qubits, have been observed to have damaged JJs due to electrostatic discharging during dicing, even with a Si substrate¹¹.

¹⁰The author helped grad students from other groups to dicing their sapphire substrate. During their process, the blade was broken more than 5 times, and the dicing speed is 1/10 of that for silicon.

¹¹They found that using an ion-fan to blow the wafer surface before dicing helps.

After dicing, we store the chips in a vacuum box to prevent further oxidation. The selected chip will be soaked in 80 °C NMP for more than 5 hours to remove the ER, and then cleaned sequentially in DI water, acetone, and IPA with sonication.

Package Box Preparation

We package the qubit in an oxygen-free copper box shown in Fig. 3.22¹². Before loading the chip in the box, we apply copper flux on the sample holder in the box to remove possible surface oxidation on the copper holder. Then, we clean the box by DI spray, followed by acetone and IPA sequentially with sonication.

Wire Bonding and Package Seal

The electrical connection between the microwave buses on the chip and the SMA ports on the package box is made by wire bonding them on a customized PCB. We bond them using a semi-automatic wirebonder from West Bond with Al wires. One example of the wire-bonded device is shown in Fig. 3.23. Beyond the bonding wires connecting the microwave buses and PCB traces, the dense wirebonds around the chip are to ensure the equal-potential of the chip ground and the package ground, and the thermalization of the chip. The on-chip wire bonds are used to ensure the equal potential of the chip ground to avoid parasitic modes like the slotline modes.

After finishing wirebonding, we close the box and seal it with indium sealing, which is perfect to isolate possible radiation and reduce the radiative loss of the device. The device is then ready to be loaded into the dilution fridge for testing.

¹²The picture is just for schematic understanding, and the label on the box does not match the device described in this thesis.

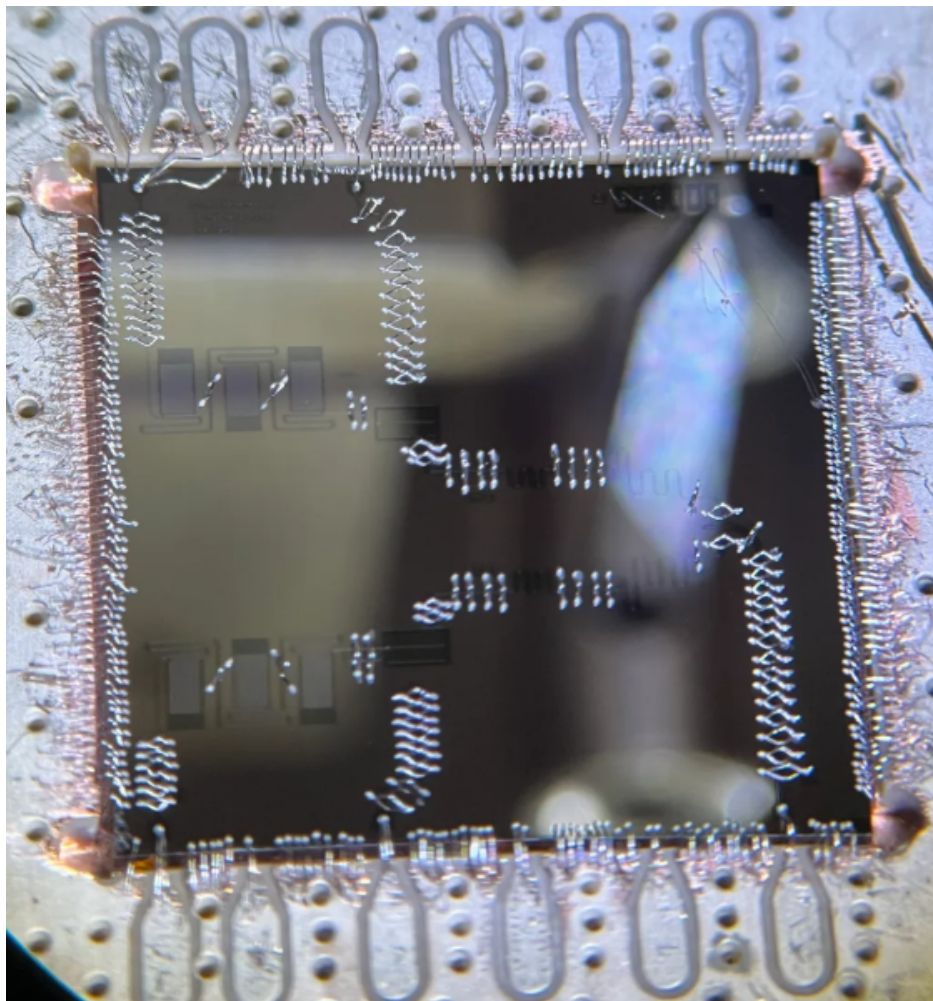


Figure 3.23: Wire-bonded device

Chapter 4

Kerr-cat Qubit Measurement

The Kerr-cat qubit device is measured in a He3/He4 dry dilution refrigerator named VeriCold in QNL, which cools the device down to 20 mK. A picture of the inner structure of VeriCold is shown in Fig. 4.1. The cooling procedure of VeriCold can be roughly separated into three processes: pre-cooling, condensing, and diluting.

The pre-cooling process will circulate a He3/He4 mixture around the fridge with a Helium compressor to cool down the fridge to around 4 K, which is the limit of most Helium compressor-based cooling processes because the He4 will start to condense at this temperature. Then, the thermal connection between the lowest three plates and the two upper plates is interrupted, so that the two upper plates will stay at their current temperatures, and the lowest three plates will be further cooled down. This thermal separation could reduce the heat mass and save the cooling power.

Next, the He3/He4 mixture is introduced around the lowest three plates, and the He4 starts to condense. During condensing, the He4 will be constantly pumped through a flow impedance to further cool down the fridge to around 900 mK through Joule–Thomson Expansion[193]. At this temperature, the He3 gas will dissolve into the He4 as two phases: dilute phase with less He3 and concentrated phase with more He3.

When the temperature is below 870 mK, a phase boundary will appear between the concentrated and dilute phases, and the phase transition from concentrated phase to dilute phase will absorb heat and further cool down the system. Therefore, through this process, the fridge will further cool down the lowest plate, named mixing chamber (MXC), to the base temperature of 20 mK. We attach our device to MXC to allow it to be thermalized with 20 mK.

4.1 Hardware Setup

As a strongly driven qubit, the Kerr-cat qubit requires multiple, strong drives at different frequency ranges with high spectral purity and fast control. To accomplish this, we use the standard techniques of up-conversion and down-conversion of RF signals that we gen-

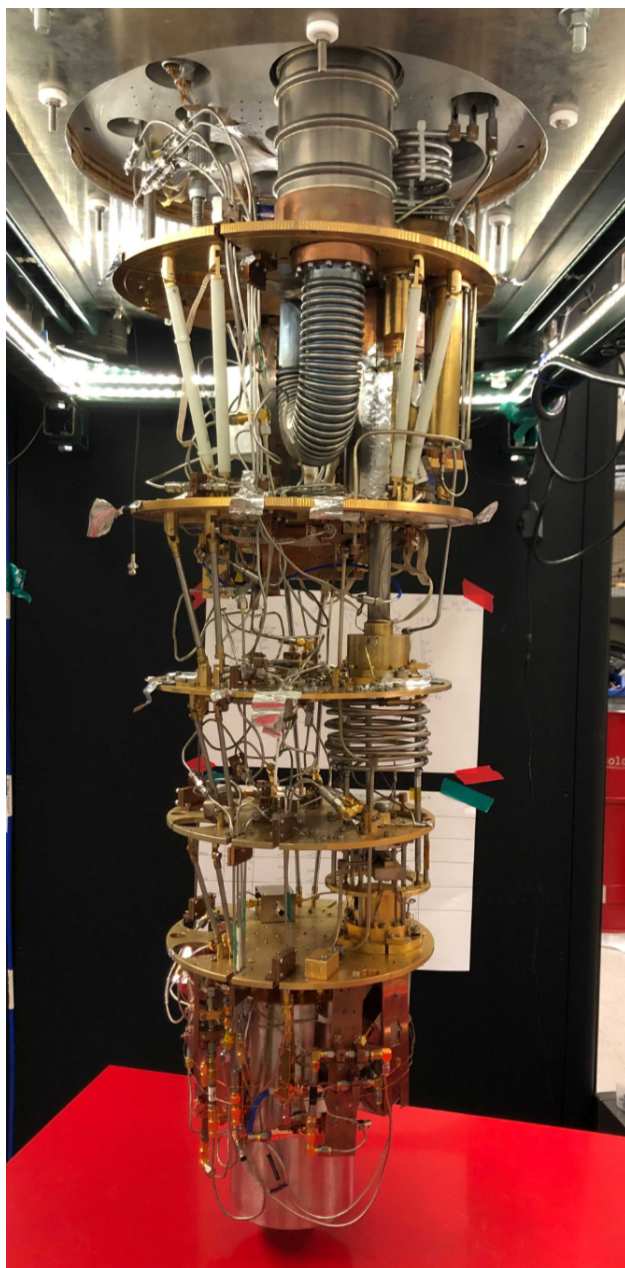


Figure 4.1: VeriCold dilution refrigerator

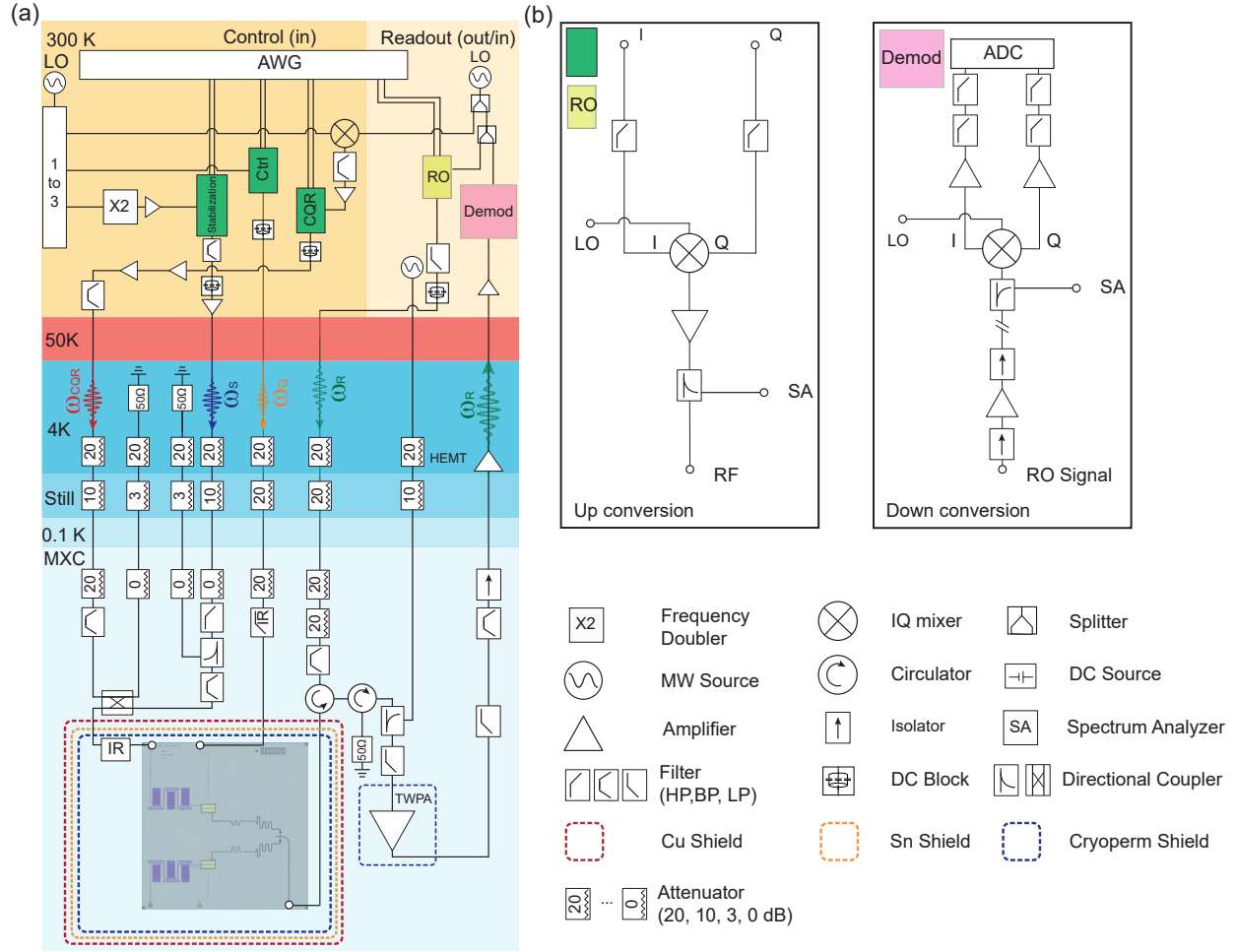


Figure 4.2: **Wiring Diagram.** (a) The microwave components used in this experiment for control and readout involve an arbitrary waveform generator (AWG), local oscillators (LOs), and an analog-to-digital converter (ADC). The rest of the microwave components are shown. (b) The two schemes of up-conversion and down-conversion of the signals using the AWG and ADC equipment at a 1GS/s sampling rate.

erate with the arbitrary waveform generator (AWG) and analyze with an analog-to-digital converter (ADC) at 1 GS/s sampling rate.

The detailed cryogenic and room-temperature wiring diagram is shown in Fig. 4.2. To make the presentation of the wiring configuration clearer, a simplified wiring diagram with more details on the room-temperature connections is shown in Fig. 4.3.

The AWG enables using the lower sidebands of the IQ mixer with IF frequencies of 70 MHz for the qubit control signals, 140 MHz for the stabilization drive, and 140 MHz for the cat quadrature readout up-conversion. For the down-conversion of the readout signal, we

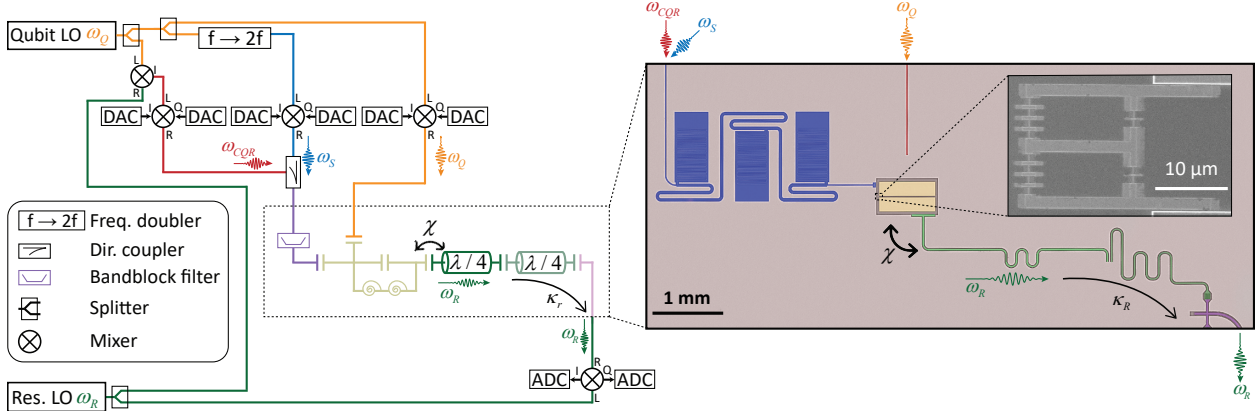


Figure 4.3: Simplified wiring diagram of the chip and the measurement hardware, where the digital-to-analog converters (DACs) are implemented by an arbitrary waveform generator (AWG), and the analog-to-digital converters (ADCs) are implemented by an Alazar sampling card.

use a 210 MHz IF frequency. The difference between the upconversion and downconversion IF frequencies is the result of the input signal being at ω_{CQR} and the output signal being at ω_R . In this experiment, we use two local oscillators around the qubit frequency and the readout frequency. To generate the carrier of the stabilization drive, we use a frequency doubler, and to generate the carrier of the cat quadrature readout, we mix the two local oscillators with an RF mixer. This arrangement eliminates the sensitivity to the phase of the local oscillators when controlling the qubit or performing the cat readout. Also, since the Kerr cat Hamiltonian is realized in the rotating frame, the readout signal picks a phase depending on the delay we use relative to the time we ramp up the stabilization drive. To account for this delay, we multiply the CQR signal by the appropriate phase to make sure the output signal always aligns with the reference we use to draw the discrimination line in the IQ plane. In this experiment, we ensured spectral purity of ≈ 40 dB suppression of the next visible side band, which requires large IF frequencies and the use of both high-pass and low-pass filters to suppress the sidebands and DC bias at each mixer to suppress the carrier.

To ensure the thermalization of the qubit, we attenuate the lines heavily and use a gold-plated cryogenic attenuator from Quantum Microwave at the mixing chamber in combination with EccoSorb filters. To avoid dumping heat at the MXC, we use directional couplers to attenuate the signal at the MXC plate while directing the signal to the 4 K with 50 ohm terminations. The output signal is first amplified with a traveling wave parametric amplifier (TWPA) near the quantum limit with a gain of around 20 dB at the mixing chamber [32], then a HEMT amplifier at 4 K, and a low-noise amplifier at room temperature.

The parameters of the device and the calibration/estimation methods are described in Tab. 3.1.

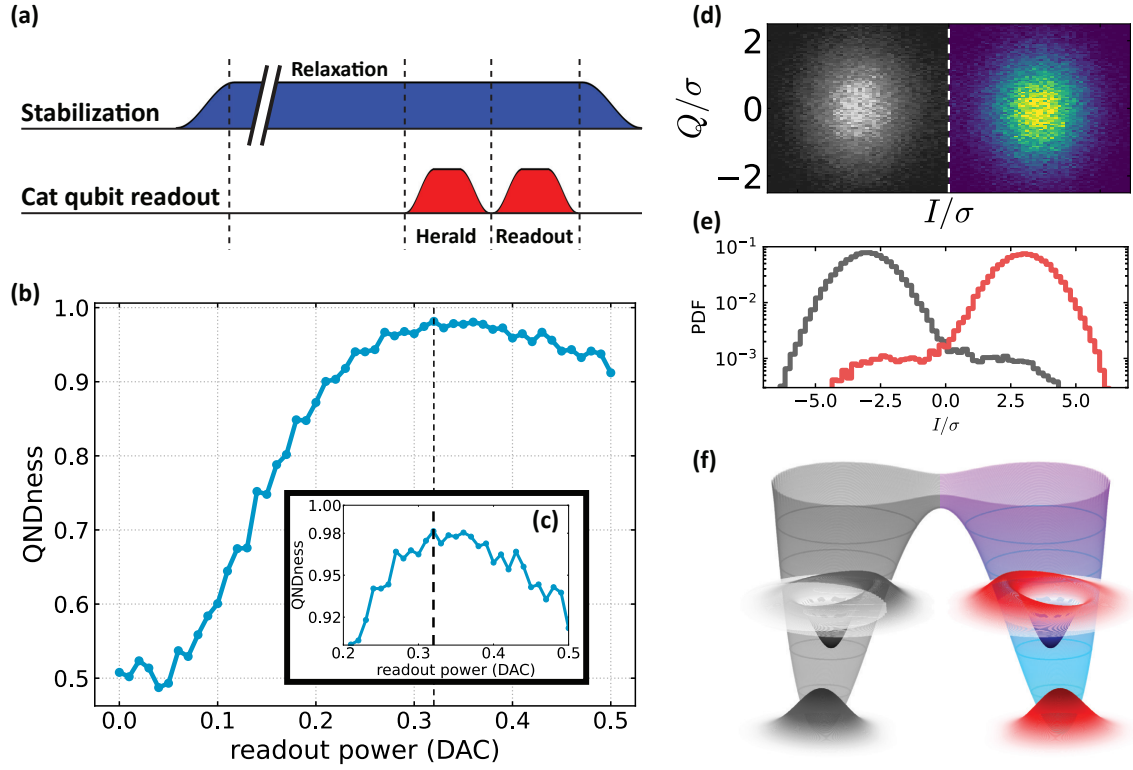


Figure 4.4: (a) The sequence to calibrate CQR drive strength. (b) The QNDness with different CQR drive strength for a cat size of 4 and a detuning of $\Delta = 2K$. (c) The zoom-in view around the peak of the QNDness. (d) The measured results of CQR, where the two blobs are separated and correspond to two Z-axis states. (e) The histogram of CQR. and (f) The collapse of wave-functions corresponding to CQR.

4.2 Cat Quadrature Readout

As mentioned before, the CQR of the Kerr-cat qubits is enabled by a beam-splitter interaction between the Kerr-cat qubits' mode and the readout resonator mode. To calibrate the CQR drive, we apply two consecutive CQR pulses with a duration of $4.5 \mu\text{s}$ for each, and define the QNDness as $(P[M_{|+\alpha\rangle}|M_{|+\alpha\rangle}] + P[M_{|-\alpha\rangle}|M_{|-\alpha\rangle}])/2$, where $P[M_A|M_B]$ represents the probability of getting result A with the second CQR pulse conditioned on having obtained result B with the first CQR pulse, as shown in Fig. 4.4 (a). The QNDness is optimized with respect to the CQR drive strengths shown in Fig. 4.4 (b), resulting in good separations shown in Fig. 4.4 (d) and (e). Such a quantum non-demolition readout can be utilized to initialize the qubit through heralding. However, as shown in Fig. 4.4 (f) and

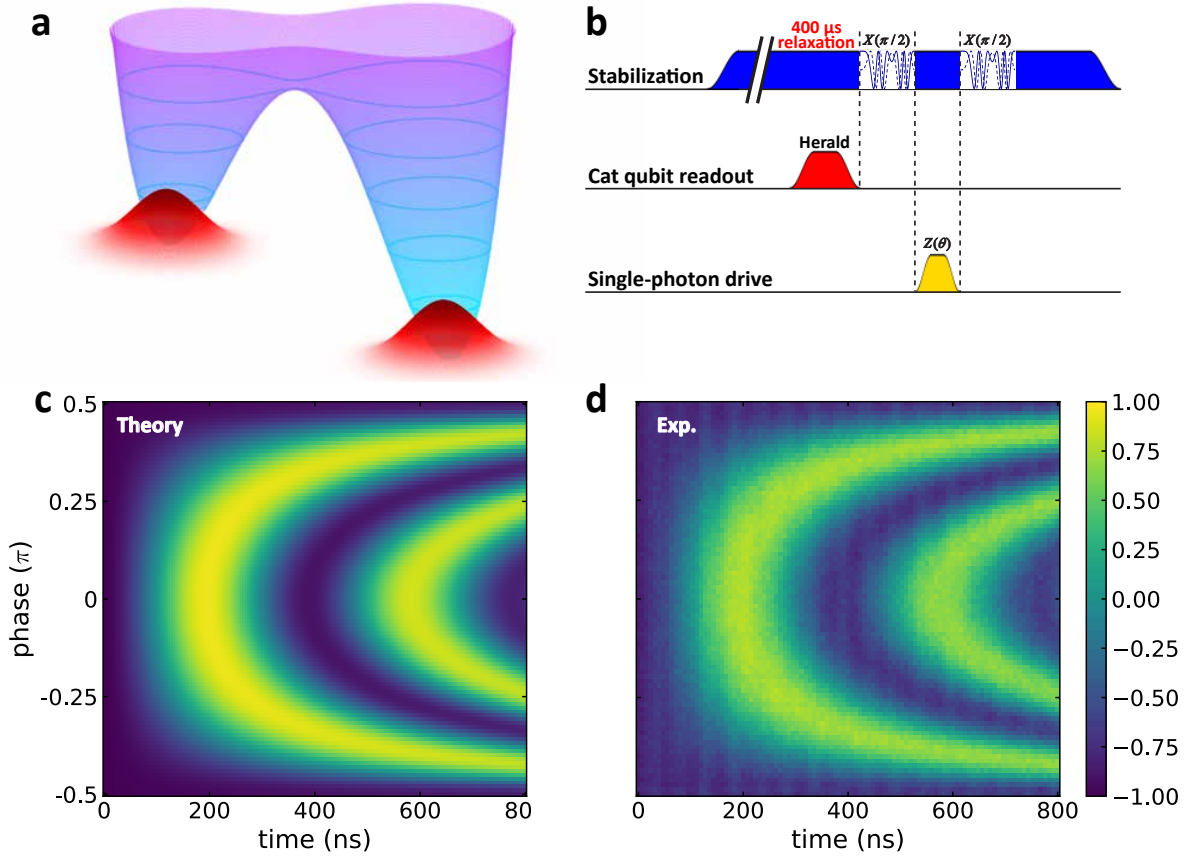


Figure 4.5: Calibration of $Z(\theta)$ gates. (a) The shifted energy profile when applying a single-photon drive. (b) The sequence for calibrating the $Z(\theta)$ gates. (c) Simulation of the measured Chevron-like plot. (d) The measured Chevron-like plot with the sequence above.

discussed in the previous section, the qubit states have to stay in the ground states before heralding because such readout cannot distinguish the excited states and the ground state in the same potential well.

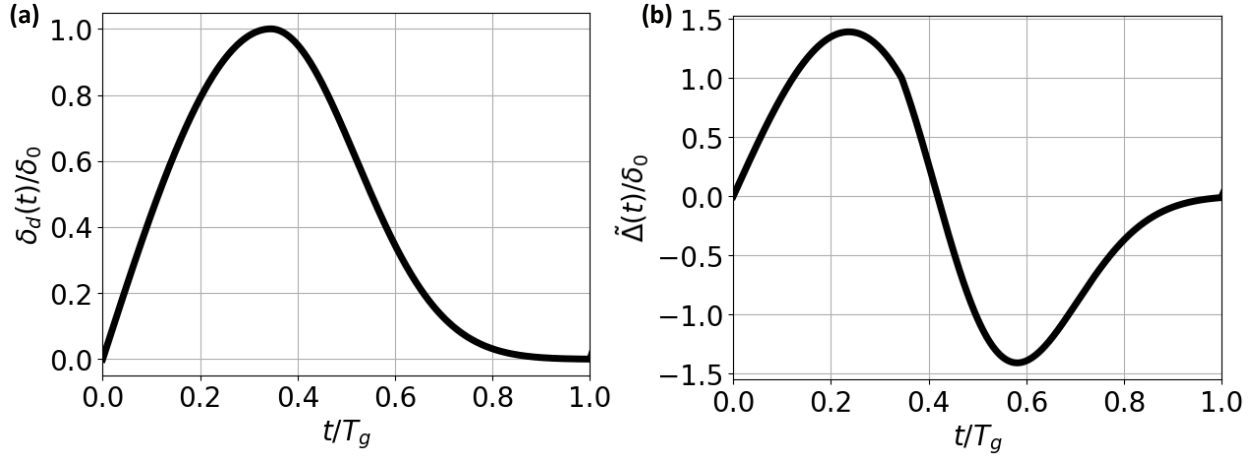


Figure 4.6: The introduced detuning during the discrete $X(\pi/2)$ gate. (a) The modulation of the two-photon stabilization drive. (b) The extra detuning term in the Hamiltonian.

4.3 Universal Single-qubit Operation

As mentioned before, the universal single-qubit gates consist of the continuous $Z(\theta)$ gates and a discrete $X(\pi/2)$ gate through pseudo-potential deformation. The continuous $Z(\theta)$ gates are realized by a single-photon drive Ω to lift the degeneracy between the two coherent states, allowing them to accumulate different dynamical phases, and the discrete $X(\pi/2)$ gate is engineered by frequency-modulating the stabilization drive to lower the energy barrier between the two coherent states, allowing them to tunnel between the two wells.

Continuous $Z(\theta)$ Gates

We calibrate $Z(\theta)$ gates following the pulse sequence shown in Fig. 4.5b. The pre-calibrated $X(\pi/2)$ gates are used to prepare and measure the state along the Y axis. The phase and duration of the single-photon drive are swept to obtain the Chevron-like plot shown in Fig. 4.5d, which is in good agreement with the numerical simulation shown in Fig. 4.5c. The $Z(\theta)$ gate speed can be increased by increasing either the cat qubit size or the single-photon drive strength. However, the large single-photon drive strength will lead to non-adiabatic error.

Discrete $X(\pi/2)$ Gate

The $X(\pi/2)$ gate, corresponding to coherent tunneling between the two pseudo-potential wells, is implemented by adiabatically lowering the energy barrier by introducing an effective detuning of the two-photon stabilization drive adiabatically. We implement such gate by

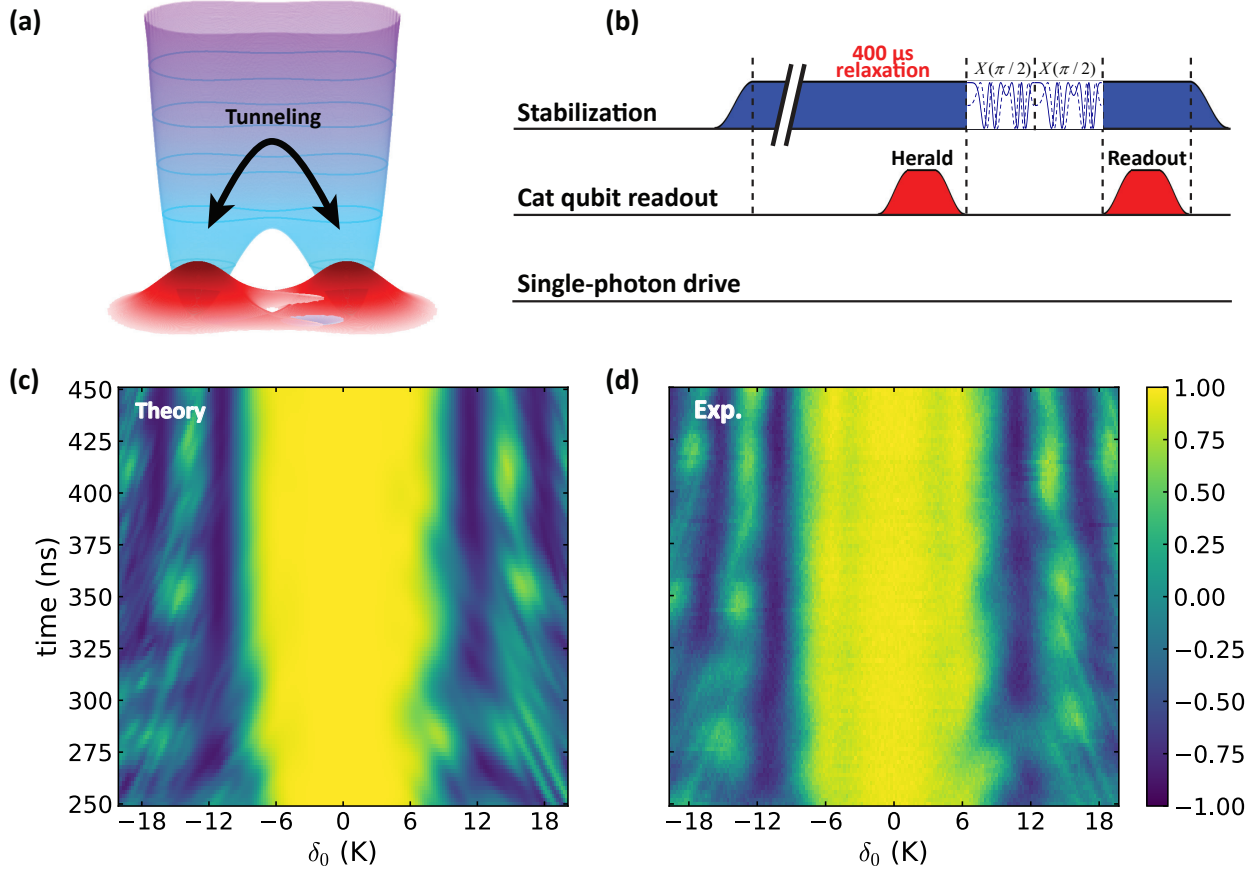


Figure 4.7: Calibration of $X(\pi/2)$ gate. (a) The energy profile with low barrier when modulating the two-photon stabilization drive. (b) The sequence for calibrating the $X(\pi/2)$ gate. (c) Simulation of the measured $\langle \hat{\sigma}_z \rangle$ (d) The measured $\langle \hat{\sigma}_z \rangle$ with the sequence above.

phase-modulating the two-photon stabilization drive $\epsilon_2(t) = \epsilon_2(0)e^{i\delta_d(t)t}$ with $\delta_d(t)$ defined as

$$\delta_d(t) = \delta_0 \times \begin{cases} -\sin\left(\frac{3\pi}{2}t/T_g\right) & t \leq T_g/3 \\ -\frac{f(t)}{1-f(T_g)}(f(t) - f(T_g)) & t > T_g/3 \end{cases} \quad (4.1)$$

where T_g is the gate time, δ_0 is the modulation depth, and $f(t) = \exp\left(-\frac{8(t-T_g/3)^2}{T_g^2}\right)$ follows a Gaussian profile. The asymmetric modulation profile is chosen to minimize the gate error. Such phase modulation adiabatically introduces a blue-detuning to the two-photon

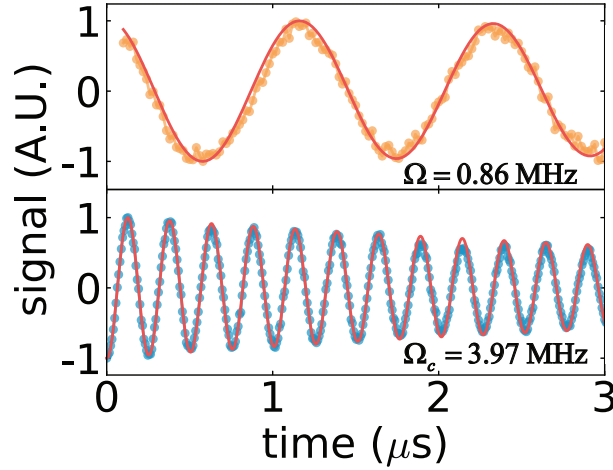


Figure 4.8: Calibration of the cat size based on Fock and cat state Rabi frequencies.

stabilization drive. As a result, it will introduce an extra detuning term

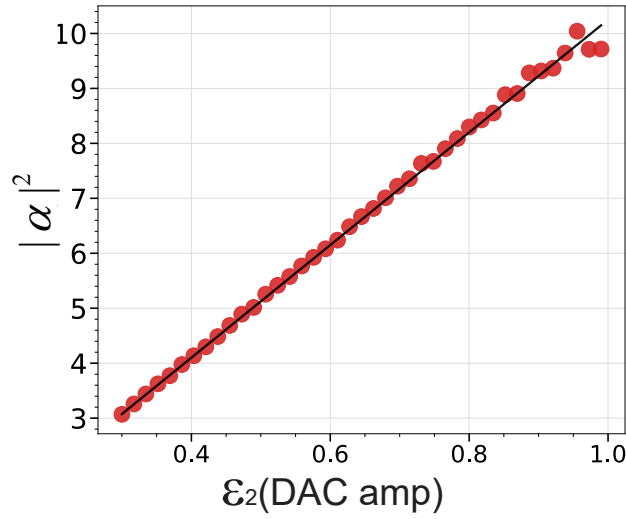
$$\tilde{\Delta}(t) = -\frac{1}{2}(\delta_d(t) + t\dot{\delta}_d(t)) \quad (4.2)$$

which lowers the energy barrier of the pseudo-potential to allow the inter-well tunneling. We plot the $\delta_d(t)$ and $\tilde{\Delta}(t)$ in Fig. 4.6 to provide a visualization of the modulation profile.

We calibrate the $X(\pi/2)$ gate with the sequence shown in Fig. 4.7 (b). A CQR pulse (labeled “Herald”) initializes the qubit into $|+Z\rangle$ state through heralding. Then, two $X(\pi/2)$ gate pulses with various modulation depth δ_0 and duration are applied. Finally, another CQR pulse (labeled “Readout”) is applied to read out the qubit state. The measured expectation of the Pauli Z operator $\langle\hat{\sigma}_z\rangle$ is shown in Fig. 4.7 (d), which clearly shows population oscillating coherently between the $|\pm Z\rangle$ states. The numerical simulation, based on solving the time-dependent Schrödinger equations, is shown in Fig. 4.7 (c) in good agreement with the experimental results. Therefore, we calibrate the $X(\pi/2)$ gate for a detuned-Kerr-cat qubit by optimizing the gate time and the modulation depth to achieve the largest $\langle\hat{\sigma}_z\rangle$ conversion. Importantly, a sufficiently long relaxation time (400 μs) is necessary for the initialization.

4.4 Cat Size Calibration

In the experiment with Kerr-cat qubits, it is important to calibrate the cat size $|\alpha|^2$. With the two-photon stabilization drive, the Kerr-cat qubit Hamiltonian supports cat states $|C_\alpha^\pm\rangle \propto |+\alpha\rangle \pm |-\alpha\rangle$ with cat size $|\alpha|^2 = (\epsilon_2 + \Delta/2)/K$. As mentioned in the previous chapter, a single photon drive $H_d = \frac{\Omega}{2}\hat{a}^\dagger + \frac{\Omega^*}{2}\hat{a}$ induces a transition between the two cat states with

Figure 4.9: Cat size calibration with different ϵ_2

the rate $\Omega_c = 2\text{Re}[\Omega]\alpha$ with $\epsilon_2 > 0$. Conversely, without the two-photon stabilization drive ($\epsilon_2 = 0$), the same drive can induce the transition between $|0\rangle$ and $|1\rangle$ states of the nonlinear oscillator with rate Ω . Therefore, by tuning the phase of the single photon drive to make Ω real and measuring the two transition rates, we can extract $\alpha = \Omega_c/(2\text{Re}[\Omega])$, as shown in Fig. 4.8. For $\Delta = 0$, we performed a cat size calibration with different strengths of two-photon stabilization drive as shown in Fig. 4.9.

4.5 Initialization

The initialization of the detuned-Kerr-cat qubit is implemented through the CQR pulse by heralding. The SNAIL nonlinear oscillator mode predominantly remains in the vacuum state $|0\rangle$ with negligible thermal noise at low temperature. Therefore, the resonant-Kerr-cat qubit can be initialized by adiabatically ramping up the two-photon stabilization drive, followed immediately by a CQR drive, which maps the vacuum state to the cat state and then collapses it into $|\pm Z\rangle$. However, this technique fails for the detuned-Kerr-cat qubit because the finite detuning changes the energies of SNAIL nonlinear oscillator states in the rotating frame. A finite Δ alters the spectrum in the rotating frame, causing $|0\rangle$ to no longer be the ground state, while the cat states remain the ground states of the detuned-Kerr-cat qubit Hamiltonian. Consequently, when the two-photon stabilization drive is ramped up, the $|0\rangle$ state is mapped to the higher excited states instead of the ground states. The subsequent CQR drive will then collapse the qubit into excited states in one of the double wells of the pseudo-potential, resulting in a failure to initialize the qubit into $|\pm Z\rangle$ states.

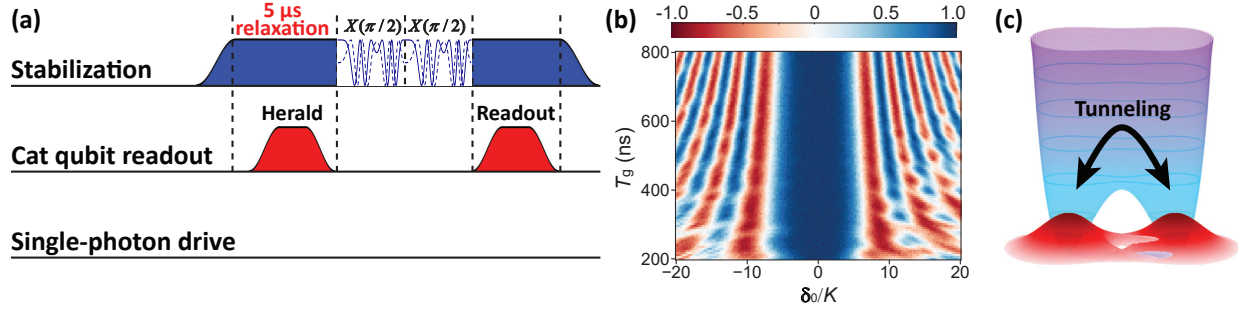


Figure 4.10: Characterization of the resonant-Kerr-cat qubit initialization. (a) The pulse sequence, noticing that there is only a short relaxation time. (b) The measured $\langle \hat{\sigma}_z \rangle$. (c) The schematic of the tunneling dynamics.

To address this, it is crucial to introduce sufficient relaxation time before applying the CQR drive, allowing the system to decay into the degenerate ground states. This ensures the validity of preparing the $|\pm Z\rangle = |\pm \alpha\rangle$ state through heralding.

In this section, we characterize the initialization process by implementing the discrete $X(\pi/2)$ gate after the initialization. When we successfully initialize the qubit into the Z-axis states, i.e., the ground states, the calibration of the discrete $X(\pi/2)$ gate will show the population transfer between two ground states $|\pm \alpha\rangle$. However, when we failed to initialize the qubit into the ground states, we observed the dynamics of the higher excited states.

Resonant-Kerr-cat Qubit

We characterize the initialization of the resonant-Kerr-cat qubit using the sequence shown in Fig. 4.10 (a). Specifically, for resonant-Kerr-cat qubits, we implement the heralding CQR pulse only 5 μ s after ramping up the two-photon stabilization drive because the vacuum state of the SNAIL nonlinear oscillator will be mapped to the ground state of the Kerr-cat qubits with $\Delta = 0$. The measured results in Fig. 4.10 (b) show the dynamics of ground states, indicating a successful initialization.

Detuned-Kerr-cat Qubit

While the resonant-Kerr-cat qubit can be initialized from a vacuum state by applying a CQR pulse immediately after ramping up the stabilization drive, a sufficiently long relaxation time before the CQR pulse is necessary to initialize the detuned-Kerr-cat qubit, because the vacuum state can be mapped to the excited states with a detuned stabilization drive with non-zero Δ . The initialization sequence for the detuned-Kerr-cat qubit is shown in Fig. 4.11 (a), along with the corresponding potential profiles and wave-functions in Fig. 4.11 (b), where a 400- μ s relaxation time is inserted before the heralding CQR drive. By ramping

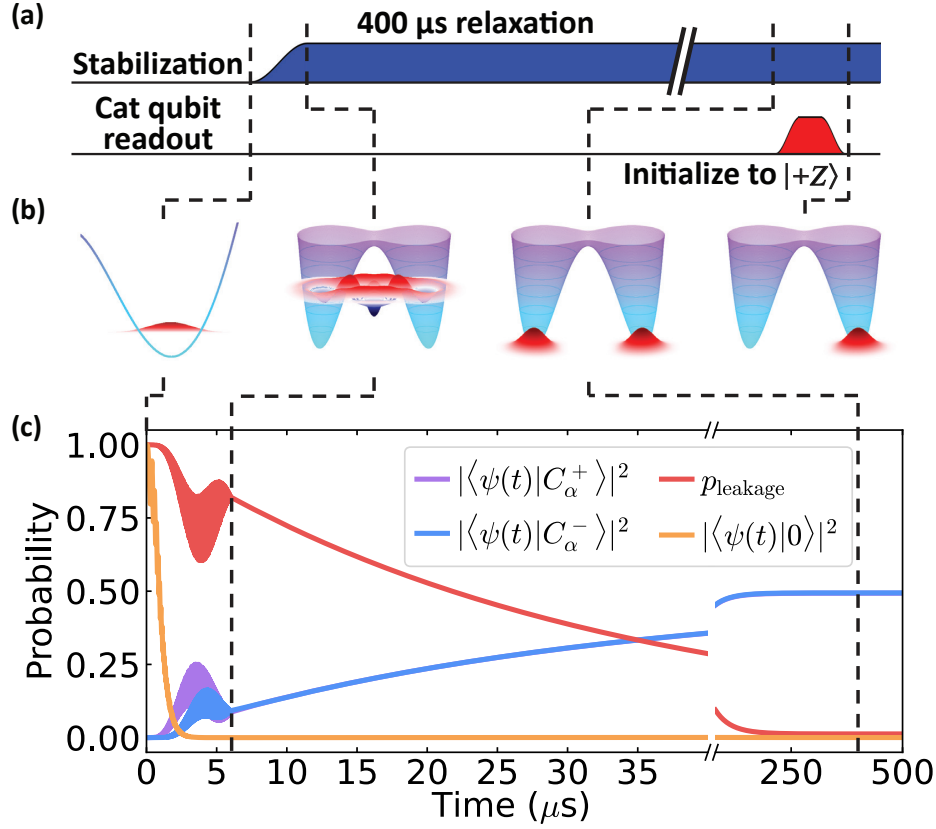


Figure 4.11: Initialization of detuned-Kerr-cat qubits. (a) The sequence for detuned-Kerr-cat qubit initialization. (b) The evolution of the potential profiles and wave-functions during the initialization. (c) The numerical simulation of the detuned-Kerr-cat qubit state population transfer during the initialization.

up the two-photon stabilization drive, the potential of the SNAIL oscillator is converted to a double-well pseudo-potential. Notably, due to the detuning Δ of the two-photon stabilization drive, the vacuum state is mapped to the even-parity excited state of the pseudo-potential, which is out of the computational manifold. Therefore, a 400- μ s relaxation time is added to allow this excited state to decay to the ground states in the computational manifold. Finally, the detuned-Kerr-cat qubit state is collapsed onto the $|+Z\rangle$ states by a CQR pulse.

Fig. 4.11 (c) shows the simulated overlaps of the detuned-Kerr-cat qubit state $|\psi(t)\rangle$ with the computational states $|C_{\alpha}^{\pm}\rangle = |\pm X\rangle$ and the vacuum state $|0\rangle$ during initialization with $\Delta = 2K$. The leakage out of the computational basis is defined as $p_{\text{leakage}} = 1 - |\langle \psi(t) | C_{\alpha}^{+} \rangle|^2 - |\langle \psi(t) | C_{\alpha}^{-} \rangle|^2$. Another fast initialization method based on a chirp pulse has been reported recently [52, 190], but our method achieves higher initialization fidelity despite the long relaxation times, as indicated by the high QNDness over 98%.

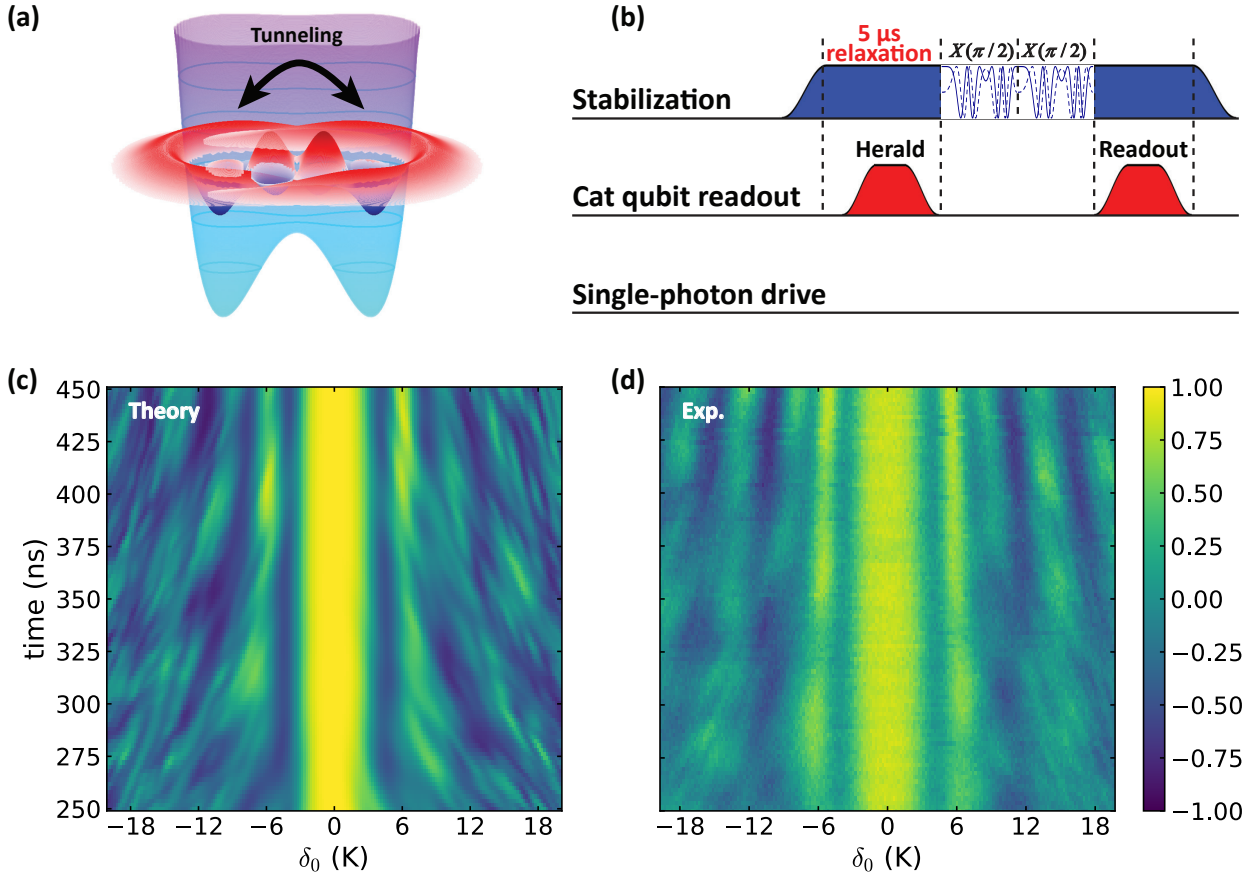


Figure 4.12: (a) The tunneling between the excited states when the relaxation time is short. (b) The sequence for observing the excited states tunneling with 5- μ s relaxation time in initialization. (c) The simulated population oscillation between excited states. (d), The measured population oscillation between excited states with the pulse above.

To observe the effect of higher excited states when the relaxation time is not long enough and the detuned-Kerr-cat qubit is initialized into some higher excited states, we applied a sequence similar to that used for calibrating the $X(\pi/2)$ gate to a detuned-Kerr-cat qubit system with $\Delta = 2K$, but with the relaxation time reduced to 5 μ s, as shown in Fig. 4.12 (b). Due to this shortened relaxation time, the detuned-Kerr-cat qubit states predominantly occupy the excited states, and the pulse sequence captures the population oscillation between these excited states within each well, as illustrated in Fig. 4.12 (a). The experimental results, presented in Fig. 4.12 (d), show a reduced contrast and tunneling occurring at a lower modulation depth due to the low lifetime and high energy of the excited states, which is in excellent agreement with the numerical simulation shown in Fig. 4.12 (c).

Chapter 5

Kerr-cat Qubit Characterization

Kerr-cat qubits typically exhibit biased noise with large Pauli Z errors and much smaller Pauli X and Y errors. Previous performance characterizations of Kerr-cat qubits are either limited to measurements of idling bit-flip and phase-flip times, which fail to capture the noise structure of quantum operations, or are susceptible to non-bias-preserving SPAM errors that obscure the operational error rates and prevent their reliable extraction.

In this chapter, we provide a comprehensive characterization of the performance of Kerr-cat qubits, including basic characterizations of Fock state coherence, cat qubit state coherence, quantum process tomography (QPT), gate set tomography (GST), and dihedral randomized benchmarking (DRB). We will start from basic characterizations of the lifetimes, showing the biased noise of the idling cat qubit with a high bit-flip time.

Next, we will move to the characterization of quantum operations. Among the native single-qubit operations of the Kerr-cat qubit, the $Z(\theta)$ gates are usually bias preserving, because they do not mix the Pauli X and Y errors with the Pauli Z error. Therefore, the operational errors of $Z(\theta)$ gates on Kerr-cat qubits should show a biased pattern with high Pauli Z error and low Pauli X and Y error. In contrast, the $X(\pi/2)$ gate is not bias-preserving because it will mix the Pauli Y and Pauli Z errors.

To characterize the operational performance of Kerr-cat qubits, we first characterize the performance of $Z(\pi/2)$ and $X(\pi/2)$ gates with QPT as reported in previous work[62], a characterization method that is susceptible to SPAM errors. As we will see, the extracted results for $Z(\pi/2)$ show no noise-bias, because the SPAM operations contain $X(\pi/2)$ gate, which does not preserve the noise-bias.

We then characterize the $Z(\pi/2)$ and $X(\pi/2)$ gates with GST, a characterization method that provides estimates of gate performance that are robust to SPAM errors. This method could show the significant difference between the noise structures of $Z(\pi/2)$ and $X(\pi/2)$ gates, indicated by the strongly biased noise of the $Z(\pi/2)$ gate. However, due to the limited classical computational power and the expensive computational cost of GST, the quantum circuit depth used in the characterization is shallow and unable to extract the small bit-flip errors in $Z(\pi/2)$ gate accurately¹.

¹One should never mix a quantum circuit with a superconducting circuit. The former one is the abstract

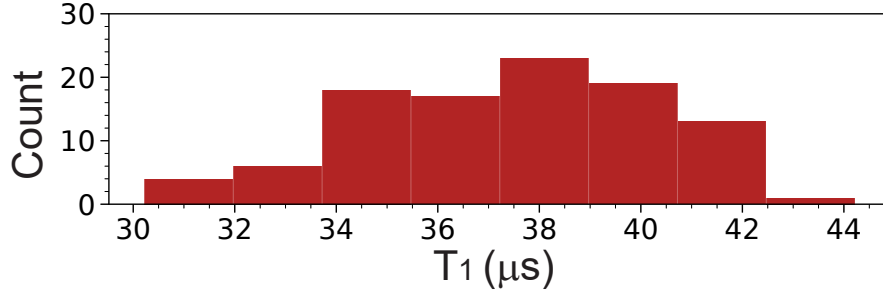


Figure 5.1: The Fock state relaxation times T_1 over 2 hours with 100 iterations.

Therefore, we apply the scalable DRB method with significantly deeper quantum circuits to accurately extract the rare bit-flip errors in $Z(\theta)$ gates. Due to the scalable nature of randomized benchmarking protocols, the computational cost of DRB is small compared to GST, allowing the characterization of qubit operations with deeper circuits. Moreover, the DRB method provides an estimates of the errors for the whole $\mathbb{D}_8$ dihedral group consisting of $\{Z(n\pi/4), X(\pi)Z(n\pi/4) \mid n \in \mathbb{N}\}$, instead of a single $Z(\pi/2)$ gate, which offers a more comprehensive characterization of the continues $Z(\theta)$ gates. One should notice that, unlike the $X(\pi/2)$ gate, the $X(\pi)$ gate for the Kerr-cat qubit is trivially a π phase shift of the reference, which can be applied virtually by adding an extra phase to the following control and readout pulses in a noise-free manner.

5.1 Lifetimes

In this section, we provide the measurement of Fock state lifetime and cat qubit state lifetime, and emphasize the advantage of finite detuning with $\Delta = 2K$ by its effect on enhancing the bit-flip time.

Fock State Lifetimes

The Fock state relaxation time T_1 is measured using standard time-domain coherence measurement as mentioned in Tab. 3.1. A statistics of the measured T_1 over 2 hours with 100 iterations is shown in Fig. 5.1². Assuming that the thermal photon population in the system is small, $n_{\text{th}} \ll 1$, we can extract the vacuum decay/heating rate as $\gamma_{\text{vac}}^{\uparrow} = \gamma_{\text{vac}}^{\downarrow} = \kappa^{(1)} = 1/T_1$, which is an important parameter in predicting the cat state bit-flip time with the theory model.

concept given by an ordered series of quantum gates, and the latter one is the physical circuits on the qubit device.

²The author noticed that the averaged T_1 increases to around 40 μs in the following experiments.

The Fock state dephasing time T_2 is measured to be $3.2 \mu\text{s}$, which is significantly lower than its coherence limit $2T_1$ primarily due to the low-frequency noise from the external flux. In the later theoretical models for the cat qubit bit-flip time, we use a pure dephasing rate γ_ϕ much smaller than that extracted from this T_2 using $1/T_2 - 1/(2T_1)$. This is because the cat qubit bit-flip error is only susceptible to high-frequency dephasing noise and exhibits resilience to low-frequency dephasing noises, as mentioned in section 3.2.

Cat Qubit State Lifetimes

We characterize the bit-flip times T_z of Kerr-cat qubits by preparing it on Z-axis using heralding, and measure the decay of the Pauli Z operator $\langle \hat{\sigma}_z \rangle$, and characterize the phase-flip times T_y of Kerr-cat qubits by preparing it on Y-axis using heralding followed by an $X(\pi/2)$ gate, and measure the decay of the Pauli Y operator $\langle \hat{\sigma}_y \rangle$.

Bit-flip Times

The bit-flip times for both resonant- and detuned-Kerr cat qubits are shown in Fig. 5.2 (a). The bit-flip time T_z increases quasi-exponentially in a staircase-like manner with the mean photon number, or cat size, $\langle \hat{n} \rangle = |\alpha|^2$.

A master equation simulation, accounting for both single- and two-photon losses, photon heating, mode dephasing, and non-Markovian noise mechanisms, captured the behavior of T_z , with discrepancies at high $\langle \hat{n} \rangle$ primarily due to increased thermal population. We illustrate the simulation of bit-flip times T_z here with the example of $\Delta = 0$. A careful analysis of the effects of non-zero detuning will be provided later in this section.

The Kerr-cat qubit inevitably disperses irretrievable information to the environment, leading to incoherent processes. We model the environment by a chain of harmonic oscillators with infinite modes and temperature T_E . While the exact dynamics of arbitrary system-environment interaction and environment evolution are intractable analytically, one can construct the master equation describing various effects of the environment on the system under assumptions that generally hold in the experiments, including weak system-environment coupling and a Markovian thermal environment. For simplicity, we consider an environment with a fixed thermal distribution and energy-independent coupling with the system. With the above approximations, the master equation has the following form,

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \sum_l \gamma_l \mathcal{D}[\hat{\mathcal{O}}_l] \hat{\rho}, \quad (5.1)$$

where $\hat{\mathcal{O}}_l$ describe three sets of incoherent processes, including $\hat{\mathcal{O}}_{1,\downarrow}$ ($\hat{\mathcal{O}}_{1,\uparrow} = \hat{a}$ ($\hat{a}^\dagger$) for the single-photon dissipation (excitation) process, $\hat{\mathcal{O}}_{2,\downarrow}$ ($\hat{\mathcal{O}}_{2,\uparrow} = \hat{a}^2$ ($\hat{a}^{\dagger 2}$) for the two-photon dissipation (excitation) process, and $\hat{\mathcal{O}}_\phi = \hat{a}^\dagger \hat{a}$ for the dephasing process. The corresponding coefficients $\gamma_{1,\downarrow}$ ($\gamma_{1,\uparrow}$) $\propto \kappa^{(1)}$, $\gamma_{2,\downarrow}$ ($\gamma_{2,\uparrow}$) $\propto \kappa^{(2)}$, and γ_ϕ are single-photon dissipation (excitation), two-photon dissipation (excitation), and dephasing rates, where $\kappa^{(1)}$ and $\kappa^{(2)}$ are the

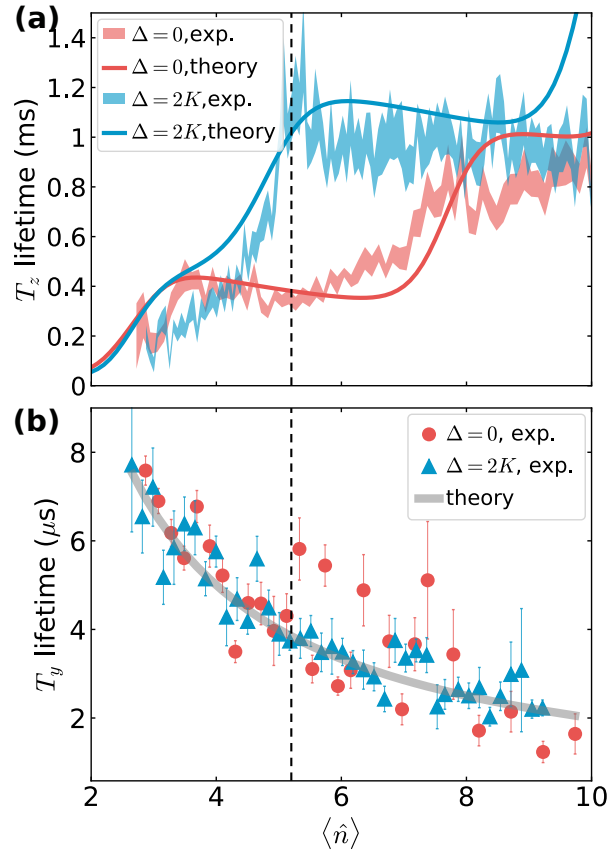


Figure 5.2: (a) The bit-flip time T_z of resonant-Kerr-cat qubit and detuned-Kerr-cat qubit as a function of mean photon numbers. (b) The phase-flip time T_y of the resonant-Kerr-cat qubit and detuned-Kerr-cat qubit. The error bars are from fitting uncertainty, and the theory predictions of the bit-flip times are calculated by master-equation simulations illustrated in [183, 194]. The dashed lines correspond to an average photon number of 5.2, which will be the operational point for QPT, GST, and DRB.

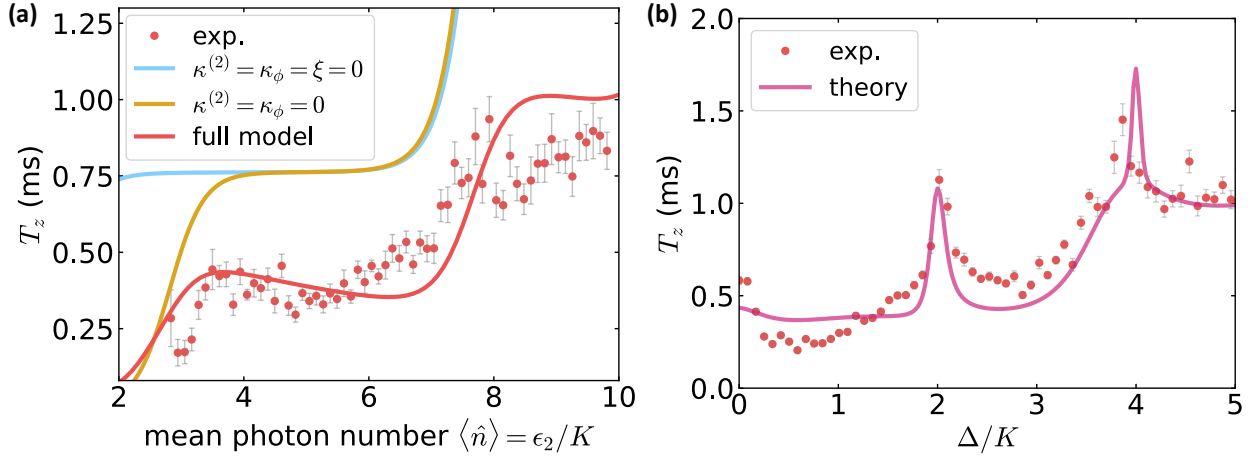


Figure 5.3: (a) The bit-flip times T_z when $\Delta = 0$ with different mean photon number and their master equation simulation with the fitting parameters taking different incoherent processes into account. (b) The bit-flip times T_z with different detuning and their master equation simulation.

single-photon and two-photon spontaneous decay rates, respectively. A detailed derivation of the master equation along with the expressions for $\gamma_{1,\downarrow}(\gamma_{1,\uparrow})$, $\gamma_{2,\downarrow}(\gamma_{2,\uparrow})$ and γ_ϕ is presented in references [53, 183, 195]. Beyond the three incoherent processes above, we also consider a non-Markovian stochastic process that leads to fluctuations in the SNAIL nonlinear oscillator frequency with a noise strength of ξ [53, 183].

With such four processes in a thermal environment with thermal photon population $n_{\text{th}} = 4\%$, and fitted parameters $\kappa^{(2)} = 7$ MHz, $\kappa_\phi = 100$ Hz, $\xi = 40$ kHz, the experimental behavior of the bit-flip times T_z is accurately modeled as shown by the agreement between the experiment data and master equation simulation results in Fig. 5.2 (a). The thermal population corresponds to a temperature of 86.9 mK, which is higher than the base temperature of the dilution fridge due to the strong microwave drives. The pure dephasing rate κ_ϕ is lower than that estimated from the T_2 values, which is also observed in all related literature [65, 179, 183], and might be due to the non-uniform power spectrum of the dephasing process under strong drive.

Here, we illustrate the effects of each process in detail. As shown in Fig. 5.3 (a), we fit the bit-flip times of a resonant-Kerr-cat qubit as a function of two-photon stabilization drive strength ϵ_2 . When only considering the single-photon dissipation and excitation, i.e. $\kappa^{(2)} = \kappa_\phi = \xi = 0$, the master equation simulation follows the blue curve, overestimating the bit-flip times T_z . By introducing the stochastic frequency fluctuation of the SNAIL nonlinear, i.e., $\kappa^{(2)} = \kappa_\phi = 0$, the master equation simulation successfully modeled the bit-flip times with small ϵ_2 following the yellow curve, but failed to model the strongly driven regime. Finally, only by including all four processes does the master equation simulation accurately

model the experimental data, as shown by the red curve.

Phase-flip Times

The phase-flip time of the resonant- and detuned-Kerr-cat qubits is shown in Fig. 5.2 (b), which decreases according to $T_y = T_1/(2\langle\hat{n}\rangle)$ [54], where $T_1 \approx 40 \mu\text{s}$ is the relaxation time of the SNAIL nonlinear oscillator. One important observation is that for the same average photon number, though the detuned-Kerr-cat qubit with $\Delta = 2K$ has the same phase-flip time T_y as the resonant-Kerr-cat qubit, it has a much higher bit-flip time T_z for moderate cat sizes. Such an advantage makes detuning a promising method to further enhance the performance of Kerr-cat qubits.

Advantages of Detuning

Due to the degeneracy of higher excited states mentioned in section 3.2, we observed increased T_z with detuning $\Delta = 2K, 4K\dots$ as shown in Fig. 5.3 (b). Specifically, T_z is higher for $\Delta = 2K$ compared to $\Delta = 0$ with the same mean photon number, and more importantly, the same T_y , as shown in Fig. 5.2 (a) and 5.2 (b). Therefore, a detuned-Kerr-cat qubit with $\Delta = 2K$ is expected to exhibit greater noise bias without additional phase-flip errors, which motivates the exploration of detuned-Kerr-cat qubits for enhanced performance [194, 196].

5.2 Quantum Process Tomography

Quantum operations, or gates, correspond to the mapping of one state of qubits to another state of qubits. In other words, the quantum operations are mappings from one density matrix to another density matrix.

There are various representations of such mapping, including Kraus representation, superoperator representation, Pauli transfer matrix (PTM) representation, Chi matrix representation, and Choi matrix representation[112]. All of those representations are physically equivalent and can be converted from one to the other. In this work, we will focus on PTM representation.

In PTM representation, the density matrix of an n-qubit state is decomposed into its Pauli components following

$$\rho = \frac{1}{2^n} \sum_{P \in \{I, X, Y, Z\}^n} \rho_P \sigma_P \quad (5.2)$$

where ρ_P is the coefficient corresponding to the Pauli matrix σ_P .

Then, we can stack the Pauli coefficients of the density matrix and form a 4^n dimensional vector given by

$$|\rho\rangle\rangle = [\rho_P, P \in \{I, X, Y, Z\}^n]^T, \quad (5.3)$$

where we use the label $|\cdot\rangle\rangle$ to distinguish this vector from usual quantum state vectors.

For a single qubit, this vector will be

$$|\rho\rangle\rangle = [\rho_I, \rho_X, \rho_Y, \rho_Z]^T \quad (5.4)$$

A quantum operation, or a mapping between different qubit states, can now be represented by the mapping between different $|\rho\rangle\rangle$ vectors. Therefore, we can form a matrix $\mathbf{\Lambda}$ which transform the $|\rho\rangle\rangle$ following the effect of the quantum operations, given by

$$|\rho'\rangle\rangle = \mathbf{\Lambda}|\rho\rangle\rangle \quad (5.5)$$

Such a matrix $\mathbf{\Lambda}$ is called a PTM and can faithfully describe the effect of the corresponding quantum operations. Moreover, since PTM is a general transformation from one density matrix to another density matrix and does not necessary corresponding to unitary operations on the density matrix, it can also be used to describe incoherent processes such as noises. Particularly, PTM is very effective for describing stochastic Pauli errors and coherent Pauli errors. However, PTM can not be any arbitrary matrix and has to satisfy several physical constraints, which ensure that ρ' is a valid density matrix. We briefly describe them below.

- Real: $\mathbf{\Lambda}$ is a real matrix, because Pauli coefficients are real and $|\rho\rangle\rangle$ is a real vector.
- Complete Positive: $\mathbf{\Lambda}$ should represent a complete positive process, because the density matrix has positive eigenvalues.

We will summarize the introduction to PTM with an expansion of the PTM and $|\rho\rangle\rangle$ for a single qubit system shown below, which should provide a clear illustration of the PTM for the readers.

$$\begin{pmatrix} \rho'_I \\ \rho'_X \\ \rho'_Y \\ \rho'_Z \end{pmatrix} = \begin{pmatrix} \Lambda_{II} & \Lambda_{IX} & \Lambda_{IY} & \Lambda_{IZ} \\ \Lambda_{XI} & \Lambda_{XX} & \Lambda_{XY} & \Lambda_{XZ} \\ \Lambda_{YI} & \Lambda_{YX} & \Lambda_{YY} & \Lambda_{YZ} \\ \Lambda_{ZI} & \Lambda_{ZX} & \Lambda_{ZY} & \Lambda_{ZZ} \end{pmatrix} \begin{pmatrix} \rho_I \\ \rho_X \\ \rho_Y \\ \rho_Z \end{pmatrix} \quad (5.6)$$

Theory

Quantum process tomography is a method to extract PTM by preparing and measuring the qubit in different Pauli matrix eigenstates³.

Suppose that we prepare the qubit into a specific state with $\rho_{\text{prep},j}$ with its Pauli vector representation $|\rho_{\text{prep},j}\rangle\rangle$, and after an unknown process G with its PTM $\mathbf{\Lambda}_G$, we measure the state population in $\rho_{\text{meas},i}$ with its Pauli vector representation $|\rho_{\text{meas},i}\rangle\rangle$, then the measured population is written as

³More generally, people can prepare and measure the qubit in any complete set of qubit states and reconstruct the process matrix

$$p_{ij} = \langle\langle \rho_{\text{meas},i} | \Lambda_G | \rho_{\text{prep},j} \rangle\rangle \quad (5.7)$$

By preparing and measuring the qubit in different states, we can stack the preparation states and measurement states into matrix and construct the measurement matrix $\mathbf{A}$ and preparation matrix $\mathbf{B}$ given by

$$\mathbf{A} = \begin{pmatrix} \langle\langle \rho_{\text{meas},1} | \\ \langle\langle \rho_{\text{meas},2} | \\ \vdots \\ \langle\langle \rho_{\text{meas},m_{\text{meas}}} | \end{pmatrix}, \mathbf{B} = [|\rho_{\text{prep},1}\rangle\rangle, |\rho_{\text{prep},2}\rangle\rangle, \dots, |\rho_{\text{prep},m_{\text{prep}}}\rangle\rangle] \quad (5.8)$$

where m_{meas} (m_{prep}) is the number of measurement (preparation) states. To ensure the validity of QPT, we require the measurement matrix $\mathbf{A}$ to have full column rank and the preparation matrix $\mathbf{B}$ to have full row rank, or equivalently, we require the measurement and preparation states to be informationally complete.

We can also arrange the measured population result into a population matrix given by

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1,m_{\text{prep}}} \\ p_{21} & p_{22} & \cdots & p_{2,m_{\text{prep}}} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m_{\text{meas}},1} & p_{m_{\text{meas}},2} & \cdots & p_{m_{\text{meas}},m_{\text{prep}}} \end{pmatrix} \quad (5.9)$$

With these matrices, we can write down all the population results and the measurement and preparation configuration into a matrix equation

$$\mathbf{P} = \mathbf{A} \Lambda_G \mathbf{B} \quad (5.10)$$

We can then solve the matrix equation above using the Moore-Penrose pseudo-inverse of the $\mathbf{A}$ and $\mathbf{B}$ matrices to get the PTM of the gate G .

$$G = \mathbf{A}^+ \mathbf{P} \mathbf{B}^+ = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{P} \mathbf{B}^T (\mathbf{B} \mathbf{B}^T)^{-1} \quad (5.11)$$

Notice that when $\mathbf{A}$ or (and) $\mathbf{B}$ is (are) square matrix (matrices), the Moore-Penrose pseudo-inverse will reduce to the conventional inverse with $\mathbf{A}^+ = \mathbf{A}^{-1}$ or (and) $\mathbf{B}^+ = \mathbf{B}^{-1}$. Notice that the full column (row) rank requirement for $\mathbf{A}$ ($\mathbf{B}$) ensures the existence of the Moore-Penrose pseudo-inverse. Even though the solution above with matrix inverse is elegant, people usually get un-physical results of G that do not satisfy the complete positive constraints. Therefore, in practice, the equation above is typically solved using optimization methods to obtain the closest physical process matrix that satisfies the completely positive.

QPT relies on a known set of state measurements and preparations, and therefore is very sensitive to the SPAM errors.

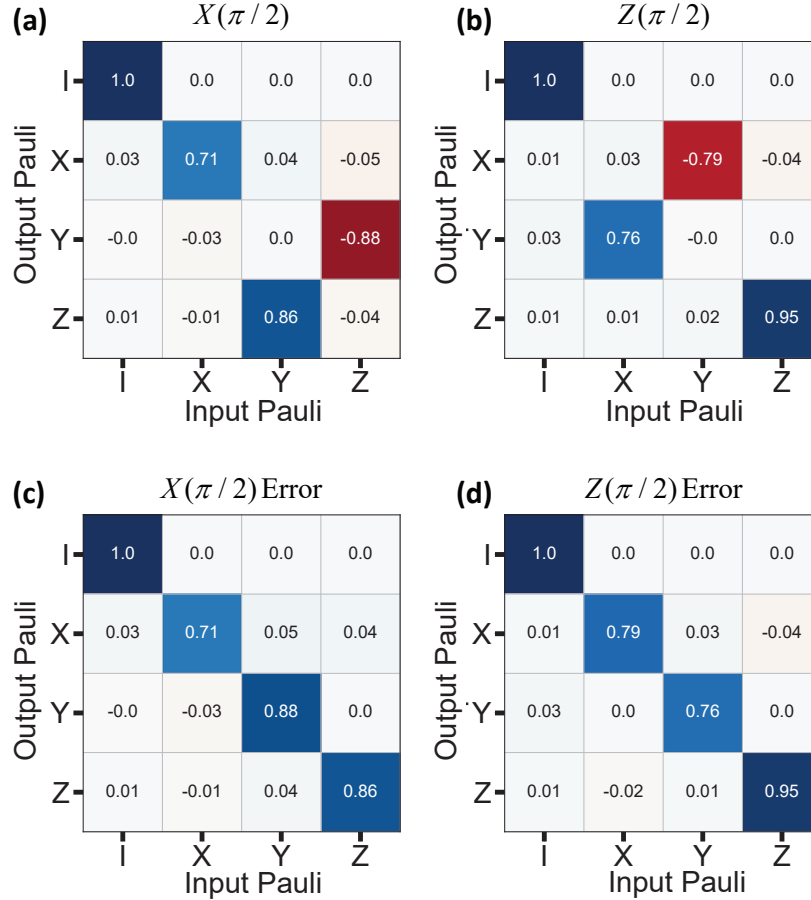


Figure 5.4: PTMs of resonant-Kerr-cat qubit extracted from QPT for (a) the $X(\pi/2)$ gate and (b) the $Z(\pi/2)$ gate. The errors extracted from the PTMs for (c) the $X(\pi/2)$ gate and (d) the $Z(\pi/2)$ gate.

Experiment

In our experiments for QPT with a single Kerr-cat qubit, we choose to prepare the qubit into $|+Z\rangle$, $|-Z\rangle$, $|+X\rangle$, $|+Y\rangle$, and measure them along the X, Y, and Z axes. The preparation and measurement of states along the X and Y axes involves $X(\pi/2)$ gate implementation, which obscures the noise-bias of the tomography gate. Instead of using the pseudo-inverse, we perform the least-squares method to find the PTMs of the gate that satisfy the physical constraints. The PTMs of the $X(\pi/2)$ gate and the $Z(\pi/2)$ gate for resonant-Kerr-cat qubits with $\Delta = 0$ are shown in Fig. 5.4 (a) and (b) with their fidelities of $\mathcal{F}_{X\pi/2} = 90.8\%$ and $\mathcal{F}_{Z\pi/2} = 91.7\%$ calculated by $\mathcal{F}_G = \frac{1}{3}(\frac{1}{2}\text{Tr}(\mathbf{\Lambda}_{\text{ideal}}^T \mathbf{\Lambda}_{\text{exp}}) + 1)$, with $\mathbf{\Lambda}_{\text{ideal}}^T$ being the ideal Pauli transfer matrix (PTM) and $\mathbf{\Lambda}_{\text{exp}}$ being the experimental value[53].

We extract the errors of the gate G using $\mathbf{\Lambda}_{\text{error,G}} = \mathbf{\Lambda}_{\text{ideal}}^T \mathbf{\Lambda}_{\text{exp}}$ shown in Fig. 5.4 (c)

and (d). There is no noise bias observed for both $Z(\pi/2)$ and $X(\pi/2)$ gates because the noise-bias structures are destroyed by the SPAM errors.

5.3 Gate Set Tomography

Unlike QPT, gate set tomography (GST) is robust to SPAM errors because it does not assume a known set of state preparation and measurement. GST is a self-consistent calibration-free protocol for characterizing the noisy implementation of an informationally complete set of quantum gates[197–199]. GST usually executes a set of circuits that consist of a state preparation and readout sequence (*fiducials*) and repeated sequences of gate operations (*germs*) that amplify specific types of error [197, 200]. The fiducial operations are included in the gate set and estimated self-consistently with the target gates. As a result, non-ideal implementations of the fiducials do not necessarily corrupt the reconstructed PTMs of the gates, leading to robustness against SPAM errors.

Moreover, GST is also capable of fitting models with physical constraints such as completely positive, trace preserving (CPTP) constraints[197–199]. This technique has been applied to various platforms of quantum computing, including ion traps[201], spin qubits[202, 203], and superconducting circuits[204].

SPAM-free PTM estimations

The local gates of Kerr-cat qubit are the discrete $X(\pi/2)$ gate and continuous $Z(\theta)$ gates, so we construct the informationally complete set of quantum gates as $\{X(\pi/2), Z(\pi/2), Y(\pi/2)\}$, where the $Y(\pi/2)$ gate is compiled using local gates as $X(\pi)Z(\pi/2)X(\pi/2)Z(\pi/2)$. Each GST circuit has the form $F_i G^l F_j$, where F_i and F_j are *fiducial sequences*, G is a *germ sequence*, and l is the maximum integer such that the length of G^l is at most some maximum length L . Using the pyGSTi python package[200], we generate the GST circuits which have a germ set with 10 germs of maximum length 5. The fiducials and germs are shown in Tab. 5.1.

Table 5.1: Gate Set Tomography Germs and Fiducials

Germs	$Z(\pi/2), X(\pi/2), Y(\pi/2), Y(\pi/2)X(\pi/2), X(\pi/2)Z(\pi/2),$ $Y(\pi/2)Z(\pi/2), Y(\pi/2)X(\pi/2)X(\pi/2)Z(\pi/2),$ $Y(\pi/2)Y(\pi/2)Y(\pi/2)Z(\pi/2), X(\pi/2)Z(\pi/2)Y(\pi/2)Z(\pi/2)Z(\pi/2),$ $X(\pi/2)X(\pi/2)X(\pi/2)Z(\pi/2)$
Preparation Fiducials	$X(\pi/2), Y(\pi/2), X(\pi/2)X(\pi/2)$, no gates
Measurement Fiducials	$X(\pi/2), Y(\pi/2), X(\pi/2)X(\pi/2)$, no gates

We perform GST with 800 circuits of maximum lengths $L \in \{0, 1, 2, 4, 8, 16, 32, 64, 128\}$ and each circuit is implemented with 1024 shots to obtain the statistics of the measurement

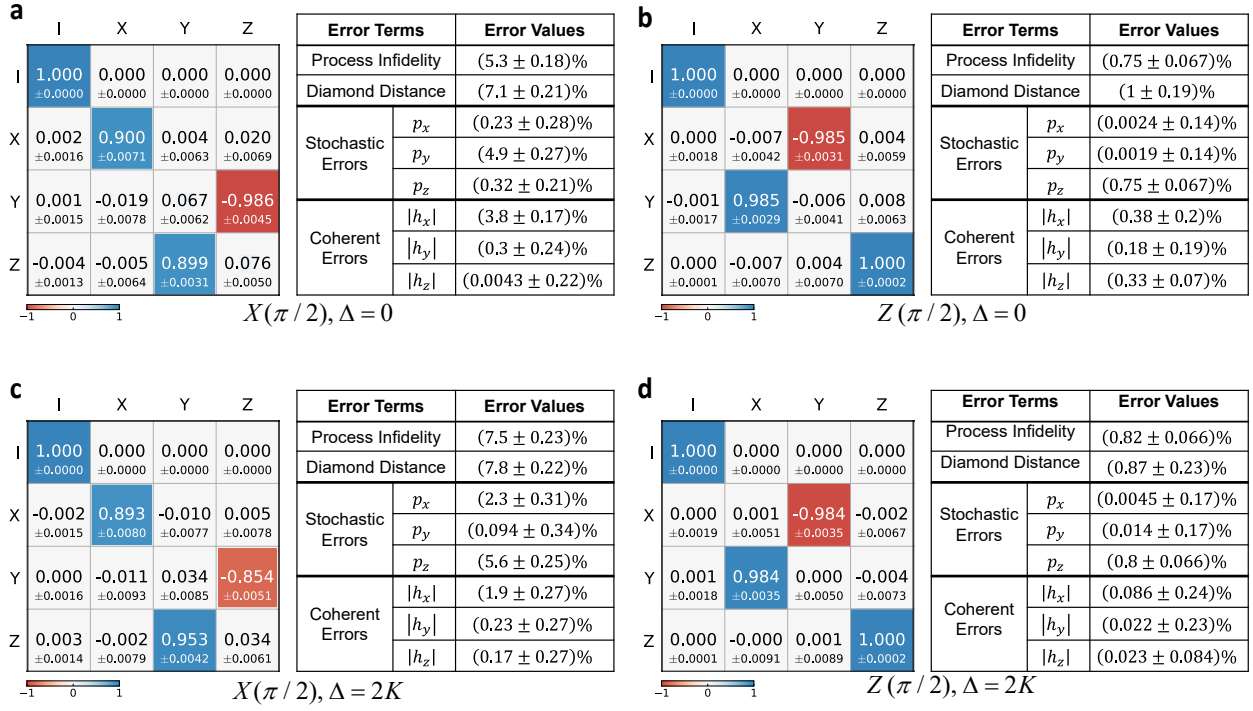


Figure 5.5: GST results. SPAM-free Pauli transfer matrices, process infidelities, diamond distances, stochastic Pauli errors, and coherent Pauli errors for gates on resonant- and detuned-KCQs. **a**, resonant-Kerr-cat qubit $X(\pi/2)$ gate with 367-ns gate time. **b**, resonant-Kerr-cat qubit $Z(\pi/2)$ gate with 70-ns gate time. **c**, detuned-Kerr-cat qubit $X(\pi/2)$ gate with 418-ns gate time. **d**, detuned-Kerr-cat qubit $Z(\pi/2)$ gate with 70-ns gate time. The uncertainty represents the standard deviation of the model fitting, and each circuit in GST is repeated by 1024 shots.

results. The process matrices and error generators of $X(\pi/2)$ and $Z(\pi/2)$ gates are extracted by fitting the measurement results to a model with CPTP constraints.

The GST results of the detuned-Kerr-cat qubit with $\Delta = 2K$ and resonant-Kerr-cat qubit with $\Delta = 0$ are shown in Fig. 5.5, including the SPAM-free estimations of the PTMs, errors, and fidelities for $X(\pi/2)$ and $Z(\pi/2)$ gates. The error channels are defined as a combination of stochastic and coherent Pauli errors, described by a PTM $\mathcal{E} = e^{\mathbf{L}}$. The error generator $\mathbf{L}$ is defined as

$$\mathbf{L} = h_x \mathbf{H}_x + h_y \mathbf{H}_y + h_z \mathbf{H}_z + p_x \mathbf{P}_x + p_y \mathbf{P}_y + p_z \mathbf{P}_z, \quad (5.12)$$

where $\mathbf{H}_x(\mathbf{P}_x)$, $\mathbf{H}_y(\mathbf{P}_y)$, $\mathbf{H}_z(\mathbf{P}_z)$ are the coherent (stochastic) error generators with respect to X , Y and Z axis and h_x (p_x), h_y (p_y), h_z (p_z) are the corresponding errors. The coherent components in the error generator are defined in Eq. 5.13, and the incoherent components

are defined in Eq. 5.14.[205, 206]

$$H_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, H_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, H_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.13)$$

$$P_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, P_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, P_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.14)$$

For $Z(\pi/2)$ gates, the stochastic Pauli X and Y errors are significantly smaller than the stochastic Pauli Z errors, confirming the bias preservation. Our KCQs exhibit low coherent errors, as evidenced by similar diamond distances and process infidelity. In fact, the effects of coherent errors are negligible after Pauli twirling, as discussed later. According to the extracted process infidelities of the quantum gates, the process fidelities of $Z(\pi/2)$ and $X(\pi/2)$ gates for the resonant-Kerr-cat qubit are $(99.25 \pm 0.067)\%$ and $(94.7 \pm 0.18)\%$, and the detuned-Kerr-cat qubit has similar performance with the fidelities of $(99.18 \pm 0.066)\%$ and $(92.5 \pm 0.23)\%$, respectively, signifying state-of-the-art performance [53, 62]. The slightly lower gate fidelity of the $X(\pi/2)$ for the detuned-Kerr-cat qubit is mainly due to the longer gate time.

Even though GST provides a comprehensive description of the noise structures of quantum operations on KCQs and confirms the noise-bias structure for bias-preserving gates, the uncertainty of extracted stochastic Pauli X and Y errors for bias-preserving gates is so large that a faithful estimation of them is not possible. This is because the Pauli X and Y errors are too small to be measured by GST with a shallow circuit depth of 128 at most.

Pauli Twirling

Pauli twirling converts a general noise channel into an effective Pauli channel by averaging the noise over random Pauli operations, thereby removing the coherent error components, or equivalently, removing the off-diagonal components of the PTMs of the errors. The Pauli twirling process randomly inserts Pauli gates while compiling circuits, and all the Pauli gates can be executed with an error-free virtual $X(\pi)$ gate, a high-fidelity bias-preserving $Z(\pi)$ gate, or their combination in Kerr-cat qubits. Therefore, Pauli twirling error channels introduce minimal extra noise with preserved the noise-bias property, and are reported to be statistically the same as the full error channels in XZZX surface code[177].

For the error process defined in Eq. 5.12, we can expand the Pauli transfer matrix as $\mathcal{E} = 1 + \mathbf{L} + 0.5\mathbf{L}^2 + o(\mathbf{L}^3)$ up to $O(p_m), O(h_n^2)$ with $m, n \in \{x, y, z\}$ and $p = p_x + p_y + p_z$ for small errors, as shown below

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - 2(p_y + p_z) - \frac{1}{2}(h_y^2 + h_z^2) & -h_z(1 + p) + h_x h_y & h_y(1 + p) + 2h_x h_z \\
0 & h_z(1 + p) + h_x h_y & 1 - 2(p_x + p_z) - \frac{1}{2}(h_x^2 + h_z^2) & -h_x(1 + p) + h_y h_z \\
0 & -h_y(1 + p) + h_x h_z & h_x(1 + p) + h_y h_z & 1 - 2(p_x + p_y) - \frac{1}{2}(h_x^2 + h_y^2)
\end{pmatrix} \quad (5.15)$$

The coherent error manifests as the off-diagonal elements of the Pauli transfer matrix, which can be effectively removed by the Pauli twirling process.

The Pauli transfer matrix of the error process after the Pauli twirling process is reduced to that of a Pauli stochastic error channel, shown in Eq. 5.16,

$$\mathcal{E} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - 2(p'_y + p'_z) & 0 & 0 \\
0 & 0 & 1 - 2(p'_x + p'_z) & 0 \\
0 & 0 & 0 & 1 - 2(p'_x + p'_y)
\end{pmatrix} \quad (5.16)$$

where the associated stochastic Pauli errors are $p'_m = p_m + 0.25h_m^2$, $m \in \{x, y, z\}$. Even with the highest coherent error rate, $h_x = 0.0038 \pm 0.002$, of the bias-preserving $Z(\pi/2)$ gate estimated from GST, the contribution of stochastic Pauli X error from this coherent error is $3.61 \times 10^{-6} \pm 3 \times 10^{-6}$, which is two orders of magnitude smaller than the stochastic Pauli X error. Therefore, the effects of coherent errors are negligible.

Model Violation

The GST model in this work is fitted with the CPTP constraints, which assume vanishing non-Markovian errors, such as leakage, in the system. Therefore, any non-zero model violation from the fitting result indicates a deviation of the measured data set from an expected distribution, implying the observed non-Markovian noises in the system. However, model violation depends on the number of shots and the depths of circuits, and does not quantify the amount of non-Markovianity in any physically meaningful units [112, 200], making it difficult to interpret the model violation $N_\sigma = 14.51$ in our GST protocol.

Therefore, we report the $2\Delta_{\text{GST}} \log \mathcal{L}$ box plots [201, 207] for the GST fittings of each sequence to describe the non-Markovianity in our system as shown in Fig. 5.6 for both resonant-Kerr-cat qubit with $\Delta = 0$ and detuned-Kerr-cat qubit with $\Delta = 2K$, which indicates the probability for detecting non-Markovian errors. The major y-axis corresponds to the germ sequences, and the major x-axis corresponds to the germ repetitions. The minor axes denote the four measurement and four preparation fiducials, respectively. Red markers indicate detections of out-of-model effects at a confidence level exceeding 95%.

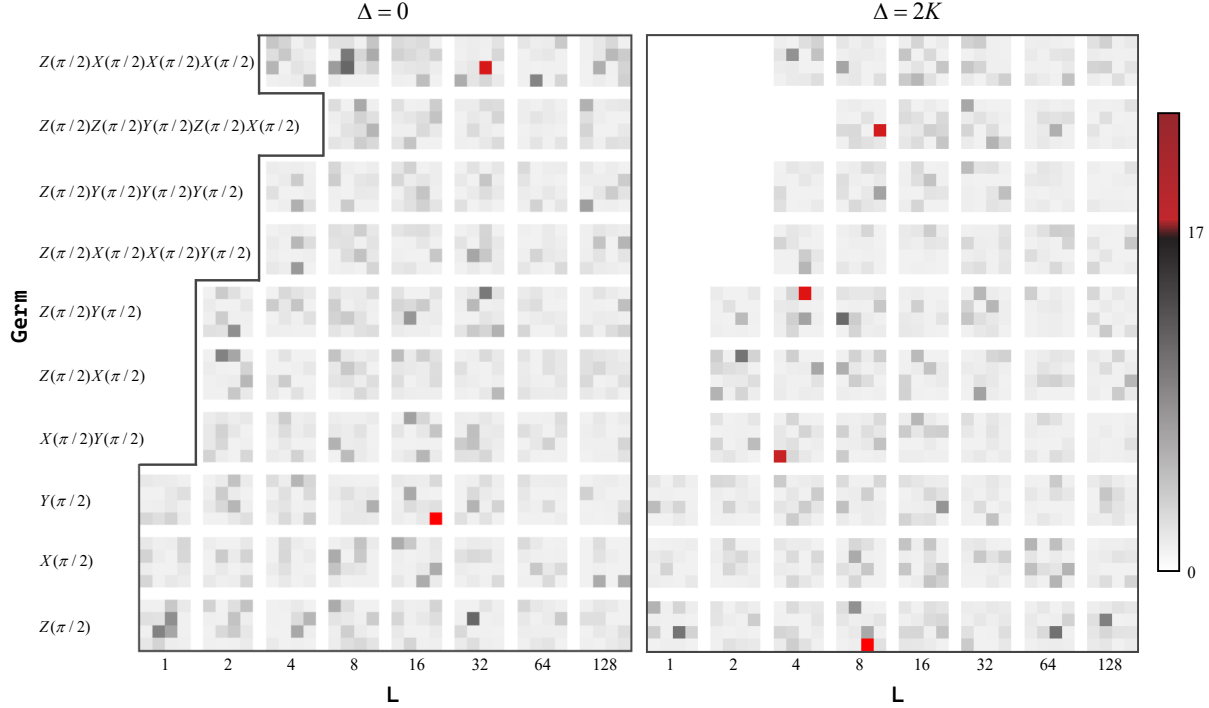


Figure 5.6: $2\Delta_{\text{GST}} \log \mathcal{L}$ box plots for GST fits of each sequence with the resonant- and detuned-Kerr-cat qubit, where red marks indicate a detection of non-Markovianity at a confidence level of more than 95%.

5.4 Dihedral Randomized Benchmarking

The estimation of the minimal Pauli X and Pauli Y errors from GST has a huge uncertainty, preventing the accurate estimation of the bit-flip errors of the Kerr-cat qubits. This is primarily because the complicated fitting procedure and expensive computational cost of GST limit the associated circuit depth to be shallow. Therefore, we will apply the noise-biased dihedral randomized benchmarking method with significantly deeper circuits to accurately extract the rare Pauli X and Pauli Y errors.

Quantum operations usually introduce additional noise, which may degrade the noise bias. Therefore, the naive estimations of noise-bias from bit-flip and phase-flip time measurements [53, 62, 194] are not accurate, and it is essential to benchmark the noise-bias of bias-preserving operations of the Kerr-cat qubit. Based on the stochastic Pauli error channel, we define the bit-flip error as $p_{\text{bit}} = p_x + p_y$, the phase-flip error as $p_{\text{ph}} = p_z$, and the noise bias as $\eta = p_{\text{ph}}/p_{\text{bit}}$. The conventional randomized benchmarking method based on Clifford gates will mix the bit-flip and phase-flip errors, thereby preventing the estimation of noise bias [208]. Therefore, we present the first experimental application of the DRB protocol on the Kerr-cat qubit to extract the noise structure of the $Z(\theta)$ gates with high precision [209,

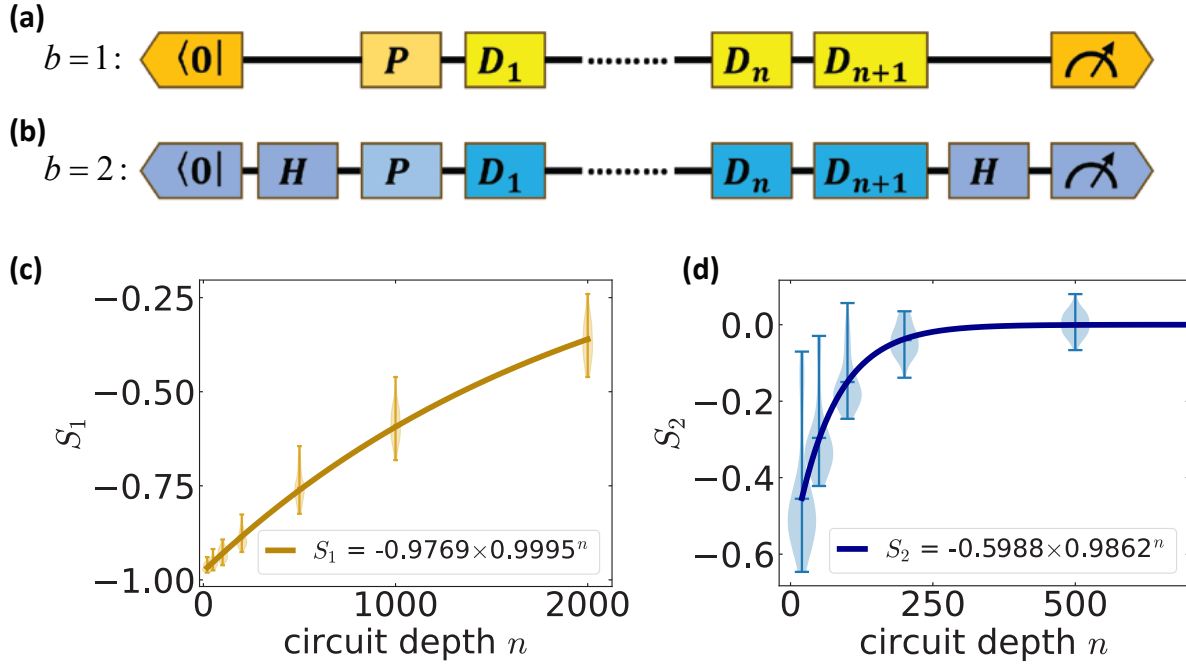


Figure 5.7: DRB survival probability fitting. Circuits and fitting curve examples of the exponential decay in the $\mathbb{D}_8$ dihedral randomized benchmarking protocol when (a), (c) preparing and measuring the states in Z axis, and (b), (d) preparing and measuring the states in X axis. The error bars indicate the shot noise from 1024 repeated measurements.

210] by applying a benchmarking circuit with a maximum depth over 2000. This protocol is also extensible to multi-qubit gate benchmarking by a recent protocol on CX dihedral groups [209].

DRB Circuits and Fitting

The noise-biased DRB protocol samples random operations from the single-qubit $\mathbb{D}_8$ dihedral group, which is generated by $X(\pi)$ and $Z(\pi/4)$ gates. The $X(\pi)$ gate for the Kerr-cat qubit is trivially a π phase shift of the reference, which can be applied virtually by adding an extra phase to the following control and readout pulses in a noise-free manner. This noise-free virtual $X(\pi)$ gate generates a bias-preserving $\mathbb{D}_8$ dihedral group with $Z(\pi/4)$ gates. Therefore, the DRB protocol avoids mixing the bit-flip and phase-flip errors, allowing them to be benchmarked separately.

The circuits of the DRB protocol consist of two modified RB sequences labeled by $b = 1$ and 2, shown in Fig. 5.7 (a) and (b), which prepare and measure the qubit along the Z and

X axis, respectively. The random gates in RB sequences consist of a gate P sampled from the single qubit Pauli group $\mathbb{P}$, followed by n gates D_k sampled from the $\mathbb{D}_8$ dihedral group, and a final gate $D_{n+1} = (D_n \dots D_2 D_1)^{-1}$ to reverse the states. The bit-flip error and phase-flip error are extracted by fitting the character-weighted survival probability $S_b(n)$ (shown in Eq. 5.17) to an exponential function.

$$S_b(n) := \mathbb{E}_{\substack{P \in \mathbb{P} \\ U_1 \dots U_n \in \mathbb{D}_8}} [\chi_b^*(P) \Gamma_{\{U_i\}}] \quad (5.17)$$

Here, $\Gamma_{\{U_i\}}$ represents the expectation value of the measurement after applying a sample of RB sequences, while $\mathbb{E}[\cdot]$ denotes the average over all samples of gates. $\chi_b^*(P)$ is a factor to track the effect of Pauli gate P on the qubit and is given by

$$\begin{aligned} \chi_1^*(X) = \chi_1^*(Y) = \chi_0^*(Z) = \chi_0^*(Y) = -1 \\ \chi_1^*(Z) = \chi_0^*(X) = 1 \end{aligned} \quad (5.18)$$

where X , Y , and Z here refer to the Pauli X , Y , and Z gates.

We fit the survival probability $S_b(n)$ into an exponential function given by [209]

$$S_b(n) \stackrel{\text{fitting}}{=} A_b \lambda_b^n \quad (5.19)$$

Using the fitted parameter λ_b , one can extract the bit-flip error p_{bit} and phase-flip error p_{ph} as

$$\begin{aligned} p_{\text{bit}} &= \frac{1}{2}(1 - \lambda_1) \\ p_{\text{ph}} &= \frac{1}{4}(1 + \lambda_1 - 2\lambda_2) \end{aligned} \quad (5.20)$$

A few typical fitting plots are shown in Fig. 5.7 (c) and Fig. 5.7 (d).

Notably, the noise-free property of the virtual $X(\pi)$ gate leads to an underestimation of the extracted errors of $Z(\theta)$ rotations because the extracted errors are weighted averages of both the noise-free virtual $X(\pi)$ gate and noisy $Z(\theta)$ rotations. Even though such underestimation is minor due to the low weight of the $X(\pi)$ gate in $\mathbb{D}_8$, we proceed to address this underestimation by introducing a scaling factor of 1.07 for p_{bit} and 1.02 for p_{ph} .

The scaling factor between the errors of $Z(\theta)$ rotations and the extracted errors from the DRB protocol is derived by numerical simulation with the qiskit [211] package. We introduce various phase-flip errors $p_{\text{pf, real}}$ ranging from 0 to 3% and bit-flip errors $p_{\text{bf, real}} = 0.02p_{\text{pf, real}}$ to $Z(\theta)$ rotations, while keeping the $X(\pi)$ gate noise-free, and then simulate the DRB experiments to extract the bit-flip $p_{\text{pf, extracted}}$ and phase-flip errors $p_{\text{bf, extracted}}$. As shown in Fig. 5.8, the extracted errors are proportional to the errors on the $Z(\theta)$ rotations. By fitting the simulation results with a linear function, we extracted the scaling factor between them, where $p_{\text{pf, real}} = 1.02p_{\text{pf, extracted}}$ and $p_{\text{bf, real}} = 1.07p_{\text{bf, extracted}}$.

As a summary, by applying the DRB protocol, we can precisely extract the bit-flip errors, phase-flip errors, and noise bias of the bias-preserving gates.

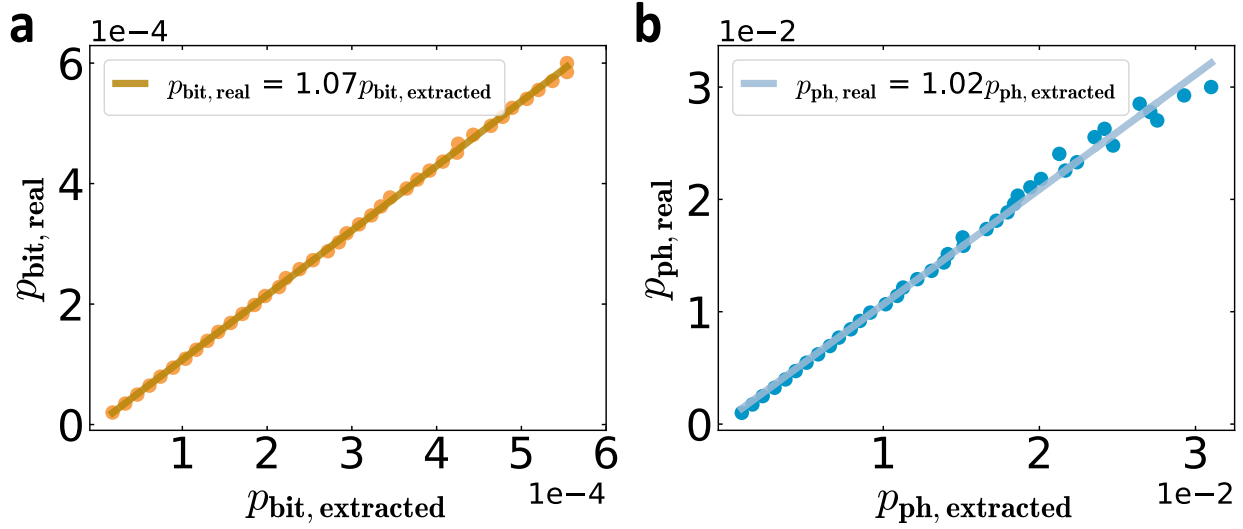


Figure 5.8: Scaling factor **a** between the extracted and real bit-flip errors, and **b** between the extracted and real phase-flip errors

Errors and Noise Bias

We presently explore the noise structure of the resonant-Kerr-cat qubit and detuned-Kerr-cat qubit as a function of $Z(\theta)$ gate times, with results shown in Fig. 5.9a, where the uncertainty of the extracted errors is significantly smaller than that from GST. We also plot the bit-flip and phase-flip errors as predicted by the bit-flip and phase-flip times based on the stochastic Pauli error model [112]. For both the resonant-Kerr-cat qubit and detuned-Kerr-cat qubit, the bit-flip error is much lower than the phase-flip error. Clearly, while the phase-flip errors align with the rates inferred from the phase-flip time (green line), the bit-flip error is typically much higher than predicted by the bit-flip time (red and blue lines for resonant- and detuned-KCQs, respectively). We attribute the high bit-flip error at short gate times to the leakage into higher excited states caused by the strong gate drive, which tunnels between the wells of the pseudo-potential. As the gate time increases, the bit-flip error approaches the error set by the bit-flip time. These observations highlight an underestimation of the bit-flip errors, and accordingly, an overestimation of the noise bias, based solely on the bit-flip and phase-flip times, underscoring the need for direct benchmarking of the operational noise characteristics. The additional structure of the bit-flip errors with several local minima is related to the details of the gate pulses and leakage discussed in the next section, which may provide insights for improving gate performance through pulse shaping.

As shown in Fig. 5.9a, for gate times longer than 60 ns, the detuned-Kerr-cat qubit has a lower bit-flip error than the resonant-Kerr-cat qubit while maintaining the same phase-flip error, indicating the advantage of the detuned-Kerr-cat qubit with higher noise bias.

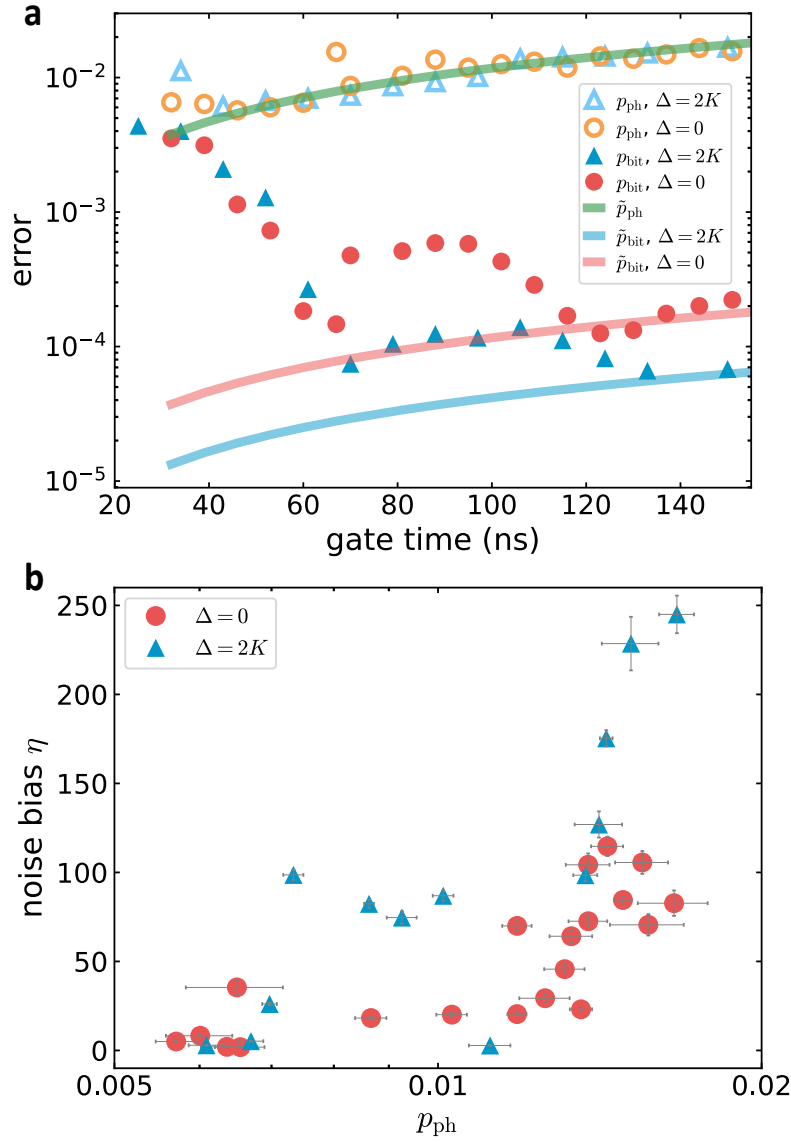


Figure 5.9: DRB results. **a**, The bit-flip p_{bit} and phase-flip p_{ph} errors of resonant- and detuned-Kerr-cat qubit, and corresponding estimation $\tilde{p}_{\text{ph}}$ and $\tilde{p}_{\text{bit}}$ based on the bit-flip and phase-flip times measured in Fig. 5.2. The error bars are smaller than the marker size and therefore not plotted. **b**, The noise bias as a function of phase-flip errors.

Fig. 5.9b further shows the noise-bias of both resonant- and detuned-KCQs with respect to their phase-flip errors. Using the detuned-Kerr-cat qubit, we can achieve a noise bias as high as 250 with phase-flip error rates below 2% for the bias-preserving gate set. This is over twice as large as that for the resonant-Kerr-cat qubit without increasing the phase-

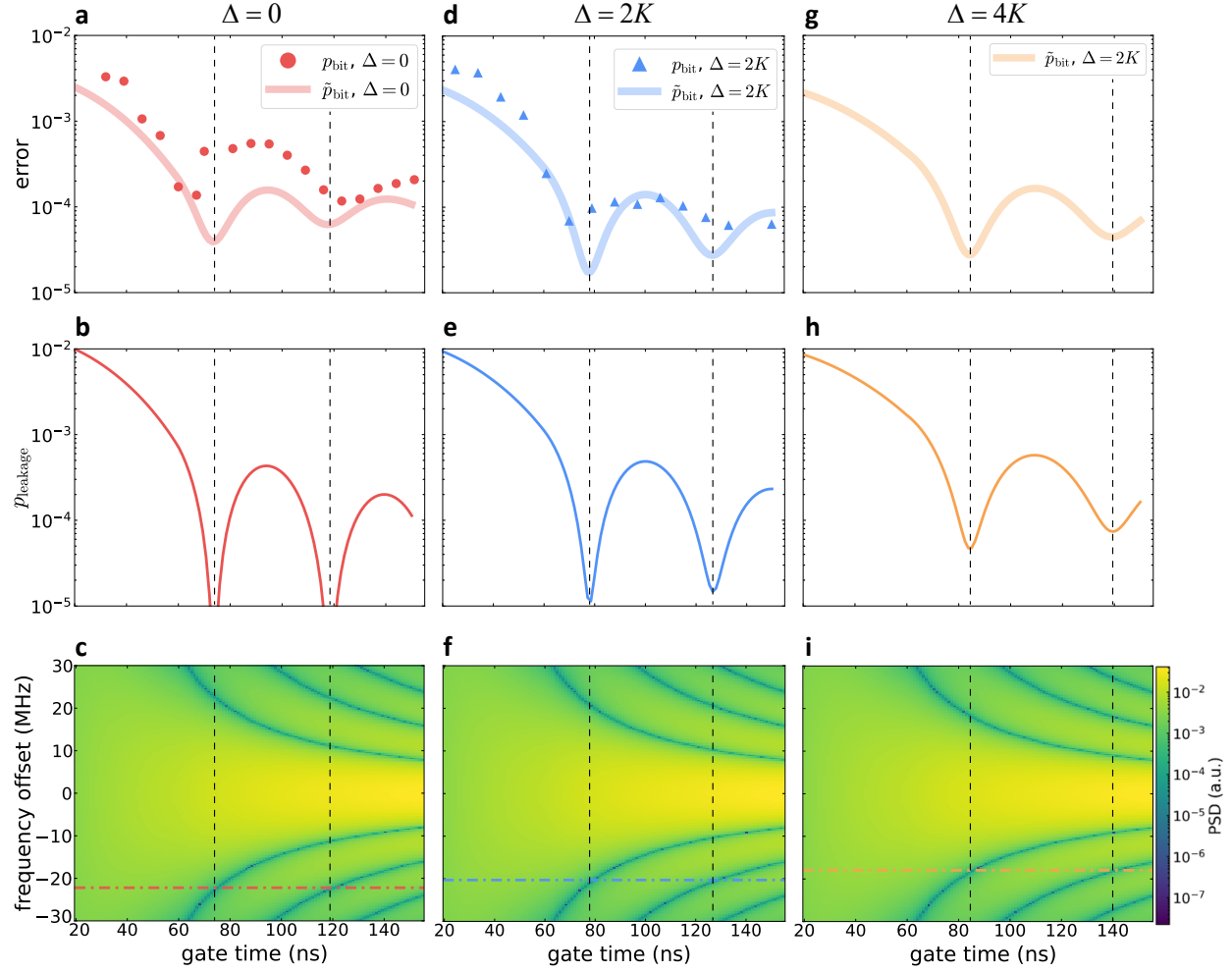


Figure 5.10: Analysis of bit-flip errors. **a** Bit-flip errors of resonant-Kerr-cat qubit from simulation $\tilde{p}_{\text{bit}}$ and experiments p_{bit} . **b** Simulated leakage of resonant-Kerr-cat qubit. **c** Power spectral density (PSD) of the gate pulse, showing that the local minima in leakage and bit-flip errors coincide with the zeros of the PSD at the transition frequency between the ground and first excited states. **d, e, f** Corresponding results for the detuned-Kerr-cat qubit with $\Delta = 2K$. **g, h, i** Corresponding results for the detuned-Kerr-cat qubit with $\Delta = 4K$, for which experimental data were not collected in this study.

flip error. Such high-fidelity bias-preserving gates with a large noise bias, together with high-fidelity quantum nondemolition readout and state initialization, represent the state-of-the-art performance and position detuned-Kerr-cat qubit as a promising platform for the future implementation of hardware-efficient quantum error correction.

Non-adiabaticity and Leakage in Bit-flip Errors

The bit-flip error p_{bit} for $Z(\theta)$ gates exhibits additional fluctuation structures with several local minima before approaching the errors set by the lifetimes. This structure originates from leakage processes induced by the gate drive, where local minima occur when the power spectral density (PSD) of the drive vanishes at the transition frequency between the ground states and the first pair of excited states, thereby minimizing drive-induced excitations.

We simulate the bit-flip errors $\tilde{p}_{\text{bit}}$ of $Z(\theta)$ rotation gates using the full detuned-Kerr-cat qubit Hamiltonian and measured bit-flip lifetime, and compared them with the experimental results p_{bit} in Fig. 5.10a and d. The simulated bit-flip errors exhibit multiple local minima, qualitatively reproducing the patterns observed experimentally. To further illustrate the correlation between bit-flip error and leakage, we also calculate the leakage probability of the $Z(\theta)$ gates, p_{leakage} , defined as the population outside the two ground states after the gate operation. As shown in Fig. 5.10b and e, the leakage errors also follow a similar trend with several local minima which appear at the same gate time for both leakage and bit-flip error as indicated by the dashed lines.

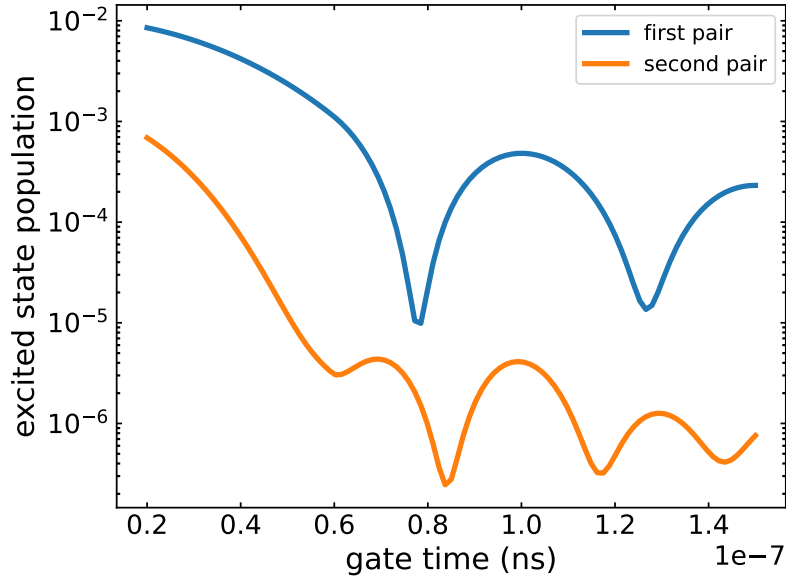
To understand the local minima, we show the PSD of the gate drive defined as

$$S(\delta f) = \lim_{T \rightarrow \infty} \frac{1}{T} \left| \int_0^T \Omega(t) e^{-i(2\pi\delta f + \omega_0)t} dt \right|^2 \quad (5.21)$$

where δf is the frequency offset from the carrier frequency ω_0 , and $\Omega(t)$ is the time domain gate pulse.

As shown in Fig. 5.10c and f, for each gate duration, the PSD exhibits several zeros, while the transition frequency between the ground states and the first pair of excited states is indicated by the red (blue) dash-dotted line for $\Delta = 0$ ($\Delta = 2K$). Importantly, the minima in both the leakage and bit-flip errors, indicated by the dashed lines, occur at gate times where the PSD vanishes at the corresponding transition frequency with minimized drive-induced excitations. This observation provides valuable insight into improving gate performance through derivative-removal techniques such as DRAG pulses [212]. The slight shift in the minima between experiment and simulation may arise from pulse distortions in the microwave wiring, which alter the PSD at the qubit.

To examine the performance of gate operations on detuned-KCQs with higher detunings, we also simulated the bit-flip error, leakage error, and PSD for the $Z(\theta)$ rotation gates on detuned-KCQs with $\Delta = 4K$, assuming a bit-flip time of 1.2 ms. The gate performance shows only minor improvement compared with the case of $\Delta = 2K$, because the $Z(\theta)$ rotation is based on a single-photon process that primarily induces leakage into the first pair of higher excited states. As shown in Fig. 5.11, the excited-state population after the $Z(\theta)$ operations resides mainly in the first pair of higher excited states, while the population in the second pair is approximately an order of magnitude lower. Therefore, the improvement from a larger detuning of $\Delta = 4K$, which makes the second pair of excited states degenerate, remains marginal.

Figure 5.11: Excited state population after applying the $Z(\theta)$ gates

5.5 Outlook and Conclusion

In summary, we have developed a complete quantum control toolbox for detuned-KCQs, including initialization, universal single-qubit gates, and quantum nondemolition readout. Moreover, we provide the first comprehensive experimental benchmarking of noise-biased qubit operations, confirming their bias-preserving properties and revealing the inaccuracy of noise bias estimation from idling bit-flip and phase-flip times. By introducing a detuned stabilization drive to enhance the bit-flip time, we demonstrate the first implementation of high-fidelity single-qubit universal quantum operations on a detuned-Kerr-cat qubit, achieving a bit-flip time of 1.2 ms, which is three times the conventional resonant-Kerr-cat qubit, without further introducing more phase-flip noises. Using GST, we extract the detailed noise structure of the noise-biased qubit operations, and measured SPAM-free gate fidelities of over $(92.5 \pm 0.23)\%$ for the $X(\pi/2)$ gate and $(99.18 \pm 0.066)\%$ for the bias-preserving $Z(\pi/2)$ gate, representing the highest performance reported for KCQs. The minimal bit-flip error and the high noise bias of the bias-preserving gates are further accurately measured by the DRB protocol. While the resonant-Kerr-cat qubit already shows a large noise bias as high as 100 with a phase-flip error rate below 2%, the detuned-Kerr-cat qubit has an even larger noise bias of 250 with a similar phase-flip error rate. With the DRB protocol, we confirmed the enhanced noise-bias without degraded phase-flip error for bias-preserving gates of detuned-Kerr-cat qubit, demonstrating the advantage of detuned-Kerr-cat qubit.

The decay rate of the SNAIL nonlinear oscillator primarily constrains the performance

of the detuned-Kerr-cat qubit. Continued advancements in superconducting circuit materials are expected to reduce this decay rate [213]. Further performance enhancement of detuned-KCQs can be achieved by engineering faster gate operations through pulse shaping to reduce leakage error [97, 214], or through optimized circuit designs to increase the excited state energy gap [183]. In addition, the model violation observed in GST under the CPTP framework for noise characterization implies some out-of-model effects, such as leakage, but a more direct and precise characterization of these errors would be beneficial for optimizing performance.

Our work exemplifies the characterization and benchmarking of noise-biased qubits, providing a framework that can be extended to other quantum systems with structured noises, such as dissipative cat qubits [55, 63, 176], GKP qubits [215–217], and binomial qubits [218]. Compared to other bosonic qubit platforms with biased noise, the detuned-Kerr-cat qubit uniquely achieves high-fidelity quantum nondemolition readout and universal gate operations simultaneously with low hardware complexity. It is also compatible with recently demonstrated two-qubit gate operations in experiment [66]. Leveraging a scalable planar superconducting architecture, our work is well-positioned for extension into multi-qubit processors, paving the way for the application of noise-biased qubits in QEC. Moreover, the enhanced bit-flip time of the detuned-Kerr-cat qubit presents it as a promising ancilla for universal control of a quantum cavity, effectively mitigating parasitic error propagation [219, 220]. As a Kerr parametric oscillator (KPO) with the double-well potential, the detuned-Kerr-cat qubit also opens new possibilities for KPO-network-based Ising machines, fostering advancements in analog quantum computing [221–223].

Bibliography

- [1] Erwin Schrödinger. “Quantisierung als eigenwertproblem”. In: *Annalen der physik* 385.13 (1926), pp. 437–490.
- [2] Jun John Sakurai and Jim Napolitano. *Modern quantum mechanics*. Cambridge University Press, 2020.
- [3] Werner Heisenberg. “Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen”. In: *Original Scientific Papers Wissenschaftliche Originalarbeiten*. Springer, 1985, pp. 382–396.
- [4] Yichen Huang. *Classical simulation of quantum many-body systems*. University of California, Berkeley, 2015.
- [5] Massimo Bernaschi et al. “The quantum transition of the two-dimensional Ising spin glass”. In: *Nature* 631.8022 (2024), pp. 749–754.
- [6] Richard P Feynman. “Simulating physics with computers”. In: *Feynman and computation*. cRc Press, 2018, pp. 133–153.
- [7] Andrew D King et al. “Quantum critical dynamics in a 5,000-qubit programmable spin glass”. In: *Nature* 617.7959 (2023), pp. 61–66.
- [8] Ashley Montanaro. “Quantum algorithms: an overview”. In: *npj Quantum Information* 2.1 (2016), pp. 1–8.
- [9] Lov K Grover. “Quantum mechanics helps in searching for a needle in a haystack”. In: *Physical review letters* 79.2 (1997), p. 325.
- [10] Leo Zhou et al. “Quantum approximate optimization algorithm: Performance, mechanism, and implementation on near-term devices”. In: *Physical Review X* 10.2 (2020), p. 021067.
- [11] Mario Motta and Julia E Rice. “Emerging quantum computing algorithms for quantum chemistry”. In: *Wiley Interdisciplinary Reviews: Computational Molecular Science* 12.3 (2022), e1580.
- [12] Aram W Harrow, Avinandan Hassidim, and Seth Lloyd. “Quantum algorithm for linear systems of equations”. In: *Physical review letters* 103.15 (2009), p. 150502.

- [13] Peter W Shor. “Algorithms for quantum computation: discrete logarithms and factoring”. In: *Proceedings 35th annual symposium on foundations of computer science*. Ieee. 1994, pp. 124–134.
- [14] Edward M Purcell, Henry Cutler Torrey, and Robert V Pound. “Resonance absorption by nuclear magnetic moments in a solid”. In: *Physical review* 69.1-2 (1946), p. 37.
- [15] F Bloch WW Hansen M Packard, F Bloch, and W Hansen. “The nuclear induction experiment”. In: *Physical Review* 70.7-8 (1946), pp. 474–485.
- [16] Isidor Isaac Rabi et al. “A new method of measuring nuclear magnetic moment”. In: *Physical review* 53.4 (1938), p. 318.
- [17] NC Menicucci and Carlton M Caves. “Local realistic model for the dynamics of bulk-ensemble NMR information processing”. In: *Physical review letters* 88.16 (2002), p. 167901.
- [18] Han-Sen Zhong et al. “Quantum computational advantage using photons”. In: *Science* 370.6523 (2020), pp. 1460–1463.
- [19] Emanuel Knill, Raymond Laflamme, and Gerald J Milburn. “A scheme for efficient quantum computation with linear optics”. In: *nature* 409.6816 (2001), pp. 46–52.
- [20] Juan I Cirac and Peter Zoller. “Quantum computations with cold trapped ions”. In: *Physical review letters* 74.20 (1995), p. 4091.
- [21] Shuqi Xu et al. “3D-printed micro ion trap technology for quantum information applications”. In: *Nature* 645.8080 (2025), pp. 362–368.
- [22] Harry Levine et al. “Parallel implementation of high-fidelity multiqubit gates with neutral atoms”. In: *Physical review letters* 123.17 (2019), p. 170503.
- [23] Dolev Bluvstein et al. “Logical quantum processor based on reconfigurable atom arrays”. In: *Nature* 626.7997 (2024), pp. 58–65.
- [24] Daniel Loss and David P DiVincenzo. “Quantum computation with quantum dots”. In: *Physical Review A* 57.1 (1998), p. 120.
- [25] Nico W Hendrickx et al. “A four-qubit germanium quantum processor”. In: *Nature* 591.7851 (2021), pp. 580–585.
- [26] Michael Freedman et al. “Topological quantum computation”. In: *Bulletin of the American Mathematical Society* 40.1 (2003), pp. 31–38.
- [27] Jens Koch et al. “Charge-insensitive qubit design derived from the Cooper pair box”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 76.4 (2007), p. 042319.
- [28] Google Quantum AI. “Quantum error correction below the surface code threshold”. In: *Nature* 638.8052 (2025), pp. 920–926.
- [29] Michel H Devoret, John M Martinis, and John Clarke. “Measurements of macroscopic quantum tunneling out of the zero-voltage state of a current-biased Josephson junction”. In: *Physical review letters* 55.18 (1985), p. 1908.

- [30] Hanhee Paik et al. “Observation of High Coherence in Josephson Junction Qubits Measured in a Three-Dimensional Circuit QED Architecture”. In: *Physical review letters* 107.24 (2011), p. 240501.
- [31] Joseph A Schreier et al. “Suppressing charge noise decoherence in superconducting charge qubits”. In: *Physical Review B—Condensed Matter and Materials Physics* 77.18 (2008), p. 180502.
- [32] Chris Macklin et al. “A near-quantum-limited Josephson traveling-wave parametric amplifier”. In: *Science* 350.6258 (2015), pp. 307–310.
- [33] Jose Aumentado. “Superconducting parametric amplifiers: The state of the art in Josephson parametric amplifiers”. In: *IEEE Microwave magazine* 21.8 (2020), pp. 45–59. URL: <https://doi.org/10.1109/MMM.2020.2993476>.
- [34] Peter W Shor. “Scheme for reducing decoherence in quantum computer memory”. In: *Physical review A* 52.4 (1995), R2493.
- [35] Victor V Albert and Philippe Faist. “Error-correction zoo”. In: *2023 APS March Meeting*. APS. 2023.
- [36] Simon J Devitt, William J Munro, and Kae Nemoto. “Quantum error correction for beginners”. In: *Reports on Progress in Physics* 76.7 (2013), p. 076001.
- [37] Austin G Fowler et al. “Surface codes: Towards practical large-scale quantum computation”. In: *Physical Review A—Atomic, Molecular, and Optical Physics* 86.3 (2012), p. 032324.
- [38] William E Ryan et al. “An introduction to LDPC codes”. In: *CRC Handbook for Coding and Signal Processing for Recording Systems* 5.2 (2004), pp. 1–23.
- [39] Nathan Lacroix et al. “Scaling and logic in the color code on a superconducting quantum processor”. In: *Nature* (2025), pp. 1–3.
- [40] Ian D Kivlichan et al. “Improved fault-tolerant quantum simulation of condensed-phase correlated electrons via trotterization”. In: *Quantum* 4 (2020), p. 296.
- [41] Joonho Lee et al. “Even more efficient quantum computations of chemistry through tensor hypercontraction”. In: *PRX Quantum* 2.3 (2021), p. 030305.
- [42] Irfan Siddiqi. “Engineering high-coherence superconducting qubits”. In: *Nature Reviews Materials* 6.10 (2021), pp. 875–891.
- [43] András Gyenis et al. “Moving beyond the transmon: Noise-protected superconducting quantum circuits”. In: *PRX quantum* 2.3 (2021), p. 030101.
- [44] Long B Nguyen et al. “High-coherence fluxonium qubit”. In: *Physical Review X* 9.4 (2019), p. 041041.
- [45] Ivan V Pechenezhskiy et al. “The superconducting quasicharge qubit”. In: *Nature* 585.7825 (2020), pp. 368–371.

- [46] András Gyenis et al. “Experimental realization of a protected superconducting circuit derived from the $0-\pi$ qubit”. In: *PRX Quantum* 2.1 (2021), p. 010339.
- [47] WC Smith et al. “Superconducting circuit protected by two-Cooper-pair tunneling”. In: *npj Quantum Information* 6.1 (2020), p. 8.
- [48] Atharv Joshi, Kyungjoo Noh, and Yvonne Y Gao. “Quantum information processing with bosonic qubits in circuit QED”. In: *Quantum Science and Technology* 6.3 (2021), p. 033001.
- [49] Ling Hu et al. “Quantum error correction and universal gate set operation on a binomial bosonic logical qubit”. In: *Nature Physics* 15.5 (2019), pp. 503–508.
- [50] Daniel Gottesman, Alexei Kitaev, and John Preskill. “Encoding a qubit in an oscillator”. In: *Phys. Rev. A* 64 (1 June 2001), p. 012310. DOI: 10.1103/PhysRevA.64.012310. URL: <https://link.aps.org/doi/10.1103/PhysRevA.64.012310>.
- [51] Philippe Campagne-Ibarcq et al. “Quantum error correction of a qubit encoded in grid states of an oscillator”. In: *Nature* 584.7821 (2020), pp. 368–372.
- [52] Daisuke Iyama et al. “Observation and manipulation of quantum interference in a superconducting Kerr parametric oscillator”. In: *Nature communications* 15.1 (2024), p. 86.
- [53] Ahmed Hajr et al. “High-coherence Kerr-Cat qubit in 2D architecture”. In: *Physical Review X* 14.4 (2024), p. 041049.
- [54] Bingcheng Qing et al. “Quantum benchmarking of high-fidelity noise-biased operations on a detuned Kerr-cat qubit”. In: *Proceedings of the National Academy of Sciences* 123.5 (2026), e2520479123.
- [55] Raphaël Lescanne et al. “Exponential suppression of bit-flips in a qubit encoded in an oscillator”. In: *Nature Physics* 16.5 (2020), pp. 509–513.
- [56] Harald Putterman et al. “Hardware-efficient quantum error correction via concatenated bosonic qubits”. In: *Nature* 638.8052 (2025), pp. 927–934.
- [57] J Pablo Bonilla Ataides et al. “The XZZX surface code”. In: *Nature communications* 12.1 (2021), p. 2172.
- [58] David K Tuckett et al. “Fault-tolerant thresholds for the surface code in excess of 5% under biased noise”. In: *Physical review letters* 124.13 (2020), p. 130501.
- [59] David K Tuckett et al. “Tailoring surface codes for highly biased noise”. In: *Physical Review X* 9.4 (2019), p. 041031.
- [60] David K Tuckett, Stephen D Bartlett, and Steven T Flammia. “Ultrahigh error threshold for surface codes with biased noise”. In: *Physical review letters* 120.5 (2018), p. 050505.
- [61] Shruti Puri. “Biased Noise Kerr-Cat Qubits for Fault-Tolerant Quantum Computing”. In: (2021).

- [62] Alexander Grimm et al. “Stabilization and operation of a Kerr-cat qubit”. In: *Nature* 584.7820 (2020), pp. 205–209.
- [63] Ulysse Réglade et al. “Quantum control of a cat qubit with bit-flip times exceeding ten seconds”. In: *Nature* 629.8013 (2024), pp. 778–783.
- [64] Jorge Chávez-Carlos et al. “Spectral kissing and its dynamical consequences in the squeeze-driven Kerr oscillator”. In: *npj Quantum Information* 9.1 (2023), p. 76.
- [65] Nicholas E Frattini et al. “Observation of pairwise level degeneracies and the quantum regime of the Arrhenius law in a double-well parametric oscillator”. In: *Physical Review X* 14.3 (2024), p. 031040.
- [66] Daisuke Hoshi et al. “Entangling Schrödinger’s cat states by bridging discrete-and continuous-variable encoding”. In: *Nature communications* 16.1 (2025), p. 1309.
- [67] Andrew N Jordan and Irfan A Siddiqi. *Quantum measurement: Theory and practice*. Cambridge University Press, 2024.
- [68] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge University Press, 2010.
- [69] John S Bell. “On the Einstein Podolsky Rosen paradox”. In: *Physics Physique Fizika* 1.3 (1964), p. 195.
- [70] Charles H Bennett et al. “Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels”. In: *Physical review letters* 70.13 (1993), p. 1895.
- [71] Charles H Bennett and Gilles Brassard. “Quantum cryptography: Public key distribution and coin tossing”. In: *Theoretical computer science* 560 (2014), pp. 7–11.
- [72] Yang Li et al. “Microsatellite-based real-time quantum key distribution”. In: *Nature* (2025), pp. 1–8.
- [73] Aram Harrow, Patrick Hayden, and Debbie Leung. “Superdense coding of quantum states”. In: *Physical review letters* 92.18 (2004), p. 187901.
- [74] David P DiVincenzo. “The physical implementation of quantum computation”. In: *Fortschritte der Physik: Progress of Physics* 48.9-11 (2000), pp. 771–783.
- [75] Marlan O Scully and M Suhail Zubairy. *Quantum optics*. Cambridge university press, 1997.
- [76] Philip Krantz et al. “A quantum engineer’s guide to superconducting qubits”. In: *Applied physics reviews* 6.2 (2019).
- [77] R Vijay, DH Slichter, and I Siddiqi. “Observation of quantum jumps in a superconducting artificial atom”. In: *Physical review letters* 106.11 (2011), p. 110502.
- [78] R Dassonneville et al. “Fast high-fidelity quantum nondemolition qubit readout via a nonperturbative cross-Kerr coupling”. In: *Physical Review X* 10.1 (2020), p. 011045.

- [79] W Dai et al. “Characterization of Drive-Induced Unwanted State Transitions in Superconducting Circuits”. In: *Physical Review X* 16.1 (2026), p. 011011.
- [80] Daniel Sank et al. “Measurement-induced state transitions in a superconducting qubit: Beyond the rotating wave approximation”. In: *Physical review letters* 117.19 (2016), p. 190503.
- [81] Neng-Chun Chiu et al. “Continuous operation of a coherent 3,000-qubit system”. In: *Nature* 646.8087 (2025), pp. 1075–1080.
- [82] Dolev Bluvstein et al. “A quantum processor based on coherent transport of entangled atom arrays”. In: *Nature* 604.7906 (2022), pp. 451–456.
- [83] Long B Nguyen et al. “Empowering a qudit-based quantum processor by traversing the dual bosonic ladder”. In: *Nature Communications* 15.1 (2024), p. 7117.
- [84] Kurtis Geerlings et al. “Demonstrating a driven reset protocol for a superconducting qubit”. In: *Physical review letters* 110.12 (2013), p. 120501.
- [85] JE Johnson et al. “Heralded state preparation in a superconducting qubit”. In: *Physical review letters* 109.5 (2012), p. 050506.
- [86] Yao-Yao Jiang et al. “Advancements in superconducting quantum computing”. In: *National Science Review* 12.8 (2025), nwaf246.
- [87] XY Jin et al. “Thermal and residual excited-state population in a 3D transmon qubit”. In: *Physical Review Letters* 114.24 (2015), p. 240501.
- [88] Dmitrii S Lvov et al. “Thermometry based on a superconducting qubit”. In: *Physical Review Applied* 23.5 (2025), p. 054079.
- [89] Aidar Sultanov et al. “Protocol for temperature sensing using a three-level transmon circuit”. In: *Applied Physics Letters* 119.14 (2021).
- [90] Charles Kittel and Herbert Kroemer. *Thermal physics*. Macmillan, 1980.
- [91] Shu Watanabe et al. “Nondemolition fluorescence readout and high-fidelity unconditional reset of a fluxonium qubit via dissipation engineering”. In: *Phys. Rev. A* 112 (1 July 2025), p. 012624. DOI: 10.1103/gcgj-srgy. URL: <https://link.aps.org/doi/10.1103/gcgj-srgy>.
- [92] William Happer, Yuan-Yu Jau, and Thad Walker. *Optically pumped atoms*. John Wiley & Sons, 2010.
- [93] Martin Ringbauer et al. “A universal qudit quantum processor with trapped ions”. In: *Nature Physics* 18.9 (2022), pp. 1053–1057.
- [94] Brian C Hall. “Lie groups, Lie algebras, and representations”. In: *Quantum Theory for Mathematicians*. Springer, 2013, pp. 333–366.
- [95] Arian Vezvaei et al. “Virtual-Z Gates and Symmetric Gate Compilation”. In: *PRX Quantum* 6.2 (2025), p. 020348.

- [96] Zhen Chen et al. “Efficient implementation of arbitrary two-qubit gates using unified control”. In: *Nature Physics* 21.9 (2025), pp. 1489–1496.
- [97] Long B Nguyen et al. “Programmable Heisenberg interactions between Floquet qubits”. In: *Nature Physics* 20.2 (2024), pp. 240–246.
- [98] Bradley K Mitchell et al. “Hardware-efficient microwave-activated tunable coupling between superconducting qubits”. In: *Physical review letters* 127.20 (2021), p. 200502.
- [99] AD Patterson et al. “Calibration of a cross-resonance two-qubit gate between directly coupled transmons”. In: *Physical Review Applied* 12.6 (2019), p. 064013.
- [100] David C McKay et al. “Universal gate for fixed-frequency qubits via a tunable bus”. In: *Physical Review Applied* 6.6 (2016), p. 064007.
- [101] SA Caldwell et al. “Parametrically activated entangling gates using transmon qubits”. In: *Physical Review Applied* 10.3 (2018), p. 034050.
- [102] Matthew P Bland et al. “Millisecond lifetimes and coherence times in 2D transmon qubits”. In: *Nature* (2025), pp. 1–6.
- [103] Aaron Somoroff et al. “Millisecond coherence in a superconducting qubit”. In: *Physical Review Letters* 130.26 (2023), p. 267001.
- [104] William K Wootters and Wojciech H Zurek. “A single quantum cannot be cloned”. In: *Nature* 299.5886 (1982), pp. 802–803.
- [105] Sihao Huang et al. “Microwave package design for superconducting quantum processors”. In: *PRX quantum* 2.2 (2021), p. 020306.
- [106] Mustafa Bal et al. “Systematic improvements in transmon qubit coherence enabled by niobium surface encapsulation”. In: *npj Quantum Information* 10.1 (2024), p. 43.
- [107] B Abdisatarov et al. “Direct measurement of microwave loss in Nb films for superconducting qubits”. In: *Applied Physics Letters* 125.12 (2024).
- [108] Larry Chen et al. “Scalable and Site-Specific Frequency Tuning of Two-Level System Defects in Superconducting Qubit Arrays”. In: *arXiv preprint arXiv:2503.04702* (2025).
- [109] Jiawei Qiu et al. “Suppressing Coherent Two-Qubit Errors via Dynamical Decoupling”. In: *Phys. Rev. Appl.* 16 (5 Nov. 2021), p. 054047. DOI: 10.1103/PhysRevApplied.16.054047. URL: <https://link.aps.org/doi/10.1103/PhysRevApplied.16.054047>.
- [110] Patrick M Harrington et al. “Synchronous detection of cosmic rays and correlated errors in superconducting qubit arrays”. In: *Nature Communications* 16.1 (2025), p. 6428.
- [111] Andreas Ketterer and Thomas Wellens. “Characterizing crosstalk of superconducting transmon processors”. In: *Physical Review Applied* 20.3 (2023), p. 034065.

- [112] Akel Hashim et al. “Practical Introduction to Benchmarking and Characterization of Quantum Computers”. In: *PRX Quantum* 6 (3 Aug. 2025), p. 030202. DOI: 10.1103/PRXQuantum.6.030202. URL: <https://link.aps.org/doi/10.1103/PRXQuantum.6.030202>.
- [113] Felix Thomsen et al. “Low-overhead quantum computing with the color code”. In: *Physical Review Research* 6.4 (2024), p. 043125.
- [114] Peter Brooks. *Quantum error correction with biased noise*. California Institute of Technology, 2013.
- [115] John Bardeen, Leon N Cooper, and John Robert Schrieffer. “Theory of superconductivity”. In: *Physical review* 108.5 (1957), p. 1175.
- [116] WJ Moore. *Quantum theory of solids (Kittel, C.)* 1964.
- [117] Ingolf V Hertel and Claus-Peter Schulz. *Atoms, Molecules and Optical Physics*. 2022.
- [118] Alain J Brizard. *Introduction To Lagrangian Mechanics*. World Scientific Publishing Company, 2014.
- [119] Bingcheng Qing et al. “Broadband coplanar-waveguide-based impedance-transformed Josephson parametric amplifier”. In: *Physical Review Research* 6.1 (2024), p. L012035.
- [120] Josh Y Mutus et al. “Strong environmental coupling in a Josephson parametric amplifier”. In: *Applied Physics Letters* 104.26 (2014).
- [121] Yapeng Lu et al. “Broadband Josephson parametric amplifier using lumped-element transmission line impedance matching architecture”. In: *Applied Physics Letters* 120.8 (2022). URL: <https://pubs.aip.org/aip/apl/article-abstract/120/8/082601/2833177/Broadband-%7BJosephson%7D-parametric-amplifier-using?redirectedFrom=fulltext>.
- [122] Leonardo Ranzani et al. “Wideband Josephson parametric amplifier with integrated transmission line transformer”. In: *2022 IEEE International Conference on Quantum Computing and Engineering (QCE)*. IEEE. 2022, pp. 314–319. URL: <https://ieeexplore.ieee.org/document/9951223>.
- [123] Maria-Lavinia Iordache et al. “Parameter identification of magnetic coupled resonators in power wireless transfer systems”. In: *2016 International Symposium on Fundamentals of Electrical Engineering (ISFEE)*. IEEE. 2016, pp. 1–6.
- [124] J David Irwin and Robert M Nelms. *Engineering circuit analysis*. John Wiley & Sons, 2021.
- [125] JL Safko, Herbert Goldstein, and C Poole. *Classical mechanics*. 2002.
- [126] John David Jackson. *Classical electrodynamics*. John Wiley & Sons, 2021.
- [127] Ted Van Duzer and Charles W Turner. *Principles of superconductive devices and circuits*. Prentice Hall PTR, 1998.

- [128] NE Frattini et al. “Optimizing the nonlinearity and dissipation of a SNAIL parametric amplifier for dynamic range”. In: *Physical Review Applied* 10.5 (2018), p. 054020. URL: <https://journals.aps.org/prapplied/abstract/10.1103/PhysRevApplied.10.054020>.
- [129] Bibek Bhandari et al. “Symmetrically threaded superconducting quantum interference devices as next-generation Kerr-cat qubits”. In: *PRX Quantum* 6.3 (2025), p. 030338.
- [130] Helmut Eschrig. “Legendre Transformation”. In: *The Fundamentals of Density Functional Theory*. Springer, 1996, pp. 99–126.
- [131] João Carlos Alves Barata and Mahir Saleh Hussein. “The Moore–Penrose pseudoinverse: A tutorial review of the theory”. In: *Brazilian Journal of Physics* 42.1 (2012), pp. 146–165.
- [132] Rukhsan Ul Haq and Keshav Singh. “A systematic method for Schrieffer-Wolff transformation and its generalizations”. In: *arXiv preprint arXiv:2004.06534* (2020).
- [133] Youngkyu Sung et al. “Realization of high-fidelity CZ and ZZ-free iSWAP gates with a tunable coupler”. In: *Physical Review X* 11.2 (2021), p. 021058.
- [134] Heinz-Peter Breuer and Francesco Petruccione. *The theory of open quantum systems*. OUP Oxford, 2002.
- [135] Ido Zuk et al. “Robust gates with spin-locked superconducting qubits”. In: *Physical Review Research* 6.1 (2024), p. 013217.
- [136] Wen Yang, Zhen-Yu Wang, and Ren-Bao Liu. “Preserving qubit coherence by dynamical decoupling”. In: *Frontiers of Physics in China* 6.1 (2011), pp. 2–14.
- [137] Carlton M Caves. “Quantum limits on noise in linear amplifiers”. In: *Physical Review D* 26.8 (1982), p. 1817. URL: <https://journals.aps.org/prd/abstract/10.1103/PhysRevD.26.1817>.
- [138] Alexandre Blais et al. “Circuit quantum electrodynamics”. In: *Rev. Mod. Phys.* 93 (2 May 2021), p. 025005. DOI: 10.1103/RevModPhys.93.025005. URL: <https://link.aps.org/doi/10.1103/RevModPhys.93.025005>.
- [139] Guido Burkard et al. “Semiconductor spin qubits”. In: *Rev. Mod. Phys.* 95 (2 June 2023), p. 025003. DOI: 10.1103/RevModPhys.95.025003. URL: <https://link.aps.org/doi/10.1103/RevModPhys.95.025003>.
- [140] Michael Hatridge et al. “Quantum back-action of an individual variable-strength measurement”. In: *Science* 339.6116 (2013), pp. 178–181. URL: <https://www.science.org/doi/10.1126/science.1226897>.
- [141] Frank Arute et al. “Quantum supremacy using a programmable superconducting processor”. In: *Nature* 574.7779 (2019), pp. 505–510. URL: <https://www.nature.com/articles/s41586-019-1666-5>.

- [142] Behrouz Arash, Jin-Wu Jiang, and Timon Rabczuk. “A review on nanomechanical resonators and their applications in sensors and molecular transportation”. In: *Applied Physics Reviews* 2.2 (Apr. 2015), p. 021301. ISSN: 1931-9401. DOI: 10.1063/1.4916728. eprint: https://pubs.aip.org/aip/apr/article-pdf/doi/10.1063/1.4916728/14572831/021301_1_online.pdf. URL: <https://doi.org/10.1063/1.4916728>.
- [143] AA Clerk et al. “Hybrid quantum systems with circuit quantum electrodynamics”. In: *Nature Physics* 16.3 (2020), pp. 257–267. URL: <https://doi.org/10.1038/s41567-020-0797-9>.
- [144] Allen Caldwell et al. “Dielectric Haloscopes: A New Way to Detect Axion Dark Matter”. In: *Phys. Rev. Lett.* 118 (9 Mar. 2017), p. 091801. DOI: 10.1103/PhysRevLett.118.091801. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.118.091801>.
- [145] T. Nitta et al. “Search for a Dark-Matter-Induced Cosmic Axion Background with ADMX”. In: *Phys. Rev. Lett.* 131 (10 Sept. 2023), p. 101002. DOI: 10.1103/PhysRevLett.131.101002. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.131.101002>.
- [146] Ananda Roy and Michel Devoret. “Introduction to parametric amplification of quantum signals with Josephson circuits”. In: *Comptes Rendus Physique* 17.7 (2016), pp. 740–755. URL: <https://www.sciencedirect.com/science/article/pii/S1631070516300640>.
- [147] VE Manucharyan et al. “Microwave bifurcation of a Josephson junction: Embedding-circuit requirements”. In: *Physical Review B* 76.1 (2007), p. 014524. URL: <https://journals.aps.org/prb/abstract/10.1103/PhysRevB.76.014524>.
- [148] MA Castellanos-Beltran and KW Lehnert. “Widely tunable parametric amplifier based on a superconducting quantum interference device array resonator”. In: *Applied Physics Letters* 91.8 (2007). URL: <https://doi.org/10.1063/1.2773988>.
- [149] Tsuyoshi Yamamoto et al. “Flux-driven Josephson parametric amplifier”. In: *Applied Physics Letters* 93.4 (2008). URL: <https://pubs.aip.org/aip/apl/article-abstract/93/4/042510/324175/Flux-driven-%7BJosephson%7D-parametric-amplifier?redirectedFrom=fulltext>.
- [150] N Bergeal et al. “Phase-preserving amplification near the quantum limit with a Josephson ring modulator”. In: *Nature* 465.7294 (2010), pp. 64–68. URL: <https://doi.org/10.1038/nature09035>.
- [151] M. Hatridge et al. “Dispersive magnetometry with a quantum limited SQUID parametric amplifier”. In: *Phys. Rev. B* 83 (13 Apr. 2011), p. 134501. DOI: 10.1103/PhysRevB.83.134501. URL: <https://link.aps.org/doi/10.1103/PhysRevB.83.134501>.

- [152] David M Pozar. *Microwave engineering*. John Wiley & sons, 2011. URL: [http://mw1.diet.uniroma1.it/people/pisa/RFELSYS/MATERIALE%20INTEGRATIVO/BOOKS/Pozar_Microwave%20Engineering\(2012\).pdf](http://mw1.diet.uniroma1.it/people/pisa/RFELSYS/MATERIALE%20INTEGRATIVO/BOOKS/Pozar_Microwave%20Engineering(2012).pdf).
- [153] Rainee N Simons. *Coplanar waveguide circuits, components, and systems*. John Wiley & Sons, 2004. URL: <https://onlinelibrary.wiley.com/doi/book/10.1002/0471224758>.
- [154] Andrew Wilson Eddins. *Superconducting Circuits for Quantum Metrology with Non-classical Light*. University of California, Berkeley, 2017. URL: <https://escholarship.org/uc/item/7z79m845>.
- [155] Samuel Boutin et al. “Effect of higher-order nonlinearities on amplification and squeezing in Josephson parametric amplifiers”. In: *Physical Review Applied* 8.5 (2017), p. 054030. URL: <https://journals.aps.org/prapplied/abstract/10.1103/PhysRevApplied.8.054030>.
- [156] J Grebel et al. “Flux-pumped impedance-engineered broadband Josephson parametric amplifier”. In: *Applied Physics Letters* 118.14 (2021). URL: <https://pubs.aip.org/aip/apl/article/118/14/142601/40146/Flux-pumped-impedance-engineered-broadband>.
- [157] GJ Dolan. “Offset masks for lift-off photoprocessing”. In: *Applied Physics Letters* 31.5 (1977), pp. 337–339. URL: <https://pubs.aip.org/aip/apl/article-abstract/31/5/337/45825/Offset-masks-for-lift-off-photoprocessing?redirectedFrom=fulltext>.
- [158] A Potts et al. “CMOS compatible fabrication methods for submicron Josephson junction qubits”. In: *IEE Proceedings-Science, Measurement and Technology* 148.5 (2001), pp. 225–228. URL: https://digital-library.theiet.org/content/journals/10.1049/ip-smt_20010395.
- [159] William Livingston. *Continuous feedback on quantum superconducting circuits*. University of California, Berkeley, 2021. URL: https://escholarship.org/content/qt2s508385/qt2s508385_noSplash_5589d2633bed25fbb4e50c1f845522c6.pdf.
- [160] Tanay Roy et al. “Broadband parametric amplification with impedance engineering: Beyond the gain-bandwidth product”. In: *Applied Physics Letters* 107.26 (2015). URL: <https://pubs.aip.org/aip/apl/article-abstract/107/26/262601/235920/Broadband-parametric-amplification-with-impedance?redirectedFrom=fulltext>.
- [161] Jay Gambetta et al. “Qubit-photon interactions in a cavity: Measurement-induced dephasing and number splitting”. In: *Physical Review A* 74.4 (2006), p. 042318. URL: <https://journals.aps.org/pr/abstract/10.1103/PhysRevA.74.042318>.
- [162] William F Kindel, MD Schroer, and KW Lehnert. “Generation and efficient measurement of single photons from fixed-frequency superconducting qubits”. In: *Physical Review A* 93.3 (2016), p. 033817. URL: <https://journals.aps.org/pr/abstract/10.1103/PhysRevA.93.033817>.

- [163] Theodore White et al. “Readout of a quantum processor with high dynamic range Josephson parametric amplifiers”. In: *Applied Physics Letters* 122.1 (2023). URL: <https://doi.org/10.1063/5.0127375>.
- [164] Ryan Kaufman et al. “Josephson parametric amplifier with Chebyshev gain profile and high saturation”. In: *arXiv preprint arXiv:2305.17816* (2023). URL: <https://arxiv.org/pdf/2305.17816.pdf>.
- [165] D Ezenkova et al. “Broadband SNAIL parametric amplifier with microstrip impedance transformer”. In: *Applied Physics Letters* 121.23 (2022). URL: <https://pubs.aip.org/aip/apl/article-abstract/121/23/232601/2834734/Broadband-SNAIL-parametric-amplifier-with?redirectedFrom=fulltext>.
- [166] John M Martinis et al. “Decoherence in Josephson qubits from dielectric loss”. In: *Phys. Rev. Lett.* 95 (2005), p. 210503.
- [167] DJ Reilly et al. “Suppressing spin qubit dephasing by nuclear state preparation”. In: *Science* 321 (2008), pp. 817–821.
- [168] Akel Hashim et al. “Randomized Compiling for Scalable Quantum Computing on a Noisy Superconducting Quantum Processor”. In: *Phys. Rev. X* 11 (2021), p. 041039.
- [169] Google Quantum AI. “Suppressing quantum errors by scaling a surface code logical qubit”. In: *Nature* 614 (2023), pp. 676–681.
- [170] Marios H Michael et al. “New class of quantum error-correcting codes for a bosonic mode”. In: *Phys. Rev. X* 6.3 (2016), p. 031006.
- [171] Harry Levine et al. “Demonstrating a long-coherence dual-rail erasure qubit using tunable transmons”. In: *Phys. Rev. X* 14 (2024), p. 011051.
- [172] Panos Aliferis et al. “Fault-tolerant computing with biased-noise superconducting qubits: a case study”. In: *New J. Phys.* 11 (2009), p. 013061.
- [173] Yue Wu et al. “Erasure conversion for fault-tolerant quantum computing in alkaline earth Rydberg atom arrays”. In: *Nat. Commun.* 13 (2022), p. 4657.
- [174] Qian Xu et al. “Engineering Kerr-cat qubits for hardware efficient quantum error correction”. In: *SPIE* 12015 (2022), pp. 50–63.
- [175] Qian Xu et al. “Tailored XZZX codes for biased noise”. In: *Phys. Rev. Res.* 5 (2023), p. 013035.
- [176] Harald Putterman et al. “Hardware-efficient quantum error correction using concatenated bosonic qubits”. In: *preprint arXiv:2409.13025* (2024).
- [177] Andrew S Darmawan et al. “Practical quantum error correction with the XZZX code and Kerr-cat qubits”. In: *PRX Quantum* 2 (2021), p. 030345.
- [178] Chen Wang et al. “A Schrödinger cat living in two boxes”. In: *Science* 352 (2016), pp. 1087–1091.

- [179] Ronan Gautier, Alain Sarlette, and Mazyar Mirrahimi. “Combined dissipative and hamiltonian confinement of cat qubits”. In: *PRX Quantum* 3 (2022), p. 020339.
- [180] Marius Bild et al. “Schrödinger cat states of a 16-microgram mechanical oscillator”. In: *Science* 380 (2023), pp. 274–278.
- [181] Timo Hillmann and Fernando Quijandría. “Quantum error correction with dissipatively stabilized squeezed-cat qubits”. In: *Phys. Rev. A* 107 (2023), p. 032423.
- [182] Xiaozhou Pan et al. “Protecting the quantum interference of cat states by phase-space compression”. In: *Phys. Rev. X* 13 (2023), p. 021004.
- [183] Bibek Bhandari et al. “Symmetrically Threaded SQUIDs As Next Generation Kerr-cat Qubits”. In: *preprint arXiv:2405.11375* (2024).
- [184] Hayato Goto. “Universal quantum computation with a nonlinear oscillator network”. In: *Phys. Rev. A* 93 (2016), p. 050301.
- [185] Jérémie Guillaud and Mazyar Mirrahimi. “Repetition cat qubits for fault-tolerant quantum computation”. In: *Phys. Rev. X* 9 (2019), p. 041053.
- [186] Shruti Puri et al. “Bias-preserving gates with stabilized cat qubits”. In: *Sci. Adv.* 6 (2020), eaay5901.
- [187] Diego Ruiz et al. “Two-photon driven Kerr quantum oscillator with multiple spectral degeneracies”. In: *Phys. Rev. A* 107 (2023), p. 042407.
- [188] Vinay Ambegaokar and Alexis Baratoff. “Tunneling between superconductors”. In: *Phys. Rev. Lett.* 10 (1963), p. 486.
- [189] Daniel A Steck. *Quantum and atom optics*. 2007.
- [190] Yifang Xu et al. “Dynamic compensation for pump-induced frequency shift in Kerr-cat qubit initialization”. In: *Physical Review Applied* 23.3 (2025), p. 034060.
- [191] Kenichi Kuroda and Masahiro Yuda. “Niobium-stress influence on Nb/Al-oxide/Nb Josephson junctions”. In: *Journal of applied physics* 63.7 (1988), pp. 2352–2357.
- [192] Florent Lecocq et al. “Junction fabrication by shadow evaporation without a suspended bridge”. In: *Nanotechnology* 22 (2011), p. 315302.
- [193] Michael J Moran et al. *Fundamentals of engineering thermodynamics*. John Wiley & Sons, 2010.
- [194] Jayameenakshi Venkatraman et al. “A driven Kerr oscillator with two-fold degeneracies for qubit protection”. In: *PNAS* 121.24 (2024), e2311241121.
- [195] Jayameenakshi Venkatraman et al. “On the static effective Lindbladian of the squeezed Kerr oscillator”. In: *preprint arXiv:2209.11193* (2022).
- [196] Luca Gravina, Fabrizio Minganti, and Vincenzo Savona. “Critical schrödinger cat qubit”. In: *PRX Quantum* 4 (2023), p. 020337.
- [197] Erik Nielsen et al. “Gate set tomography”. In: *Quantum* 5 (2021), p. 557.

- [198] Raphael Brieger, Ingo Roth, and Martin Kliesch. “Compressive gate set tomography”. In: *PRX quantum* 4 (2023), p. 010325.
- [199] Shuxiang Cao et al. “Efficient Characterization of Qudit Logical Gates with Gate Set Tomography Using an Error-Free Virtual Z Gate Model”. In: *Phys. Rev. Lett.* 133 (2024), p. 120802.
- [200] Erik Nielsen et al. *Python GST Implementation (PyGSTi) v. 0.9*. Tech. rep. Sandia National Lab.(SNL-NM), Albuquerque, NM (United States), 2019.
- [201] Robin Blume-Kohout et al. “Robust, self-consistent, closed-form tomography of quantum logic gates on a trapped ion qubit”. In: *preprint arXiv:1310.4492* (2013).
- [202] Holly G Stemp et al. “Tomography of entangling two-qubit logic operations in exchange-coupled donor electron spin qubits”. In: *Nat. Commun.* 15 (2024), p. 8415.
- [203] HP Bartling et al. “Universal high-fidelity quantum gates for spin-qubits in diamond”. In: *preprint arXiv:2403.10633* (2024).
- [204] Akel Hashim et al. “Benchmarking quantum logic operations relative to thresholds for fault tolerance”. In: *npj Quantum Inf.* 9 (2023), p. 109.
- [205] Robin Blume-Kohout et al. “A taxonomy of small markovian errors”. In: *PRX Quantum* 3 (2022), p. 020335.
- [206] Arnaud Carignan-Dugas, Shashank Kumar Ranu, and Patrick Dreher. “Estimating Coherent Contributions to the Error Profile Using Cycle Error Reconstruction”. In: *Quantum* 8 (2024), p. 1367.
- [207] Jaemin Park et al. “Passive and active suppression of transduced noise in silicon spin qubits”. In: *Nature Communications* 16.1 (2025), p. 78.
- [208] Emanuel Knill et al. “Randomized benchmarking of quantum gates”. In: *Phys. Rev. A* 77.1 (2008), p. 012307.
- [209] Jahan Claes and Shruti Puri. “Estimating the bias of CX gates via character randomized benchmarking”. In: *PRX Quantum* 4 (2023), p. 010307.
- [210] Arnaud Carignan-Dugas, Joel J Wallman, and Joseph Emerson. “Characterizing universal gate sets via dihedral benchmarking”. In: *Phys. Rev. A* 92 (2015), p. 060302.
- [211] Robert Wille, Rod Van Meter, and Yehuda Naveh. “IBM’ s Qiskit Tool Chain: Working with and Developing for Real Quantum Computers”. In: *2019 Design, Automation and Test in Europe Conference and Exhibition (DATE)*. 2019, pp. 1234–1240.
- [212] Eric Hyyppä et al. “Reducing leakage of single-qubit gates for superconducting quantum processors using analytical control pulse envelopes”. In: *PRX Quantum* 5.3 (2024), p. 030353.
- [213] Alexander PM Place et al. “New material platform for superconducting transmon qubits with coherence times exceeding 0.3 milliseconds”. In: *Nat. Commun.* 12 (2021), p. 1779.

- [214] Qian Xu et al. “Engineering fast bias-preserving gates on stabilized cat qubits”. In: *Phys. Rev. Res.* 4 (2022), p. 013082.
- [215] Lisa Häggli, Margret Heinze, and Robert König. “Enhanced noise resilience of the surface–Gottesman-Kitaev-Preskill code via designed bias”. In: *Phys. Rev. A* 102.5 (2020), p. 052408.
- [216] Arne L Grimsmo and Shruti Puri. “Quantum error correction with the Gottesman-Kitaev-Preskill code”. In: *PRX Quantum* 2.2 (2021), p. 020101.
- [217] Dany Lachance-Quirion et al. “Autonomous quantum error correction of Gottesman-Kitaev-Preskill states”. In: *Phys. Rev. Lett.* 132.15 (2024), p. 150607.
- [218] Zhongchu Ni et al. “Beating the break-even point with a discrete-variable-encoded logical qubit”. In: *Nature* 616.7955 (2023), pp. 56–60.
- [219] Shruti Puri et al. “Stabilized cat in a driven nonlinear cavity: a fault-tolerant error syndrome detector”. In: *Phys. Rev. X* 9.4 (2019), p. 041009.
- [220] Andy Z Ding et al. “Quantum Control of an Oscillator with a Kerr-cat Qubit”. In: *preprint arXiv:2407.10940* (2024).
- [221] Pablo Álvarez et al. “Biased Ising Model Using Two Coupled Kerr Parametric Oscillators with External Force”. In: *Phys. Rev. Lett.* 132.20 (2024), p. 207401.
- [222] Hayato Goto, Zhirong Lin, and Yasunobu Nakamura. “Boltzmann sampling from the Ising model using quantum heating of coupled nonlinear oscillators”. In: *Sci. Rep.* 8.1 (2018), p. 7154.
- [223] T Yamaji et al. “Correlated oscillations in Kerr parametric oscillators with tunable effective coupling”. In: *Phys. Rev. Appl.* 20.1 (2023), p. 014057.