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Lentfer, John

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Combinatorics of Bosonic-Fermionic Coinvariant Rings

by

John Lentfer

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University of California, Berkeley

Committee in charge:

Professor Mark D. Haiman, Chair
Professor Sylvie Corteel, Co-chair
Professor David Nadler
Professor Vera Serganova

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Abstract

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The (k, j)-bosonic-fermionic coinvariant ring $R_n^{(k,j)}$, defined for k sets of commuting variables and j sets of anticommuting variables, generalizes many previously-studied coinvariant rings. As special cases, $R_n^{(k,j)}$ includes the classical coinvariant ring $R_n^{(1,0)}$, the diagonal coinvariant ring $R_n^{(2,0)}$ studied by Haiman, the superspace coinvariant ring $R_n^{(1,1)}$, Zabrocki's module for the Delta conjecture $R_n^{(2,1)}$, and the diagonal fermionic coinvariant ring $R_n^{(0,2)}$. In this dissertation, we prove results on $R_n^{(k,j)}$ for specific small values of k, j, along with structural results for all k, j.

We give the first conjectural construction of a monomial basis for the coinvariant ring $R_n^{(1,2)}$, for the symmetric group $\mathfrak{S}_n$ acting on one set of commuting and two sets of anticommuting variables. Our construction interpolates between the modified Motzkin path basis for $R_n^{(0,2)}$ of Kim–Rhoades (2022) and the super-Artin basis for $R_n^{(1,1)}$ conjectured by Sagan–Swanson (2024) and proven by Angarone et al. (2025). We prove that our proposed basis has cardinality $2^{n-1}n!$, aligning with a conjecture of Zabrocki (2020) on the dimension of $R_n^{(1,2)}$, and show how it gives a combinatorial expression for the Hilbert series.

We also conjecture a Frobenius series for $R_n^{(1,2)}$. We show that these proposed Hilbert and Frobenius series on $R_n^{(1,2)}$ are equivalent to conjectures of Iraci, Nadeau, and Vanden Wyngaerd (2024) in terms of segmented Smirnov words, by exhibiting a weight-preserving bijection between our proposed basis and their segmented permutations. We extend some of their results on the sign character to hook characters, and give a formula for the m_μ coefficients of the conjectural Frobenius series.

We conjecture a monomial basis for $R_{\mathfrak{B}_n}^{(1,2)}$, the analogous ring for the hyperoctahedral group $\mathfrak{B}_n$, and show that it has cardinality $4^n n!$.

We determine the trigraded multiplicity of the sign character of the triagonal fermionic coinvariant ring $R_n^{(0,3)}$. As a corollary, this proves a conjecture of Bergeron (2020) that the multiplicity of the sign character of $R_n^{(0,3)}$ is $n^2 - n + 1$.

We also give an explicit formula for double hook characters in the diagonal fermionic coinvariant ring $R_n^{(0,2)}$, and discuss methods towards calculating the sign character of $R_n^{(0,4)}$.

We give a multigraded refinement of a conjecture of Bergeron (2020) that the multiplicity of the sign character of the $(1,3)$-bosonic-fermionic coinvariant ring $R_n^{(1,3)}$ is $\frac{1}{2}F_{3n}$, where F_n is the n-th Fibonacci number.

For finite groups G, we show that bosonic-fermionic coinvariant rings have a natural $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module structure. In particular, we show that their character series are sums of super Schur functions $s_\lambda(\mathbf{q}/\mathbf{u})$ times irreducible characters of G with universal coefficients (independent of k, j). In the case where G is the symmetric group with diagonal action, this proves the "Diagonal Supersymmetry" conjecture of Bergeron (2020).

Alfano (1994) determined a monomial basis, bigraded Hilbert series, and bigraded Frobenius series for the ring of diagonal dihedral group coinvariants $R_{\mathfrak{I}_2(n)}^{(2,0)}$. Using diagonal supersymmetry, we determine a universal multigraded character series, universal multigraded Hilbert series, and monomial basis for the generalization to any k sets of bosonic variables and j sets of fermionic variables $R_{\mathfrak{I}_2(n)}^{(k,j)}$.

We determine that the standard character of all (k,j)-bosonic-fermionic coinvariant rings $R_n^{(k,j)}$ has multigraded multiplicity $s_{(1)}(\mathbf{q}/\mathbf{u}) + s_{(2)}(\mathbf{q}/\mathbf{u}) + \cdots + s_{(n-1)}(\mathbf{q}/\mathbf{u})$.

We note a conjectural relationship between multichains in the Tamari lattice and the multiplicity of the sign character in multivariate coinvariant rings $R_n^{(k,0)}$.

We extend the operator conjecture of Haiman (1994) to bosonic-fermionic coinvariant rings $R_n^{(k,j)}$.

To Mom and Dad

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Chapter 1

Introduction

This dissertation is on algebraic combinatorics, which is the field that explores the deep connections between algebraic and combinatorial structures. Many problems in geometry and representation theory can be translated into a combinatorial language, revealing further structure and enabling explicit computations. A central object in algebraic combinatorics is the classical coinvariant algebra

$$R_n^{(1,0)} = \mathbb{C}[x_1, \ldots, x_n] / \langle \mathbb{C}[x_1, \ldots, x_n]_+^{\mathfrak{S}_n} \rangle, \tag{1.1}$$

the quotient of a polynomial ring by symmetric functions without constant term. The symmetric group $\mathfrak{S}_n$ permutes the variables and gives $R_n^{(1,0)}$ a graded $\mathfrak{S}_n$-module structure, where the grading is by total degree. Borel [22] showed that $R_n^{(1,0)}$ is the cohomology ring of the flag variety. As an $\mathfrak{S}_n$-module, it affords the regular representation, and Stanley [91] gave a formula for the graded structure of $R_n^{(1,0)}$ in terms of statistics on standard Young tableaux.

This dissertation investigates a generalization of coinvariant algebras, called the *bosonic-fermionic coinvariant algebra* $R_n^{(k,j)}$, which involves k sets of commuting and j sets of anticommuting variables (defined in equation (1.2)). Studying $R_n^{(k,j)}$ is extremely challenging for general k and j, but we prove one of the first structural results for all k and j (Theorem 1.0.5 below).

A well-known example is the diagonal coinvariant algebra $R_n^{(2,0)}$, studied in seminal work of Haiman [57, 59], which is related to his proof of the Macdonald positivity conjecture [58]. The diagonal coinvariant algebra is isomorphic to the diagonal harmonic space, which is a solution to a system of partial differential equations. The diagonal coinvariant algebra has been involved in several breakthroughs in algebraic combinatorics, including the proofs of the longstanding Shuffle Conjecture [20, 51, 81] and the Loehr–Warrington Conjecture [18, 77]. It is connected to the elliptic Hall algebra [21, 26, 46], to link homology [48, 49, 83, 100], and to theoretical physics [10, 68].

The study of the coinvariant ring with one set of commuting variables and one set of anticommuting variables $R_n^{(1,1)}$ was initiated around 2018. Eventually, the multiplicity of the sign character of $R_n^{(1,1)}$ [96], its dimension and Hilbert series [88], a monomial basis [3, 90] and its Frobenius series [84] were all determined. Zabrocki [101] introduced a module with two sets of commuting variables and one set of anticommuting variables, $R_n^{(2,1)}$, which he conjecturally related to the Delta Theorem [19, 39, 53].

Extending coinvariant algebras beyond the well-studied cases where $k+j \leq 2$ introduces dramatic new complexity but also reveals intriguing patterns. A general understanding of $R_n^{(k,j)}$ remains elusive, especially once $k+j \geq 3$. In fact, our result on the sign character of $R_n^{(0,3)}$ (Theorem 1.0.4) is the first to be established for any $k+j \geq 3$. Recent conjectures of Zabrocki and Bergeron [10, 11, 102] suggest that these algebras possess rich behavior across broader values of k and j; see Figure 1.1 for known and conjectural dimensions in small cases. Among these conjectures, Bergeron's diagonal supersymmetry conjecture plays a central role by identifying a deep relationship between the commutative and anticommutative variables. We prove Bergeron's diagonal supersymmetry conjecture (Theorem 1.0.5) using the representation theory of $\mathfrak{gl}(k|j)$ and Howe duality [74], establishing one of the first structural results across all $R_n^{(k,j)}$.

Building on this, much current research aims to study more of these algebras $R_n^{(k,j)}$ systematically by determining their dimensions, monomial bases, Hilbert series, and Frobenius series.

Problem 1.0.1. Give a monomial basis, dimension formula, multigraded Hilbert series, and multigraded Frobenius series for $R_n^{(k,j)}$.

$k \backslash j$	0	1	2
0	1	2^{n-1}	$\binom{2n-1}{n}$
1	$n!$	$\sum_{i=1}^{n} i!S(n,i)$	$2^{n-1}n!^*$
2	$(n+1)^{n-1}$	$\sum_{i=0}^{n+1} \binom{n+1}{i} \frac{i^n}{2(n+1)}^*$	–
3	$2^n(n+1)^{n-2*}$	–	–

Figure 1.1: Known or conjectured* dimension formulas for $R_n^{(k,j)}$.

Fix k, j as nonnegative integers or infinity. Denote k sets of n commuting (bosonic) variables by $\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}$, where $\boldsymbol{x}^{(i)} = \{x_1^{(i)}, \dots, x_n^{(i)}\}$, and j sets of n anticommuting (fermionic) variables by $\boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}$, where $\boldsymbol{\theta}^{(i)} = \{\theta_1^{(i)}, \dots, \theta_n^{(i)}\}$. Define the (k,j)-*bosonic-fermionic coinvariant ring* by

$$R_n^{(k,j)} = \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}] / \langle \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]_+^{\mathfrak{S}_n} \rangle, \tag{1.2}$$

where $\mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]_+^{\mathfrak{S}_n}$ denotes polynomials without constant term, which are invariant under the diagonal action of the symmetric group on each set of variables (see [9–11, 102]).

$R_n^{(k,j)}$ is a symmetric group module, so it decomposes into a sum of simple modules. The Frobenius characteristic map associates to a symmetric group module a symmetric function, such that each simple module corresponds to a Schur function s_μ. This correspondence allows us to translate between symmetric group modules and the combinatorics of symmetric functions.

The multigraded $\mathfrak{S}_n$-module structure is encoded by the multigraded Frobenius series, denoted by $\mathrm{Frob}(R_n^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j)$. It is obtained by applying the Frobenius characteristic map to each multigraded component of $R_n^{(k,j)}$. Let $\mathbf{q} = \{q_1, \ldots, q_k\}$ denote the gradings for the commuting variables and $\mathbf{u} = \{u_1, \ldots, u_j\}$ denote the gradings for the anti-commuting variables. In symmetric function theory, a symmetric function can be superized. In particular, the superization of a Schur function is a super Schur function, denoted by $s_\lambda(\mathbf{q}/\mathbf{u})$.

The main results and organization of this dissertation follow. Background is discussed in Chapter 2.

Chapter 3: The $(1,2)$-bosonic-fermionic coinvariant ring

In [73], we give the first conjectural construction of a monomial basis $B_n^{(1,2)}$ for the coinvariant ring $R_n^{(1,2)}$. Our construction interpolates between the modified Motzkin path basis $\Pi(n)_{>0}$ of Kim–Rhoades [69] for $R_n^{(0,2)}$ and the super-Artin basis for $R_n^{(1,1)}$ conjectured by Sagan–Swanson [90] and proven by Angarone et al. [3]. We prove that the proposed basis $B_n^{(1,2)}$ has cardinality $2^{n-1}n!$, aligning with a conjecture of Zabrocki [102] on the dimension of $R_n^{(1,2)}$.

Showing that the conjectural basis for $R_n^{(1,2)}$ is indeed a basis would imply the following formula for the Hilbert series, which is a multigraded version of the dimension.

Conjecture 1.0.2 (Lentfer [73])**.** *For $n \geq 1$,*

$$\mathrm{Hilb}(R_n^{(1,2)}; q; u, v) = \sum_{\pi \in \Pi(n)_{>0}} u^{\deg_\theta(\pi)} v^{\deg_\xi(\pi)} \mathrm{stair}_q(\pi). \tag{1.3}$$

Here we set $q := q_1$, $u := u_1$, $v := u_2$, and $\mathrm{stair}_q(\pi)$ is a certain polynomial in $\mathbb{N}[q]$. In particular, this conjecture would imply that $\dim R_n^{(1,2)} = 2^{n-1}n!$.

We also conjecture a Frobenius series for $R_n^{(1,2)}$ in terms of fundamental quasisymmetric functions $Q_{S,n}$ and a statistic $\mathrm{Asc}(b)$ on $B_n^{(1,2)}$.

Conjecture 1.0.3 (Lentfer [73])**.**

$$\mathrm{Frob}(R_n^{(1,2)}; q; u, v) = \sum_{b \in B_n^{(1,2)}} u^{\deg_\theta(b)} v^{\deg_\xi(b)} q^{\deg_x(b)} Q_{\mathrm{Asc}(b),n}. \tag{1.4}$$

We show in [73] that these proposed Hilbert and Frobenius series imply conjectures of Iraci, Nadeau, and Vanden Wyngaerd [65] on the Hilbert and Frobenius series of $R_n^{(1,2)}$ in terms of segmented Smirnov words.

Chapter 4: The sign character of the triagonal fermionic coinvariant ring

In [75], we determine the trigraded multiplicity of the sign character in the triagonal fermionic coinvariant ring $R_n^{(0,3)}$. Note that $s_{(1^n)}$ is the Schur function corresponding to the sign character in the Frobenius series.

Theorem 1.0.4 (Lentfer [75]). *The trigraded multiplicity of the sign character of $R_n^{(0,3)}$, denoted by $\langle \mathrm{Frob}(R_n^{(0,3)}; u_1, u_2, u_3), s_{(1^n)} \rangle$, is*

$$\langle \mathrm{Frob}(R_n^{(0,3)}; u_1, u_2, u_3), s_{(1^n)} \rangle = s_{(n-1)}(u_1, u_2, u_3) + s_{(n-2,1,1)}(u_1, u_2, u_3). \tag{1.5}$$

As a corollary, this proves a conjecture of Bergeron [10] that the multiplicity of the sign character of $R_n^{(0,3)}$ is $n^2 - n + 1$. In [75], we also give an explicit formula for double hook characters in the diagonal fermionic coinvariant ring $R_n^{(0,2)}$.

In [75], we give a multigraded refinement of a conjecture of Bergeron [10] that the multiplicity of the sign character of $R_n^{(1,3)}$ is $\frac{1}{2}F_{3n}$, where F_n is the n-th Fibonacci number.

Chapter 5: Diagonal supersymmetry for coinvariant rings

We show that bosonic-fermionic coinvariant rings have a natural $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[\mathfrak{S}_n]$-module structure, where $U(\mathfrak{gl}(k|j))$ is the universal enveloping algebra of the Lie superalgebra $\mathfrak{gl}(k|j)$. As a consequence, we prove the following “diagonal supersymmetry” conjecture of Bergeron [10].

Theorem 1.0.5 (Lentfer [74]). *Fix a positive integer n. For partitions λ with $\ell(\lambda) \leq n$, and partitions $\mu \vdash n$, there exist nonnegative integer coefficients $c_{\lambda,\mu}$ such that for any (k, j), the multigraded Frobenius series of $R_n^{(k,j)}$ is*

$$\mathrm{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda} \sum_{\mu \vdash n} c_{\lambda,\mu} s_\lambda(\mathbf{q}/\mathbf{u}) s_\mu. \tag{1.6}$$

Chapter 6: Universal series for dihedral group coinvariant rings

In fact, a more general version of Theorem 1.0.5 holds for all finite groups $G \subset \mathrm{GL}(n)$ in place of $\mathfrak{S}_n$. In the case of the symmetric group, it is computationally very difficult to determine the coefficients $c_{\lambda,\mu}$ for general λ and μ. For example, having a combinatorial description of just $c_{\lambda,(1^n)}$ where $\ell(\lambda) \leq 2$ would give us enough information to solve a well-known open problem of Haglund: find a combinatorial proof of the q,t-symmetry of the q,t-Catalan numbers via Dyck paths [54]. However, for the dihedral and cyclic groups, we determine a monomial basis, universal multigraded Hilbert series, and universal multigraded character series for all k, j [76], which is the subject of this chapter.

Chapter 7: Further directions

As previously mentioned, it is quite challenging to determine the coefficients $c_{\lambda,\mu}$ for general λ and μ. The trivial character is known to have multiplicity 1 for all $R_n^{(k,j)}$. We are able to show that the multiplicity of the standard character, that is, the character indexed by the partition $\mu = (n-1,1)$, is $s_{(1)}(\mathbf{q}/\mathbf{u}) + s_{(2)}(\mathbf{q}/\mathbf{u}) + \cdots + s_{(n-1)}(\mathbf{q}/\mathbf{u})$ for all $n \geq 2$.

We also explain the connection between the Tamari lattice and $R_n^{(3,0)}$, and include a conjecture on the multiplicity of the sign character of $R_n^{(4,0)}$.

Finally, we conjecture an extension of Haiman's operator conjecture to $R_n^{(k,j)}$.

Chapter 2

Background

In this chapter, we briefly overview some of the background material from algebraic combinatorics, symmetric function theory, and representation theory. We also give an introduction to the (k, j)-bosonic-fermionic coinvariant ring $R_n^{(k,j)}$, and briefly summarize some of the results on the classical coinvariant ring $R_n^{(1,0)}$ and the diagonal coinvariant ring $R_n^{(2,0)}$.

The reader who is familiar with the basic tenets of combinatorics and representation theory but not familiar with coinvariant algebras should skip ahead to Section 2.4. The reader who is also familiar with coinvariant algebras may skip this chapter entirely, as each of Chapters 3, 4, 5, and 6 begins with an introduction specific to that chapter.

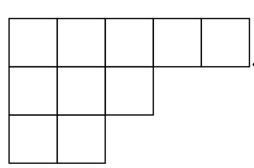
2.1 Combinatorial basics

Most of the information in this section can be found in [8].

Partitions and tableaux

A *partition* λ of a nonnegative integer n is a weakly decreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. Each λ_i is called a *part* of λ, and the number of parts, $\ell = \ell(\lambda)$, is called the *length* of the partition. We call $n = |\lambda|$ the *size* of the partition.

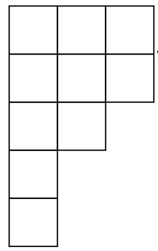
A *Ferrers diagram*[1] is an arrangement of left-justified square boxes such that the top row has λ_1 boxes, the second row has λ_2 boxes, etc. For example, the Ferrers diagram for the partition $(5, 3, 2)$ of 10 is



. (2.1)

[1]This is drawn in English notation. French notation would result from flipping each diagram over a horizontal line.

The *transpose* of a partition, denoted by λ', is the partition obtained from flipping the partition's Ferrers diagram across a line of slope -1 starting from the upper left corner. For example, if $\lambda = (5,3,2)$, then $\lambda' = (3,3,2,1,1)$, with Ferrers diagram



(2.2)

A *standard Young tableau* is a Ferrers diagram in which each box is filled with an integer in $\{1,\ldots,n\}$ exactly once, where n is the size of the partition according to the following rules:

- the numbers increase to the right along each row;
- the numbers increase down each column.

For example, all standard Young tableaux of the partition $(3,1)$ are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}. \tag{2.3}$$

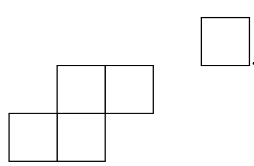
A *semistandard Young tableau* is a Ferrers diagram in which each box is filled with a positive integer according to the following rules:

- the numbers weakly increase to the right along each row;
- the numbers strictly increase down each column.

Now, there are infinitely many semistandard Young tableaux of any (nonempty) shape. For example, some semistandard Young tableaux of the partition $(3,1)$ are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 5 & 6 & 8 \\ \hline 6 \\ \cline{1-1}\end{array}. \tag{2.4}$$

A *skew shape* λ/μ is defined for two partitions λ, μ, when the Ferrers diagram of λ contains the Ferrers diagram of μ. Then the boxes of the Ferrers diagram of λ/μ are those which are in λ, but not in μ. For example, $(5,3,2)$ contains $(4,1)$, so we have the skew shape $(5,3,2)/(4,1)$:



(2.5)

Standard and semistandard tableaux can be similarly defined for skew shapes.

q-analogues

A *q-analogue* of a combinatorial expression O is an expression in terms of a parameter q, which we call $O(q)$, such that upon setting $q = 1$ (or taking a limit as $q \to 1$), we recover O. That is to say,

$$\lim_{q\to 1} O(q) = O. \tag{2.6}$$

This definition does not prescribe what the q-analogue of O must be, in fact, there can be multiple q-analogues of the same thing. In combinatorics, we typically desire to have a q-analogue generalize some desirable combinatorial properties of the original expression.

Define the *q-integer* $[n]_q$ by

$$[n]_q = 1 + q + \cdots + q^{n-1}. \tag{2.7}$$

Define the *q-factorial number* $[n]_q!$ by

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \tag{2.8}$$

and set $[0]_q! = 1$.

Define the *q-binomial coefficients* by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{2.9}$$

Just like the binomial coefficients, the q-binomial coefficients satisfy many identities including Pascal's rule:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q, \tag{2.10}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{2.11}$$

We also have a combinatorial interpretation (see for example [93, Chapter 1.7])

$$\sum_{\lambda \subseteq (n-k)^k} q^{|\lambda|} = \begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{2.12}$$

where $(n-k)^k$ denotes the rectangular partition with k parts each of size $n-k$.

2.2 Symmetric functions

Invariant theory

The material in this section can be found in [8].

Consider the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. The degree of a polynomial $f(x_1, \ldots, x_n)$ is the maximum degree of all of its monomials. A ring R is graded with respect to degree if

$$R \cong \bigoplus_{d \geq 0} R_d \tag{2.13}$$

such that $R_d R_i \subseteq R_{d+i}$, where R_d denotes the $\mathbb{C}$-span of all degree d monomials. Note that $\mathbb{C}[x_1, \ldots, x_n]$ is graded by total degree.

We define the *Hilbert series* of a graded ring R by

$$\mathrm{Hilb}(R; q) = \sum_{d \geq 0} \dim_{\mathbb{C}}(R_d) q^d. \tag{2.14}$$

Note that the dimension of R as a vector space over $\mathbb{C}$ is recovered by setting $q = 1$ (if R is finite-dimensional).

Next we describe how groups act on polynomials. Let G be a finite group which is a subgroup of the group of orthogonal matrices $O_n(\mathbb{C})$. For $g \in G$, we say that

$$g \cdot f(x_1, \ldots, x_n) = f((x_1, \ldots, x_n)g), \tag{2.15}$$

where $(x_1, \ldots, x_n)g$ is the product of the vector $(x_1, \ldots, x_n)$ with the $n \times n$ matrix g.

We say that a polynomial $f(x_1, \ldots, x_n) \in R = \mathbb{C}[x_1, \ldots, x_n]$ is *G-invariant* if $g \cdot f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$ for all $g \in G$. Let R^G denote the subspace of all G-invariant polynomials in R. For groups G generated by reflections, it is well-known that there exist sets of n algebraically independent elements $f_1, \ldots, f_n$, with degrees $d_1, \ldots, d_n$, which generate R^G.

For $G = \mathfrak{S}_n$ the symmetric group, we can take $f_i = p_i(x_1, \ldots, x_n)$, where

$$p_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j^i \tag{2.16}$$

is the i-th power sum symmetric polynomial.

One can then show that we have the Hilbert series

$$\mathrm{Hilb}(R^G; q) = \prod_{i=1}^{n} \frac{1}{1 - q^{d_i}}. \tag{2.17}$$

See [64] for more on the topic.

A *symmetric polynomial* is a polynomial $f(x_1, \ldots, x_n) \in R$ which is $\mathfrak{S}_n$-invariant. That is, for each permutation $\sigma \in \mathfrak{S}_n$, we have that

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n). \tag{2.18}$$

The power sum symmetric polynomials are examples of symmetric polynomials.

Ring of symmetric functions

The material in this section can be found in [92, Chapter 7] or in [78].

A *weak composition* of a nonnegative integer n is a countably infinite sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of nonnegative integers α_i such that the sum of the finitely-many nonzero entries is n.

Let $x = (x_1, x_2, \ldots)$ be an infinite set of indeterminates. A *symmetric function* which is homogeneous of degree n over a commutative unital ring R is the formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha, \tag{2.19}$$

where the sum ranges over all weak compositions $\alpha = (\alpha_1, \alpha_2, \ldots)$ of n, the c_α are some coefficients in the ring R, the notation x^α means $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, and importantly, $f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots)$ for every permutation σ in $\mathfrak{S}_\infty = \bigcup_{i \geq 1} \mathfrak{S}_i$.

A note on terminology: symmetric functions are not really functions, but they are formal power series. Sometimes, *symmetric polynomial* is used when the set of indeterminates x is finite.

The *ring of symmetric functions* Λ_R consists of all symmetric functions over R. This is graded by homogeneous degree. That is, let Λ_R^n consist of all symmetric functions over R of homogeneous degree n. Then it is easy to check that if $f, g \in \Lambda_R^n$ and $a, b \in R$, then $af + bg \in \Lambda_R^n$, so it has the structure of an R-module. Also, if $f \in \Lambda_R^n$ and if $g \in \Lambda_R^m$, then $fg \in \Lambda_R^{n+m}$ and if R contains $\mathbb{Q}$, then Λ_R is an R-algebra. Let $\Lambda_{\mathbb{Q}} = \Lambda$ and $\Lambda_{\mathbb{Q}}^n = \Lambda^n$ for short.

Define the *monomial symmetric function* by

$$m_\lambda = \sum_{\alpha} x^\alpha, \tag{2.20}$$

where the sum ranges over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$ of the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.

Define the *elementary symmetric function* by

$$e_n = m_{(1^n)} = \sum_{1 \leq i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} \tag{2.21}$$

for $n \geq 1$, by $e_0 = 1$, and for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, by

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots . \tag{2.22}$$

Define the *complete homogeneous symmetric function* by

$$h_n = \sum_{\lambda \vdash n} m_\lambda = \sum_{1 \leq i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} \tag{2.23}$$

for $n \geq 1$, by $h_0 = 1$, and for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, by

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots . \tag{2.24}$$

Define the *power sum symmetric function* by

$$p_n = m_n = \sum_{i \geq 1} x_i^n \tag{2.25}$$

for $n \geq 1$, by $p_0 = 1$, and for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, by

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots . \tag{2.26}$$

The fundamental theorem of symmetric functions says that $\{m_\lambda \,|\, \lambda \vdash n\}$, $\{e_\lambda \,|\, \lambda \vdash n\}$, $\{h_\lambda \,|\, \lambda \vdash n\}$, and $\{p_\lambda \,|\, \lambda \vdash n\}$ are all bases for Λ^n. Note that the power sum symmetric functions are not a basis over the ring $\mathbb{Z}$, but the others are.

Schur functions

Denote the set of all semistandard Young tableaux of shape λ/μ by $\mathrm{SSYT}(\lambda/\mu)$. For a semistandard Young tableaux $T \in \mathrm{SSYT}(\lambda/\mu)$, by x^T, we mean $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, where α_i is the number of times that i appears in the filling of the tableau T.

Define the *Schur function* by

$$s_{\lambda/\mu} = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} x^T. \tag{2.27}$$

The Schur functions are symmetric functions, and $\{s_\lambda \,|\, \lambda \vdash n\}$ is also a basis for Λ^n.

We have already defined Schur functions in an alphabet of infinitely many indeterminates $x = (x_1, x_2, \ldots)$. Upon specializing to finitely many indeterminates, the Schur functions may also be defined as the ratio of two alternating polynomials, that is,

$$s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta}, \tag{2.28}$$

where $a_{\lambda+\delta} = \det(x_i^{\lambda_j+n-j})_{i,j=1}^n$ and $a_\delta = \det(x_i^{n-j})_{i,j=1}^n = \prod_{1 \leq i < j \leq n}(x_i - x_j)$ is the Vandermonde determinant.

Omega involution and Hall inner product

The following all define the same algebra endomorphism $\omega : \Lambda \to \Lambda$:

- $\omega(e_\lambda) = h_\lambda$;
- $\omega(h_\lambda) = e_\lambda$;

- $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$, where $\varepsilon_\lambda = (-1)^{|\lambda|-\ell(\lambda)}$;
- $\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$.

The endomorphism ω is an involution, that is, $\omega^2 = id$.

The *Hall inner product* may be defined by any of:

- $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$;
- $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$, where $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ where m_i is the number of parts of size i in λ;
- $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.

The Hall inner product is symmetric and positive definite. The omega involution ω is an isometry, that is $\langle \omega f, \omega g \rangle = \langle f, g \rangle$ for all $f, g \in \Lambda$.

2.3 Representation theory

The basic information in this section can be found in [92, Chapter 7] or [42].

Representation theory of the symmetric group

The *symmetric group* $\mathfrak{S}_n$ is the group of all permutations of n elements.

A *representation* of the symmetric group $\mathfrak{S}_n$ is a group homomorphism $\varphi : \mathfrak{S}_n \to \mathrm{GL}(V)$, where V is a finite-dimensional complex vector space. A *subrepresentation* is a subspace $W \subseteq V$ that is invariant under the group action of $\mathfrak{S}_n$. An *irreducible representation* is a representation which has only the 0 vector space and itself as subrepresentations.

By Maschke's theorem, every representation of the symmetric group $\mathfrak{S}_n$ over $\mathbb{C}$ is a direct sum of irreducible representations. The irreducible representations of the symmetric group $\mathfrak{S}_n$ are indexed by the set of partitions $\{\lambda \vdash n\}$. Furthermore, $\mathfrak{S}_n$ representations are equivalent to $\mathbb{C}[\mathfrak{S}_n]$-modules. A concrete realization of these irreducible representations is given by the *Specht module* S^λ, an irreducible $\mathbb{C}[\mathfrak{S}_n]$-module (see [42] for details of the construction).

A *class function* is a function which is constant on conjugacy classes in the symmetric group. Let CF^n denote the set of all class functions $f : \mathfrak{S}_n \to \mathbb{Q}$. There is a natural scalar product on CF^n given by

$$\langle f, g \rangle_{\mathrm{CF}^n} = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} f(w) g(w). \tag{2.29}$$

The character of a representation $\varphi : \mathfrak{S}_n \to \mathrm{GL}(V)$ is defined by

$$\chi^\varphi(w) = \mathrm{tr}(\varphi(w)), \tag{2.30}$$

where tr denotes the trace of the matrix in $\mathrm{GL}(V)$.

The *Frobenius characteristic map* is a linear function $F : \mathrm{CF}^n \to \Lambda^n$ defined by

$$F(f) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} f(w) p_{\rho(w)} = \sum_{\mu \vdash n} f(\mu) \frac{p_\mu}{z_\mu}, \tag{2.31}$$

where $\rho(w)$ is the partition obtained from rearranging the cycle type of w, and $f(\mu)$ is $f(w)$ for any w such that $\rho(w) = \mu$.

The Frobenius characteristic map is an isometry, that is, $\langle f, g \rangle_{\mathrm{CF}^n} = \langle F(f), F(g) \rangle_{\Lambda^n}$.

Two operations governing the relationship between the symmetric group $\mathfrak{S}_n$ and its subgroups are restriction, which takes a representation of $\mathfrak{S}_n$ back to a subgroup, and induction, which builds a representation of $\mathfrak{S}_n$ from one defined on a subgroup.

The natural subgroups in this context are the Young subgroups. Given a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of n (a sequence of positive integers summing to n), the corresponding *Young subgroup* is

$$\mathfrak{S}_\mu := \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_k}, \tag{2.32}$$

embedded in $\mathfrak{S}_n$ as the subgroup of permutations that separately permute the blocks of consecutive integers $\{1, \ldots, \mu_1\}$, $\{\mu_1 + 1, \ldots, \mu_1 + \mu_2\}$, etc.

Using the language of the group algebra, given a $\mathbb{C}[\mathfrak{S}_n]$-module V, its *restriction* to $\mathfrak{S}_\mu$ is the module $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_\mu} V$, which has the same underlying vector space as V but with the action restricted to the subgroup $\mathfrak{S}_\mu \subset \mathfrak{S}_n$. In the reverse direction, given a $\mathbb{C}[\mathfrak{S}_\mu]$-module W, its *induction* is

$$\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\mu} W \;:=\; \mathbb{C}[\mathfrak{S}_n] \otimes_{\mathbb{C}[\mathfrak{S}_\mu]} W, \tag{2.33}$$

the $\mathbb{C}[\mathfrak{S}_n]$-module formed by extending scalars from $\mathbb{C}[\mathfrak{S}_\mu]$ to $\mathbb{C}[\mathfrak{S}_n]$.

These two operations are adjoint by *Frobenius reciprocity*:

$$\mathrm{Hom}_{\mathfrak{S}_n}\left(\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\mu} W,\, V\right) \;\cong\; \mathrm{Hom}_{\mathfrak{S}_\mu}\left(W,\, \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_\mu} V\right). \tag{2.34}$$

Finally, let R^n denote the set of all *virtual characters* of the symmetric group $\mathfrak{S}_n$, which consists of all $\mathbb{Z}$-linear combinations of the irreducible characters of $\mathfrak{S}_n$. Then it follows that $F : R^n \to \Lambda^n_{\mathbb{Z}}$ is a ring isomorphism. Furthermore, the image of the irreducible symmetric group character χ^λ under the Frobenius characteristic map is the Schur function s_λ, that is,

$$F(\chi^\lambda) = s_\lambda. \tag{2.35}$$

This is important because it allows us to translate between problems on symmetric group representations and on symmetric functions. In particular, because the multiplicity of an irreducible representation in any representation must be a nonnegative integer, we can show that a symmetric function is *Schur-positive* (expands nonnegatively in the Schur basis) by exhibiting a symmetric group module that it is the Frobenius image of.

Representation theory of the general linear group

The *general linear group* $\mathrm{GL}_n(\mathbb{C})$ is the set of all $n \times n$ invertible matrices, with the group operation of matrix multiplication.

A finite-dimensional continuous *representation* of $\mathrm{GL}_n(\mathbb{C})$ is a group homomorphism $\rho : \mathrm{GL}_n(\mathbb{C}) \to GL(V)$, where V is a finite-dimensional complex vector space. We will focus on rational or polynomial representations, wherein the matrix entries of $\rho(g)$ are respectively rational functions or polynomials in the matrix entries of $\mathrm{GL}_n(\mathbb{C})$.

The category of finite-dimensional rational representations of $\mathrm{GL}_n(\mathbb{C})$ is semisimple, so any such representation exhibits complete reducibility, uniquely decomposing into a direct sum of irreducible subrepresentations up to isomorphism.

The classification of these irreducible polynomial modules is governed by the theory of highest weights. Every irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ is uniquely parameterized by a dominant integral weight, which corresponds to an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. The algebraic structure of finite-dimensional rational representations is completely encoded by their characters. Here, the corresponding character is the Schur function $s_\lambda(x_1, \ldots, x_n)$. Notice that a Schur function $s_\lambda(x_1, \ldots, x_n)$ is 0 if λ has more than n parts. Thus the Schur functions $s_\lambda(x_1, \ldots, x_n)$ for partitions of length at most n form an orthonormal basis for the ring of symmetric functions in the variables $(x_1, \ldots, x_n)$. Hence the decomposition of any $\mathrm{GL}_n(\mathbb{C})$ representation into irreducible components is equivalent to expanding its character as a nonnegative linear combination of Schur functions.

Schur–Weyl duality connects the representation theory of the group $\mathrm{GL}_n(\mathbb{C})$ and the symmetric group $\mathfrak{S}_k$ via their natural commuting actions on the k-fold tensor product space $V^{\otimes k}$. It states that under the joint action of $\mathrm{GL}_n(\mathbb{C}) \times \mathfrak{S}_k$, this tensor space decomposes as a multiplicity-free direct sum:

$$V^{\otimes k} \cong \bigoplus_\lambda V_\lambda \otimes S^\lambda, \tag{2.36}$$

where the sum runs over partitions λ of k (with at most n parts), V_λ is an irreducible $\mathrm{GL}_n(\mathbb{C})$-module, and S^λ is the corresponding irreducible $\mathfrak{S}_k$-module.

The general linear Lie superalgebra

The information in this section can be found in [32].

A *vector superspace* V is a vector space with a $\mathbb{Z}_2$ grading, that is, $V = V_0 \oplus V_1$. Elements in V_0 are called *even* and elements in V_1 are called *odd*. The dimension of V is $(\dim V_0 | \dim V_1)$. By $\mathbb{C}^{k|j}$, we mean the superspace with even subspace $\mathbb{C}^k$ and odd subspace $\mathbb{C}^j$ of dimension $(k|j)$.

A *superalgebra* A is a vector superspace $A = A_0 \oplus A_1$ whose bilinear multiplication satisfies $A_0A_0 \subseteq A_0$, $A_0A_1 \subseteq A_1$, $A_1A_0 \subseteq A_1$, and $A_1A_1 \subseteq A_0$. Modules, subalgebras, and ideals are all $\mathbb{Z}_2$-graded.

A *Lie superalgebra* $\mathfrak{g}$ is a superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with bilinear multiplication $[\cdot,\cdot]$ which satisfies, for all homogeneous elements $a, b, c \in \mathfrak{g}$:

- skew-supersymmetry: $[a, b] = -(-1)^{|a|\cdot|b|}[b, a]$;
- the super Jacobi identity: $[a, [b, c]] = [[a, b], c] + (-1)^{|a|\cdot|b|}[b, [a, c]]$.

Our only use of Lie superalgebras will be the general linear Lie superalgebra, denoted $\mathfrak{gl}(k|j)$. This consists of all linear endomorphisms of the superspace $\mathbb{C}^{k|j} = V$, that is, $\mathfrak{gl}(k|j) = \mathrm{End}(V)$.

Choose a basis for V_0 and for V_1, parametrizing the basis for V by the set $\{1, \dots, k, 1', \dots, j'\} =: I(k|j)$. The elementary matrices may be denoted by $E_{h,i}$ where $h, i \in I(k|j)$. Then $\mathfrak{gl}(k|j) = \mathrm{End}(V)$ can be realized as all $(k + j) \times (k + j)$ matrices with entries in $\mathbb{C}$, with a block form of

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{2.37}$$

where A is a $k \times k$ matrix, B is a $k \times j$ matrix, C is a $j \times k$ matrix, and D is a $j \times j$ matrix. Then the even subalgebra $\mathfrak{gl}(k|j)_0$ is comprised of all matrices with B and C blocks of all 0's. In other words, $\mathfrak{gl}(k|j)_0 \cong \mathfrak{gl}(k) \oplus \mathfrak{gl}(j)$. On the other hand, the odd subalgebra $\mathfrak{gl}(k|j)_1$ is comprised of all matrices with A and D blocks of all 0's. Then $\mathfrak{gl}(k|j)_1 \cong (\mathbb{C}^k \otimes \mathbb{C}^{j*}) \oplus (\mathbb{C}^{k*} \otimes \mathbb{C}^j)$, where $\mathbb{C}^{j*}$ denotes the dual space of $\mathbb{C}^j$.

The theory of characters and highest weight theory have extensions to the Lie superalgebra setting. In general, a $\mathfrak{gl}(k|j)$-module is not semisimple. However, there exist certain irreducible $\mathfrak{gl}(k|j)$-modules, which have character $s_\lambda(\mathbf{q}/\mathbf{u})$, a super Schur function. Define a *super Schur function* by

$$s_\lambda(\mathbf{q}/\mathbf{u}) = \sum_{\nu \subseteq \lambda} s_\nu(\mathbf{q}) s_{\lambda'/\nu'}(\mathbf{u}), \tag{2.38}$$

where $\nu \subseteq \lambda$ means that the Ferrers diagram of ν is contained in that of λ. Super Schur functions may also be defined in terms of super tableaux (see [85, Chapter 12.3]). The super Schur functions $s_\lambda(\mathbf{q}/\mathbf{u})$ are characters of the simple $\mathfrak{gl}(k|j)$-modules $U^\lambda_{k|j}$ which appear in the decomposition of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$ via Howe duality (see [61, 62]). This will be discussed more in Chapter 5.

Super Schur functions are uniquely characterized by the following four properties (see [78, I.3 Example 23] but with a different sign convention):

- Homogeneity: The polynomial $s_\lambda(q_1, \dots, q_k/u_1, \dots, u_j)$ is homogeneous of degree $|\lambda|$;
- Factorization: If $\lambda_k \geq j \geq \lambda_{k+1}$ so that $\lambda = (j^k + \tau) \cup \eta$, then

$$s_\lambda(q_1, \dots, q_k/u_1, \dots, u_j) = s_\tau(q_1, \dots, q_k) s_{\eta'}(u_1, \dots, u_j) \prod_{h=1}^{k} \prod_{i=1}^{j} (q_h + u_i); \tag{2.39}$$

- Cancellation:

$$s_\lambda(q_1,\ldots,q_{k-1},q_k/u_1,\ldots,u_{j-1},u_j)|_{u_j=-q_k} = s_\lambda(q_1,\ldots,q_{k-1}/u_1,\ldots,u_{j-1}); \tag{2.40}$$

- Restriction:

$$s_\lambda(q_1,\ldots,q_{k-1},q_k/u_1,\ldots,u_j)|_{q_k=0} = s_\lambda(q_1,\ldots,q_{k-1}/u_1,\ldots,u_j), \tag{2.41}$$

$$s_\lambda(q_1,\ldots,q_k/u_1,\ldots,u_{j-1},u_j)|_{u_j=0} = s_\lambda(q_1,\ldots,q_k/u_1,\ldots,u_{j-1}). \tag{2.42}$$

2.4 Bosonic-fermionic coinvariant rings

For a complex vector space $V=\mathbb{C}^n$, let $\operatorname{Sym}\mathbb{C}^n$ denote the symmetric algebra and $\wedge\mathbb{C}^n$ denote the exterior algebra. For nonnegative integers k,j, define the (k,j)-*bosonic-fermionic coinvariant ring for a finite group* $G\subset \mathrm{GL}(n)$ by

$$R_G^{(k,j)} = ((\operatorname{Sym}\mathbb{C}^n)^{\otimes k}\otimes(\wedge\mathbb{C}^n)^{\otimes j})/\langle((\operatorname{Sym}\mathbb{C}^n)^{\otimes k}\otimes(\wedge\mathbb{C}^n)^{\otimes j})_+^G\rangle, \tag{2.43}$$

where $I_G^{(k,j)} := \langle((\operatorname{Sym}\mathbb{C}^n)^{\otimes k}\otimes(\wedge\mathbb{C}^n)^{\otimes j})_+^G\rangle$ is generated by the G-invariants for the simultaneous action of $k+j$ copies of G on $(\operatorname{Sym}\mathbb{C}^n)^{\otimes k}\otimes(\wedge\mathbb{C}^n)^{\otimes j}$, without constant term.

By placing coordinates, we can identify this with the quotient of a polynomial ring in both commutative and anticommutative variables. Let $\boldsymbol{x}^{(i)}=\{x_1^{(i)},\ldots,x_n^{(i)}\}$ be a set of n commutative (bosonic) variables, and let $\boldsymbol{\theta}^{(i)}=\{\theta_1^{(i)},\ldots,\theta_n^{(i)}\}$ be a set of n anticommutative (fermionic) variables. Commutative variables commute with all variables, while anticommutative variables anticommute only with all anticommutative variables. This means that $x_h^{(a)}x_i^{(b)}=x_i^{(b)}x_h^{(a)}$ and $x_h^{(a)}\theta_i^{(b)}=\theta_i^{(b)}x_h^{(a)}$, while $\theta_h^{(a)}\theta_i^{(b)}=-\theta_i^{(b)}\theta_h^{(a)}$.

Then we may write that

$$R_G^{(k,j)} = \mathbb{C}[\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(k)},\boldsymbol{\theta}^{(1)},\ldots,\boldsymbol{\theta}^{(j)}]/\langle\mathbb{C}[\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(k)},\boldsymbol{\theta}^{(1)},\ldots,\boldsymbol{\theta}^{(j)}]_+^G\rangle, \tag{2.44}$$

where $\mathbb{C}[\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(k)},\boldsymbol{\theta}^{(1)},\ldots,\boldsymbol{\theta}^{(j)}]_+^G$ denotes the polynomials without constant term, which are invariant under the diagonal action of G on each set of variables. In the case where G is the symmetric group $\mathfrak{S}_n$ acting diagonally by permuting the indices of the variables within each set, we denote this by $R_n^{(k,j)}$.

To define $R_G^{(k,j)}$ where k or j is infinite, take the direct limit of the $R_G^{(k,j)}$ as k or j goes to infinity.

Consider the ring $\mathbb{C}[\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(k)},\boldsymbol{\theta}^{(1)},\ldots,\boldsymbol{\theta}^{(j)}]$. We may arrange these variables into a pair M of a $k\times n$ matrix and a $j\times n$ matrix:

$$M_0 = \begin{bmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(k)} & x_2^{(k)} & \cdots & x_n^{(k)} \end{bmatrix}, M_1 = \begin{bmatrix} \theta_1^{(1)} & \theta_2^{(1)} & \cdots & \theta_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1^{(j)} & \theta_2^{(j)} & \cdots & \theta_n^{(j)} \end{bmatrix} \tag{2.45}$$

For each pair A of a $k \times k$ invertible matrix A_0 in GL_k and a $j \times j$ invertible matrix A_1 in GL_j multiply M on the left by A to obtain the product $A \cdot M = (A_0 \cdot M_0, A_1 \cdot M_1)$. Then, in any polynomial $f \in \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$, replace each variable by the corresponding entry of $A \cdot M$. This defines a left $\mathrm{GL}_k \times \mathrm{GL}_j$-action on $\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$.

The ring $\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$ is naturally multigraded by $\mathbf{q} = \{q_1, \ldots, q_k\}$ where q_i is the degree of the variables in the set $\boldsymbol{x}^{(i)}$ and by $\mathbf{u} = \{u_1, \ldots, u_j\}$ where u_i is the degree of the variables in the set $\boldsymbol{\theta}^{(i)}$.

Denote by $(\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}])_{r_1,\ldots,r_k;s_1,\ldots,s_j}$ the homogeneous component of multidegree $q_1, \ldots, q_k; u_1, \ldots, u_j$. Since $\mathrm{GL}_k \times \mathrm{GL}_j$ acts by linear combinations of the k and j rows (separately), the $\mathrm{GL}_k \times \mathrm{GL}_j$-action preserves the total degree $d = r_1 + \cdots + r_k + s_1 + \cdots + s_j$. Hence each total degree d component

$$\bigoplus_{r_1+\cdots+r_k+s_1+\cdots+s_j=d} (\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}])_{r_1,\ldots,r_k;s_1,\ldots,s_j} \tag{2.46}$$

is $\mathrm{GL}_k \times \mathrm{GL}_j$-invariant.

Each permutation $\sigma \in \mathfrak{S}_n$ may be associated to a permutation matrix P_σ, defined by $(P_\sigma)_{i,j} = 1$ if $i = \sigma(j)$ and 0 otherwise. Then the right multiplication $M \cdot P_\sigma = (M_0 \cdot P_\sigma, M_1 \cdot P_\sigma)$ has the effect on M of letting σ act on the indices of all variables: $x_h^{(i)} \mapsto x_{\sigma(h)}^{(i)}$ and $\theta_h^{(i)} \mapsto \theta_{\sigma(h)}^{(i)}$. Then, in any polynomial $f \in \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$, replace each variable by the corresponding entry of the pair $M \cdot P_\sigma$. This defines a right $\mathfrak{S}_n$-action on $\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$.

Because left multiplication by A and right multiplication by P_σ commute, these $\mathrm{GL}_k \times \mathrm{GL}_j$ and $\mathfrak{S}_n$-actions commute, so $\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$ is a $\mathrm{GL}_k \times \mathrm{GL}_j \times \mathfrak{S}_n$-module. By replacing the matrices for the symmetric group $\mathfrak{S}_n$ with another group G which is a matrix subgroup of GL_n, we may generalize this module structure.

Furthermore, by combining both the commuting and anticommuting variables into one matrix, instead of using a pair of matrices, we can act with $\mathfrak{gl}(k|j)$ to put on a $\mathfrak{gl}(k|j)$-module structure. This is discussed in Chapter 5, where we propose that a natural setting for studying $R_G^{(k,j)}$ is as a $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module.

The coinvariant ring $R_G^{(k,j)}$ decomposes as a direct sum of multihomogeneous components, which are G-modules:

$$R_G^{(k,j)} = \bigoplus_{r_1,\ldots,r_k,s_1,\ldots,s_j \geq 0} (R_G^{(k,j)})_{r_1,\ldots,r_k,s_1,\ldots,s_j}. \tag{2.47}$$

We denote the *multigraded Hilbert series* by

$$\begin{aligned}&\mathrm{Hilb}(R_G^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j)\\ &:= \sum_{r_1,\ldots,r_k,s_1,\ldots,s_j \geq 0} \dim\left((R_G^{(k,j)})_{r_1,\ldots,r_k,s_1,\ldots,s_j}\right) q_1^{r_1} \cdots q_k^{r_k} u_1^{s_1} \cdots u_j^{s_j}.\end{aligned}$$

Denote the *multigraded G-character series* by

$$\begin{aligned}&\mathrm{Char}(R_G^{(k,j)};q_1,\ldots,q_k;u_1,\ldots,u_j)\\ &:=\sum_{r_1,\ldots,r_k,s_1,\ldots,s_j\geq 0}\mathrm{Char}\left((R_G^{(k,j)})_{r_1,\ldots,r_k,s_1,\ldots,s_j}\right)q_1^{r_1}\cdots q_k^{r_k}u_1^{s_1}\cdots u_j^{s_j},\end{aligned}$$

where Char denotes the G-character. Since $R_G^{(k,j)}$ is a $\mathrm{GL}_k\times\mathrm{GL}_j\times G$-module its multigraded character series is a sum of products of two Schur functions, which are irreducible characters of polynomial representations of GL_k and GL_j, along with a G-character.

In the special case of the symmetric group $\mathfrak{S}_n$, we may further apply the Frobenius characteristic map, yielding the *multigraded Frobenius series:*

$$\begin{aligned}&\mathrm{Frob}(R_n^{(k,j)};q_1,\ldots,q_k;u_1,\ldots,u_j)\\ &:=\sum_{r_1,\ldots,r_k,s_1,\ldots,s_j\geq 0}F\,\mathrm{Char}\left((R_n^{(k,j)})_{r_1,\ldots,r_k,s_1,\ldots,s_j}\right)q_1^{r_1}\cdots q_k^{r_k}u_1^{s_1}\cdots u_j^{s_j},\end{aligned}$$

where F denotes the Frobenius characteristic map and Char denotes the character. We often omit the word "graded" or "multigraded" when the gradings are clear from the notation or context.

For simplicity, if $k\leq 2$, we will use q,t for q_1,q_2 and if $j\leq 3$, we will use u,v,w for u_1,u_2,u_3.

Recall that $\langle\mathrm{Frob}(R_n^{(k,j)};q_1,\ldots,q_k;u_1,\ldots,u_j),h_1^n\rangle=\mathrm{Hilb}(R_n^{(k,j)};q_1,\ldots,q_k;u_1,\ldots,u_j)$, i.e., the Hilbert series can be recovered from the Frobenius series.

Harmonic spaces

Recall the definition of derivative of anticommuting variables is (see for example [97, Section 1.5]):

$$\partial_{\theta_j^{(a)}}\theta_{i_1}^{(a_1)}\cdots\theta_{i_p}^{(a_p)}=\begin{cases}(-1)^{\ell-1}\theta_{i_1}^{(a_1)}\cdots\hat{\theta}_{i_\ell}^{(a)}\cdots\theta_{i_p}^{(a_p)} & \text{if } j=i_\ell,\\ 0 & \text{otherwise.}\end{cases}\tag{2.48}$$

Then we have that the partial derivatives with respect to commutative variables commute with all partial derivatives, while the partial derivative with respect to anticommutative variables anticommute only with all the partial derivatives of anticommutative variables. This means that $\partial_{x_h^{(a)}}\partial_{x_i^{(b)}}=\partial_{x_i^{(b)}}\partial_{x_h^{(a)}}$ and $\partial_{x_h^{(a)}}\partial_{\theta_i^{(b)}}=\partial_{\theta_i^{(b)}}\partial_{x_h^{(a)}}$, while $\partial_{\theta_h^{(a)}}\partial_{\theta_i^{(b)}}=-\partial_{\theta_i^{(b)}}\partial_{\theta_h^{(a)}}$.

Write $S=\mathbb{C}[\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(k)},\boldsymbol{\theta}^{(1)},\ldots,\boldsymbol{\theta}^{(j)}]$ for short. Define the space of (k,j)-*bosonic-fermionic harmonics* by

$$H_n^{(k,j)}:=\left\{f\in S\;\middle|\;\sum_{i=1}^n\partial_{x_i^{(1)}}^{a_1}\cdots\partial_{x_i^{(k)}}^{a_k}\partial_{\theta_i^{(1)}}^{b_1}\cdots\partial_{\theta_i^{(j)}}^{b_j}f=0\text{ for all }\sum_{i=1}^k a_i+\sum_{i=1}^j b_i>0\right\}.\tag{2.49}$$

Looking at the infinitesimal action of the Lie algebra $\mathfrak{sl}_k \times \mathfrak{sl}_j$ on $H_n^{(k,j)}$ allows us to study $H_n^{(k,j)}$ as a $\mathrm{GL}_k \times \mathrm{GL}_j$-module. Similarly to [57, Section 3.1], we define polarization operators. For $1 \le a < b \le k$ and $1 \le c < d \le j$, we have

$$E_{a,b} = \sum_{p=1}^{n} x_p^{(a)} \partial_{x_p^{(b)}},\ E_{c',d'} = \sum_{p=1}^{n} \theta_p^{(c)} \partial_{\theta_p^{(d)}}, \tag{2.50}$$

and for $1 \le b < a \le k$ and $1 \le d < c \le j$, we have

$$F_{a,b} = \sum_{p=1}^{n} x_p^{(a)} \partial_{x_p^{(b)}},\ F_{c',d'} = \sum_{p=1}^{n} \theta_p^{(c)} \partial_{\theta_p^{(d)}}. \tag{2.51}$$

Along with H operators given by taking the commutators of the E and F operators, these generate the Lie algebra $\mathfrak{sl}_k \times \mathfrak{sl}_j$.

Furthermore $H_n^{(k,j)}$ is a $\mathrm{GL}_k \times \mathrm{GL}_j \times \mathfrak{S}_n$-module. One can show that $H_n^{(k,j)} \cong R_n^{(k,j)}$ as $\mathrm{GL}_k \times \mathrm{GL}_j \times \mathfrak{S}_n$-modules. This was shown in the seemingly special case of $(k, j) = (1, 1)$ in [96, Section 3] and [97, Section 4.1].

$H_n^{(k,j)}$ may also be described as the orthogonal complement to $I_n^{(k,j)}$ with respect to a certain form. We leave the details to [95], where the case of $(k, j) = (1, 1)$ is worked out in great detail, but the general case is essentially the same.

Therein, they show that for certain finite groups G, $R_G^{(1,1)} \cong H_G^{(1,1)}$. We wish to show that $R_G^{(k,j)} \cong H_G^{(k,j)}$. Without loss of generality, say $k \ge j$ and then this would follow from $R_G^{(k,k)} \cong H_G^{(k,k)}$. Rewrite the known isomorphism as $R_{\tilde{G}}^{(1,1)} \cong H_{\tilde{G}}^{(1,1)}$, where

$$\tilde{G} = \underbrace{G \times \cdots \times G}_{k} \tag{2.52}$$

acts on sets of $\tilde{n}$ variables, where $\tilde{n} = nk$. Thus $R_G^{(k,k)} \cong H_G^{(k,k)}$ follows.

To get a $\mathfrak{gl}(k|j)$ action, we describe the action of $\mathfrak{gl}(k|j)$ by superderivations. Notably, these include operators that switch between the commutative and anticommutative variables. For $1 \le a, b \le k$ and $1 \le c, d \le j$, we have

$$E_{a,b} = \sum_{p=1}^{n} x_p^{(a)} \partial_{x_p^{(b)}},\ E_{a,d'} = \sum_{p=1}^{n} x_p^{(a)} \partial_{\theta_p^{(d)}},\ E_{c',b} = \sum_{p=1}^{n} \theta_p^{(c)} \partial_{x_p^{(b)}},\ E_{c',d'} = \sum_{p=1}^{n} \theta_p^{(c)} \partial_{\theta_p^{(d)}}, \tag{2.53}$$

which realize the action of $\mathfrak{gl}(k|j)$ on $\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$ [32, Lemma 5.13].

A polynomial $p \in \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$ is called antisymmetric if $\sigma(p) = \mathrm{sgn}(\sigma)p$ for all $\sigma \in \mathfrak{S}_n$, where σ acts by simultaneously permuting the variables. Let $(H_n^{(k,j)})^\epsilon$ denote the antisymmetric subspace of $H_n^{(k,j)}$, which consists of all elements $p \in H_n^{(k,j)}$ which are antisymmetric. This corresponds to the multigraded multiplicity of the sign character in $R_n^{(k,j)}$, that is,

$$\mathrm{Hilb}((H_n^{(k,j)})^\epsilon; \mathbf{q}; \mathbf{u}) = \langle \mathrm{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}), s_{(1^n)} \rangle. \tag{2.54}$$

2.5 The classical coinvariant ring

This section serves to summarize some of what is known about $R_n^{(1,0)}$, especially to see the contrast with the more challenging cases of $R_n^{(k,j)}$. This material is found in [57] and [60].

Although we have already seen the following more generally for $R_n^{(k,j)}$, it is repeated here for $R_n^{(1,0)}$ for clarity and concreteness. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and consider the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$ by permuting variables. The *classical coinvariant algebra* is the quotient ring

$$R_n^{(1,0)} = \mathbb{C}[x_1, \ldots, x_n]/I_n^{(1,0)}, \tag{2.55}$$

where $I_n^{(1,0)}$ is the ideal generated by all $\mathfrak{S}_n$-invariant polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ of strictly positive degree. Equivalently, $I_n^{(1,0)}$ is generated by the elementary symmetric polynomials $e_1(\boldsymbol{x}), \ldots, e_n(\boldsymbol{x})$, or by the power sum symmetric polynomials $p_1(\boldsymbol{x}), \ldots, p_n(\boldsymbol{x})$. This is a homogeneous ideal, so $R_n^{(1,0)}$ inherits a grading $R_n^{(1,0)} = \bigoplus_{d \geq 0} (R_n^{(1,0)})_d$.

The harmonics perspective is quite useful. The *space of* $\mathfrak{S}_n$*-harmonics* is $H_n^{(1,0)} = (I_n^{(1,0)})^{\perp}$, the orthogonal complement of $I_n^{(1,0)}$ with respect to the *apolar form*

$$\langle f, g \rangle = f(\partial_{x_1}, \ldots, \partial_{x_n}) \cdot g(x_1, \ldots, x_n)\big|_{x_1, \ldots, x_n = 0}. \tag{2.56}$$

The spaces $R_n^{(1,0)}$ and $H_n^{(1,0)}$ are isomorphic as graded $\mathfrak{S}_n$-modules, with an explicit isomorphism given from $H_n^{(1,0)} \to R_n^{(1,0)}$ by $\alpha \mapsto \alpha + I_n^{(1,0)}$.

The space of harmonics $H_n^{(1,0)}$ contains the *Vandermonde determinant*

$$\Delta(\mathbf{x}) = \prod_{1 \leq i < j \leq n} (x_i - x_j). \tag{2.57}$$

$H_n^{(1,0)}$ is precisely the $\mathbb{C}$-span of $\Delta(\mathbf{x})$ and all of its partial derivatives $\partial_{x_1}, \ldots, \partial_{x_n}$. In particular, $\Delta(\mathbf{x})$ lies in the top degree component $(H_n^{(1,0)})_{\binom{n}{2}}$, which is one-dimensional and affords the sign representation of $\mathfrak{S}_n$.

Dimension and Hilbert series

Classical work of Chevalley, Shephard–Todd, and Steinberg asserts that $\mathbb{C}[\boldsymbol{x}]$ is free of rank $n!$ as a module over $\mathbb{C}[\boldsymbol{x}]^{\mathfrak{S}_n}$, so

$$\dim_{\mathbb{C}} R_n^{(1,0)} = n!. \tag{2.58}$$

Moreover, $R_n^{(1,0)}$ is isomorphic to the *regular representation* of $\mathfrak{S}_n$ as an $\mathfrak{S}_n$-module, which is obtained when $\mathfrak{S}_n$ acts on itself on the left. The Hilbert series of $R_n^{(1,0)}$ is

$$\mathrm{Hilb}(R_n^{(1,0)}; q) = [n]_q!. \tag{2.59}$$

Frobenius series

The Frobenius series of $R_n^{(1,0)}$ encodes its graded $\mathfrak{S}_n$-character. One has

$$\mathrm{Frob}(R_n^{(1,0)};q) = \sum_{\lambda \vdash n} \big((1-q)(1-q^2)\cdots(1-q^n)\cdot s_\lambda(1,q,q^2,\ldots)\big)\, s_\lambda. \tag{2.60}$$

This is equal to Stanley's formula [91]:

$$\sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}\; s_\lambda, \tag{2.61}$$

where the inner sum runs over standard Young tableaux of shape λ and $\mathrm{maj}(T)$ is the major index statistic.

2.6 The diagonal coinvariant ring

Let $\mathbb{C}[\boldsymbol{x},\boldsymbol{y}] = \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$. The symmetric group $\mathfrak{S}_n$ acts diagonally: $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_i) = y_{\sigma(i)}$. Let $I_n^{(2,0)} \subset \mathbb{C}[\boldsymbol{x},\boldsymbol{y}]$ be the ideal generated by all $\mathfrak{S}_n$-invariant polynomials of strictly positive degree. The *diagonal coinvariant ring* is

$$R_n^{(2,0)} = \mathbb{C}[\boldsymbol{x},\boldsymbol{y}]/I_n^{(2,0)}. \tag{2.62}$$

By a theorem of Weyl [99], $I_n^{(2,0)}$ is generated by the *polarized power sums* $p_{r,s}(\boldsymbol{x},\boldsymbol{y}) = \sum_i x_i^r y_i^s$ for $1 \le r+s \le n$. Since $I_n^{(2,0)}$ is bihomogeneous, $R_n^{(2,0)} = \bigoplus_{h,i\ge 0} (R_n^{(2,0)})_{h,i}$ is a bigraded $\mathfrak{S}_n$-module.

The *space of diagonal harmonics* is $H_n^{(2,0)} = (I_n^{(2,0)})^\perp$ with respect to the apolar form extended to two sets of variables. We have that $H_n^{(2,0)}$ is isomorphic to $R_n^{(2,0)}$ as a bigraded $\mathfrak{S}_n$-module. Furthermore, $R_n^{(2,0)}$ contains a copy of $R_n^{(1,0)}$ in $\boldsymbol{x}$-degree 0 and also in $\boldsymbol{y}$-degree 0.

Dimension

The "$(n+1)^{n-1}$ conjecture" was conjectured by Haiman in 1994 [57] and proved by Haiman in 2002 [59] via the geometry of the Hilbert scheme of points. It states that

$$\dim_{\mathbb{C}} R_n^{(2,0)} = (n+1)^{n-1}. \tag{2.63}$$

This conjecture attracted attention in part because of the appearance of the delightful combinatorially-meaningful number $(n+1)^{n-1}$. For example $(n+1)^{n-1}$ is the number of rooted labeled forests on vertex set $\{1,\ldots,n\}$, or equivalently the number of labeled rooted trees on $\{0,1,\ldots,n\}$. It also equals the number of *parking functions* $f:\{1,\ldots,n\}\to\{1,\ldots,n\}$ (functions satisfying $|\{i : f(i)\le k\}| \ge k$ for all k) [70]. It turns out that the module structure of parking functions is also connected with the diagonal coinvariants, as we shall see shortly.

The conjecture was also more difficult to prove than it might appear at a first glance. Some of the key techniques which worked on $R_n^{(1,0)}$, such as Gröbner basis theory, were not successfully extended to this setting. Specifically, the polarized power sums are not algebraically independent when there are two or more sets of variables.

The Frobenius series

The ungraded module structure is given by the following.

Theorem 2.6.1 ([59])**.** *As an $\mathfrak{S}_n$-module, $R_n^{(2,0)} \cong \varepsilon \otimes V_n$, where V_n is the permutation representation of $\mathfrak{S}_n$ on $\mathbb{Z}_{n+1}^n/\mathbb{Z}_{n+1}$, and ε is the sign representation.*

There is a natural action of the symmetric group $\mathfrak{S}_n$ on the set of parking functions for a given n, giving it the structure of an $\mathfrak{S}_n$-module. Furthermore, it is graded according to the weight, where the weight $\mathrm{wt}(f)$ of a parking function f is

$$\mathrm{wt}(f) = \binom{n+1}{2} - \sum_i f(i). \tag{2.64}$$

The generating function for $\mathrm{wt}(f)$ over all parking functions f is the inversion enumerator for trees [71].

The singly graded structure, in either the $\boldsymbol{x}$ or in the $\boldsymbol{y}$ variables, is given by the following.

Theorem 2.6.2 ([59])**.** *As a graded $\mathfrak{S}_n$-module, $R_n^{(2,0)} \cong \varepsilon \otimes V_n$, where V_n is the module of parking functions, graded by the weight* wt*, and ε is the sign representation.*

In order to give the bigraded Frobenius series of $R_n^{(2,0)}$, we need a few more definitions. Previously we have discussed symmetric functions over the base ring $\mathbb{Q}$. Extending to the ring $\mathbb{Q}(q,t)$, the *modified Macdonald polynomials* $\widetilde{H}_\lambda$ form a basis for $\Lambda_{\mathbb{Q}(q,t)}$ (see [60] for their definition). The *nabla operator* ∇ [12] acts on symmetric functions in $\Lambda_{\mathbb{Q}(q,t)}$ by $\nabla \widetilde{H}_\mu = t^{n(\mu)} q^{n(\mu')} \widetilde{H}_\mu$, where $n(\mu) = \sum_i (i-1)\mu_i$.

Now we can state Haiman's theorem.

Theorem 2.6.3 ([59])**.** *For any positive integer n,*

$$\mathrm{Frob}(R_n^{(2,0)}; q, t) = \nabla e_n. \tag{2.65}$$

Regarding using this theorem computationally, one must first expand e_n into the Macdonald basis $\widetilde{H}_\lambda$, then apply nabla, before converting to the basis of choice for the answer. Since this is the bigraded Frobenius series of a symmetric group module, we know that when expanded in the Schur basis, the coefficients of each s_λ will be polynomials in $\mathbb{Z}[q,t]$ with nonnegative coefficients.

It follows that the Hilbert series of $R_n^{(2,0)}$ is given by

$$\mathrm{Hilb}(R_n^{(2,0)}; q, t) = \langle h_{(1^n)}, \nabla e_n \rangle. \tag{2.66}$$

More recently, the resolution of the Shuffle conjecture [51] by [28] gives further combinatorial formulas for the Frobenius and Hilbert series of $R_n^{(2,0)}$.

Catalan numbers

The multiplicity of the sign representation ε in $R_n^{(2,0)}$ is the *Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{2.67}$$

A bivariate q,t-Catalan number

$$C_n(q,t) = \langle s_{(1^n)}, \operatorname{Frob}(R_n^{(2,0)}; q, t)\rangle \tag{2.68}$$

refines C_n, and satisfies $C_n(q,t) = C_n(t,q)$. See [54] for more on the q,t-Catalan numbers, and for various combinatorial formulas.

Chapter 3

A conjectural basis for the $(1, 2)$-bosonic-fermionic coinvariant ring

We give the first conjectural construction of a monomial basis for the coinvariant ring $R_n^{(1,2)}$, for the symmetric group $\mathfrak{S}_n$ acting on one set of bosonic (commuting) and two sets of fermionic (anticommuting) variables. Our construction interpolates between the modified Motzkin path basis for $R_n^{(0,2)}$ of Kim–Rhoades (2022) and the super-Artin basis for $R_n^{(1,1)}$ conjectured by Sagan–Swanson (2024) and proven by Angarone et al. (2025). We prove that our proposed basis has cardinality $2^{n-1}n!$, aligning with a conjecture of Zabrocki (2020) on the dimension of $R_n^{(1,2)}$, and show how it gives a combinatorial expression for the Hilbert series.

We also conjecture a Frobenius series for $R_n^{(1,2)}$. We show that these proposed Hilbert and Frobenius series are equivalent to conjectures of Iraci, Nadeau, and Vanden Wyngaerd (2024) on $R_n^{(1,2)}$ in terms of segmented Smirnov words, by exhibiting a weight-preserving bijection between our proposed basis and their segmented permutations. We extend some of their results on the sign character to hook characters, and give a formula for the m_μ coefficients of the conjectural Frobenius series.

Finally, we conjecture a monomial basis for the analogous ring in type B_n, and show that it has cardinality $4^n n!$.

This chapter appears in [73].

3.1 Introduction

The classical coinvariant ring $R_n^{(1,0)} = \mathbb{C}[\boldsymbol{x}_n]/\langle \mathbb{C}[\boldsymbol{x}_n]_+^{\mathfrak{S}_n} \rangle$ is the quotient ring of a polynomial ring in n variables $\boldsymbol{x}_n = \{x_1, \ldots, x_n\}$ by $\mathfrak{S}_n$-invariant polynomials with no constant term. It is well-known (see for example [57, Section 1.5]) to have dimension $n!$, Hilbert series $[n]_q!$, and Frobenius series

$$\sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} s_\lambda.$$

An important basis of the classical coinvariant ring is the Artin basis $\{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \mid 0 \le \alpha_i \le i-1\}$.

In 1994, Haiman [57] introduced the diagonal coinvariant ring $R_n^{(2,0)} = \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{y}_n]/\langle \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{y}_n]_+^{\mathfrak{S}_n}\rangle$, which extends the classical coinvariant ring to two sets of n variables. Here and throughout, $\mathfrak{S}_n$ acts diagonally by permuting the indices of the variables. Given a polynomial ring R, the notation $\langle R_+^{\mathfrak{S}_n}\rangle$ denotes the ideal generated by all polynomials in R which are invariant under the diagonal action of $\mathfrak{S}_n$, with no constant term. In 2002, Haiman [59] proved that $R_n^{(2,0)}$ has dimension $(n+1)^{n-1}$, Hilbert series $\langle \nabla e_n, h_1^n\rangle$, and Frobenius series ∇e_n, using several deep results in algebraic geometry. A combinatorial formula for its Hilbert series was conjectured by Haglund and Loehr [52], and eventually was proven as a consequence of the more general shuffle theorem, conjectured by Haglund, Haiman, Loehr, Remmel, and Ulyanov [51] and proven by Carlsson and Mellit [28], which gives a combinatorial formula for ∇e_n. A monomial basis was given by Carlsson and Oblomkov [29].

Recently, there has been interest (see [6, 10, 11, 102]) in extending the setting to include coinvariant rings with k sets of n commuting variables $\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots$ and j sets of n anticommuting variables $\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots$. We denote this by

$$R_n^{(k,j)} = \mathbb{C}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}]/\langle \mathbb{C}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}]_+^{\mathfrak{S}_n}\rangle. \tag{3.1}$$

Commuting variables commute with all variables. Anticommuting variables anticommute with all anticommuting variables, that is, $\theta_i\theta_j = -\theta_j\theta_i$, which also implies that $\theta_i^2 = 0$.

We briefly overview some recent results on the special cases of $R_n^{(k,j)}$ for which there has been significant progress. Kim and Rhoades [69] showed that the fermionic diagonal coinvariant ring $R_n^{(0,2)} = \mathbb{C}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n]/\langle \mathbb{C}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n]_+^{\mathfrak{S}_n}\rangle$ has dimension $\binom{2n-1}{n}$, confirming a conjecture of Zabrocki [102]. They gave a combinatorial formula for its Hilbert series:

$$\operatorname{Hilb}(R_n^{(0,2)}; u, v) = \sum_{\pi \in \Pi(n)_{>0}} u^{\deg_\theta(\pi)} v^{\deg_\xi(\pi)}, \tag{3.2}$$

where $\Pi(n)_{>0}$ denotes the set of modified Motzkin paths[1] of length n. They also gave a monomial basis for $R_n^{(0,2)}$, and found its Frobenius series [69, Theorem 6.1]:

$$\operatorname{Frob}(R_n^{(0,2)}; u, v) = \sum_{i+j<n} u^i v^j \big(s_{(n-i,1^i)} * s_{(n-j,1^j)} - s_{(n-i+1,1^{i-1})} * s_{(n-j+1,1^{j-1})}\big), \tag{3.3}$$

where $*$ denotes the Kronecker product and $s_{(n+1,1^{-1})}$ is interpreted as 0.

The superspace coinvariant ring is $R_n^{(1,1)} = \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{\theta}_n]/\langle \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n}\rangle$. Sagan and Swanson [90] conjectured, and Rhoades and Wilson [88] proved, that its Hilbert series is

$$\operatorname{Hilb}(R_n^{(1,1)}; q; u) = \sum_{k=1}^{n} u^{n-k}[k]_q! \operatorname{Stir}_q(n,k), \tag{3.4}$$

[1] This and other undefined terms which are needed will be defined in the subsequent sections.

where the q-Stirling number $\mathrm{Stir}_q(n,k)$ is defined by the recurrence relation $\mathrm{Stir}_q(n,k) = [k]_q \mathrm{Stir}_q(n-1,k) + \mathrm{Stir}_q(n-1,k-1)$ with initial conditions $\mathrm{Stir}_q(0,k) = \delta_{k,0}$. This implies that the dimension of $R_n^{(1,1)}$ is the ordered Bell number, which counts the number of ordered set partitions of $\{1,\ldots,n\}$. Sagan and Swanson [90] conjectured and Angarone, Commins, Karn, Murai, and Rhoades [3] proved that a certain super-Artin set is a basis for $R_n^{(1,1)}$. A Frobenius series has been conjectured for $R_n^{(1,1)}$ [6, Equation 5.2]:

$$\mathrm{Frob}(R_n^{(1,1)};q;u) = \sum_{k=0}^{n-1}\sum_{\lambda\vdash n}\sum_{T\in\mathrm{SYT}(\lambda)} u^k q^{\mathrm{maj}(T)-k\,\mathrm{des}(T)+\binom{k}{2}} \begin{bmatrix}\mathrm{des}(T)\\ k\end{bmatrix}_q s_\lambda. \tag{3.5}$$

In general, much less is known about $R_n^{(k,j)}$ when $k+j\geq 3$. In 1994, Haiman [57] conjectured that the dimension of $R_n^{(3,0)}$ is given by $2^n(n+1)^{n-2}$, which is still open. A partially-graded Frobenius series has been conjectured by Bergeron and Préville-Ratelle (originally [14]; phrased as in [6]):

$$\mathrm{Frob}(R_n^{(3,0)};q,t,1) = \sum_{\alpha\preceq\beta} q^{\mathrm{dist}(\alpha,\beta)}\mathbb{L}_\beta(t), \tag{3.6}$$

where $\alpha\preceq\beta$ are pairs of elements (Dyck paths) of the Tamari lattice, $\mathrm{dist}(\alpha,\beta)$ is the length of the longest chain from α to β, and $\mathbb{L}_\beta(t)$ denotes the LLT polynomial associated to the Dyck path β.

There is no conjecture for the dimension of $R_n^{(0,3)}$.

Zabrocki [101] conjectured the following Frobenius series for $R_n^{(2,1)}$:

$$\mathrm{Frob}(R_n^{(2,1)};q,t;u) = \Delta'_{e_{n-1}+ue_{n-2}+\cdots+u^{n-1}}(e_n), \tag{3.7}$$

where Δ'_f is a Macdonald eigenoperator defined by $\Delta'_f\tilde{H}_\mu[X;q,t] = f[B_\mu(q,t)-1]\tilde{H}_\mu[X;q,t]$.[2] Finally, its dimension is conjectured by Zabrocki [102] to be given by $\frac{1}{2}\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{k^n}{n+1}$ (see OEIS sequence A201595 [86]). Haglund, Rhoades, and Shimozono showed that this implies the conjecture in equation (3.5) [55].

D'Adderio, Iraci, and Vanden Wyngaerd [37] introduced certain symmetric function operators Θ_f, called Theta operators, where f is any symmetric function, in order to extend Zabrocki's conjecture on the Frobenius series of $R_n^{(2,1)}$ to a conjectural Frobenius series for $R_n^{(2,2)}$. They conjectured that

$$\mathrm{Frob}(R_n^{(2,2)};q,t;u,v) = \sum_{\substack{k,\ell\geq 0,\\ k+\ell<n}} u^k v^\ell \Theta_{e_k}\Theta_{e_\ell}\nabla e_{n-k-\ell}, \tag{3.8}$$

which is known as the Theta conjecture. Since they showed that $\Delta'_{n-k-1}e_n = \Theta_{e_k}\nabla e_{n-k}$, Zabrocki's conjecture is recovered by setting $v=0$ (equivalently, $\ell=0$).

[2]For definitions of the terms in this expression, see [53].

The ring $R_n^{(1,2)} = \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n]/\langle \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n]_+^{\mathfrak{S}_n}\rangle$ is the main object of study in this paper. Zabrocki [102] conjectured that the dimension of $R_n^{(1,2)}$ is $2^{n-1}n!$, and an ungraded Frobenius series was conjectured by Bergeron [10]:

$$\mathrm{Frob}(R_n^{(1,2)}; q; u, v)|_{q=u=v=1} = \frac{1}{2}\sum_{\mu \vdash n} 2^{\ell(\mu)}(-1)^{n-\ell(\mu)} \binom{\ell(\mu)}{d_1(\mu), \ldots, d_n(\mu)} p_\mu, \tag{3.9}$$

where $\ell(\mu)$ denotes the length of the partition μ and $d_i(\mu)$ denotes the number of parts of size i in μ. Another formulation of the conjecture was given in [6]:

$$\mathrm{Frob}(R_n^{(1,2)}; q; u, v)|_{q=u=v=1} = \sum_{\mu \vdash n} e_\mu \sum_{\alpha \sim \mu} \alpha_1(2\alpha_2 - 1)\cdots(2\alpha_{\ell(\alpha)} - 1), \tag{3.10}$$

where the second sum is over all compositions α whose parts rearrange to the partition μ.

The Theta conjecture specialized at $t = 0$ immediately gives a conjectural Frobenius series for $R_n^{(1,2)}$, however, the specialization can only be done after applying both of the Theta operators, so the resulting formula does not easily simplify. Recently, Iraci, Nadeau, and Vanden Wyngaerd [65] have given a conjectural Hilbert series (equation (3.17)) and conjectural Frobenius series (equation (3.16) and Proposition 3.5.4) using the combinatorics of segmented Smirnov words. However, they did not propose a monomial basis. Their results which will be used in this paper are surveyed in Section 3.5.

Our first main contribution is a combinatorial construction of a set of monomials, denoted by $B_n^{(1,2)}$ (Definition 3.3.1), which we conjecture to give a basis of $R_n^{(1,2)}$ (Conjecture 3.3.2). If the conjecture holds, it implies the following combinatorial Hilbert series (Proposition 3.3.6):

$$\mathrm{Hilb}(R_n^{(1,2)}; q; u, v) = \sum_{\pi \in \Pi(n)_{>0}} u^{\deg_\theta(\pi)} v^{\deg_\xi(\pi)} \mathrm{stair}_q(\pi). \tag{3.11}$$

In support of this conjecture, we show that the cardinality of $B_n^{(1,2)}$ is $2^{n-1}n!$ (Theorem 3.3.7). By establishing a weight-preserving bijection between $B_n^{(1,2)}$ and the set of segmented permutations $\mathrm{SW}(1^n)$ (Theorem 3.6.2), we are able to show that our conjectural Hilbert series is equivalent to the conjectural Hilbert series of Iraci, Nadeau, and Vanden Wyngaerd [65] (Corollary 3.6.4).

Our second main contribution is a simple combinatorial formula for the conjectural Frobenius series of $R_n^{(1,2)}$ (Conjecture 3.4.1):

$$\mathrm{Frob}(R_n^{(1,2)}; q; u, v) = \sum_{b \in B_n^{(1,2)}} u^{\deg_\theta(b)} v^{\deg_\xi(b)} q^{\deg_x(b)} Q_{\mathrm{Asc}(b),n}, \tag{3.12}$$

where $Q_{S,n}$ denotes the fundamental quasisymmetric function. We show that this is equivalent to the conjectural Frobenius series of Iraci, Nadeau, and Vanden Wyngaerd (Theorem 3.6.6). A benefit of using $B_n^{(1,2)}$ instead of segmented permutations is that determining $\mathrm{Asc}(b)$ and

$\deg_x(b)$ for $b \in B_n^{(1,2)}$ is typically more direct than determining $\operatorname{Split}(\sigma)$ and $\operatorname{sminv}(\sigma)$ for $\sigma \in \mathrm{SW}(1^n)$, which fulfill analogous roles.

In Section 3.7, we give a formula for the m_μ coefficient of our conjectural Frobenius series. In Section 3.8, we extend some results of Iraci, Nadeau, and Vanden Wyngaerd on the sign character to hook shapes. Specifically, we give two formulas, one in terms of the proposed basis (Theorem 3.8.1) and one in terms of q-binomial coefficients (Theorem 3.8.4):

$$\begin{aligned}&\langle \operatorname{Frob}(R_n^{(1,2)}; q; u, v), s_{(d+1,1^{n-d-1})}\rangle \\ &\quad = \sum_{k+\ell<n} u^k v^\ell q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q.\end{aligned}$$

In Section 3.9, we consider some of the same questions in type B. Let

$$R_{\mathfrak{B}_n}^{(k,j)} = \mathbb{C}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}] / \langle \mathbb{C}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}]_+^{\mathfrak{B}_n} \rangle, \tag{3.13}$$

where $\mathfrak{B}_n$ denotes the Weyl group of type B and rank n acting diagonally, that is, by permuting variables with sign.

The Hilbert series for $R_{\mathfrak{B}_n}^{(1,0)}$ is well-known to be $[2n]_q!!$, and unknown in general for $R_{\mathfrak{B}_n}^{(2,0)}$, although Gordon [47] constructed a notable singly-graded $\mathfrak{B}_n$-quotient module of $R_{\mathfrak{B}_n}^{(2,0)}$ with dimension $(2n+1)^n$, as conjectured by Haiman [57]. Gordon's construction shows that $\dim R_{\mathfrak{B}_n}^{(2,0)} \geq (2n+1)^n$. Equality holds only for $n = 2, 3$; for $n \geq 4$, Haiman [57] conjectured and Ajila and Griffeth [1] proved that $\dim R_{\mathfrak{B}_n}^{(2,0)} > (2n+1)^n$.

The Hilbert series for $R_{\mathfrak{B}_n}^{(0,2)}$ was found by Kim and Rhoades (in fact, their results were type-independent). The Hilbert series for $R_{\mathfrak{B}_n}^{(1,1)}$ was conjectured by Sagan and Swanson [90] to be:

$$\operatorname{Hilb}(R_{\mathfrak{B}_n}^{(1,1)}; q; u) = \sum_{k=1}^{n} u^{n-k} [2k]_q!! \operatorname{Stir}_q^B(n,k), \tag{3.14}$$

where the type B q-Stirling number $\operatorname{Stir}_q^B(n,k)$ is defined by the recurrence relation $\operatorname{Stir}_q^B(n,k) = [2k+1]_q \operatorname{Stir}_q^B(n-1,k) + \operatorname{Stir}_q^B(n-1,k-1)$ with initial conditions $\operatorname{Stir}_q^B(0,k) = \delta_{k,0}$.

We interpolate between the work of Kim–Rhoades and Sagan–Swanson to give a combinatorial construction of a set of monomials, denoted by $B_{\mathfrak{B}_n}^{(1,2)}$ (Definition 3.9.5), which we conjecture to be a basis of $R_{\mathfrak{B}_n}^{(1,2)}$ (Conjecture 3.9.6). If the conjecture holds, this implies the following combinatorial Hilbert series:

$$\operatorname{Hilb}(R_{\mathfrak{B}_n}^{(1,2)}; q; u) = \sum_{\pi \in \Pi_{\geq 0}(n)} u^{\deg_\theta(\pi)} v^{\deg_\xi(\pi)} \operatorname{stair}_q^B(\pi). \tag{3.15}$$

We also conjecture that $\dim R_{\mathfrak{B}_n}^{(1,2)} = 4^n n!$ (Conjecture 3.9.8), and conclude by showing that the cardinality of $B_{\mathfrak{B}_n}^{(1,2)}$ agrees with this conjecture (Theorem 3.9.10).

3.2 Background

In this section, we describe the Kim–Rhoades basis for $R_n^{(0,2)}$ [69] and the super-Artin basis for $R_n^{(1,1)}$ conjectured by Sagan–Swanson [90] and proven by Angarone, Commins, Karn, Murai, and Rhoades [3].

Define the set of *modified Motzkin paths* of length n, $\Pi(n)_{>0}$, to be the set of all paths $\pi = (p_1, \ldots, p_n)$ in $\mathbb{Z}^2$ such that each step p_i is one of:

(a) an up-step $(1,1)$,

(b) a horizontal step $(1,0)$ with decoration θ_i,

(c) a horizontal step $(1,0)$ with decoration ξ_i,

(d) or a down-step $(1,-1)$ with decoration $\theta_i\xi_i$,

where the first step must be an up-step, and subsequently, the path never goes below the horizontal line $y = 1$.[3]

Define the *weight* $\mathrm{wt}(p_i)$ of a step p_i of a modified Motzkin path to be its decoration, or 1 if it does not have a decoration. Then define the *weight* $\mathrm{wt}(\pi)$ of a modified Motzkin path $\pi \in \Pi(n)_{>0}$ to be the product of the weights of each step p_i, that is,

$$\mathrm{wt}(\pi) := \prod_{p_i \in (p_1, \ldots, p_n) = \pi} \mathrm{wt}(p_i).$$

Definition 3.2.1 ([69])**.** The *Kim–Rhoades basis* $B_n^{(0,2)}$ is the set of all weights of the modified Motzkin paths $\pi \in \Pi(n)_{>0}$, that is,

$$B_n^{(0,2)} := \{\mathrm{wt}(\pi) \mid \pi \in \Pi(n)_{>0}\}.$$

For an example, see Figure 3.1. We are justified in calling it a basis because of the following result of Kim and Rhoades.

Theorem 3.2.2 ([69])**.** *The Kim–Rhoades basis $B_n^{(0,2)}$ is a basis for $R_n^{(0,2)}$.*

Next, we recall the super-Artin basis, defined by Sagan and Swanson. Let $\chi(P)$ be 1 if the proposition P is true, and 0 if the proposition P is false. Let θ_T denote the ordered product $\theta_{t_1} \cdots \theta_{t_k}$ for any subset $T = \{t_1 < \cdots < t_k\} \subseteq \{1, \ldots, n\}$. For any $T \subseteq \{2, \ldots, n\}$, define the *$\alpha$-sequence* $\alpha(T) = (\alpha_1(T), \ldots, \alpha_n(T))$ recursively by the initial condition $\alpha_1(T) = 0$ and for $2 \leq i \leq n$,

$$\alpha_i(T) = \alpha_{i-1}(T) + \chi(i \notin T).$$

Let x^α denote $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for a sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$.

[3]Kim and Rhoades defined the modified Motzkin paths in a slightly different manner, where there are no decorations on the down-steps, but they still contribute $\theta_i\xi_i$ to the weight. Furthermore, they defined the decorations to be just θ or ξ instead of θ_i and ξ_i. Converting between these conventions is straightforward.

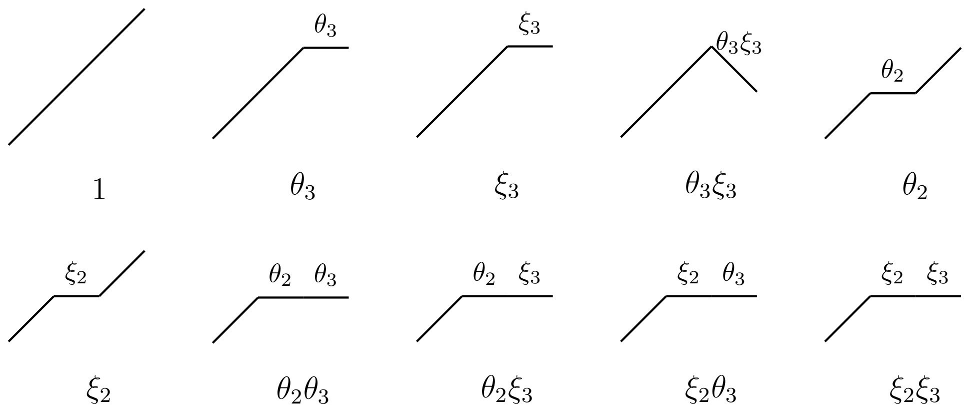


Figure 3.1: The basis $B_3^{(0,2)}$. Each modified Motzkin path is labeled with its corresponding monomial.

Definition 3.2.3 ([90])**.** The super-Artin set is

$$B_n^{(1,1)} := \{x^\alpha \theta_T \,|\, T \subseteq \{2,\dots,n\} \text{ and } \alpha \leq \alpha(T) \text{ componentwise}\}.$$

See Figure 3.2 for an example. The following result was conjectured by Sagan and Swanson, and was proven by Angarone, Commins, Karn, Murai, and Rhoades.

Theorem 3.2.4 ([3])**.** *The super-Artin set $B_n^{(1,1)}$ is a basis for $R_n^{(1,1)}$.*

In this paper, we assume familiarity with the basics of symmetric function theory (see for example [78, Chapter I] or [92, Chapter 7]). Let m_λ, e_λ, h_λ, p_λ, and s_λ denote respectively the monomial, elementary, complete homogeneous, power-sum, and Schur symmetric functions in infinitely many variables.

Finally, we record the definitions of the multigraded Hilbert and Frobenius series of $R_n^{(k,j)}$ (see for example [10]). Fix integers $k, j \geq 0$. $R_n^{(k,j)}$ decomposes as a direct sum of multihomogenous components, which are $\mathfrak{S}_n$-modules:

$$R_n^{(k,j)} = \bigoplus_{r_1,\dots,r_k,s_1,\dots,s_j \geq 0} (R_n^{(k,j)})_{r_1,\dots,r_k,s_1,\dots,s_j}.$$

We denote the multigraded Hilbert series by

$$\begin{aligned}&\mathrm{Hilb}(R_n^{(k,j)}; q_1,\dots,q_k; u_1,\dots,u_j)\\ &\quad := \sum_{r_1,\dots,r_k,s_1,\dots,s_j \geq 0} \dim\left((R_n^{(k,j)})_{r_1,\dots,r_k,s_1,\dots,s_j}\right) q_1^{r_1}\cdots q_k^{r_k} u_1^{s_1}\cdots u_j^{s_j},\end{aligned}$$

and the multigraded Frobenius series by

$$\begin{aligned}&\mathrm{Frob}(R_n^{(k,j)}; q_1,\dots,q_k; u_1,\dots,u_j)\\ &\quad := \sum_{r_1,\dots,r_k,s_1,\dots,s_j \geq 0} F\,\mathrm{Char}\left((R_n^{(k,j)})_{r_1,\dots,r_k,s_1,\dots,s_j}\right) q_1^{r_1}\cdots q_k^{r_k} u_1^{s_1}\cdots u_j^{s_j},\end{aligned}$$

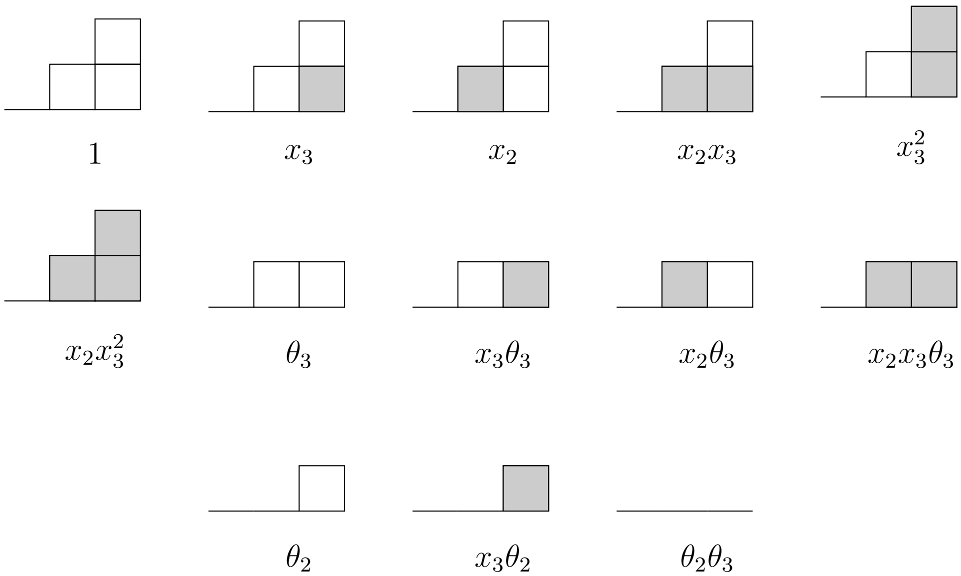


Figure 3.2: The basis $B_3^{(1,1)}$. The α-sequence is shown as the outline of all boxes, and those x_i used for a particular basis element are shaded in gray.

where F denotes the Frobenius characteristic map and Char denotes the character. For simplicity, if $k \le 2$, we will use q, t for q_1, q_2 and if $j \le 2$, we will use u, v for u_1, u_2. Recall that $\langle \mathrm{Frob}(R_n^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j), h_1^n \rangle = \mathrm{Hilb}(R_n^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j)$.

3.3 The conjectural monomial basis

The goal of this section is to interpolate between the Kim–Rhoades basis and the super-Artin basis to construct a new set $B_n^{(1,2)}$, which we conjecture to be a basis for $R_n^{(1,2)}$.

We generalize the α-sequence as follows. Let ξ_S denote the ordered product $\xi_{s_1} \cdots \xi_{s_k}$ for any subset $S = \{s_1 < \cdots < s_k\} \subseteq \{1, \ldots, n\}$. For any $T, S \subseteq \{2, \ldots, n\}$, define the *generalized α-sequence* $\alpha(T, S) = (\alpha_1(T, S), \ldots, \alpha_n(T, S))$ recursively by the initial condition $\alpha_1(T, S) = 0$ and for $2 \le i \le n$,

$$\alpha_i(T, S) = \alpha_{i-1}(T, S) - 1 + \chi(i \notin T) + \chi(i \notin S).$$

Definition 3.3.1. We let

$$B_n^{(1,2)} := \{x^\alpha \theta_T \xi_S \,|\, \theta_T \xi_S \in B_n^{(0,2)} \text{ and } 0 \le \alpha_i \le \alpha_i(T, S) \text{ for all } i \in \{1, \ldots, n\}\}.$$

See Figure 3.3 for an example of $B_n^{(1,2)}$ at $n = 3$.

Conjecture 3.3.2. *The set $B_n^{(1,2)}$ is a basis for $R_n^{(1,2)}$.*

Remark 3.3.3. The conjecture has been verified for $n \le 4$.

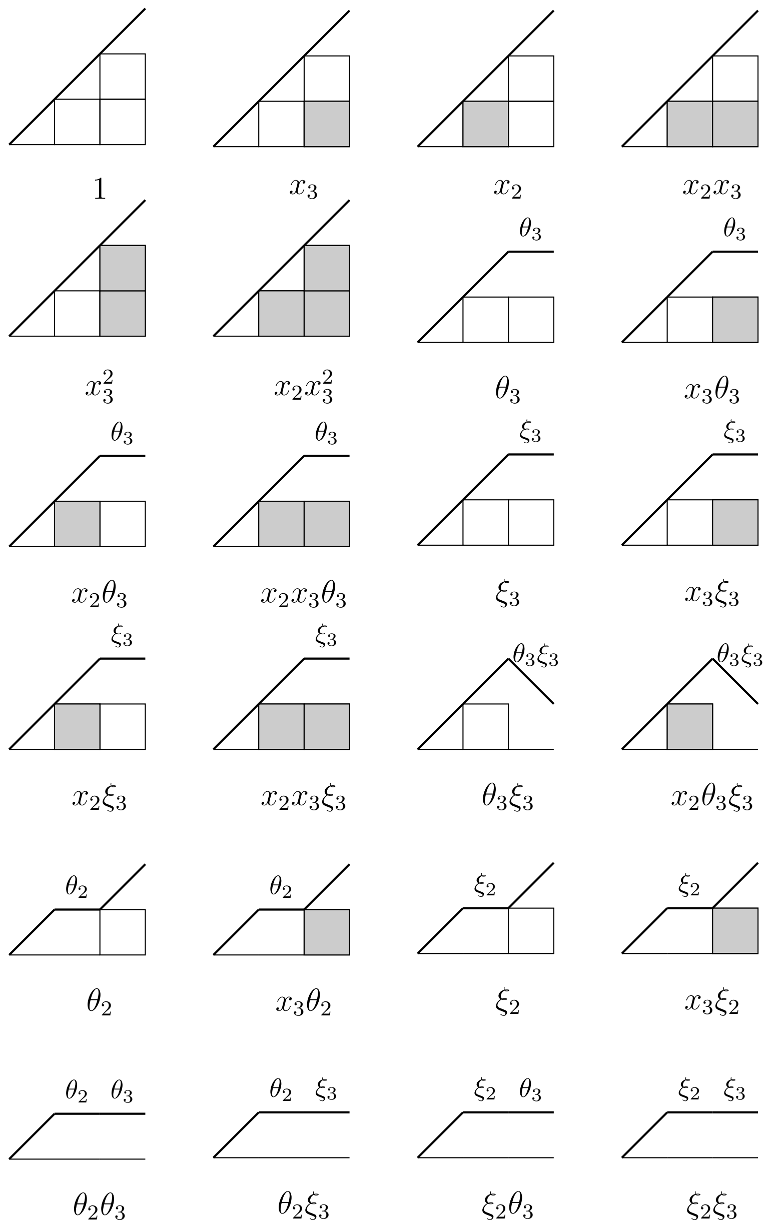


Figure 3.3: The basis $B_3^{(1,2)}$. Below each modified Motzkin path is the outline of the generalized α-sequence, and those x_i used for a particular basis element are shaded in gray. Note that the outline of the generalized α-sequence can be determined by the following rule: in each position i, the column of boxes extends up until its right side is one unit below the path above it.

Remark 3.3.4. $B_n^{(1,2)}$ specializes to the super-Artin basis $B_n^{(1,1)}$ by setting all $\xi_i = 0$. Then the Kim–Rhoades basis elements permitted in the definition will always have $S = \varnothing$, so that condition becomes just $T \subseteq \{2, \ldots, n\}$. Thus $\alpha(T, S) = \alpha(T, \varnothing) = \alpha(T)$.

Remark 3.3.5. $B_n^{(1,2)}$ specializes to the Kim–Rhoades basis $B_n^{(0,2)}$ by setting all $x_i = 0$.

Next, we consider the implications of Conjecture 3.3.2 on the Hilbert series of $R_n^{(1,2)}$. To do so, define

$$\mathrm{stair}_q(\pi) := \prod_{k \in \alpha(T(\pi), S(\pi))} [k+1]_q,$$

where $T(\pi)$ and $S(\pi)$ are determined by which elements in $\{2, \ldots, n\}$ appear as indices for θ_i and ξ_i respectively in the weight of the modified Motzkin path π.

For a modified Motzkin path π, let $\deg_\theta(\pi) = |T(\pi)|$ and let $\deg_\xi(\pi) = |S(\pi)|$.

Proposition 3.3.6. *Assuming Conjecture 3.3.2, it follows that the Hilbert series of $R_n^{(1,2)}$ is*

$$\mathrm{Hilb}(R_n^{(1,2)}; q; u, v) = \sum_{\pi \in \Pi(n)_{>0}} u^{\deg_\theta(\pi)} v^{\deg_\xi(\pi)} \mathrm{stair}_q(\pi).$$

Proof. The number of θ_i weights contributes to the u-degree and the number of ξ_i weights contributes to the v-degree. Then regarding the q-degree, for each index $i \in \{1, \ldots, n\}$, let $k = \alpha_i(T, S)$. Then there will be $k+1$ choices for the exponent of x_i in the monomial, from 0 to k. So the contribution of x_i to the Hilbert series is $[k+1]_q$, and across all indices i, it contributes $\mathrm{stair}_q(\pi)$. Summing over the contributions of all elements $\pi \in \Pi(n)_{>0}$ gives the result. □

Finally, we show that the proposed basis $B_n^{(1,2)}$ has Zabrocki's conjectured dimension.

Theorem 3.3.7. *The cardinality of $B_n^{(1,2)}$ is $2^{n-1}n!$.*

Proof. Let $P(n, r)$ denote the subset of elements of $B_n^{(1,2)}$ corresponding to modified Motzkin paths $\pi \in \Pi(n)_{>0}$ which end at height r. We claim that $p(n, r) := |P(n, r)| = n!\binom{n-1}{r}$, which we show by induction on n. The base cases consist of

$$p(0, r) = \begin{cases} 1 & \text{if } r = n, \\ 0 & \text{otherwise.} \end{cases}$$

Assume the claim holds for arbitrary, fixed $n-1$ (for all r), and we wish to show it holds for n (for all r). Note that $p(n, r) = 0$ whenever $r < 0$ or $r > n$.

If the final step of π is an up-step, then for each element in $P(n-1, r-1)$, we can multiply it by one of the $r+1$ choices $1, x_n, x_n^2, \ldots, x_n^r$ to get such an element in $P(n, r)$.

If the final step of π is a horizontal step, then for each element in $P(n-1, r)$, we can multiply it by one of the $r+1$ choices $1, x_n, x_n^2, \ldots, x_n^r$, and by one of the two weights θ_n, ξ_n, depending on the decoration, to get such an element in $P(n, r)$.

If the final step of π is a down-step, then for each element in $P(n-1,r+1)$, we can multiply it by one of the $r+1$ choices $1, x_n, x_n^2, \ldots, x_n^r$, along with $\theta_n\xi_n$, to get such an element in $P(n,r)$.

Thus

$$\begin{aligned}
p(n,r) &= (r+1)p(n-1,r-1) + 2(r+1)p(n-1,r) + (r+1)p(n-1,r+1) \\
&= (r+1)\left((n-1)!\binom{n-2}{r-1} + 2(n-1)!\binom{n-2}{r} + (n-1)!\binom{n-2}{r+1}\right) \\
&= (r+1)(n-1)!\binom{n}{r+1} \\
&= n!\binom{n-1}{r},
\end{aligned}$$

where we used the inductive hypothesis and some binomial coefficient identities. Finally, sum over all possible heights r to get that $|B_n^{(1,2)}| = \sum_r n!\binom{n-1}{r} = 2^{n-1}n!$, as desired. □

Remark 3.3.8. The argument in this enumeration is adapted from a similar argument of Corteel–Nunge [35, Lemma 17] on marked Laguerre histories, which also are enumerated by $2^{n-1}n!$.

3.4 A conjectural Frobenius series

The goal of this section is to demonstrate how the conjectural basis can be used to propose a Frobenius series for $R_n^{(1,2)}$.

Recall that the *fundamental quasisymmetric function* $Q_{S,n}$ is defined by

$$Q_{S,n} = \sum_{\substack{a_1 \le a_2 \le \cdots \le a_n, \\ a_i < a_{i+1} \text{ if } i \in S}} z_{a_1} z_{a_2} \cdots z_{a_n},$$

for any subset $S \subseteq \{1, \ldots, n-1\}$. Note that the fundamental quasisymmetric function can also be indexed by a composition $\alpha \vDash n$, which we denote by F_α. To convert between the two, if $S = \{s_1 < s_2 < \cdots < s_k\}$, then use $\mathrm{co}(S) := (s_1, s_2 - s_1, s_3 - s_2, \ldots, s_k - s_{k-1}, n - s_k)$ for α. For the other direction, if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, then use $\mathrm{Set}(\alpha) := (\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell-1})$ for S (see for example [92, Chapter 7.19]).

We will also need the following definitions, which are motivated by related definitions of Iraci, Nadeau, and Vanden Wyngaerd on segmented permutations. We will recall their definitions in Section 3.5 and see the relationship between their definitions and the present ones in Proposition 3.6.5. For any $b \in B_n^{(1,2)}$, we may write

$$b = \pm \prod_{i=1}^{n} x_i^{\alpha_i} \theta_i^{\beta_i} \xi_i^{\gamma_i},$$

for some exponents $\alpha_i \in \mathbb{Z}_{\geq 0}$ and $\beta_i, \gamma_i \in \{0,1\}$. Since this is an ordered product, reordering the factors may change the sign, however, for our purposes the sign does not matter. Then define i to be an *ascent of* b if and only if one of the following occurs:

- $\beta_i < \beta_{i+1}$;
- $\beta_i = \beta_{i+1} = 1$ and $\alpha_i \geq \alpha_{i+1} + \gamma_{i+1}$; or
- $\beta_i = \beta_{i+1} = 0$ and $\alpha_i < \alpha_{i+1} + \gamma_{i+1}$.

For $b \in B_n^{(1,2)}$, we say that

$$\mathrm{Asc}(b) := \{i \in \{1, \ldots, n-1\} \mid i \text{ is an ascent of } b\}.$$

Now we can state a conjectural Frobenius series for $R_n^{(1,2)}$ in terms of the set $B_n^{(1,2)}$.

Conjecture 3.4.1.

$$\mathrm{Frob}(R_n^{(1,2)}; q; u, v) = \sum_{b \in B_n^{(1,2)}} u^{\deg_\theta(b)} v^{\deg_\xi(b)} q^{\deg_x(b)} Q_{\mathrm{Asc}(b),n}.$$

This conjecture has been verified for $n \leq 6$. We will see further evidence for this conjecture with Theorem 3.6.6.

3.5 Background on segmented Smirnov words

The goal of this section is to provide the necessary background on the theory of segmented Smirnov words, recently advanced by Iraci, Nadeau, and Vanden Wyngaerd [65], so that we can discuss the relationship between their work and the present work in Section 3.6.

We recall the following definitions from [65, Section 1]. A *Smirnov word* of length n is a word $w \in \mathbb{Z}_{\geq 0}^n$ such that $w_i \neq w_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. A *segmented Smirnov word* of shape $\alpha = (\alpha_1, \ldots, \alpha_s) \vDash n$ is a word $w \in \mathbb{Z}_{\geq 0}^n$ such that $w = w^{(1)} \cdots w^{(s)}$ is a concatenation of s Smirnov words, where each subword $w^{(i)}$, called a *block*, is a Smirnov word of length α_i. We write segmented Smirnov words with a vertical bar between each block. For example, $121|13$ is a segmented Smirnov word of shape $(3,2)$. Let $\mathrm{SW}(n)$ denote the set of segmented Smirnov words of length n. Let $\mu \vDash_0 n$ denote that μ is a weak composition of n, i.e., $\mu = (\mu_1, \mu_2, \ldots)$, where $\mu_i \in \mathbb{Z}_{\geq 0}$ and $\sum_{i \in \mathbb{Z}_{\geq 0}} \mu_i = n$. Given $\mu \vDash_0 n$, let $\mathrm{SW}(\mu)$ denote the set of segmented Smirnov words with content μ, i.e., there are μ_1 many 1's, μ_2 many 2's, etc.

An *ascent*[4] of a Smirnov word w is an index i such that $w_i < w_{i+1}$ and a *descent* of a Smirnov word w is an index i such that $w_i > w_{i+1}$. For segmented Smirnov words, ascents and descents are those that occur strictly within its blocks. Let $\mathrm{SW}(n, k, \ell)$ denote the set

[4]It should be clear from the context whether we are referring to the ascents of a (segmented) Smirnov word, or the ascents of an element $b \in B_n^{(1,2)}$, which are different.

of segmented Smirnov words of length n with exactly k ascents and ℓ descents. Since each index $i \in \{1, \ldots, n\}$ is exactly one of: the index of an ascent, the index of a descent, or the last index in a block, we have that the number of blocks is $n - k - \ell$. Finally, for $\mu \vDash_0 n$, let $\mathrm{SW}(\mu, k, \ell) := \mathrm{SW}(\mu) \cap \mathrm{SW}(n, k, \ell)$.

Definition 3.5.1 ([65, Definition 1.7])**.** For a segmented Smirnov word w, the ordered pair (i, j) with $1 \leq i < j \leq n$ is a *sminversion* (short for Smirnov inversion) if $w_i > w_j$ and one of the following conditions holds:

(1) w_j is the first letter of its block;

(2) $w_{j-1} > w_i$;

(3) $i \neq j - 1$, $w_{j-1} = w_i$, and w_{j-1} is the first letter of its block; or

(4) $i \neq j - 1$ and $w_{j-2} > w_{j-1} = w_i$.

Denote $\mathrm{SF}(n, k, \ell) := (\Theta_{e_k} \Theta_{e_\ell} \nabla e_{n-k-\ell})|_{t=0}$.[5] Define the statistic $\mathrm{sminv}(w)$ on a segmented Smirnov word to be the number of sminversions of w. Denote

$$\mathrm{SW}_{z;q}(n, k, \ell) := \sum_{w \in \mathrm{SW}(n,k,\ell)} q^{\mathrm{sminv}(w)} z_w.$$

Now we can state one of the main results of [65].

Theorem 3.5.2 ([65, Theorem 3.1])**.** *For any $0 \leq k + \ell < n$,*

$$\mathrm{SF}(n, k, \ell) = \mathrm{SW}_{z;q}(n, k, \ell).$$

By summing over all possible k, ℓ, we get their conjectural Frobenius series for $R_n^{(1,2)}$:

$$\sum_{k+\ell<n} u^k v^\ell (\Theta_{e_k} \Theta_{e_\ell} \nabla e_{n-k-\ell})|_{t=0} = \sum_{k+\ell<n} u^k v^\ell \sum_{w \in \mathrm{SW}(n,k,\ell)} q^{\mathrm{sminv}(w)} z_w. \tag{3.16}$$

By taking the inner product of this conjectural Frobenius series with h_1^n, Iraci, Nadeau, and Vanden Wyngaerd are able to derive the following recursive formula for the conjectural Hilbert series of $R_n^{(1,2)}$. When $\mu = (1^n)$, we call the corresponding segmented Smirnov words *segmented permutations*, since they have each of $\{1, \ldots, n\}$ appearing exactly once. They also note that when we restrict to the case of segmented permutations, only conditions (1) and (2) in the definition of sminversion (Definition 3.5.1) will apply. Let

$$\mathrm{SW}_q(\mu, k, \ell) := \sum_{w \in \mathrm{SW}(\mu,k,\ell)} q^{\mathrm{sminv}(w)}.$$

[5] In [65], $\mathrm{SF}(n, k, \ell)$ is first defined by $(\Theta_{e_k} \Theta_{e_\ell} \tilde{H}_{n-k-\ell})|_{t=0}$, where $\tilde{H}_{n-k-\ell}$ is the modified Macdonald polynomial. These two definitions are equivalent by [66, Lemma 3.6].

Proposition 3.5.3 ([65, Proposition 3.2]). *For any $k+\ell < n$, the polynomials $\mathrm{SW}_q(1^n, k, \ell) \in \mathbb{Z}[q]$ satisfy the recursion*

$$\begin{aligned}\mathrm{SW}_q(1^n, k, \ell) = [n-k-\ell]_q \big(&\mathrm{SW}_q(1^{n-1}, k, \ell) + \mathrm{SW}_q(1^{n-1}, k, \ell-1) \\ &+ \mathrm{SW}_q(1^{n-1}, k-1, \ell) + \mathrm{SW}_q(1^{n-1}, k-1, \ell-1)\big),\end{aligned}$$

with initial conditions $\mathrm{SW}_q(\varnothing, k, \ell) = \delta_{k,0}\delta_{\ell,0}$. Furthermore, $\langle \mathrm{SF}(n,k,\ell), h_1^n \rangle = \mathrm{SW}_q(1^n, k, \ell)$.

To obtain a conjectural Hilbert series of $R_n^{(1,2)}$, take the sum

$$\sum_{k+\ell<n} \langle \mathrm{SF}(n,k,\ell), h_1^n \rangle u^k v^\ell = \sum_{k+\ell<n} \mathrm{SW}_q(1^n, k, \ell) u^k v^\ell. \tag{3.17}$$

We recall some additional definitions from [65, Section 5]. Given a segmented Smirnov word w, for each index $i \in \{1, \ldots, n\}$, we say:

- i is *thick* if i is initial (in a block) or $w_{i-1} > w_i$;
- i is *thin* if i is not initial and $w_{i-1} < w_i$.

Note that i is thick is equivalent to i is initial or i is the end of a descent. Also i is thin is equivalent to i is not initial and i is the end of an ascent. Let σ be a segmented permutation of length n, $\sigma_i < n$, and let j be defined by $\sigma_j = \sigma_i + 1$. If any of the following hold, we say that σ_i is *splitting* for σ:

(a) i is thick and j is thin;

(b) i and j are both thin and $i < j$;

(c) i and j are both thick and $j < i$.

Define $\mathrm{Split}(\sigma) = \{m \in \{1, \ldots, n-1\} \mid m \text{ is splitting for } \sigma\}$. Then Split can be used to give the following quasisymmetric expansion.

Proposition 3.5.4 ([65, Proposition 5.3]).

$$\mathrm{SW}_{z;q}(n,k,\ell) = \sum_{\sigma \in \mathrm{SW}(1^n,k,\ell)} q^{\mathrm{sminv}(\sigma)} Q_{\mathrm{Split}(\sigma),n}.$$

3.6 The proposed basis and segmented permutations

In this section, we establish a bijection between our proposed basis $B_n^{(1,2)}$ and the set of segmented permutations $\mathrm{SW}(1^n)$ which is q, u, v-weight preserving. This implies the equivalence of two conjectural Hilbert series for $R_n^{(1,2)}$ (equations (3.11) and (3.17)).

Define a map $\psi : B_n^{(1,2)} \longrightarrow \mathrm{SW}(1^n)$ as follows. For any $b \in B_n^{(1,2)}$, we can write it as

$$b = \pm \prod_{i=1}^{n} x_i^{\alpha_i} \theta_i^{\beta_i} \xi_i^{\gamma_i},$$

for some $\alpha_i \in \mathbb{Z}_{\geq 0}$ and $\beta_i, \gamma_i \in \{0, 1\}$. Each $b \in B_n^{(1,2)}$ is associated with a modified Motzkin path π. Each π starts with an up-step: this implies that $\alpha_1 = \beta_1 = \gamma_1 = 0$, so we start by writing the corresponding segmented permutation as 1. Then, as i ranges from 2 up through n, do exactly one of the following for each i:

(a) if $\beta_i = \gamma_i = 0$ (this corresponds to an up-step): insert "$|i$" or "$i|$" in such a way as to create a new block consisting of only i at position $\alpha_i + 1$ from the rightmost block in the permutation (indexing starting at 1);

(b) if $\beta_i = 1$ and $\gamma_i = 0$ (this corresponds to a horizontal step with decoration θ_i): insert i as the last element of an existing block, at position $\alpha_i + 1$ from the rightmost block in the permutation;

(c) if $\beta_i = 0$ and $\gamma_i = 1$ (this corresponds to a horizontal step with decoration ξ_i): insert i as the first element of an existing block, at position $\alpha_i + 1$ from the rightmost block in the permutation;

(d) if $\beta_i = \gamma_i = 1$ (this corresponds to a down-step with decoration $\theta_i \xi_i$): insert i to replace a "$|$" and thus merge two adjacent blocks into one block, which is now at position $\alpha_i + 1$ from the rightmost block in the permutation.

Upon completion of this process, observe that the output is some segmented permutation σ in $\mathrm{SW}(1^n)$. Also, note that the height of the path is equal to the number of blocks in the corresponding segmented permutation.

Example 3.6.1. Consider $b = x_3\theta_3\theta_4\xi_4\xi_5 \in B_5^{(1,2)}$. We rewrite this as $b = \pm \prod_{i=1}^{n} x_i^{\alpha_i} \theta_i^{\beta_i} \xi_i^{\gamma_i}$, with $(\alpha_1, \beta_1, \gamma_1) = (0, 0, 0)$, $(\alpha_2, \beta_2, \gamma_2) = (0, 0, 0)$, $(\alpha_3, \beta_3, \gamma_3) = (1, 1, 0)$, $(\alpha_4, \beta_4, \gamma_4) = (0, 1, 1)$, $(\alpha_5, \beta_5, \gamma_5) = (0, 0, 1)$. We start building the segmented permutation by writing 1. Then, for $i = 2$, we are in case (a) and insert $|2$ to get $1|2$ so that the new block is in position 1 from the right. Next, for $i = 3$, we are in case (b) and insert 3 as the last element in the existing block in position 2 from the right, yielding $13|2$. Next, for $i = 4$, we are in case (d) and insert 4 to replace the "$|$" to merge the two blocks to create just one block in position 1 from the right, yielding 1342. Finally, for $i = 5$, we are in case (c) and insert 5 as the first element in the existing block in position 1 from the right, yielding 51342.

For more examples, see the tables in Appendix 3.10.

Theorem 3.6.2. *The map $\psi : B_n^{(1,2)} \longrightarrow \mathrm{SW}(1^n)$ is a q, u, v-weight preserving bijection.*

Proof. First, we can check that ψ is an injection by exhibiting a left inverse. In other words, we will show that $B_n^{(1,2)}$ is in bijection with a subset of $\mathrm{SW}(1^n)$. Consider any $\sigma \in \psi(B_n^{(1,2)}) \subseteq \mathrm{SW}(1^n)$. We will algorithmically construct a $b \in B_n^{(1,2)}$ such that $\psi(b) = \sigma$. Initialize the value of b at 1. Each iteration of the algorithm will update the values of σ and b; the values of b and σ which satisfy $\psi(b) = \sigma$ will be the initial value of σ and the final value of b. Take i to be the highest number present in the segmented word σ (i will first be n and then decrease by 1 for each iteration of the algorithm). We determine how i must have been inserted when σ was constructed from an element of $B_n^{(1,2)}$:

(a) if i occurs alone in a block, then $\beta_i = \gamma_i = 0$, and α_i is determined by the block which consists of i being $\alpha_i + 1$ from the rightmost block in the permutation (indexing starting at 1);

(b) if i occurs as the last element of a block with at least two elements, then $\beta_i = 1$, $\gamma_i = 0$, and α_i is determined by the block which contains i being $\alpha_i + 1$ from the rightmost block in the permutation;

(c) if i occurs as the first element of a block with at least two elements, then $\beta_i = 0$, $\gamma_i = 1$, and α_i is determined by the block which contains i being $\alpha_i + 1$ from the rightmost block in the permutation;

(d) if i occurs within a block with at least one element to the right and to the left, then $\beta_i = \gamma_i = 1$, and α_i is determined by the block which contains i being $\alpha_i + 1$ from the rightmost block in the permutation.

Having found the values of $\alpha_i, \beta_i, \gamma_i$, multiply the current value of b by $x_i^{\alpha_i}\theta_i^{\beta_i}\xi_i^{\gamma_i}$, updating it with the new value. Then update σ by undoing the insertion of i according to the same cases:

(a) delete i and if two $|$ become adjacent, delete one of them;

(b) delete i;

(c) delete i;

(d) replace i with a $|$.

Then the highest value in σ is now $i-1$. Repeat this process until the segmented word σ becomes 1, and then b is fully constructed.

By Theorem 3.3.7, the cardinality of $B_n^{(1,2)}$ is $2^{n-1}n!$. To enumerate the segmented permutations $\sigma \in \mathrm{SW}(1^n)$, consider that there are $n!$ permutations and $n-1$ binary choices for whether or not there is a vertical bar between any two adjacent letters. Thus the cardinality of $\mathrm{SW}(1^n)$ is also $2^{n-1}n!$, and ψ is a bijection, and the left inverse just described is an inverse function.

It only remains to check that ψ is weight-preserving. We show that the θ-degree and the ξ-degree is preserved. Consider the four cases used in the construction of a segmented permutation under ψ from $b \in B_n^{(1,2)}$:

(a) no ascents or descents are created, so the θ-degree and the ξ-degree are unchanged;

(b) one ascent is created (which ends with i) and no descents are created, so the θ-degree increases by 1 and the ξ-degree is unchanged;

(c) no ascents are created and one descent is created (which begins with i), so the θ-degree is unchanged and the ξ-degree increases by 1;

(d) one ascent is created and one descent is created (the ascent ends with i and the descent begins with i, as i is the highest number in the segmented permutation at this point), so the θ-degree increases by 1 and the ξ-degree increases by 1.

In each case, the number of ascents remains the same as the θ-degree and the number of descents remains the same as the ξ-degree. Regarding the x-degree, when i is inserted in the construction, it creates a sminversion with exactly all initial elements of blocks to its right, which corresponds exactly to the increase in x-degree by α_i (cf. [37, proof of Proposition 3.2]). □

We record an important consequence of this bijection, which follows since q-weights are preserved.

Corollary 3.6.3. *Let $b \in B_n^{(1,2)}$. Then* $\deg_x(b) = \operatorname{sminv}(\psi(b))$.

As another consequence of this bijection, we are able to conclude that the conjectural Hilbert series of Iraci, Nadeau, and Vanden Wyngaerd is equal to ours.

Corollary 3.6.4.

$$\sum_{k+\ell<n} u^k v^\ell \sum_{w \in \mathrm{SW}(1^n,k,\ell)} q^{\operatorname{sminv}(w)} = \sum_{\pi \in \Pi(n)_{>0}} u^{\deg_\theta(\pi)} v^{\deg_\xi(\pi)} \operatorname{stair}_q(\pi).$$

This result can also be shown directly, by showing that

$$\sum_{\pi \in \Pi(n)_{>0}} u^{\deg_\theta(\pi)} v^{\deg_\xi(\pi)} \operatorname{stair}_q(\pi)$$

satisfies the recurrence relation given in Proposition 3.5.3.

With the bijection $\psi : B_n^{(1,2)} \longrightarrow \mathrm{SW}(1^n)$ in hand, the following result relates $\operatorname{Asc}(b)$ with $\operatorname{Split}(\sigma)$.

Proposition 3.6.5. *For any $b \in B_n^{(1,2)}$,*

$$\operatorname{Asc}(b) = \operatorname{Split}(\psi(b)).$$

Proof. We wish to relate $\mathrm{Asc}(b)$ for elements $b \in B_n^{(1,2)}$ with $\mathrm{Split}(\sigma)$ for $\psi(b) = \sigma \in \mathrm{SW}(1^n)$. Both definitions reduce to determining which indices m are ascents of b or splitting for σ, respectively. For σ, this depends on whether indices are thick or thin, so we wish to determine the analogous conditions on b. Consider each of the four ways to insert m into a segmented permutation in $\mathrm{SW}(1^{m-1})$ to yield one in $\mathrm{SW}(1^m)$, and how each affects the modified Motzkin path used to construct the proposed basis elements:

(a) ("$|m$" or "$m|$" is inserted to create a new block $\longleftrightarrow$ append up-step): this will be thick since it is the beginning of a block;

(b) (m is inserted as the last element of an existing block $\longleftrightarrow$ append horizontal step with decoration θ_m): this will be thin since it is preceded by a smaller number;

(c) (m is inserted as the first element of an existing block $\longleftrightarrow$ append horizontal step with decoration ξ_m): this will be thick since it is the beginning of a block (and note that the next index remains thick since it was previously the first element in a block and is now the end of a descent);

(d) (m is inserted to replace a "$|$" $\longleftrightarrow$ append down-step with decoration $\theta_m \xi_m$): this will be thin, as it is the end of an ascent, as the number which precedes it must be smaller (and note that the next index remains thick since it was previously the first element in a block and is now the end of a descent).

From this discussion, we conclude that thin indices m in a segmented permutation σ correspond exactly to the weights θ_{σ_m} which appear as factors in $b \in B_n^{(1,2)}$.

Now we desire a specific classification for when $m = \sigma_i$ is splitting. Let j be defined by $\sigma_j = \sigma_i + 1$. We have that:

- if $\theta_m \notin b$ (i thick) and $\theta_{m+1} \in b$ (j thin), then m is splitting for σ;
- if $\theta_m, \theta_{m+1} \in b$ (i, j thin) and $i < j$, then m is splitting for σ;
- if $\theta_m, \theta_{m+1} \notin b$ (i, j thick) and $i > j$, then m is splitting for σ;

and otherwise, m is not splitting for σ. However, we have not yet described what i and j are without appealing to the segmented permutation. But we only need the relative position of i and j for this purpose, which we can deduce directly from $b \in B_n^{(1,2)}$, which we write as

$$b = \pm \prod_{m=1}^{n} x_m^{\alpha_m} \theta_m^{\beta_m} \xi_m^{\gamma_m},$$

for some $\alpha_m \in \mathbb{Z}_{\geq 0}$ and $\beta_m, \gamma_m \in \{0, 1\}$.

Each α_m indicates that when building up the corresponding segmented permutation, just after m has been inserted, it is in block $\alpha_m + 1$ from the right. If step m in the path associated to b is:

(a) an up-step: if $\alpha_m \geq \alpha_{m+1}$, then $i < j$, else $i > j$;

(b) a horizontal step with decoration θ_m: if $\alpha_m \geq \alpha_{m+1}$, then $i < j$, else $i > j$;

(c) a horizontal step with decoration ξ_m: if $\alpha_m > \alpha_{m+1}$, then $i < j$, else $i > j$;

(d) a down-step with decoration $\theta_m\xi_m$: if $\alpha_m > \alpha_{m+1}$, then $i < j$, else $i > j$.

We determined whether the condition $\alpha_m \geq \alpha_{m+1}$ or $\alpha_m > \alpha_{m+1}$ is used in each case depending on how the insertion affects the relative order of m and $m+1$.

Recall that for $b \in B_n^{(1,2)}$, $\mathrm{Asc}(b) = \{m \in \{1, \ldots, n-1\} \mid m \text{ is an ascent of } b\}$. At this point, we have that m is splitting for σ if and only if one of the following occurs:

- if $\theta_m \nmid b$ and $\theta_{m+1} \mid b$;
- if $\theta_m, \theta_{m+1} \mid b$ and
  - if b has an up-step or horizontal step with decoration θ_m in position m, we have $\alpha_m \geq \alpha_{m+1}$, or
  - if b has a horizontal step with decoration ξ_m or a down-step with decoration $\theta_m\xi_m$ in position m, we have $\alpha_m > \alpha_{m+1}$;
- if $\theta_m, \theta_{m+1} \nmid b$ and
  - if b has an up-step or horizontal step with decoration θ_m, we have $\alpha_m < \alpha_{m+1}$, or
  - if b has a horizontal step with decoration ξ_m or a down-step with decoration $\theta_m\xi_m$, we have $\alpha_m \leq \alpha_{m+1}$.

Finally, we can convert this to a criteria on only the exponents of b. That is, m is splitting for σ if and only if one of the following occurs:

- $\beta_m < \beta_{m+1}$;
- $\beta_m = \beta_{m+1} = 1$ and $\alpha_m \geq \alpha_{m+1} + \gamma_{m+1}$; or
- $\beta_m = \beta_{m+1} = 0$ and $\alpha_m < \alpha_{m+1} + \gamma_{m+1}$,

which is exactly the definition of when m is an ascent of b, as desired. □

Now, we are able to establish the equivalence of our conjectural Frobenius series (Conjecture 3.4.1) with that of Iraci, Nadeau, and Vanden Wyngaerd (equation (3.16)).

Theorem 3.6.6.

$$\sum_{b \in B_n^{(1,2)}} u^{\deg_\theta(b)} v^{\deg_\xi(b)} q^{\deg_x(b)} Q_{\mathrm{Asc}(b),n} = \sum_{k+\ell<n} u^k v^\ell \sum_{\sigma \in \mathrm{SW}(1^n,k,\ell)} q^{\mathrm{sminv}(\sigma)} Q_{\mathrm{Split}(\sigma),n}.$$

Proof. This follows from Theorem 3.6.2, establishing that $\psi : B_n^{(1,2)} \longrightarrow \mathrm{SW}(1^n)$ is a bijection, along with Corollary 3.6.3 and Proposition 3.6.5, which state, respectively, that if $\psi(b) = \sigma$, then $\deg_x(b) = \mathrm{sminv}(\sigma)$ and $\mathrm{Asc}(b) = \mathrm{Split}(\sigma)$. $\square$

We conclude this section with the discussion of some specializations. Iraci, Rhoades, and Romero [66, Theorem 1.3] showed that

$$\mathrm{Frob}(R_n^{(0,2)}; u, v) = \sum_{k+\ell<n} u^k v^\ell \Theta_{e_k} \Theta_{e_\ell} \nabla e_{n-k-\ell}|_{q=t=0}.$$

As discussed in [65, Section 6.3], the conjectural Frobenius for $R_n^{(1,2)}$ in equation (3.16) recovers the Frobenius series for $R_n^{(0,2)}$ (equation (3.3)) by further specialization of the Theta conjecture at $q = 0$. This implies by Theorem 3.6.6 that Conjecture 3.4.1 also specializes to the Frobenius series for $R_n^{(0,2)}$.

The conjecture in equation (3.5) can be recovered from Conjecture 3.4.1. The Theta conjecture (equation (3.8)) can be specialized at $v = 0$ to Zabrocki's conjecture (equation (3.7)), which can be further specialized at $t = 0$ to equation (3.5). That is,

$$\begin{aligned} &\sum_{k+\ell<n} u^k v^\ell \Theta_{e_k} \Theta_{e_\ell} \nabla e_{n-k-\ell}|_{v=0,t=0} \\ &\quad = \sum_{k=0}^{n-1} \sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} u^k q^{\mathrm{maj}(T) - k\,\mathrm{des}(T) + \binom{k}{2}} \begin{bmatrix} \mathrm{des}(T) \\ k \end{bmatrix}_q s_\lambda. \end{aligned} \tag{3.18}$$

On the other hand, starting with the Theta conjecture and specializing at $t = 0$ gives Conjecture 3.4.1; upon further specialization at $v = 0$, since the order of specialization commutes, implies that equation (3.18) is also equal to

$$\sum_{b \in B_n^{(1,2)}} u^{\deg_\theta(b)} v^{\deg_\xi(b)} q^{\deg_x(b)} Q_{\mathrm{Asc}(b),n}|_{v=0} = \sum_{b \in B_n^{(1,1)}} u^{\deg_\theta(b)} q^{\deg_x(b)} Q_{\mathrm{Asc}(b),n}|_{v=0},$$

where $\mathrm{Asc}(b)$ for $b \in B_n^{(1,1)}$ is determined in the same way as for $b \in B_n^{(1,2)}$.

It would be interesting to see if the conjectures in equations (3.9) or (3.10), can be recovered by specializing from either form of the conjectural Frobenius series given in Theorem 3.6.6. It would also be interesting to have direct combinatorial proofs of the two specializations just discussed.

3.7 Refining the fundamental quasisymmetric function

In this section, we establish a formula for $\langle \mathrm{SF}(n,k,\ell), h_\mu \rangle = [m_\mu]\,\mathrm{SF}(n,k,\ell)$. Recall from Section 3.4 that $\mathrm{Set}(\mu) = \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_{\ell(\mu)-1}\}$, where $\ell(\mu)$ is the length of the partition μ.

Theorem 3.7.1. *Let $\mu \vdash n$. For any fixed k, ℓ, we have that*

$$\langle \mathrm{SF}(n,k,\ell), h_\mu \rangle = \sum_{\substack{b \in B_n^{(1,2)}, \\ \deg_\theta(b)=k, \\ \deg_\xi(b)=\ell, \\ \mathrm{Asc}(b) \subseteq \mathrm{Set}(\mu)}} q^{\deg_x(b)}.$$

We make use of the following lemma, which is a modification of [54, Lemma 6.14], a similar result on (non-segmented) permutations, and is proven in the same manner.

Lemma 3.7.2. *Let $\mu \vdash n$. If f is a symmetric function, then for any family of constants $c(b)$, and any set B such that elements $b \in B$ have a function $\mathrm{Asc} : B \longrightarrow \{1, \ldots, n-1\}$,*

$$f = \sum_{b \in B} c(b) Q_{\mathrm{Asc}(b),n} \tag{3.19}$$

if and only if

$$\langle f, h_\mu \rangle = \sum_{\substack{b \in B, \\ \mathrm{Asc}(b) \subseteq \mathrm{Set}(\mu)}} c(b). \tag{3.20}$$

Proof of Theorem 3.7.1. Fix k and ℓ. Let $c(b) = q^{\deg_x(b)}$. Then we can use

$$f = \sum_{\substack{b \in B_n^{(1,2)}, \\ \deg_\theta(b)=k, \\ \deg_\xi(b)=\ell}} Q_{\mathrm{Asc}(b),n}$$

in Lemma 3.7.2. It follows that

$$\langle f, h_\mu \rangle = \sum_{\substack{b \in B_n^{(1,2)}, \\ \deg_\theta(b)=k, \\ \deg_\xi(b)=\ell, \\ \mathrm{Asc}(b) \subseteq \mathrm{Set}(\mu)}} q^{\deg_x(b)}.$$

Proposition 3.5.4 and Theorem 3.5.2 imply $\mathrm{SF}(n,k,\ell) = \sum_{\sigma \in \mathrm{SW}(1^n,k,\ell)} q^{\mathrm{sminv}(\sigma)} Q_{\mathrm{Split}(\sigma),n}$. We can restrict the bijection ψ from Theorem 3.6.2 to one on $\{b \in B_n^{(1,2)} \mid \deg_\theta(b) = k \text{ and } \deg_\xi(b) = \ell\}$ and $\mathrm{SW}(1^n,k,\ell)$ since it is q,u,v-weight preserving. Corollary 3.6.3 tells us that $\deg_x(b) = \mathrm{sminv}(\psi(b))$ and Proposition 3.6.5 tells us that $\mathrm{Asc}(b) = \mathrm{Split}(\psi(b))$, so $f = \mathrm{SF}(n,k,\ell)$, and the claim follows. □

In the special case where $\mu = (d+1, 1^{n-d-1})$ is a hook shape, we have the following result. Denote by $B_n^{(1,2)}|_{(d+1)-\mathrm{up}}$ the subset of $b \in B_n^{(1,2)}$ where the corresponding modified Motzkin path begins with $d+1$ up-steps, with no x-degree contribution from those steps. Equivalently, if $b = \pm \prod_{m=1}^n x_m^{\alpha_m} \theta_m^{\beta_m} \xi_m^{\gamma_m}$, then $\alpha_m = \beta_m = \gamma_m = 0$ for all $m \in \{1, \ldots, d+1\}$.

Corollary 3.7.3.

$$\langle \mathrm{SF}(n,k,\ell), h_{(d+1,1^{n-d-1})}\rangle = \sum_{\substack{b\in B_n^{(1,2)}|_{(d+1)-\mathrm{up}},\\ \deg_\theta(b)=k,\\ \deg_\xi(b)=\ell.}} q^{\deg_x(b)}.$$

Proof. From Theorem 3.7.1 specialized to hook shapes, we have that

$$\langle \mathrm{SF}(n,k,\ell), h_{(d+1,1^{n-d-1})}\rangle = \sum_{\substack{b\in B_n^{(1,2)},\\ \deg_\theta(b)=k,\\ \deg_\xi(b)=\ell,\\ \mathrm{Asc}(b)\subseteq\{d+1,\ldots,n-1\}}} q^{\deg_x(b)}.$$

It only remains to show that for any $b\in B_n^{(1,2)}$, the condition $b\in B_n^{(1,2)}|_{(d+1)-\mathrm{up}}$ is equivalent to $\mathrm{Asc}(b)\subseteq\{d+1,\ldots,n-1\}$.

First suppose that $\mathrm{Asc}(b)\subseteq\{d+1,\ldots,n-1\}$, so then m is not an ascent of b for all $m\in\{1,\ldots,d\}$. Every modified Motzkin path must start with an up-step, so $\alpha_1=\beta_1=\gamma_1=0$. Then for 1 to not be an ascent, $\beta_2\neq 1$, so $\beta_2=0$. Furthermore, since $\beta_1=\beta_2=0$, then for 1 to not be an ascent, we must have $\alpha_1\geq\alpha_2+\gamma_2$. Since α_2,γ_2 are both nonnegative, this implies that $\alpha_2=\gamma_2=0$. This means that the second step must be an up-step with no x-degree contribution. Repeating this argument proves this direction.

On the other hand, suppose that $b\in B_n^{(1,2)}|_{(d+1)-\mathrm{up}}$. Then it follows that $\alpha_m=\beta_m=\gamma_m=0$ for all $m\in\{1,\ldots,d+1\}$, so $\mathrm{Asc}(b)\subseteq\{d+1,\ldots,n-1\}$. ☐

3.8 Characters for hook shapes

In this section, we derive a formula for $\langle \mathrm{SF}(n,k,\ell), s_\lambda\rangle$, when λ is a hook shape $(d+1,1^{n-d-1})\vdash n$ for $0\leq d\leq n-1$.

First, we prove the following proposition, which extends [65, Proposition 5.7] for column shapes (at $d=0$) to hook shapes.

Theorem 3.8.1.

$$\langle \mathrm{SF}(n,k,\ell), s_{(d+1,1^{n-d-1})}\rangle = \sum_{\substack{\sigma\in\mathrm{SW}(1^n,k,\ell),\\ \mathrm{Split}(\sigma)=\{d+1,\ldots,n-1\}}} q^{\mathrm{sminv}(\sigma)} = \sum_{\substack{b\in B_n^{(1,2)},\\ \deg_\theta(b)=k,\\ \deg_\xi(b)=\ell,\\ \mathrm{Asc}(b)=\{d+1,\ldots,n-1\}}} q^{\deg_x(b)}.$$

Proof. Since $\mathrm{SF}(n,k,\ell)$ is a symmetric function, its expansion into fundamental quasisymmetric functions $\mathrm{SF}(n,k,\ell)=\sum_{\alpha\vDash n}c_\alpha F_\alpha$ determines its Schur expansion $\mathrm{SF}(n,k,\ell)=\sum_{\alpha\vDash n}c_\alpha s_\alpha$ (see for example [44, 45]). Note that the Schur function s_α is indexed by a composition, and it may be simplified to a Schur function indexed by a partition or 0, using the "slinky rule" (see [40]), which we describe briefly here. Draw a compositional Young diagram for α in French notation, and fix all left endpoints of each row, so they can never move. Then let each row fall down as far as possible, such that each row remains a ribbon (hence the term "slinky"). If this forms a partition Young diagram λ, then $s_\alpha = \pm s_\lambda$, but if not, then $s_\alpha = 0$. (See Figure 3.4 for examples.)

Since $S = \mathrm{Split}(\sigma) = \{d+1, \dots, n-1\}$, then $\alpha = (d+1, 1^{n-d-1})$. We claim that the only composition α such that $s_\alpha = \pm s_{(d+1,1^{n-d-1})}$ is $\alpha = (d+1, 1^{n-d-1})$. Indexing the positions of the boxes in the compositional Young diagram with Cartesian coordinates starting from $(0,0)$ in the lower left corner. When applying the slinky rule to any other α with a row longer than 1 except for the bottom row would fall in such a way that a box would be either created in position $(1,1)$, and λ would not be a hook shape, or would not fall all the way to $(1,1)$, and not be a partition Young diagram so the resulting Schur function would be 0.

Consider the coefficient of $s_{(d+1,1^{n-d-1})}$ in $\mathrm{SF}(n,k,\ell)$. We write:

$$\begin{aligned}
\langle \mathrm{SF}(n,k,\ell), s_{(d+1,1^{n-d-1})} \rangle &= \left\langle \sum_{\alpha \vDash n} c_\alpha F_\alpha, s_{(d+1,1^{n-d-1})} \right\rangle \\
&= \left\langle \sum_{\alpha \vDash n} c_\alpha s_\alpha, s_{(d+1,1^{n-d-1})} \right\rangle \\
&= c_{(d+1,1^{n-d-1})}.
\end{aligned}$$

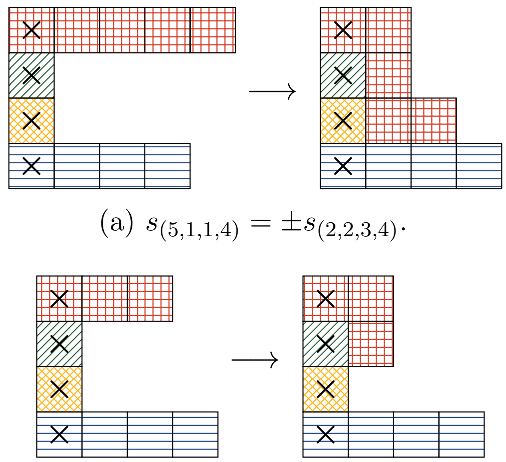
Let $[F_\alpha]X$ denote the coefficient of F_α in X. From the discussion on the slinky rule, we have that

$$\begin{aligned}
c_{(d+1,1^{n-d-1})} &= [F_{(d+1,1^{n-d-1})}]\, \mathrm{SF}(n,k,\ell) \\
&= [F_{(d+1,1^{n-d-1})}]\, \mathrm{SW}_{z;q}(n,k,\ell) \\
&= [Q_{\{d+1,\dots,n-1\},n}] \sum_{\sigma \in \mathrm{SW}(1^n,k,\ell)} q^{\mathrm{sminv}(\sigma)} Q_{\mathrm{Split}(\sigma),n} \\
&= \sum_{\substack{\sigma \in \mathrm{SW}(1^n,k,\ell), \\ \mathrm{Split}(\sigma) = \{d+1,\dots,n-1\}}} q^{\mathrm{sminv}(\sigma)},
\end{aligned}$$

where we used Theorem 3.5.2 and Proposition 3.5.4.

Finally, to convert into the formula in terms of $B_n^{(1,2)}$, we can restrict the bijection ψ from Theorem 3.6.2 to one on $\{b \in B_n^{(1,2)} \mid \deg_\theta(b) = k \text{ and } \deg_\xi(b) = \ell\}$ and $\mathrm{SW}(1^n,k,\ell)$ since it is q,u,v-weight preserving. Then the result follows from Proposition 3.6.5. □

We can characterize when $\mathrm{Asc}(b) = \{d+1, \dots, n-1\}$. Via the weight-preserving bijection between $B_n^{(1,2)}$ and $\mathrm{SW}(1^n)$, this generalizes a characterization in the special case of $d = 0$ [65, Proposition 5.8].



(a) $s_{(5,1,1,4)} = \pm s_{(2,2,3,4)}$.

(b) $s_{(3,1,1,4)} = 0$.

Figure 3.4: Examples of applying the slinky rule.

Proposition 3.8.2. *An element $b \in B_n^{(1,2)}$ satisfies $\mathrm{Asc}(b) = \{d+1, \ldots, n-1\}$ if and only if when b is written as*

$$b = \pm \prod_{m=1}^{n} x_m^{\alpha_m} \theta_m^{\beta_m} \xi_m^{\gamma_m},$$

for some $\alpha_m \in \mathbb{Z}_{\geq 0}$ and $\beta_m, \gamma_m \in \{0,1\}$, we have that for some $a \in \{d+1, \ldots, n\}$,

(a) $\alpha_m = \beta_m = \gamma_m = 0$ for all $m \in \{1, \ldots, d+1\}$;

(b) $\beta_m = 0$ and $\alpha_{m-1} < \alpha_m + \gamma_m$ for all $m \in \{d+2, \ldots, a\}$;

(c) $\beta_a = 0$ and $\beta_{a+1} = 1$; and

(d) $\beta_m = 1$ and $\alpha_{m-1} \geq \alpha_m + \gamma_m$ for all $m \in \{a+2, \ldots, n\}$.

Note that if $a = d+1$, then criterion (b) is skipped. If $a = n$, then criterion (c) is skipped. If $a = n-1$ or $a = n$, then criterion (d) is skipped.

Proof. First, observe that if b satisfies all of the given criteria, then $\mathrm{Asc}(b) = \{d+1, \ldots, n-1\}$, by applying the definition of an ascent of b.

On the other hand, suppose that $\mathrm{Asc}(b) = \{d+1, \ldots, n-1\}$. From the characterization of $\mathrm{Asc}(b) \subseteq \{d+1, \ldots, n-1\}$ given in the proof of Corollary 3.7.3, criterion (a) is satisfied for all $m \in \{1, \ldots, d+1\}$. Next, since $d+1$ is an ascent, either $\beta_{d+2} = 0$ with $\alpha_{d+1} < \alpha_{d+2} + \gamma_{d+2}$ (satisfying criterion (b) for $m = d+2$), or $\beta_{d+2} = 1$ (satisfying criterion (c) for $a = d+1$). In the former case, we can continue satisfying criterion (b) for m in an interval $\{d+1, \ldots, a\}$. Eventually, if we have not reached the end already, we will hit criterion (c) when some $\beta_{a+1} = 1$. At no point once we have reached any $\beta_i = 1$ will we be able to return to any $\beta_j = 0$ for $j > i$, as an immediate consequence of the definition of an ascent of b. This implies that if we have not reached the end, since we continue to have ascents, we will continue to have $\beta_m = 1$ and $\alpha_{m-1} \geq \alpha_m + \gamma_m$ for the remaining m, satisfying criterion (d). □

As a consequence of [38, Theorem 8.2], the following explicit formula was given by Iraci, Nadeau, and Vanden Wyngaerd for $\lambda = (1^n)$, which indexes the sign character.

Theorem 3.8.3 ([65, Theorem 5.6])**.**

$$\langle \mathrm{SF}(n,k,\ell), s_{(1^n)}\rangle = q^{\binom{n-k-\ell}{2}} \begin{bmatrix} n-1 \\ k+\ell \end{bmatrix}_q \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q .$$

Using the characterization given in Proposition 3.8.2, we give the following generalization to hook shapes.

Theorem 3.8.4. *For $0 \leq d \leq n-1$,*

$$\langle \mathrm{SF}(n,k,\ell), s_{(d+1,1^{n-d-1})}\rangle = q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q .$$

To prove the theorem, we will make use of the following lemma, which is an immediate consequence of the q-Chu-Vandermonde identity.

Lemma 3.8.5. *For any positive integer n and nonnegative integers $k, \ell < n-d$,*

$$q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-d-1 \\ \ell \end{bmatrix}_q = \sum_{f=0}^{\ell} q^{\binom{n-d-k-f}{2}+\binom{\ell-f}{2}} \begin{bmatrix} n-d-1-k \\ f \end{bmatrix}_q \begin{bmatrix} k \\ \ell-f \end{bmatrix}_q .$$

Proof. Recall the q-Chu-Vandermonde identity (see for example [65, Proposition 2.4] or [93, Chapter 1, Solution to Exercise 100]):

$$\begin{bmatrix} j \\ a \end{bmatrix}_q = \sum_{i=0}^{a} q^{(r-i)(a-i)} \begin{bmatrix} r \\ i \end{bmatrix}_q \begin{bmatrix} j-r \\ a-i \end{bmatrix}_q .$$

Substituting $i \mapsto f$, $a \mapsto \ell$, $j \mapsto n-d-1$, and $r \mapsto n-d-1-k$ gives

$$\begin{bmatrix} n-d-1 \\ \ell \end{bmatrix}_q = \sum_{f=0}^{\ell} q^{(n-d-1-k-f)(\ell-f)} \begin{bmatrix} n-d-1-k \\ f \end{bmatrix}_q \begin{bmatrix} k \\ \ell-f \end{bmatrix}_q ,$$

and the result follows from $\binom{n-d-k-f}{2} + \binom{\ell-f}{2} - \binom{n-d-k-\ell}{2} = (\ell-f)(n-d-1-k-f)$. □

Proof of Theorem 3.8.4. By Theorem 3.8.1, it remains to show that

$$\sum_{\substack{b \in B_n^{(1,2)}, \\ \deg_\theta(b)=k, \\ \deg_\xi(b)=\ell, \\ \mathrm{Asc}(b)=\{d+1,\ldots,n-1\}}} q^{\deg_x(b)} = q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q .$$

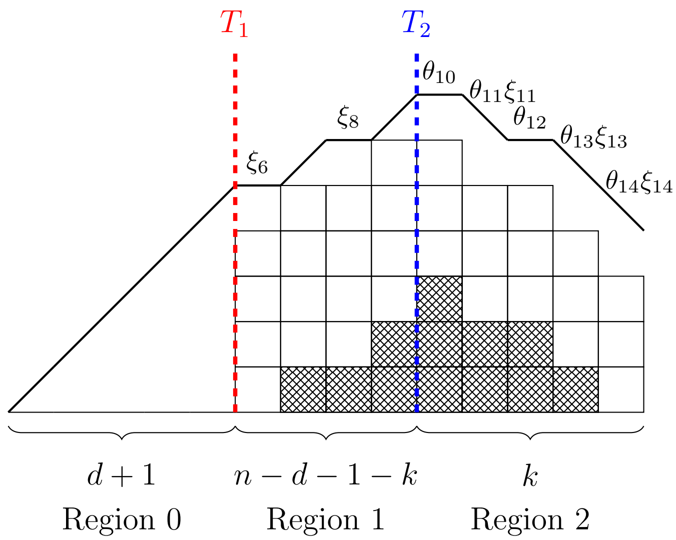


Figure 3.5: A diagram showing the three regions for some b with $\mathrm{Asc}(b) = \{d+1, \ldots, n-1\}$. The boxes shaded in crosshatch indicate where there is a minimal structure of boxes which must be filled in, and then above the minimal structure, boxes may be filled in, subject to certain increasing/decreasing conditions.

Recall the characterization of when $\mathrm{Asc}(b) = \{d+1, \ldots, n-1\}$ from Proposition 3.8.2. Throughout the proof, we refer to Figure 3.5 for a guiding example of what such a b will look like. We record some facts on such a b. Region 0 will consist of only up-steps with no contribution to the x-degree. This corresponds to condition (a) of Proposition 3.8.2. Region 0 cannot be empty, as $d+1 \geq 1$.

At line T_1, we change to Region 1. This corresponds to condition (b) of Proposition 3.8.2. The path in this region can be any sequence of up-steps or horizontal steps with decoration ξ_i. Region 1 may be empty. Throughout Region 1, when there is a horizontal step with decoration ξ_i, the part of the staircase that is filled in must weakly increase, and when there is an up-step, it must strictly increase. This implies that there exists some part of the staircase that must always be filled in, which we call the *minimal structure for Region 1* and encode as a vector of column heights: $\mathrm{MS}(1) = (\mathrm{MS}(1)_1, \ldots, \mathrm{MS}(1)_{n-d-1-k})$. We determine $\mathrm{MS}(1)$ recursively: for $1 \leq i \leq n-d-1-k$,

$$\mathrm{MS}(1)_i = \begin{cases} \mathrm{MS}(1)_{i-1} + 1 & \text{if } p_{d+1+i} \text{ is an up-step,} \\ \mathrm{MS}(1)_{i-1} & \text{if } p_{d+1+i} \text{ is a horizontal step with decoration } \xi_{d+1+i}, \end{cases}$$

where we use the initial condition $\mathrm{MS}(1)_0 = 0$, although that is not considered part of the minimal structure. Recall that p_{d+1+i} denotes the $(d+1+i)$-th step in the path, which is the i-th step in Region 1.

Changing from Region 1 to Region 2 at line T_2 corresponds to condition (c) of Proposition 3.8.2, and then the rest of Region 2 corresponds to condition (d). The path in this region can be any sequence of horizontal steps with decoration θ_i or down-steps with decoration $\theta_i\xi_i$. Region 2 may be empty. Throughout Region 2, when there is a horizontal step with decoration θ_i, the part of the staircase that is filled must weakly decrease, and when there is a down-step with decoration $\theta_i\xi_i$, it must strictly decrease. Again, this implies that there exists some part of the staircase that must always be filled in, which we call the *minimal structure for Region 2* and encode as a vector of column heights: $\mathrm{MS}(2) = (\mathrm{MS}(2)_1, \dots, \mathrm{MS}(2)_k)$. We determine $\mathrm{MS}(2)$ recursively: for $k-1 \geq i \geq 1$,

$$\mathrm{MS}(2)_i = \begin{cases} \mathrm{MS}(2)_{i+1} & \text{if } p_{n-k+i} \text{ is a horizontal step with decoration } \xi_{n-k+i}, \\ \mathrm{MS}(2)_{i+1}+1 & \text{if } p_{n-k+i} \text{ is a down-step with decoration } \theta_{n-k+i}\xi_{n-k+i}, \end{cases}$$

with initial condition $\mathrm{MS}(2)_k = 0$. Recall that p_{n-k+i} denotes the $(n-k+i)$-th step in the path, which is the i-th step in Region 2.

Now we are ready to prove the claimed q-enumeration. First, note that since k, ℓ, d, n are all fixed, the location of the lines T_1 and T_2, and thus the regions, are fixed. It only remains to count the possible paths and fillings in Regions 1 and 2. Let f denote the number of horizontal steps with decoration ξ_i, all of which occur in Region 1. Note that there are $\binom{n-d-1-k}{f}$ ways to pick which of the steps in Region 1 will be horizontal steps with decoration ξ_i.

Consider $\mathrm{MS}(1)$: there is a classical staircase shape with heights $(1, 2, 3, \dots, n-d-1-k-f)$ (corresponding to the up-steps) which is interleaved with f additional components (corresponding to the horizontal steps with decoration ξ_i) which are equal in height to whatever immediately precedes them. So the contribution to the q-enumeration is $q^{\binom{n-d-k-f}{2}}$ by the classical staircase and $\begin{bmatrix} n-d-1-k \\ f \end{bmatrix}_q$ by the additional interleaving components. See Figure 3.6 for an example.

Similarly, consider $\mathrm{MS}(2)$: there is a reversed classical staircase shape with heights $(\ell-f-1, \dots, 2, 1, 0)$ (corresponding to the down-steps with decoration $\theta_i\xi_i$) which is interleaved with $\ell - f$ additional components (corresponding to the horizontal steps with decoration θ_i) which are equal in height to whatever immediately precedes them (or height $\ell - f$ if nothing in Region 2 precedes it). So the contribution to the q-enumeration is $q^{\binom{\ell-f}{2}}$ by the classical staircase and $\begin{bmatrix} k \\ \ell-f \end{bmatrix}_q$ by the additional interleaving components.

By Lemma 3.8.5, by summing over all possible values for f, we get that the contribution of the minimal structures $\mathrm{MS}(1)$ and $\mathrm{MS}(2)$ to the q-enumeration is $q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-d-1 \\ \ell \end{bmatrix}_q$.

All that remains is to consider how we can fill in the region above the minimal structure. First, observe that by construction, the boxes for which we have choice to fill in are equinumerous within each column in Region 1 (there are d boxes) and within Region 2 (there are $n-k-\ell-1$). There is one more choice for filling a column than the number of boxes, since it is possible to leave all empty. In Region 1, the criteria that:

- when there is a horizontal step with decoration ξ_i, the part of the staircase that is filled must weakly increase, and when there is an up-step, it must strictly increase

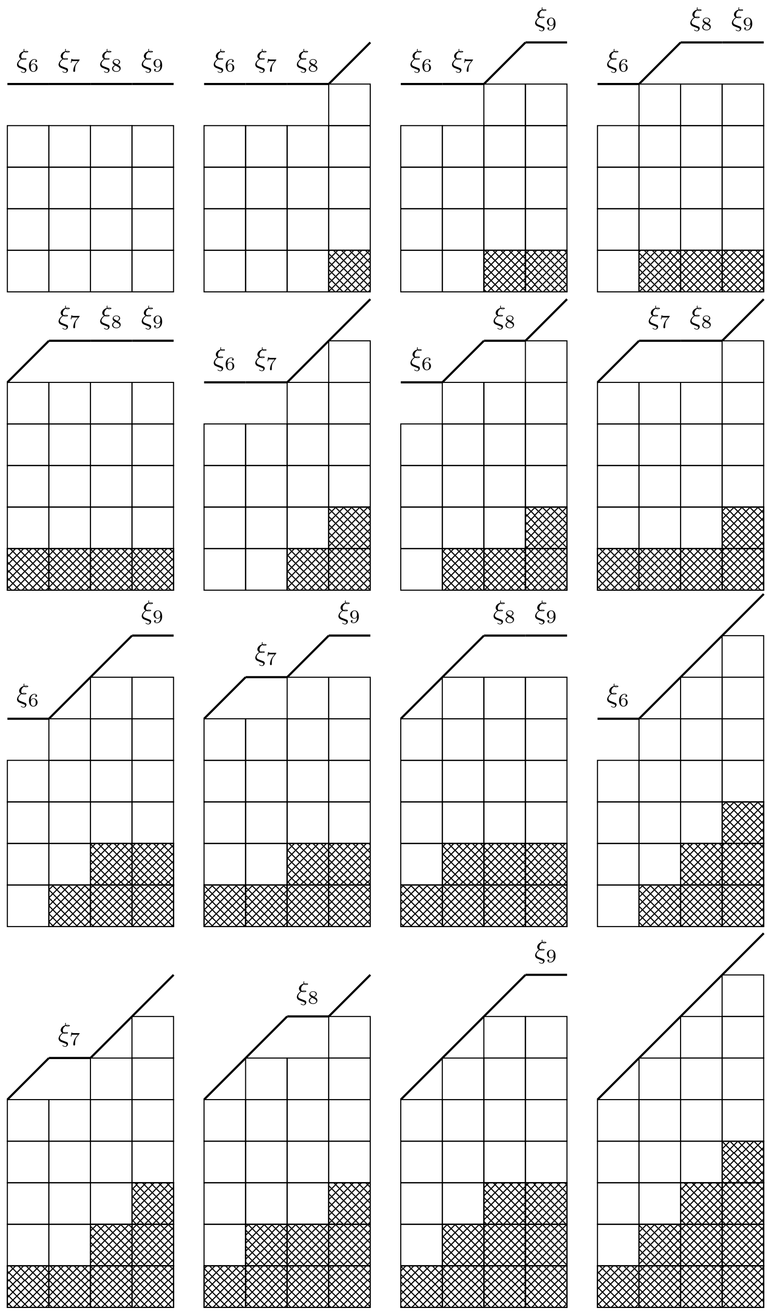


Figure 3.6: All possible paths for Region 1, when $d+1=5$ and $n-d-1-k=4$, along with each path's minimal structure shaded in crosshatch. The contribution of $f=4$ is $q^{\binom{1}{2}}\begin{bmatrix}4\\4\end{bmatrix}_q = 1$, of $f=3$ is $q^{\binom{2}{2}}\begin{bmatrix}4\\3\end{bmatrix}_q = q+q^2+q^3+q^4$, of $f=2$ is $q^{\binom{3}{2}}\begin{bmatrix}4\\2\end{bmatrix}_q = q^3+q^4+2q^5+q^6+q^7$, of $f=1$ is $q^{\binom{4}{2}}\begin{bmatrix}4\\1\end{bmatrix}_q = q^6+q^7+q^8+q^9$, and of $f=0$ is $q^{\binom{5}{2}}\begin{bmatrix}4\\0\end{bmatrix}_q = q^{10}$.

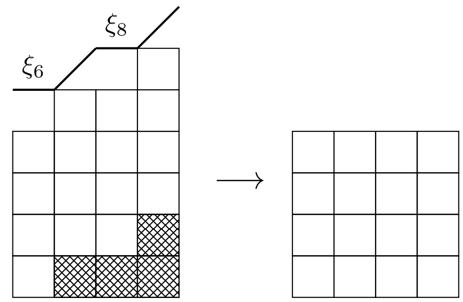


Figure 3.7: A demonstration of deleting the minimal structure from the maximal staircase, and letting the remaining boxes fall by gravity. This will always result in a rectangular shape.

upon deletion of the minimal structure MS(1), and letting the remaining boxes fall by gravity becomes:

- the part of the staircase that is filled must weakly increase.

See Figure 3.7. Then this is counted by the number of length $n-d-1-k$ sequences on $d+1$ letters, which is $\left(\!\binom{d+1}{n-d-1-k}\!\right) = \binom{n-1-k}{d}$.[6] Keeping track of the effect of the fillings on the q-weights gives $\begin{bmatrix} n-1-k \\ d \end{bmatrix}_q$.

Following similar analysis for Region 2, where we delete the minimal structure MS(2), we find what we need to count is the number of length k sequences on $n-k-\ell$ letters, which is $\left(\!\binom{n-k-\ell}{k}\!\right) = \binom{n-1-\ell}{k}$. Keeping track of the effect of the fillings on the q-weights gives $\begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q$.

Finally, multiplying $q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-d-1 \\ \ell \end{bmatrix}_q$ by $\begin{bmatrix} n-1-k \\ d \end{bmatrix}_q$ and $\begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q$ proves the theorem. $\square$

3.9 The conjectural basis in type B

In this section, we describe analogous results and conjectures where the Weyl group of type $\mathfrak{B}_n$ replaces the symmetric group $\mathfrak{S}_n$. We denote the ring in question by $R^{(1,2)}_{\mathfrak{B}_n}$. We describe the Kim–Rhoades basis for $R^{(0,2)}_{\mathfrak{B}_n}$ and Sagan–Swanson's conjectural super-Artin basis for $R^{(1,1)}_{\mathfrak{B}_n}$.

Define the set of *(type B) modified Motzkin paths* of length n, $\Pi(n)_{\geq 0}$,[7] to be the set of all paths $\pi = (p_1, \dots, p_n)$ in $\mathbb{Z}^2$ such that each step p_i is one of:

(a) an up-step $(1,1)$,

(b) a horizontal step $(1,0)$ with decoration θ_i,

[6] $\left(\!\binom{a}{b}\!\right)$ denotes the multiset coefficient (see for example [93, Chapter 1.2]).

[7] Note the difference in notation between this and in Section 2. The difference between the definitions is that paths either must not go below $y=0$ in $\Pi(n)_{\geq 0}$, or $y=1$ after the first step in $\Pi(n)_{>0}$.

(c) a horizontal step $(1,0)$ with decoration ξ_i,

(d) or a down-step $(1,-1)$ with decoration $\theta_i\xi_i$,

where the path never goes below the horizontal line $y=0$. The weight of $\pi \in \Pi(n)_{\geq 0}$ is defined in the same way as in Section 3.2.

Definition 3.9.1 ([69])**.** The *type B Kim–Rhoades basis* $B^{(0,2)}_{\mathfrak{B}_n}$ is the set of all weights of the modified Motzkin paths $\pi \in \Pi(n)_{\geq 0}$, that is,

$$B^{(0,2)}_{\mathfrak{B}_n} := \{\mathrm{wt}(\pi) \,|\, \pi \in \Pi(n)_{\geq 0}\}.$$

We are justified in calling it a basis because of the following result, which is a special case of a more general, type-independent result of Kim and Rhoades.

Theorem 3.9.2 ([69])**.** *The type B Kim–Rhoades basis $B^{(0,2)}_{\mathfrak{B}_n}$ is a basis for $R^{(0,2)}_{\mathfrak{B}_n}$.*

Next, we recall the conjectural type B super-Artin basis, due to Sagan and Swanson. Recall that $\chi(P)$ is 1 if the proposition P is true, and 0 if the proposition P is false. For any $T \subseteq \{1,\ldots,n\}$, define the *$\beta$-sequence* $\beta(T) = (\beta_1(T),\ldots,\beta_n(T))$ recursively by the initial condition $\beta_1(T) = \chi(1 \notin T)$ and for $2 \leq i \leq n$,

$$\beta_i(T) = \beta_{i-1}(T) + \chi(i \notin T) + \chi(i-1 \notin T).$$

Definition 3.9.3 ([90])**.** The type B super-Artin set is

$$B^{(1,1)}_{\mathfrak{B}_n} := \{x^\beta\theta_T \,|\, T \subseteq \{1,\ldots,n\} \text{ and } \beta \leq \beta(T) \text{ componentwise}\}.$$

Conjecture 3.9.4 ([90])**.** *The type B super-Artin set $B^{(1,1)}_{\mathfrak{B}_n}$ is a basis for $R^{(1,1)}_{\mathfrak{B}_n}$.*

We generalize the β-sequence as follows. For any $T, S \subseteq \{1,\ldots,n\}$, define the *generalized β-sequence* by $\beta(T,S) = (\beta_1(T,S),\ldots,\beta_n(T,S))$ recursively by the initial condition $\beta_1(T,S) = -1 + \chi(1 \notin T) + \chi(1 \notin S)$ and for $2 \leq i \leq n$,

$$\beta_i(T,S) = \beta_{i-1}(T,S) - 2 + \chi(i \notin T) + \chi(i-1 \notin T) + \chi(i \notin S) + \chi(i-1 \notin S).$$

Definition 3.9.5. We let

$$B^{(1,2)}_{\mathfrak{B}_n} := \{x^\beta\theta_T\xi_S \,|\, \theta_T\xi_S \in B^{(0,2)}_{\mathfrak{B}_n} \text{ and } 0 \leq \beta_i \leq \beta_i(T,S) \text{ for all } i \in \{1,\ldots,n\}\}.$$

For example, consider $B^{(1,2)}_{\mathfrak{B}_6}$. To construct this set, we start with $B^{(0,2)}_{\mathfrak{B}_6}$. One such element in $B^{(0,2)}_{\mathfrak{B}_6}$ is $\theta_{\{3,4\}}\xi_{\{3,6\}} = \theta_3\theta_4\xi_3\xi_6$. We compute the generalized β-sequence $\beta(\{3,4\},\{3,6\}) = (1,3,3,2,3,4)$. Then to construct the elements in $B^{(1,2)}_{\mathfrak{B}_6}$ associated to $\theta_3\theta_4\xi_3\xi_6$, multiply by x^β, where β satisfies $0 \leq \beta_i \leq \beta_i(\{3,4\},\{3,6\})$. See Figure 3.8 for one of the $2\cdot 4\cdot 4\cdot 3\cdot 4\cdot 5$, elements associated to the modified Motzkin path, and observe that in type B, the *maximal staircase* given by $\beta(T,S)$ in general does not fit underneath the modified Motzkin path.

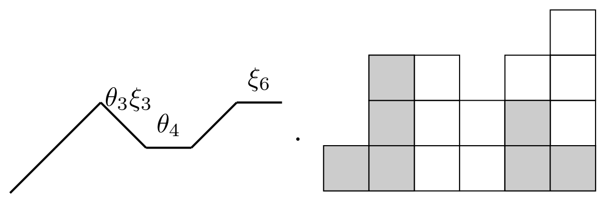


Figure 3.8: An element $x_1x_2^3x_5^2x_6\theta_3\theta_4\xi_3\xi_6$ in $B^{(1,2)}_{\mathfrak{B}_6}$, represented by a type B modified Motzkin path and partially filled-in staircase.

Conjecture 3.9.6. *The set $B^{(1,2)}_{\mathfrak{B}_n}$ is a basis for $R^{(1,2)}_{\mathfrak{B}_n}$.*

Define

$$\mathrm{stair}^B_q(\pi) := \prod_{k\in\beta(T(\pi),S(\pi))} [k+1]_q,$$

where $T(\pi)$ and $S(\pi)$ are determined by which elements in $\{1,\ldots,n\}$ appear as indices for θ_i and ξ_i respectively in the weight of the modified Motzkin path π. As before, $\deg_\theta(\pi) = |T(\pi)|$ and $\deg_\xi(\pi) = |S(\pi)|$. The following result follows from a similar argument as that used to show Proposition 3.3.6.

Proposition 3.9.7. *Assuming Conjecture 3.9.6, it follows that the Hilbert series of $R^{(1,2)}_{\mathfrak{B}_n}$ is*

$$\sum_{\pi\in\Pi_{\geq 0}(n)} u^{\deg_\theta(\pi)}v^{\deg_\xi(\pi)}\,\mathrm{stair}^B_q(\pi).$$

Based on computations for $n \leq 5$, we make the following conjecture on the dimension of $R^{(1,2)}_{\mathfrak{B}_n}$, which to our knowledge, has not previously appeared.

Conjecture 3.9.8. *The dimension of $R^{(1,2)}_{\mathfrak{B}_n}$ is $4^n n!$.*

Remark 3.9.9. Note that the conjectural dimension of $R^{(1,2)}_n$ and $R^{(1,2)}_{\mathfrak{B}_n}$ are both given by $2^r|W|$, where r is the rank of W. It would be interesting to know for which finite Coxeter groups this holds, and for which it fails. Type-independent work in the case of one set of bosonic and one set of fermionic variables has been advanced by Swanson and Wallach [96,97].

Theorem 3.9.10. *The cardinality of $B^{(1,2)}_{\mathfrak{B}_n}$ is $4^n n!$.*

Proof. Let $P_B(n,r)$ denote the subset of elements of $B^{(1,2)}_{\mathfrak{B}_n}$ which have maximal staircase ending at height r.[8]

Since in type B, both the current step and the previous step in π affect the shape of the maximal staircase at the current position, it will be convenient to introduce the following

[8] Unlike in the case of $B^{(1,2)}_n$, as in the proof of Theorem 3.3.7, here the height of the modified Motzkin path does not in general correspond to the height of the maximal staircase.

notation. Let $P_E(n,r)$ denote the subset of $P_B(n,r)$ whose current step (that is, step n) of the corresponding path is a horizontal step. Let $P_U(n,r)$ denote the subset of $P_B(n,r)$ whose current step is an up-step. Let $P_D(n,r)$ denote the subset of $P_B(n,r)$ whose current step is a down-step. It follows that $P_B(n,r) = P_E(n,r) \sqcup P_U(n,r) \sqcup P_D(n,r)$. Also denote $p_B(n,r) := |P_B(n,r)|$, $p_E(n,r) := |P_E(n,r)|$, $p_U(n,r) := |P_U(n,r)|$, and $p_D(n,r) := |P_D(n,r)|$. Thus $p_B(n,r) = p_E(n,r) + p_U(n,r) + p_D(n,r)$.

Consider $\pi \in P_E(n,r)$: since step n is a horizontal step, it does not affect the height of the maximal staircase. But whether step $n-1$ in π is either a horizontal step, an up-step, or a down-step does affect the height of the maximal staircase: specifically, we get $p_E(n-1,r) + p_U(n-1,r-1) + p_D(n-1,r+1)$ as the possibilities, and then we multiply by $2(r+1)$ to account for the two choices of weight on n, either θ_n or ξ_n, along with the $r+1$ choices for the exponent of x_n. This implies that

$$p_E(n,r) = 2(r+1)\left(p_E(n-1,r) + p_U(n-1,r-1) + p_D(n-1,r+1)\right).$$

Similarly, we find that

$$p_U(n,r) = (r+1)\left(p_E(n-1,r-1) + p_U(n-1,r-2) + p_D(n-1,r)\right),$$

and

$$p_D(n,r) = (r+1)\left(p_E(n-1,r+1) + p_U(n-1,r) + p_D(n-1,r+2)\right).$$

The initial conditions are given by: $p_E(0,r) = \delta_{0,r}$, $p_U(0,r) = 0$, and $p_D(0,r) = 0$. It follows that $p_E(n,2r+1) = 0$, $p_U(n,2r) = 0$, and $p_D(n,2r) = 0$ for any nonnegative integers r.

Next, we claim that for $n \geq 1$,

$$p_E(n,2r) = (n-1)!2^n\binom{n-1}{r}(2r+1),$$

$$p_U(n,2r+1) = (n-1)!2^n\binom{n-1}{r}(r+1),$$

$$p_D(n,2r+1) = (n-1)!2^n\binom{n-1}{r}(n-r-1).$$

We show these by induction on n. For the base case of $n=1$,

$$p_E(1,2r) = \begin{cases} 2 & \text{if } r = 0, \\ 0 & \text{otherwise,} \end{cases}$$

since there are two elements in $B^{(1,2)}_{\mathfrak{B}_1}$ with maximal staircase of height 0: θ_1 and ξ_1, and none with maximal staircase of height greater than 0. Similarly, we check that

$$p_U(1,2r+1) = \begin{cases} 2 & \text{if } r = 0, \\ 0 & \text{otherwise,} \end{cases}$$

since there are two elements in $B^{(1,2)}_{\mathfrak{B}_1}$, 1 and x_1, each with maximal staircase of height 1, and none with maximal staircase of height greater than 1. Finally, $p_U(1, 2r+1) = 0$, as there are no paths which begin with a down-step.

Now assume that the claim holds for fixed $n \geq 1$. Using the previously established recurrence relations, we have that

$$\begin{aligned} &p_E(n+1, 2r) \\ &\quad = 2(2r+1)\left[(n-1)!2^n\left(\binom{n-1}{r}(2r+1) + \binom{n-1}{r-1}r + \binom{n-1}{r}(n-r-1)\right)\right] \\ &\quad = (2r+1)(n-1)!2^{n+1}\left[n\binom{n-1}{r} + r\left(\binom{n-1}{r} + \binom{n-1}{r-1}\right)\right] \\ &\quad = n!2^{n+1}\binom{n}{r}(2r+1). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &p_U(n+1, 2r+1) \\ &\quad = 2(r+1)\left[(n-1)!2^n\left(\binom{n-1}{r}(2r+1) + \binom{n-1}{r-1}r + \binom{n-1}{r}(n-r-1)\right)\right] \\ &\quad = (r+1)(n-1)!2^{n+1}\left[n\binom{n-1}{r} + r\left(\binom{n-1}{r} + \binom{n-1}{r-1}\right)\right] \\ &\quad = n!2^{n+1}\binom{n}{r}(r+1), \end{aligned}$$

and

$$\begin{aligned} &p_D(n+1, 2r+1) \\ &\quad = 2(r+1) \\ &\qquad \cdot\left[(n-1)!2^n\left(\binom{n-1}{r+1}(2r+3) + \binom{n-1}{r}(r+1) + \binom{n-1}{r+1}(n-r-2)\right)\right] \\ &\quad = (r+1)(n-1)!2^{n+1}\left[n\binom{n-1}{r+1} + (r+1)\left(\binom{n-1}{r+1} + \binom{n-1}{r}\right)\right] \\ &\quad = n!2^{n+1}\binom{n}{r+1}(r+1) \\ &\quad = n!2^{n+1}\binom{n}{r}(n-r). \end{aligned}$$

This completes our induction.

Considering $p_B(n, s)$, depending on the parity of s, we get that

$$p_B(n, 2r) = p_E(n, 2r) = (n-1)!2^n\binom{n-1}{r}(2r+1),$$

and

$$p_B(n, 2r+1) = p_U(n, 2r+1) + p_D(n, 2r+1) = n!2^n \binom{n-1}{r}.$$

Putting everything together, we have that

$$\begin{aligned}\sum_{s=0}^{2n+1} p_B(n,s) &= \sum_{r=0}^{n} p_B(n,2r) + \sum_{r=0}^{n} p_B(n,2r+1)\\ &= \sum_{r=0}^{n}(n-1)!2^n\binom{n-1}{r}(2r+1) + \sum_{r=0}^{n} n!2^n\binom{n-1}{r}\\ &= (n-1)!2^n\sum_{r=0}^{n}\binom{n-1}{r}(2r+1+n)\\ &= 4^n n!,\end{aligned}$$

where the last line follows from the identity $n2^n = \sum_{r=0}^{n}\binom{n-1}{r}(2r+1+n)$, which can be shown by induction. $\square$

3.10 The bases $B_n^{(1,2)}$ and $\mathrm{SW}(1^n)$ for $n = 1, 2, 3$

This appendix consists of tables which show the bijection between segmented permutations $\sigma \in \mathrm{SW}(1^n)$ and basis elements $b \in B_n^{(1,2)}$. In each segmented permutation, the thin indices are underlined.

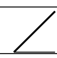
Segmented Permutation σ	**Basis Element** b	**Path Model**	$k(\sigma) = \deg_\theta(b)$	$\ell(\sigma) = \deg_\xi(b)$	$\mathrm{sminv}(\sigma) = \deg_x(b)$	$\mathrm{sdinv}(\sigma)$[9]	$\mathrm{Split}(\sigma) = \mathrm{Asc}(b)$
1	1		0	0	0	0	$\varnothing$

Table 3.1: Conversion between $B_1^{(1,2)}$ and $\mathrm{SW}(1)$ with statistics.

[9]We do not define sdinv in this paper because we do not otherwise use it. It is equidistributed with sminv and is defined in [65, Section 4.3].

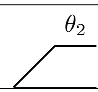
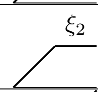
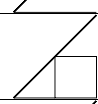
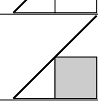
Segmented Permutation σ	**Basis Element** b	**Path Model**	$k(\sigma) = \deg_\theta(b)$	$\ell(\sigma) = \deg_\xi(b)$	$\mathrm{sminv}(\sigma) = \deg_x(b)$	$\mathrm{sdinv}(\sigma)$	$\mathrm{Split}(\sigma) = \mathrm{Asc}(b)$
$1\underline{2}$	θ_2		1	0	0	0	$\{1\}$
21	ξ_2		0	1	0	0	$\{1\}$
$1\|2$	1		0	0	0	0	$\varnothing$
$2\|1$	x_2		0	0	1	1	$\{1\}$

Table 3.2: Conversion between $B_2^{(1,2)}$ and $\mathrm{SW}(1^2)$ with statistics.

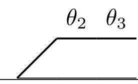
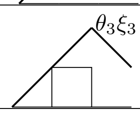
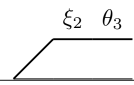
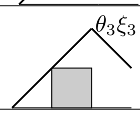
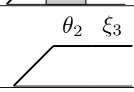
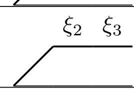
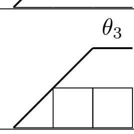
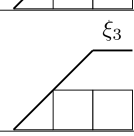
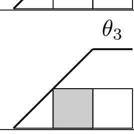
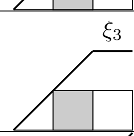
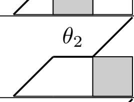
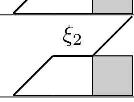
Segmented Permutation σ	**Basis Element** b	**Path Model**	$k(\sigma) = \deg_\theta(b)$	$\ell(\sigma) = \deg_\xi(b)$	$\mathrm{sminv}(\sigma) = \deg_x(b)$	$\mathrm{sdinv}(\sigma)$	$\mathrm{Split}(\sigma) = \mathrm{Asc}(b)$
$1\underline{23}$	$\theta_2\theta_3$		2	0	0	0	$\{1,2\}$
$1\underline{3}2$	$\theta_3\xi_3$		1	1	0	0	$\{2\}$
$21\underline{3}$	$\xi_2\theta_3$		1	1	0	0	$\{1,2\}$
$2\underline{3}1$	$x_2\theta_3\xi_3$		1	1	1	1	$\{1,2\}$
$31\underline{2}$	$\theta_2\xi_3$		1	1	0	0	$\{1\}$
321	$\xi_2\xi_3$		0	2	0	0	$\{1,2\}$
$1\|2\underline{3}$	θ_3		1	0	0	1	$\{2\}$
$1\|32$	ξ_3		0	1	0	0	$\{2\}$
$2\|1\underline{3}$	$x_2\theta_3$		1	0	1	2	$\{1,2\}$
$2\|31$	$x_2\xi_3$		0	1	1	1	$\{1\}$
$3\|1\underline{2}$	$x_3\theta_2$		1	0	1	1	$\{1\}$
$3\|21$	$x_3\xi_2$		0	1	1	1	$\{1,2\}$

Table 3.3: Conversion between $B_3^{(1,2)}$ and $\mathrm{SW}(1^3)$ with statistics (first half).

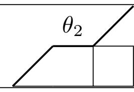
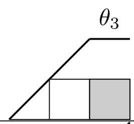
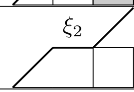
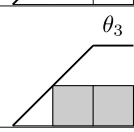
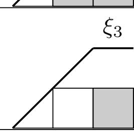
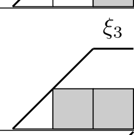
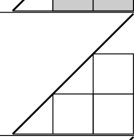
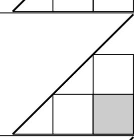
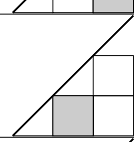
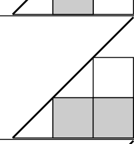
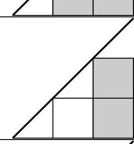
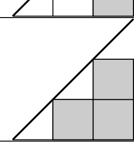
Segmented Permutation σ	**Basis Element** b	**Path Model**	$k(\sigma) = \deg_\theta(b)$	$\ell(\sigma) = \deg_\xi(b)$	$\operatorname{sminv}(\sigma) = \deg_x(b)$	$\operatorname{sdinv}(\sigma)$	$\operatorname{Split}(\sigma) = \operatorname{Asc}(b)$
$1\underline{2}\|3$	θ_2		1	0	0	0	$\{1\}$
$1\underline{3}\|2$	$x_3\theta_3$		1	0	1	0	$\{2\}$
$21\|3$	ξ_2		0	1	0	0	$\{1\}$
$2\underline{3}\|1$	$x_2x_3\theta_3$		1	0	2	1	$\{1,2\}$
$31\|2$	$x_3\xi_3$		0	1	1	1	$\{2\}$
$32\|1$	$x_2x_3\xi_3$		0	1	2	2	$\{1,2\}$
$1\|2\|3$	1		0	0	0	0	$\varnothing$
$1\|3\|2$	x_3		0	0	1	1	$\{2\}$
$2\|1\|3$	x_2		0	0	1	1	$\{1\}$
$2\|3\|1$	x_2x_3		0	0	2	2	$\{1\}$
$3\|1\|2$	x_3^2		0	0	2	2	$\{2\}$
$3\|2\|1$	$x_2x_3^2$		0	0	3	3	$\{1,2\}$

Table 3.4: Conversion between $B_3^{(1,2)}$ and $\mathrm{SW}(1^3)$ with statistics (second half).

3.11 Examples of Frobenius and Hilbert series

$$\mathrm{Frob}(R_1^{(1,2)};q;u,v) = s_1$$

$$\mathrm{Hilb}(R_1^{(1,2)};q;u,v) = 1$$

$$\mathrm{Frob}(R_2^{(1,2)};q;u,v) = (q+u+v)s_{11} + s_2$$

$$\mathrm{Hilb}(R_2^{(1,2)};q;u,v) = q+u+v+1$$

$$\begin{aligned}\mathrm{Frob}(R_3^{(1,2)};q;u,v) &= \Big(q^3+(q^2+q)u+(q^2+q)v+u^2+(q+1)uv+v^2\Big)s_{111}\\ &+\Big((q^2+q)+(q+1)u+(q+1)v+uv\Big)s_{21}+s_3\end{aligned}$$

$$\begin{aligned}\mathrm{Hilb}(R_3^{(1,2)};q;u,v) &= (q^3+2q^2+2q+1)+(q^2+3q+2)u+(q^2+3q+2)v+u^2\\ &+(q+3)uv+v^2\end{aligned}$$

$$\begin{aligned}\mathrm{Frob}(R_4^{(1,2)};q;u,v) &= \Big(q^6+(q^5+q^4+q^3)u+(q^5+q^4+q^3)v+(q^3+q^2+q)u^2\\ &\quad +(q^4+2q^3+2q^2+q)uv+(q^3+q^2+q)v^2+(q^2+q+1)u^2v\\ &\quad +(q^2+q+1)uv^2+u^3+v^3\Big)s_{1111}\\ &+\Big((q^5+q^4+q^3)+(q^4+2q^3+2q^2+q)u+(q^4+2q^3+2q^2+q)v\\ &\quad +(q^2+q+1)u^2+(q^3+3q^2+3q+1)uv+(q^2+q+1)v^2\\ &\quad +(q+1)u^2v+(q+1)uv^2\Big)s_{211}\\ &+\Big((q^4+q^2)+(q^3+q^2+q)u+(q^3+q^2+q)v+(q^2+2q+1)uv\\ &\quad +qu^2+qv^2+u^2v+uv^2\Big)s_{22}\\ &+\Big((q^3+q^2+q)+(q^2+q+1)u+(q+1)uv+(q^2+q+1)v\Big)s_{31}\\ &+s_4\end{aligned}$$

$$\begin{aligned}\mathrm{Hilb}(R_4^{(1,2)};q;u,v) = {} & (q^6+3q^5+5q^4+6q^3+5q^2+3q+1)\\ & + (q^5+4q^4+9q^3+11q^2+8q+3)u\\ & + (q^5+4q^4+9q^3+11q^2+8q+3)v + (q^3+4q^2+6q+3)u^2\\ & + (q^4+5q^3+13q^2+17q+8)uv + (q^3+4q^2+6q+3)v^2\\ & + (q^2+4q+6)u^2v + (q^2+4q+6)uv^2 + u^3 + v^3\end{aligned}$$

Chapter 4

The sign character of the triagonal fermionic coinvariant ring

We determine the trigraded multiplicity of the sign character of the triagonal fermionic coinvariant ring $R_n^{(0,3)}$. As a corollary, this proves a conjecture of Bergeron (2020) that the multiplicity of the sign character of $R_n^{(0,3)}$ is $n^2 - n + 1$.

We also give an explicit formula for double hook characters in the diagonal fermionic coinvariant ring $R_n^{(0,2)}$, and discuss methods towards calculating the sign character of $R_n^{(0,4)}$.

Finally, we give a multigraded refinement of a conjecture of Bergeron (2020) that the multiplicity of the sign character of the $(1,3)$-bosonic-fermionic coinvariant ring $R_n^{(1,3)}$ is $\frac{1}{2}F_{3n}$, where F_n is a Fibonacci number.

This chapter appears in [75].

4.1 Introduction

The diagonal coinvariant ring

$$R_n^{(2,0)} := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]/\langle \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]_+^{\mathfrak{S}_n} \rangle \tag{4.1}$$

was introduced by Haiman in 1994 [57], and since then has been studied extensively. Its defining ideal is generated by all polynomials in $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, with no constant term, which are invariant under the diagonal action of $\mathfrak{S}_n$:

$$\sigma \cdot p(x_1, \ldots, x_n, y_1, \ldots, y_n) = p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}). \tag{4.2}$$

Haiman found the dimension, bigraded Hilbert series, and bigraded Frobenius series of $R_n^{(2,0)}$ [59].

There has been much recent interest (see [6, 10, 11, 102]) in studying a more general class of coinvariant rings $R_n^{(k,j)}$ with k sets of n commuting (bosonic) variables $\boldsymbol{x}_n := \{x_1, \ldots, x_n\}$, $\boldsymbol{y}_n := \{y_1, \ldots, y_n\}$, $\boldsymbol{z}_n := \{z_1, \ldots, z_n\}$, etc., and j sets of n anticommuting (fermionic) variables $\boldsymbol{\theta}_n := \{\theta_1, \ldots, \theta_n\}$, $\boldsymbol{\xi}_n := \{\xi_1, \ldots, \xi_n\}$, $\boldsymbol{\rho}_n := \{\rho_1, \ldots, \rho_n\}$, etc. We define the (k, j)-bosonic-fermionic coinvariant ring by

$$R_n^{(k,j)} := \mathbb{Q}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}]/\langle \mathbb{Q}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}]_+^{\mathfrak{S}_n} \rangle, \tag{4.3}$$

where its defining ideal is generated by all polynomials in $\mathbb{Q}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}]$, without constant term, which are invariant under the diagonal action of the symmetric group $\mathfrak{S}_n$, given by permuting the indices of the variables. Commuting variables commute with all variables. Anticommuting variables anticommute with all anticommuting variables. That is, $\theta_i\theta_j = -\theta_j\theta_i$ for all i, j, and mixed products between different sets of fermionic variables likewise anticommute. Note that this implies that $\theta_i^2 = 0$.

Note that $R_n^{(k,j)}$ also can be defined in terms of symmetric and exterior algebras:

$$R_n^{(k,j)} = ((\operatorname{Sym} \mathbb{Q}^n)^{\otimes k} \otimes (\wedge \mathbb{Q}^n)^{\otimes j})/\langle ((\operatorname{Sym} \mathbb{Q}^n)^{\otimes k} \otimes (\wedge \mathbb{Q}^n)^{\otimes j})_+^{\mathfrak{S}_n} \rangle. \tag{4.4}$$

We recall the definitions of the multigraded Hilbert and Frobenius series of $R_n^{(k,j)}$ (see for example [10]). For fixed integers $k, j \geq 0$, $R_n^{(k,j)}$ decomposes as a direct sum of multihomogeneous components, which are $\mathfrak{S}_n$-modules:

$$R_n^{(k,j)} = \bigoplus_{r_1,\ldots,r_k,s_1,\ldots,s_j \geq 0} (R_n^{(k,j)})_{r_1,\ldots,r_k,s_1,\ldots,s_j}. \tag{4.5}$$

We denote the multigraded Hilbert series by

$$\begin{aligned} &\operatorname{Hilb}(R_n^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j) \\ &\quad := \sum_{r_1,\ldots,r_k,s_1,\ldots,s_j \geq 0} \dim\left((R_n^{(k,j)})_{r_1,\ldots,r_k,s_1,\ldots,s_j}\right) q_1^{r_1} \cdots q_k^{r_k} u_1^{s_1} \cdots u_j^{s_j}, \end{aligned} \tag{4.6}$$

and the multigraded Frobenius series by

$$\begin{aligned} &\operatorname{Frob}(R_n^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j) \\ &\quad := \sum_{r_1,\ldots,r_k,s_1,\ldots,s_j \geq 0} F \operatorname{Char}\left((R_n^{(k,j)})_{r_1,\ldots,r_k,s_1,\ldots,s_j}\right) q_1^{r_1} \cdots q_k^{r_k} u_1^{s_1} \cdots u_j^{s_j}, \end{aligned} \tag{4.7}$$

where F denotes the Frobenius characteristic map and Char denotes the character. For simplicity, if $k \leq 2$, we will use q, t for q_1, q_2 and if $j \leq 4$, we will use u, v, w, z for u_1, u_2, u_3, u_4. Recall that $\langle \operatorname{Frob}(R_n^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j), h_1^n \rangle = \operatorname{Hilb}(R_n^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j)$. Furthermore, $R_n^{(k,j)}$ is a $\mathrm{GL}_k \times \mathrm{GL}_j \times \mathfrak{S}_n$-module (see [10, Section 2]), so its multigraded Frobenius character is a sum of products of three Schur functions, which are irreducible characters of polynomial representations of GL_k and GL_j, along with a Frobenius character. The main focus of this paper is on the case $(k, j) = (0, 3)$.

Observe that the Schur function $s_{(\ell-2)}(u,v,w)$ is a u,v,w-analogue of the binomial coefficient $\binom{\ell}{2}$; denote it by

$$\binom{\ell}{2}_{u,v,w} := s_{(\ell-2)}(u,v,w). \tag{4.8}$$

Recall the notation $[n]_{u,v} := u^{n-1}+u^{n-2}v+\cdots+uv^{n-2}+v^{n-1}$. It follows that as a polynomial in w with coefficients in u and v we have

$$\binom{\ell}{2}_{u,v,w} = [\ell-1]_{u,v} + [\ell-2]_{u,v}w + [\ell-3]_{u,v}w^2 + \cdots + [1]_{u,v}w^{\ell-2}, \tag{4.9}$$

and there are two similar expressions for $\binom{\ell}{2}_{u,v,w}$ as a polynomial in u or in v.

Our main result is the following.

Theorem 4.1.1.

$$\begin{aligned} \langle \mathrm{Frob}(R_n^{(0,3)}; u,v,w), s_{(1^n)}\rangle &= s_{(n-1)}(u,v,w) + s_{(n-2,1,1)}(u,v,w) \\ &= \binom{n+1}{2}_{u,v,w} + uvw\binom{n-1}{2}_{u,v,w}. \end{aligned} \tag{4.10}$$

The theorem immediately implies the following corollary, which was conjectured by Bergeron [10, Table 3].

Corollary 4.1.2.

$$\langle \mathrm{Frob}(R_n^{(0,3)}; 1,1,1), s_{(1^n)}\rangle = n^2-n+1. \tag{4.11}$$

Proof. Recall that $\binom{\ell}{2}_{u,v,w}|_{u=v=w=1} = \binom{\ell}{2}$. By evaluating Theorem 4.1.1 at $u=v=w=1$, the multiplicity is $\binom{n+1}{2}+\binom{n-1}{2} = n^2-n+1$. ☐

The organization of the paper is as follows. In Section 4.2, we detail the setting of $R_n^{(0,3)}$ from the perspective of both coinvariants and harmonics. In Section 4.3, we recall some results of Haglund–Sergel [56] and Kim–Rhoades [69], along with proving some preliminary results. In Section 4.4, building on the work of Haglund–Sergel and Kim–Rhoades, we give an upper bound on the multiplicity of the sign character of $R_n^{(0,3)}$ (Corollary 4.4.2). In Section 4.5, we construct two elements in the ring of triagonal fermionic harmonics T_n (Proposition 4.5.4), and study the GL_3-representations that they generate (Proposition 4.5.6), which constructs enough elements in T_n to show that the upper bound on the multiplicity of the sign character is achieved with equality, proving the main theorem.

In Section 4.6, we derive a formula for double hook characters of $R_n^{(0,2)}$ (Theorem 4.6.2). In Section 4.7, we discuss methods to analyze the sign character of $R_n^{(0,4)}$. In Section 4.8, we provide a q,u,v,w-refinement of a conjecture of Bergeron [10] on the sign character of $R_n^{(1,3)}$ (Conjecture 4.8.6).

4.2 Coinvariants and harmonics

We now describe the setting in more detail, from the perspective of both coinvariants and, isomorphically, harmonics.

Specialize to $R_n^{(0,3)} := \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]/\langle \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]_+^{\mathfrak{S}_n}\rangle$, which we call the *triagonal fermionic coinvariant ring*. Its defining ideal $\langle \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]_+^{\mathfrak{S}_n}\rangle$ is the ideal generated by polynomials in $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$, without constant term, invariant under the diagonal action of $\mathfrak{S}_n$. A generating set for the ideal $\langle \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]_+^{\mathfrak{S}_n}\rangle$ is given by all the nonzero monomial symmetric functions in any of the three sets of variables:

$$\begin{aligned}\{&\theta_1 + \cdots + \theta_n, \xi_1 + \cdots + \xi_n, \rho_1 + \cdots + \rho_n, \theta_1\xi_1 + \cdots + \theta_n\xi_n,\\ &\theta_1\rho_1 + \cdots + \theta_n\rho_n, \xi_1\rho_1 + \cdots + \xi_n\rho_n, \theta_1\xi_1\rho_1 + \cdots + \theta_n\xi_n\rho_n\}.\end{aligned} \tag{4.12}$$

Consider the ring $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$ in three sets of n anticommuting variables, which we arrange into the $3 \times n$ matrix

$$M := \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_n \\ \xi_1 & \xi_2 & \cdots & \xi_n \\ \rho_1 & \rho_2 & \cdots & \rho_n \end{bmatrix}. \tag{4.13}$$

For each 3×3 invertible matrix A in GL_3, multiply M on the left by A to obtain the product $A \cdot M$. Then, in any polynomial $f \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$, replace each variable θ_i, ξ_i, ρ_i by the corresponding entry of the matrix $A \cdot M$. This defines a left GL_3-action on $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$.

The ring $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$ is naturally trigraded. Denote by $(\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n])_{a,b,c}$ the homogeneous component spanned by all monomials that contain exactly a variables in $\boldsymbol{\theta}_n$, b variables in $\boldsymbol{\xi}_n$, and c variables in $\boldsymbol{\rho}_n$. Since GL_3 acts by linear combinations of the three rows, the GL_3-action preserves the total degree $d = a + b + c$. Hence each total degree d component $\bigoplus_{a+b+c=d}(\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n])_{a,b,c}$ is GL_3-invariant.

Each permutation $\sigma \in \mathfrak{S}_n$ corresponds to a permutation matrix P_σ, defined by $(P_\sigma)_{i,j} = 1$ if $i = \sigma(j)$ and 0 otherwise. Then the right multiplication $M \cdot P_\sigma$ has the effect on M of letting σ act on the indices of all variables: $\theta_i \mapsto \theta_{\sigma(i)}$, $\xi_i \mapsto \xi_{\sigma(i)}$, and $\rho_i \mapsto \rho_{\sigma(i)}$. Then, in any polynomial $f \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$, replace each variable θ_i, ξ_i, ρ_i by the corresponding entry of the matrix $M \cdot P_\sigma$. This defines a right $\mathfrak{S}_n$-action on $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$.

Because left multiplication by A and right multiplication by P_σ commute, these GL_3 and $\mathfrak{S}_n$-actions commute, so $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$ is a $\mathrm{GL}_3 \times \mathfrak{S}_n$-module.

Recall the definition of derivative of anticommuting variables (where here each θ_i could be from any of the three sets of anticommuting variables) is (see for example [97, Section 1.5]):

$$\partial_{\theta_j}\theta_{i_1}\cdots\theta_{i_k} = \begin{cases} (-1)^{\ell-1}\theta_{i_1}\cdots\hat{\theta}_{i_\ell}\cdots\theta_{i_k} & \text{if } j = i_\ell, \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

Define the space of *triagonal fermionic harmonics* by

$$T_n := \left\{ f \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n] \;\middle|\; \sum_{i=1}^n \partial_{\theta_i}^h \partial_{\xi_i}^k \partial_{\rho_i}^\ell f = 0 \text{ for all } h + k + \ell > 0 \right\}. \tag{4.15}$$

Since any $\theta_i^2 = 0$, we need not check any second derivatives, hence

$$T_n = \left\{ f \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n] \;\middle|\; \sum_{i=1}^{n} \partial_{\theta_i}^h \partial_{\xi_i}^k \partial_{\rho_i}^\ell f = 0 \text{ for all } h+k+\ell > 0 \text{ and } h,k,\ell \in \{0,1\} \right\}. \tag{4.16}$$

Looking at the infinitesimal action of the Lie algebra $\mathfrak{sl}_3$ on T_n allows us to study T_n as a GL_3-module.[1] Similarly to [57, Section 3.1], we define the operators

$$F^{\theta\to\xi} := \sum_{i=1}^{n} \xi_i \partial_{\theta_i}, \text{ and } F^{\xi\to\rho} := \sum_{i=1}^{n} \rho_i \partial_{\xi_i}, \tag{4.17}$$

and

$$E^{\theta\leftarrow\xi} := \sum_{i=1}^{n} \theta_i \partial_{\xi_i} \text{ and } E^{\xi\leftarrow\rho} := \sum_{i=1}^{n} \xi_i \partial_{\rho_i}. \tag{4.18}$$

Along with two H operators given by taking the commutators of the E and F operators, these generate the Lie algebra $\mathfrak{sl}_3$.

Furthermore T_n is a $\mathrm{GL}_3 \times \mathfrak{S}_n$-module. Write a trigraded component of T_n as $(T_n)_{a,b,c}$ and $R_n^{(0,3)}$ as $(R_n^{(0,3)})_{a,b,c}$. Then we have that $T_n \cong R_n^{(0,3)}$ as trigraded $\mathfrak{S}_n$-modules. Working with harmonics was advocated by Garsia: a benefit of working with harmonics over coinvariants is that one works with polynomials instead of equivalence classes.

A polynomial $p \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$ is called antisymmetric if $\sigma(p) = \mathrm{sgn}(\sigma)p$ for all $\sigma \in \mathfrak{S}_n$, where σ acts diagonally by permuting the variables. Let $(T_n)^\epsilon$ denote the antisymmetric subspace[2] of T_n, which consists of all elements $p \in T_n$ which are antisymmetric. This corresponds to the u,v,w-graded multiplicity of the sign character in $R_n^{(0,3)}$, that is, $\mathrm{Hilb}((T_n)^\epsilon; u,v,w) = \langle \mathrm{Frob}(R_n^{(0,3)}; u,v,w), s_{(1^n)} \rangle$.

4.3 Preliminary results

In this section, we recall or prove some preliminary results. Haglund and Sergel gave a formula for the graded Frobenius series of the fermionic coinvariants $R_n^{(0,1)} := \mathbb{Q}[\boldsymbol{\theta}_n]/\langle \mathbb{Q}[\boldsymbol{\theta}_n]_+^{\mathfrak{S}_n} \rangle$.

Lemma 4.3.1 ([56, Lemma 4.10])**.** *For $n \geq 1$,*

$$\mathrm{Frob}(R_n^{(0,1)}; w) = \sum_{k=0}^{n-1} w^k s_{(n-k,1^k)}. \tag{4.19}$$

Kim and Rhoades gave a formula for the bigraded Frobenius series of the diagonal fermionic coinvariants $R_n^{(0,2)} := \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n]/\langle \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n]_+^{\mathfrak{S}_n} \rangle$.

[1] See for example [43, Section 15.5] for more on the relationship between SL_3 and GL_3-representations.
[2] This is sometimes called the alternating subspace.

Theorem 4.3.2 ([69, Theorem 6.1]). *For $n \geq 1$,*

$$\mathrm{Frob}(R_n^{(0,2)}; u, v) = \sum_{0 \leq i+j < n} u^i v^j \left(s_{(n-i,1^i)} * s_{(n-j,1^j)} - s_{(n-i+1,1^{i-1})} * s_{(n-j+1,1^{j-1})} \right), \tag{4.20}$$

where $*$ denotes the Kronecker product and $s_{(n+1,1^{-1})}$ is interpreted as 0.

In particular, they found bigraded multiplicities for the trivial, sign, and hook characters.

Proposition 4.3.3 ([69, Proposition 6.2]). *In $R_n^{(0,2)}$, the multiplicity of the trivial character is*

$$\langle \mathrm{Frob}(R_n^{(0,2)}; u, v), s_{(n)} \rangle = 1, \tag{4.21}$$

the bigraded multiplicity of the sign character is

$$\langle \mathrm{Frob}(R_n^{(0,2)}; u, v), s_{(1^n)} \rangle = [n]_{u,v}, \tag{4.22}$$

and for $0 < k < n-1$, the bigraded multiplicity of a hook character is

$$\langle \mathrm{Frob}(R_n^{(0,2)}; u, v), s_{(n-k,1^k)} \rangle = [k+1]_{u,v} + uv[k]_{u,v}. \tag{4.23}$$

As a consequence of their result on the sign character, we conclude the following.

Corollary 4.3.4.

1. *If a nonzero harmonic polynomial in $T_n \cap \mathbb{Q}[\boldsymbol{\theta}_n]$ is antisymmetric, then it must be of degree exactly $n-1$.*

2. *If a nonzero harmonic polynomial in $T_n \cap \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n]$ is antisymmetric, then it must be of degree exactly $n-1$.*

Proof. The first claim is implied by the second. To show the second, notice that Proposition 4.3.3 shows that the multiplicity of each component of the sign character is $n-1$. □

There is a useful condition for when antisymmetrizing a polynomial results in zero.

Lemma 4.3.5. *Let p be a polynomial in $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$. If there exists a transposition $s \in \mathfrak{S}_n$ such that $s(p) = p$, then*

$$\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\sigma(p) = 0. \tag{4.24}$$

Proof. Consider the symmetric group $\mathfrak{S}_n$ as a set, and partition it $\mathfrak{S}_n = \mathfrak{S}_n' \sqcup \mathfrak{S}_n''$ into two equal-sized subsets such that for each $\sigma' \in \mathfrak{S}_n'$, we have that $\sigma' \cdot s = \sigma''$, for some $\sigma'' \in \mathfrak{S}_n''$. Since $\mathrm{sgn}(s) = -1$, then

$$\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\sigma(p) = \sum_{\sigma' \in \mathfrak{S}_n'} \mathrm{sgn}(\sigma')\sigma'(p) + \sum_{\sigma' \in \mathfrak{S}_n'} -\mathrm{sgn}(\sigma')\sigma'(p) = 0. \tag{4.25}$$

□

As a consequence, we establish that an antisymmetric polynomial must use enough distinct indices of variables to be nonzero.

Lemma 4.3.6. *If a polynomial in $\mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$ is antisymmetric, and for the variables that appear, at most $n-2$ unique indices are used, then the polynomial is identically 0.*

Proof. Call the polynomial in question r. Without loss of generality, say that the indices used in r are $\{1, 2, \ldots, n-2\}$. Since r is antisymmetric, it can be obtained by applying an antisymmetrizing operator $\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma$ to some polynomial $p \in \mathbb{Q}[\boldsymbol{\theta}_{n-2}, \boldsymbol{\xi}_{n-2}, \boldsymbol{\rho}_{n-2}]$. Then p is invariant under the action of s_{n-1}. Apply Lemma 4.3.5 to conclude that $r = 0$. □

4.4 An upper bound on degree

In this section, we prove an upper bound on the trigraded degree of the sign character. The following result is inspired by a similar result of Haglund–Sergel (on a different ring, $R_n^{(2,1)}$) [56, Theorem 4.11].

Proposition 4.4.1. *For $n \geq 1$,*

$$\langle \operatorname{Frob}(R_n^{(0,2)} \otimes R_n^{(0,1)}; u, v, w), s_{(1^n)} \rangle = \binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w} + uv[n-1]_{u,v}. \tag{4.26}$$

Proof. As noted in [56, equation (4.9)], following from [16], for any $\mathfrak{S}_n$-modules A and B tensored under the diagonal action of $\mathfrak{S}_n$, we have that

$$\operatorname{Frob}(A \otimes B) = \sum_{\nu \vdash n} s_\nu \sum_{\lambda, \mu \vdash n} \langle \operatorname{Frob}(A), s_\lambda \rangle \langle \operatorname{Frob}(B), s_\mu \rangle \langle s_\lambda * s_\mu, s_\nu \rangle, \tag{4.27}$$

where $\langle s_\lambda * s_\mu, s_\nu \rangle = g(\lambda, \mu, \nu)$ are the Kronecker coefficients. In our case, we are interested in

$$\begin{aligned} &\langle \operatorname{Frob}(R_n^{(0,2)} \otimes R_n^{(0,1)}; u, v, w), s_{(1^n)} \rangle \\ &\quad = \sum_{\lambda, \mu \vdash n} \langle \operatorname{Frob}(R_n^{(0,2)}; u, v), s_\lambda \rangle \langle \operatorname{Frob}(R_n^{(0,1)}; w), s_\mu \rangle g(\lambda, \mu, 1^n) \\ &\quad = \sum_{\lambda \vdash n} \langle \operatorname{Frob}(R_n^{(0,2)}; u, v), s_{\lambda'} \rangle \langle \operatorname{Frob}(R_n^{(0,1)}; w), s_\lambda \rangle, \end{aligned} \tag{4.28}$$

since $g(\lambda, \mu, 1^n) = \delta_{\mu, \lambda'}$. By Lemma 4.3.1, which states that only hook Schur functions appear in $\operatorname{Frob}(R_n^{(0,1)}; w)$, we reduce to

$$\begin{aligned} \langle \operatorname{Frob}(R_n^{(0,2)} \otimes R_n^{(0,1)}; u, v, w), s_{(1^n)} \rangle &= \sum_{k=0}^{n-1} w^k \langle \operatorname{Frob}(R_n^{(0,2)}; u, v), s_{(k+1, 1^{n-k-1})} \rangle \\ &= \sum_{k=0}^{n-1} w^k ([n-k]_{u,v} + uv[n-k-1]_{u,v}), \end{aligned} \tag{4.29}$$

where we applied Proposition 4.3.3. After a bit of algebra, we obtain the claimed formula. □

Now the following result gives us an upper bound on the occurrences of the sign character in $R_n^{(0,3)}$. For multivariate polynomials A and B, the notation $A \leq B$ means that $B - A$ is a sum of monomials with only nonnegative coefficients.

Corollary 4.4.2. *For $n \geq 1$,*

$$\langle \mathrm{Frob}(R_n^{(0,3)}; u, v, w), s_{(1^n)} \rangle \leq \binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w}. \tag{4.30}$$

Proof. First note that any occurrence of the sign character in $R_n^{(0,3)}$ is predicated on it occurring in $R_n^{(0,2)} \otimes R_n^{(0,1)}$, since $R_n^{(0,3)}$ is a quotient of $R_n^{(0,2)} \otimes R_n^{(0,1)}$ under the diagonal action of $\mathfrak{S}_n$ (see [56]).

By permuting which sets of variables out of $\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n$ are assigned to $R_n^{(0,2)}$ and to $R_n^{(0,1)}$ in Proposition 4.4.1, we conclude that a sign character in $R_n^{(0,3)}$ must have trigraded multiplicity bounded above by

$$\binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w} + uv[n-1]_{u,v}, \tag{4.31}$$

$$\binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w} + uw[n-1]_{u,w}, \tag{4.32}$$

and

$$\binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w} + vw[n-1]_{v,w}. \tag{4.33}$$

Note that if a polynomial $A \geq 0$ satisfies $A \leq uv[n-1]_{u,v}$, $A \leq uw[n-1]_{u,w}$, and $A \leq vw[n-1]_{v,w}$, then $A = 0$. Hence by taking the bounds given by equations (4.31-4.33) together, we conclude that the sign character in $R_n^{(0,3)}$ must have trigraded multiplicity bounded above by

$$\binom{n+1}{2}_{u,v,w} + uvw \binom{n-1}{2}_{u,v,w}. \tag{4.34}$$

$\square$

4.5 Construction of basis elements

Now we will work towards the proof of the main theorem, by constructing two explicit elements in T_n, which are highest weight vectors for certain GL_3-representations. We ultimately show that the upper bound given in Corollary 4.4.2 is obtained with equality.

Definition 4.5.1. For $n \geq 1$, define the *primary theta-seed* by

$$\Delta_1(\boldsymbol{\theta}_n) := \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\sigma(\theta_1 \theta_2 \cdots \theta_{n-1}). \tag{4.35}$$

Definition 4.5.2. For $n \geq 3$, define the *secondary theta-seed* by

$$\Delta_2(\boldsymbol{\theta}_n) := \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma((\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\theta_4\cdots\theta_{n-1}). \tag{4.36}$$

When $n = 1$ or 3 in the definitions of the primary and secondary theta-seed, respectively, the empty product of θ_i's is interpreted as 1.

We now prove a technical lemma.

Lemma 4.5.3.

1. *The antisymmetrization operator* $\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sigma$ *commutes with the differential operator* $\sum_{i=1}^n \partial_{\theta_i}^h\partial_{\xi_i}^k\partial_{\rho_i}^\ell$.
2. *The antisymmetrization operator* $\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sigma$ *commutes with the operators* $F^{\theta\to\xi}$, $F^{\xi\to\rho}$, $E^{\theta\leftarrow\xi}$, *and* $E^{\xi\leftarrow\rho}$.

Proof. First we show (1). Let $f \in \mathbb{Q}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n]$. Observe that for any $\sigma \in \mathfrak{S}_n$,

$$\sum_{i=1}^n \partial_{\theta_i}^h\partial_{\xi_i}^k\partial_{\rho_i}^\ell = \sum_{i=1}^n \partial_{\sigma(\theta_i)}^h\partial_{\sigma(\xi_i)}^k\partial_{\sigma(\rho_i)}^\ell, \tag{4.37}$$

since addition is commutative. Thus,

$$\begin{aligned}
\sum_{i=1}^n \partial_{\theta_i}^h\partial_{\xi_i}^k\partial_{\rho_i}^\ell\left(\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sigma(f)\right) &= \sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sum_{i=1}^n \partial_{\theta_i}^h\partial_{\xi_i}^k\partial_{\rho_i}^\ell(\sigma(f)) \\
&= \sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sum_{i=1}^n \partial_{\sigma(\theta_i)}^h\partial_{\sigma(\xi_i)}^k\partial_{\sigma(\rho_i)}^\ell(\sigma(f)) \qquad (4.38) \\
&= \sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sigma\left(\sum_{i=1}^n \partial_{\theta_i}^h\partial_{\xi_i}^k\partial_{\rho_i}^\ell f\right),
\end{aligned}$$

so the operators commute as claimed.

Next we show (2). For any $\sigma \in \mathfrak{S}_n$,

$$\sum_{i=1}^n \xi_i\partial_{\theta_i} = \sum_{i=1}^n \sigma(\xi_i)\partial_{\sigma(\theta_i)}, \tag{4.39}$$

since addition is commutative. Thus for $F^{\theta\to\xi} = \sum_{i=1}^n \xi_i\partial_{\theta_i}$,

$$\begin{aligned}
\sum_{i=1}^n \xi_i\partial_{\theta_i}\left(\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sigma(f)\right) &= \sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sum_{i=1}^n \xi_i\partial_{\theta_i}\sigma(f) \\
&= \sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sum_{i=1}^n \sigma(\xi_i)\partial_{\sigma(\theta_i)}\sigma(f) \qquad (4.40) \\
&= \sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\sigma\left(\sum_{i=1}^n \xi_i\partial_{\theta_i} f\right),
\end{aligned}$$

and similarly for the other operators. □

Now we show that $\Delta_1(\boldsymbol{\theta}_n)$ and $\Delta_2(\boldsymbol{\theta}_n)$ are harmonic.

Proposition 4.5.4.

1. *The primary theta-seed* $\Delta_1(\boldsymbol{\theta}_n)$ *is in* $(T_n)^\epsilon$.
2. *The secondary theta-seed* $\Delta_2(\boldsymbol{\theta}_n)$ *is in* $(T_n)^\epsilon$.

Proof. We show (1). Consider the primary theta-seed

$$\Delta_1(\boldsymbol{\theta}_n) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(\theta_1\theta_2 \cdots \theta_{n-1}). \tag{4.41}$$

By equation (4.16), to show it is in T_n, we only need to show that $\sum_{i=1}^n \partial_{\theta_i}^h \partial_{\xi_i}^k \partial_{\rho_i}^\ell \Delta_1(\boldsymbol{\theta}_n) = 0$ for $(h,k,\ell) \in \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$. Since only θ_i variables appear in $\Delta_1(\boldsymbol{\theta}_n)$, it only remains to check $(1,0,0)$. Consider $\theta_1 \cdots \theta_{n-1}$, which is what we will antisymmetrize to get the primary theta-seed. By Lemma 4.5.3, the differential operator commutes with the antisymmetrization operator. We write

$$\sum_{i=1}^n \partial_{\theta_i}\theta_1 \cdots \theta_{n-1} = \sum_{i=1}^n (-1)^{i-1}\theta_1 \cdots \hat{\theta}_i \cdots \theta_{n-1}, \tag{4.42}$$

where each monomial on the right hand side is in only $n-2$ variables. By Lemma 4.3.6, this becomes 0 upon antisymmetrization. This shows $\Delta_1(\boldsymbol{\theta}_n)$ is in T_n, and since it is constructed via an antisymmetrization operator, it is in $(T_n)^\epsilon$.

We show (2). For $n \geq 3$, consider the secondary theta-seed

$$\Delta_2(\boldsymbol{\theta}_n) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma((\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\theta_4 \cdots \theta_{n-1}). \tag{4.43}$$

We must check that $\Delta_2(\boldsymbol{\theta}_n)$ is in T_n. We only need to show that $\sum_{i=1}^n \partial_{\theta_i}^h \partial_{\xi_i}^k \partial_{\rho_i}^\ell \Delta_2(\boldsymbol{\theta}_n) = 0$ for the following seven choices of (h,k,ℓ):

$$\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}. \tag{4.44}$$

Case 1: $(h,k,\ell) = (1,0,0)$. Let $f = (\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\theta_4 \cdots \theta_{n-1}$, which is what we will antisymmetrize to get the secondary theta-seed. For concision, write $\prod_{i=3}^{n-1}\theta_i$ for the ordered product $\theta_3\theta_4 \cdots \theta_{n-1}$. We write that

$$\begin{aligned} \sum_{i=1}^n \partial_{\theta_i} f &= \partial_{\theta_1} f + \partial_{\theta_2} f + \sum_{i=3}^{n-1} \partial_{\theta_i} f \\ &= \xi_2\rho_2 \prod_{i=3}^{n-1}\theta_i + (\xi_1\rho_2 + \xi_2\rho_1)\prod_{i=3}^{n-1}\theta_i \\ &\quad + \sum_{i=3}^{n-1}(-1)^i(\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3 \cdots \hat{\theta}_i \cdots \theta_{n-1}. \end{aligned} \tag{4.45}$$

Then upon antisymmetrization, both of the terms $\xi_2\rho_2\prod_{i=3}^{n-1}\theta_i$ and $\sum_{i=3}^{n-1}(-1)^i(\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\cdots\hat{\theta}_i\cdots\theta_{n-1}$ will be 0 due to Lemma 4.3.6, since they only contain $n-2$ indices for variables. Since the simple transposition s_1 leaves the term $(\xi_1\rho_2 + \xi_2\rho_1)\prod_{i=3}^{n-1}\theta_i$ invariant, by Lemma 4.3.5, it becomes 0 upon antisymmetrization.

Case 2: $(h,k,\ell) = (0,1,0)$. Before antisymmetrization, we write that

$$\begin{aligned}\sum_{i=1}^{n}\partial_{\xi_i} f &= \partial_{\xi_1} f + \partial_{\xi_2} f \\ &= -\theta_2\rho_2\prod_{i=3}^{n-1}\theta_i - (\theta_1\rho_2 + \theta_2\rho_1)\prod_{i=3}^{n-1}\theta_i.\end{aligned} \tag{4.46}$$

By Lemma 4.3.6, $\theta_2\rho_2\prod_{i=3}^{n-1}\theta_i$ becomes 0 upon antisymmetrization. By Lemma 4.3.5, $(\theta_1\rho_2 + \theta_2\rho_1)\prod_{i=3}^{n-1}\theta_i$ becomes 0 upon antisymmetrization, since it is invariant under s_1.

Case 3: $(h,k,\ell) = (0,0,1)$. The same argument as in Case 2 applies.

Case 4: $(h,k,\ell) = (1,1,0)$. Before antisymmetrization, we write that

$$\begin{aligned}\sum_{i=1}^{n}\partial_{\theta_i}\partial_{\xi_i} f &= \partial_{\theta_1}\partial_{\xi_1} f + \partial_{\theta_2}\partial_{\xi_2} f + \sum_{i=3}^{n-1}\partial_{\theta_i}\partial_{\xi_i} f \\ &= 0 - \rho_1\prod_{i=3}^{n-1}\theta_i + 0,\end{aligned} \tag{4.47}$$

which becomes 0 upon antisymmetrization by Lemma 4.3.6.

Case 5: $(h,k,\ell) = (1,0,1)$. The same argument as in Case 4 applies.

Case 6: $(h,k,\ell) = (0,1,1)$. The same argument as in Case 4 applies.

Case 7: $(h,k,\ell) = (1,1,1)$. Before antisymmetrization, we write that

$$\begin{aligned}\sum_{i=1}^{n}\partial_{\theta_i}\partial_{\xi_i}\partial_{\rho_i} f &= \partial_{\theta_1}\partial_{\xi_1}\partial_{\rho_1} f + \partial_{\theta_2}\partial_{\xi_2}\partial_{\rho_2} f + \sum_{i=3}^{n-1}\partial_{\theta_i}\partial_{\xi_i}\partial_{\rho_i} f \\ &= 0,\end{aligned} \tag{4.48}$$

which remains 0 upon antisymmetrization.

This completes the proof that $\Delta_2(\boldsymbol{\theta}_n)$ is in T_n, and since it is constructed via an antisymmetrization operator, it is in $(T_n)^\epsilon$. □

A priori, a harmonic polynomial in T_n could equal zero if certain term cancellations occur. However, the following result demonstrates that this does not happen for $\Delta_1(\boldsymbol{\theta}_n)$ and $\Delta_2(\boldsymbol{\theta}_n)$.

Proposition 4.5.5.

1. *The primary theta-seed $\Delta_1(\boldsymbol{\theta}_n)$ is nonzero in T_n.*

2. *The secondary theta-seed $\Delta_2(\boldsymbol{\theta}_n)$ is nonzero in T_n.*

Proof. We show (1). Consider in T_n,

$$\Delta_1(\boldsymbol{\theta}_n) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(\theta_1\theta_2 \cdots \theta_{n-1}). \tag{4.49}$$

When the antisymmetrization operator $\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma$ is applied to $\theta_1\theta_2\cdots\theta_{n-1}$, consider which σ will output the monomial $\theta_1\theta_2\cdots\theta_{n-1}$. Such a σ must fix n, but $1,\ldots,n-1$ can be permuted. For any σ in the symmetric group on letters $1,\ldots,n-1$, we have that $\operatorname{sgn}(\sigma)\sigma(\theta_1\theta_2\cdots\theta_{n-1}) = \theta_1\theta_2\cdots\theta_{n-1}$ (see for example [23, Chapter III, Section 7.3, Proposition 5]). Hence antisymmetrization produces $(n-1)!$ copies of $\theta_1\theta_2\cdots\theta_{n-1}$, so $\Delta_1(\boldsymbol{\theta}_n)$ is nonzero in T_n, as desired.

Next we show (2). Consider in T_n,

$$\Delta_2(\boldsymbol{\theta}_n) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma((\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\theta_4 \cdots \theta_{n-1}). \tag{4.50}$$

When the operator $\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma$ is applied to $(\theta_1\xi_2\rho_2+\theta_2\xi_1\rho_2+\theta_2\xi_2\rho_1)\theta_3\theta_4\cdots\theta_{n-1}$, consider which σ will include the monomial $\theta_1\xi_2\rho_2\theta_3\theta_4\cdots\theta_{n-1}$ in its output. Since the ξ_i and ρ_i have the same index, it must come from an antisymmetrization of $\theta_1\xi_2\rho_2\theta_3\theta_4\cdots\theta_{n-1}$ (and not of $\theta_2\xi_1\rho_2\theta_3\theta_4\cdots\theta_{n-1}$ nor $\theta_2\xi_2\rho_1\theta_3\theta_4\cdots\theta_{n-1}$). Thus such a σ must fix 2 and n, but $1,3,4,\ldots,n-1$ can be permuted. For any σ in the symmetric group on letters $1,3,4,\ldots,n-1$, it follows that $\operatorname{sgn}(\sigma)\sigma(\theta_1\theta_3\theta_4\cdots\theta_{n-1}) = \theta_1\theta_3\theta_4\cdots\theta_{n-1}$. This implies that $\operatorname{sgn}(\sigma)\sigma(\theta_1\xi_2\rho_2\theta_3\theta_4\cdots\theta_{n-1}) = \theta_1\xi_2\rho_2\theta_3\theta_4\cdots\theta_{n-1}$. Hence antisymmetrization produces $(n-2)!$ copies of $\theta_1\xi_2\rho_2\theta_3\theta_4\cdots\theta_{n-1}$, so $\Delta_2(\boldsymbol{\theta}_n)$ is nonzero in T_n, as desired. □

Starting with $\Delta_1(\boldsymbol{\theta}_n)$ and $\Delta_2(\boldsymbol{\theta}_n)$, the operators defined in equation (4.17) $F^{\theta\to\xi}$ and $F^{\xi\to\rho}$ create representations of GL_3.

Proposition 4.5.6.

1. *The primary theta-seed $\Delta_1(\boldsymbol{\theta}_n)$ is the highest weight vector for the representation of* GL_3 *with character $s_{(n-1)}(u,v,w)$.*

2. *The secondary theta-seed $\Delta_2(\boldsymbol{\theta}_n)$ is the highest weight vector for the representation of* GL_3 *with character $s_{(n-2,1,1)}(u,v,w)$.*

Proof. We show (1). Start with the primary theta-seed

$$\Delta_1(\boldsymbol{\theta}_n) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(\theta_1\theta_2 \cdots \theta_{n-1}). \tag{4.51}$$

By Proposition 4.5.4, it is in $(T_n)^\epsilon$, and by Proposition 4.5.5, it is nonzero. Observe that $E^{\theta\leftarrow\xi}$ and $E^{\xi\leftarrow\rho}$ both kill $\Delta_1(\boldsymbol{\theta}_n)$. Thus $\Delta_1(\boldsymbol{\theta}_n)$ is a highest weight vector for the irreducible

GL_3-representation with highest weight $(n-1,0,0)$, where the weight of any $p \in T_n$ is given by $(\deg_\theta p, \deg_\xi p, \deg_\rho p)$ (see for example [43, Section 14-15] for a reference on the Lie theory used). This GL_3-representation is in $(T_n)^\epsilon$ since Lemma 4.5.3 implies that $(T_n)^\epsilon$ is closed under $F^{\theta\to\xi}$ and $F^{\xi\to\rho}$. The GL_3-character of this representation is $s_{(n-1)}(u,v,w)$.

We show (2). Consider the secondary theta-seed

$$\Delta_2(\boldsymbol{\theta}_n) = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma)\sigma((\theta_1\xi_2\rho_2 + \theta_2\xi_1\rho_2 + \theta_2\xi_2\rho_1)\theta_3\theta_4\cdots\theta_{n-1}). \tag{4.52}$$

This is antisymmetric, as it is constructed using an antisymmetrization operator. By Corollary 4.3.4, there are no antisymmetric elements of degree n solely in any one or two sets of variables. This implies that $E^{\theta\leftarrow\xi}$ and $E^{\xi\leftarrow\rho}$ both kill $\Delta_2(\boldsymbol{\theta}_n)$. By Proposition 4.5.4, it is in $(T_n)^\epsilon$, and by Proposition 4.5.5, it is nonzero. Thus $\Delta_2(\boldsymbol{\theta}_n)$ is a highest weight vector for the irreducible GL_3-representation with highest weight $(n-2,1,1)$. Again, this GL_3-representation is in $(T_n)^\epsilon$ by Lemma 4.5.3. The GL_3-character of this representation is $s_{(n-2,1,1)}(u,v,w)$. □

Now we can prove the main theorem.

Theorem 4.1.1.

$$\begin{aligned}\langle \mathrm{Frob}(R_n^{(0,3)};u,v,w), s_{(1^n)}\rangle &= s_{(n-1)}(u,v,w) + s_{(n-2,1,1)}(u,v,w)\\ &= \binom{n+1}{2}_{u,v,w} + uvw\binom{n-1}{2}_{u,v,w}.\end{aligned} \tag{4.53}$$

Proof. The representations constructed in Proposition 4.5.6 are sufficient to force the bound in Corollary 4.4.2 to be achieved with equality. A bit of algebra verifies that the formulation in terms of Schur functions is equivalent to the formulation in terms of u,v,w-binomial coefficients. □

Remark 4.5.7. While it is not necessarily true in general that the Lie algebra operators F and E will correspond to crystal operators, in the present case, it is true since every weight space for the representations studied in Proposition 4.5.6 is one-dimensional. The representation with highest weight $(n-1,0,0)$ has crystal structure isomorphic to a crystal of tableaux $\mathcal{B}_{(n-1)}$ and the representation with highest weight $(n-2,1,1)$ has crystal structure isomorphic to a crystal of tableaux $\mathcal{B}_{(n-2,1,1)}$ (see for example [25, Chapter 3]).

4.6 Double hook characters of the diagonal fermionic coinvariant ring

With Proposition 4.3.3, Kim and Rhoades gave explicit formulas for the trivial, sign, and hook characters of $R_n^{(0,2)}$. In this section, we extend the analysis to give an explicit formula for double hook characters of $R_n^{(0,2)}$, where a *double hook* is a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots)$

such that $\lambda_3 \leq 2$ and $\lambda_2 \geq 2$. All characters of $R_n^{(0,2)}$ indexed by shapes not contained in a double hook have multiplicity 0.

Rosas gave a combinatorial formula for the Kronecker coefficients $g(\lambda,(n-e,1^e),(n-f,1^f))$, for any shape λ. These Kronecker coefficients are only nonzero if λ is contained in a double hook shape. Here, we recall her formulas for double hook shapes, rows, and hook shapes.

Theorem 4.6.1 ([89, Theorem 3])**.** *Let $(n-e,1^e)$ and $(n-f,1^f)$ be hook shapes.*

1. *If λ is not contained in a double hook, then $g(\lambda,(n-e,1^e),(n-f,1^f)) = 0$.*
2. *Let $\lambda = (\lambda_1,\lambda_2,2^\ell,1^{n-2\ell-\lambda_1-\lambda_2})$ where $\lambda_1 \geq \lambda_2 \geq 2$ be a double hook. Then*

$$\begin{aligned}
&g(\lambda,(n-e,1^e),(n-f,1^f)) \\
&\quad = \chi(\lambda_2 - 1 \leq \tfrac{e+f-n+\lambda_1+\lambda_2}{2} \leq \lambda_1)\cdot\chi(|f-e| \leq n-2\ell-\lambda_1-\lambda_2) \\
&\quad + \chi(\lambda_2 \leq \tfrac{e+f-n+\lambda_1+\lambda_2+1}{2} \leq \lambda_1)\cdot\chi(|f-e| \leq n-2\ell-\lambda_1-\lambda_2+1),
\end{aligned} \tag{4.54}$$

   where $\chi(P)$ is 1 if the proposition P is true and 0 if it is false.
3. *If $\lambda = (n)$, then $g(\lambda,(n-e,1^e),(n-f,1^f)) = \chi(e=f)$.*
4. *If λ is a hook shape $(n-d,1^d)$, then*

$$g(\lambda,(n-e,1^e),(n-f,1^f)) = \chi(|e-f| \leq d)\cdot\chi(d \leq e+f \leq 2n-d-2). \tag{4.55}$$

This implies that these coefficients are in $\{0,1,2\}$ [89, Corollary 4]. We are ready to prove the following result, establishing a formula for double hook characters in $R_n^{(0,2)}$.

Theorem 4.6.2. *Let $\lambda \vdash n$ be a double hook, that is, $\lambda = (\lambda_1,\lambda_2,2^\ell,1^{n-2\ell-\lambda_1-\lambda_2})$ where $\lambda_1 \geq \lambda_2 \geq 2$. Let $m := n-2\ell-\lambda_1-\lambda_2+2$. Then if $\lambda_1 = \lambda_2$, we have that*

$$\langle \mathrm{Frob}(R_n^{(0,2)};u,v), s_\lambda\rangle = (uv)^{\ell+\lambda_2-1}(uv[m-2]_{u,v} + [m]_{u,v} + [m-1]_{u,v}). \tag{4.56}$$

If $\lambda_1 > \lambda_2$, we have that

$$\langle \mathrm{Frob}(R_n^{(0,2)};u,v), s_\lambda\rangle = (uv)^{\ell+\lambda_2-1}(uv[m-1]_{u,v} + uv[m-2]_{u,v} + [m]_{u,v} + [m-1]_{u,v}). \tag{4.57}$$

Proof. Recall from Theorem 4.3.2 that

$$\mathrm{Frob}(R_n^{(0,2)};u,v) = \sum_{0\leq i+j<n} u^i v^j \left(s_{(n-i,1^i)} * s_{(n-j,1^j)} - s_{(n-i+1,1^{i-1})} * s_{(n-j+1,1^{j-1})}\right). \tag{4.58}$$

Thus

$$\begin{aligned}
&\langle \mathrm{Frob}(R_n^{(0,2)};u,v), s_\lambda\rangle \\
&\quad = \sum_{0\leq i+j<n} u^i v^j \left(g(\lambda,(n-i,1^i),(n-j,1^j)) - g(\lambda,(n-i+1,1^{i-1}),(n-j+1,1^{j-1}))\right).
\end{aligned} \tag{4.59}$$

Since $\lambda = (\lambda_1, \lambda_2, 2^\ell, 1^{n-2\ell-\lambda_1-\lambda_2})$ is a double hook, by Theorem 4.6.1,

$$\begin{aligned}
&g(\lambda,(n-i,1^i),(n-j,1^j)) - g(\lambda,(n-i+1,1^{i-1}),(n-j+1,1^{j-1}))\\
&\quad = \big(\chi(\lambda_2 - 1 \le \tfrac{i+j-n+\lambda_1+\lambda_2}{2} \le \lambda_1) - \chi(\lambda_2 - 1 \le \tfrac{i+j-n+\lambda_1+\lambda_2-2}{2} \le \lambda_1)\big)\\
&\qquad \cdot \chi(|j-i| \le n - 2\ell - \lambda_1 - \lambda_2)\\
&\qquad + \big(\chi(\lambda_2 \le \tfrac{i+j-n+\lambda_1+\lambda_2+1}{2} \le \lambda_1) - \chi(\lambda_2 \le \tfrac{i+j-n+\lambda_1+\lambda_2-1}{2} \le \lambda_1)\big)\\
&\qquad\quad \cdot \chi(|j-i| \le n - 2\ell - \lambda_1 - \lambda_2 + 1).
\end{aligned} \tag{4.60}$$

Define $d := \lambda_1 - \lambda_2$. We analyze a component of equation (4.60):

$$\chi(\lambda_2 - 1 \le \tfrac{i+j-n+\lambda_1+\lambda_2}{2} \le \lambda_1) - \chi(\lambda_2 - 1 \le \tfrac{i+j-n+\lambda_1+\lambda_2-2}{2} \le \lambda_1), \tag{4.61}$$

which is equivalent to

$$\chi(|i+j-(n-1)| \le d+1) - \chi(|i+j-(n+1)| \le d+1). \tag{4.62}$$

Since the bigraded component $(R_n^{(0,2)})_{i,j} = 0$ whenever $i + j \ge n$, we exclude these cases from our analysis. Thus $|i + j - (n-1)| \le d + 1$ is satisfied if $i + j = n - 1$ or $n - 2$ (if $d \ge 0$), if $i + j = n - 3$ (if $d \ge 1$), if $i + j = n - 4$ (if $d \ge 2$), etc. On the other hand, $|i + j - (n + 1)| \le d + 1$ is satisfied if $i + j = n - 1$ (if $d \ge 1$), if $i + j = n - 2$ (if $d \ge 2$), etc. Putting these together, we get that equation (4.62) is 1 when $i + j \in \{n - 1 - d, n - 2 - d\}$ and 0 otherwise.

We analyze when condition $\chi(|j - i| \le n - 2\ell - \lambda_1 - \lambda_2)$ is satisfied. At $i + j = n - 1 - d$, the condition is true exactly for integers j which satisfy

$$\ell + \lambda_2 \le j \le n - d - \ell - \lambda_2 - 1. \tag{4.63}$$

Summing up $u^{n-1-d-j}v^j$ over such j gives

$$(uv)^{\ell+\lambda_2}[n - d - 2\ell - 2\lambda_2]_{u,v}. \tag{4.64}$$

At $i + j = n - 2 - d$, the condition is true exactly for integers j which satisfy

$$\ell + \lambda_2 - 1 \le j \le n - d - \ell - \lambda_2 - 1. \tag{4.65}$$

Summing up $u^{n-2-d-j}v^j$ over such j gives

$$(uv)^{\ell+\lambda_2-1}[n - d - 2\ell - 2\lambda_2 + 1]_{u,v}. \tag{4.66}$$

Next we analyze another component of equation (4.60):

$$\chi(\lambda_2 \le \tfrac{i+j-n+\lambda_1+\lambda_2+1}{2} \le \lambda_1) - \chi(\lambda_2 \le \tfrac{i+j-n+\lambda_1+\lambda_2-1}{2} \le \lambda_1), \tag{4.67}$$

which is equivalent to

$$\chi(|i+j-(n-1)| \le d) - \chi(|i+j-(n+1)| \le d). \tag{4.68}$$

We have that $|i+j-(n-1)| \leq d$ is satisfied if $i+j = n-1$ (if $d \geq 0$), if $i+j = n-2$ (if $d \geq 1$), if $i+j = n-3$ (if $d \geq 2$), etc. On the other hand, $|i+j-(n+1)| \leq d$ is satisfied if $i+j = n-1$ (if $d \geq 2$), if $i+j = n-2$ (if $d \geq 3$), etc. Putting these together, when $d = 0$, we get that equation (4.68) is 1 when $i+j = n-1$ and 0 otherwise. When $d \geq 1$, we get that equation (4.68) is 1 when $i+j \in \{n-d, n-1-d\}$ and 0 otherwise.

We analyze when condition $\chi(|j-i| \leq n-2\ell-\lambda_1-\lambda_2+1)$ is satisfied. At $i+j = n-1-d$, the condition is true exactly for integers j which satisfy

$$\ell+\lambda_2-1 \leq j \leq n-d-\ell-\lambda_2. \tag{4.69}$$

Summing up $u^{n-1-d-j}v^j$ over such j gives

$$(uv)^{\ell+\lambda_2-1}[n-d-2\ell-2\lambda_2+2]_{u,v}. \tag{4.70}$$

When $d \geq 1$, at $i+j = n-d$, the condition is true exactly for integers j which satisfy

$$\ell+\lambda_2 \leq j \leq n-d-\ell-\lambda_2. \tag{4.71}$$

Summing up $u^{n-d-j}v^j$ over such j gives

$$(uv)^{\ell+\lambda_2}[n-d-2\ell-2\lambda_2+1]_{u,v}. \tag{4.72}$$

For concision, we use $m := n-2\ell-\lambda_1-\lambda_2+2 = n-d-2\ell-2\lambda_2+2$. When $d = 0$, i.e., $\lambda_1 = \lambda_2$, summing equations (4.64), (4.66), and (4.70) proves equation (4.56). When $d \geq 1$, i.e., $\lambda_1 > \lambda_2$, summing equations (4.64), (4.66), (4.70), and (4.72) proves equation (4.57). $\square$

4.7 Four sets of fermions

A natural further question is to determine $\langle \mathrm{Frob}(R_n^{(0,j)}; u_1, \ldots, u_j), s_{(1^n)}\rangle$ for $j \geq 4$, of which the simplest next case is $\langle \mathrm{Frob}(R_n^{(0,4)}; u, v, w, z), s_{(1^n)}\rangle$. If the hook characters in $R_n^{(0,3)}$ are all known, then a similar process as in Proposition 4.4.1 using $\mathrm{Frob}(R_n^{(0,3)} \otimes R_n^{(0,1)}; u, v, w, z)$ can be attempted. Using Theorem 4.6.1, we can first bound $\langle \mathrm{Frob}(R_n^{(0,3)}; u, v, w), s_{(n-d,1^d)}\rangle$.

Example 4.7.1. Let $n = 3$. Suppose we want to calculate $\langle \mathrm{Frob}(R_3^{(0,3)}; u, v, w), s_{(2,1)}\rangle$. Consider that any $s_{(2,1)}$ in $\mathrm{Frob}(R_3^{(0,3)}; u, v, w)$ must appear in $\mathrm{Frob}(R_3^{(0,2)} \otimes R_3^{(0,1)}; u, v, w)$. We compute that

$$\begin{aligned}&\langle \mathrm{Frob}(R_3^{(0,2)} \otimes R_3^{(0,1)}; u, v, w), s_{(2,1)}\rangle \\ &\qquad = \sum_{\lambda \vdash 3} \sum_{d=0}^{2} \langle \mathrm{Frob}(R_3^{(0,2)}; u, v), s_\lambda\rangle \langle \mathrm{Frob}(R_3^{(0,1)}; w), s_{(3-d,1^d)}\rangle g(\lambda, (3-d, 1^d), (2,1)).\end{aligned} \tag{4.73}$$

Using Rosas' formula for Kronecker coefficients of two hooks, we determine that the only pairs of λ and $(3-d,1^d)$ which do not have multiplicity of 0 are $(3),(2,1)$; $(2,1),(3)$; $(2,1),(2,1)$; $(2,1),(1^3)$; and $(1^3),(2,1)$, which all have multiplicity of 1. Using Proposition 4.3.3 and Lemma 4.3.1, we obtain

$$w + ([2]_{u,v} + uv) + ([2]_{u,v} + uv)w + ([2]_{u,v} + uv)w^2 + [3]_{u,v}w. \tag{4.74}$$

By permuting which sets of variables are assigned to $R_n^{(0,2)}$ and $R_n^{(0,1)}$, we can similarly obtain

$$v + ([2]_{u,w} + uw) + ([2]_{u,w} + uw)v + ([2]_{u,w} + uw)v^2 + [3]_{u,w}v, \tag{4.75}$$

$$u + ([2]_{w,v} + wv) + ([2]_{w,v} + wv)u + ([2]_{w,v} + wv)u^2 + [3]_{w,v}u. \tag{4.76}$$

Summing the monomials which appear in all three equations, we obtain

$$u + v + w + uv + uw + vw + 2uvw. \tag{4.77}$$

Now a computer calculation finds that

$$\langle \mathrm{Frob}(R_3^{(0,3)}; u, v, w), s_{(2,1)} \rangle = u + v + w + uv + uw + vw, \tag{4.78}$$

which shows that the bound is not tight in this case due to the monomial $2uvw$. It will require additional work to determine how to cut out the extra monomials in general.

Example 4.7.2. Let $n = 3$. Suppose we know all of the hook characters in $R_3^{(0,3)}$; in this case $\mathrm{Frob}(R_3^{(0,3)}; u, v, w) = (uvw + u^2 + uv + v^2 + uw + vw + w^2)s_{(1^3)} + (uv + uw + vw + u + v + w)s_{(2,1)} + s_{(3)}$. Consider that any sign character in $R_3^{(0,4)}$ must appear in $R_3^{(0,3)} \otimes R_3^{(0,1)}$. We compute that

$$\begin{aligned}\langle \mathrm{Frob}(R_3^{(0,3)} \otimes R_3^{(0,1)}; u, v, w, z), s_{(1^3)} \rangle &= \sum_{\lambda \vdash 3} \langle \mathrm{Frob}(R_3^{(0,3)}; u, v, w), s_{\lambda'} \rangle \langle \mathrm{Frob}(R_3^{(0,1)}; z), s_\lambda \rangle \\ &= \sum_{d=0}^{2} \langle \mathrm{Frob}(R_3^{(0,3)}; u, v, w), s_{(d+1,1^{2-d})} \rangle z^d,\end{aligned} \tag{4.79}$$

using Lemma 4.3.1. This simplifies to

$$u^2 + v^2 + w^2 + z^2 + uv + uw + uz + vw + vz + wz + uvw + uvz + uwz + vwz, \tag{4.80}$$

which is unchanged under changing which sets of variables are assigned to $R_n^{(0,3)}$ and $R_n^{(0,1)}$. In this example, equation (4.80) is equal to $\langle \mathrm{Frob}(R_3^{(0,4)}; u, v, w, z), s_{(1^3)} \rangle$ determined by computer calculation.

Example 4.7.3. The previous example also gives tight bounds for $n = 4$, with graded multiplicity $s_{(3)}(u, v, w, z) + s_{(2,1,1)}(u, v, w, z) + s_{(2,1,1,1)}(u, v, w, z)$, and for $n = 5$, with graded multiplicity $s_{(4)}(u, v, w, z) + s_{(3,1,1)}(u, v, w, z) + s_{(3,1,1,1)}(u, v, w, z) + s_{(2,2,1,1)}(u, v, w, z) + s_{(2,2,2,1)}(u, v, w, z)$.

Another approach is to use $\mathrm{Frob}(R_n^{(0,2)} \otimes R_n^{(0,2)}; u, v, w, z)$ to bound the sign character in $\mathrm{Frob}(R_n^{(0,4)}; u, v, w, z)$ directly. As Kim and Rhoades note, only Schur functions of shapes contained within double hooks occur in $\mathrm{Frob}(R_n^{(0,2)}; u, v)$, so the shapes which index sign characters are limited.

Example 4.7.4. Let $n = 3$. Consider that any sign character in $R_3^{(0,4)}$ must appear in $R_3^{(0,2)} \otimes R_3^{(0,2)}$. We compute that

$$\langle \mathrm{Frob}(R_3^{(0,2)} \otimes R_3^{(0,2)}; u, v, w, z), s_{(1^3)} \rangle = \sum_{\lambda \vdash 3} \langle \mathrm{Frob}(R_3^{(0,2)}; u, v), s_{\lambda'} \rangle \langle \mathrm{Frob}(R_3^{(0,2)}; w, z), s_\lambda \rangle. \tag{4.81}$$

In this case, there are only three partitions of 3: $(1^3), (2,1), (3)$. Using Proposition 4.3.3, we get

$$\langle \mathrm{Frob}(R_3^{(0,2)}; u, v), s_{(3)} \rangle \langle \mathrm{Frob}(R_3^{(0,2)}; w, z), s_{(1^3)} \rangle = [3]_{w,z}, \tag{4.82}$$

$$\langle \mathrm{Frob}(R_3^{(0,2)}; u, v), s_{(2,1)} \rangle \langle \mathrm{Frob}(R_3^{(0,2)}; w, z), s_{(2,1)} \rangle = ([2]_{u,v} + uv)([2]_{w,z} + wz), \tag{4.83}$$

$$\langle \mathrm{Frob}(R_3^{(0,2)}; u, v), s_{(1^3)} \rangle \langle \mathrm{Frob}(R_3^{(0,2)}; w, z), s_{(3)} \rangle = [3]_{u,v}. \tag{4.84}$$

Putting these together, we obtain

$$[3]_{w,z} + ([2]_{u,v} + uv)([2]_{w,z} + wz) + [3]_{u,v}. \tag{4.85}$$

By changing which sets of variables are assigned to each $R_n^{(0,2)}$, we can similarly obtain

$$[3]_{v,z} + ([2]_{u,w} + uw)([2]_{v,z} + vz) + [3]_{u,w}, \tag{4.86}$$

$$[3]_{u,z} + ([2]_{w,v} + wv)([2]_{u,z} + uz) + [3]_{w,v}. \tag{4.87}$$

In this case, all three are equal to each other, and expand to give

$$uvwz + uvw + uvz + uwz + vwz + u^2 + uv + v^2 + uw + vw + w^2 + uz + vz + wz + z^2. \tag{4.88}$$

Now a computer calculation finds that

$$\begin{aligned} &\langle \mathrm{Frob}(R_3^{(0,4)}; u, v, w, z), s_{(1^3)} \rangle \\ &\qquad = uvw + uvz + uwz + vwz + u^2 + uv + v^2 + uw + vw + w^2 + uz + vz + wz + z^2, \end{aligned} \tag{4.89}$$

which shows that the bound is not tight in this case due to the monomial $uvwz$. It will require additional work to determine how to cut out the extra monomials in general.

4.8 One set of bosons and three sets of fermions

In this section, we study the sign character in $R_n^{(1,3)}$, the $(1,3)$-bosonic-fermionic coinvariant ring. We first recall the "Theta conjecture" of D'Adderio, Iraci, and Vanden Wyngaerd, which expresses the multigraded Frobenius series of $R_n^{(2,2)}$ in terms of certain Theta operators and the nabla operator.

Conjecture 4.8.1 ([37, Conjecture 8.2]). *For all $n \geq 1$,*

$$\mathrm{Frob}(R_n^{(2,2)}; q, t; u, v) = \sum_{k+\ell<n} u^k v^\ell \Theta_{e_k} \Theta_{e_\ell} \nabla e_{n-k-\ell}. \tag{4.90}$$

The Theta conjecture specialized at $t = 0$ is the following.

Conjecture 4.8.2 (D'Adderio, Iraci, and Vanden Wyngaerd). *For all $n \geq 1$,*

$$\mathrm{Frob}(R_n^{(1,2)}; q; u, v) = \sum_{k+\ell<n} u^k v^\ell \left(\Theta_{e_k} \Theta_{e_\ell} \nabla e_{n-k-\ell}\right)|_{t=0}. \tag{4.91}$$

We recall the following result on the conjectural hook characters in $R_n^{(1,2)}$.

Theorem 4.8.3 ([73, Theorem 8.4]). *If the Theta conjecture specialized at $t = 0$ (Conjecture 4.8.2) is true, then*

$$\begin{aligned}
&\langle \mathrm{Frob}(R_n^{(1,2)}; q; u, v), s_{(d+1,1^{n-d-1})} \rangle \\
&\quad = \sum_{k+\ell<n} u^k v^\ell q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q.
\end{aligned} \tag{4.92}$$

Now we compute an example.

Example 4.8.4. Let $n = 3$. Consider that any sign character in $R_3^{(1,3)}$ must appear in $R_3^{(1,2)} \otimes R_3^{(0,1)}$. We compute that

$$\begin{aligned}
&\langle \mathrm{Frob}(R_3^{(1,2)} \otimes R_3^{(0,1)}; q; u, v, w), s_{(1^3)} \rangle \\
&\quad = \sum_{\lambda \vdash 3} \langle \mathrm{Frob}(R_3^{(1,2)}; q; u, v), s_{\lambda'} \rangle \langle \mathrm{Frob}(R_3^{(0,1)}; w), s_\lambda \rangle \\
&\quad = \sum_{d=0}^{2} \langle \mathrm{Frob}(R_3^{(1,2)}; q; u, v), s_{(d+1,1^{2-d})} \rangle w^d \\
&\quad = (q^3 + vq[2]_q + uq[2]_q + uv[2]_q + v^2 + u^2) + (uv + q[2]_q + v[2]_q + u[2]_q)w + w^2.
\end{aligned} \tag{4.93}$$

In this example, this is equal to $\langle \mathrm{Frob}(R_3^{(1,3)}; q; u, v, w), s_{(1^3)} \rangle$ determined by computer calculation.

Proposition 4.8.5. *If the Theta conjecture specialized at $t = 0$ (Conjecture 4.8.2) is true, then we have the following upper-bound:*

$$\begin{aligned}&\langle \mathrm{Frob}(R_n^{(1,3)}; q; u, v, w), s_{(1^n)}\rangle \\ &\qquad \leq \sum_{k,\ell,d\geq 0} u^k v^\ell w^d q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q.\end{aligned} \tag{4.94}$$

Proof. Any sign character in $R_n^{(1,3)}$ must appear in $R_n^{(1,2)} \otimes R_n^{(0,1)}$. Assume the Theta conjecture specialized at $t = 0$ (Conjecture 4.8.2) is true. By Theorem 4.8.3, we compute that

$$\begin{aligned}&\langle \mathrm{Frob}(R_n^{(1,2)} \otimes R_n^{(0,1)}; q; u, v, w), s_{(1^n)}\rangle \\ &\quad = \sum_{\lambda\vdash n} \langle \mathrm{Frob}(R_n^{(1,2)}; q; u, v), s_{\lambda'}\rangle \langle \mathrm{Frob}(R_n^{(0,1)}; w), s_\lambda\rangle \\ &\quad = \sum_{d=0}^{n-1} w^d \langle \mathrm{Frob}(R_n^{(1,2)}; q; u, v), s_{(d+1,1^{n-d-1})}\rangle \\ &\quad = \sum_{d=0}^{n-1} w^d \sum_{k+\ell<n} u^k v^\ell q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q \\ &\quad = \sum_{k,\ell,d\geq 0} u^k v^\ell w^d q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q,\end{aligned} \tag{4.95}$$

where the last line follows by considering when the q-binomial coefficients must be 0 (specifically, one could take $k + \ell < n$, $\ell + d < n$, and $d + k < n$). $\square$

Based on data for $n \leq 5$, we propose the following conjecture.

Conjecture 4.8.6.

$$\begin{aligned}&\langle \mathrm{Frob}(R^{(1,3)}; q; u, v, w), s_{(1^n)}\rangle \\ &\qquad = \sum_{k,\ell,d\geq 0} u^k v^\ell w^d q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q.\end{aligned} \tag{4.96}$$

Notice that upon specializing q, u, v, w all to 1, the conjecture becomes

$$\langle \mathrm{Frob}(R^{(1,3)}; 1; 1, 1, 1), s_{(1^n)}\rangle = \sum_{k,\ell,d\geq 0} \binom{n-1-d}{\ell} \binom{n-1-k}{d} \binom{n-1-\ell}{k}. \tag{4.97}$$

Let F_n be the nth Fibonacci number, defined by the initial conditions $F_0 = 0$, $F_1 = 1$, and recurrence $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Inspired by a conjecture of Zabrocki on $\mathrm{Frob}(R_n^{(2,1)}; q, t; u)$ [101], Bergeron made a general conjecture on how the Frobenius series of bosonic-fermionic coinvariant rings could be derived from the Frobenius series of purely bosonic coinvariant rings [10, Conjecture 1]. While collecting computational evidence, Bergeron conjectured the following.

Conjecture 4.8.7 ([10, Table 3])**.** *For all $n \geq 1$,*

$$\langle \mathrm{Frob}(R_n^{(1,3)}; 1; 1, 1, 1), s_{(1^n)} \rangle = \frac{1}{2} F_{3n}. \tag{4.98}$$

Beginning at $n = 1$, the sequence $\frac{1}{2}F_{3n}$ is $1, 4, 17, 72, 305, 1292, 5473, 23184, \ldots$ (see [86, Sequence A001076]). The two proposed enumerations are connected by the following result.

Proposition 4.8.8 ([27]; see [5] for a bijective proof)**.** *For $n \geq 1$,*

$$\sum_{k,\ell,d \geq 0} \binom{n-1-d}{\ell} \binom{n-1-k}{d} \binom{n-1-\ell}{k} = \frac{1}{2} F_{3n}. \tag{4.99}$$

Remark 4.8.9. If the Theta conjecture specialized at $t = 0$ (Conjecture 4.8.2) is true, and the conjecture of Bergeron (Conjecture 4.8.7) that $\langle \mathrm{Frob}(R^{(1,3)}; 1; 1, 1, 1), s_{(1^n)} \rangle = \frac{1}{2} F_{3n}$ is true, then by Proposition 4.8.8, the sign character has multiplicity large enough to force equality in the upper bound given in Proposition 4.8.5. This would prove Conjecture 4.8.6.

Chapter 5

Diagonal supersymmetry for coinvariant rings

For finite groups G, we show that bosonic-fermionic coinvariant rings have a natural $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module structure. In particular, we show that their character series are a sum of super Schur functions $s_\lambda(\mathbf{q}/\mathbf{u})$ times irreducible characters of G with universal coefficients, which do not depend on k, j. In the case where G is the symmetric group with diagonal action, this proves the "Diagonal Supersymmetry" conjecture of Bergeron (2020).

This chapter appears in [74].

The diagonal coinvariant ring $R_n^{(2,0)} = \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]/\langle \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]_+^{\mathfrak{S}_n} \rangle$ was introduced by Haiman in 1994 [57], and has played a central role in contemporary algebraic combinatorics (see for example [19, 20, 28, 39, 53, 81]). Here $\boldsymbol{x}$ and $\boldsymbol{y}$ denote two sets of variables $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$, respectively. The defining ideal $\langle \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]_+^{\mathfrak{S}_n} \rangle$ is generated by all polynomials in $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$ without constant term, which are invariant under the diagonal action of the symmetric group $\mathfrak{S}_n$. Haiman conjectured [57] and then later proved [59] formulas for the dimension, bigraded Hilbert series, and Frobenius series of $R_n^{(2,0)}$ using novel results on the Hilbert scheme of points in $\mathbb{C}^2$.

There has been previous work on coinvariant rings $R_n^{(k,0)}$ with k sets of commuting variables [9, 11], however a proof of just the dimension for $R_n^{(3,0)}$, the case with three sets of variables, has been elusive [14].

The study of the coinvariant ring with one set of commuting variables and one set of anticommuting variables $R_n^{(1,1)}$ was initiated by N. Bergeron, Colmenarejo, Li, Machacek, Sulzgruber, and Zabrocki in a working seminar in algebraic combinatorics at the Fields Institute in 2018. Eventually, the multiplicity of the sign character of $R_n^{(1,1)}$ [96], its dimension and Hilbert series [88], a monomial basis [3, 90] and its Frobenius series [84] were all determined. Zabrocki introduced a coinvariant ring with two sets of commuting variables and one set of anticommuting variables [101], which is conjecturally related to a symmetric function appearing in the Delta conjecture of Haglund, Remmel, and Wilson [53].[1] Then D'Adderio,

[1] One version of the Delta conjecture has been proven by [39] and [19], but the connection between that

Iraci, and Vanden Wyngaerd [37] considered a coinvariant ring with two sets of commuting variables and two sets of anticommuting variables, related to their work on Theta operators.

All of these are instances of coinvariant rings $R_n^{(k,j)}$ with k sets of n commuting (bosonic, even) variables $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(k)}$, where $\boldsymbol{x}^{(i)} = \{x_1^{(i)}, \ldots, x_n^{(i)}\}$, and j sets of n anticommuting (fermionic, odd) variables $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \ldots, \boldsymbol{\theta}^{(j)}$, where $\boldsymbol{\theta}^{(i)} = \{\theta_1^{(i)}, \ldots, \theta_n^{(i)}\}$, for k, j nonnegative integers (see equation (5.1) for the complete definition). It has been proposed to study coinvariant rings in this more general setting (see [6, 10, 11, 102]).

We first describe the construction of the polynomial ring $\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$. Let $G \subset \mathrm{GL}(V) = \mathrm{GL}(n)$ be a finite group acting on $V = \mathbb{C}^n$. Let $\mathbb{C}^{k|j}$ denote the complex superspace of even dimension k and odd dimension j. Then $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) = (\mathrm{Sym}(V))^{\otimes k} \otimes (\bigwedge(V))^{\otimes j}$, and upon choosing a basis of V, we number the coordinates so that they correspond to the tensor factors. That is, $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}$ correspond to coordinates on k copies of the symmetric algebra $\mathrm{Sym}(V)$ and $\boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}$ correspond to coordinates on j copies of the exterior algebra $\bigwedge(V)$. Thus we write $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) = \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$, where commuting variables commute with all variables; anticommuting variables anticommute with all anticommuting variables. That is, $\theta_i^{(\ell)}\theta_h^{(m)} = -\theta_h^{(m)}\theta_i^{(\ell)}$, which implies that $(\theta_i^{(\ell)})^2 = 0$.

We now define the (k, j)-*bosonic-fermionic coinvariant ring for a finite group* $G \subset \mathrm{GL}(n)$ by

$$R_G^{(k,j)} = \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}] / \langle \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]_+^G \rangle, \tag{5.1}$$

where $\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]_+^G$ denotes the polynomials without constant term, which are invariant under the diagonal action of G on each set of variables. In the case where G is the symmetric group $\mathfrak{S}_n$ acting diagonally by permuting the indices of the variables within each set, we denote this by $R_n^{(k,j)}$.

In this paper, we propose that a natural setting for studying $R_G^{(k,j)}$ is as a $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module, where $U(\mathfrak{gl}(k|j))$ is the universal enveloping algebra of the Lie superalgebra $\mathfrak{gl}(k|j)$. Recall that a G-representation $G \to \mathrm{GL}(V)$ is equivalent to a $\mathbb{C}[G]$-module [24, §21.1]. In general, given any Lie superalgebra $\mathfrak{g}$, a $\mathfrak{g}$-module is equivalent to a $U(\mathfrak{g})$-module, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ [31, Chapter 1.5.1]. We will frequently utilize these equivalences.

We describe the relevant actions on $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$. First, $\mathfrak{gl}(k|j)$ acts on $\mathbb{C}^{k|j}$ as the defining representation. Since $\mathfrak{gl}(k|j)$ acts trivially on $V = \mathbb{C}^n$, then $\mathfrak{gl}(k|j)$ acts on $\mathbb{C}^{k|j} \otimes V$. As $\mathbb{C}^{k|j} \otimes V$ is the degree 1 (in total degree) part of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$, then the action of $\mathfrak{gl}(k|j)$ extends to the whole ring $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$ by left superderivations (see equation (5.20)). Next, since $G \subset \mathrm{GL}(V)$, then G acts in a natural way on V, in addition to trivially on $\mathbb{C}^{k|j}$. Combining these, G acts on $\mathbb{C}^{k|j} \otimes V$, which extends to $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$. The actions of $\mathfrak{gl}(k|j)$ and G commute with each other (cf. [32, Section 5.2.1]).

With this action in hand, we then show that $R_G^{(k,j)}$ is a quotient module of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$. Since $\mathfrak{gl}(k|j)$ is not semisimple, we are not a priori guaranteed that $R_G^{(k,j)}$ will decompose into simple $U(\mathfrak{gl}(k|j))$-modules. Our first main result is that the desired $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module

and Zabrocki's coinvariant ring remains conjectural.

structure exists for $R_G^{(k,j)}$, and we describe how it decomposes into simple modules. Denote the set of partitions λ with length $\ell(\lambda) \leq n$ which satisfy $\lambda_{k+1} \leq j$ by $P(k,j,n)$. Let μ index the irreducible characters of G. Let $s_\lambda(\mathbf{q}/\mathbf{u})$ denote a super Schur function (see Section 5.1 for definitions), where $\mathbf{q}$ denotes $(q_1, \dots, q_k)$, and $\mathbf{u}$ denotes $(u_1, \dots, u_j)$.

Theorem 5.0.1. *Fix a positive integer n and a finite group $G \subset \mathrm{GL}(n)$. There is an isomorphism of $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-modules:*

$$R_G^{(k,j)} \cong \bigoplus_{\lambda \in P(k,j,n)} \bigoplus_{\mu} \left(U_{k|j}^{\lambda} \otimes N^{\mu}\right)^{\oplus c_{\lambda\mu}}, \tag{5.2}$$

where the $U_{k|j}^{\lambda}$ are simple $U(\mathfrak{gl}(k|j))$-modules with character $s_\lambda(\mathbf{q}/\mathbf{u})$ and the N^μ are simple $\mathbb{C}[G]$-modules, for some nonnegative integer coefficients $c_{\lambda\mu}$.

Let $\mathrm{IrrChar}(G)$ denote the set of irreducible characters of G. From the module structure in Theorem 5.0.1, taking $\mathfrak{gl}(k|j)$ and G-characters yields a character series in terms of super Schur functions and G-characters. A priori, for fixed $G \subset \mathrm{GL}(V)$, these coefficients are functions of (k,j). However, in our second main result we show that they are independent of (k,j).

Theorem 5.0.2. *Fix a positive integer n and a finite group $G \subset \mathrm{GL}(n)$. For partitions λ with $\ell(\lambda) \leq n$, and indices μ of irreducible G-characters, there exist nonnegative integer coefficients $c_{\lambda\mu}$ such that for any (k,j), the multigraded character series of $R_G^{(k,j)}$ is*

$$\mathrm{Char}(R_G^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,n)} \sum_{\chi^\mu \in \mathrm{IrrChar}(G)} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) \chi^\mu. \tag{5.3}$$

When G is the symmetric group acting by permuting variables, we may apply the Frobenius characteristic map to the symmetric group characters, which are indexed by partitions of n. Let $\mathbf{z}$ be an (infinite) auxiliary set of variables for the Frobenius characters. Then the following result is an immediate consequence of Theorem 5.0.2. Thus we prove a conjecture of Bergeron.

Corollary 5.0.3 (Diagonal Supersymmetry [10, Conjecture 1])**.** *Fix a positive integer n. For partitions λ with $\ell(\lambda) \leq n$, and $\mu \vdash n$, there exist nonnegative integer coefficients $c_{\lambda\mu}$ such that for any (k,j), the multigraded Frobenius series of $R_n^{(k,j)}$ is*

$$\mathrm{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,n)} \sum_{\mu \vdash n} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) s_\mu(\mathbf{z}). \tag{5.4}$$

Remark 5.0.4. Given fixed n, k, j, it is important to note that equation (5.4) does not determine $c_{\lambda\mu}$ for all partitions λ of length at most n, but only for those which satisfy $\lambda_{k+1} \leq j$.

Previously, Bergeron showed [10, equation (2.1)] that $R_n^{(k,j)}$ is a $\mathrm{GL}(k) \times \mathrm{GL}(j) \times \mathfrak{S}_n$-module, without taking advantage of the relationship between the even and odd parts, which in the present setting comes from the Lie superalgebra structure. In this case, the irreducible characters of polynomial representations of $\mathrm{GL}(k)$ are Schur functions s_λ where $\ell(\lambda) \leq k$ (see for example [78, I.A.8]). Taking $\mathrm{GL}(k) \times \mathrm{GL}(j)$ characters and Frobenius characters, one obtains a series which is a product of Schur functions in three sets of variables. That is, for partitions λ with $\ell(\lambda) \leq n$, and $\mu \vdash n$, there exist nonnegative integers $c_{\lambda\nu\mu}$ such that for any (k, j)

$$\operatorname{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\substack{\lambda,\nu, \\ \ell(\lambda)\leq k, \ell(\nu)\leq j}} \sum_{\mu\vdash n} c_{\lambda\nu\mu} s_\lambda(\mathbf{q}) s_\nu(\mathbf{u}) s_\mu(\mathbf{z}). \tag{5.5}$$

Since a super Schur function $s_\lambda(\mathbf{q}/\mathbf{u})$ can always be written as a sum of products of Schur functions $s_\lambda(\mathbf{q})s_\nu(\mathbf{u})$, but the converse does not hold, equation (5.4) implies equation (5.5), but equation (5.5) does not imply equation (5.4).

Define $R_n^{(\infty,\infty)}$ as the direct limit of the modules $R_n^{(k,j)}$ over the directed set $\mathbb{N} \times \mathbb{N}$ ordered componentwise. The limit $R_n^{(\infty,\infty)}$ has an $\mathbb{N}^\infty \times \mathbb{N}^\infty$ multigrading, where $\mathbb{N}^\infty$ denotes the set of nonnegative integer sequences with finitely many nonzero terms. Let $\Lambda(\mathbf{z})$ denote the ring of symmetric functions in the variables $\mathbf{z}$. For any $\mathbb{N}^\infty \times \mathbb{N}^\infty$ multigraded $\mathfrak{S}_n$-module V, one can define a multigraded Frobenius series

$$\operatorname{Frob}(V; \mathbf{q}; \mathbf{u}) = \sum_{(\alpha,\beta)\in\mathbb{N}^\infty\times\mathbb{N}^\infty} \mathbf{q}^\alpha \mathbf{u}^\beta \operatorname{Frob}(V_{(\alpha,\beta)}) \in \mathbb{Z}[[\mathbf{q}, \mathbf{u}]] \otimes \Lambda(\mathbf{z}). \tag{5.6}$$

Here $\mathbf{q} = (q_1, q_2, \ldots)$ and $\mathbf{u} = (u_1, u_2, \ldots)$ are infinite formal alphabets, and $\mathbf{q}^\alpha \mathbf{u}^\beta$ records the multidegree of the component $V_{(\alpha,\beta)}$. This formula recovers the standard definition of the multigraded Frobenius series in the finite case.

If $V = \varinjlim V^{(k,j)}$ for some finite-dimensional, $\mathbb{N}^k \times \mathbb{N}^j$ multigraded modules $\{V^{(k,j)}\}$ with graded transition maps, then

$$\operatorname{Frob}(V; \mathbf{q}; \mathbf{u}) = \varprojlim_{k,j} \operatorname{Frob}(V^{(k,j)}; q_1, \ldots, q_k; u_1, \ldots, u_j), \tag{5.7}$$

where the inverse limit is taken with respect to the standard projections.

Bergeron showed [10] that there is a well-defined universal series

$$\operatorname{Frob}(R_n^{(\infty,\infty)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda,\nu} \sum_{\mu\vdash n} c_{\lambda\nu\mu} s_\lambda(\mathbf{q}) s_\nu(\mathbf{u}) s_\mu(\mathbf{z}), \tag{5.8}$$

which determines all $c_{\lambda\nu\mu}$ which could appear in the expansion in equation (5.5) for any (k, j).

We strengthen equation (5.8) by showing that there is a well-defined universal series in terms of super Schur functions.

Proposition 5.0.5. *Fix a positive integer n. There is a well-defined universal series*

$$\mathrm{Frob}(R_n^{(\infty,\infty)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda,\ \ell(\lambda)\leq n} \sum_{\mu\vdash n} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) s_\mu(\mathbf{z}) \tag{5.9}$$

which determines all $c_{\lambda\mu}$ which could appear in the expansion in equation (5.4) for any (k, j).

In general, $c_{\lambda\mu}$ may be zero or nonzero. For some finite k or j in the expansion of $\mathrm{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u})$ (equation (5.4)), not all nonzero $c_{\lambda\mu}$ from the universal series (equation (5.9)) need to appear, since they are multiplied by $s_\lambda(\mathbf{q}/\mathbf{u})$, which equals 0 when k or j is too small.

This raises the question of when there are enough nonzero coefficients $c_{\lambda\mu}$ determined by the multigraded Frobenius series of $R_n^{(k,j)}$ to determine the multigraded Frobenius series of $R_n^{(k',j')}$. By Proposition 5.0.5, the multigraded Frobenius series of $R_n^{(k,j)}$ determines the multigraded Frobenius series of $R_n^{(k',j')}$ whenever $k' \leq k$ and $j' \leq j$. In Proposition 5.3.2, we show that neither of the graded Frobenius series of $R_n^{(1,0)}$ and $R_n^{(0,1)}$ determines enough nonzero $c_{\lambda\mu}$ to determine the other; similarly none of the bigraded Frobenius series of $R_n^{(2,0)}$, $R_n^{(1,1)}$, and $R_n^{(0,2)}$ determine enough nonzero $c_{\lambda\mu}$ to determine another.

The outline of the paper is as follows. After reviewing the background in Section 5.1, we prove Theorems 5.0.1 and 5.0.2 in Section 5.2. In Section 5.3, we study the universal series for the symmetric group, where we prove Propositions 5.0.5 and 5.3.2. In Section 5.4, we conclude by discussing further combinatorial consequences of cancellation on the Hilbert and Frobenius series of certain $R_n^{(k,j)}$.

5.1 Background

We assume basic familiarity with symmetric functions (see [92, Chapter 7] or [78, Chapter I]), and representation theory of finite groups and Lie algebras (see [24]). Assume all tensor products are taken over $\mathbb{C}$. If a ring R decomposes as a direct sum of multihomogenous $\mathbb{C}[G]$-modules, for G a finite group:

$$R = \bigoplus_{r_1,\ldots,r_k,s_1,\ldots,s_j\geq 0} (R)_{r_1,\ldots,r_k,s_1,\ldots,s_j}, \tag{5.10}$$

then its *multigraded character series* is

$$\mathrm{Char}(R; \mathbf{q}; \mathbf{u}) := \sum_{r_1,\ldots,r_k,s_1,\ldots,s_j\geq 0} \mathrm{ch}\left((R)_{r_1,\ldots,r_k,s_1,\ldots,s_j}\right) q_1^{r_1}\cdots q_k^{r_k} u_1^{s_1}\cdots u_j^{s_j}, \tag{5.11}$$

where $\mathrm{ch}(S)$ denotes the character of a G-module S.

Recall that the Frobenius characteristic map F is an isomorphism between the ring of virtual $\mathfrak{S}_n$-characters and the ring of symmetric functions $\Lambda_{\mathbb{Z}}$, given by $F(\chi^\mu) = s_\mu(\mathbf{z})$, where χ^μ is the irreducible $\mathfrak{S}_n$-character indexed by μ, and $s_\mu(\mathbf{z})$ is a Schur function (see [92, Chapter

7.18]). In the special case where G is the symmetric group $\mathfrak{S}_n$, such as when $R = R_n^{(k,j)}$, we can apply the Frobenius characteristic map F to the character of each multigraded component, and obtain the *multigraded Frobenius series:*

$$\mathrm{Frob}(R_n^{(k,j)};\mathbf{q};\mathbf{u}) := \sum_{r_1,\ldots,r_k,s_1,\ldots,s_j\geq 0} F\,\mathrm{ch}\left((R_n^{(k,j)})_{r_1,\ldots,r_k,s_1,\ldots,s_j}\right) q_1^{r_1}\cdots q_k^{r_k} u_1^{s_1}\cdots u_j^{s_j}. \tag{5.12}$$

We often will use q,t for q_1,q_2 and u,v for u_1,u_2 when $k,j\leq 2$.

We have that $\mathrm{Sym}(\mathbb{C}^{k|j}\otimes V)$ has a multigrading with respect to $\mathbf{q}$ and $\mathbf{u}$. That is, each variable in $\boldsymbol{x}^{(i)}$ has q_i-degree of 1 and each variable in $\boldsymbol{\theta}^{(i)}$ has u_i-degree of 1. Furthermore, $\mathrm{Sym}(\mathbb{C}^{k|j}\otimes V)$ is graded by total degree, where each variable has degree 1. That is,

$$\mathrm{Sym}(\mathbb{C}^{k|j}\otimes V) = \bigoplus_{d\geq 0}\mathrm{Sym}^d(\mathbb{C}^{k|j}\otimes V), \tag{5.13}$$

where $\mathrm{Sym}^d(\mathbb{C}^{k|j}\otimes V)$ is the dth symmetric power.

Define a *super Schur function*[2] (see [32, Appendix A.2.2]) by

$$s_\lambda(\mathbf{q}/\mathbf{u}) = \sum_{\nu\subseteq\lambda} s_\nu(\mathbf{q}) s_{\lambda'/\nu'}(\mathbf{u}). \tag{5.14}$$

Super Schur functions can also be defined in terms of super tableaux (see [85, Chapter 12.3]). While we will not otherwise use it, we note that in [10] the plethystic notation $s_\lambda[\mathbf{q}-\varepsilon\mathbf{u}]$ is used. Since (see for example [10, Section 2])

$$s_\lambda[\mathbf{q}-\varepsilon\mathbf{u}] = \sum_{\nu\subseteq\lambda} s_\nu(\mathbf{q}) s_{\lambda'/\nu'}(\mathbf{u}), \tag{5.15}$$

then $s_\lambda(\mathbf{q}/\mathbf{u}) = s_\lambda[\mathbf{q}-\varepsilon\mathbf{u}]$. We will exclusively use $s_\lambda(\mathbf{q}/\mathbf{u})$.

Super Schur functions satisfy the following key properties [78, I.3 Example 23]:

- Cancellation:

$$s_\lambda(q_1,\ldots,q_{k-1},q_k/u_1,\ldots,u_{j-1},u_j)|_{u_j=-q_k} = s_\lambda(q_1,\ldots,q_{k-1}/u_1,\ldots,u_{j-1}), \tag{5.16}$$

- Restriction:

$$s_\lambda(q_1,\ldots,q_{k-1},q_k/u_1,\ldots,u_j)|_{q_k=0} = s_\lambda(q_1,\ldots,q_{k-1}/u_1,\ldots,u_j), \tag{5.17}$$

$$s_\lambda(q_1,\ldots,q_k/u_1,\ldots,u_{j-1},u_j)|_{u_j=0} = s_\lambda(q_1,\ldots,q_k/u_1,\ldots,u_{j-1}). \tag{5.18}$$

The super Schur functions $s_\lambda(\mathbf{q}/\mathbf{u})$ are characters of the simple $\mathfrak{gl}(k|j)$-modules $U_{k|j}^\lambda$ which appear in the decomposition of $\mathrm{Sym}(\mathbb{C}^{k|j}\otimes V)$ via Howe duality (see [61, 62]). The following result is due to Howe [61] (see also [32, Theorem 5.19; Chapter 5: Notes]).

[2] Also called a hook Schur function–not to be confused with a Schur function of a hook shaped partition.

Theorem 5.1.1 (($\mathfrak{gl}(k|j), \mathrm{GL}(n)$)-Howe Duality)**.** *For all $d \geq 0$, we have that*

$$\mathrm{Sym}^d(\mathbb{C}^{k|j} \otimes V) \cong \bigoplus_{\substack{\lambda \in P(k,j,n) \\ \lambda \vdash d}} U_{k|j}^{\lambda} \otimes U_n^{\lambda}, \tag{5.19}$$

where for each $\lambda \in P(k, j, n)$, $U_{k|j}^{\lambda}$ is a simple $\mathfrak{gl}(k|j)$-module with character the super Schur function $s_\lambda(\mathbf{q}/\mathbf{u})$, and U_n^λ is a simple $\mathrm{GL}(n)$-module with character the Schur function s_λ.

We note that taking characters of both sides of ($\mathfrak{gl}(k|j), \mathrm{GL}(n)$)-Howe Duality gives a super Cauchy identity, which can also be used to compute the super Schur functions (see [32, Chapter 5.2.2]).

We describe the action of $\mathfrak{gl}(k|j)$ by superderivations. For $1 \leq a, b \leq k$ and $1 \leq c, d \leq j$, we have

$$E_{a,b}^{+} = \sum_{p=1}^{n} x_p^{(a)} \partial_{x_p^{(b)}}, \; E_{a,d}^{\pm} = \sum_{p=1}^{n} x_p^{(a)} \partial_{\theta_p^{(d)}}, \; E_{c,b}^{\mp} = \sum_{p=1}^{n} \theta_p^{(c)} \partial_{x_p^{(b)}}, \; E_{c,d}^{-} = \sum_{p=1}^{n} \theta_p^{(c)} \partial_{\theta_p^{(d)}}, \tag{5.20}$$

which realize the action of $\mathfrak{gl}(k|j)$ on $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$ [32, Lemma 5.13]. In the special case of $(k, j) = (1, 1)$, see also [96, Section 3] and [97, Section 4.1]. Note that the action of $\mathfrak{gl}(k|j)$ by superderivations preserves each degree d component (with respect to the total degree) $\mathrm{Sym}^d(\mathbb{C}^{k|j} \otimes V)$.

We also have that $\mathrm{GL}(V)$ acts by matrix multiplication on V and trivially on $\mathbb{C}^{k|j}$. This action also preserves $\mathrm{Sym}^d(\mathbb{C}^{k|j} \otimes V)$. Since $G \subset \mathrm{GL}(V)$, G acts by matrix multiplication on V and trivially on $\mathbb{C}^{k|j}$, and the action preserves $\mathrm{Sym}^d(\mathbb{C}^{k|j} \otimes V)$.

We conclude this section with a consequence of the Jordan-Hölder theorem, on the multiplicities of simple modules occurring in a composition series of a quotient module (see [24, §11.3]).

Lemma 5.1.2. *Let A be an associative algebra over $\mathbb{C}$. Let M be a finite-dimensional A-module, and let N be an A-submodule of M. Then for any simple A-module S in a composition series of M/N, its multiplicity in M/N is less than or equal to its multiplicity in M.*

5.2 Proofs of the main results

We prove the first main theorem from the introduction.

Proof of Theorem 5.0.1. We continue with the setup from the introduction, where $V = \mathbb{C}^n$ and

$$\begin{aligned} \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}] &= (\mathrm{Sym}(V))^{\otimes k} \otimes (\textstyle\bigwedge(V))^{\otimes j} \\ &= \mathrm{Sym}(V \otimes \mathbb{C}^{k|0} \oplus V \otimes \mathbb{C}^{0|j}) \\ &= \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V). \end{aligned} \tag{5.21}$$

Our strategy is to first study $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$ and then consider the quotient $R_G^{(k,j)}$. We will work with $\mathrm{Sym}^d(\mathbb{C}^{k|j} \otimes V)$, the part of homogeneous total degree d.

Consider the simple $\mathrm{GL}(n)$-modules U_n^λ which occur in Howe duality (Theorem 5.1.1). Since $V = \mathbb{C}^n$, $\mathrm{GL}(V)$ is $\mathrm{GL}(n)$, and we identify their modules. From $\mathrm{GL}(V)$ we may restrict to any finite subgroup G of $\mathrm{GL}(V)$. Upon restricting the representations U_n^λ, since any finite-dimensional G-representation is semisimple, we have that

$$\mathrm{Res}_G^{\mathrm{GL}(V)}(U_n^\lambda) = \bigoplus_\mu (N^\mu)^{\oplus d_{\lambda\mu}}, \tag{5.22}$$

where the sum is over an indexing set for the simple G-representations N^μ, and the $d_{\lambda\mu}$ are some nonnegative integers. In particular, the $d_{\lambda\mu}$ are finite because U_n^λ is finite-dimensional.

Since the actions preserve $\mathrm{Sym}^d(\mathbb{C}^{k|j} \otimes V)$, it is a $(\mathfrak{gl}(k|j), G)$-module (equivalently, a $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module). By Howe duality (Theorem 5.1.1) and equation (5.22), $\mathrm{Sym}^d(\mathbb{C}^{k|j} \otimes V)$ has decomposition:

$$\begin{aligned} \mathrm{Sym}^d(\mathbb{C}^{k|j} \otimes V) &\cong \bigoplus_{\substack{\lambda \in P(k,j,n) \\ \lambda \vdash d}} U_{k|j}^\lambda \otimes \left(\bigoplus_\mu (N^\mu)^{\oplus d_{\lambda\mu}} \right) \\ &\cong \bigoplus_{\substack{\lambda \in P(k,j,n) \\ \lambda \vdash d}} \bigoplus_\mu U_{k|j}^\lambda \otimes (N^\mu)^{\oplus d_{\lambda\mu}} . \end{aligned} \tag{5.23}$$

Let $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)_d^G$ denote the total degree d polynomials in $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)^G$. Since the $\mathfrak{gl}(k|j)$-action is degree-preserving and the $\mathfrak{gl}(k|j)$ and G-actions commute, $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)_d^G$ is a $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-submodule of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$. Then

$$\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)_+^G = \bigoplus_{d \geq 1} \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)_d^G \tag{5.24}$$

is a $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-submodule of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$.

We turn our attention to $R_G^{(k,j)}$. From its definition, we have that

$$R_G^{(k,j)} = \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)/\langle \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)_+^G \rangle. \tag{5.25}$$

Consider the multiplication map $m : \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) \otimes \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) \to \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$. Restrict to the map $m' : \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) \otimes \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)_+^G \to \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$. The image of m', which is $\langle \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)_+^G \rangle$, is a $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-submodule of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$. Thus $R_G^{(k,j)}$ is a quotient module of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$.

Since $U_{k|j}^\lambda$ is a finite-dimensional simple $U(\mathfrak{gl}(k|j))$-module and N^μ is a finite-dimensional simple $\mathbb{C}[G]$-module, then $U_{k|j}^\lambda \otimes N^\mu$ is a finite-dimensional simple $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module, by [24, §12.1 Theorem 1(a)]. This implies that

$$\mathrm{Sym}^d(\mathbb{C}^{k|j} \otimes V) \cong \bigoplus_{\substack{\lambda \in P(k,j,n) \\ \lambda \vdash d}} \bigoplus_\mu \left(U_{k|j}^\lambda \otimes N^\mu \right)^{\oplus d_{\lambda\mu}} \tag{5.26}$$

is a decomposition into simple $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-modules, and is semisimple. Then $(R_G^{(k,j)})_d$, the homogeneous part of total degree d of $R_G^{(k,j)}$, is semisimple, by [24, §4.1 Corollary 3]. In the decomposition in equation (5.26), there are finitely many λ indexing the sum and the decomposition in equation (5.22) is into finitely many simple modules. Thus $\mathrm{Sym}^d(\mathbb{C}^{k|j} \otimes V)$ has a direct sum decomposition into finitely many simple modules, so it is of finite length. Then by Lemma 5.1.2,

$$(R_G^{(k,j)})_d \cong \bigoplus_{\substack{\lambda \in P(k,j,n) \\ \lambda \vdash d}} \bigoplus_{\mu} \left(U_{k|j}^{\lambda} \otimes N^{\mu}\right)^{\oplus c_{\lambda\mu}}, \tag{5.27}$$

for some integers $0 \leq c_{\lambda\mu} \leq d_{\lambda\mu}$. By summing over all d, we get that

$$R_G^{(k,j)} \cong \bigoplus_{\lambda \in P(k,j,n)} \bigoplus_{\mu} \left(U_{k|j}^{\lambda} \otimes N^{\mu}\right)^{\oplus c_{\lambda\mu}}. \tag{5.28}$$

□

Next, we prove our second main theorem.

Proof of Theorem 5.0.2. Given the decomposition in Theorem 5.0.1, take the $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$ character to conclude that

$$\mathrm{Char}(R_G^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,n)} \sum_{\chi^{\mu} \in \mathrm{IrrChar}(G)} c_{\lambda\mu} s_{\lambda}(\mathbf{q}/\mathbf{u}) \chi^{\mu}. \tag{5.29}$$

It remains to show that as long as $c_{\lambda\mu}$ appears in a Frobenius series expansion, it does not depend on the choice of (k, j).

Note that there is a canonical isomorphism between the quotient ring $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)/\langle \boldsymbol{x}^{(k)} \rangle = \mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)|_{\boldsymbol{x}^{(k)}=0}$ and the subring of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$ consisting of polynomials with zero q_k-degree.

For any G-invariant polynomial p, since G acts diagonally on each set of variables, every monomial in p has the same q_k-degree. Thus

$$\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]_+^G|_{\boldsymbol{x}^{(k)}=0} = \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k-1)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]_+^G. \tag{5.30}$$

Restrict the image of the multiplication map m' (from the proof of Theorem 5.0.1) to the subring $\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k-1)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$. Observe that a polynomial which has zero q_k-degree in the image of m' must have zero q_k-degree in both tensor factors in the preimage. We conclude that

$$\langle \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]_+^G \rangle|_{\boldsymbol{x}^{(k)}=0} = \langle \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k-1)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]_+^G \rangle. \tag{5.31}$$

It follows that $R_G^{(k-1,j)} = R_G^{(k,j)}|_{\boldsymbol{x}^{(k)}=0}$. Since only the variables in $\boldsymbol{x}^{(k)}$ contribute to the q_k-degree,

$$\mathrm{Char}(R_G^{(k-1,j)}; q_1, \ldots, q_{k-1}; \mathbf{u}) = \mathrm{Char}(R_G^{(k,j)}; q_1, \ldots, q_{k-1}, q_k; \mathbf{u})|_{q_k=0}. \tag{5.32}$$

On the one hand, suppose we are given equation (5.29); then by equation (5.32),

$$\begin{aligned}\mathrm{Char}(R_G^{(k-1,j)}; q_1, \ldots, q_{k-1}; \mathbf{u}) &= \sum_{\lambda \in P(k,j,n)} \sum_{\chi^\mu \in \mathrm{IrrChar}(G)} c_{\lambda\mu} s_\lambda(q_1, \ldots, q_{k-1}, q_k/\mathbf{u})|_{q_k=0} \cdot \chi^\mu \\ &= \sum_{\lambda \in P(k-1,j,n)} \sum_{\chi^\mu \in \mathrm{IrrChar}(G)} c_{\lambda\mu} s_\lambda(q_1, \ldots, q_{k-1}, 0/\mathbf{u}) \chi^\mu,\end{aligned} \tag{5.33}$$

where the last equality follows because when $q_k = 0$, the restriction on which super Schur functions are nonzero changes from $\lambda_{k+1} \leq j$ to $\lambda_k \leq j$.

On the other hand, we can compute directly that for some nonnegative integer coefficients $\tilde{c}_{\lambda\mu}$,

$$\begin{aligned}\mathrm{Char}(R_G^{(k-1,j)}; q_1, \ldots, q_{k-1}; \mathbf{u}) &= \sum_{\lambda \in P(k-1,j,n)} \sum_{\chi^\mu \in \mathrm{IrrChar}(G)} \tilde{c}_{\lambda\mu} s_\lambda(q_1, \ldots, q_{k-1}/\mathbf{u}) \chi^\mu \\ &= \sum_{\lambda \in P(k-1,j,n)} \sum_{\chi^\mu \in \mathrm{IrrChar}(G)} \tilde{c}_{\lambda\mu} s_\lambda(q_1, \ldots, q_{k-1}, 0/\mathbf{u}) \chi^\mu,\end{aligned} \tag{5.34}$$

where we applied restriction (equation (5.17)). Thus we conclude that $\tilde{c}_{\lambda\mu} = c_{\lambda\mu}$ for all $\lambda \in P(k-1, j, n)$, i.e., which appear in both expansions (equations (5.33) and (5.34)). Checking the character series for $R_G^{(k,j-1)}$ is analogous, and we conclude that the $c_{\lambda\mu}$ do not depend on (k, j). □

For the symmetric group $\mathfrak{S}_n$, once k, j have been fixed, in the special case that $n \leq k + j$, then the purely bosonic case or the purely fermionic case completely determines the mixed bosonic-fermionic case. For the general situation, see Proposition 5.3.2.

Corollary 5.2.1. *For fixed $n \leq k + j$, the multigraded Frobenius series of $R_n^{(k,j)}$ may be calculated from the multigraded Frobenius series of $R_n^{(k+j,0)}$ or $R_n^{(0,k+j)}$. In particular, the coefficients $c_{\lambda\mu}$ in equation (5.4) are determined by*

$$\mathrm{Frob}(R_n^{(k+j,0)}; q_1, \ldots, q_{k+j}) = \sum_{\lambda \in P(k+j,0,n)} \sum_{\mu \vdash n} c_{\lambda\mu} s_\lambda(q_1, \ldots, q_{k+j}) s_\mu(\mathbf{z}), \tag{5.35}$$

or by

$$\mathrm{Frob}(R_n^{(0,k+j)}; u_1, \ldots, u_{k+j}) = \sum_{\lambda \in P(0,k+j,n)} \sum_{\mu \vdash n} c_{\lambda\mu} s_{\lambda'}(u_1, \ldots, u_{k+j}) s_\mu(\mathbf{z}). \tag{5.36}$$

Proof. By Corollary 5.0.3, it only remains to check that equation (5.35) or equation (5.36) can be used to determine the $c_{\lambda\mu}$ if $n \leq k + j$. By [9], the multigraded Frobenius series for $R_n^{(k+j,0)}$ is coefficient stable once $k + j \geq n$. Therefore all of the coefficients are determined by equation (5.35). Using the other equation is analogous. □

If we take the limit as k or j goes to infinity, we conclude the following result, which was a motivation for [10]. Let $R_n^{(\infty,0)}$ be the direct limit of the $R_n^{(k,0)}$, and let $R_n^{(0,\infty)}$ be the direct limit of the $R_n^{(0,j)}$. The result says that the mixed bosonic-fermionic case may be determined from the exclusively bosonic or the exclusively fermionic case.

Corollary 5.2.2. *For all n, the multigraded Frobenius series of $R_n^{(\infty,\infty)}$ may be calculated from the multigraded Frobenius series of $R_n^{(\infty,0)}$ or $R_n^{(0,\infty)}$. In particular, the coefficients $c_{\lambda\mu}$ in equation (5.4) are determined by*

$$\operatorname{Frob}(R_n^{(\infty,0)}; q_1, q_2, \ldots) = \sum_{\lambda,\ \ell(\lambda)\leq n} \sum_{\mu\vdash n} c_{\lambda\mu} s_\lambda(q_1, q_2, \ldots) s_\mu(\mathbf{z}), \tag{5.37}$$

or by

$$\operatorname{Frob}(R_n^{(0,\infty)}; u_1, u_2, \ldots) = \sum_{\lambda,\ \ell(\lambda)\leq n} \sum_{\mu\vdash n} c_{\lambda\mu} s_{\lambda'}(u_1, u_2, \ldots) s_\mu(\mathbf{z}). \tag{5.38}$$

5.3 Universal series for the symmetric group

Currently, combinatorial or algebraic formulas have been proven for the Frobenius series of $R_n^{(k,j)}$ only in the cases of $R_n^{(1,0)}$ [33], $R_n^{(2,0)}$ [59], $R_n^{(0,1)}$,[3] $R_n^{(0,2)}$ [69], and $R_n^{(1,1)}$ [84]. All cases where $k + j \geq 3$ are outstanding.

A natural question is if the graded Frobenius series for $R_n^{(1,0)}$ or $R_n^{(0,1)}$ determines the other; similarly if the bigraded Frobenius series for $R_n^{(2,0)}$, $R_n^{(1,1)}$, or $R_n^{(0,2)}$ determines any of the others. We will answer these in the negative in Proposition 5.3.2.

More generally, fix $k, j \geq 0$ and consider the following. Suppose we know the following multigraded Frobenius series expansion for fixed n:

$$\operatorname{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda\in P(k,j,n)} \sum_{\mu\vdash n} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) s_\mu(\mathbf{z}). \tag{5.39}$$

For any $0 \leq k' \leq k$ and $0 \leq j' \leq j$, because of restriction (equations (5.17) and (5.18)), this determines the $c_{\lambda\mu}$ necessary to expand

$$\operatorname{Frob}(R_n^{(k',j')}; q_1, \ldots, q_{k'}; u_1, \ldots, u_{j'}) = \sum_{\lambda\in P(k',j',n)} \sum_{\mu\vdash n} c_{\lambda\mu} s_\lambda(q_1, \ldots, q_{k'}/u_1, \ldots, u_{j'}) s_\mu(\mathbf{z}). \tag{5.40}$$

Now we can prove Proposition 5.0.5.

Proof of Proposition 5.0.5. The above discussion justifies calling

$$\operatorname{Frob}(R_n^{(\infty,\infty)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda,\ \ell(\lambda)\leq n} \sum_{\mu\vdash n} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) s_\mu(\mathbf{z}) \tag{5.41}$$

[3] Since $R_n^{(0,1)} = \mathbb{C}[\theta_1, \ldots, \theta_n]/\langle\theta_1 + \cdots + \theta_n\rangle \cong \mathbb{C}[\theta_1, \ldots, \theta_{n-1}]$, this just reduces to an exterior algebra.

the universal series, since this determines all $c_{\lambda\mu}$ which could appear in such an expansion for any finite or infinite k, j. □

For fixed k, j, to show that it is not possible to determine multigraded Frobenius series expansions for k', j' where at least one of $k' > k$ or $j' > j$ would follow from identifying some specific $\hat{\lambda}, \hat{\mu}$ such that:

- $c_{\hat{\lambda}\hat{\mu}} > 0$,
- $\hat{\lambda}_{k'+1} \leq j'$, and
- $\hat{\lambda}_{k+1} > j$.

We make the following natural conjecture.

Conjecture 5.3.1. *There are nonzero $c_{\lambda\mu}$ in the expansion of the multigraded Frobenius series of $R_n^{(k',j')}$ (equation (5.40)) which are not determined by the expansion of the multigraded Frobenius series of $R_n^{(k,j)}$ (equation (5.39)) when (k, j) and (k', j') are incomparable in the componentwise order.*

Then we verify some small cases of this conjecture with the following result.

Proposition 5.3.2. *There are nonzero $c_{\lambda\mu}$ in the expansion of the multigraded Frobenius series of $R_n^{(k',j')}$ (equation (5.40)) which are not determined by the expansion of the multigraded Frobenius series of $R_n^{(k,j)}$ (equation (5.39)) when (k, j) and (k', j') are either permutation of $(1, 0)$ and $(0, 1)$, or when (k, j) and (k', j') are any two distinct choices of $(2, 0)$, $(1, 1)$, and $(0, 2)$.*

Before proving Proposition 5.3.2, we will find formulas for certain $c_{\lambda\mu}$ to use in the proof. The following result classifies all $c_{\lambda\mu}$ which we can determine from $\mathrm{Frob}(R_n^{(0,2)}; u, v)$.

Proposition 5.3.3. *For λ a partition where each part is at most 2, and $\mu \vdash n$, the coefficients $c_{\lambda\mu}$ defined by*

$$\mathrm{Frob}(R_n^{(\infty,\infty)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(\infty,\infty,n)} \sum_{\mu \vdash n} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) s_\mu(\mathbf{z}) \tag{5.42}$$

are 1 if:

(i) $\mu = (n)$ *and* $\lambda = \varnothing$;

(ii) $\mu = (1^n)$ *and* $\lambda = (1^{n-1})$;

(iii) $\mu = (n-k, 1^k)$ *and* $\lambda = (1^k)$ *or* $(2, 1^{k-1})$ *for* $k \in \{1 \ldots, n-2\}$;

(iv) $\mu = (\mu_1, \mu_1, 2^\ell, 1^{n-2\ell-2\mu_1})$ *where* $\mu_1 \geq 2$ *and* $\lambda = (2^{\ell+\mu_1}, 1^{n-2\ell-2\mu_1-1})$, $(2^{\ell+\mu_1-1}, 1^{n-2\ell-2\mu_1+1})$, *or* $(2^{\ell+\mu_1-1}, 1^{n-2\ell-2\mu_1})$;

(v) $\mu = (\mu_1, \mu_2, 2^\ell, 1^{n-2\ell-\mu_1-\mu_2})$ *where* $\mu_1 > \mu_2 \geq 2$ *and* $\lambda = (2^{\ell+\mu_2}, 1^{n-2\ell-\mu_1-\mu_2})$, $(2^{\ell+\mu_2-1}, 1^{n-2\ell-\mu_1-\mu_2+1})$, $(2^{\ell+\mu_2}, 1^{n-2\ell-\mu_1-\mu_2-1})$, *or* $(2^{\ell+\mu_2-1}, 1^{n-2\ell-\mu_1-\mu_2})$,

and 0 *otherwise.*

Proof. Consider

$$\mathrm{Frob}(R_n^{(0,2)}; u, v) = \sum_{\lambda \in P(0,2,n)} \sum_{\mu \vdash n} c_{\lambda\mu} s_\lambda(0/u, v) s_\mu(\mathbf{z}). \tag{5.43}$$

Recall that $s_\lambda(\mathbf{q}/\mathbf{u}) = s_{\lambda'}(\mathbf{u}/\mathbf{q})$. From [69, Proposition 6.2], we have that the coefficient of $s_{(n)}(\mathbf{z})$ is $1 = s_\varnothing(u, v) = s_\varnothing(0/u, v)$, the coefficient of $s_{(1^n)}(\mathbf{z})$ is $[n]_{u,v} = s_{(n-1)}(u, v) = s_{(1^{n-1})}(0/u, v)$, and for any $k \in \{1, \ldots, n-2\}$, the coefficient of $s_{(n-k,1^k)}(\mathbf{z})$ is $[k+1]_{u,v} + uv[k]_{u,v} = s_{(k)}(u, v) + s_{(k,1)}(u, v) = s_{(1^k)}(0/u, v) + s_{(2,1^{k-1})}(0/u, v)$. Similarly, from [75, Theorem 5.2] we determine (iv) and (v). ☐

Swanson–Wallach [96, Corollary 5.8] found the bigraded multiplicity of the sign character in $R_n^{(1,1)}$, which can be expressed in the form (see [10, equation (3.14)]):

$$\langle \mathrm{Frob}(R_n^{(1,1)}; q; u), s_{(1^n)}(\mathbf{z}) \rangle = \sum_{d=0}^{n-1} q^{\binom{n-d}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q u^d. \tag{5.44}$$

We wish to write this in terms of super Schur functions $s_\lambda(q/u)$.

Lemma 5.3.4. *If* λ *is* $(a, 1^b)$ *for some* $a \geq 1$ *and* $b \geq 0$, *then* $s_\lambda(q/u) = q^a u^b + q^{a-1} u^{b+1}$. *If* λ *is* $\varnothing$, *then* $s_\lambda(q/u) = 1$. *If* λ *is not contained in a hook shape, then* $s_\lambda(q/u) = 0$.

Proof. Direct calculation. ☐

For a polynomial $p \in \mathbb{C}[q]$, let $\langle q^i \rangle p$ denote the coefficient of q^i in p. Define

$$g_{i,d,n} := \langle q^i \rangle \begin{bmatrix} n-2 \\ d \end{bmatrix}_q = |\{\lambda \vdash i : \lambda \subseteq (n-2-d)^d\}|, \tag{5.45}$$

where $(n-2-d)^d$ is the rectangular shape of width $n-2-d$ and height d.

Proposition 5.3.5. *For all* $n \geq 1$,

$$\langle \mathrm{Frob}(R_n^{(1,1)}; q; u), s_{(1^n)}(\mathbf{z}) \rangle = \sum_{d=0}^{n-1} \sum_{i \geq 0} g_{i,d,n} \, s_{(\binom{n-d}{2}+i, 1^d)}(q/u). \tag{5.46}$$

Proof. By Lemma 5.3.4, we write

$$\begin{aligned}\sum_{d=0}^{n-1}\sum_{i\geq 0} g_{i,d,n}\, s_{(\binom{n-d}{2}+i,1^d)}(q/u) &= \sum_{d=0}^{n-1}\sum_{i\geq 0}\left(q^{\binom{n-d}{2}+i}u^d + q^{\binom{n-d}{2}+i-1}u^{d+1}\right)\langle q^i\rangle \begin{bmatrix} n-2 \\ d \end{bmatrix}_q \\ &= \sum_{d=0}^{n-1} u^d\left(\sum_{i\geq 0} q^{\binom{n-d}{2}+i}\langle q^i\rangle \begin{bmatrix} n-2 \\ d \end{bmatrix}_q + \sum_{i\geq 0} q^{\binom{n-d+1}{2}+i-1}\langle q^i\rangle \begin{bmatrix} n-2 \\ d-1 \end{bmatrix}_q\right) \\ &= \sum_{d=0}^{n-1} u^d q^{\binom{n-d}{2}}\left(\begin{bmatrix} n-2 \\ d \end{bmatrix}_q + q^{n-d-1}\begin{bmatrix} n-2 \\ d-1 \end{bmatrix}_q\right) \\ &= \sum_{d=0}^{n-1} u^d q^{\binom{n-d}{2}}\begin{bmatrix} n-1 \\ d \end{bmatrix}_q,\end{aligned} \tag{5.47}$$

where the last line is Pascal's identity. Then apply equation (5.44). □

Corollary 5.3.6. *For λ a partition contained in a hook shape, the coefficients $c_{\lambda,(1^n)}$ defined by*

$$\mathrm{Frob}(R_n^{(\infty,\infty)};\mathbf{q};\mathbf{u}) = \sum_{\lambda\in P(\infty,\infty,n)}\sum_{\mu\vdash n} c_{\lambda\mu}s_\lambda(\mathbf{q}/\mathbf{u})s_\mu(\mathbf{z}) \tag{5.48}$$

are given by:

$$c_{\lambda,(1^n)} = \begin{cases} g_{i,d,n} & \text{if there exist } i,d\geq 0 \text{ such that } \lambda = (\binom{n-d}{2}+i,1^d), \\ 0 & \text{otherwise.}\end{cases} \tag{5.49}$$

Proof of Proposition 5.3.2.

1. $R_n^{(1,0)} \twoheadrightarrow R_n^{(0,1)}$: By Proposition 5.3.3(*ii*), $c_{\hat\lambda\hat\mu} = c_{(1^{n-1}),(1^n)} = 1$. For all $n\geq 3$, $\hat\lambda = (1^{n-1})$ satisfies $\hat\lambda_1 \leq 1$ and $\hat\lambda_2 > 0$.
2. $R_n^{(0,1)} \twoheadrightarrow R_n^{(1,0)}$: By Corollary 5.3.6, $c_{\hat\lambda\hat\mu} = c_{(\binom{n}{2}),(1^n)} = 1$. For all $n\geq 3$, $\hat\lambda = (\binom{n}{2})$ satisfies $\hat\lambda_2 \leq 0$ and $\hat\lambda_1 > 1$.
3. $R_n^{(2,0)} \twoheadrightarrow R_n^{(0,2)}$: By Proposition 5.3.3(*ii*), $c_{\hat\lambda\hat\mu} = c_{(1^{n-1}),(1^n)} = 1$. For all $n\geq 4$, $\hat\lambda = (1^{n-1})$ satisfies $\hat\lambda_1 \leq 2$ and $\hat\lambda_3 > 0$.
4. $R_n^{(0,2)} \twoheadrightarrow R_n^{(2,0)}$: By Corollary 5.3.6, $c_{\hat\lambda\hat\mu} = c_{(\binom{n}{2}),(1^n)} = 1$. For all $n\geq 3$, $\hat\lambda = (\binom{n}{2})$ satisfies $\hat\lambda_3 \leq 0$ and $\hat\lambda_1 > 2$.
5. $R_n^{(2,0)} \twoheadrightarrow R_n^{(1,1)}$: By Corollary 5.3.6, $c_{\hat\lambda\hat\mu} = c_{(\binom{n-2}{2},1,1),(1^n)} = 1$. For all $n\geq 4$, $\hat\lambda = (\binom{n-2}{2},1,1)$ satisfies $\hat\lambda_2 \leq 1$ and $\hat\lambda_3 > 0$.

6. $R_n^{(0,2)} \not\twoheadrightarrow R_n^{(1,1)}$: By Corollary 5.3.6, $c_{\hat{\lambda}\hat{\mu}} = c_{((\binom{n}{2})),(1^n)} = 1$. For all $n \geq 3$, $\hat{\lambda} = (\binom{n}{2})$ satisfies $\hat{\lambda}_2 \leq 1$ and $\hat{\lambda}_1 > 2$.
7. $R_n^{(1,1)} \not\twoheadrightarrow R_n^{(0,2)}$: By Proposition 5.3.3$(iv)$, $c_{\hat{\lambda}\hat{\mu}} = c_{(2,2,1^{n-5}),(2,2,1^{n-4})} = 1$. For all $n \geq 5$, $\hat{\lambda} = (2,2,1^{n-5})$ satisfies $\hat{\lambda}_1 \leq 3$ and $\hat{\lambda}_2 > 1$.
8. $R_n^{(1,1)} \not\twoheadrightarrow R_n^{(2,0)}$: Lee, Li, and Loehr [72, Appendix] determine the $\mathfrak{sl}(2)$-string structure for the homogeneous components of the q,t-Catalan number $C_n(q,t) = \langle \mathrm{Frob}(R_n^{(2,0)}; q, t), s_{(1^n)}(\mathbf{z})\rangle$ of total degree greater than or equal to $\binom{n}{2} - 9$. In particular, the homogeneous component of total degree $\binom{n}{2} - 2$ is shown to be $s_{(\binom{n}{2}-3,1)}(q,t) + s_{(\binom{n}{2}-4,2)}(q,t)$ for large enough n. This implies that $c_{\hat{\lambda}\hat{\mu}} = c_{(\binom{n}{2}-4,2),(1^n)} = 1$. For large enough n, $\hat{\lambda} = (\binom{n}{2}-4, 2)$ satisfies $\hat{\lambda}_3 \leq 0$ and $\hat{\lambda}_2 > 1$.

□

5.4 Cancellation and the Hilbert series for the symmetric group

In this section, we discuss further applications of diagonal supersymmetry for the symmetric group. In particular, we consider the Hilbert series, and obtain a diagonal supersymmetry result for Hilbert series in Proposition 5.4.1. Super Schur functions satisfy a cancellation property (equation (5.16)) which has no analogue for classical Schur functions. We are able to leverage cancellation in Proposition 5.4.2 and then explain how several conjectures or results on symmetric functions in the literature relate to conjectures or results on the Frobenius and Hilbert series of coinvariant rings.

The multigraded Hilbert series is defined by

$$\mathrm{Hilb}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) := \sum_{r_1,\ldots,r_k,s_1,\ldots,s_j \geq 0} \dim\left((R_n^{(k,j)})_{r_1,\ldots,r_k,s_1,\ldots,s_j}\right) q_1^{r_1} \cdots q_k^{r_k} u_1^{s_1} \cdots u_j^{s_j}. \tag{5.50}$$

Recall that we can recover the Hilbert series from the Frobenius series via:

$$\langle \mathrm{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}), h_{(1^n)}(\mathbf{z})\rangle = \mathrm{Hilb}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}). \tag{5.51}$$

We give a Hilbert series version of Corollary 5.0.3.

Proposition 5.4.1. *Fix a positive integer n. For partitions λ with $\ell(\lambda) \leq n$, there exist nonnegative integer coefficients c_λ such that for any (k,j), the multigraded Hilbert series of $R_n^{(k,j)}$ is*

$$\mathrm{Hilb}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,n)} c_\lambda s_\lambda(\mathbf{q}/\mathbf{u}). \tag{5.52}$$

Proof. By Corollary 5.0.3,

$$\begin{aligned}\operatorname{Hilb}(R_n^{(k,j)};\mathbf{q};\mathbf{u}) &= \langle \operatorname{Frob}(R_n^{(k,j)};\mathbf{q};\mathbf{u}), h_{(1^n)}(\mathbf{z})\rangle \\ &= \left\langle \sum_{\lambda\in P(k,j,n)} \sum_{\mu\vdash n} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) s_\mu(\mathbf{z}), h_{(1^n)}(\mathbf{z}) \right\rangle \\ &= \sum_{\lambda\in P(k,j,n)} s_\lambda(\mathbf{q}/\mathbf{u}) \sum_{\mu\vdash n} c_{\lambda\mu} \langle s_\mu(\mathbf{z}), h_{(1^n)}(\mathbf{z})\rangle\end{aligned} \tag{5.53}$$

and the formula follows upon setting $c_\lambda := \sum_{\mu\vdash n} c_{\lambda\mu}\langle s_\mu(\mathbf{z}), h_{(1^n)}(\mathbf{z})\rangle$, which is independent of μ. Since $\langle s_\mu(\mathbf{z}), h_{(1^n)}(\mathbf{z})\rangle$ and $c_{\lambda\mu}$ are both nonnegative integers and independent of (k, j), the description of the c_λ follows. □

Now we show a general result on Frobenius and Hilbert series involving certain evaluations.

Proposition 5.4.2. *For any $n \geq 1$ and for any $0 \leq m \leq \min(k, j)$,*

(i) $\operatorname{Frob}(R_n^{(k,j)};\mathbf{q};\mathbf{u})|_{q_k=-u_j,\ldots,q_{k-m+1}=-u_{j-m+1}} = \operatorname{Frob}(R_n^{(k-m,j-m)}; q_1,\ldots,q_{k-m},u_1,\ldots,u_{j-m})$,

(ii) $\operatorname{Hilb}(R_n^{(k,j)};\mathbf{q};\mathbf{u})|_{q_k=-u_j,\ldots,q_{k-m+1}=-u_{j-m+1}} = \operatorname{Hilb}(R_n^{(k-m,j-m)}; q_1,\ldots,q_{k-m},u_1,\ldots,u_{j-m})$.

Proof. The first claim follows immediately from Corollary 5.0.3 and m applications of cancellation (equation (5.16)), along with restriction (equations (5.17) and (5.18)). The second claim follows from the first after applying $\langle -, h_{(1^n)}(\mathbf{z})\rangle$. □

We now discuss some connections with other results and conjectures in the literature. Sagan–Swanson conjectured [90] and Rhoades–Wilson proved [88] that

$$\operatorname{Hilb}(R_n^{(1,1)};q;u) = \sum_{d=0}^{n} [d]_q! \operatorname{Stir}_q(n,d) u^{n-d}, \tag{5.54}$$

where the q-number $[d]_q$ is defined by $1 + q + q^2 + \cdots + q^{d-1}$, the q-factorial $[d]_q!$ is defined by $[d]_q[d-1]_q \cdots [2]_q[1]_q$, and the q-Stirling number $\operatorname{Stir}_q(n, d)$ is defined by the recurrence relation $\operatorname{Stir}_q(n, d) = [d]_q \operatorname{Stir}_q(n-1, d) + \operatorname{Stir}_q(n-1, d-1)$ with initial conditions $\operatorname{Stir}_q(0, d) = \delta_{d,0}$.

We give a short proof of the following result, originally proven by Sagan—Swanson with a sign-reversing involution.

Corollary 5.4.3 ([90, Theorem 5.5])**.** *For $n \geq 0$, we have*

$$\sum_{d=0}^{n} [d]_q! \operatorname{Stir}_q(n,d)(-q)^{n-d} = 1. \tag{5.55}$$

Proof. By Proposition 5.4.2, $\operatorname{Hilb}(R_n^{(1,1)};q;u)|_{q=-u} = \operatorname{Hilb}(R_n^{(0,0)}) = 1$. The result follows by equation (5.54). □

We briefly recall the Delta and nabla operators. The modified Macdonald polynomials $\tilde{H}_\lambda$ form a basis for $\Lambda_{\mathbb{Q}(q,t)}$, the ring of symmetric function with coefficients in $\mathbb{Q}(q,t)$ (see [60] for example). Let

$$B_\mu = \sum_{c\in\mu} q^{a'_\mu(c)} t^{l'_\mu(c)} \text{ and } T_\mu = \prod_{c\in\mu} q^{a'_\mu(c)} t^{l'_\mu(c)}, \tag{5.56}$$

where the sum or product is taken over all boxes c in the Ferrers diagram of the partition μ. The co-arm statistic $a'_\mu(c)$ is the number of boxes strictly to the left of c in μ and the co-leg statistic $l'_\mu(c)$ is the number of boxes strictly below c in μ (in the French notation for partitions: parts are left-justified with the largest part on the bottom).

Nabla ∇ [12] is defined to be an eigenoperator on the Macdonald basis:

$$\nabla \tilde{H}_\mu = T_\mu \tilde{H}_\mu \text{ for all } \mu. \tag{5.57}$$

Let f be any symmetric function in $\Lambda_{\mathbb{Q}(q,t)}$. The more general Delta operators Δ_f and Δ'_f [13, 53] are eigenoperators on the Macdonald basis defined by

$$\Delta_f \tilde{H}_\mu = f[B_\mu]\tilde{H}_\mu \text{ and } \Delta'_f \tilde{H}_\mu = f[B_\mu - 1]\tilde{H}_\mu \text{ for all } \mu. \tag{5.58}$$

Here, $f[\cdot]$ denotes plethystic substitution (see for example [60]).

Zabrocki conjectured [101] that

$$\mathrm{Frob}(R_n^{(2,1)}; q,t;u) = \sum_{d=0}^{n-1} u^d \Delta'_{e_{n-d-1}} e_n(\mathbf{z}). \tag{5.59}$$

D'Adderio, Iraci, and Vanden Wyngaerd [36, Theorem 4.11] proved that

$$\sum_{d=0}^{n-1} (-q)^d \Delta'_{e_{n-d-1}} e_n(\mathbf{z}) = \nabla e_n(\mathbf{z})|_{q=0}, \tag{5.60}$$

Haiman [59] showed that $\mathrm{Frob}(R_n^{(2,0)}; q,t) = \nabla e_n(\mathbf{z})$. Combining this with Proposition 5.4.2, we have

$$\mathrm{Frob}(R_n^{(2,1)}; q,t;u)|_{q=-u} = \mathrm{Frob}(R_n^{(1,0)}; t) = \nabla e_n(\mathbf{z})|_{q=0}. \tag{5.61}$$

So if equation (5.59) is true, then equation (5.60) follows from equation (5.61).

Moving to the level of Hilbert series, Zabrocki's conjecture implies that

$$\mathrm{Hilb}(R_n^{(2,1)}; q,t;u) = \sum_{d=0}^{n-1} \langle u^d \Delta'_{e_{n-d-1}} e_n(\mathbf{z}), h_{(1^n)}(\mathbf{z})\rangle. \tag{5.62}$$

From equation (5.60), Corteel, Josuat-Vergès, and Vanden Wyngaerd [34, equation (6)] derived

$$\sum_{d=0}^{n-1} \langle u^d \Delta'_{e_{n-d-1}} e_n(\mathbf{z}), h_{(1^n)}(\mathbf{z})\rangle \Big|_{q=-1=-u} = [n]_t!. \tag{5.63}$$

From Proposition 5.4.2, we have

$$\mathrm{Hilb}(R_n^{(2,1)}; q, t; u)|_{q=-u} = \mathrm{Hilb}(R_n^{(1,0)}; t) = [n]_t!, \tag{5.64}$$

where the second equality is due to Artin [4]. So if equation (5.62) is true, then equation (5.63) follows from equation (5.64).

D'Adderio, Iraci, and Vanden Wyngaerd [37] introduced the Theta operators $\Theta_f : \Lambda_{\mathbb{Q}(q,t)} \to \Lambda_{\mathbb{Q}(q,t)}$, which, like Δ_f, depend on a choice of a symmetric function $f \in \Lambda_{\mathbb{Q}(q,t)}$. Unlike Δ_f, they are are not eigenoperators for the Macdonald polynomials $\tilde{H}_\lambda$. Corteel, Josuat-Vergès, and Vanden Wyngaerd [34, Section 6.4] conjectured that

$$\langle \Theta_{e_d}\Theta_{e_\ell}\nabla e_{n-d-\ell}(\mathbf{z}), h_{(1^n)}(\mathbf{z})\rangle|_{q=-1} \tag{5.65}$$

is t-positive. D'Adderio, Iraci, and Vanden Wyngaerd [37] conjectured that

$$\mathrm{Frob}(R_n^{(2,2)}; q, t; u, v) = \sum_{\substack{d,\ell\geq 0,\\ d+\ell<n}} u^d v^\ell \Theta_{e_d}\Theta_{e_\ell}\nabla e_{n-d-\ell}(\mathbf{z}), \tag{5.66}$$

which implies that

$$\mathrm{Hilb}(R_n^{(2,2)}; q, t; u, v) = \sum_{\substack{d,\ell\geq 0,\\ d+\ell<n}} u^d v^\ell \langle \Theta_{e_d}\Theta_{e_\ell}\nabla e_{n-d-\ell}(\mathbf{z}), h_{(1^n)}(\mathbf{z})\rangle. \tag{5.67}$$

From Proposition 5.4.2, we have

$$\mathrm{Hilb}(R_n^{(2,2)}; q, t; u, v)|_{q=-1=-v} = \mathrm{Hilb}(R_n^{(1,1)}; t; u), \tag{5.68}$$

which is clearly t-positive. Thus if equation (5.67) is true, then

$$\sum_{\substack{d,\ell\geq 0,\\ d+\ell<n}} u^d \langle \Theta_{e_d}\Theta_{e_\ell}\nabla e_{n-d-\ell}(\mathbf{z}), h_{(1^n)}(\mathbf{z})\rangle|_{q=-1} \tag{5.69}$$

is t-positive. By considering the components graded by u-degree, this would imply that equation (5.65) is t-positive.

Chapter 6

Universal series for dihedral group coinvariant rings

In 1994, Alfano determined a monomial basis, bigraded Hilbert series, and bigraded Frobenius series for the ring of diagonal dihedral group coinvariants $R^{(2,0)}_{\mathfrak{I}_2(n)}$. Using diagonal supersymmetry, we determine a universal multigraded character series, universal multigraded Hilbert series, and monomial basis for the generalization to any k sets of bosonic variables and j sets of fermionic variables $R^{(k,j)}_{\mathfrak{I}_2(n)}$.

This chapter appears in [76].

6.1 Introduction

The diagonal coinvariant ring $R^{(2,0)}_n = \mathbb{C}[\boldsymbol{x},\boldsymbol{y}]/\langle \mathbb{C}[\boldsymbol{x},\boldsymbol{y}]^{\mathfrak{S}_n}_+\rangle$ was introduced by Haiman in 1994 [57], and has been of much interest in algebraic combinatorics since. Here $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively denote two sets of variables $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_n\}$. The defining ideal $\langle \mathbb{C}[\boldsymbol{x},\boldsymbol{y}]^{\mathfrak{S}_n}_+\rangle$ is generated by all polynomials in $\mathbb{C}[\boldsymbol{x},\boldsymbol{y}]$ which are invariant under the diagonal action of $\mathfrak{S}_n$, with no constant term. In 1994, Haiman conjectured the dimension, bigraded Hilbert series, and Frobenius series of $R^{(2,0)}_n$, however these resisted proof until 2002, where he proved them using new results on the Hilbert scheme of points in the complex plane [59].

In the interim, Alfano [2] studied the related ring $R^{(2,0)}_{\mathfrak{I}_2(n)} = \mathbb{C}[x_1,x_2,y_1,y_2]/\langle \mathbb{C}[x_1,x_2,y_1,y_2]^{\mathfrak{I}_2(n)}_+\rangle$, where $\mathfrak{I}_2(n)$ denotes the dihedral group of order $2n$. This ring is easier to study than $R^{(2,0)}_n$ since every dihedral group is of rank 2, while the rank of the symmetric group $\mathfrak{S}_n$ grows with n. Alfano found the dimension, bigraded Hilbert series, and bigraded character series of $R^{(2,0)}_{\mathfrak{I}_2(n)}$ using combinatorial constructions, linear algebra, and Lie algebra operators, without any new results in algebraic geometry.

There has been recent interest (see [6,7,10,11,102]) in considering coinvariant rings with k sets of commuting (bosonic) variables $\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(k)}$ and j sets of anticommuting (fermionic) variables $\boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}$. We denote this by

$$R_G^{(k,j)} = \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]/\langle \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]_+^G \rangle, \tag{6.1}$$

where $G \subseteq \mathrm{GL}(n)$ is a finite group acting diagonally on all sets of variables. In the case where $G = \mathfrak{S}_n$ acts by permuting variables, we write $R_n^{(k,j)} := R_{\mathfrak{S}_n}^{(k,j)}$. Commuting variables commute with all variables. Anticommuting variables anticommute with all anticommuting variables, that is, $\theta_i^{(\ell)}\theta_j^{(m)} = -\theta_j^{(m)}\theta_i^{(\ell)}$, which implies that $(\theta_i^{(\ell)})^2 = 0$. Note that $\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]$ comes from taking coordinates on $(\mathrm{Sym}\,\mathbb{C}^n)^{\otimes k} \otimes (\wedge \mathbb{C}^n)^{\otimes j}$.

In [74], it was shown that $R_G^{(k,j)}$ is a $\mathcal{U}(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[G]$-module, where $\mathcal{U}(\mathfrak{gl}(k|j))$ is the universal enveloping algebra of the Lie superalgebra $\mathfrak{gl}(k|j)$. As a consequence, for fixed n and $G \subseteq \mathrm{GL}(n)$, the multigraded character series of $R_G^{(k,j)}$ can be written in terms of universal coefficients $c_{\lambda,\mu}$ and super Schur functions as

$$\mathrm{Char}(R_G^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,n)} \sum_{\chi^\mu \in \mathrm{IrrChar}(G)} c_{\lambda,\mu} s_\lambda(\mathbf{q}/\mathbf{u}) \chi^\mu, \tag{6.2}$$

where $P(k,j,n)$ denotes the set of partitions λ with length $\ell(\lambda) \le n$ which satisfy $\lambda_{k+1} \le j$, and $\mathrm{IrrChar}(G)$ denotes the set of irreducible characters χ^μ of G. This is a general, but abstract result, since there is no known explicit formula for all of these coefficients. In the important case of $G = \mathfrak{S}_n$ acting by permuting variables, the universal series is currently known explicitly only for $n \le 6$ [11, Section 2.3].

The goal of this paper is to give explicit universal series coefficients and an explicit monomial basis when $G = \mathfrak{I}_2(n)$. Our main object of study is

$$\begin{aligned} R_{\mathfrak{I}_2(n)}^{(k,j)} = \mathbb{C}[&x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)}, \ldots, x_1^{(k)}, x_2^{(k)}, \theta_1^{(1)}, \theta_2^{(1)}, \theta_1^{(2)}, \theta_2^{(2)}, \ldots, \theta_1^{(j)}, \theta_2^{(j)}] \\ &/\langle \mathbb{C}[x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)}, \ldots, x_1^{(k)}, x_2^{(k)}, \theta_1^{(1)}, \theta_2^{(1)}, \theta_1^{(2)}, \theta_2^{(2)}, \ldots, \theta_1^{(j)}, \theta_2^{(j)}]_+^{\mathfrak{I}_2(n)} \rangle, \end{aligned} \tag{6.3}$$

where, as before, the defining ideal is generated by all polynomials which are invariant under the diagonal action of $\mathfrak{I}_2(n)$, with no constant term.

In Section 6.2, we review background. In the next three sections, we study $R_{\mathfrak{I}_2(n)}^{(k,j)}$ for arbitrary non-negative integers (k,j), and prove our main results. In Section 6.3, using diagonal supersymmetry and a theorem of Alfano, we derive an explicit multigraded character series for $R_{\mathfrak{I}_2(n)}^{(k,j)}$ (see Section 6.2 for the definitions of the dihedral group characters χ_i and χ^i).

Theorem 6.1.1. *Let $\mathbf{q} = (q_1, \ldots, q_k)$ and $\mathbf{u} = (u_1, \ldots, u_j)$. The $\mathbf{q}, \mathbf{u}$-graded character series* *for* $R^{(k,j)}_{\mathfrak{I}_2(n)}$ *is given by*

$$\mathrm{Char}(R^{(k,j)}_{\mathfrak{I}_2(n)};\mathbf{q};\mathbf{u}) = \chi_1 + s_{(1,1)}(\mathbf{q}/\mathbf{u})\chi_2 + s_{(n)}(\mathbf{q}/\mathbf{u})\chi_2 + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}(s_{(i)}(\mathbf{q}/\mathbf{u}) + s_{(n-i)}(\mathbf{q}/\mathbf{u}))\chi^i$$
$$+\begin{cases} s_{(\frac{n}{2})}(\mathbf{q}/\mathbf{u})(\chi_3+\chi_4) & \textit{if } n \textit{ even,}\\ 0 & \textit{if } n \textit{ odd.}\end{cases} \tag{6.4}$$

In Section 6.4, as a consequence, we obtain an explicit multigraded Hilbert series for $R^{(k,j)}_{\mathfrak{I}_2(n)}$ (Theorem 6.4.1), and determine the dimension of $R^{(k,j)}_{\mathfrak{I}_2(n)}$ (Corollary 6.4.2). In Section 6.5, we derive a monomial basis for $R^{(k,j)}_{\mathfrak{I}_2(n)}$ (Theorem 6.5.2) using straightening relations.

Section 6.6 briefly remarks on the sign character and the connection with Catalan numbers of type I. In Section 6.7, we discuss some numerological implications of these results for specializations of (k,j) to certain small values. Appendix 6.8 contains analogous results for the much simpler case of cyclic groups $\mathbb{Z}_n$.

6.2 Preliminaries

The dihedral group

The setting of the paper is $G = \mathfrak{I}_2(n)$, the dihedral group of order $2n$, and we require that $n \geq 2$.[1] $\mathfrak{I}_2(n)$ is the symmetry group of a regular n-gon in the real plane $\mathbb{R}^2$, and is generated by the Euclidean transformations ρ, rotation by $2\pi/n$ radians counterclockwise, and ϕ, reflection about the x-axis. Matrices for these are given by:

$$\rho = \begin{bmatrix}\cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n}\\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n}\end{bmatrix} \text{ and } \phi = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{6.5}$$

Since these matrices are unitary matrices, we may take $\{x_1^{(1)}, x_2^{(1)}\}$ to an orthonormal basis for $\mathbb{R}^2$. The action of $\mathfrak{I}_2(n)$ can be extended homomorphically to all $\mathbb{R}[\boldsymbol{x}^{(1)}]$. Since we will use the character theory of $\mathfrak{I}_2(n)$, and its irreducible real characters are not necessarily all irreducible over $\mathbb{C}$, we will work with complex characters. So we complexify $\mathbb{R}^2$ to $\mathbb{C}^2$, and work over $\mathbb{C}$ throughout. We remark that we could work more minimally over $\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive nth root of unity.

Let $\Re(P)$ denote taking the real part of a complex polynomial P. As our fundamental $\mathfrak{I}_2(n)$-invariants, we use $(x_1^{(1)})^2 + (x_2^{(1)})^2$ and from [2, equation (4.6)],

$$\Re(x_1^{(1)} + ix_2^{(1)})^n = \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2m}(-1)^m(x_1^{(1)})^{n-2m}(x_2^{(1)})^{2m}, \tag{6.6}$$

[1]Sometimes the case where $n = 2$ is treated separately from $n > 2$ (for example, $\mathfrak{I}_2(2) \cong \mathfrak{S}_2 \times \mathfrak{S}_2$ is reducible).

where here $i = \sqrt{-1}$.

Dihedral group coinvariants

Fix $V = \mathbb{C}^2$. Note that $\mathfrak{I}_2(n) \subseteq \mathrm{GL}(V) = \mathrm{GL}(2)$. Let $\mathrm{Sym}\, V$ denote the symmetric algebra on V over $\mathbb{C}$ and let $\wedge V$ denote the exterior algebra on V over $\mathbb{C}$.

Here and throughout, let

$$\mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}] \tag{6.7}$$

denote the polynomial ring $\mathbb{C}[x_1^{(1)}, x_2^{(1)}, \dots, x_1^{(k)}, x_2^{(k)}, \theta_1^{(1)}, \theta_2^{(1)}, \dots, \theta_1^{(j)}, \theta_2^{(j)}]$. The diagonal action (in k sets of bosonic variables and j sets of fermionic variables) of $\mathfrak{I}_2(n)$ on $\mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]$ is defined for any $g \in \mathfrak{I}_2(n)$ and polynomial $P(x_1^{(1)}, x_2^{(1)}, \dots, x_1^{(k)}, x_2^{(k)}, \theta_1^{(1)}, \theta_2^{(1)}, \dots, \theta_1^{(j)}, \theta_2^{(j)}) \in \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]$ by

$$\begin{aligned} &g \cdot P(x_1^{(1)}, x_2^{(1)}, \dots, x_1^{(k)}, x_2^{(k)}, \theta_1^{(1)}, \theta_2^{(1)}, \dots, \theta_1^{(j)}, \theta_2^{(j)}) \\ &= P(g \cdot x_1^{(1)}, g \cdot x_2^{(1)}, \dots, g \cdot x_1^{(k)}, g \cdot x_2^{(k)}, g \cdot \theta_1^{(1)}, g \cdot \theta_2^{(1)}, \dots, g \cdot \theta_1^{(j)}, g \cdot \theta_2^{(j)}). \end{aligned} \tag{6.8}$$

Note that matrices for this action are generated by block diagonal matrices with $k + j$ blocks of the matrices for ρ and ϕ given in equation (6.5).

Dihedral group characters

We recall the dihedral group irreducible characters (see for example [67, Chapter 18.3]). We use the group presentation $\mathfrak{I}_2(n) = \langle \rho, \phi \mid \rho^n = \phi^2 = 1, \phi^{-1}\rho\phi = \rho^{-1} \rangle$. The 1-dimensional irreducible characters are:

- χ_1 defined by $\chi_1(\rho) = 1$ and $\chi_1(\phi) = 1$,
- χ_2 defined by $\chi_2(\rho) = -1$ and $\chi_2(\phi) = 1$,

and if and only if n is even,

- χ_3 defined by $\chi_3(\rho) = 1$ and $\chi_3(\phi) = -1$,
- χ_4 defined by $\chi_4(\rho) = -1$ and $\chi_4(\phi) = -1$.

The 2-dimensional irreducible characters are, for $1 \leq h \leq \lfloor \frac{n-1}{2} \rfloor$:

- χ^h defined by $\chi^h(\rho) = 2\cos(\frac{2h\pi}{n})$ and $\chi^h(\phi) = 0$.

These constitute the complete list of irreducible characters.

Remark 6.2.1. The definition of the 2-dimensional characters may be extended to for $1 \leq h \leq n - 1$:

- χ^h defined by $\chi^h(\rho) = 2\cos(\frac{2h\pi}{n})$ and $\chi^h(\phi) = 0$.

Then χ^i and χ^{n-i} are isomorphic, and when n is even, the character $\chi^{\frac{n}{2}}$ is reducible with $\chi^{\frac{n}{2}} = \chi_3 + \chi_4$.

Dihedral group polarization operators

Define polarization operators for $\ell \in \{1, \ldots, k-1\}$ by

$$E_\ell := \sum_{i=1}^{2} x_i^{(\ell+1)} \partial_{x_i^{(\ell)}}, \tag{6.9}$$

for k,

$$E_k := \sum_{i=1}^{2} \theta_i^{(1)} \partial_{x_i^{(k)}}, \tag{6.10}$$

and for $\ell \in \{k+1, \ldots, k+j-1\}$,

$$E_\ell := \sum_{i=1}^{2} \theta_i^{(\ell-k+1)} \partial_{\theta_i^{(\ell-k)}}, \tag{6.11}$$

which are all diagonal invariant under the action of $\mathfrak{I}_2(n)$; this is shown for E_1 in [2, Section IV.C], and the same argument holds for the others.

The operators in equations (6.9-6.11) generate the Lie superalgebra $\mathfrak{sl}(k|j)$ [32, Section 5.2.1].

Super Schur functions

A super Schur function $s_\lambda(\mathbf{q}/\mathbf{u})$ is defined by

$$s_\lambda(\mathbf{q}/\mathbf{u}) = \sum_{\nu \subseteq \lambda} s_\nu(\mathbf{q}) s_{\lambda'/\nu'}(\mathbf{u}), \tag{6.12}$$

where λ' denotes the transpose of the partition λ (see for example [32, Sections A.2.2 and 3.2]).

A special case that we use often is

$$s_{(n)}(\mathbf{q}/\mathbf{u}) = \sum_{i=0}^{n} s_{(n-i)}(\mathbf{q}) s_{(1^i)}(\mathbf{u}), \tag{6.13}$$

where (0) means the empty partition $\varnothing$.

Diagonal supersymmetry

We recall the "diagonal supersymmetry" theorem, which was previewed in equation (6.2).

Theorem 6.2.2 ([74, Theorem 1.2])**.** *Fix a positive integer n and a finite group $G \subset \mathrm{GL}(n)$. For partitions λ with $\ell(\lambda) \leq n$, and indices μ of irreducible G-characters, there exist nonnegative integer coefficients $c_{\lambda,\mu}$ such that for any (k, j), the multigraded character series of $R_G^{(k,j)}$ is*

$$\mathrm{Char}(R_G^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,n)} \sum_{\chi^\mu \in \mathrm{IrrChar}(G)} c_{\lambda,\mu} s_\lambda(\mathbf{q}/\mathbf{u}) \chi^\mu. \tag{6.14}$$

By specializing to dihedral groups, the multigraded character series of $R_{\mathfrak{I}_2(n)}^{(k,j)}$ is a sum of super Schur functions times $\mathfrak{I}_2(n)$-characters.

Theorem 6.2.3. *Fix an integer $n \geq 2$. For partitions λ with $\ell(\lambda) \leq 2$, and irreducible $\mathfrak{I}_2(n)$-characters χ^μ, there exist nonnegative integer coefficients $c_{\lambda,\mu}$ such that for any (k, j), the multigraded character series of $R_{\mathfrak{I}_2(n)}^{(k,j)}$ is*

$$\mathrm{Char}(R_{\mathfrak{I}_2(n)}^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,2)} \sum_{\chi^\mu \in \mathrm{IrrChar}(\mathfrak{I}_2(n))} c_{\lambda,\mu} s_\lambda(\mathbf{q}/\mathbf{u}) \chi^\mu. \tag{6.15}$$

In particular, the condition $\lambda \in P(k, j, 2)$ means that $\ell(\lambda) \leq 2$ and $\lambda_{k+1} \leq j$.

6.3 The multigraded character series

In this section, we derive a multigraded character series for $R_{\mathfrak{I}_2(n)}^{(k,j)}$. Alfano [2, equation (4.18)] recorded the singly graded character of $R_{\mathfrak{I}_2(n)}^{(1,0)}$ as the following graded version of the regular representation:

$$\mathrm{Char}(R_{\mathfrak{I}_2(n)}^{(1,0)}; q) = \chi_1 + q^n \chi_2 + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (q^i + q^{n-i}) \chi^i + \begin{cases} q^{\frac{n}{2}}(\chi_3 + \chi_4) & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases} \tag{6.16}$$

Alfano [2, Chapter IV.C] applied the polarization operators $E_1^0, E_1^1, \ldots, E_1^i$ to any bigraded component $(R_{\mathfrak{I}_2(n)}^{(2,0)})_{i,0}$ to generate an $\mathfrak{sl}(2)$-string of modules $(R_{\mathfrak{I}_2(n)}^{(2,0)})_{i,0}, (R_{\mathfrak{I}_2(n)}^{(2,0)})_{i-1,1}, \ldots, (R_{\mathfrak{I}_2(n)}^{(2,0)})_{0,i}$. By looking at dimensions, the only bidegree not fully accounted for by this process is $(1, 1)$. The extra basis element in bidegree $(1, 1)$ spans a 1-dimensional module with bigraded character $qt\chi_2$. Thus Alfano concluded the following.

Theorem 6.3.1 ([2, equation (4.19)]). *The bigraded character of $R^{(2,0)}_{\mathfrak{I}_2(n)}$ is given by*

$$\begin{aligned}\mathrm{Char}(R^{(2,0)}_{\mathfrak{I}_2(n)};q,t) = \chi_1 + qt\chi_2 + \sum_{i=0}^{n} q^{n-i}t^i\chi_2 + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{h=0}^{i} q^{i-h}t^h\chi^i + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{h=0}^{n-i} q^{n-i-h}t^h\chi^i \\ + \begin{cases}\sum_{i=0}^{\frac{n}{2}} q^{\frac{n}{2}-i}t^i(\chi_3+\chi_4) & \text{if } n \text{ even},\\ 0 & \text{if } n \text{ odd}.\end{cases}\end{aligned} \tag{6.17}$$

We are ready to prove our first main result.

Theorem 6.1.1. *Let $\mathbf{q} = (q_1, \ldots, q_k)$ and $\mathbf{u} = (u_1, \ldots, u_j)$. The $\mathbf{q}, \mathbf{u}$-graded character series for $R^{(k,j)}_{\mathfrak{I}_2(n)}$ is given by*

$$\begin{aligned}\mathrm{Char}(R^{(k,j)}_{\mathfrak{I}_2(n)};\mathbf{q};\mathbf{u}) = \chi_1 + s_{(1,1)}(\mathbf{q}/\mathbf{u})\chi_2 + s_{(n)}(\mathbf{q}/\mathbf{u})\chi_2 + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}(s_{(i)}(\mathbf{q}/\mathbf{u}) + s_{(n-i)}(\mathbf{q}/\mathbf{u}))\chi^i \\ + \begin{cases} s_{(\frac{n}{2})}(\mathbf{q}/\mathbf{u})(\chi_3+\chi_4) & \text{if } n \text{ even},\\ 0 & \text{if } n \text{ odd}.\end{cases}\end{aligned} \tag{6.18}$$

Remark 6.3.2. Using the notation from Remark 6.2.1, we can write the character series more succinctly as

$$\mathrm{Char}(R^{(k,j)}_{\mathfrak{I}_2(n)};\mathbf{q};\mathbf{u}) = \chi_1 + s_{(1,1)}(\mathbf{q}/\mathbf{u})\chi_2 + s_{(n)}(\mathbf{q}/\mathbf{u})\chi_2 + \sum_{i=1}^{n-1} s_{(i)}(\mathbf{q}/\mathbf{u})\chi^i. \tag{6.19}$$

Proof of Theorem 6.1.1. First observe that equation (6.17) can be rewritten in terms of Schur functions (which are a special case of super Schur functions) as

$$\begin{aligned}\mathrm{Char}(R^{(2,0)}_{\mathfrak{I}_2(n)};q,t) = \chi_1 + s_{(1,1)}(q,t)\chi_2 + s_{(n)}(q,t)\chi_2 + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}(s_{(i)}(q,t) + s_{(n-i)}(q,t))\chi^i \\ + \begin{cases} s_{(\frac{n}{2})}(q,t)(\chi_3+\chi_4) & \text{if } n \text{ even},\\ 0 & \text{if } n \text{ odd}.\end{cases}\end{aligned} \tag{6.20}$$

This determines the coefficients $c_{\lambda,\mu}$ in equation (6.15) for all $\lambda \in P(2,0,2)$, i.e., for all λ which satisfy $\ell(\lambda) \leq 2$, and for all χ^μ. Recall that we require $n \geq 2$. Explicitly, for $\ell(\lambda) \leq 2$ we have:

- if $\lambda = \varnothing$ and $\chi^\mu = \chi_1$, then $c_{\lambda,\mu} = 1$;
- if $\lambda = (1,1)$ and $\chi^\mu = \chi_2$, then $c_{\lambda,\mu} = 1$;

- if $\lambda = (n)$ and $\chi^\mu = \chi_2$, then $c_{\lambda,\mu} = 1$;
- if n is even, if $\lambda = (\frac{n}{2})$ and $\chi^\mu = \chi_3$, then $c_{\lambda,\mu} = 1$;
- if n is even, if $\lambda = (\frac{n}{2})$ and $\chi^\mu = \chi_4$, then $c_{\lambda,\mu} = 1$;
- if $\lambda = (i)$ for $i \in \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ and $\chi^\mu = \chi^i$, then $c_{\lambda,\mu} = 1$;
- if $\lambda = (n-i)$ for $i \in \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ and $\chi^\mu = \chi^i$, then $c_{\lambda,\mu} = 1$;
- otherwise $c_{\lambda,\mu} = 0$.

The question which remains is if we have found all of the coefficients $c_{\lambda,\mu}$ which appear in the universal series in equation (6.15). The condition $\lambda \in P(k, j, 2)$ means λ such that $\ell(\lambda) \leq 2$ and $\lambda_{k+1} \leq j$, so since we already determined all $c_{\lambda,\mu}$ for all λ which satisfy $\ell(\lambda) \leq 2$, there are no additional coefficients left to determine. Hence expanding equation (6.15) with the coefficients we have determined completes the proof. $\square$

6.4 The multigraded Hilbert series

In this section, we derive the Hilbert series of $R^{(k,j)}_{\mathfrak{I}_2(n)}$. The following theorem was first shown by Alfano [2, equation (4.17)] for $k = 2$ and $j = 0$. It was then generalized by Bergeron [9, Theorem 2.3 and equation (30)] to arbitrary k with $j = 0$.

Theorem 6.4.1. *Let $\mathbf{q} = (q_1, \ldots, q_k)$ and $\mathbf{u} = (u_1, \ldots, u_j)$. The $\mathbf{q}, \mathbf{u}$-graded Hilbert series for $R^{(k,j)}_{\mathfrak{I}_2(n)}$ is given by*

$$\mathrm{Hilb}(R^{(k,j)}_{\mathfrak{I}_2(n)}; \mathbf{q}; \mathbf{u}) = 1 + s_{(1,1)}(\mathbf{q}/\mathbf{u}) + s_{(n)}(\mathbf{q}/\mathbf{u}) + 2\sum_{i=1}^{n-1} s_{(i)}(\mathbf{q}/\mathbf{u}). \tag{6.21}$$

Proof. This follows immediately from Theorem 6.1.1 and the fact that the characters $\chi_1, \chi_2, \chi_3, \chi_4$ are 1-dimensional, and the characters χ^i are 2-dimensional. $\square$

By specializing the Hilbert series at $q_1, \ldots, q_k, u_1, \ldots, u_j = 1$, we obtain a formula for the dimension of $R^{(k,j)}_{\mathfrak{I}_2(n)}$.

Corollary 6.4.2. *The dimension of $R^{(k,j)}_{\mathfrak{I}_2(n)}$ is given by*

$$1 + \binom{k}{2} + kj + \binom{j+1}{2} + \sum_{h=0}^{n} \binom{j}{h}\binom{k+n-h-1}{n-h} + 2\sum_{i=1}^{n-1}\sum_{h=0}^{i} \binom{j}{h}\binom{k+i-h-1}{i-h}. \tag{6.22}$$

Proof. Consider the Hilbert series given in Theorem 6.4.1. Recall the Schur function specializations

$$s_{(1^m)}(\mathbf{q})|_{q_1,\ldots,q_k=1} = \binom{k}{m} \text{ and } s_{(m)}(\mathbf{q})|_{q_1,\ldots,q_k=1} = \binom{k+m-1}{m}. \tag{6.23}$$

Then we obtain the super Schur function specializations

$$s_{(m)}(\mathbf{q}/\mathbf{u})|_{q_1,\ldots,q_k,u_1,\ldots,u_j=1} = \sum_{\ell=0}^{m} \binom{k+\ell-1}{\ell}\binom{j}{m-\ell}, \tag{6.24}$$

and

$$s_{(1^m)}(\mathbf{q}/\mathbf{u})|_{q_1,\ldots,q_k,u_1,\ldots,u_j=1} = \sum_{\ell=0}^{m} \binom{k}{\ell}\binom{j+m-\ell-1}{m-\ell}. \tag{6.25}$$

Specializing the Hilbert series at $q_1,\ldots,q_k,u_1,\ldots,u_j = 1$ proves the claim. □

6.5 A monomial basis

In this section, we derive a monomial basis for $R^{(k,j)}_{\mathfrak{I}_2(n)}$. In lieu of determining the $\mathfrak{I}_2(n)$-invariants of the polynomial ring $\mathbb{C}[\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(k)},\boldsymbol{\theta}^{(1)},\ldots,\boldsymbol{\theta}^{(j)}]$, we list some elements of $I^{(k,j)}_{\mathfrak{I}_2(n)}$ which we will use as straightening relations in the quotient ring $R^{(k,j)}_{\mathfrak{I}_2(n)}$.

Proposition 6.5.1. *The following elements are in the ideal $I^{(k,j)}_{\mathfrak{I}_2(n)}$:*

$$\begin{aligned}
&\{x_1^{(h)}x_1^{(i)} + x_2^{(h)}x_2^{(i)} \mid 1 \le h \le i \le k\} \cup \{x_1^{(h)}\theta_1^{(i)} + x_2^{(h)}\theta_2^{(i)} \mid 1 \le h \le k, 1 \le i \le j\}\\
&\cup \{\theta_1^{(h)}\theta_1^{(i)} + \theta_2^{(h)}\theta_2^{(i)} \mid 1 \le h < i \le j\}\\
&\cup \{(x_1^{(1)})^{\alpha_1}\cdots(x_1^{(k)})^{\alpha_k}(\theta_1^{(1)})^{\beta_1}\cdots(\theta_1^{(j)})^{\beta_j} \mid \alpha_1+\cdots+\alpha_k+\beta_1+\cdots+\beta_j = n\},
\end{aligned} \tag{6.26}$$

where the α_i are nonnegative integers and the β_i are 0 or 1.

Proof. The set $\{(x_1^{(1)})^2 + (x_2^{(1)})^2, \Re(x_1^{(1)} + ix_2^{(1)})^n\}$ is a minimal set of generators of $I^{(1,0)}_{\mathfrak{I}_2(n)}$. Using equation (6.6), since $(x_1^{(1)})^2 \equiv -(x_2^{(1)})^2 \pmod{I^{(1,0)}_{\mathfrak{I}_2(n)}}$, the invariant $\Re(x_1^{(1)} + ix_2^{(1)})^n$ can be replaced by $(x_1^{(1)})^n$. Thus $\{(x_1^{(1)})^2 + (x_2^{(1)})^2, (x_1^{(1)})^n\}$ is also a minimal set of generators for $I^{(1,0)}_{\mathfrak{I}_2(n)}$.[2]

Applying all sequences of operators E_ℓ from equations (6.9-6.11) to $(x_1^{(1)})^2 + (x_2^{(1)})^2$ gives the sets

$$\{x_1^{(h)}x_1^{(i)} + x_2^{(h)}x_2^{(i)} \mid 1 \le h \le i \le k\}, \tag{6.27}$$

$$\{x_1^{(h)}\theta_1^{(i)} + x_2^{(h)}\theta_2^{(i)} \mid 1 \le h \le k, 1 \le i \le j\}, \tag{6.28}$$

$$\{\theta_1^{(h)}\theta_1^{(i)} + \theta_2^{(h)}\theta_2^{(i)} \mid 1 \le h < i \le j\}. \tag{6.29}$$

[2]Here and throughout, whenever we are considering $I^{(0,j)}_{\mathfrak{I}_2(n)}$ or $R^{(0,j)}_{\mathfrak{I}_2(n)}$, if we use $I^{(1,0)}_{\mathfrak{I}_2(n)}$ or $R^{(1,0)}_{\mathfrak{I}_2(n)}$ in a proof, we can first construct $I^{(1,j)}_{\mathfrak{I}_2(n)}$ or $R^{(1,j)}_{\mathfrak{I}_2(n)}$ and then restrict to the desired ideal or ring.

Regarding equation (6.29), the operators must be applied in a certain order to avoid squaring any fermionic variables. Specifically, apply $E_{k+i-1}\cdots E_1$ to $(x_1^{(1)})^2+(x_2^{(1)})^2$ to yield $x_1^{(1)}\theta_1^{(i)}+x_2^{(1)}\theta_2^{(i)}$. Then apply $E_{k+h-1}\cdots E_1$ to yield $\theta_1^{(h)}\theta_1^{(i)}+\theta_2^{(h)}\theta_2^{(i)}$.

Similarly, applying all sequences of operators E_ℓ to $(x_1^{(1)})^n$ gives the set

$$\{(x_1^{(1)})^{\alpha_1}\cdots(x_1^{(k)})^{\alpha_k}(\theta_1^{(1)})^{\beta_1}\cdots(\theta_1^{(j)})^{\beta_j}\mid \alpha_1+\cdots+\alpha_k+\beta_1+\cdots+\beta_j=n, \alpha_i\in\mathbb{Z}_{\geq 0}, \beta_i\in\{0,1\}\}. \tag{6.30}$$

Since $(x_1^{(1)})^2+(x_2^{(1)})^2$ and $(x_1^{(1)})^n$ are in $I^{(k,j)}_{\mathfrak{I}_2(n)}$ and the operators E_ℓ are $\mathfrak{I}_2(n)$-invariant, the polynomials in equations (6.27-6.30) are in the ideal $I^{(k,j)}_{\mathfrak{I}_2(n)}$. □

Now we find a monomial basis for $R^{(k,j)}_{\mathfrak{I}_2(n)}$.

Theorem 6.5.2. *A monomial basis $B^{(k,j)}_{\mathfrak{I}_2(n)}$ for the coinvariant ring $R^{(k,j)}_{\mathfrak{I}_2(n)}$ is given by:*

$$\begin{aligned}
&\{(x_1^{(1)})^{\alpha_1}\cdots(x_1^{(k)})^{\alpha_k}(\theta_1^{(1)})^{\beta_1}\cdots(\theta_1^{(j)})^{\beta_j}\mid 0\leq \alpha_1+\cdots+\alpha_k+\beta_1+\cdots+\beta_j\leq n-1\}\\
&\cup\bigcup_{i=1}^{k}\{(x_1^{(i)})^{\alpha_i}\cdots(x_1^{(k)})^{\alpha_k}(\theta_1^{(1)})^{\beta_1}\cdots(\theta_1^{(j)})^{\beta_j}x_2^{(i)}\mid 0\leq \alpha_i+\cdots+\alpha_k+\beta_1+\cdots+\beta_j\leq n-1\}\\
&\cup\bigcup_{i=1}^{j}\{(\theta_1^{(i+1)})^{\beta_{i+1}}\cdots(\theta_1^{(j)})^{\beta_j}\theta_2^{(i)}\mid 0\leq\beta_{i+1}+\cdots+\beta_j\leq n-1\}\\
&\cup\{x_1^{(h)}x_2^{(i)}\mid 1\leq h<i\leq k\}\cup\{x_1^{(h)}\theta_2^{(i)}\mid 1\leq h\leq k, 1\leq i\leq j\}\cup\{\theta_1^{(h)}\theta_2^{(i)}\mid 1\leq h\leq i\leq j\},
\end{aligned} \tag{6.31}$$

where the α_i are nonnegative integers and the β_i are 0 *or* 1.

Proof. The main idea of the proof is to describe straightening relations, which will allow us to construct the desired spanning set. Then we show that the spanning set is a basis by showing that it has the correct dimension.

The following identities are elements in the ideal $I^{(k,j)}_{\mathfrak{I}_2(n)}$, by Proposition 6.5.1:

$$x_1^{(\ell)}(x_2^{(h)}x_1^{(i)}-x_1^{(h)}x_2^{(i)})=x_2^{(h)}(x_1^{(\ell)}x_1^{(i)}+x_2^{(\ell)}x_2^{(i)})-x_2^{(i)}(x_1^{(\ell)}x_1^{(h)}+x_2^{(\ell)}x_2^{(h)}), \tag{6.32}$$

$$\theta_1^{(\ell)}(x_2^{(h)}x_1^{(i)}-x_1^{(h)}x_2^{(i)})=x_2^{(h)}(\theta_1^{(\ell)}x_1^{(i)}+\theta_2^{(\ell)}x_2^{(i)})-x_2^{(i)}(\theta_1^{(\ell)}x_1^{(h)}+\theta_2^{(\ell)}x_2^{(h)}), \tag{6.33}$$

$$x_1^{(\ell)}(x_2^{(h)}\theta_1^{(i)}-x_1^{(h)}\theta_2^{(i)})=x_2^{(h)}(x_1^{(\ell)}\theta_1^{(i)}+x_2^{(\ell)}\theta_2^{(i)})-\theta_2^{(i)}(x_1^{(\ell)}x_1^{(h)}+x_2^{(\ell)}x_2^{(h)}), \tag{6.34}$$

$$\theta_1^{(\ell)}(x_2^{(h)}\theta_1^{(i)}-x_1^{(h)}\theta_2^{(i)})=x_2^{(h)}(\theta_1^{(\ell)}\theta_1^{(i)}+\theta_2^{(\ell)}\theta_2^{(i)})-(\theta_1^{(\ell)}x_1^{(h)}+\theta_2^{(\ell)}x_2^{(h)})\theta_2^{(i)}, \tag{6.35}$$

$$x_1^{(\ell)}(\theta_2^{(h)}\theta_1^{(i)}-\theta_1^{(h)}\theta_2^{(i)})=\theta_2^{(h)}(x_1^{(\ell)}\theta_1^{(i)}+x_2^{(\ell)}\theta_2^{(i)})-(x_1^{(\ell)}\theta_1^{(h)}+x_2^{(\ell)}\theta_2^{(h)})\theta_2^{(i)}, \tag{6.36}$$

$$\theta_1^{(\ell)}(\theta_2^{(h)}\theta_1^{(i)}-\theta_1^{(h)}\theta_2^{(i)})=-\theta_2^{(h)}(\theta_1^{(\ell)}\theta_1^{(i)}+\theta_2^{(\ell)}\theta_2^{(i)})-(\theta_1^{(\ell)}\theta_1^{(h)}+\theta_2^{(\ell)}\theta_2^{(h)})\theta_2^{(i)}. \tag{6.37}$$

We establish the following straightening rules:

(A) $\{x_1^{(h)}x_1^{(i)} + x_2^{(h)}x_2^{(i)} \mid 1 \le h \le i \le k\} \cup \{x_1^{(h)}\theta_1^{(i)} + x_2^{(h)}\theta_2^{(i)} \mid 1 \le h \le k, 1 \le i \le j\} \cup \{\theta_1^{(h)}\theta_1^{(i)} + \theta_2^{(h)}\theta_2^{(i)} \mid 1 \le h < i \le j\} \subseteq I_{\mathfrak{I}_2(n)}^{(k,j)}$ allows us to re-express any monomial $M \in \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]$ modulo $I_{\mathfrak{I}_2(n)}^{(k,j)}$ as a monomial of total degree no greater than 1 in $\{x_2^{(1)}, \dots, x_2^{(k)}, \theta_2^{(1)}, \dots, \theta_2^{(j)}\}$.

(B) $\{(x_1^{(1)})^{\alpha_1} \cdots (x_1^{(k)})^{\alpha_k} (\theta_1^{(1)})^{\beta_1} \cdots (\theta_1^{(j)})^{\beta_j} \mid \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_j = n, \alpha_i \in \mathbb{Z}_{\ge 0}, \beta_i \in \{0,1\}\} \subseteq I_{\mathfrak{I}_2(n)}^{(k,j)}$ allows us to re-express any monomial $M \in \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]$ modulo $I_{\mathfrak{I}_2(n)}^{(k,j)}$ as a monomial of total degree no greater than $n-1$ in $\{x_1^{(1)}, \dots, x_1^{(k)}, \theta_1^{(1)}, \dots, \theta_1^{(j)}\}$.

(C) For any monomial $M \in \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]$ which has as a factor $x_1^{(\ell)}(x_1^{(h)}x_2^{(i)})$ for some ℓ and $h < i$, then modulo $I_{\mathfrak{I}_2(n)}^{(k,j)}$ we can re-express that factor within M as $x_1^{(\ell)}(x_2^{(h)}x_1^{(i)})$, by equation (6.32). Similarly, for any monomial M which has as a factor $\theta_1^{(\ell)}(x_1^{(h)}x_2^{(i)})$ for some ℓ and $h < i$, then modulo $I_{\mathfrak{I}_2(n)}^{(k,j)}$ we can re-express that factor within M as $\theta_1^{(\ell)}(x_2^{(h)}x_1^{(i)})$, by equation (6.33).

(D) For any monomial $M \in \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]$ which has as a factor $x_1^{(\ell)}(x_1^{(h)}\theta_2^{(i)})$ for some ℓ, then modulo $I_{\mathfrak{I}_2(n)}^{(k,j)}$ we can re-express that factor within M as $x_1^{(\ell)}(x_2^{(h)}\theta_1^{(i)})$, by equation (6.34). Similarly, for any monomial M which has as a factor $\theta_1^{(\ell)}(x_1^{(h)}\theta_2^{(i)})$ for some ℓ, then modulo $I_{\mathfrak{I}_2(n)}^{(k,j)}$ we can re-express that factor within M as $\theta_1^{(\ell)}(x_2^{(h)}\theta_1^{(i)})$, by equation (6.35).

(E) For any monomial $M \in \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]$ which has as a factor $x_1^{(\ell)}(\theta_1^{(h)}\theta_2^{(i)})$ for some ℓ and $h < i$, then modulo $I_{\mathfrak{I}_2(n)}^{(k,j)}$ we can re-express that factor within M as $x_1^{(\ell)}(\theta_2^{(h)}\theta_1^{(i)})$, by equation (6.36). Similarly, for any monomial M which has as a factor $\theta_1^{(\ell)}(\theta_1^{(h)}\theta_2^{(i)})$ for some ℓ and $h < i$, then modulo $I_{\mathfrak{I}_2(n)}^{(k,j)}$ we can re-express that factor within M as $\theta_1^{(\ell)}(\theta_2^{(h)}\theta_1^{(i)})$, by equation (6.37).

(F) For any monomial $M \in \mathbb{C}[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]$ which has as a factor $\theta_1^{(\ell)}(\theta_1^{(i)}\theta_2^{(i)})$ for some ℓ, then modulo $I_{\mathfrak{I}_2(n)}^{(k,j)}$ we can re-express that factor within M as $\theta_1^{(\ell)}(\theta_2^{(i)}\theta_1^{(i)})$, by equation (6.37) specialized at $h = i$. On the other hand, $\theta_1^{(\ell)}(\theta_1^{(i)}\theta_2^{(i)}) = -\theta_1^{(\ell)}(\theta_2^{(i)}\theta_1^{(i)})$ by directly swapping the order of the anticommuting variables. Together, these imply that $\theta_1^{(\ell)}(\theta_2^{(i)}\theta_1^{(i)})$ must be 0 modulo $I_{\mathfrak{I}_2(n)}^{(k,j)}$, so $M = 0$ modulo $I_{\mathfrak{I}_2(n)}^{(k,j)}$.

First, note the spanning set should contain a monomial $(x_1^{(1)})^{\alpha_1} \cdots (x_1^{(k)})^{\alpha_k} (\theta_1^{(1)})^{\beta_1} \cdots (\theta_1^{(j)})^{\beta_j}$ for each choice of nonnegative integers $\alpha_1, \dots, \alpha_k$ and $\beta_1, \dots, \beta_j \in \{0,1\}$ such that $0 \le \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_j \le n-1$ because of (B). While there might be no variables from $\{x_2^{(1)}, \dots, x_2^{(k)}, \theta_2^{(1)}, \dots, \theta_2^{(j)}\}$ present, there could also be one, say $x_2^{(1)}$, of degree 1, due to (A).

Thus in the spanning set we include

$$\{(x_1^{(1)})^{\alpha_1}\cdots(x_1^{(k)})^{\alpha_k}(\theta_1^{(1)})^{\beta_1}\cdots(\theta_1^{(j)})^{\beta_j}\,|\,0\leq\alpha_1+\cdots+\alpha_k+\beta_1+\cdots+\beta_j\leq n-1\},\tag{6.38}$$

and

$$\{(x_1^{(1)})^{\alpha_1}\cdots(x_1^{(k)})^{\alpha_k}(\theta_1^{(1)})^{\beta_1}\cdots(\theta_1^{(j)})^{\beta_j}x_2^{(1)}\,|\,0\leq\alpha_1+\cdots+\alpha_k+\beta_1+\cdots+\beta_j\leq n-1\}.\tag{6.39}$$

In most circumstances, we will be able to swap for a different choice of the variable $x_2^{(i)}$ (to lower i) due to (C), (D), and (E). The exceptions to (C) occur when $x_1^{(h)}x_2^{(i)}$ appears alone, so we must also include these monomials in the spanning set:

$$\{x_1^{(h)}x_2^{(i)}\,|\,1\leq h<i\leq k\}.\tag{6.40}$$

The exceptions to (D) occur when $x_1^{(h)}\theta_2^{(i)}$ appears alone, so we must also include these monomials in the spanning set:

$$\{x_1^{(h)}\theta_2^{(i)}\,|\,1\leq h\leq k, 1\leq i\leq j\}.\tag{6.41}$$

The exceptions to (E) occur when $\theta_1^{(h)}\theta_2^{(i)}$ appears alone, so we must also include these monomials in the spanning set:

$$\{\theta_1^{(h)}\theta_2^{(i)}\,|\,1\leq h<i\leq j\}.\tag{6.42}$$

Now we can reduce the problem repeatedly, as there are no more monomials needed left with a variable $x_1^{(1)}$ or $x_2^{(1)}$. As we already took care of all monomials with no $x_2^{(i)}$ variables, we are left to consider

$$\begin{aligned}
&\{(x_1^{(2)})^{\alpha_2}\cdots(x_1^{(k)})^{\alpha_k}(\theta_1^{(1)})^{\beta_1}\cdots(\theta_1^{(j)})^{\beta_j}x_2^{(2)}\,|\,0\leq\alpha_2+\cdots+\alpha_k+\beta_1+\cdots+\beta_j\leq n-1\},\\
&\{(x_1^{(3)})^{\alpha_3}\cdots(x_1^{(k)})^{\alpha_k}(\theta_1^{(1)})^{\beta_1}\cdots(\theta_1^{(j)})^{\beta_j}x_2^{(3)}\,|\,0\leq\alpha_3+\cdots+\alpha_k+\beta_1+\cdots+\beta_j\leq n-1\},\\
&\qquad\cdots\\
&\{(x_1^{(k)})^{\alpha_k}(\theta_1^{(1)})^{\beta_1}\cdots(\theta_1^{(j)})^{\beta_j}x_2^{(k)}\,|\,0\leq\alpha_k+\beta_1+\cdots+\beta_j\leq n-1\}.
\end{aligned}\tag{6.43}$$

Now at this point, there are no more monomials needed left with a variable $x_1^{(\ell)}$ or $x_2^{(\ell)}$ for any ℓ. We are left to consider monomials consisting of only fermionic variables which must contain some variable $\theta_2^{(\ell)}$. By (F), when $\theta_1^{(i)}\theta_2^{(i)}$ appears alone, we must also include these monomials in the spanning set:

$$\{\theta_1^{(i)}\theta_2^{(i)}\,|\,1\leq i\leq j\}.\tag{6.44}$$

Going forwards, we cannot have $\theta_1^{(\ell)}\theta_2^{(\ell)}$ appearing in a monomial, as by (F) it will reduce to 0 or be one of the monomials we already included. Thus we are left to progressively consider

$$\begin{aligned}
&\{(\theta_1^{(2)})^{\beta_2}\cdots(\theta_1^{(j)})^{\beta_j}\theta_2^{(1)}\,|\,0\leq\beta_2+\cdots+\beta_j\leq n-1\},\\
&\{(\theta_1^{(3)})^{\beta_3}\cdots(\theta_1^{(j)})^{\beta_j}\theta_2^{(2)}\,|\,0\leq\beta_3+\cdots+\beta_j\leq n-1\},\\
&\qquad\cdots\\
&\{(\theta_1^{(j)})^{\beta_j}\theta_2^{(j-1)}\,|\,0\leq\beta_j\leq n-1\},\\
&\{\theta_2^{(j)}\}.
\end{aligned}\tag{6.45}$$

Taking the union of all of these sets in equations (6.38-6.45) shows that the claimed basis is a spanning set.

We will show that the spanning set is a basis by showing that its cardinality matches the dimension of $R^{(k,j)}_{\mathfrak{I}_2(n)}$ from Corollary 6.4.2. We will show this by induction on n. For fixed k, j, let P_n denote the cardinality given in equation (6.22), and the induction hypothesis is that the cardinality of the spanning set in equation (6.31) is equal to P_n.

For the base case of $n = 2$, a direct computation shows that both P_2 and the cardinality of the spanning set are

$$1 + 2k + 2j + 2kj + \binom{k}{2} + \binom{k+1}{2} + \binom{j}{2} + \binom{j+1}{2}. \tag{6.46}$$

For the inductive step, we calculate

$$P_{n+1} - P_n = \sum_{h=0}^{n+1} \binom{j}{h}\binom{k+n-h}{n+1-h} + \sum_{h=0}^{n} \binom{j}{h}\binom{k+n-h-1}{n-h}. \tag{6.47}$$

Let us label the constituent sets in equation (6.31) which depend on n:

$$A_n = \{(x_1^{(1)})^{\alpha_1} \cdots (x_1^{(k)})^{\alpha_k} (\theta_1^{(1)})^{\beta_1} \cdots (\theta_1^{(j)})^{\beta_j} \mid 0 \le \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_j \le n-1\} \tag{6.48}$$

$$B_n = \bigcup_{i=1}^{k} \{(x_1^{(i)})^{\alpha_i} \cdots (x_1^{(k)})^{\alpha_k} (\theta_1^{(1)})^{\beta_1} \cdots (\theta_1^{(j)})^{\beta_j} x_2^{(i)} \mid 0 \le \alpha_i + \cdots + \alpha_k + \beta_1 + \cdots + \beta_j \le n-1\} \tag{6.49}$$

$$C_n = \bigcup_{i=1}^{j} \{(\theta_1^{(i+1)})^{\beta_{i+1}} \cdots (\theta_1^{(j)})^{\beta_j} \theta_2^{(i)} \mid 0 \le \beta_{i+1} + \cdots + \beta_j \le n-1\} \tag{6.50}$$

We compute the set differences between $n+1$ and n, finding that

$$A_{n+1} \setminus A_n = \{(x_1^{(1)})^{\alpha_1} \cdots (x_1^{(k)})^{\alpha_k} (\theta_1^{(1)})^{\beta_1} \cdots (\theta_1^{(j)})^{\beta_j} \mid \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_j = n\}, \tag{6.51}$$

$$B_{n+1} \setminus B_n = \bigcup_{i=1}^{k} \{(x_1^{(i)})^{\alpha_i} \cdots (x_1^{(k)})^{\alpha_k} (\theta_1^{(1)})^{\beta_1} \cdots (\theta_1^{(j)})^{\beta_j} x_2^{(i)} \mid \alpha_i + \cdots + \alpha_k + \beta_1 + \cdots + \beta_j = n\}, \tag{6.52}$$

$$C_{n+1} \setminus C_n = \bigcup_{i=1}^{j} \{(\theta_1^{(i+1)})^{\beta_{i+1}} \cdots (\theta_1^{(j)})^{\beta_j} \theta_2^{(i)} \mid \beta_{i+1} + \cdots + \beta_j = n\}. \tag{6.53}$$

Counting with stars and bars arguments, their cardinalities are

$$|A_{n+1} \setminus A_n| = \sum_{h=0}^{n} \binom{j}{h}\binom{k-1+n-h}{n-h}, \tag{6.54}$$

$$|B_{n+1} \setminus B_n| = \sum_{i=1}^{k} \sum_{h=0}^{n} \binom{j}{h} \binom{k-i+n-h}{n-h}, \tag{6.55}$$

which by reversing the order of summation and the hockey-stick identity simplifies to

$$|B_{n+1} \setminus B_n| = \sum_{h=0}^{n} \binom{j}{h} \binom{k+n-h}{n-h-1}. \tag{6.56}$$

We compute that

$$|C_{n+1} \setminus C_n| = \sum_{i=1}^{j} \binom{j-i}{n} = \binom{j}{n+1}. \tag{6.57}$$

Note that equations (6.56) and (6.57) can be combined to give

$$\sum_{h=0}^{n+1} \binom{j}{h} \binom{k+n-h}{n-h-1}, \tag{6.58}$$

hence their sum with equation (6.54) equals equation (6.47). Thus the spanning set has cardinality P_n for all $n \geq 2$, and hence must be a basis. □

Remark 6.5.3. It is possible to prove the main results without using diagonal supersymmetry, generalizing the method in [2, Chapter IV].

Using a certain form where multiplication is adjoint to differentiation, one can define a space of diagonal harmonics in k sets of bosonic variables and j sets of fermionic variables (see [96, Section 2] and [97, Section 2]). This harmonic space is isomorphic to the coinvariant ring $R^{(k,j)}_{\mathfrak{I}_2(n)}$ as a multigraded $\mathfrak{I}_2(n)$-module.

First, derive the monomial basis for $R^{(k,j)}_{\mathfrak{I}_2(n)}$ (Theorem 6.5.2) by constructing a spanning set, and then construct enough linearly independent harmonics in the harmonic space isomorphic to $R^{(k,j)}_{\mathfrak{I}_2(n)}$. Second, enumerate each graded component of the basis to determine the explicit multigraded Hilbert series for $R^{(k,j)}_{\mathfrak{I}_2(n)}$ (Theorem 6.4.1). Finally, apply the polarization operators E_ℓ to Alfano's formula for the bigraded character series of $R^{(2,0)}_{\mathfrak{I}_2(n)}$ in equation (6.17) to derive the explicit multigraded character series for $R^{(k,j)}_{\mathfrak{I}_2(n)}$ (Theorem 6.1.1).

6.6 The sign character

The sign character, which is χ_2 in our notation, is of particular interest, so we briefly record its multigraded multiplicity in $R^{(k,j)}_{\mathfrak{I}_2(n)}$. The dimension of the sign character of $R^{(2,0)}_W$ defines the Catalan numbers of type W, for any finite Coxeter group W, and the q,t-graded dimension of the sign character defines the q,t-Catalan numbers of type W. Denote the latter by $\mathrm{Cat}(W;q,t)$. See [57, Section 7] and [94] for a discussion of what is conjectured and known for different groups W.

Define

$$[n]_{q,t} := q^{n-1} + q^{n-2}t + \cdots + qt^{n-2} + t^{n-1}. \tag{6.59}$$

From Alfano's formula for the q,t-graded character series of $R^{(2,0)}_{\mathfrak{I}_2(n)}$ [2], we can extract the q,t-graded multiplicity of the sign character as

$$\mathrm{Cat}(\mathfrak{I}_2(n); q, t) = [n+1]_{q,t} + qt = s_{(n)}(q,t) + s_{(1,1)}(q,t), \tag{6.60}$$

which is also given in [94, Corollary 3].

More generally, for arbitrary (k, j), denote the $\mathbf{q}, \mathbf{u}$-graded multiplicity of the sign character of $R^{(k,j)}_W$ by $\mathrm{Cat}(W; \mathbf{q}; \mathbf{u})$,[3] where as usual, $\mathbf{q} = q_1, \ldots, q_k$ and $\mathbf{u} = u_1, \ldots, u_j$. Then from Theorem 6.1.1, we immediately get that

$$\mathrm{Cat}(\mathfrak{I}_2(n); \mathbf{q}; \mathbf{u}) = s_{(n)}(\mathbf{q}/\mathbf{u}) + s_{(1,1)}(\mathbf{q}/\mathbf{u}). \tag{6.61}$$

6.7 Numerology

In this section, we discuss various specializations of the dimension of $R^{(k,j)}_G$ to some small values of (k, j), using Corollary 6.4.2 to get formulas for the dimension of $R^{(k,j)}_{\mathfrak{I}_2(n)}$. See Figure 6.1 for some values of $\dim(R^{(k,j)}_{\mathfrak{I}_2(n)})$.

There is interest in establishing formulas that are type-invariant (either for all types, or for at least some types) depending on information on G. For example, it is a classical result that $\dim(R^{(1,0)}_G) = |G|$ if and only if G is a finite reflection group. See Figure 6.2 for some example dimension formulas, which are known or conjectured to hold for both types A and I.

Haiman [57] observed that $\dim(R^{(2,0)}_W) \geq (h+1)^r$, where h denotes the Coxeter number and r denotes the rank for many finite Coxeter groups W. This inequality was proven by Gordon [47] for all Weyl groups W. Alfano showed that equality holds for type I, where $\dim(R^{(2,0)}_{\mathfrak{I}_2(n)}) = (n+1)^2$. When $W = \mathfrak{B}_n$, equality holds for $n = 2, 3$, while for $n \geq 4$, Haiman conjectured and Ajila–Griffeth [1] proved that $\dim(R^{(2,0)}_{\mathfrak{B}_n}) > (2n+1)^n$. For W an irreducible complex reflection group with order greater than 2 reflections, the lower bound was further improved by Griffeth [50].

Zabrocki [102] observed that $\dim(R^{(1,1)}_W)$ is the order of $\Sigma(W, S)$, the Coxeter complex of W, for small n for $W = \mathfrak{S}_n$, $W = \mathfrak{B}_n$, $W = \mathfrak{D}_n$, and $W = \mathfrak{I}_2(n)$. This was proven in type A by Rhoades and Wilson [88] and in type B by Bhattacharya [17]. Rhoades and Swanson calculated that this is not true for $W = \mathfrak{F}_4$. Specializing Corollary 6.4.2, we find that $\dim(R^{(1,1)}_{\mathfrak{I}_2(n)}) = 4n + 1$, which is the order of the Coxeter complex of $\mathfrak{I}_2(n)$.

It is not hard to see that $\dim(R^{(0,1)}_W) = 2^r$; for example, this was noted by Machacek and Zabrocki [79] for type A. A type-independent proof follows as a corollary from the

[3] Note that $\mathrm{Cat}(W; q, t)$ is recovered, when $k \geq 2$, by setting all but two of the q_i to zero and all of the u_i to zero. Specializing this with all variables to 1 is not in general a Catalan number of type W.

$k \backslash j$	0	1	2
0		4	≤ 10
1	$2n$	$4n+1$	$8n$
2	$(n+1)^2$	$2(n^2+n+2)$	$4n^2+9$
3	$\frac{2n^3+9n^2+13n+18}{6}$	$\frac{2n^3+6n^2+7n+21}{3}$	$\frac{4n^3+6n^2+8n+36}{3}$
4	$\frac{n^4+8n^3+23n^2+28n+72}{12}$	$\frac{n^4+6n^3+14n^2+15n+66}{6}$	$\frac{n^4+4n^3+8n^2+8n+51}{3}$

$k \backslash j$	3	4
0	≤ 21	≤ 41
1	$\geq 16n-8$	$\geq 32n-35$
2	$8n^2-8n+20$	$\leq 16n^2-32n+50$
3	$\frac{8n^3+16n+51}{3}$	$\frac{16n^3-24n^2+56n+48}{3}$
4	$\frac{2n^4+4n^3+10n^2+8n+72}{3}$	$\frac{4n^4+20n^2+99}{3}$

Figure 6.1: Some polynomial dimension formulas for $R^{(k,j)}_{\mathfrak{I}_2(n)}$ for $0 \leq k, j \leq 4$. When $j - k \geq 2$, the dimensions eventually stabilize to the listed values as $n \to \infty$. All formulas are obtained by specializing Corollary 6.4.2.

$k \backslash j$	0	1	2
0		2^r	$\binom{2r+1}{r}$
1	$\lvert W\rvert$	$\lvert\Sigma(W,S)\rvert$	$2^r\lvert W\rvert$
2	$(h+1)^r$		

Figure 6.2: Known or conjectured dimension formulas for $R^{(k,j)}_W$ for types A and I together. The entries for $(1,0)$, $(0,1)$, and $(0,2)$ are type invariant for all finite Coxeter groups. The entries for $(2,0)$, $(1,1)$, and $(1,2)$ are known or expected to not necessarily hold for all types, but are known or expected to hold for both types A and I.

work of Kim and Rhoades [69] on $R_W^{(0,2)}$, by setting one set of variables to zero. Specializing Corollary 6.4.2, we find that $\dim(R^{(0,1)}_{\mathfrak{I}_2(n)}) = 4$.

Kim and Rhoades [69] showed that $\dim(R_W^{(0,2)}) = \binom{2r+1}{r}$, for any irreducible complex reflection group W. Since $\mathfrak{I}_2(2)$ is reducible, when $n > 2$, this agrees with specializing Corollary 6.4.2, which says that $\dim(R^{(k,j)}_{\mathfrak{I}_2(n)}) = 10 = \binom{5}{2}$.

In [73], it was observed that for small n, $\dim(R_W^{(1,2)}) = 2^r|W|$ for $W = \mathfrak{S}_n$ or $\mathfrak{B}_n$ and conjectural bases were constructed which give those enumerations. Specializing Corollary 6.4.2, we find that $\dim(R^{(1,2)}_{\mathfrak{I}_2(n)}) = 8n = 2^2|\mathfrak{I}_2(n)|$. Contrast with [87], where the Hilbert series of the defining ideal $I^{(1,2)}_{\mathfrak{I}_2(n)}$ was found: since with multiple sets of variables $\mathfrak{I}_2(n)$ does not act by reflections, neither the Hilbert series of $R^{(1,2)}_{\mathfrak{I}_2(n)}$ nor $I^{(1,2)}_{\mathfrak{I}_2(n)}$ directly implies the other.

Haiman observed that for small n, $\dim(R_n^{(3,0)}) = 2^n(n+1)^{n-2}$. Note that for $W = \mathfrak{I}_2(6) = \mathfrak{G}_2$, the factorization $\dim(R^{(3,0)}_{\mathfrak{I}_2(6)}) = 142 = 2 \cdot 71$ does not seem to readily mesh with Haiman's conjecture on $R_n^{(3,0)}$ in a type-independent way.

6.8 Cyclic groups

In this appendix we concisely derive analogous results for the cyclic group. Let $\mathbb{Z}_n$ denote the cyclic group of order n. As this is a rank 1 complex reflection group, each set of variables only has one variable, so we dispense with the superscripts, and consider the ring $R^{(k,j)}_{\mathbb{Z}_n} = \mathbb{C}[x_1, \ldots, x_k, \theta_1, \ldots, \theta_j]/\langle \mathbb{C}[x_1, \ldots, x_k, \theta_1, \ldots, \theta_j]_+^{\mathbb{Z}_n}\rangle$. The action of $\mathbb{Z}_n$ on x_i is given by $g \cdot x_i = \zeta_n^g x_i$ and on θ_i by $g \cdot \theta_i = \zeta_n^g \theta_i$ for $g \in \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, where $\zeta_n = e^{2\pi i/n}$ is a primitive nth root of unity. The generators and relations for $\mathbb{Z}_n$ may be written as $\langle a \mid a^n = 1\rangle$. The irreducible characters of $\mathbb{Z}_n$ are:

- χ_i defined by $\chi_i(a^\ell) = \zeta_n^{i\ell}$ for $i \in \{0, \ldots, n-1\}$.

We determine the multigraded character series.

Theorem 6.8.1. *Let $\mathbf{q} = (q_1, \ldots, q_k)$ and $\mathbf{u} = (u_1, \ldots, u_j)$. The $\mathbf{q}, \mathbf{u}$-graded character series for $R^{(k,j)}_{\mathbb{Z}_n}$ is given by*

$$\mathrm{Char}(R^{(k,j)}_{\mathbb{Z}_n}; \mathbf{q}; \mathbf{u}) = \sum_{i=0}^{n-1} s_{(i)}(\mathbf{q}/\mathbf{u})\chi_i. \tag{6.62}$$

Proof. It is straightforward to show that $R^{(1,0)}_{\mathbb{Z}_n} = \mathbb{C}[x_1]/(x_1^n)$, with monomial basis $\{1, x_1, x_1^2, \ldots, x_1^{n-1}\}$. Then the singly graded character of $R^{(1,0)}_{\mathbb{Z}_n}$ is the following graded version of the regular representation:

$$\mathrm{Char}(R^{(1,0)}_{\mathbb{Z}_n}; q) = \sum_{i=0}^{n-1} q^i \chi_i. \tag{6.63}$$

Since each set of variables has only one variable in it, then diagonal supersymmetry (Theorem 6.2.2) says that for partitions λ with $\ell(\lambda) \leq 1$, and indices μ of irreducible $\mathbb{Z}_n$-characters, there exist nonnegative integer coefficients $c_{\lambda,\mu}$ such that for any (k,j), the multigraded character series of $R_{\mathbb{Z}_n}^{(k,j)}$ is

$$\mathrm{Char}(R_{\mathbb{Z}_n}^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,1)} \sum_{\chi^\mu \in \mathrm{IrrChar}(\mathbb{Z}_n)} c_{\lambda,\mu} s_\lambda(\mathbf{q}/\mathbf{u}) \chi^\mu. \tag{6.64}$$

Thus equation (6.63) determines all coefficients $c_{\lambda,\mu}$ where $\lambda \in P(1,0,1)$, i.e., $\ell(\lambda) \leq 1$. Explicitly, for $n \geq 1$, and $\ell(\lambda) \leq 1$ we have:

- if $\lambda = (i)$ for $i \in \{0, \ldots, n-1\}$ and $\chi^\mu = \chi_i$, then $c_{\lambda,\mu} = 1$;
- otherwise $c_{\lambda,\mu} = 0$.

Thus we have determined $c_{\lambda,\mu}$ for any $\lambda \in P(k,j,1)$. Hence expanding equation (6.64) with the coefficients we have determined completes the proof. □

We then conclude the following formula for the Hilbert series. This recovers a result of Bergeron for the purely bosonic case (at $j = 0$) [9, Theorem 2.3].

Theorem 6.8.2. *Let $\mathbf{q} = (q_1, \ldots, q_k)$ and $\mathbf{u} = (u_1, \ldots, u_j)$. The $\mathbf{q}, \mathbf{u}$-graded Hilbert series for $R_{\mathbb{Z}_n}^{(k,j)}$ is given by*

$$\mathrm{Hilb}(R_{\mathbb{Z}_n}^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{i=0}^{n-1} s_{(i)}(\mathbf{q}/\mathbf{u}). \tag{6.65}$$

Proof. This follows immediately from Theorem 6.8.1 since each character is 1-dimensional. □

As a corollary, we obtain the dimension of $R_{\mathbb{Z}_n}^{(k,j)}$.

Corollary 6.8.3. *The dimension of $R_{\mathbb{Z}_n}^{(k,j)}$ is given by*

$$\sum_{i=0}^{n-1} \sum_{\ell=0}^{i} \binom{k+\ell-1}{\ell} \binom{j}{i-\ell}. \tag{6.66}$$

Proof. This follows from Theorem 6.8.2 specialized using equation (6.24). □

Finally, we determine a monomial basis for $R_{\mathbb{Z}_n}^{(k,j)}$.

Theorem 6.8.4. *A basis $B_{\mathbb{Z}_n}^{(k,j)}$ for the coinvariant ring $R_{\mathbb{Z}_n}^{(k,j)}$ is given by*

$$\{x_1^{\alpha_1} \cdots x_k^{\alpha_k} \theta_1^{\beta_1} \cdots \theta_j^{\beta_j} \mid 0 \leq \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_j \leq n-1\}, \tag{6.67}$$

where $\beta_i \in \{0, 1\}$ for all $i \in \{1, \ldots, j\}$.

Proof. It is not difficult to see that the invariant ideal $\langle \mathbb{C}[x_1, \ldots, x_k, \theta_1, \ldots, \theta_j]_+^{\mathbb{Z}_n} \rangle$ contains all monomials $x_1^{\alpha_1} \cdots x_k^{\alpha_k} \theta_1^{\beta_1} \cdots \theta_j^{\beta_j}$ where $\alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_j = n$ and $\beta_i \in \{0, 1\}$ for all $i \in \{1, \ldots, j\}$. Furthermore, there are no non-constant invariants of lower total degree. We can use these as straightening relations to show that the claimed basis is a spanning set.

To show that the spanning set is a basis, it suffices to show that its enumeration matches that in Corollary 6.8.3. Let $i = \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_j$ and use a stars and bars argument to complete the proof. $\square$

Chapter 7

Further directions

7.1 The standard character

In general, it is quite difficult to compute the universal series coefficients of the Frobenius series of a (k, j)-bosonic-fermionic coinvariant algebra.

In this section, we determine that the standard character of all (k, j)-bosonic-fermionic coinvariant algebras has multigraded multiplicity $s_{(1)}(\mathbf{q}/\mathbf{u}) + s_{(2)}(\mathbf{q}/\mathbf{u}) + \cdots + s_{(n-1)}(\mathbf{q}/\mathbf{u})$, where $s_\lambda(\mathbf{q}/\mathbf{u})$ is a super Schur function in grading variables $\mathbf{q} = \{q_1, \ldots, q_k\}$ and $\mathbf{u} = \{u_1, \ldots, u_j\}$.

Define the (k, j)-*bosonic-fermionic coinvariant ring* by

$$R_n^{(k,j)} = \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]/\langle \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]_+^{\mathfrak{S}_n} \rangle, \tag{7.1}$$

where $\mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]_+^G$ denotes the polynomials without constant term, which are invariant under the diagonal action of the symmetric group $\mathfrak{S}_n$ on each set of variables.

The *standard character* of the symmetric group $\mathfrak{S}_n$ is the irreducible character indexed by the partition $(n-1, 1)$. The main theorem of this note is a formula for the multigraded multiplicity of the standard character of $R_n^{(k,j)}$. This is the first character, aside from the trivial character, to be determined for all $R_n^{(k,j)}$.

Theorem 7.1.1. *Fix $n \geq 2$. For any k, j nonnegative integers or infinity,*

$$\langle \mathrm{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}), s_{(n-1,1)}(\mathbf{z}) \rangle = s_{(1)}(\mathbf{q}/\mathbf{u}) + s_{(2)}(\mathbf{q}/\mathbf{u}) + \cdots + s_{(n-1)}(\mathbf{q}/\mathbf{u}). \tag{7.2}$$

We recall the following result from [74], which was conjectured in [10, Conjecture 1]. Let $\mathbf{z}$ be an (infinite) auxiliary set of variables for the Frobenius characters.

Theorem 7.1.2 ([74])**.** *Fix a positive integer n. For partitions λ with $\ell(\lambda) \leq n$, and $\mu \vdash n$, there exist nonnegative integer coefficients $c_{\lambda\mu}$ such that for any (k, j), the multigraded Frobenius series of $R_n^{(k,j)}$ is*

$$\mathrm{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,n)} \sum_{\mu \vdash n} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) s_\mu(\mathbf{z}). \tag{7.3}$$

Define a *super Schur function* (see [32, Appendix A.2.2]) by

$$s_\lambda(\mathbf{q}/\mathbf{u}) = \sum_{\nu \subseteq \lambda} s_\nu(\mathbf{q}) s_{\lambda'/\nu'}(\mathbf{u}). \tag{7.4}$$

The following is well-known, but we include a proof for completeness. It has essentially the same proof as that of [57, Proposition 1.2.2].

Lemma 7.1.3. *Fix $n \geq 1$. For any k, j nonnegative integers or infinity,*

$$\langle \mathrm{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}), s_{(n)}(\mathbf{z}) \rangle = 1. \tag{7.5}$$

Proof. Let $(R_n^{(k,j)})_{r_1,\ldots,r_k;s_1,\ldots,s_j}$ denote the homogeneous subspace of $R_n^{(k,j)}$ of multidegree $r_1, \ldots, r_k; s_1, \ldots, s_j$. Since $(R_n^{(k,j)})_{0,\ldots,0;0,\ldots,0}$ is spanned by 1, an $\mathfrak{S}_n$-invariant polynomial, it affords the trivial representation. However, any $\mathfrak{S}_n$-invariant polynomial of any multidegree $(r_1, \ldots, r_k; s_1, \ldots, s_j)$ not equal to $(0, \ldots, 0; 0, \ldots, 0)$ is in the defining ideal $\langle \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(j)}]_+^{\mathfrak{S}_n} \rangle$, and thus cannot be in $(R_n^{(k,j)})_{r_1,\ldots,r_k;s_1,\ldots,s_j}$. □

We have that $\mathfrak{gl}(k|j)$ acts by superderivations. For $1 \leq a, b \leq k$ and $1 \leq c, d \leq j$, we have polarization operators

$$E_{a,b} = \sum_{i=1}^n x_i^{(a)} \partial_{x_i^{(b)}},\ E_{a,d'} = \sum_{i=1}^n x_i^{(a)} \partial_{\theta_i^{(d)}},\ E_{c',b} = \sum_{i=1}^n \theta_i^{(c)} \partial_{x_i^{(b)}},\ E_{c',d'} = \sum_{i=1}^n \theta_i^{(c)} \partial_{\theta_i^{(d)}}, \tag{7.6}$$

which realize the action of $\mathfrak{gl}(k|j)$ on $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$ [32, Lemma 5.13].

Since each polarization operator is a sum over all indices $\{1, \ldots, n\}$, we have the following.

Lemma 7.1.4. *The polarization operators from equation (7.6) commute with the action of $\mathfrak{S}_n$.*

We recall the following formula for the Frobenius series of $R_n^{(1,0)}$.

Theorem 7.1.5 ([91])**.** *For any positive integer n,*

$$\mathrm{Frob}(R_n^{(1,0)}; q) = \sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} s_\lambda. \tag{7.7}$$

Proposition 7.1.6.

$$\langle \mathrm{Frob}(R_n^{(1,0)}; q), s_{(n-1,1)}(\mathbf{z}) \rangle = s_{(1)}(q) + s_{(2)}(q) + \cdots + s_{(n-1)}(q). \tag{7.8}$$

Proof. We only consider standard Young tableaux of shape $(n-1, 1)$:

$$\begin{array}{|c|c|c|c|} \hline 1 & & \cdots & \\ \hline i \\ \cline{1-1} \end{array}.$$

Such a standard Young tableaux must have a 1 in the upper left corner, and then may have any value $i \in \{2, \ldots, n\}$ in the lower left corner. The choice of i then determines the rest of the tableau. Then i is part of a descent with $i-1$, and this is the only descent in the tableau. Recall that the major index statistic of a standard Young tableaux is the sum of all descents. As a result, we write

$$\langle \mathrm{Frob}(R_n^{(1,0)}; q), s_{(n-1,1)}(\mathbf{z})\rangle = \sum_{T \in \mathrm{SYT}((n-1,1))} q^{\mathrm{maj}(T)} = q + q^2 + \cdots + q^{n-1}, \tag{7.9}$$

by Theorem 7.1.5, which is equivalent to the claimed formula. □

Let the *polarized power sum* be defined by

$$p_{r_1,\ldots,r_k}(\boldsymbol{x}_n^{(1)}, \ldots, \boldsymbol{x}_n^{(k)}) = \sum_{i=1}^{n} x_1^{r_1} \cdots x_k^{r_k}. \tag{7.10}$$

Note that we added the subscript n to $\boldsymbol{x}_n^{(1)}$ to record how many variables are in each set.

Proposition 7.1.7 ([99]). *The defining ideal of $R_n^{(k,0)}$,*

$$\langle \mathbb{C}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}]_+^{\mathfrak{S}_n}\rangle \tag{7.11}$$

has as a generating set

$$\{p_{r_1,\ldots,r_k}(\boldsymbol{x}_n^{(1)}, \ldots, \boldsymbol{x}_n^{(k)}) | 1 \leq r_1 + \cdots + r_k \leq n\}. \tag{7.12}$$

Now we are ready to prove Theorem 7.1.1.

Proof of Theorem 7.1.1. Fix $n \geq 2$. By Lemma 7.1.3, we have that

$$\begin{aligned} \langle \mathrm{Frob}(R_n^{(k,0)}; \mathbf{q}; \mathbf{u}), s_{(n-1)}s_{(1)}\rangle &= \langle \mathrm{Frob}(R_n^{(k,0)}; \mathbf{q}; \mathbf{u}), s_{(n-1,1)} + s_{(n)}\rangle & (7.13)\\ &= \langle \mathrm{Frob}(R_n^{(k,0)}; \mathbf{q}; \mathbf{u}), s_{(n-1,1)}\rangle + 1. & (7.14)\end{aligned}$$

Now since $\mathfrak{S}_{n-1} \times \mathfrak{S}_1 \cong \mathfrak{S}_{n-1}$, we can write

$$s_{(n-1)}s_{(1)} = \mathrm{Frob}(\mathrm{Ind}_{\mathfrak{S}_{n-1}\times\mathfrak{S}_1}^{\mathfrak{S}_n}(\mathbb{1}_{\mathfrak{S}_{n-1}} \otimes \mathbb{1}_{\mathfrak{S}_1})) = \mathrm{Frob}(\mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(\mathbb{1}_{\mathfrak{S}_{n-1}})), \tag{7.15}$$

where $\mathbb{1}$ denotes the trivial representation.

We have that

$$\begin{aligned} \langle \mathrm{Frob}(R_n^{(k,0)}), s_{(n-1)}s_{(1)}\rangle &= \langle R_n^{(k,0)}, \mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(\mathbb{1}_{\mathfrak{S}_{n-1}})\rangle & (7.16)\\ &= \langle \mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(R_n^{(k,0)}), \mathbb{1}_{\mathfrak{S}_{n-1}}\rangle, & (7.17)\end{aligned}$$

by Frobenius reciprocity.

It remains to compute the multiplicity of the trivial representation $\mathbb{1}_{\mathfrak{S}_{n-1}}$ inside of $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(R_n^{(k,0)})$.

But this is equivalent to determining $\langle \mathrm{Frob}(R_n^{(k,0)}), s_{(n-1)}s_{(1)}\rangle$, which is all $\mathfrak{S}_{n-1}$-invariants inside $R_n^{(k,0)}$.

Every $\mathfrak{S}_{n-1}$-invariant polynomial in $\mathbb{C}[\boldsymbol{x}_n^{(1)}, \ldots, \boldsymbol{x}_n^{(k)}]$ can be written as $g(\boldsymbol{x}_{n-1}^{(1)}, \ldots, \boldsymbol{x}_{n-1}^{(k)}) \cdot f(x_n^{(1)}, \ldots, x_n^{(k)})$, where g is an $\mathfrak{S}_{n-1}$-invariant polynomial in $\mathbb{C}[\boldsymbol{x}_{n-1}^{(1)}, \ldots, \boldsymbol{x}_{n-1}^{(k)}]$ and f is any polynomial in just the nth variable from each set, that is, $\mathbb{C}[x_n^{(1)}, \ldots, x_n^{(k)}]$.

By Proposition 7.1.7, the defining ideal of $R_n^{(k,0)}$ imposes the relations $\{p_{r_1,\ldots,r_k}(\boldsymbol{x}_n^{(1)}, \ldots, \boldsymbol{x}_n^{(k)}) = 0 \,|\, 1 \le r_1 + \cdots + r_k \le n\}$, equivalently

$$\{p_{r_1,\ldots,r_k}(\boldsymbol{x}_{n-1}^{(1)}, \ldots, \boldsymbol{x}_{n-1}^{(k)}) = -(x_n^{(1)})^{r_1} \cdots (x_n^{(k)})^{r_k} | 1 \le r_1 + \cdots + r_k \le n\}. \tag{7.18}$$

Hence the $\mathfrak{S}_{n-1}$-invariant polynomial g can be expressed as a function of just the variables $x_n^{(1)}, \ldots, x_n^{(k)}$ in the quotient ring. Thus any $\mathfrak{S}_{n-1}$-invariant polynomial in $\mathbb{C}[\boldsymbol{x}_n^{(1)}, \ldots, \boldsymbol{x}_n^{(k)}]$ can be expressed as a function of just the variables $x_n^{(1)}, \ldots, x_n^{(k)}$ in the quotient ring.

Now from the $k = 1$ case, we can compute that $(x_n^{(1)})^n = 0$ in $R_n^{(1,0)}$. This can be seen because the elementary symmetric functions also form a generating set for the defining ideal of $R_n^{(1,0)}$, specifically $\{e_1(\boldsymbol{x}_n^{(1)}), \ldots, e_n(\boldsymbol{x}_n^{(1)})\}$. Consider the polynomial $P(t) = \prod_{i=1}^n (t - x_i^{(1)})$ in $R_n^{(1,0)}[t]$. Expanding gives $P(t) = t^n - e_1(\boldsymbol{x}_n^{(1)})t^{n-1} + e_2(\boldsymbol{x}_n^{(1)}) - \cdots + (-1)^n e_n(\boldsymbol{x}_n^{(1)})$. In $R_n^{(1,0)}[t]$, we have $e_i(\boldsymbol{x}_n^{(1)}) = 0$ for all $i \ge 1$, so $P(t) = t^n$. Since $x_n^{(1)}$ is a root of $\prod_{i=1}^n (t - x_i^{(1)}) = t^n$, then $(x_n^{(1)})^n = 0$.

By Lemma 7.1.4, the defining ideal $I_n^{(k,0)}$ is invariant under the action of the operators in equation 7.6 which only act on commutative variables. Thus if indices $i_1, \ldots, i_k$ satisfy $i_1 + \cdots + i_k = n$, then $E_{1,k}^{i_k} \cdots E_{1,2}^{i_2}(x_n^{(1)})^{i_1} \cdots (x_n^{(k)})^{i_k} = c(x_n^{(1)})^n = 0$, for some nonzero scalar c. We conclude that all monomials in $\mathbb{C}[x_n^{(1)}, \ldots, x_n^{(k)}]$ of total degree n vanish.

Thus the $\mathfrak{S}_{n-1}$-invariant subspace is isomorphic as a vector space to a subspace of

$$\mathbb{C}[x_n^{(1)}, \ldots, x_n^{(k)}] / \langle (x_n^{(1)})^{i_1} \cdots (x_n^{(k)})^{i_k} \,|\, i_1 + \cdots + i_k = n\rangle, \tag{7.19}$$

which has a basis of

$$\{(x_n^{(1)})^{i_1} \cdots (x_n^{(k)})^{i_k} \,|\, 0 \le i_1 + \cdots + i_k \le n - 1\}. \tag{7.20}$$

The basis has multigraded Hilbert series

$$1 + s_{(1)}(\mathbf{q}) + s_{(2)}(\mathbf{q}) + \cdots + s_{(n-1)}(\mathbf{q}). \tag{7.21}$$

We claim that the $\mathfrak{S}_{n-1}$-invariant subspace is isomorphic exactly to equation (7.19). Suppose not. Then at least one element will be missing from the basis in equation (7.20), say, $(x_n^{(1)})^{i_1} \cdots (x_n^{(k)})^{i_k}$ for some $i_1, \ldots, i_k$ such that $i_1 + \cdots + i_k = d < n$. Then use polarization operators to push all variables into the first set, that is, $E_{1,k}^{i_k} \cdots E_{1,2}^{i_2}(x_n^{(1)})^{i_1} \cdots (x_n^{(k)})^{i_k} =$

$c(x_n^{(1)})^d$, for some nonzero scalar c. However, since $R_n^{(k,0)}$ contains $R_n^{(1,0)}$ as a subring, this contradicts that $(x_n^{(1)})^d \neq 0$ unless $d \geq n$.

Finally, take the limit as $k \to \infty$. This shows that

$$\langle \mathrm{Frob}(R_n^{(\infty,0)}; \mathbf{q}), s_{(n-1,1)}(\mathbf{z})\rangle = s_{(1)}(\mathbf{q}) + s_{(2)}(\mathbf{q}) + \cdots + s_{(n-1)}(\mathbf{q}). \tag{7.22}$$

Then by [74], we conclude that

$$\langle \mathrm{Frob}(R_n^{(\infty,\infty)}; \mathbf{q}; \mathbf{u}), s_{(n-1,1)}(\mathbf{z})\rangle = s_{(1)}(\mathbf{q}/\mathbf{u}) + s_{(2)}(\mathbf{q}/\mathbf{u}) + \cdots + s_{(n-1)}(\mathbf{q}/\mathbf{u}), \tag{7.23}$$

and all finite cases follow from this.

□

7.2 The Tamari lattice

The *Tamari lattice* Y_n is a partial order on the $C_n = \frac{1}{n+1}\binom{2n}{n}$ Catalan objects of order n, introduced by Tamari [98]. It may be defined by a covering relation on a set of Catalan objects.

The Tamari lattice Y_n is simple to describe in terms of binary trees (see [15] for the following exposition). Binary trees can be defined recursively so that a binary tree is either a leaf, or an ordered pair of binary trees $B = (B_1, B_2)$. The number of binary trees with n internal nodes is C_n. Next, we define the covering relations for the Tamari lattice. A binary tree B which contains $((B_1, B_2), B_3)$ as a subtree is covered by B', which is obtained from B by replacing the subtree $((B_1, B_2), B_3)$ with $(B_1, (B_2, B_3))$. It follows that Y_n can also be succinctly described on non-associative parenthesizations. For other Catalan objects, for example Dyck paths, to describe the Tamari lattice amounts to determining the right characterization of the covering relations under some bijection with binary trees.

The fact that Y_n is indeed a lattice was established in [41, 63].

An *interval* in a lattice L is a pair $x, y \in L$ such that $x \preceq y$ in the partial order. The number of intervals in the Tamari lattice was found by Chapoton [30] to be

$$\frac{2(4n+1)!}{(3n+2)!(n+1)!}. \tag{7.24}$$

Now equation (7.24) is the conjectural multiplicity of the sign character of the triagonal coinvariant ring, according to a conjecture of Haiman.

Conjecture 7.2.1 ([57, Fact 2.8.1])**.** *For positive integers n,*

$$\langle \mathrm{Frob}(R_n^{(3,0)}; 1, 1, 1), s_{(1^n)}\rangle = \frac{2(4n+1)!}{(3n+2)!(n+1)!}. \tag{7.25}$$

$n \backslash k$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	5	13	26	45	71
4	1	14	68	217	549	1196
5	1	42	399	2110	8067	25007

Table 7.1: The multiplicity $\langle \mathrm{Frob}(R_n^{(k,0)}; 1, \ldots, 1), s_{(1^n)} \rangle$.

Haiman proved that

$$\langle \mathrm{Frob}(R_n^{(2,0)}; 1, 1), s_{(1^n)} \rangle = C_n \tag{7.26}$$

using results on the Hilbert scheme of n points in $\mathbb{C}^2$, however, the framework does not extend to $\mathbb{C}^3$, which would be needed to extend the proof methods to $R_n^{(3,0)}$.

Further work on the connection between the Tamari lattice and $R_n^{(3,0)}$ was advanced by Bergeron and Préville-Ratelle [14], where they both generalized this story to the m-Tamari lattice (which is related to the higher triagonal harmonics) and have a conjectural Frobenius series for $R_n^{(3,0)}$ with two out of the three gradings.

Define a *length k multichain* of a lattice L to be a sequence $x_1 \preceq x_2 \preceq \cdots \preceq x_k$ of k elements $x_i \in L$ which are weakly increasing in the partial order. Now we see that the number of length 1 multichains of Y_n is $\langle \mathrm{Frob}(R_n^{(2,0)}; 1, 1), s_{(1^n)} \rangle$ and, conjecturally, the number of length 2 multichains of Y_n is $\langle \mathrm{Frob}(R_n^{(3,0)}; 1, 1, 1), s_{(1^n)} \rangle$. One could then make the false conjecture that the number of length $k-1$ multichains of Y_n is $\langle \mathrm{Frob}(R_n^{(k,0)}; 1, \ldots, 1), s_{(1^n)} \rangle$. However, data shows that this starts to fail at $k = 4$.

We compute that the sequence of the number of length 3 multichains of Y_n begins

$$\{1, 4, 26, 218, 2141\}. \tag{7.27}$$

The sequence of the multiplicity of the sign character of $R_n^{(4,0)}$ begins

$$\{1, 4, 26, 217, 2110\}. \tag{7.28}$$

We summarize the experimental data for $\langle \mathrm{Frob}(R_n^{(k,0)}; 1, \ldots, 1), s_{(1^n)} \rangle$ in Table 7.1.

Contrast this with the experimental data for the number of length $k-1$ multichains of Y_n in Table 7.2.

It is interesting to note how similar these tables are. The data suggests the following conjecture.

Conjecture 7.2.2. $\langle \mathrm{Frob}(R_n^{(k,0)}; 1, \ldots, 1), s_{(1^n)} \rangle$ *is less than or equal to the number of length $k-1$ multichains of Y_n.*

Furthermore, it would be interesting to establish a combinatorial condition to exclude some $k-1$ multichains of Y_n to make the conjecture tight.

$n \backslash k$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	5	13	26	45	71
4	1	14	68	218	556	1224
5	1	42	399	2141	8401	26947

Table 7.2: The number of length $k-1$ multichains of Y_n.

A sequence

$$\{T_{d+1}\}_{d\geq 1} = \{1, 4, 26, 217, 2110, 22744, 264057, \dots\} \tag{7.29}$$

is introduced in [80]. T_d is the number of degree d rational curves in $\mathbb{CP}^2$ that satisfy an order $3d-1$ local tangency constraint (see OEIS sequence A364973 [86]). In [82], this sequence is studied further and given a recursive formula. Furthermore, therein it is noted that Q. Ren found that this sequence may be computed as follows. Let $F(x) = 1 + \sum_{d=1}^{\infty} \tilde{T}_d x^d$ and let the coefficient of x^d in $(F(x))^{3d}$ be $(3d)!/(d!)^3$. Solve for $\tilde{T}_d$ and then divide by $3d-1$ to obtain T_d.

Now this exactly matches the known data from equation (7.28) on the multiplicity of the sign character of $R_n^{(4,0)}$. Hence it would be interesting to investigate the following.

Conjecture 7.2.3. *The multiplicity of the sign character of $R_n^{(4,0)}$ is given by T_{n+1}.*

7.3 Higher operator conjecture

The harmonic space $H_n^{(1,0)}$ can be described as the space which consists of the Vandermonde determinant $\prod_{1\leq i<j\leq n}(x_j - x_i)$ and is closed under the action of the operators ∂_{x_i} for all $i \in \{1, \dots, n\}$.

Haiman conjectured the following [57, Conjecture 5.1.1], which was later proven as a consequence of his work in [59]. Here let $H_n^{(2,0)}$ use $x_1, \dots, x_n$ as the first set of variables and $y_1, \dots, y_n$ as the second.

Theorem 7.3.1. *The diagonal harmonic space $H_n^{(2,0)}$ is the smallest space which:*

- *contains the Vandermonde determinant $\prod_{1\leq i<j\leq n}(x_j - x_i)$;*
- *is closed under the action of the operators*

$$E_p = \sum_{i=1}^{n} y_i \partial_{x_i}^p \tag{7.30}$$

for all $p \in \{1, \dots, n-1\}$;

- *is closed under the action of the operators ∂_{x_i} for all $i \in \{1, \dots, n\}$.*

There is also an analogue for the super harmonic space $H_n^{(1,1)}$, which was conjectured by [97].

Theorem 7.3.2 ([88]). *The harmonic space $H_n^{(1,1)}$ is the smallest space which:*

- *contains the Vandermonde determinant $\prod_{1 \le i < j \le n}(x_j - x_i)$;*
- *is closed under the action of the operators d_p which are defined by*

$$d_p(f) = \sum_{i=1}^{n} \partial_{x_i}^p (f) \theta_i \tag{7.31}$$

  for all $p \in \{1, \dots, n-1\}$;
- *is closed under the action of the operators ∂_{x_i} for all $i \in \{1, \dots, n\}$.*

For $1 \le a, b \le k$ and $1 \le c, d \le j$, define operators, which generalize those from equation (7.6) by:

$$E_{a,b}^{(p)} = \sum_{i=1}^{n} x_i^{(a)} \partial^p_{x_i^{(b)}}, \ E_{a,d'} = \sum_{i=1}^{n} x_i^{(a)} \partial_{\theta_i^{(d)}}, \ E_{c',b}^{(p)} = \sum_{i=1}^{n} \theta_i^{(c)} \partial^p_{x_i^{(b)}}, \ E_{c',d'} = \sum_{i=1}^{n} \theta_i^{(c)} \partial_{\theta_i^{(d)}}, \tag{7.32}$$

Some of the operators are partial inverses to each other, or up to a scalar can be obtained by composing operators, so consider a subset of them:

$$\begin{aligned} E_{2,1}^{(p)} &= \sum_{i=1}^{n} x_i^{(2)} \partial^p_{x_i^{(1)}}, \dots, E_{k,k-1}^{(p)} = \sum_{i=1}^{n} x_i^{(k)} \partial^p_{x_i^{(k-1)}}, \\ E_{1',k}^{(p)} &= \sum_{i=1}^{n} \theta_i^{(1)} \partial^p_{x_i^{(k)}}, \\ E_{2',1'} &= \sum_{i=1}^{n} \theta_i^{(2)} \partial_{\theta_i^{(1)}}, \dots, E_{j',j-1'} = \sum_{i=1}^{n} \theta_i^{(j)} \partial_{\theta_i^{(j-1)}}. \end{aligned} \tag{7.33}$$

It would be interesting to know for which choices of k, j the following holds.

Problem 7.3.3 (Higher operator conjecture)**.** Let $k \ge 1$ and $j \ge 0$. Determine if the harmonic space $H_n^{(k,j)}$ is the smallest space which:

- contains the Vandermonde determinant $\prod_{1 \le i < j \le n}(x_j - x_i)$;
- is closed under the action of the operators given in equation (7.33) for all $p \in \{1, \dots, n-1\}$;

- is closed under the action of the operators ∂_{x_i} for all $i \in \{1, \ldots, n\}$.

We have checked that it holds for $(k, j) = (k, 0)$ for $k \leq 6$ and $n \leq 4$. Generalizing in a different direction, it is known that the analogue of the operator conjecture for type B is slightly incorrect when $(k, j) = (2, 0)$ [57].

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