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Abstract

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The main theme of this thesis is to develop de Finetti type theorems in noncommutative probability. In noncommutative area, there are independence relations other than classical independence, e.g. Voiculescu's free independence, Boolean independence and Muraki's monotone independence. Free analogues of de Finetti type theorems were discovered by Köstler and Speicher and were developed by Banica, Curran and Speicher. Here, we will define noncommutative distributional symmetries for Boolean and monotone independence and we will prove de Finetti type theorems for them. These distributional symmetries are defined via coactions of quantum structures including Woronowicz C^*-algebra and Sołtan's quantum families of maps. We show that the joint distribution of an infinite sequence of noncommutative random variables satisfies boolean exchangeability is equivalent to the fact that the sequence of the random variables is identically distributed and boolean independent with respect to the conditional expectation onto its tail algebra. Then, we define noncommutative versions of spreadability and show Ryll-Nardzewski type theorems for monotone independence and boolean independence. We will show that, roughly speaking, an infinite bilateral sequence of random variables is monotonically(boolean) spreadable if and only if the variables are identically distributed and monotone(boolean) with respect to the conditional expectation onto its tail algebra. In the end of this thesis, we will prove general de Finetti theorems for classical, free and boolean independence. Our general de Finetti theorems work for non-easy quantum groups, which generalizes a recent work of Banica, Curran and Speicher. For infinite sequences, we determine maximal distributional symmetries which means the corresponding de Finetti theorem fails if the sequence satisfies more symmetries other than the maximal one.

To My Family

I love you all.

Contents

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Chapter 1

Introduction

In classical probability, the study of random variables with probabilistic symmetries was started by the pioneering work of de Finetti on 2-point valued random variables. One of the most general versions of de Finetti's work states that an infinite sequence of random variables, whose joint distribution is invariant under all finite permutations, is conditionally independent and identically distributed. One can see e.g. [20] for an exposition on the classical de Finetti theorem for more details. Also, see [18], Hewitt and Savage considered the probabilistic symmetries of random variables which are distributed on $X = E \times E \times E \times \cdots$, where E is a compact Hausdorff space. Later, in [36], an early noncommutative version of de Finetti theorem was given by Størmer. His work focused on exchangeable states on the infinite reduced tensor product of C^*-algebras. Roughly speaking, in noncommutative probability, Størmer studied symmetric states on commuting noncommutative random variables. Recently, in [22], without the commuting relation, Köstler studied exchangeable sequences of noncommutative random variables in W^*-probability spaces with normal faithful states. In classical probability, if the second moment of a real valued random variable is 0, then the random variable is 0 a.e.. Faithfulness is a natural generalization of this property in noncommutative probability, readers are refered to [38]. Köstler showed that exchangeable sequences of random variables possess some kind of factorization property, but the exchangeability does not imply any kind of universal relation. In other words, we can not expect to determine mixed moments of an exchangeable sequence of random variables in Speicher's universal sense [34]. By strengthening "exchangeability" to invariance under certain coactions of the free quantum permutations, in [23], Köstler and Speicher discovered that the de Finetti theorem has a natural analogue in Voiculescu's free probability theory(see [38]). Here, free quantum permutations refer to Wang's quantum groups $\mathcal{A}_s(n)$ in [41].

Köstler and Speicher's work starts a systematic study of the probabilistic symmetries on noncommutative probability theory. Most of the further projects are developed by Banica, Curran and Speicher, see [2], [9], [8]. They showed their de Finetti type theorems in both of the classical(commutative) probability theory and the noncommutative probability theory under the invariance conditions of easy groups and easy quantum groups, respectively. All these works in noncommutative case were proceeded under the assumption that the state of a probability space is faithful. This is a natural assumption in free probability theory, because in [11], Dykema showed that the free product of a family of W^*-probability spaces with normal faithful states is also a W^*-probability space with a normal faithful state. Thus the category of W^*-probability spaces with faithful states is closed under the free product construction. Since a normal state on W^*-probability space is not necessarily faithful, one may need to consider what happens to probability spaces with states which are not faithful. More specific, what are de Finetti type theorems for noncommutative probability spaces with normal states which are not necessarily faithful?

Recall that in the noncommutative realm, besides the freeness and the classical independence, there are many other kinds of independence relations, e.g. monotone independence [27], boolean independence [35], type B independence [4] and more recently two-face freeness for pairs of random variables [39]. All these types of independence are associated with certain products on probability spaces. Among these products, in [34], Speicher showed that there are only two universal products on the unital noncommutative probability spaces, namely the tensor product and the free product. The corresponding independent relations associated with these two universal products are the classical independence and the free independence. It was also shown in [34] that there is a unique universal product in the non-unital framework which is called boolean product. This non-unital universal product provides a way to construct probability spaces with non-faithful states from probability spaces with faithful states. By modifying the faithfulness, we will consider a more general noncommutative probability space which is a noncommutative probability space with a non-degenerated state. We would expect that boolean independence plays the same role in noncommutative probability spaces with non-degenerated states as the classical independence and the freeness play in commutative probability spaces and noncommutative probability spaces with faithful states, respectively. Then, we will prove our de Finetti type theorem for boolean independence.

On the other hand, compared with exchangeability, there is a weaker condition of spreadability: $(\xi_1, ..., \xi_n)$ is said to be spreadable if for any $k < n$, we have

$$(\xi_1, ..., \xi_k) \overset{d}{=} (\xi_{l_1}, ..., \xi_{l_k}), \quad \forall 1 \leq l_1 < l_2 < \cdots < l_k \leq n.$$

An infinite sequence of random variables is said to be spreadable if all its finite subsequences have this property. In [30], Ryll-Nardzewski showed that de Finetti theorem hold under the weaker condition of spreadability. Therefore, for infinite sequences of random variables in classical probability, spreadability is equivalent to exchangeability. The second purpose of this thesis is to study noncommutative versions of spreadability and extended de Finetti type theorems associated with them.

Some other objects come into our consideration when we study spreadable sequences of random objects. It was shown in [27], there are two other universal products in noncommutative probability if people do not require the universal construction to be commutative. We call the two universal products monotone and anti-monotone product. As tensor product, free product and boolean product, we can define monotone and anti-monotone independence associated with monotone and anti-monotone product. Monotone independence and anti-monotone independence are essentially the same but with different orders, i.e. if a is monotone with b, then b is anti-monotone with a. For more details on monotone independence, the reader is referred to [26], [29]. It is well known that a sequence of monotone random variables is not exchangeable but spreadable. Therefore, there should be a noncommutative spreadability which can characterize conditionally monotone independence.

We will define noncommutative distributional symmetries in analogue with spreadability and partial exchangeability. Recall that in [2] [7], noncommutative distributional symmetries are defined via invariance conditions associated with certain quantum structures. For instance, Curran's quantum spreadability is described by a family of quantum increasing sequences and their quantum family of maps in sense of Soltan. The family of quantum increasing sequences are universal C^*-algebras $A_i(n,k)$ generated by the entries of a $n \times k$ matrix which satisfy certain relations R. Following the idea in [**Liu1**], to construct a boolean type of spaces of increasing sequences $B_i(n,k)$, we replace the unit partition condition in R by an invariant projection condition. Recall that in In [14], Franz studied relations between freeness, monotone independence and boolean independence via Bożejko, Marek and Speicher's two-state free products[5]. In his construction, monotone product is something "between" free product and boolean product. Thereby, we construct the noncommutative spreadability for monotone independence by modifying quantum spreadability and our boolean spreadability. We will study simple relations between those distributional symmetries, i.e. which one is stronger.

As the situation for boolean independence, there is no nontrivial pair of monotonically independent random variables in W^*-probability spaces with faithful states. Therefore, the framework we use in this paper is a W^*-probability space with a non-degenerated normal state which gives a faithful GNS representation of the probability space. In this framework, we will see that spreadability is too weak to ensure the existence of a conditional expectation. Recall that, in W^*-probability spaces with faithful states, we can define a normal shift on a unilateral infinite sequence of spreadable random variables. Here, "unilateral"means the sequence is indexed by natural numbers $\mathbb{N}$. An important property of this shift is that its norm is one. Therefore, given an operator, we can construct a WOT convergent sequence of bounded variables via shifts. This is the key step to construct a normal conditional expectation in previous works. But, in W^*-probability spaces with non-degenerated normal states, the unilateral shift of spreadable random variables is not necessarily norm one. An example is provided in the beginning of section 5.2. Actually, the sequence of random variables are monotonically spreadable which is an invariance condition stronger than classical spreadability. Therefore, we can not construct a conditional expectation, for unilateral sequences, via shifts under the condition of spreadability. To fix this issue, we will consider bilateral sequences of random variables instead of unilateral sequences. "bilateral"means that the sequences are indexed by integers $\mathbb{Z}$. In this framework, we will see that the shift of spreadable random variables is norm one so that we can define a conditional expectation via shifts by following Köstler's construction. Notice that the index set $\mathbb{Z}$ has two infinities, i.e. the positive infinity and the negative infinity. Therefore, we will have two tail algebras with respect to the two infinities and will define two conditional expectations consequently. We denote by E^+ the conditional expectation which shifts indices to the positive infinity and E^- the conditional expectation which shifts indices to negative infinity. We will see that the two tail algebras are subsets of fixed points of the shift and the conditional expectations may not be extended normally to the whole algebra. In general, the two tail algebras are different and the conditional expectation may have different properties. Then, we will prove Ryll-Nardzewski type theorems for boolean and monotone independence.

In [15], Freedman considered rotatable random variables and showed a de Finetti type theorem which characterizes conditional central limit law. Recall that exchangeability and rotatability are classical symmetries associate with permutation groups and orthogonal groups. The quantum analogue of permutation and orthogonal groups were given by Wang in [40, 41]. Following the idea of free de Finetti theorem, in [9], Curran proved a free analogue of Freedman's work on quantum rotatability that an infinite sequence of noncommutative random variables are invariant under quantum orthogonal groups is equivalent to the fact that the random variables satisfy operator-valued free central limit law(semicircular) and free with respect to the conditional expectation onto their tail algebra. Later, in [5], both classical symmetries and quantum symmetries are studied in the "easiness"formalism. Roughly speaking, those structures are quantum groups associated tensor categories of partitions. For each n, it was shown that there are six easy groups which are denoted by S_n, O_n, B_n, H_n, B'_n, S'_n. We will denote the algebras of continuous functions on these groups by $C_s(n)$, $C_o(n)$, $C_b(n)$, $C_h(n)$, $C_{b'}(n)$, $C_{s'}(n)$, respectively. In the quantum aspect, for each n, together with the work of Weber [42], there are seven easy quantum groups which are denoted by $A_s(n)$, $A_o(n)$, $A_b(n)$, $A_h(n)$, $A_{s'}(n)$, $A_{b'}(n)$, $A_{b\#}(n)$. All these algebras are generated by n^2 matrix coordinates $u_{i,j}$'s which satisfy certain relation R. The relations R for $C_*(n)$ and $A_*(n)$ are suitable such that all these algebras are Hopf algebras in the sense of Woronowicz[43]. The distributional symmetries associated with Woronowicz's are defined via coactions of quantum groups on noncommutative polynomials in the sense of Sołtan [31]. Among these symmetries, in [2], Banica, Curran and Speicher studied de Finetti theorems for $C_s(n)$, $C_o(n)$, $C_b(n)$, $C_h(n)$ and $A_s(n)$, $A_o(n)$, $A_b(n)$, $A_h(n)$. In short, these symmetries can characterize independence relations which are classical or free, and can characterize some special distributions which are symmetric, shifted central limit and centered central limit laws. One goal of this paper is to study de Finetti theorems for all compact quantum groups, for classical and free independence, which are either between $C_s(n)$ and $C_o(n)$ or between $A_s(n)$ and $A_o(n)$.

Recall that Ryll-Nardzewski's theorem holds under the weaker condition of spreadability. Therefore, for infinite sequences of random variables, different symmetries may characterize a same property. Another goal of this paper is to determine that under what conditions the symmetries characterize a same property for infinite sequences. In our compact quantum group framework, we will show that there is no characterization other than what $C_s(n)$, $C_o(n)$, $C_b(n)$, $C_h(n)$ and $A_s(n)$, $A_o(n)$, $A_b(n)$, $A_h(n)$ can characterize. On the other hand, we will show that these symmetries are maximal which means the corresponding de Finetti theorem fails if a sequence satisfies more symmetries other than a maximal one.

In [34, 35], it was shown that there is a unique non-unital independence, which is called boolean independence, in noncommutative probability. The study of distributional symmetries for boolean independence was started in [25]. We constructed a family of quantum semigroups in analogue of Wang's quantum permutation groups and defined their coactions on joint distributions of sequences. It was shown that the distributional symmetries associated those coactions can be used to characterize boolean independence in a proper framework. In a recent work of Hayase [17], by following the idea of Banica and Speicher, many distributional symmetries related to boolean independence were constructed via the category of interval partitions. By using those distributional symmetries, Haysase find de Finetti theorems for a boolean analogue of easy quantum groups. In this paper, we will defined quantum semigroups, which are related to boolean independence in analogue of easy quantum groups via some universal conditions, $B_s(n)$, $B_o(n)$, $B_b(n)$, $B_h(n)$, $B_{s'}(n)$, $B_{b'}(n)$. Our quantum semigroups are quotient algebras of Hayase's. We do not have maximal distributional symmetries for boolean independence, but we provide a way to check de Finetti theorems for some quantum semigroups other than these universal ones. The last topic of the thesis is to prove a general de Finetti type theorem for all classical, free and boolean independence.

Chapter 2

Preliminaries and Notation

2.1 Notation in noncommutative probability

In this section, we recall some necessary definitions and notation in noncommutative probability. For further details, see texts [23], [28], [3], [38].

Definition 2.1.1. A non-commutative probability space $(\mathcal{A}, \phi)$ consists of a unital algebra $\mathcal{A}$ and a linear functional $\phi : \mathcal{A} \to \mathbb{C}$. $(\mathcal{A}, \phi)$ is called a $*$-probability space if $\mathcal{A}$ is a $*$-algebra and $\phi(xx^*) \geq 0$ for all $x \in \mathcal{A}$. $(\mathcal{A}, \phi)$ is called a W^*-probability space if $\mathcal{A}$ is a W^*-algebra and ϕ is a normal state on it. We say $(\mathcal{A}, \phi)$ is tracial if

$$\phi(xy) = \phi(yx), \quad \forall x, y \in \mathcal{A}.$$

The elements of $\mathcal{A}$ are called random variables. Let $x \in \mathcal{A}$ be a random variable, the distribution of x is a linear functional μ_x on $\mathbb{C}[X]$ such that $\mu_x(P) = \phi(P(x))$ for all $P \in \mathbb{C}[x]$, where $\mathbb{C}[x]$ is the set of complex polynomials in one variable.

Note that we do not require the state on W^*-probability space to be tracial. We will specify the probability spaces we are concerned with in section 4.1 and section 5.2.

Definition 2.1.2. Let I be an index set. The algebra of noncommutative polynomials in $|I|$ variables, $\mathbb{C}\langle X_i | i \in I\rangle$, is the linear span of 1 and noncommutative monomials of the form $X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_n}^{k_n}$ with $i_1 \neq i_2 \neq \cdots \neq i_n \in I$ and all k_j's are positive integers. For convenience, we use $\mathbb{C}\langle X_i | i \in I\rangle_0$ to denote the set of noncommutative polynomials without a constant term.

Let $(x_i)_{i \in I}$ be a family of random variables in a noncommutative probability space $(\mathcal{A}, \phi)$. Their joint distribution is a linear functional $\mu : \mathbb{C}\langle X_i | i \in I\rangle \to \mathbb{C}$ defined by

$$\mu(X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_n}^{k_n}) = \phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}),$$

and $\mu(1) = 1$.

Remark 2.1.3. In general, the joint distribution depends on the order of the random variables. For example, let $I = \{1, 2\}$, then μ_{x_1,x_2} may not equal μ_{x_2,x_1}. According to our notation, $\mu_{x_1,x_2}(X_1X_2) = \phi(x_1x_2)$, but $\mu_{x_2,x_1}(X_1X_2) = \phi(x_2x_1)$.

Definition 2.1.4. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. A family of unital subalgebras $(\mathcal{A}_i)_{i\in I}$ is said to be free if

$$\phi(a_1 \cdots a_n) = 0,$$

whenever $a_k \in \mathcal{A}_{i_k}$, $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\phi(a_k) = 0$ for all k. Let $(x_i)_{i\in I}$ be a family of random variables and $\mathcal{A}_i$'s be the unital subalgebras generated by x_i's, respectively. We say the family of random variables $(x_i)_{i\in I}$ is free if the family of unital subalgebras $(\mathcal{A}_i)_{i\in I}$ is free.

Definition 2.1.5. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. A family of (not necessarily unital) subalgebras $\{\mathcal{A}_i | i \in I\}$ of $\mathcal{A}$ are said to be boolean independent if

$$\phi(x_1x_2 \cdots x_n) = \phi(x_1)\phi(x_2) \cdots \phi(x_n)$$

whenever $x_k \in \mathcal{A}_{i_k}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$. The family of subalgebras $\{\mathcal{A}_i | i \in I\}$ are said to be monotonically independent if

$$\phi(x_1 \cdots x_{k-1}x_kx_{k+1} \cdots x_n) = \phi(x_k)\phi(x_1 \cdots x_{k-1}x_{k+1} \cdots x_n)$$

whenever $x_j \in \mathcal{A}_{i_j}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$ and $i_{k-1} < i_k > i_{k+1}$. A set of random variables $\{x_i \in \mathcal{A} | i \in I\}$ are said to be boolean(monotonically) independent if the family of non-unital subalgebras $\mathcal{A}_i$, which are generated by x_i's respectively, is boolean(monotonically) independent.

One refers to [13] for more details on boolean product of random variables. Since the framework for boolean independence is a non-unital algebra in general, we will not require our operator valued probability spaces to be unital:

Definition 2.1.6. An operator valued probability space $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ consists of an algebra $\mathcal{A}$, a subalgebra $\mathcal{B}$ of $\mathcal{A}$ and a $\mathcal{B} - \mathcal{B}$ bimodule linear map $E : \mathcal{A} \to \mathcal{B}$ i.e.

$$E[b_1ab_2] = b_1E[a]b_2, \quad E[b] = b$$

for all $b_1, b_2, b \in \mathcal{B}$ and $a \in \mathcal{A}$. According to the definition in [37], we call E a conditional expectation from $\mathcal{A}$ to $\mathcal{B}$ if E is onto, i.e. $E[\mathcal{A}] = \mathcal{B}$. The elements of $\mathcal{A}$ are called random variables.

In operator valued free probability theory, $\mathcal{A}$ and $\mathcal{B}$ are unital and have the same unit

Definition 2.1.7. Given an algebra $\mathcal{B}$, we denote by $\mathcal{B}\langle X\rangle$ the algebra which is freely generated by $\mathcal{B}$ and the indeterminant X. Let 1_X be the identity of $\mathbb{C}\langle X\rangle$, then $\mathcal{B}\langle X\rangle$ is set of linear combinations of the elements in $\mathcal{B}$ and the noncommutative monomials $b_0Xb_1Xb_2\cdots b_{n-1}Xb_n$ where $b_k\in\mathcal{B}\cup\{\mathbb{C}1_X\}$ and $n\geq 0$. The elements in $\mathcal{B}\langle X\rangle$ are called $\mathcal{B}$-polynomials. In addition, $\mathcal{B}\langle X\rangle_0$ denotes the subalgebra of $\mathcal{B}\langle X\rangle$ which does not contain a constant term i.e. the linear span of the noncommutative monomials $b_0Xb_1Xb_2\cdots b_{n-1}Xb_n$ where $b_k\in\mathcal{B}\cup\{\mathbb{C}1_X\}$ and $n\geq 1$.

Definition 2.1.8. Given an operator valued probability space $(\mathcal{A},\mathcal{B},E:\mathcal{A}\to\mathcal{B})$ such that $\mathcal{A}$ and $\mathcal{B}$ are unital. A family of unital subalgebras $\{\mathcal{A}_i\supset\mathcal{B}\}_{i\in I}$ is said to be freely independent with respect to E if

$$E[a_1\cdots a_n]=0,$$

whenever $i_1\neq i_2\neq\cdots\neq i_n$, $a_k\in\mathcal{A}_{i_k}$ and $E[a_k]=0$ for all k. A family of $(x_i)_{i\in I}$ is said to be freely independent over $\mathcal{B}$, if the unital subalgebras $\{\mathcal{A}_i\}_{i\in I}$ which are generated by x_i and B respectively are free, or equivalently

$$E[p_1(x_{i_1})p_2(x_{i_2})\cdots p_n(x_{i_n})]=0,$$

whenever $i_1\neq i_2\neq\cdots\neq i_n$, $p_1,...,p_n\in\mathcal{B}\langle X\rangle$ and $E[p_k(x_{i_k})]=0$ for all k.
Let $\{x_i\}_{i\in I}$ be a family of random variables in an operator valued probability space $(\mathcal{A},\mathcal{B},E:\mathcal{A}\to\mathcal{B})$. $\mathcal{A}$, $\mathcal{B}$ are not necessarily unital. $\{x_i\}_{i\in I}$ is said to be boolean independent over $\mathcal{B}$ if for all $i_1,...,i_n\in I$, with $i_1\neq i_2\neq\cdots\neq i_n$ and all $\mathcal{B}$-valued polynomials $p_1,...,p_n\in\mathcal{B}\langle X\rangle_0$ we have

$$E[p_1(x_{i_1})p_2(x_{i_2})\cdots p_n(x_{i_n})]=E[p_1(x_{i_1})]E[p_2(x_{i_2})]\cdots E[p_n(x_{i_n})].$$

$\{x_i\}_{i\in I}$ are said to be monotonically independent over $\mathcal{B}$ if

$$\begin{aligned}&E[p_1(x_{i_1})\cdots p_{k-1}(x_{i_{k-1}})p_k(x_{i_k})p_{k+1}(x_{i_{k+1}})\cdots p_n(x_{i_n})]\\ =\;&E[p_1(x_{i_1})\cdots p_{k-1}(x_{i_{k-1}})E[p_k(x_{i_k})]p_{k+1}(x_{i_{k+1}})\cdots p_n(x_{i_n})]\end{aligned},$$

whenever $i_1,\cdots,i_n\in I$, $i_1\neq i_2\neq\cdots\neq i_n$, $i_{k-1}<i_k>i_{k+1}$ and $p_1,\cdots,p_n\in\mathcal{B}\langle X\rangle_0$.

2.2 Combinatorics in noncommutative probability

All these three independence relations have rich combinatorial theories which we will recall in the follows. One can see [1, 24, 33] for details.

Definition 2.2.1. Let S be an ordered set:

1. A partition π of a set S is a collection of disjoint, nonempty sets $V_1,...,V_r$ such that the union of them is S. $V_1,...,V_r$ are blocks of π. The collection of all partitions of S will be denoted by $P(S)$.

2. Given two partitions π, σ, we say $\pi \leq \sigma$ if each block of π is contained in a block of σ.

3. A partition $\pi \in P(S)$ is noncrossing if there is no quadruple (s_1, s_2, r_1, r_2) such that $s_1 < r_1 < s_2 < r_2$, $s_1, s_2 \in V$, $r_1, r_2 \in W$ and V, W are two different blocks of π.

4. A partition $\pi \in P(S)$ is interval if there is no triple (s_1, s_2, r) such that $s_1 < r < s_2$, $s_1, s_2 \in V$, $r \in W$ and V, W are two different blocks of π.

5. Let $\mathbf{i} = (i_1, ..., i_k)$ be a sequence of indices of I and $[k] = \{1, ..., k\}$. We denote by ker $\mathbf{i}$ the element of $P([k])$ whose blocks are the equivalence classes of the relation

$$s \sim t \Leftrightarrow i_s = i_t$$

Remark 2.2.2. In this paper, we are interested in $S = \{1, ..., k\}$ for some $k \in \mathbb{N}$. It is easy to see that interval partitions are noncrossing.

Definition 2.2.3. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator valued probability space:

1. A $\mathcal{B}$-functional is a n-linear map $\rho : \mathcal{A}^n \to \mathcal{B}$ such that

$$\rho(b_0 a_1 b_1, a_2 b_2, ..., a_n b_n) = b_0 \rho(a_1, b_1 a_2, ..., b_{n-1} a_n) b_n$$

for all $b_0, ..., b_n \in \mathcal{B} \cup \{1_A\}$.

2. For $k \in \mathbb{N}$, let $\rho^{(k)}$ be a $\mathcal{B}$-functional from $\mathcal{A}^k$ to $\mathcal{B}$.

3. If $\mathcal{B}$ is commutative. Given $\pi \in P(n)$, we define a $\mathcal{B}$-functional $\rho^{(\pi)} : \mathcal{A}^n \to \mathcal{B}$ by the formula:

$$\rho^{(\pi)}(a_1, ..., a_n) = \prod_{V \in \pi} \rho(V)(a_1, ..., a_n),$$

where if $V = (i_1 < i_2 < \cdots < i_s)$ is a block of π then

$$\rho(V)(a_1, ..., a_n) = \rho^{(s)}(a_{i_1}, ..., a_{i_s}).$$

4. Given $\pi \in NC(n)$, then a $\rho^{(\pi)} : \mathcal{A}^n \to \mathcal{B}$ can be defined recursively as follows:

$$\rho^{(\pi)}(a_1, ..., a_n) = \rho^{(\pi \setminus V)}(a_1, ..., a_l \rho^{(s)}(a_{l+1, ..., a_{l+s}}), a_{l+s+1}, ..., a_n)$$

where $V = (l+1, l+2, ..., l+s)$ is an interval block of π.

Remark 2.2.4. If $\mathcal{B}$ is noncommutative, there is no natural way to compute $\rho^{(\pi)}(a_1, ..., a_n)$ for $\pi \notin NC(n)$.

Definition 2.2.5. Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space:

1. If $\mathcal{A}$ is commutative, then the operator-valued classical cumulants $c_E^{(n)} : \mathcal{A}^n \to \mathcal{B}$ are defined by the classical moment-cumulant formula:

$$E[a_1 \cdots a_n] = \sum_{\pi \in P(n)} c_E^{(\pi)}(a_1, ..., a_n),$$

   for all $a_1, ..., a_n \in \mathcal{A}$.

2. The operator-valued free cumulants $\kappa_E^{(k)} : \mathcal{A}^n \to \mathcal{B}$ are defined by the free moment-cumulant formula:

$$E[a_1 \cdots a_n] = \sum_{\pi \in NC(n)} \kappa_E^{(\pi)}(a_1, ..., a_n),$$

   for all $a_1, ..., a_n \in \mathcal{A}$.

3. The operator-valued boolean cumulants $b_E^{(k)} : \mathcal{A}^n \to \mathcal{B}$ are defined by the boolean moment-cumulant formula:

$$E[a_1 \cdots a_n] = \sum_{\pi \in I(n)} b_E^{(\pi)}(a_1, ..., a_n),$$

   for all $a_1, ..., a_n \in \mathcal{A}$.

Note that all these three types of cumulants can be resolved recursively, e.g.

$$c_E^{(1)}(a_1) = E[a_1]$$

and

$$c_E^{(n)}(a_1, ..., a_n) = E[a_1 \cdots a_n] - \sum_{\pi \in P(n), \pi \neq 1_n} c_E^{(\pi)}(a_1, ..., a_n),$$

where $c_E^{(\pi)}(a_1, ..., a_n)$ depends on $c_E^{(k)}(a_1, ..., a_n)$ for $k = 1, ..., n-1$ if $\pi \neq 1_n$. The same, to determine $\kappa_E^{(n)}$ and $b_E^{(n)}$ we just need to replace $P(n)$ by $NC(n)$ and $I(n)$, respectively.

Theorem 2.2.6. *Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space and $(x_i)_{i \in I}$ be a family of random variables in $\mathcal{A}$:*

1. *If $\mathcal{A}$ is is commutative, then $(x_i)_{i \in I}$ are conditionally independent with respect to E iff*

$$c_E^{(n)}(b_0 x_{i_1} b_1, ..., x_{i_n} b_n) = 0,$$

   whenever $i_k \neq i_l$ for some $1 \leq k, l \leq n$.

2. *$(x_i)_{i \in I}$ are free independent with respect to E iff*

$$\kappa_E^{(n)}(b_0 x_{i_1} b_1, ..., x_{i_n} b_n) = 0,$$

   whenever $i_k \neq i_l$ for some $1 \leq k, l \leq n$.

3. $(x_i)_{i\in I}$ *are boolean independent with respect to* E *iff*

$$b_E^{(n)}(b_0x_{i_1}b_1,...,x_{i_n}b_n)=0,$$

whenever $i_k\neq i_l$ *for some* $1\leq k,l\leq n$*.*

Proof. The classical case is well know, the free case is due to Speicher and the scalar boolean case is due to Lehner. For completeness, we provide a sketch of proof to operator-valued boolean case:

If $i_k\neq i_l$ for some $1\leq k,l\leq n$, then there exists l such that $i_l\neq i_{l+1}$. Therefore, we have

$$\begin{aligned}\sum_{\pi\in I(n)} b_E^{(\pi)}(x_{i_1}b_1,...,x_{i_n}b_n) &= E[x_{i_1}b_1,...,x_{i_n}b_n]\\ &= E[x_{i_1}b_1,...,x_{i_l}b_l]E[x_{i_{l+1}}b_{l+1},...,x_{i_n}b_n]\\ &= \sum_{\pi_1\in I(l)} b_E^{(\pi_1)}(x_{i_1}b_1,...,x_{i_l}b_l)\sum_{\pi_2\in I(n-l)} b_E^{(\pi_2)}(x_{i_{l+1}}b_{l+1},...,x_{i_n}b_n)\end{aligned}$$

We see that the coefficient of $b_E^{(n}(x_{i_1}b_1,...,x_{i_n}b_n)$ on the right is 0 which implies that $b_E^{(n}(x_{i_1}b_1,...,x_{i_n}b_n)=$ 0 and vice verse.

□

Definition 2.2.7. Let $(\mathcal{A},\mathcal{B},E:\mathcal{A}\to\mathcal{B})$ be an operator-valued probability space. Two random variables $x_1,x_2\in\mathcal{A}$ are said to be conditionally(free, boolean) i.i.d. respect to E if they are conditionally(free, boolean) independent and have a same distribution. Suppose $x_1,x_2\in\mathcal{A}$ are conditionally(free boolean) i.i.d. x_1 is said to be symmetric if x_1 and $-x_1$ have a same distribution. x_1 is said to be Gaussian (semicircular, Bernoulli) distributed if x_1 and $\alpha x_1+\beta x_2$ have a same distribution whenever α,β are real numbers such that $\alpha^2+\beta^2=1$. x_1 is shifted Gaussian (semicircular, Bernoulli) distributed if x_1-b is Gaussian (semicircular, Bernoulli) distributed for some $b\in\mathcal{B}$.

Remark 2.2.8. Gaussian (semicircular, Bernoulli) distribution in Definition2.2.7 is equivalent to the usual definition which is also equivalent to the following cumulants definition. In scalar case for free independence and classical independence, the tail algebra can be considered as the commutative algebra generated by the unit of the probability space. Therefore, the shifted constant commutes with random variables. Graphically, density functions of shifted scalar Gaussian(Semicircular) laws are density functions of centered Gaussian(Semicircular) laws translated by a constant. For example, the density function of the centered semicircular law with variance 1 is

$$\frac{1}{2\pi}\sqrt{4-x^2}$$

on $[-2,2]$, where the density function of shifted semicircular law with variance 1 are in the form

$$\frac{1}{2\pi}\sqrt{4-(x-a)^2}$$

on $[-2+a, 2+a]$. But, for boolean independence, the tail algebra does not necessarily contain the unit of the space. Therefore, the shifted constant may not commute with random variables. Graphically, density functions of shifted scalar Bernoulli laws are not simply density functions of centered Bernoulli laws translated by a constant. For example, the density function of the centered semicircular law with variance 1 is

$$1/2\delta_{-1} + 1/2\delta_1,$$

where the density function of shifted Bernoulli law are in the form

$$\frac{a\delta_a + b\delta_{-b}}{a+b}$$

for $a, b > 0$.

Theorem 2.2.9. Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and $(x_i)_{i\in I}$ be a family of random variables in $\mathcal{A}$:

1. If $\mathcal{A}$ is is commutative, then the $\mathcal{B}$-valued joint distribution of $(x_i)_{i\in I}$ has the property corresponding to D in the table below iff for any $\pi \in P(n)$.

$$c_E^{(\pi)}(b_0 x_{i_1} b_1, ..., x_{i_n} b_n) = 0,$$

unless $\pi \in D(n)$ and $\pi \leq ker\ \mathbf{i}$ where $\mathbf{i} = (i_1, ..., i_n)$.

Partitions D	**Joint distribution**
P: All partitions	Classical independent
P_h: Partitions with even block sizes	Classical independent and symmetric
P_b: Partitions with block size 1 or 2	Classical independent and Gaussian
P_2: Pair partitions	Classical independent and centered Gaussian

2. The $\mathcal{B}$-valued joint distribution of $(x_i)_{i\in I}$ has the property corresponding to D in the table below iff for any $\pi \in P(n)$.

$$\kappa_E^{(\pi)}(b_0 x_{i_1} b_1, ..., x_{i_n} b_n) = 0,$$

unless $\pi \in D(n)$ and $\pi \leq ker\ \mathbf{i}$.

Partitions D	**Joint distribution**
P: Noncrossing partitions	Free independent
P_h: Noncrossing Partitions with even block sizes	Free and symmetric
P_b: Noncrossing Partitions with block size 1 or 2	Free and semicircular
P_2: Noncrossing Pair partitions	Free and centered semicircular

3. The $\mathcal{B}$-valued joint distribution of $(x_i)_{i\in I}$ has the property corresponding to D in the table below iff for any $\pi \in P(n)$.

$$b_E^{(\pi)}(b_0 x_{i_1} b_1, ..., x_{i_n} b_n) = 0,$$

unless $\pi \in D(n)$ and $\pi \leq ker\ \mathbf{i}$.

Partitions D	**Joint distribution**
I: Interval partitions	Boolean independent
I_h: Interval partitions with even block sizes	Boolean and symmetric
I_b: Interval partitions with block size 1 or 2	Boolean and Bernoulli
I_2: Interval pair partitions	Boolean and centered Bernoulli

Proof. These results are well know for free case and classical case. For boolean case, one just need to follow the proof for free case and replace noncrossing partitions by interval partitions. □

Chapter 3

Distributional symmetries in noncommutative probability

3.1 Quantum Exchangeability

In [41], Wang introduced the following quantum groups $A_s(n)$'s.

Definition 3.1.1. $A_s(n)$ is defined as the universal unital C^*-algebra generated by elements $u_{i,j}$ $(i, j = 1, \cdots n)$ such that we have

- each $u_{i,j}$ is an orthogonal projection, i.e. $u_{i,j}^* = u_{i,j} = u_{i,j}^2$ for all $i, j = 1, ..., n$.
- the elements in each row and column of $u = (u_{i,j})_{i,j=1,...,n}$ form a partition of unit, i.e. are orthogonal and sum up to 1: for each $i = 1, \cdots, n$ and $k \neq l$ we have

$$u_{i,k}u_{i,l} = 0 \quad \text{and} \quad u_{k,i}u_{l,i} = 0;$$

  and for each $i = 1, \cdots, n$ we have

$$\sum_{k=1}^{n} u_{i,k} = 1 = \sum_{k=1}^{n} u_{k,i}.$$

$A_s(n)$ is a compact quantum group in the sense of Woronowicz [43], with comultiplication, counit and antipode given by the formulas:

$$\Delta u_{i,j} = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j}$$

$$\epsilon(u_{i,j}) = \delta_{i,j}$$

$$S(u_{i,j}) = u_{j,i}.$$

In [23], the right coaction of $A_s(n)$ on $\mathbb{C}\langle X_1,...,X_n\rangle$ is a linear map $\alpha : \mathbb{C}\langle X_1,...,X_n\rangle \to \mathbb{C}\langle X_1,...,X_n\rangle \otimes A_s(n)$ given by:

$$\alpha(X_{i_1}X_{i_2}\cdots X_{i_m}) = \sum_{j_1,\dots,j_m=1}^{n} X_{j_1}X_{j_2}\cdots X_{j_m} \otimes u_{j_1,i_1}u_{j_2,i_2}\cdots u_{j_m,i_m},$$

where $\otimes$ denotes the algebraic tensor product.

In the earlier papers, α is defined as an algebraic homomorphism. We put emphasis on the linearity here because we will define some coactions of our quantum semigroups on noncommutative polynomials in a similar way. The right coaction has the following property:

$$(\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha.$$

Let $(x_i)_{i\in\mathbb{N}}$ be an infinite sequence of random variables in a noncommutative probability space $(\mathcal{A}, \phi)$. $(x_i)_{i\in\mathbb{N}}$ is said to be quantum exchangeable if their joint distribution is invariant under Wang's quantum permutation groups, i.e. for all n, we have

$$\mu_{x_1,\dots x_n}(p)1_{A_s(n)} = \mu_{x_1,\dots,x_n} \otimes id_{A_s(n)}(\alpha(p)),$$

where $\mu_{x_1,\dots,x_n}$ is the joint distribution of $x_1,...,x_n$ with respect to ϕ and $p \in \mathbb{C}\langle X_1,...,X_n\rangle$. For example, if $p = X_{i_1}X_{i_2}\cdots X_{i_m}$, then the equation above can be written as:

$$\begin{aligned}
\phi(x_{i_1}x_{i_2}\cdots x_{i_m})1_{A_s(n)} &= \mu_{x_1,\dots x_n}((X_{i_1}X_{i_2}\cdots X_{i_m})1_{A_s(n)} \\
&= \mu_{x_1,\dots,x_n} \otimes id_{A_s(n)}(\sum_{j_1,\dots,j_m=1}^{n} X_{j_1}X_{j_2}\cdots X_{j_m} \otimes u_{j_1,i_1}u_{j_2,i_2}\cdots u_{j_m,i_m}) \\
&= \sum_{j_1,\dots,j_m=1}^{n} \phi(x_{j_1}x_{j_2}\cdots x_{j_m})u_{j_1,i_1}u_{j_2,i_2}\cdots u_{j_m,i_m}.
\end{aligned}$$

whenever $i_1,...,i_n \in \{1,...,n\}$.
Let S_n be the permutation group on $\{1,...,n\}$. The joint distribution of $(x_i)_{i\in\mathbb{N}}$ is said be exchangeable if for all n, $\sigma \in S_n$, we have

$$\mu_{x_1,\dots x_n} = \mu_{x_{\sigma(1)},\dots,x_{\sigma(n)}},$$

where $\mu_{x_1,\dots,x_n}$ is the joint distribution of $x_1,...,x_n$ with respect to ϕ . It was shown, in [23], that quantum exchangeability implies classical exchangeability.

3.2 Quantum semigroups $\mathcal{B}_s(n)$

In this section, we will introduce quantum semigroups $\mathcal{B}_s(n)$'s. Our probabilistic symmetries are described by linear coactions of $\mathcal{B}_s(n)$'s. First, we recall the related definitions and notation of quantum semigroups. Then, we will introduce our boolean quantum semigroups and their coactions on the joint distribution of random variables. In the end of this section, we will show that the invariance conditions associated with boolean quantum semigroups are stronger than the quantum exchangeability defined in [23].

A quantum space is an object of the category dual to the category of C^*-algebras([44]). For any C^*-algebras A and B, the set of morphisms Mor(A, B) consists of all C^*-algebra homomorphisms acting from A to $M(B)$, where $M(B)$ is the multiplier algebra of B, such that $\phi(A)B$ is dense in B. If A and B are unital C^*-algebras, then all unital C^*-homomorphisms from A to B are in Mor(A,B). In [32],

Definition 3.2.1. *By a quantum semigroup we mean a C^*-algebra $\mathcal{A}$ endowed with an additional structure described by a morphism $\Delta \in Mor(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ such that*

$$(\Delta \otimes id_{\mathcal{A}})\Delta = (id_{\mathcal{A}} \otimes \Delta)\Delta.$$

In other words, Δ defines a comultiplication on $\mathcal{A}$. Here the tensor product $\otimes$ denotes the minimal tensor product $\otimes_{min}$.

Before introducing $\mathcal{B}_s(n)$, we need to define a quantum semigroup $B_s(n)$ of which $\mathcal{B}_s(n)$ is a sub quantum semigroup. We need the help of $B_s(n)$ because it is easy to define a coassociative comultiplication on it. The definition of $B_s(n)$ is close to Wang's quantum group $A_s(n)$:

Quantum semigroup ($B_s(n)$, Δ): The algebra $B_s(n)$ is defined as the universal unital C^*-algebra generated by elements $u_{i,j}$ $(i, j = 1, \cdots n)$ and a projection $\mathbf{P}$ such that we have

- each $u_{i,j}$ is an orthogonal projection, i.e. $u_{i,j}^* = u_{i,j} = u_{i,j}^2$ for all $i, j = 1, \cdots, n$.
- $$u_{i,k}u_{i,l} = 0 \quad \text{and} \quad u_{k,i}u_{l,i} = 0,$$
  whenever $k \neq l$.
- For all $1 \leq i \leq n$, $\mathbf{P} = \sum_{k=1}^{n} u_{k,i}\ \mathbf{P}$.

We will denote the unit of $B_s(n)$ by I, the projection $\mathbf{P}$ is called the invariant projection of $B_s(n)$. On this unital C^*-algebra, we can define a unital C^*-homomorphism

$$\Delta : B_s(n) \to B_s(n) \otimes B_s(n)$$

by the following formulas:

$$\Delta u_{i,j} = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j}$$

and

$$\Delta\ \mathbf{P} = \mathbf{P} \otimes \mathbf{P}, \quad \Delta I = I \otimes I.$$

We will see that $(B_s(n), \Delta)$ is a quantum semigroup. To show this we need to check that Δ defines a unital C^*-homomorphism from $B_s(n)$ to $B_s(n) \otimes B_s(n)$ and satisfies the coassociative condition :
Because $u_{i,k}$, $u_{k,j}$ are orthogonal projections and $u_{i,k}u_{i,l} = 0$ if $k \neq l$, $\{u_{i,k} \otimes u_{k,j}\}_{k=1,\dots,n}$ is a set of orthogonal projections which are orthogonal to each other. Therefore, $\Delta u_{i,j} = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j}$ is an orthogonal projection. $\Delta \mathbf{P} = \mathbf{P} \otimes \mathbf{P}$ is a projection as well. Let $l \neq m$, then

$$\begin{aligned} \Delta(u_{i,l})\Delta u_{i,m} &= (\sum_{k=1}^{n} u_{i,k} \otimes u_{k,l})(\sum_{j=1}^{n} u_{i,j} \otimes u_{j,m}) \\ &= \sum_{k,j=1}^{n} u_{i,k}u_{i,j} \otimes u_{k,l}u_{j,m} \\ &= \sum_{k=1}^{n} u_{i,k} \otimes u_{k,l}u_{k,m} \\ &= 0. \end{aligned}$$

The same, we have $\Delta(u_{l,i})\Delta u_{m,i} = 0$, for $m \neq l$. Moreover, we have

$$\begin{aligned} \Delta(\sum_{l=1}^{n} u_{l,i})\Delta \mathbf{P} &= (\sum_{l,k=1}^{n} u_{l,k} \otimes u_{k,i}) \mathbf{P} \otimes \mathbf{P} \\ &= \sum_{l,k=1}^{n} u_{l,k} \mathbf{P} \otimes u_{k,i} \mathbf{P} \\ &= \sum_{k=1}^{n} \mathbf{P} \otimes u_{k,i} \mathbf{P} \\ &= \mathbf{P} \otimes \mathbf{P}. \end{aligned}$$

and Δ sends the unit of $B_s(n)$ to the unit of $B_s(n) \otimes B_s(n)$. Therefore, Δ defines a unital C^*-homomorphism on $B_s(n)$ by the universality of $B_s(n)$.

The coassociative condition holds, because on the generators we have:

$$(\Delta \otimes id_{\mathcal{A}})\Delta u_{i,j} = \sum_{k,l=1}^{n} u_{ik} \otimes u_{k,l} \otimes u_{l,j} = (id_{\mathcal{A}} \otimes \Delta)\Delta u_{i,j}$$

$$(\Delta \otimes id_{\mathcal{A}})\Delta \mathbf{P} = \mathbf{P} \otimes \mathbf{P} \otimes \mathbf{P} = (id_{\mathcal{A}} \otimes \Delta)\Delta \mathbf{P}$$

$$(\Delta \otimes id_{\mathcal{A}})\Delta I = I \otimes I \otimes I = (id_{\mathcal{A}} \otimes \Delta)\Delta I.$$

Therefore, $(B_s(n), \Delta)$ is a quantum semigroup.

Remark 3.2.2. If we let the invariant projection to be the identity, then we get Wang's free quantum permutation group. Therefore, $A_s(n)$ is a quotient C^*-algebra of $B_s(n)$, i.e. there exists a unital C^*-homomorphism $\beta : B_s(n) \to A_s(n)$ such that β is surjective.

The following two examples are nontrivial representations of $B_s(n)$'s:

Example 1. Let $\mathbb{C}^6$ be the standard 6-dimensional complex Hilbert space with orthonormal basis $v_1, ..., v_6$. Let

$$\begin{array}{lll} P_{11} = P_{v_1+v_2}, & P_{21} = P_{v_3+v_4}, & P_{13} = P_{v_5+v_6}, \\ P_{21} = P_{v_3+v_6}, & P_{22} = P_{v_5+v_2}, & P_{23} = P_{v_1+v_4}, \\ P_{31} = P_{v_4+v_5}, & P_{32} = P_{v_1+v_6}, & P_{33} = P_{v_2+v_3}. \end{array}$$

and $P = P_{v_1+v_2+v_3+v_4+v_5+v_6}$, where P_v denotes the one dimensional orthogonal projection onto the subspace spaned by v. Then the unital algebra generated by $P_{i,j}$ and P gives a representation π of $B_s(3)$ on $\mathbb{C}^6$ by the following formulas on the generators of $\mathcal{B}_s(3)$:

$$\pi(I) = I_{\mathbb{C}^6}, \ \ \pi(u_{i,j}) = P_{ij}, \ \ \pi(\mathbf{P}) = P.$$

π is well defined by the universality of $B_s(3)$. In this example, we see that sums of columns are different, e.g. $P_{11} + P_{21} + P_{31} \neq P_{12} + P_{22} + P_{32}$. Therefore, we can not construct $B_s(3)$ from $A_s(3)$ by simply adding a special projection $\mathbf{P}$. In general, $B_s(n)$ is not quit different from $\mathcal{A}_s(n)$ for all $n > 1$.

Example 2. Again,let $\mathbb{C}^6$ be the standard 6-dimensional complex Hilbert space with orthonormal basis $v_1, ..., v_6$. Let

$$\begin{array}{lll} P_{11} = P_{v_1+v_2}, & P_{12} = P_{v_4+v_5}, & P_{13} = P_{v_3+v_6}, \\ P_{21} = P_{v_3+v_6}, & P_{22} = P_{v_1+v_2}, & P_{23} = P_{v_4+v_5}, \\ P_{31} = P_{v_4+v_5}, & P_{32} = P_{v_3+v_6}, & P_{33} = P_{v_1+v_2}. \end{array}$$

and $P = P_{v_1+v_2+v_3+v_4+v_5+v_6}$, where P_v denotes the one dimensional orthogonal projection onto the subspace spaned by v. Then the unital algebra generated by $P_{i,j}$ and P gives a representation π of $B_s(3)$ on $\mathbb{C}^6$ by the following formulas on the generators of $\mathcal{B}_s(3)$:

$$\pi(I) = I_{\mathbb{C}^6}, \ \ \pi(u_{i,j}) = P_{ij}, \ \ \pi(\mathbf{P}) = P.$$

π is well defined by the universality of $B_s(3)$.
Moreover, the matrix form for $P_{1,1}$ and P with respect to the basis are

$$P_{11} = 1/2 \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{and } P = 1/6 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

then we have

$$PP_{11}P = 1/18 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = 1/3P.$$

In general,we have

Lemma 3.2.3. *Let $v_1, ..v_{2n}$ be an orthonormal basis of the standard $2n$-dimensional Hilbert space $\mathbb{C}^{2n}$, and let $v_k = v_{k+2n}$ for all $k \in \mathbb{Z}$, let*

$$P_{i,j} = P_{v_{2(i-j)+1}+v_{2(j-i)+2}},$$

where P_v is the orthogonal projection the one dimensional subspace generated by the vector v and $P = P_{v_1+v_2+\cdots+v_{2n}}$, $\mathbf{1}$ is the identity of $B(\mathbb{C}^{2n})$. Then $\{P_{i,j}\}_{i,j=1,\ldots,n}$, P and $\mathbf{1}$ satisfy the defining conditions of the algebra $B_s(n)$. In addition, $PP_{i,j}P = \frac{1}{n}P$ for all $i, j = 1, ..., n$.

Proof. It is easy to see that the inner product

$$\langle v_{2(i-j)+1} + v_{2(j-i)+2}, v_{2(i-k)+1} + v_{2(k-i)+2}\rangle = 2\delta_{j,k},$$

so $P_{ik}P_{ij} = 0$ if $j \neq k$. The same $P_{ki}P_{ji} = 0$ if $k \neq j$. Fix i, we see that $v_1 + v_2 + \cdots + v_{2n} \in$ *span*$\{v_{2(i-j)+1} + v_{2(j-i)+2} | j = 1, ...n\}$, so $\sum\limits_{k=1}^{n} P_{ik}P = P$. By a direct computation, we have

$$\langle PP_{i,j}P \sum_{i=1}^{2n} v_i, \sum_{i=1}^{2n} v_i\rangle = 2$$

and

$$\| \sum_{i=1}^{2n} v_i\| = 2n.$$

Since $PP_{i,j}P$ is a selfadjoint operator with rank 1 and $\sum\limits_{i=1}^{2n} v_i$ is in the range of $PP_{i,j}P$, we have

$$PP_{i,j}P = \frac{\langle PP_{i,j}P \sum\limits_{i=1}^{2n} v_i, \sum\limits_{i=1}^{2n} v_i\rangle}{\| \sum\limits_{i=1}^{2n} v_i\|} P = \frac{1}{n}P$$

The proof is complete. □

Therefore, by lemma 3.2.3, there exists a representation π of $B_s(n)$ on $\mathbb{C}^{2n}$ which is defined by the following formulas:

$$\pi(1_{B_s(n)}) = \mathbf{1}, \quad \pi(\mathbf{P}) = P$$

and

$$\pi(u_{i,j}) = P_{i,j},$$

for all $i, j = 1, ..., n$.

Now, we turn to introduce a sub quantum semigroup of $(B_s(n),\Delta)$. Since $\mathbf{P}\neq I$ is a projection in $B_s(n)$, $\mathcal{B}_s(n)=\mathbf{P}B_s(n)\mathbf{P}$ is a C^*-algebra with identity $\mathbf{P}$ and generators

$$\{\mathbf{P}u_{i_1,j_1}\cdots u_{i_k,j_k}\mathbf{P}|i_1,j_1,...i_k,j_k\in\{1,...n\},k\geq 0\}.$$

If we restrict the comultiplication Δ onto $\mathcal{B}_s(n)$, then we have

$$\Delta(\mathbf{P}u_{i_1,j_1}\cdots u_{i_k,j_k}\mathbf{P})=(\mathbf{P}\otimes\mathbf{P})(\sum_{l_1,...l_k=1}^{n}u_{i_1,l_1}\cdots u_{i_k,l_k}\otimes u_{l_1,j_1}\cdots u_{l_k,j_k})(\mathbf{P}\otimes\mathbf{P}),$$

which is contained in $\mathcal{B}_s(n)\otimes\mathcal{B}_s(n)$. Therefore, $(\mathcal{B}_s(n),\Delta)$ is also a quantum semigroup and $\mathbf{P}$ is the identity of $\mathcal{B}_s(n)$. We will call $\mathcal{B}_s(n)$ the boolean permutation quantum semigroup of n.

Remark 3.2.4. *If we require $\boldsymbol{P}u_{i,j}=u_{i,j}\boldsymbol{P}$ for all $i,j=1,...,n$, then the universal algebra we constructed in the above way is exactly Wang's quantum permutation group. Therefore, $A_s(n)$ is also a quotient algebra of $\mathcal{B}_s(n)$.*

In the following definition, $\otimes$ denotes the tensor product for linear spaces:

Definition 3.2.5. *Let $\mathcal{S}=(\mathcal{A},\Delta)$ be a quantum semigroup and $\mathcal{V}$ be a complex vector space, by a (right) coaction of the quantum group $\mathcal{S}$ on $\mathcal{V}$ we mean a linear map $Ł:\mathcal{V}\to\mathcal{V}\otimes\mathcal{A}$ such that*

$$(Ł\otimes id)Ł=(id\otimes\Delta)Ł.$$

We say a linear functional $\omega:\mathcal{V}\to\mathbb{C}$ is invariant under $Ł$ if

$$(\omega\otimes id)Ł(v)=\omega(v)I_{\mathcal{A}},$$

where $I_{\mathcal{A}}$ is the identity of $\mathcal{A}$.
Given a complex vector space $\mathcal{W}$, We say a linear map $T:\mathcal{V}\to\mathcal{W}$ is invariant under $Ł$ if

$$(T\otimes id)Ł(v)=T(v)\otimes I_{\mathcal{A}}.$$

Remark 3.2.6. *This definition is about coactions on linear spaces but not coactions on algebras.*

Let $\mathbb{C}\langle X_1,...,X_n\rangle$ be the set of noncommutative polynomials in n variables, which is a linear space over $\mathbb{C}$ with basis $X_{i_1}\cdots X_{i_k}$ for all integer $k\geq 0$ and $i_1,...,i_k\in\{1,...n\}$.
Now, we define a right coaction $Ł_n$ of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1,...,X_n\rangle$ as follows:

$$Ł_n(X_{i_1}\cdots X_{i_k})=\sum_{j_1,...j_k=1}^{n}X_{j_1}\cdots X_{j_k}\otimes\mathbf{P}u_{j_1,i_1}\cdots u_{j_n,i_n}\mathbf{P}$$

and

$$Ł_n(1)=1\otimes\mathbf{P}.$$

It is a well defined coaction of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1,...,X_n\rangle_0$, because:

$$\begin{aligned}
& (Ł_n\otimes id)Ł_n(X_{i_1}\cdots X_{i_k}) \\
= & (Ł_n\otimes id)\sum_{j_1,...j_k=1}^{n} X_{j_1}\cdots X_{j_k}\otimes \mathbf{P}u_{j_1,i_1}\cdots u_{j_n,i_n}\mathbf{P} \\
= & \sum_{j_1,...j_k=1}^{n}\sum_{l_1,...l_k=1}^{n} X_{l_1}\cdots X_{l_k}\otimes \mathbf{P}u_{l_1,j_1}\cdots u_{l_n,j_n}\mathbf{P}\otimes \mathbf{P}u_{j_1,i_1}\cdots u_{j_n,i_n}\mathbf{P} \\
= & \sum_{l_1,...l_k=1}^{n} X_{l_1}\cdots X_{l_k}\otimes (\sum_{j_1,...j_k=1}^{n}\mathbf{P}u_{l_1,j_1}\cdots u_{l_n,j_n}\mathbf{P}\otimes \mathbf{P}u_{j_1,i_1}\cdots u_{j_n,i_n}\mathbf{P}) \\
= & \sum_{l_1,...l_k=1}^{n} X_{l_1}\cdots X_{l_k}\otimes (\Delta \mathbf{P}u_{l_1,i_1}\cdots u_{l_n,i_n}\mathbf{P}) \\
= & (id\otimes\Delta)\sum_{j_1,...j_k=1}^{n} X_{l_1}\cdots X_{l_k}\otimes (\mathbf{P}u_{l_1,i_1}\cdots u_{l_n,i_n}\mathbf{P}) \\
= & (id\otimes\Delta)Ł_n(X_{i_1}\cdots X_{i_k}).
\end{aligned}$$

We will call $Ł_n$ the linear coaction of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1,...,X_n\rangle$. The algebraic coaction will be defined in section 4.3.

Lemma 3.2.7. *Let $Ł_n$ be the linear coaction of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1,...,X_n\rangle$, $\{u_{i,j}\}_{i,j=1,...n}$ and $\boldsymbol{P}$ be the standard generators of $\mathcal{B}_s(n)$. Then,*

$$Ł_n(p_1(X_{i_1})\cdots p_k(X_{i_k}))=\sum_{j_1,...j_k=1}^{n} p_1(X_{j_1})\cdots p_k(X_{j_k})\otimes \boldsymbol{P}u_{j_1,i_1}\cdots u_{j_k,i_k}\boldsymbol{P},$$

for all $i_1\neq i_2\neq\cdots\neq i_k$ and $p_1,...p_k\in\mathbb{C}\langle X\rangle_0$.

Proof. Since the map is linear, it suffices to show that the equation holds by assuming $p_l(X)=X^{t_l}$ where $t_l\geq 1$ for all $l=1,...k$. Then, we have

$$\begin{aligned}
& Ł_n(\underbrace{x_{i_1}\cdots x_{i_1}}_{t_1\ \text{times}}\cdots\underbrace{x_{i_1}\cdots x_{i_k}}_{t_k\ \text{times}}) \\
= & \sum_{j_{1,1},...j_{1,t_1},...,j_{k,1},...j_{k,t_k}=1}^{n} x_{j_{1,1}}\cdots x_{j_{1,t_1}}\cdots x_{j_{k,1}}\cdots x_{j_{k,t_k}}\otimes \mathbf{P}u_{j_{1,1}i_1}\cdots u_{j_{1,t_1}i_1}\cdots\mathbf{P}.
\end{aligned}$$

Notice that $u_{j_{m,s}i_m}u_{j_{m,s+1}i_m}=\delta_{j_{m,s},j_{m,s+1}}u_{j_{m,s}i_m}$, the right hand side of the above equation becomes

$$\sum_{j_1,...,j_k=1}^{n} x_{j_1}^{t_1}\cdots x_{j_k}^{t_k}\otimes \mathbf{P}u_{j_1i_1}\cdots u_{j_k,j_k}\mathbf{P}.$$

The proof is now completed $\square$

We will be using the following invariance condition to characterize conditionally boolean independence.

Definition 3.2.8. *Let $(\mathcal{A}, \phi)$ be a noncommutative probability space and $(x_i)_{i\in\mathbb{N}}$ be an infinite sequence of random variables in $\mathcal{A}$. We say that the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is boolean exchangeable if for all n, we have*

$$\mu_{x_1,...,x_n}(p)\boldsymbol{P} = \mu_{x_1,...,x_n} \otimes id_{\mathcal{B}_s(n)}(Ł_n p)$$

for all $p \in \mathbb{C}\langle X_1, ..., X_n\rangle$, where $\mu_{x_1,...,x_n}$ is the joint distribution of $x_1, ..., x_n$.

Let $\{\bar{u}_{ij}\}_{i,j=1,..,n}$ be the standard generators of $\mathcal{A}_s(n)$, and $\{u_{ij}\}_{i,j=1,..,n} \cup \{\mathbf{P}\}$ be the standard generators of $B_s(n)$, then there exists a unital C^*-homomorphism $\beta : B_s(n) \to \mathcal{A}_s(n)$ such that:

$$\beta(u_{ij}) = \bar{u}_{ij}, \quad \beta(\mathbf{P}) = 1_{\mathcal{A}_s(n)}.$$

The C^*-homomorphism is well defined because of the universality of $B_s(n)$. Let $p = X_{i_1} \cdots X_{i_k} \in \mathbb{C}\langle X_1, ..., X_n\rangle$, then

$$\mu_{x_1,...,x_n}(p)\mathbf{P} = \mu_{x_1,...,x_n} \otimes id_{\mathcal{B}_s(n)}(Ł_n p)$$

implies

$$\mu_{x_1,...,x_n}(X_{i_1} \cdots X_{i_k})\mathbf{P} = \sum_{j_1,...j_k=1}^{n} (\mu_{x_1,...,x_n} \otimes id_{\mathcal{B}_s(n)})(X_{j_1} \cdots X_{j_k} \otimes \mathbf{P}u_{j_1,i_1} \cdots u_{j_n,i_n}\mathbf{P}).$$

Now, apply β on both sides of the above equation, we get

$$\mu_{x_1,...,x_n}(X_{i_1} \cdots X_{i_k})1_{\mathcal{A}_s(n)} = \sum_{j_1,...j_k=1}^{n} (\mu_{x_1,...,x_n} \otimes id_{\mathcal{A}_s(n)})(X_{j_1} \cdots X_{j_k} \otimes \bar{u}_{j_1,i_1} \cdots \bar{u}_{j_n,i_n}),$$

which is the free quantum invariance condition. Since p is arbitrary, we have:

Proposition 3.2.9. *Let $(\mathcal{A}, \phi)$ be a noncommutative probability space and $(x_i)_{i=1,...,n}$ be a sequence of random variables in $\mathcal{A}$, the joint distribution of $(x_i)_{i=1,...,n}$ is invariant under the free quantum permutations $\mathcal{A}_s(n)$ if it satisfies the invariance condition associated with the linear coaction of the boolean quantum permutation semigroup $\mathcal{B}_s(n)$.*

3.3 Distributional symmetries for finite sequences of random variables

In this section, we will review two kinds of distributional symmetries which are spreadability and partial exchangeability, in classical probability. In [20], we see that the distributional symmetries can be defined for either finite sequences or infinite sequences. Moreover, each kind of distributional symmetry for infinite sequences of random objects is determined by distributional symmetries on all its finite subsequences. For example, an infinite sequence of random variables is exchangeable iff all its finite subsequences are exchangeable. We will present distributional symmetries for finite sequences and then introduce their counterparts in noncommutative case. In the first subsection, we recall notions of spreadability and partial exchangeability in classical probability and rephrase these notions in words of quantum maps. In the second subsection, we will introduce counterparts of spreadability and partial exchangeability in noncommutative case. Even though there are many interesting properties of partial exchangeability, we are not going to study it too much here because the main problem we concern is about extended de Finetti type theorems for noncommutative spreadable sequences.

3.3.1 Spreadability and partial exchangeability

Recall that in [21], a finite sequence of random variables $(x_1, ..., x_n)$ is said to be spreadable if for any $k < n$, we have

$$(x_1, ..., x_k) \stackrel{d}{=} (x_{l_1}, ..., x_{l_k}), \quad l_1 < l_2 < \cdots < l_k. \tag{3.1}$$

For fixed natural numbers $n > k$, it is mentioned in [7], the above relation can be described in words of quantum family of maps in sense of Soltan [31]: Considering the space $I_{k,n}$ of increasing sequences $\mathcal{I} = (1 \leq i_1 < \cdots < i_k \leq n)$. For $1 \leq i \leq n$, $1 \leq j \leq k$, define $f_{i,j} : I_{k,n} \to \mathbb{C}$ by:

$$f_{i,j}(\mathcal{I}) = \begin{cases} 1, & i_j = i \\ 0, & i_j \neq i \end{cases}.$$

If we consider $I_{n,k}$ as a discrete space, then the functions $f_{i,j}$ generate $C(I_{n,k})$ by the Stone-Weierstrass theorem. Let $\mathbb{C}[X_1, ..., X_m]$ be the set of commutative polynomials in m variables. The algebra $C(I_{n,k})$ together with an algebraic homomorphism $\alpha : \mathbb{C}[X_1, ..., X_k] \to \mathbb{C}[X_1, ..., X_n] \otimes C(I_{k,n})$ define by:

$$\alpha : X_j = \sum_{i=1}^{n} X_i \otimes f_{i,j}, \quad \alpha(1) = 1 \otimes 1_{C(I_{k,n})},$$

which defines a quantum family of maps from $\{1, ..., k\}$ to $\{1, ..., n\}$.

Equation (3.1) can be rephrased in the following way: For fixed natural numbers $n > k$,

$$\mu_{x_1,...,x_k}(p) 1_{C(I_{n,k})} = \mu_{x_1,...,x_n} \otimes id_{C(I_{n,k})}(\alpha(p)) \tag{3.2}$$

for all $p \in \mathbb{C}[x_1, ..., x_k]$, where $\mu_{x_1,....,x_n}$ is the joint distribution of $(x_1, ..., x_n)$.
For completeness, we provide a sketch of proof here: Suppose equation (3.1) holds. Let $p = X_{j_1}^{i_1} \cdots X_{j_m}^{i_m}$ be a monomial in $\mathbb{C}[X_1, ..., X_k]$ such that $1 \leq j_1 < j_2 < \cdots < j_m \leq k$ and $i_1, ..., i_m$ are positive integers. Let $\mathcal{I} = (1 \leq l_1 < \cdots < l_k \leq n)$ be a point in $I_{k,n}$. Then, the $\mathcal{I}$-th component of $\mu_{x_1,...,x_k}(p) 1_{C(I_{n,k})}$ is $E[x_{j_1}^{i_1} \cdots x_{j_m}^{i_m}]$. On the other hand, the

$\mathcal{I}$-th component of $\mu_{x_1,...,x_n} \otimes id_{C(I_{n,k})}(\alpha(p))$ is

$$\sum_{s_1,...,s_m=1}^{n} E[x_{s_1}^{i_1} \cdots x_{s_m}^{i_m}](f_{s_1,j_1} \cdots f_{s_m,j_m})(\mathcal{I}).$$

According the definition of $f_{i,j}$, $(f_{s_1,j_1} \cdots f_{s_m,j_m})(\mathcal{I})$ is not vanished only if $s_t = l_{j_t}$ for all $1 \leq t \leq m$. Therefore,

$$\sum_{s_1,...,s_m=1}^{n} E[x_{s_1}^{i_1} \cdots x_{s_l}^{i_l}](f_{s_1,j_1} \cdots f_{s_m,j_m})(\mathcal{I}) = E[x_{l_{j_1}}^{i_1} \cdots x_{l_{j_m}}^{i_m}].$$

Since $1 \leq j_1 < j_2 < \cdots < j_m \leq k$ and $\mathcal{I}$ is an in creasing sequence, we have $1 \leq l_{j_1} < \cdots < l_{j_m} \leq n$. Hence, the $\mathcal{I}$-components of the two sides of equation (3.2) are equal to each other. Since $\mathcal{I}$ is arbitrary, equation (3.2) holds. By checking the $\mathcal{I}$-th component of equation (3.2), we can also show that (3.2) implies (3.1). We will say that $(\xi_1, ..., \xi_n)$ is (n,k)-spreadable if $(x_1, ..., x_n)$ satisfies equation (3.2).

Remark 3.3.1. We see that the above (n,k)-spreadability describes limited relations between the mixed moments of $(x_1, ..., x_n)$. For fixed n, k, the (n,k)-spreadability gives no information about mixed moments which involve $k+1$ variables. For example, let $n = 4$, $k = 2$ and assume that $(x_1, ..., x_4)$ is a $(4,2)$-spreadable sequence. According to equation (3.1), we know nothing about the relation between $E[x_1x_2x_3]$ and $E[x_2x_3x_4]$. We will call this kind of distributional symmetries partial symmetries because they just provide information of part of mixed moments but not all.

By using the idea of partial symmetries, we can define another family of distributional symmetries which is stronger than (n,k)-spreadability but weaker than exchangeability.

Definition 3.3.2. For fixed natural numbers $n > k$, we say a sequence of random variables $(x_1, ..., x_n)$ is (n,k)-exchangeable if

$$(x_1, ..., x_k) \stackrel{d}{=} (x_{\sigma(1)}, ..., x_{\sigma(k)}), \quad \forall \sigma \in S_n,$$

where S_n is the permutation group of n elements.

This kind of distributional symmetries are called partial exchangeability. See [10] for more details. As well as (n,k)-spreadability, we can rephrase partial exchangeability in words of quantum family of maps: Considering the space $E_{n,k}$ of length k sequences $\{\mathcal{I} = (i_1, ..., i_k) | 1 \leq i_1, ..., i_k \leq n, i_j \neq i_{j'} \text{ for } j \neq j'\}$. For $1 \leq i \leq n$, $1 \leq j \leq k$, define $g_{i,j} : I_{n,k} \to \mathbb{C}$ by:

$$g_{i,j}(\mathcal{I}) = \begin{cases} 1, & i_j = i, \\ 0, & i_j \neq i. \end{cases}$$

Given two different sequences $\mathcal{I} = (i_1, ..., i_k)$ and $\mathcal{I}' = (i'_1, ..., i'_k)$, there must exists a number j such that $i_j \neq i'_j$. Then, we have that $g_{i,i_j}(\mathcal{I}) = 1 \neq 0 = g_{i,i'_j}(\mathcal{I})$. Therefore, the set of functions $\{g_{i,j}|i = 1,...,n; j = 1,...,k\}$ separates $E_{n,k}$. According to Stone Weierstrass theorem, the functions $g_{i,j}$ generate $C(E_{n,k})$. Again, we can define a homomorphism α' : $\mathbb{C}[X_1,...,X_k] \to \mathbb{C}[X_1,...,X_n] \otimes C(E_{n,k})$ by the following formulas:

$$\alpha' : X_j = \sum_{i=1}^{n} X_i \otimes g_{i,j}, \quad \alpha'(1) = 1_{C(I_{k,n})}.$$

Lemma 3.3.3. *Let $\mu_{x_1,...,x_n}$ be the joint distribution of $x_1,...,x_n$. Then*

$$\mu_{x_1,...,x_k}(p)1_{C(I_{n,k})} = \mu_{x_1,...,x_n} \otimes id_{C(I_{n,k})}(\alpha(p))$$

for all $p \in \mathbb{C}[X_1,...,X_k]$ if and only if $x_1,...,x_n$ is (n,k)-exchangeable.

The proof is similar the proof of (n,k)-spreadability, we just need to check the values at all components of $E_{n,k}$.

3.3.2 Noncommutative analogue of partial symmetries

Now, we turn to introduce noncommutative versions of spreadability and partial exchangeability. The pioneering work was done by Curran [7]. He defined a quantum version of $C(I_{n,k})$ in analogue of Wang's quantum permutation groups as following:

Definition 3.3.4. For $k,n \in \mathbb{N}$ with $k \leq n$, the quantum increasing space $A(n,k)$ is the universal unital C^*-algebra generated by elements $\{u_{i,j}|1 \leq i \leq n, 1 \leq j \leq k\}$ such that

1. Each $u_{i,j}$ is an orthogonal projection: $u_{i,j} = u_{i,j}^* = u_{i,j}^2$ for all $i = 1,...,n; j = 1,...,k$.
2. Each column of the rectangular matrix $u = (u_{i,j})_{i=1,...,n;j=1,...,k}$ forms a partition of unity: for $1 \leq j \leq k$ we have $\sum\limits_{i=1}^{n} u_{i,j} = 1$.
3. Increasing sequence condition: $u_{i,j}u_{i',j'} = 0$ if $j < j'$ and $i \geq i'$.

Remark 3.3.5. Our notation is different from Curran's, we use $A_i(n,k)$ instead of his $A_i(k,n)$ for our convenience.

For any natural numbers $k < n$, in analogue of coactions of $A_s(n)$, there is a unital $*$-homomorphism $\alpha_{n,k} : \mathbb{C}\langle X_1,...,X_k\rangle \to \mathbb{C}\langle X_1,...,X_n\rangle \otimes A_i(n,k)$ determined by:

$$\alpha_{n,k}(X_j) = \sum_{i=1}^{n} X_i \otimes u_{i,j}.$$

The quantum spreadability of random variables is defined as the following:

Definition 3.3.6. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. A finite ordered sequence of random variables $(x_i)_{i=1,...,n}$ in $\mathcal{A}$ is said to be $A_i(n,k)$-spreadable if their joint distribution $\mu_{x_1,...,x_n}$ satisfies:

$$\mu_{x_1,...,x_k}(p)1_{A_i(n,k)} = \mu_{x_1,...,x_n} \otimes id_{A_i(n,k)}(\alpha_{n,k}(p)),$$

for all $p \in \mathbb{C}\langle X_1,...,X_k\rangle$. $(x_i)_{i=1,...,n}$ is said to be quantum spreadable if $(x_i)_{i=1,...,n}$ is $A_i(n,k)$-spreadable for all $k = 1,...,n-1$.

Remark 3.3.7. In [7], Curran studied sequences of C^*-homomorphisms which are more general than random variables. For consistency, we state his definitions in words of random variables. It is a routine to extend our work to the framework of sequences of C^*-homomorphisms.

Recall that in [**Liu1**], by replacing the condition associated with partitions of the unity of Wang's quantum permutation groups, we defined a family of quantum semigroups with invariant projections. With a natural family of coactions, we defined invariance conditions which can characterize conditional boolean independence. Here, we can modify Curran's quantum increasing spaces in the same way:

Definition 3.3.8. For $k, n \in \mathbb{N}$ with $k \leq n$, the noncommutative increasing space $B_i(n,k)$ is the unital universal C^*-algebra generated by elements $\{u_{i,j} | 1 \leq i \leq n, 1 \leq j \leq k\}$ and an invariant projection $\mathbf{P}$ such that

1. Each $u_{i,j}$ is an orthogonal projection:$u_{i,j} = (u_{i,j})^* = (u_{i,j})^2$ for all $i = 1,...,n; j = 1,...,k$.

2. For $1 \leq j \leq k$ we have $\sum\limits_{i=1}^{n} u_{i,j}\mathbf{P} = \mathbf{P}$.

3. Increasing sequence condition: $u_{i,j}u_{i',j'} = 0$ if $j < j'$ and $i \geq i'$.

The same as $A_i(n,k)$, there is a unital $*$-homomorphism $\alpha^{(b)}_{n,k} : \mathbb{C}\langle X_1,...,X_k\rangle \to \mathbb{C}\langle X_1,...,X_n\rangle \otimes B_i(n,k)$ determined by:

$$\alpha^{(b)}_{n,k}(x_j) = \sum_{i=1}^{n} x_i \otimes u_{i,j}$$

As boolean exchangeability defined in [**Liu1**], we have

Definition 3.3.9. A finite ordered sequence of random variables $(x_i)_{i=1,...,n}$ in $(\mathcal{A}, \phi)$ is said to be $B_i(n,k)$-spreadable if their joint distribution $\mu_{x_1,...,x_n}$ satisfies:

$$\mu_{x_1,...,x_k}(p)\mathbf{P} = \mathbf{P}\mu_{x_1,...,x_n} \otimes id_{B_i(n,k)}(\alpha^{(b)}_{n,k}(p))\mathbf{P},$$

for all $p \in \mathbb{C}\langle X_1,...,X_k\rangle$. $(x_i)_{i=1,...,n}$ is said to be boolean spreadable if $(x_i)_{i=1,...,n}$ is $B_i(n,k)$-spreadable for all $k = 1,...,n-1$.

We will see that $B_i(k,n)$ is an increasing space of boolean type, because we can derive an extended de Finetti type theorem for boolean independence.

Recall that, in [14], Franz showed some relations between free independence, monotone independence and boolean independence via Bożejko, Marek and Speicher's two-states free products[5]. We can see that monotone product is "between" free product and boolean product. From this viewpoint of Franz's work, we may hope to define a kind of "spreadability" for monotone independence by modifying quantum spreadability and boolean spreadability. Notice that there are at least two ways to get quotient algebras of $B_i(k,n)$'s such that the **P**-invariance condition of the quotient algebras is equivalent quantum spreadability:

1. Require **P** to be the unit of the algebra.
2. Let $P_j = \sum\limits_{i=1}^{n} u_{i,j}$, require $P_{j'}u_{ij} = u_{ij}P_{j'}$ for all $1 \leq j, j' \leq k$ and $1 \leq i \leq n$.

To define our monotone increasing spaces, we will modify the second condition a little:

Definition 3.3.10. For fixed $n, k \in \mathbb{N}$ and $k < n$, a monotone increasing sequence space $M_i(n,k)$ is the universal unital C^*-algebra generated by elements $\{u_{i,j}\}_{i=1,...,n;j=1,...,k}$

1. Each $u_{i,j}$ is an orthogonal projection;
2. Monotone condition: Let $P_j = \sum\limits_{i=1}^{n} u_{i,j}$, $P_j u_{i'j'} = u_{i'j'}$ if $j' \leq j$.
3. $\sum\limits_{i=1}^{n} u_{i,j}P_1 = P_1$ for all $1 \leq j \leq k$.
4. Increasing condition: $u_{i,j}u_{i',j'} = 0$ if $j < j'$ and $i \geq i'$.

We see that P_1 plays the role as the invariant projection **P** in the boolean case. For consistency, we denote P_1 by **P**. Then, we can define a **P**-invariance condition associated with $M_i(n,k)$ in analogy with $B_i(n,k)$: For fixed $n, k \in \mathbb{N}$ and $k < n$, there is a unique unital $*$- isomorphism $\alpha_{n,k}^{(m)} : \mathbb{C}\langle X_1, ..., X_k\rangle \to \mathbb{C}\langle X_1, ..., X_n\rangle \otimes M_i(n,k)$ such that

$$\alpha_{n,k}^{(m)}(X_j) = \sum_{i=1}^{n} X_i \otimes u_{i,j}.$$

The existence of such a homomorphism is given by the universality of $\mathbb{C}\langle X_1, ..., X_k\rangle$.

Definition 3.3.11. A finite ordered sequence of random variables $(x_i)_{i=1,...,n}$ in $(\mathcal{A}, \phi)$ is said to be $M_i(n,k)$-invariant if their joint distribution $\mu_{x_1,...,x_n}$ satisfies:

$$\mu_{x_1,...,x_k}(p)\mathbf{P} = \mathbf{P}\mu_{x_1,...,x_n} \otimes id_{M_i(n,k)}(\alpha_{n,k}^{(m)}(p))\mathbf{P},$$

for all $p \in \mathbb{C}\langle X_1, ..., X_k\rangle$. $(x_i)_{i=1,...,n}$ is said to be monotonically spreadable if it is $M_i(n,k)$-invariant for all $k = 1, ..., n-1$.

We will see that these invariance conditions can characterize conditionally Monotone independence in a proper framework.

As remark 2.3 in [7], a first question to our definitions is whether $A_i(n,k)$, $B_i(n,k)$, $M_i(n,k)$ exist. In [7], Curran has showed several nontrivial representations of $A_i(n,k)$. In the following, we provide a family of representations of $A_i(n,k)$, $B_i(n,k)$, $M_i(n,k)$ for $n > k$. Fix natural numbers $n > k$, let $l_1, ..., l_k \in \mathbb{N}$ such that

$$l_1 + \cdots + l_k = n,$$

consider the following matrix:

$$\begin{pmatrix} P_{1,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_{1,l_1} & 0 & \cdots & 0 \\ 0 & P_{2,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & P_{2,l_2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{k,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{k,l_k} \end{pmatrix}.$$

We see that the entries of the matrix satisfy the increasing condition of spaces of increasing sequences. By choosing proper projections $P_{i,j}$, we will get representations for our universal algebras:
We denote by $\mathcal{H}_i$ a l_i-dimensional Hilbert spaces with orthonormal basis $\{e_j^{(i)}|j = 1, ..., l_i\}$. Let I_{l_i} be the unit of the algebra $B(\mathcal{H}_{l_i})$, $P_{e_j^{(l_i)}}$ be the one dimensional orthogonal projection onto $\mathbb{C}e_j^{(l_i)}$, P_i be the one dimensional projection onto $\mathbb{C}\sum_j e_j^{(l_i)}$.Then, some representations of $A_i(n,k)$, $B_i(n,k)$, $M_i(n,k)$ can be constructed in the following way:
A representation of $A_i(n,k)$: For each $1 \leq j \leq k$, the algebra generated by $\{P_{e_j^i}|i = 1, ..., l_j\}$ is isomorphic to $C^*(\mathbb{Z}_{l_j})$. The reduced free product $*_{j=1}^k \mathbb{Z}_{l_i}$ is a quotient algebra of $A_i(n,k)$. One can define a C^*-homomorphism π from $A_i(n,k)$ to $*_{j=1}^k C^*(\mathbb{Z}_{l_j})$ such that

$$\pi(u_{i,j}) = \begin{cases} \text{the image of } P_{e_{j'}^{(l_i)}} \text{ in } *_{j=1}^k C^*(\mathbb{Z}_{l_j}) & \text{if} \quad 0 < j' = j - \sum\limits_{l=m}^{i-1} l_m \leq l_i \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

A representation of $B_i(n,k)$: One can define a C^*-homomorphism π from $B_i(n,k)$ into

$B(\bigotimes_{i=1}^{k}\mathcal{H}_i)$ such that

$$\pi(u_{i,j}) = \begin{cases} \bigotimes_{m_1=1}^{i-1} P_{l_{m_1}} \otimes P_{e_{j'}^{(l_i)}} \bigotimes_{m_2=i+1}^{k} P_{l_{m_2}} & \text{if} \quad 0 < j' = j - \sum_{l=m}^{i-1} l_m \leq l_i \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

A representation of $M_i(n,k)$: One can define a C^*-homomorphism π from $M_i(n,k)$ into $B(\bigotimes_{i=1}^{k}\mathcal{H}_i)$

$$\pi(u_{i,j}) = \begin{cases} \bigotimes_{m_1=1}^{i-1} I_{l_{m_1}} \otimes P_{e_{j'}^{(l_i)}} \bigotimes_{m_2=i+1}^{k} P_{l_{m_2}} & \text{if} \quad 0 < j' = j - \sum_{l=m}^{i-1} l_m \leq l_i \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

The existence of these homomorphisms are given by the universal conditions for $A_i(n,k)$, $B_i(n,k)$ and $M_i(n,k)$ respectively. The of $M_i(n,k)$ plays an important role in our work, we summarize it as the following proposition:

Proposition 3.3.12. *For fixed natural numbers $n > k$. Let $l_1, ..., l_k \in \mathbb{N}$ such that $l_1 + \cdots + l_k = n$. Let $\mathcal{H}_i$ be a l_i-dimensional Hilbert spaces with orthonormal basis $\{e_j^{(i)} | j = 1, ..., l_i\}$ and I_{l_i} be the unit of the algebra $B(\mathcal{H}_{l_i})$, $P_{e_j^{(l_i)}}$ be the one dimensional orthogonal projection onto $\mathbb{C}e_j^{(l_i)}$, P_i be the one dimensional projection onto $\mathbb{C}\sum_j e_j^{(l_i)}$. Then, there is a C^*-homomorphism $\pi : M_i(n,k) \to B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k)$ defined as follows:*

$$\pi(u_{i,j}) = \begin{cases} \bigotimes_{m_1=1}^{i-1} I_{l_{m_1}} \otimes P_{e_{j'}^{(l_i)}} \bigotimes_{m_2=i+1}^{k} P_{l_{m_2}} & \textit{if} \quad 0 < j' = j - \sum_{l=m}^{i-1} l_m \leq l_i \\ 0 & \textit{if} \quad \textit{otherwise} \end{cases}$$

In addition, we need the following property:

Lemma 3.3.13. *Given natural numbers $n_1, n_2, n, k \in \mathbb{N}$ such that $n > k$. Let $(u_{i,j})_{i=1,...,n;j=1,...,k}$ be the standard generators of $M_i(n,k)$ and $(u'_{i,j})_{i=1,...,n+n_1+n_2;j=1,...,k+n_1+n_2}$ be the standard generators of $M_i(n+n_1+n_2, k+n_1+n_2)$. Then, there exists a C^*-homomorphism $\pi : M_i(n+n_1+n_2, k+n_1+n_2) \to M_i(n,k)$ such that*

$$\pi(u'_{i,j}) = \begin{cases} \delta_{i,j}\boldsymbol{P} & \textit{if} \quad 1 \leq i \leq n_1 \\ \delta_{i,j} & \textit{if} \quad n_1+1 \leq i \leq n+n_1, \ n_1 \leq j \leq n_1+k \\ 0 & \textit{if} \quad n_1+1 \leq i \leq n+n_1, \ j \leq n_1 \textit{ or } j > n_1+k \\ \delta_{i-n,j-k} I & \textit{if} \quad i \geq n+n_1+1 \end{cases}$$

where $\boldsymbol{P} = P_1 = \sum_{i=1}^{n} u_{i,1}$ and I is the identity of $M_i(n,k)$.

Proof. We can see that the matrix form of $(\pi(u'_{i,j}))_{i=1,...,n+n_1+n_2;j=1,...,k+n_1+n_2}$ is

$$\begin{pmatrix} \mathbf{P} & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{P} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & u_{1,1} & \cdots & u_{1,k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{n,1} & \cdots & u_{n,k} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & I & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & I \end{pmatrix}$$

It is easy to check that the coordinates of the above matrix satisfy the universal conditions of $M_i(n+n_1+n_2, k+n_1+n_2)$. The proof is complete. □

In analogue of the (n,k)-partial exchangeability, we can define noncommutative versions of partial exchangeability for free independence and boolean independence:

Definition 3.3.14. For $k, n \in \mathbb{N}$ with $k \leq n$, the quantum space $A_l(n,k)$ is the universal unital C^*-algebra generated by elements $\{u_{ij} | 1 \leq i \leq n, 1 \leq j \leq k\}$ such that

1. Each u_{ij} is an orthogonal projection:$u_{ij} = u_{ij}^* = u_{ij}^2$.
2. Each column of the rectangular matrix $u = (u_{ij})$ forms a partition of unity: for $1 \leq j \leq k$ we have $\sum_{i=1}^{n} u_{ij} = 1$.

Remark 3.3.15. $A_i(n,k)$ is a quotient algebra of $A_l(n,k)$, because the definition of $A_i(n,k)$ has one more restriction than $A_l(n,k)$'s. $A_l(n,n)$ is exactly Wang's quantum permutation group $A_s(n)$.

There is a well defined unital algebraic homomorphism

$$\alpha_{n,k}^{(fp)} : \mathbb{C}\langle X_1, ...X_k\rangle \to \mathbb{C}\langle X_1, ...X_n\rangle \otimes A_l(n,k)$$

such that

$$\alpha_{n,k}^{(fp)} X_j = \sum_{i=1}^{n} X_i \otimes u_{i,j}$$

where $1 \leq j \leq k$.

Definition 3.3.16. Let $x_1, ..., x_n \in (\mathcal{A}, \phi)$ be a sequence of n-noncommutative random variables, $k \leq n$ be a positive integer. We say the sequence is (n,k)-quantum exchangeable if

$$\mu_{x_1,...x_k}(p) = \mu_{x_1,...x_n} \otimes id_{A_l(n,k)}(\alpha_{n,k}^{(fp)}(p)),$$

for all $p \in \mathbb{C}\langle X_1, ..., X_k\rangle$, where $\mu_{x_1,...,x_j}$ is the joint distribution of $x_1, ...x_j$ with respect to ϕ for $j = k, n$.

By modifying the second universal condition of $A_l(n,k)$, we can define a boolean version of partial exchangeability:

Definition 3.3.17. *For natural numbers $k \leq n$, $B_l(n,k)$ is the non-unital universal C^*-algebra generated by the elements $\{u_{i,j}\}_{i=1,...,n;j=1,...,k}$ and an orthogonal projection $\boldsymbol{P}$, such that*

1. *$u_{i,j}$ is an orthogonal projection, i.e. $u_{i,j} = u_{i,j}^* = u_{i,j}^2$.*
2. *$\sum\limits_{i=1}^{n} u_{i,j}\boldsymbol{P} = \boldsymbol{P}$ for all $1 \leq j \leq k$.*

Remark 3.3.18. *$B_l(n,n)$ is exactly the boolean exchangeable quantum semigroup $B_s(n)$.*

There is a well defined unital algebraic homomorphism

$$\alpha_{n,k}^{(bp)} : \mathbb{C}\langle X_1, ...X_k\rangle \to \mathbb{C}\langle X_1, ...X_n\rangle \otimes B_l(n,k)$$

such that

$$\alpha_{n,k}^{(bp)} X_j = \sum_{i=1}^{n} X_i \otimes u_{i,j}$$

where $1 \leq j \leq k$.

Definition 3.3.19. Let $x_1, ..., x_n \in (\mathcal{A}, \phi)$ be a sequence of n-noncommutative random variables, $k \leq n$ be a positive integer. We say the sequence is (n,k)-boolean exchangeable if

$$\mu_{x_1,...x_k}(p)\mathbf{P} = \mathbf{P}\mu_{x_1,...x_n} \otimes id_{B_l(n,k)}(\alpha_{n,k}^{(bp)}(p))\mathbf{P}$$

for all $p \in \mathbb{C}\langle X_1, ..., X_k\rangle$, where $\mu_{x_1,...,x_j}$ is the joint distribution of $x_1, ...x_j$ with respect to ϕ.

Now, we turn to define our noncommutative distributional symmetries for infinite sequences. In this paper, our infinite ordered index set I would be either $\mathbb{N}$ or $\mathbb{Z}$.

Definition 3.3.20. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space, I be an ordered index set and $(x_i)_{i\in I}$ a sequence of random variables in $\mathcal{A}$. $(x_i)_{i\in I}$ is said to be monotonically (boolean) spreadable if all its finite ordered subsequences $(x_{i_1}, ..., x_{i_l})$ are monotonically(boolean) spreadable.

Proposition 3.3.21. *Let $(x_1, ..., x_{n+1})$ be a monotonically spreadable sequence of random variables in $(\mathcal{A}, \phi)$. Then, all its subsequences are monotonically spreadable.*

Proof. By, induction, it suffices to show that the subsequence $(x_1, ..., x_{l-1}, x_{l+1}, ..., x_{n+1})$ is monotonically spreadable for all $1 \leq l \leq n$. If we denote $(x_1, ..., x_{l-1}, x_{l+1}, ..., x_{n+1})$ by $(y_1, ..., y_n)$, then we need to show that $(y_1, ..., y_n)$ is $M_i(n,k)$-spreadable for all $k < n$.

Fix $k < n$, let $\{u_{i,j}\}_{i=1,...,n;j=1,...,k}$ be the set of generators of $M_i(n,k)$ and $\{P_{i,j}\}_{i=1,...,n+1;j=1,...,k+1}$ be an $n+1$ by $k+1$ matrix with entries in $M_i(n,k)$ such that

$$P_{i,j} = \begin{cases} u_{i,j} & \text{if } 1 \leq i,j < l \\ u_{i-1,j} & \text{if } 1 \leq j < l,\, i \geq l \\ u_{i,j-1} & \text{if } 1 \leq i < l,\, j \geq l \\ u_{i-1,j-1} & \text{if } i,j \geq l \\ 0 & \text{otherwise.} \end{cases}$$

It is a routine to check that the set $\{P_{i,j}\}_{i=1,...,n+1;j=1,...,k+1}$ satisfies the universal conditions of $M_i(n+1,k+1)$. Thus, there exists a C^*-homomorphism $\psi : M_i(n+1,k+1) \to M_i(n,k)$ such that

$$\psi(u'_{i,j}) = P_{i,j},$$

where $\{u'_{i,j}\}$ is the set of generators of $M_i(n+1,k+1)$. For convenience, we will use the following notation:

$$\sigma(i) = \begin{cases} i & \text{if } 1 \leq i < l \\ i+1 & \text{if } i \geq l. \end{cases}$$

Then, $P_{\sigma(i),\sigma(j)} = u_{i,j}$ and $y_i = x_{\sigma(i)}$ for all $i = 1,...,n$ and $j = 1,....,k+1$. For all monomial $X_{j_1}\cdots X_{j_m} \in \mathbb{C}\langle X_1,...,X_k\rangle$, let $P'_1 = \sum\limits_{i=1}^{n} u'_{i,1}$ and $\mathbf{P}$ be the invariance projection of $M_i(n,k)$, we have

$$\begin{aligned} & \mu_{y_1,...,y_n}(X_{j_1}\cdots X_{j_m})\mathbf{P} \\ = \ & \mathbf{P}\mu_{x_1,...,x_{n+1}}(X_{\sigma(j_1)}\cdots X_{\sigma(j_m)})\psi(P'_1)\mathbf{P} \\ = \ & \mathbf{P}\psi(\mu_{x_1,...,x_{n+1}}(X_{\sigma(j_1)}\cdots X_{\sigma(j_m)})P'_1)\mathbf{P} \\ = \ & \mathbf{P}\psi(\mu_{x_1,...,x_{n+1}} \otimes id_{M_i(n+1,k+1)}(\sum_{i_1,...,i_m=1}^{n+1} X_{i_1}\cdots X_{i_m} \otimes u'_{i_1,\sigma(j_1)}\cdots u'_{i_m,\sigma(j_m)}))\mathbf{P}. \end{aligned}$$

Notice that $u'_{l,\sigma(j)} = 0$ since $\sigma(j)$ never equals l, it follows that:

$$\begin{aligned} & \mu_{y_1,...,y_n}(X_{j_1}\cdots X_{j_m})\mathbf{P} \\ = \ & \mathbf{P}\psi(\mu_{x_1,...,x_{n+1}} \otimes id_{M_i(n+1,k+1)}(\sum_{i_1,...,i_m=1}^{n} X_{\sigma(i_1)}\cdots X_{\sigma(i_m)} \otimes u'_{\sigma(i_1),\sigma(j_1)}\cdots u'_{\sigma(i_m),\sigma(j_m)}))\mathbf{P} \\ = \ & \mathbf{P}\sum_{i_1,...,i_m=1}^{n} \mu_{x_1,...,x_{n+1}}(X_{\sigma(i_1)}\cdots X_{\sigma(i_m)})\psi(u'_{\sigma(i_1),\sigma(j_1)}\cdots u'_{\sigma(i_m),\sigma(j_m)})\mathbf{P} \\ = \ & \sum_{i_1,...,i_m=1}^{n} \mu_{y_1,...,y_n}(X_{i_1}\cdots X_{i_m})\mathbf{P}u_{i_1,j_1}\cdots u_{i_m,j_m}\mathbf{P}, \end{aligned}$$

which completes the proof. □

Proposition 3.3.22. *Let $(\mathcal{A},\phi)$ be a noncommutative probability space and $(x_i)_{i\in\mathbb{Z}}$ be a sequence of random variables in $\mathcal{A}$. Then, $(x_i)_{i\in\mathbb{Z}}$ is monotonically (quantum, boolean) spreadable if and only if $(x_i)_{i=-n,-n+1,...,n-1,n}$ is monotonically (quantum, boolean) spreadable for all n.*

Proof. It is sufficient to prove "$\Leftarrow$". Given a subsequence $(x_{i_1}, ..., x_{i_l})$ of $(x_i)_{i\in\mathbb{Z}}$, there exits an n such that $-n < i_1, ..., i_l < n$. Since $(x_i)_{i=-n,-n+1,...,n-1,n}$ is monotonically spreadable, by Proposition 3.3.21, we have that $(x_{i_1}, ..., x_{i_l})$ is monotonically spreadable. The same to quantum spreadability and boolean spreadability. □

3.4 Relations between noncommutative probabilistic symmetries

In this section, we will study some simple relations between noncommutative distributional symmetries introduced in the previous section.

It is well known that every C^*-algebra admits a faithful representation. Fix $n, k \in \mathbb{N}$, such that $1 \le k \le n-1$. Let Φ be a faithful representation of $B_l(n,k)$ into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For convenience, we denote $\Phi(u_{i,j})$ by $u_{i,j}$ and $\Phi(\mathbf{P})$ by $\mathbf{P}$.

According to the definition of $B_l(k,n)$, $u_{i,j}$'s and $\mathbf{P}$ are orthogonal projections in $B(\mathcal{H})$. Let $Q_i = \sum\limits_{j=1}^{k} u_{i,j}$ for $1 \le i \le n$. In [19], we know that the set $P(\mathcal{H})$ of orthogonal projections on $\mathcal{H}$ is a lattice with respect to the usual order $\le$ on the set of selfadjoint operators, i.e. two selfadjoint operators A and B, $A \le B$ iff $B - A$ is a positive operator.

Now, we need the following notation in our construction. Given two projections E and F, we denote by $E \vee F$ the minimal orthogonal projection in $P(\mathcal{H})$, such that $E \vee F$ is greater or equal to E and F. $E \vee F$ is well define and unique, we call it the supreme of E and F. It is easy to see that $(E \vee F)E = E$ and $(E \vee F)F = F$

We turn to define a sequence of orthogonal projections $\{P'_i\}_{i=1,...,n}$ in $P(\mathcal{H})$ as follows:

$$P'_1 = I - Q_1,$$

$$P'_i = I - P'_1 \vee \cdots \vee P'_{i-1} \vee Q_i$$

for $2 \le i \le n$.

To proceed our work, we need the following well know lemma:

Lemma 3.4.1. *Given a nonzero vector $v \in \mathcal{H}$, E and F are two orthogonal projections on $\mathcal{H}$. If $(E \vee F)x = x$ and $Ex = 0$, then $Fx = x$.*

According the construction of $\{P'_i\}_{1\le i\le n}$, we have

$$P'_i P'_j = \delta_{i,j} P'_i$$

and

$$P'_i u_{i,j} = 0$$

for all $1 \le i \le n$ and $1 \le j \le k$.

Lemma 3.4.2. $\sum\limits_{i=1}^{n} P'_i = I$, *where I is the identity in $B(\mathcal{H})$.*

Proof. Since the orthogonal projections P_i' are orthogonal to each other, $\sum_{i=1}^{n} P_i'$ is an orthogonal projection which is less than or equal to the identity I. If $\sum_{i=1}^{n} P_i' < I$, then there exists a nonzero vector $v \in \mathcal{H}$ such that

$$\sum_{i=1}^{n} P_i' v = 0.$$

Then, we have

$$0 = P_i' x = (I - P_1' \vee \cdots \vee P_{i-1}' \vee P_i)x$$

or say

$$(P_1' \vee \cdots \vee P_{i-1}' \vee P_i)x = x$$

for all i. Since $P_m' x = 0$ for all $1 \leq m \leq i-1$, by Lemma 3.4.1, $P_i x = x$. Then, we have

$$\begin{aligned} nx &= \sum_{i=1}^{n} P_i x \\ &= \sum_{i=1}^{n} \sum_{j=1}^{k} u_{i,j} x \\ &= \sum_{j=1}^{k} (\sum_{i=1}^{n} u_{i,j} x), \end{aligned}$$

which implies that n is in the spectrum of $\sum_{j=1}^{k} \sum_{i=1}^{n} u_{i,j}$. Notice that, to every $1 \leq j \leq k$, $\sum_{i=1}^{n} u_{i,j} \leq I$ since they are orthogonal projections and orthogonal to each other. Therefore,

$$0 \leq \sum_{j=1}^{k} \sum_{i=1}^{n} u_{i,j} \leq \sum_{j=1}^{k} I \leq kI.$$

It contradicts to the implication above. The proof is complete. □

Corollary 3.4.3. $\sum_{i=1}^{n} P_i' \boldsymbol{P} = \boldsymbol{P}$.

Now, we turn to show some relations between partial distributional symmetries. The above construction can be applied quantum partial exchangeability:

Proposition 3.4.4. *Let $(\mathcal{A}, \phi)$ be a noncommutative probability space, $(x_i)_{i=1,\ldots,n}$ is a finite ordered sequence of random variables in $\mathcal{A}$. For fixed $n > k$, the joint distribution $\mu_{x_1,\ldots,x_n}$ is $A_l(n,k)$-invariant if it is $A_l(n,k+1)$-invariant*

Proof. Let $\{u_{ij}|1 \leq i \leq n, 1 \leq j \leq k\}$ be the set of standard generators of $A_l(n,k)$, Φ be a faithful representation of $A_l(n,k)$ into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. With the above construction, we can define $\{u'_{i,j}\}_{i=1,...,n;j=1,...,k+1}$ as following:

$$u'_{i,j} = \begin{cases} \Phi(u_{i,j}) & \text{if} \quad j \leq k \\ P'_j & \text{if} \quad j = k. \end{cases}$$

By Lemma 3.4.2, $\{u'_{i,j}\}_{i=1,...,n;j=1,...,k+1}$ satisfies the universal conditions for $A_l(n,k+1)$. Let $\{u''_{ij}|1 \leq i \leq n, 1 \leq j \leq k+1\}$ be the set of standard generators of $A_l(n,k+1)$. Then, there exists a C^*-homomorphism $\Phi : A_l(n,k+1) \to B(\mathcal{H})$ such that:

$$\Phi'(u''_{ij}) = u'_{i,j}.$$

Therefore, $\Phi^{-1}\Phi'$ defines a unital C^*-homomorphism

$$\Phi^{-1}\Phi' : C^* - alg\{u'_{i,j}|1 \leq i \leq n, 1 \leq j \leq k\} \to A_l(n,k)$$

such that

$$\Phi^{-1}\Phi'(u'_{i,j}) = u_{i,j}$$

for all $1 \leq i \leq n, 1 \leq j \leq k$.

If $\mu_{x_1,...,x_n}$ is $A_l(n,k+1)$-invariant, then

$$\mu_{x_1,...,x_{k+1}}(p) = \mu_{x_1,...,x_k} \otimes id_{A_l(n,k+1)}(\alpha^{(fp)}_{n,k+1}(p))$$

for all $p \in \mathbb{C}\langle X_1,...,X_{k+1}\rangle$. Let $p = X_{j_1}\cdots X_{j_l} \in \mathbb{C}\langle X_1,...,X_k\rangle$. Then, we have

$$\begin{aligned}
& \mu_{x_1,...,x_k}(p)1_{A(n,k)} \\
& \Phi^{-1}\Phi'(\mu_{x_1,...,x_{k+1}}(p)1_{A(n,k+1)})) \\
= \; & \Phi^{-1}\Phi'(\mu_{x_1,...,x_n} \otimes id_{A_l(n,k+1)}(\alpha^{(fp)}_{n,k+1}(X_{j_1}\cdots X_{j_l})) \\
= \; & \Phi^{-1}\Phi'(\mu_{x_1,...,x_n} \otimes id_{A_l(n,k+1)}(\sum_{i_1,...,i_l}^{n} X_{i_1}\cdots X_{i_l} \otimes u'_{i_1,j_1}\cdots u'_{i_l,j_l}) \\
= \; & \mu_{x_1,...,x_n} \otimes id_{A_l(n,k)}(\sum_{i_1,...,i_l}^{n} X_{i_1}\cdots X_{i_l} \otimes u_{i_1,j_1}\cdots u_{i_l,j_l}) \\
= \; & \mu_{x_1,...,x_n} \otimes id_{A_l(n,k)}(\alpha^{(fp)}_{n,k}(p)).
\end{aligned}$$

Since p is an arbitrary monomial, the proof is complete. □

The same, by comparing universal conditions , we have

Corollary 3.4.5. *$\mu_{x_1,...,x_n}$ is $B_l(n,k)$-invariant if it is $B_l(n,k+1)$-invariant*

Lemma 3.4.6. *$\mu_{x_1,...,x_n}$ is (n,k)-quantum spreadable if it is $A_l(n,k)$-invariant.*

Proof. Let $\{u_{i,j}\}_{i=1,...,n;j=1,...,k}$ be generators of $A_i(n,k)$ and $\{u'_{i,j}\}_{i=1,...,n;j=1,...,k}$ be generators of $A_l(n,k)$. Then, there is a well defined C^*-homomorphism $\beta : A_l(n,k) \to A_i(n,k)$ such that $\beta(u'_{i,j}) = u_{i,j}$. The existence of β is given by the universality of $A_l(n,k)$. Since $\mu_{x_1,...,x_n}$ is $A_l(n,k)$-invariant, for all monomials $p = X_{i_1} \cdots X_{i_m} \in \mathbb{C}\langle X_1, ..., X_k\rangle$, we have

$$\mu_{x_1,...,x_k}(p)1_{A_l(n,k)} = \mu_{x_1,...x_n} \otimes id_{A_l(n,k)}(\alpha^{(fp)}_{n,k}(p)) = \sum_{j_1,...,j_m} \phi(x_{j_1} \cdots x_{j_m})u'_{j_1,i_1} \cdots u'_{j_m,i_m}.$$

Apply β on both sides of the above equation, we have

$$\mu_{x_1,...,x_k}(p)1_{A_i(n,k)} = \sum_{j_1,...,j_m} \phi(x_{j_1} \cdots x_{j_m})u_{j_1,i_1} \cdots u_{j_m,i_m} = \mu_{x_1,...x_n} \otimes id_{A_i(n,k)}(\alpha_{n,k}(p)).$$

The proof is complete. □

The same, we have

Corollary 3.4.7. *$\mu_{x_1,...,x_n}$ is (n,k)-boolean spreadable if it is $B_l(n,k)$-invariant.*

Corollary 3.4.8. *$(x_1, ..., x_n)$ is boolean spreadable if it is boolean exchangeable. $(x_1, ..., x_n)$ is quantum spreadable if it is quantum exchangeable.*

In summary, for fixed $n, k \in \mathbb{N}$ such that $k < n$, we have the following diagrams:

$$\begin{array}{ccccc}
B(n,n)_{\text{inv}} & \longrightarrow & B_l(n,k)_{\text{inv}} & \longrightarrow & B_i(n,k)_{\text{inv}} \\
\downarrow & & \downarrow & & \downarrow \\
 & & & & M_i(n,k)_{\text{inv}} \\
 & & & & \downarrow \\
A(n,n)_{\text{inv}} & \longrightarrow & A_l(n,k)_{\text{inv}} & \longrightarrow & A_i(n,k)_{\text{inv}}
\end{array}$$

and

$$\begin{array}{ccc}
\textit{Booolean exchangeability} & \longrightarrow & \textit{Boolean spreadability} \\
\downarrow & & \downarrow \\
 & & \textit{Monotone spreadability} \\
 & & \downarrow \\
\textit{Quantum exchangeability} & \longrightarrow & \textit{Quantum spreadability.}
\end{array}$$

The arrow "condtion a) → condition b)" means that condition a) implies condition b).

Chapter 4

De Finetti type theorems in noncommutative probability

4.1 Boolean independence and freeness

In this section, we will show that operator valued boolean independent variables are sometimes operator valued freely independent. Therefore, we should not be surprised that the joint distribution of any sequence of identically boolean independent random variables is invariant under the coaction of the free quantum permutations. The main idea we will use is the unitalization of C^*-algebras. Hence, we provide a brief review of the unitalization of C^*-algebras here:
To every C^* algebra $\mathcal{A}$ one can associate a unital C^* algebra $\bar{\mathcal{A}}$ which contains $\mathcal{A}$ as a two-sided ideal and with the property that the quotient C^*-algebra $\bar{\mathcal{A}}/\mathcal{A}$ is isomorphic to $\mathbb{C}$. Actually, $\bar{\mathcal{A}} = \{x\bar{I} + a | x \in \mathbb{C}, a \in \mathcal{A}\}$, where $\bar{I}$ is the unit of $\bar{\mathcal{A}}$. We will denote $x\bar{I} + a$ by (x, a) where $x \in \mathbb{C}$ and $a \in \mathcal{A}$, then we have

$$(x, a) + (y, b) = (x + y, a + b), \quad (x, a)(y, b) = (xy, ab + xb + ya), \quad (x, a)^* = (\bar{x}, a^*),$$

for all $x, y \in \mathbb{C}$ and $a, b \in \mathcal{A}$.

Let $(\mathcal{A}, \mathcal{B}, E)$ be an operator-valued probability space where $\mathcal{A}$ and $\mathcal{B}$ are not necessarily unital. Let $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ be the unitalization defined above, then we can extend E to $\bar{E}$ s.t $(\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{E})$ is also an operator-valued probability space where $\bar{E}$ is a conditional expectation on $\bar{\mathcal{A}}$.
It is natural to define $\bar{E}$ as

$$\bar{E}[x, a] = (x, E[a]).$$

$\bar{E}[(1, 0)] = (1, 0)$, so $\bar{E}$ is unital. The linear property is easy to check.

Take $(x_1, b_1), (x_2, b_2) \in \bar{\mathcal{B}}$ and $(y, a) \in \bar{\mathcal{A}}$, we have

$$\begin{aligned}
\bar{E}[(x_1, b_1)(y, a)(x_2, b_2)] &= \bar{E}[x_1 y x_2, x_1 x_2 a + y x_2 b + x_2 b_1 a + x_1 a b_2 + y b_1 b_2 + b_1 a b_2] \\
&= (x_1 y x_2, E[x_1 x_2 a + y x_2 b + x_2 b_1 a + x_1 a b_2 + y b_1 b_2 + b_1 a b_2)] \\
&= (x_1 y x_2, x_1 x_2 E[a] + y x_2 b + x_2 b_1 E[a] + x_1 E[a] b_2 + y b_1 b_2 + b_1 E[a] b_2) \\
&= (x_1, b_1)(y, E[a])(x_2, b_2) \\
&= (x_1, b_1)\bar{E}[(y, a)](x_2, b_2).
\end{aligned}$$

It is obvious that $\bar{E}^2 = \bar{E}$. Hence, $\bar{E}$ is a $\bar{\mathcal{B}}$-$\bar{\mathcal{B}}$ bimodule from the unital algebra $\bar{\mathcal{A}}$ to the unital subalgebra $\bar{\mathcal{B}}$, i.e. a conditional expectation.

Proposition 4.1.1. *Let $(\mathcal{A}, \mathcal{B}, E) : \mathcal{A} \to \mathcal{B}$ be an operator valued probability space, $\{\mathcal{A}_i\}_{i \in I}$ be a $\mathcal{B}$-boolean independent family of sub-algebras and $\mathcal{B} \subset \mathcal{A}_i$ for all i. Then, in the unitalization operator probability space $(\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{E})$, $\{\bar{\mathcal{A}}_i\}_{i \in I}$ is a $\bar{\mathcal{B}}$-freely independent family of sub-algebras.*

Proof. Let $(x, a) \in \bar{\mathcal{A}}$, where $a \in \mathcal{A}$ and x is a complex number, then $\bar{E}[(x, a)] = (x, E[a])$, thus $\bar{E}[(x, a)] = 0$ iff $x = 0$ and $E[a] = 0$.
Now, we can check the freeness directly. Let $(x_k, a_k) \in \bar{\mathcal{A}}_{i_k}$, i.e $a_k \in \mathcal{A}_{i_k}$ and x_i's are complex numbers, for $k = 1, \cdots, n$ and $\bar{E}[x_k, a_k] = 0$ and $i_1 \neq i_2 \neq \cdots \neq i_n$, then we have $x_k = 0$ for all $k = 1, \cdots, n$ and

$$\begin{aligned}
\bar{E}[(x_1, a_1)(x_2, a_2) \cdots (x_n, a_n)] &= \bar{E}[(0, a_1)(0, a_2) \cdots (0, a_n)] \\
&= \bar{E}[(0, a_1 a_2 \cdot a_n)] \\
&= (0, E[a_1 a_2 \cdots a_n)] \\
&= (0, E[a_1]E[a_2] \cdots E[a_n]) \\
&= (0, 0) = 0.
\end{aligned}$$

and $\bar{\mathcal{B}} \subset \bar{\mathcal{A}}_i$ for all i. □

By checking the conditions for operator valued freeness directly as we did in the above theorem, we also have

Corollary 4.1.2. *Let $(\mathcal{A}, \mathcal{B}, E) : \mathcal{A} \to \mathcal{B}$ be an operator valued probability space, $\{\mathcal{B} \subset \mathcal{A}_i\}_{i \in I}$ be a $\mathcal{B}$-freely independent family of sub-algebras. Then, in their unitalization operator probability space $(\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{E})$, $\{\bar{\mathcal{A}}_i\}_{i \in I}$ is a $\bar{\mathcal{B}}$-freely independent family of sub-algebras.*

4.2 Operator valued boolean random variables are boolean exchangeable

In this section, we prove that the joint distribution of n boolean independent operator valued random variables are invariant under the linear coactions of $\mathcal{B}_s(n)$. The proof of the main theorem in this section involves combinatorial structure of the mixed moments of random variables. For boolean independence, the mixed moments of the random variables can be easily described by interval partitions. Therefore, we give an introduction below and we show some properties of this kind of partitions.

Given a set S, a collection of disjoint nonempty sets $P = \{V_i | i \in I\}$ is called a partition of S if $\bigcup_{i\in I} V_i = S$, $V_i \in P$ is called a block of the partition P. Let S be a finite ordered set, then all the partitions of S have finite blocks. A partition $P = \{V_1, \cdots V_r\}$ of S is interval if there are no two distinct blocks V_i and V_j and elements $a, c \in V_i$ and $b, d \in V_j$ s.t. $a < b < c$ or $b < c < d$. An interval partition $P = \{W_s | 1 \leq s \leq r\}$ is ordered if $a < b$ for all $a \in W_s$, $b \in W_t$ and $s < t$. We denote by $P_I(S)$ the collection of ordered interval partitions of S.

For convenience, we need to introduce an equivalence relation on indices sequences. Let I be an index set, $[k] = \{1, \cdots, k\}$ is an ordered set with the natural order. Let $I^k = I \times I \times \cdots \times I$ be the k-fold Cartesian product of the index set I. A sequence of indices $(i_m)_{m=1,\cdots,k} \in I^k$ is said to be compatible with an ordered interval partition $P = \{W_1, \cdots, W_r\} \in P_I([k])$ if $i_a = i_b$ whenever a, b are in the same block and $i_a \neq i_b$ whenever a, b are in two consecutive blocks, i.e. W_s and W_{s+1} for some $1 \leq s \leq r$. One should pay attention that $i_a = i_b$ is allowed for $a \in W_s$ and $b \in W_{s+2}$ for some $1 \leq s \leq r$.

Now, we define an equivalence relation $\sim_{P_I([k])}$ on I^k: two sequences of indices

$$(i_m)_{m=1,\cdots,k} \sim_{P_I([k])} (j_m)_{m=1,\cdots,k}$$

if the two sequences are both compatible with an ordered interval partition $P \in P_I([k])$.

Given $\mathcal{J} = (i_m)_{m=1,\cdots,k}, \mathcal{J}' = (j_m)_{m=1,\cdots,k} \in \{1, ..., n\}^k$, we denote $\mathbf{P}u_{i_1,j_1}u_{i_2,j_2}\cdots u_{i_k,j_k}\mathbf{P}$ by $U_{\mathcal{J},\mathcal{J}'}$.

Lemma 4.2.1. *Fix $k \in \mathbb{N}$, let $\mathcal{B}_s(n)$ be the boolean permutation quantum semigroup with standard generators $\{u_{i,j}\}_{i,j=1,\cdots,n}$ and $\boldsymbol{P}$. Let $\mathcal{J}_1 = (i_1, \cdots, i_k), \mathcal{J}_2 = (j_1, \cdots, j_k) \in [n]^k$ be two sequences if indices. Then, the product $U_{J_1,\mathcal{J}_2}$ is not vanishing if $\mathcal{J}_1 \sim_{P_I([k])} \mathcal{J}_2$.*

Proof. Suppose that each $\mathcal{J}_i$ is compatible with an ordered interval partition P_i for $i = 1, 2$. Let $P_1 = \{W_1, \cdots, W_{r_1}\}$ and $P_2 = \{W'_1, \cdots, W'_{r_2}\}$, then $P_1 \neq P_2$ implies that there exists a t such that $W_t \neq W'_t$ for some $1 \leq t \leq \min\{r_1, r_2\}$. Take the smallest t, then $W_s = W'_s$ whenever $s < t$ and $W_t \neq W'_t$. Then, these two intervals begin with the same number but end with different numbers. In other words , we have either $W_t \subsetneqq W'_t$ or $W'_t \subsetneqq W_t$. Without loss of generality, we assume $W_t \subsetneqq W'_t$, then there is a number q s.t $q \in W_t$ but $q + 1 \notin W_t$ and $q, q + 1 \in W'_t$. We have $i_q \neq i_{q+1}$ and $j_q = j_{q+1}$. Thus

$$U_{\mathcal{J}_1,\mathcal{J}_2} = \mathbf{P}u_{i_1,j_1} \cdots u_{i_q,j_q}u_{i_{q+1},j_{q+1}} \cdots u_{i_k,j_k}\mathbf{P} = 0.$$

□

Lemma 4.2.2. *Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ be an operator valued probability space. Let $(x_i)_{i=1,...,n}$ be a sequence of n random variables which are identically distributed and boolean independent* *with respect to E. Given two sequences of indices $\mathcal{J} = (i_q)_{q=1,\cdots,k}$, $\mathcal{J}' = (j_q)_{q=1,\cdots,k} \in [n]^k$ and $\mathcal{J} \sim_{P_I([n])} \mathcal{J}'$, then*

$$E[x_{i_1}b_1x_{i_2}b_2\cdots b_{k-1}x_{i_k})] = E[x_{j_1}b_1x_{j_2}b_2\cdots b_{k-1}x_{j_k}],$$

where $b_1, \cdots, b_{k-1} \in \mathcal{B} \cup \{I_{\mathcal{A}}\}$.

Proof. Suppose that $\mathcal{J}$ and $\mathcal{J}'$ are compatible with an ordered interval partition $P = \{W_1, \cdots, W_r\}$. Assume that $W_1 = \{1, \cdots, k_1\}$, $W_2\{k_1+1, \cdots, k_2\}$,...,$W_r = \{k_{r-1}+1, \cdots, k)\}$, then $i_{k_t} \neq i_{k_t+1}$ and $j_{k_t} \neq j_{k_t+1}$ for $t = 1, ..., r$. For convenience, we let $k_r = k$, $k_0 = 0$ and $b_k = I_{\mathcal{A}}$, we have

$$\begin{aligned}
& E[x_{i_1}b_1x_{i_2}b_2\cdots b_{k-1}x_{i_k}] \\
= & E[x_{i_1}b_1x_{i_2}b_2\cdots b_{n-1}x_{i_k}b_k] \\
= & E[\prod_{s=1}^{r}(\prod_{t=n_{s-1}+1}^{n_s} x_{i_t}b_t)] \\
= & \prod_{s=1}^{r} E[\prod_{t=n_{s-1}+1}^{n_s} x_{i_t}b_t)] \\
= & \prod_{s=1}^{r} E[\prod_{t=n_{s-1}+1}^{n_s} x_{j_t}b_t] \\
= & E[\prod_{s=1}^{r}\prod_{t=n_{s-1}+1}^{n_s} x_{j_t}b_t] \\
= & E[x_{j_1}b_1x_{j_2}b_2\cdots b_{k-1}x_{j_k}].
\end{aligned}$$

□

We denote $\sim_{P_I([k])}$ by $\sim_{P_I}$ when there is no confusion.

Let $\mathcal{B}_s(n)$ be the boolean permutation quantum semigroup with standard generators $\{u_{i,j}\}_{i,j=1,\cdots,n}$ and $\mathbf{P}$. We have

Lemma 4.2.3. *Fix k and $1 \leq i_1, \cdots, i_k \leq n$, then*

$$\sum_{j_1,\cdots,j_k=1}^{n} \boldsymbol{P}u_{i_1,j_1}\cdots u_{i_k,j_k}\boldsymbol{P} = \boldsymbol{P}$$

Proof. The proof is straightforward:

$$\begin{aligned}
& \sum_{j_1,\cdots,j_k=1}^{n} \mathbf{P}u_{i_1,j_1}\cdots u_{i_k,j_k}\mathbf{P} \\
= & \sum_{j_1,\cdots,j_{k-1}=1}^{n} \mathbf{P}u_{i_1,j_1}\cdots u_{i_{k-1},j_k-1}(\sum_{j_k=1}^{n} u_{i_k,j_k}\mathbf{P}) \\
= & \sum_{j_1,\cdots,j_{k-1}=1}^{n} \mathbf{P}u_{i_1,j_1}\cdots u_{i_{k-1},j_k-1}\mathbf{P} \\
= & \cdots = \mathbf{P}.
\end{aligned}$$

□

According to the definition of $\mathcal{B}_s(n)$, it follows that the product $u_{i_1,j_1}\cdots u_{i_k,j_k}$ is not vanishing only if it satisfies that $i_t \neq i_{t+1}$ whenever $j_t \neq j_{t+1}$ for all $1 \leq t \leq k-1$. Now, we can turn to prove the main theorem in this section.

Theorem 4.2.4. *Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ be an operator valued probability space, $\mathcal{A}$ be unital and $\{x_i\}_{i=1,\ldots,n}$ be a sequence of n random variables in $\mathcal{A}$ which is identically distributed and boolean independent with respect to E. Let ϕ be a linear functional on $\mathcal{B}$ and $\bar{\phi}$ is a linear functional on $\mathcal{A}$ where $\bar{\phi}(\cdot) = \phi(E[\cdot])$. Then, the joint distribution of the sequence $\{x_i\}_{i=1,\ldots,n}$ with respect to $\bar{\phi}$ is invariant under the linear coaction of the boolean permutation quantum semigroup $\mathcal{B}_s(n)$.*

Proof. Fix $k \in \mathbb{N}$, and indices $1 \leq i_1, \cdots, i_k \leq n$, and $b_1, \cdots, b_{k-1} \in \mathcal{B} \cup \{I_\mathcal{A}\}$, where $I_\mathcal{A}$ is the unit of $\mathcal{A}$, by the two lemmas above we have

$$\begin{aligned}
& \sum_{j_1,j_2,\cdots,j_k=1}^{n} E[x_{j_1}b_1x_{j_2}b_2\cdots b_{k-1}x_{j_k}] \otimes \mathbf{P}u_{i_1,j_1}\cdots u_{i_k,j_k}\ \mathbf{P} \\
= & \sum_{\substack{j_1,j_2,\cdots,j_k=1 \\ (j_s)_{s=1,\ldots,k}\sim_{P_I}(i_t)_{t=1,\ldots,k}}}^{n} E[x_{j_1}b_1x_{j_2}b_2\cdots b_{k-1}x_{j_k}] \otimes \mathbf{P}u_{i_1,j_1}\cdots u_{i_k,j_k}\ \mathbf{P} \\
= & \sum_{\substack{j_1,j_2,\cdots,j_k=1 \\ (j_s)_{s=1,\ldots,k}\sim_{P_I}(i_t)_{t=1,\ldots,k}}}^{n} E[x_{i_1}b_1x_{i_2}b_2\cdots b_{k-1}x_{i_k}] \otimes \mathbf{P}u_{i_1,j_1}\cdots u_{i_k,j_k}\ \mathbf{P} \\
= & \sum_{j_1,j_2,\cdots,j_n=1}^{k} E[x_{i_1}b_1x_{i_2}b_2\cdots b_{k-1}x_{i_k}] \otimes \mathbf{P}u_{i_1,j_1}\cdots u_{i_k,j_k}\ \mathbf{P} \\
= & E[x_{i_1}b_1x_{i_2}b_2\cdots b_{k-1}x_{i_k}] \otimes\ \mathbf{P}.
\end{aligned}$$

The last equality comes from Lemma 4.2.3. Let $b_1, ..., b_{k-1} = 1_\mathcal{A}$ and let $\phi \otimes id_{\mathcal{B}_s(n)}$ act on the two sides of the above equation then we have

$$\begin{aligned}
& \bar{\phi}(x_{i_1}x_{i_2}\cdots x_{i_k})\ \mathbf{P} \\
= & \bar{\phi}(x_{i_1}x_{i_2}\cdots x_{i_k})\ \mathbf{P} \\
= & \sum_{j_1,j_2,\cdots,j_k=1}^{n} \bar{\phi}(x_{j_1}x_{j_2}\cdots x_{j_n})\mathbf{P}u_{i_1,j_1}\cdots u_{i_k,j_k}\ \mathbf{P},
\end{aligned}$$

This is our desired conclusion. □

4.3 Tail algebra

In order to study boolean exchangeable sequences of random variables, we need to choose a suitable kind of noncommutative probability spaces. It is pointed by Hasebe [16] that a W^*-probability with a faithful normal state does not contain a pair of boolean independent random variables with Bernoulli law. Moreover, in the end of the next section, we will show that an infinite sequence of boolean exchangeable random variables in a W^*-probability space with a faithful normal state are equal to each other. Therefore, in boolean situation, it is necessary to consider W^* probability spaces with more general states rather than faithful states:

Definition 4.3.1. Let $\mathcal{A}$ be a von Neumann algebra. A normal state ϕ on $\mathcal{A}$ is said to be non-degenerated if $x = 0$ whenever $\phi(axb) = 0$ for all $a, b \in \mathcal{A}$.

Remark 4.3.2. By proposition 7.1.15 in [19], if ϕ is a non-degenerated normal state on $\mathcal{A}$ then the GNS representation associated to ϕ is faithful.

Let $(\mathcal{A}, \phi)$ be a W^*-probability space with a non-degenerated normal state ϕ. Suppose that $\mathcal{A}$ is generated by an infinite sequence of exchangeable random variables $\{x_i\}_{i\in\mathbb{N}}$. In the usual way, the tail algebra $\mathcal{A}_{tail}$ of $\{x_i\}_{i\in\mathbb{N}}$ is defined by:

$$\mathcal{A}_{tail} = \bigcap_{n=1}^{\infty} vN\{x_k | k \geq n\},$$

where $vN\{x_k|k \geq n\}$ is the von Neumann algebra generated by $\{x_k|k \geq n\}$. We call $\mathcal{A}_{tail}$ the unital tail algebra of $\{x_i\}_{i\in\mathbb{N}}$ because it contains the unit of the original algebra. In a W^*-probability space with a non-degenerated normal state, we need to consider another kind of tail algebra $\mathcal{T}$ which is defined by the following formula:

$$\mathcal{T} = \bigcap_{n=1}^{\infty} W^*\{x_k | k \geq n\},$$

where $W^*\{x_k|k \geq n\}$ is the WOT closure of the non-unital algebra generated by $\{x_k|k \geq n\}$. We call $\mathcal{T}$ the non-unital tail algebra of $\{x_i\}_{i\in\mathbb{N}}$. If the unit of $\mathcal{A}$ is contained in $\mathcal{T}$, then $\mathcal{T}$ is also the unital tail-algebra of $\{x_i\}_{i\in\mathbb{N}}$. Notice that the WOT closure of a non-unital algebra is different from the von Neumann algebra generated a non-unital algebra. The WOT closure of a non-unital algebra may not contain the unit of the original algebra. For example, let $P \in B(\mathbb{C}^2)$ be a one dimensional orthogonal projection. The weak operator closure of the algebra generated by P is $\mathbb{C}P$, but the vN-algebra generated by P is $\mathbb{C}1_{B(\mathbb{C}^2)} + \mathbb{C}P$.

In W^*-probability spaces with faithful states, the normal conditional expectation is constructed via the shift, i.e. a $*$-homomorphism which sends x_i to x_{i+1} for all $i \in \mathbb{N}$. In our situation, we should carefully choose the tail algebra so that we can construct a ϕ-preserving normal conditional expectation via shifts. To illustrate this phenomena, we provide two examples here. For the details of the examples, see [6] and [12].

Non-unital tail algebra case: Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_i\}_{i\in\mathbb{N}\cup\{0\}}$. We define a sequence of operators $\{x_n\}_{n\in\mathbb{N}}$ as follows:

$$x_n e_0 = e_n, \text{and} \;\; x_n e_i = \delta_{n,i} e_0 \text{ for } \; i \in \mathbb{N}.$$

Let $\mathcal{A}$ be the von Neumann algebra generated by $\{x_n\}_{n\in\mathbb{N}}$, then e_0 is cyclic for $\mathcal{A}$. Since $\mathcal{A}$ is WOT closed and contains all finite-rank operators, $\mathcal{A}$ is actually $B(\mathcal{H})$. On the other hand, if we denote by P_{e_i} the orthogonal projection onto the one dimensional generated by e_i for $i \in \mathbb{N} \cup \{0\}$, then

$$x_1^2 x_2^2 = P_{e_0}$$

and

$$\sum_{i=1}^{n} x_i^2 - (n-1)x_1^2 x_2^2 = \sum_{i=0}^{n} P_{e_i}.$$

Since $\lim_{n\to\infty} \sum_{i=0}^{n} P_{e_i} = I_{B(\mathcal{H})}$ in WOT, $I_{B(\mathcal{H})}$ is contained in the WOT closure of the non-unital algebra generated by $(x_i)_{i\in\mathbb{N}}$. Let ϕ be the vector state $\phi(\cdot) = \langle \cdot e_0, e_0 \rangle$. We can see that the random variables x_i's are identically distributed and boolean independent. Since e_0 is cyclic for $B(\mathcal{H})$, the probability space $(\mathcal{A}, \phi)$ is non-degenerated. To construct a ϕ-preserving conditional expectation, we need to use the non-unital tail algebra here. We have

$$\mathcal{T} = \bigcap_{n=1}^{\infty} W^*\{x_k | k \geq n\} = \mathbb{C}P_{e_0}.$$

The tail algebra $\mathcal{T} = \mathbb{C}P_{e_0}$ does not contain the unit of $B(\mathcal{H})$. The conditional expectation $E : \mathcal{A} \to \mathcal{T}$ is given by the following formula:

$$E[x] = P_{e_0} x P_{e_0},$$

for all $x \in \mathcal{A}$. One can check that the sequence $(x_i)_{i\in\mathbb{N}}$ is boolean independent in the operator valued probability space $(\mathcal{A}, \mathcal{T}, E : \mathcal{A} \to \mathcal{T})$.
If we use unital tail algebra in this situation, then we have

$$\mathcal{A}_{tail} = \bigcap_{n=1}^{\infty} vN\{x_k | k \geq n\} \supset \{I_{B(\mathcal{H}), P_{e_0}}\}.$$

Notice that

$$P_{e_0} = E[\sum_{i=1}^{n} x_i^2 - (n-1)x_1^2 x_2^2]$$

for all n. We have

$$w^* - \lim_{n\to\infty} E[\sum_{i=0}^{n} P_{e_i}] = P_{e_0} \neq I_{B(\mathcal{H})},$$

but $\lim_{n\to\infty} \sum_{i=0}^{n} P_{e_i} = I_{B(\mathcal{H})}$ in WOT. In conclusion, if the conditional expectation is normal, then it may not be unital. In other words, the normal map E is not a conditional expectation in this case.

Unital tail algebra case: Let $\mathcal{H}_1 = \mathcal{H} \oplus \mathbb{C}e_{-1}$ be the direct sum of the Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_i\}_{i\in\mathbb{N}\cup\{0\}}$ and $\mathbb{C}e_{-1}$. As we constructed in the previous example, we define a sequence of operators $\{x_n\}_{n\in\mathbb{N}}$ as follows:

$$x_n e_0 = e_n, \text{and } \ x_n e_i = \delta_{n,i} e_0 \text{ for } \ i \in \mathbb{N}, \quad x_n e_{-1} = 0.$$

Let $\mathcal{A}$ be the von Neumann algebra generated by $\{x_n\}_{n\in\mathbb{N}}$, then $\mathcal{A} = B(\mathcal{H}) \oplus \mathbb{C}P_{e_{-1}}$. Therefore, the WOT-closure of the non-unital algebra generated by $\{x_n\}_{n\in\mathbb{N}}$ is $B(\mathcal{H}) \oplus 0$ does not contain the unit of $\mathcal{A}$. Let ϕ be the vector state $\phi(\cdot) = \frac{1}{2}\langle \cdot(e_0 + e_{-1}), e_0 + e_{-1}\rangle$, then the random variables x_i's are identically distributed and boolean independent. In this case, we need to use the unital tail algebra here. The unital tail algebra is

$$\mathcal{A}_{tail} = \bigcap_{n=1}^{\infty} vN\{x_k | k \geq n\} = \mathbb{C}I_{\mathcal{H}_1} \oplus \mathbb{C}P_{e_0}.$$

According to our construction, the conditional expectation E is given by the following formula:

$$E[x] = P_{e_0} x P_{e_0} + \langle x e_{-1}, e_{-1}\rangle (I_{\mathcal{H}_1} - P_{e_{-1}}),$$

for all $a \in \mathcal{A}$.

We see that the the non-unital tail algebra here is

$$\mathcal{T} = \bigcap_{n=1}^{\infty} W^*\{x_k | k \geq n\} = \mathbb{C}P_{e_0}.$$

If E is a conditional expectation from $\mathcal{A}$ to $\mathcal{T}$, then it sends the unit of $\mathcal{A}$ to the unit of $\mathcal{T}$. We have

$$E[I_{\mathcal{H}}] = P_{e_0}.$$

But,

$$\phi(I_{\mathcal{H}}) \neq \frac{1}{2} = \phi(P_{e_0}).$$

Therefore, there is no ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{T}$ in this situation.

Now, we turn to define the conditional expectation for our de Finetti type theorem. In the rest of this section, we suppose that the joint distribution of $\{x_i\}_{i\in\mathbb{N}}$ is boolean exchangeable. Let $\mathcal{A}_0$ be the non-unital algebra generated by $\{x_i\}_{i\in\mathbb{N}}$. In addition, we assume that the unit $1_{\mathcal{A}}$ of $\mathcal{A}$ is contained in the WOT closure of $\mathcal{A}_0$. We will denote the GNS construction associated to ϕ by $(\mathcal{H}, \xi, \pi)$, then there is a linear map $\hat{\ } : \mathcal{A}_0 \to \mathcal{H}$ such that $\hat{a} = \pi(a)\xi$ for all $a \in \mathcal{A}_0$. For convenience, we denote by $\mathcal{A}_n$ the non-unital algebra generated by $\{x_k | k > n\}$. Now, we turn to define our $\mathcal{T}$-linear map. Recall that in [22], the normal conditional expectation is defined as the WOT limit of shifts. The construction works because the shift of an exchangeable sequence is automatically a normal isomorphism. This fact relies on the property that a faithful normal state on $\mathcal{A}$ is faithful on all its subalgebras. When we consider W^*-probability spaces with non-degenerated normal states, we can see that a non-degenerated normal state on $\mathcal{A}$ is not necessarily non-degenerated on $\mathcal{A}$'s subalgebras. Therefore, in our situation , the shift of exchangeable sequence is not automatically a normal isomorphism. In consequence, the conditional expectation defined by taking WOT limit of shifts could not be well defined. Indeed, we are not sure that if we can construct a normal conditional expectation under the assumption that the random variables are only exchangeable. But in our situation, this is not a problem, since boolean exchangeability is much stronger than classical exchangeability. For this reason, we check the details of our conditional expectation in the rest of this section:

Lemma 4.3.3. *Let $\mathcal{A}$ be a von Neumann algebra generated by an infinite sequence of self-adjoint random variables $(x_i)_{i\in\mathbb{N}}$, ϕ be a non-degenerated normal state on $\mathcal{A}$. If the sequence $(x_i)_{i\in\mathbb{N}}$ is exchangeable in $(\mathcal{A},\phi)$, then there is a C^*-isomorphism $\alpha : \mathcal{A}_0^{\|\cdot\|} \to \mathcal{A}_1^{\|\cdot\|}$ such that,*

$$\alpha(x_i) = x_{i+1},$$

for all $i \in \mathbb{N}$, where $\mathcal{A}_i^{\|\cdot\|}$ is the C^*-algebra generated by $\mathcal{A}_i$.

Proof. Let $(\mathcal{H}, \xi, \pi)$ be the GNS construction associated to ϕ, it follows that $\{\hat{a}|a \in \mathcal{A}_0\}$ is dense in $\mathcal{H}$. For each $n \in \mathbb{N}$, denote by $A_{[n]}$ the non-unital algebra generated by $\{x_i|i \le n\}$. Then $\bigcup_{n=1}^{\infty}\{\pi(a)\xi|a \in A_{[n]}\}$ is dense in $\mathcal{H}$. Given $y \in \bigcup_{n=1}^{\infty} A_{[n]}$, there exists $N \in \mathbb{N}$ such that $y \in A_{[N]}$. We can assume $y = p(x_1, ..., x_N)$ for some $p \in \mathbb{C}\langle X_1, ..., X_N\rangle_0$, then we have

$$\begin{aligned} & \|\pi(p(x_1, ..., x_N))\xi\|^2 = \phi(\pi(p(x_1, ..., x_N)^*(p(x_1, ..., x_N))) \\ = & \ \phi(p(x_2, ..., x_{N+1})^*(p(x_2, ..., x_{N+1})) \\ = & \ \|\pi(p(x_2, ..., x_{N+1}))\xi\|^2 \end{aligned}$$

We can define an isometry U from $\mathcal{H}$ to its subspace $\mathcal{H}_1$ which is generated by $\{\hat{a}|a \in A_1\}$ by the following formula:

$$U\pi(x_{i_1}\cdots x_{i_k})\xi = \pi(x_{i_1+1}\cdots x_{i_k+1})\xi,$$

for all $i_1, ..., i_k \in \mathbb{N}$.
Since ϕ gives a faithful representation to $\mathcal{A}$, it gives a faithful representation to $\mathcal{A}_0^{\|\cdot\|}$. For all $y \in \mathcal{A}_1$, according to the faithfulness, we have

$$\|y\|^2 = \sup\{\frac{\langle y^*y\hat{a}, \hat{a}\rangle}{\langle \hat{a}, \hat{a}\rangle}|a \in \mathcal{A}_0, \hat{a} \neq 0\} = \sup\{\frac{\phi(a^*y^*ya)}{\phi(a^*a)}|a \in \mathcal{A}_0, \phi(a^*a) \neq 0\}.$$

Denote by $(\mathcal{H}', \xi', \pi')$ the GNS representation of $\mathcal{A}_1$ associated to ϕ. Indeed, $\mathcal{H}'$ can treated as $\mathcal{H}_1$. Because the identity of $\mathcal{A}$ is contained in the weak*-closure of the non-unital algebra generated by $(x_i)_{i\in\mathbb{N}}$, by the Kaplansky density theorem, there exists a bounded sequence $\{y_i|\|y_i\| \le 1\} \in \bigcup_{n=1}^{\infty} A_{[n]}$ such that y_i converges to $1_{\mathcal{A}}$ in WOT. Therefore, $\pi(y_i)\xi$

converges to ξ in norm. Again, by the exchangeability of $(x_i)_{i\in\mathbb{N}}$ and $U\pi(y_i)\xi \in \{\hat{b}|b \in \mathcal{A}_1\}$ for all i, we have

$$\|U\pi(y_i)\xi\| = \|\pi(y_i)\xi\| \leq 1$$

and

$$\langle U\pi(y_i)\xi, \xi\rangle = \langle \pi(y_i)\xi, \xi\rangle \to 1.$$

Therefore, $U\pi(y_i)\xi$ converges to ξ in norm, namely, $\xi \in \mathcal{H}_1$.

Let $x \in \mathcal{A}_1$, then $x = p(x_2, ...x_{N+1})$ for some N and $p \in \mathbb{C}\langle X_1, ..., X_N\rangle_0$. For every $y \in \mathcal{A}_0$ there exists an M, such that $y = p'(x_1, ..., x_M)$ for some $p' \in \mathbb{C}\langle X_1, ..., X_M\rangle_0$. By the exchangeability, we send x_1 to x_{N+M}. Then

$$\begin{aligned}
\|\pi(x)\hat{y}\|^2_{\mathcal{H}} &= \phi(p'(x_1, ..., x_M)^* p(x_2, ...x_{N+1})^* p(x_2, ...x_{N+1}) p'(x_1, ..., x_M)) \\
&= \phi(p'(x_{M+N}, ..., x_M)^* p(x_2, ...x_{N+1})^* p(x_2, ...x_{N+1}) p'(x_{N+M}, x_2, x_3..., x_M)) \\
&= \|\pi'(x)p'(\widehat{x_{M+N}, x_2..., x_M}))\|^2_{\mathcal{H}'}
\end{aligned}$$

and

$$\|p'(\widehat{x_1, ..., x_M}))\|_{\mathcal{H}} = \|p'(\widehat{x_{M+N}, x_2..., x_M}))\|_{\mathcal{H}'}.$$

Therefore, we get

$$\{\frac{\|\pi(x)\hat{a}\|_{\mathcal{H}}}{\|\hat{a}\|_{\mathcal{H}}}|a \in \mathcal{A}_0, \hat{a} \neq 0\} \subseteq \{\frac{\|\pi'(x)\hat{a}\|_{\mathcal{H}'}}{\|\hat{a}\|_{\mathcal{H}'}}|a \in \mathcal{A}_1, \hat{a} \neq 0\},$$

which implies

$$\|x\| = \|\pi(x)\| = \sup\{\frac{\|\pi x\hat{a}\|_{\mathcal{H}}}{\|\hat{a}\|_{\mathcal{H}}}|a \in \mathcal{A}_0, \hat{a} \neq 0\} \leq \sup\{\frac{\|\pi'(x)\hat{a}\|_{\mathcal{H}'}}{\|\hat{a}\|_{\mathcal{H}'}}|a \in \mathcal{A}_1, \hat{a} \neq 0\} = \|\pi'(x)\|.$$

It follows that $\|x\| = \|\pi'(x)\|$ for all $x \in \mathcal{A}_1$. By taking the norm limit, we have $\|x\| = \|\pi'(x)\|$ for all $x \in \mathcal{A}_1^{\|\cdot\|}$, so the GNS representation of $\mathcal{A}_1^{\|\cdot\|}$ associated to ϕ is faithful.

Now, we turn to define our C^*-isomorphism α:

Since U is an isometric isomorphism from $\mathcal{H}$ to $\mathcal{H}'$, we define a homomorphism $\alpha' : \pi(\mathcal{A}_0) \to B(\mathcal{H}')$ by the following formula

$$\alpha'(y) = UyU^*,$$

for $y \in \pi(\mathcal{A}_0)$. Let $y \in \pi(\mathcal{A}_{[n]})$, then $y = \pi(p(x_1, ..., x_n))$ for some $p \in \mathbb{C}\langle X_1, ..., X_n\rangle_0$. For all $v \in \bigcup_{n=2}^{\infty}\{\pi(a)\xi|a \in A_{[n]} \subset \mathcal{H}'$, there exists $N \in \mathbb{N}$ and $p_1 \in \mathbb{C}\langle X_1, ...X_N\rangle_0$ such that $v = \pi(p_1(x_2, ..., x_{N+1}))\xi$. We have

$$\begin{aligned}
\alpha'(y)v &= U\pi(p(x_1, ..., x_n)U^*\pi(p_1(x_2, ..., x_{N+1}))\xi \\
&= U\pi(p(x_1, ..., x_n)\pi(p_1(x_1, ..., x_N))\xi \\
&= U\pi(p(x_1, ..., x_n)p_1(x_1, ..., x_N))\xi \\
&= \pi(p(x_2, ..., x_{n+1})p_1(x_1, ..., x_{N+1}))\xi
\end{aligned} .$$

Since $\bigcup_{n=2}^{\infty}\{\pi(a)\xi|a \in A_{[n]}$ is dense in $\mathcal{H}_1$, we get $\alpha'(\pi(p(x_1,...,x_n))) = \pi(p(x_2,...,x_{n+1}))$. Because $(\mathcal{H},\xi,\pi)$ and $(\mathcal{H}',\xi',\pi')$ are faithful GNS representations for $\mathcal{A}_0$ and $\mathcal{A}_1$ respectively, there is a well defined norm preserving homomorphism $\alpha : \mathcal{A}_0 \to \mathcal{A}_1$, such that $\alpha(x_i) = x_{i+1}$ for all $i \in \mathbb{N}$. Therefore, α extends to a C^*-isomorphism from $\mathcal{A}_0^{\|\cdot\|}$ to $\mathcal{A}_1^{\|\cdot\|}$. □

Since $W^*\{x_k|k \geq n\}$'s are WOT closed, their intersection is a WOT closed subset of $\mathcal{A}$. Following the proof of proposition 4.2 in [23], we have

Lemma 4.3.4. *For each $a \in \mathcal{A}_0$, $\{\alpha^n(a)\}_{n\in\mathbb{N}}$ is a bounded WOT convergent sequence. Therefore, there exists a well defined ϕ-preserving linear map $E : \mathcal{A}_0 \to \mathcal{T}$ by the following formula:*

$$E[a] = w^* - \lim_{n\infty} \alpha^n(a)$$

for $a \in \mathcal{A}_0$

Proof. By lemma 4.3.3, there is a norm preserving endomorphism α of $\mathcal{A}_0$ such that

$$\phi \circ \alpha = \phi \quad \text{and} \quad \alpha(x_i) = x_{i+1}.$$

For $I \subset \mathbb{N}$, denote by $\mathcal{A}_I$ the non-unital algebra generated by $\{x_i|i \in I\}$. Suppose $a, b, c \in \bigcup_{|I|<\infty} A_I$, we can assume $a \in \mathcal{A}_I$,$b \in \mathcal{A}_J$ and $c \in \mathcal{A}_K$ for some finite sets $I, J, K \subset \mathbb{N}$. Because I, J, K are finite, there exists an N such that $(I \cup K) \cap (J + n) = \emptyset$, for all $n > N$. We infer from the exchangeability that $\phi(a\alpha^n(b)c) = \phi(a\alpha^{n+1}(b)c)$ for all $n > N$. This establishes the limit

$$\lim_{n\to\infty} \phi(a\alpha^n(b)c)$$

on the weak*-dense algebra $\bigcup_{|I|<\infty} A_I$. We conclude from this and $\{\alpha^n(b)\}_{n\in\mathbb{N}}$ is bounded that the pointwise limit of the sequence α defines a linear map $E : \mathcal{A}_0 \to \mathcal{A}$ such that $E(\mathcal{A}_0) \subset \mathcal{T}$. □

To extend E to the W^*-algebra $\mathcal{A}$, we need to make use of the boolean invariance conditions.

Lemma 4.3.5. *Let $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{A}$ be an infinite sequence of random variables whose joint distribution is invariant under the linear coactions of the quantum semigroups $\mathcal{B}_s(k)$'s, then*

$$\phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}) = \phi(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}),$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, and $k_1, ..., k_n \in \mathbb{N}$

Proof. If $i_l \neq i_m$ for all $l \neq m$, then the statement holds by the exchangeability of the sequence. Suppose the number i_l appears m times in the sequence, which are $\{i_{l_j}\}_{j=1,...,m}$ such that $i_{l_j} = i_l$ and $l_1 < l_2 < \cdots < l_m$. Without loss of generality, we can assume that

$i_1, ..., i_n \leq N+1$ and $i_l = N+1$ for some N by the exchangeability.

For each $M \in \mathbb{N}$, by lemma 3.3, we have the following representation π_M of the quantum semigroup $\mathcal{B}_s(M+N)$:

$$\pi_M(u_{i,j}) = \begin{cases} P_{i-N,j-N}, & \text{if} \quad \min\{i,j\} > N \\ \delta_{i,j} P, & \text{if} \quad \min\{i,j\} \leq N \end{cases},$$

and $\pi(\mathbf{P}) = P$, where $p_{i,j}$ and p are projections in $B(\mathbb{C}^{2M})$given by lemma 3.3. Then we have

$$PP_{i,j}P = \frac{1}{M}P,$$

for $1 \leq i, j \leq N$.
According to the boolean invariance condition, we have:

$$\begin{aligned}
& \phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n})P \\
= & \sum_{j_1, j_2, \ldots j_n = 1}^{M+N} \phi(x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_n}^{k_n}) P u_{j_1,i_1} \cdots u_{j_n,i_n} P \\
= & \sum_{j_{l_1}, j_{l_2}, \ldots j_{l_m} = 1}^{N} \phi(x_{i_1}^{k_1} \cdots x_{j_{l_1}}^{k_{l_1}} \cdots x_{j_{l_2}}^{k_{l_2}} \cdots x_{i_n}^{k_n}) P P_{j_{l_1}, i_{l_1}} P P_{j_{l_2}, i_{l_2}} P \cdots u_{j_{l_m}, i_{l_m}} P \\
= & \tfrac{1}{M^m} \sum_{j_{l_1}, j_{l_2}, \ldots j_{l_m} = 1}^{N} \phi(x_{i_1}^{k_1} \cdots x_{j_{l_1}}^{k_{l_1}} \cdots x_{j_{l_2}}^{k_{l_2}} \cdots x_{i_n}^{k_n}) P \\
= & \tfrac{1}{M^m} [\sum_{j_{l_s} \neq j_{l_t} \text{ if } s \neq t}^{N} \phi(x_{i_1}^{k_1} \cdots x_{j_{l_1}}^{k_{l_1}} \cdots x_{j_{l_2}}^{k_{l_2}} \cdots x_{i_n}^{k_n}) P + \sum_{j_{l_s} = j_{l_t} \text{ for some } s \neq t}^{N} \phi(x_{i_1}^{k_1} \cdots x_{j_{l_1}}^{k_{l_1}} \cdots x_{j_{l_2}}^{k_{l_2}} \cdots x_{i_n}^{k_n}) P].
\end{aligned}$$

In the first part of the sum, by the exchangeability, it follows that

$$\phi(x_{i_1}^{k_1} \cdots x_{j_{l_1}}^{k_{l_1}} \cdots x_{j_{l_2}}^{k_{l_2}} \cdots x_{i_n}^{k_n}) = \phi(x_{i_1}^{k_1} \cdots x_{N+1}^{k_{l_1}} \cdots x_{N+2}^{k_{l_2}} \cdots x_{i_n}^{k_n}),$$

where we sent j_{l_s} to $N+s$. Then, we have

$$\frac{1}{M^m} \sum_{j_{l_s} \neq j_{l_t} \text{ if } s \neq t}^{N} \phi(x_{i_1}^{k_1} \cdots x_{j_{l_1}}^{k_{l_1}} \cdots x_{j_{l_2}}^{k_{l_2}} \cdots x_{i_n}^{k_n}) P = \frac{\prod_{s=0}^{m-1} (M-s)}{M^m} \phi(x_{i_1}^{k_1} \cdots x_{N+1}^{k_{l_1}} \cdots x_{N+2}^{k_{l_2}} \cdots x_{i_n}^{k_n}) P,$$

which converges to $\phi(x_{i_1}^{k_1} \cdots x_{N+1}^{k_{l_1}} \cdots x_{N+2}^{k_{l_2}} \cdots x_{i_n}^{k_n})P$ as M goes to ∞.
To the second part of the sum, we have

$$\phi(x_{i_1}^{k_1} \cdots x_{j_{l_1}}^{k_{l_1}} \cdots x_{j_{l_2}}^{k_{l_2}} \cdots x_{i_n}^{k_n}) \leq \| x_{i_1}^{k_1} \cdots x_{j_{l_1}}^{k_{l_1}} \cdots x_{j_{l_2}}^{k_{l_2}} \cdots x_{i_n}^{k_n} \| \leq \| x_1^{k_1 + \cdots + k_n} \|,$$

which is bounded, therefore,

$$|\frac{1}{M^m} \sum_{j_{l_s} = j_{l_t} \text{ for some } s \neq t}^{N} \phi(x_{i_1}^{k_1} \cdots x_{j_{l_1}}^{k_{l_1}} \cdots x_{j_{l_2}}^{k_{l_2}} \cdots x_{i_n}^{k_n})| \leq (1 - \frac{\prod_{s=0}^{m-1} (M-s)}{M^m}) \| x_1^{k_1 + \cdots + k_n} \|$$

goes to 0 as M goes to ∞. By now, we have showed that if there are indices $i_s = i_t$ for $s \neq t$ in the the sequence, we can, without changing the value of the mixed moments, change them to two different large numbers j_s, j_t such that j_s, j_t differ the other indices. After finite steps, we will have

$$\phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}) = \phi(x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_n}^{k_n}),$$

such that all the j_l' are not equal to any of the other indices. By the exchangeability, the proof is complete. □

Corollary 4.3.6. *Let $\{x_i\}_{i\in\mathbb{N}} \subset (\mathcal{A}, \phi)$ be an infinite sequence of random variables whose joint distribution is invariant under the linear coactions of the quantum semigroups $\mathcal{B}_s(k)$'s, then*

$$\phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}) = \phi(x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_n}^{k_n}),$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $j_1 \neq j_2 \neq \cdots \neq j_n$, $i_1, ..., i_n, j_1, ...j_n \in \mathbb{N}$ and $k_1, ..., k_n \in \mathbb{N}$. Moreover, we have

$$\phi(a x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n} b) = \phi(a x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_n}^{k_n} b),$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $j_1 \neq j_2 \neq \cdots \neq j_n$, $i_1, ..., i_n$, $j_1, ...j_n > M$, $k_1, ..., k_n \in \mathbb{N}$ and $a, b \in \mathcal{A}_{[M]}$ for some $M \in \mathbb{N}$.

Lemma 4.3.7. *For all $a, b, y \in \mathcal{A}_0$, we have*

$$\langle E(y)\hat{a}, \hat{b}\rangle = \langle y\widehat{E(a)}, \widehat{E[b]}\rangle.$$

Proof. Because an element in $\mathcal{A}_0$ is a finite linear combination of the noncommutative monomials, it suffices to show the property in the case: $b^* = x_{i_1}^{r_1} \cdots x_{i_l}^{r_l}$, $y = x_{j_1}^{s_1} \cdots x_{j_m}^{s_m}$, $a = x_{k_1}^{t_1} \cdots x_{k_n}^{t_n}$, where $i_1 \neq i_2 \neq \cdots \neq i_l$, $j_1 \neq ... \neq j_m$, $k_1 \neq ... \neq k_n$ and all the power indices are positive integers. Let $N = \max\{i_1, ..., i_l, j_1, ..., j_m, k_1, ..., k_n\}$, for all $L > N$, we have $i_l \neq j_1 + L$ and $j_m + L \neq k_1$. Therefore, we have

$$\begin{aligned}
\langle E(y)\hat{a}, \hat{b}\rangle &= \lim_{M\to\infty} \langle \alpha^M(y)\hat{a}, \hat{b}\rangle \\
&= \langle \alpha^L(y)\hat{a}, \hat{b}\rangle \\
&= \phi(x_{i_1}^{r_1} \cdots x_{i_l}^{r_1} x_{j_1+L}^{s_1} \cdots x_{j_m+L}^{s_m} x_{k_1}^{t_1} \cdots x_{k_n}^{t_n}),
\end{aligned}$$

by corollary 4.3.6,

$$\begin{aligned}
&= \phi(x_1^{r_1} \cdots x_l^{r_l} x_{l+1}^{s_1} \cdots x_{l+m}^{s_m} x_{l+m+1}^{t_1} \cdots x_{l+m+n}^{t_n}) \\
&= \phi(x_1^{r_1} \cdots x_l^{r_l} x_{l+1}^{s_1} \cdots x_{l+m}^{s_m} x_{l+m+1}^{t_1} \cdots x_{l+m+n}^{t_n}) \\
&= \phi(x_{i_1+L}^{r_1} \cdots x_{i_l+L}^{r_1} x_{j_1}^{s_1} \cdots x_{j_m}^{s_m} x_{k_1+2L}^{t_1} \cdots x_{k_n+2L}^{t_n}) \\
&= \phi(\alpha^L(x_{i_1}^{r_1} \cdots x_{i_l}^{r_1}) x_{j_1}^{s_1} \cdots x_{j_m}^{s_m} \alpha^{2L}(x_{k_1}^{t_1} \cdots x_{k_n}^{t_n})) \\
&= \lim_{M\to\infty} \phi(\alpha^N(b^*) y \alpha^{2L+M}(a)) \\
&= \phi(\alpha^L(b^*) y E[a]).
\end{aligned}$$

Notice that $\{\alpha^L(b)|L \leq N\}$ is a bounded sequence of random variables which converges to $E[b^*]$ in WOT and $\phi(\cdot yE[a])$ is a normal linear functional on $\mathcal{A}$, we have

$$\begin{aligned}\phi(\alpha^L(b^*)yE[a]) &= \lim_{M\to\infty}\phi(\alpha^M(b^*)yE[a]) \\ &= \phi(E[b]^*yE[a]) \\ &= \langle y\widehat{E[a]}, \widehat{E[b]}\rangle.\end{aligned}$$

□

Lemma 4.3.8. *Let* $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_0$ *be a bounded sequence of random variables such that* $w^* - \lim y_n = 0$*, then* $w^* - \lim E[y_n] = 0$.

Proof. For all $a, b \in \mathcal{A}_0$, we have

$$\lim_n \langle E[y_n]\hat{a}, \widehat{E[b]}\rangle = \lim_n \langle y_n\widehat{E[a]}, \widehat{E[b]}\rangle = 0.$$

Since $\{\hat{a}|a \in \mathcal{A}_0\}$ is dense in $\mathcal{H}_\xi$, we get our desired conclusion. □

Let $y \in \mathcal{A}$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_0$ be a bounded sequence such that y_n converges to y in WOT. For all $a, b \in \mathcal{A}_0$, we have

$$\lim_n \langle E[y_n]\hat{a}, \hat{b}\rangle = \lim_n \langle y_n\widehat{E[a]}, \widehat{E[b]}\rangle = \langle y\widehat{E[a]}, \widehat{E[b]}\rangle.$$

Therefore, $\{E[y_n]\}_{n\in\mathbb{N}}$ converges to an element y' in pointwise weak topology, by the lemma above, we see that y' is independent of the choice of $\{y_n\}_{n\in\mathbb{N}}$. Since $\{E[y_n]\}_{n\in\mathbb{N}} \subset \mathcal{T}$, we have $y' \in \mathcal{T}$. By now, we have defined a linear map $E : \mathcal{A} \to \mathcal{T}$ and we have

Lemma 4.3.9. E *is normal.*

Proof. Let $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ be a bounded WOT convergent sequence of random variables such that $w^* - \lim\limits_{n\to\infty} y_n = y$. Then, we have

$$\lim_{n\to\infty}\langle E[y_n]\hat{a}, \hat{b}\rangle = \lim_{n\to\infty}\langle y_n\widehat{E[a]}, \widehat{E[b]}\rangle = \langle y\widehat{E[a]}, \widehat{E[b]}\rangle = \langle E[y]\hat{a}, \hat{b}\rangle,$$

for all $a, b \in \mathcal{A}_0$. Therefore, E is normal. □

Now, we can turn to show that E is a conditional expectation from $\mathcal{A}$ to ϕ:

Lemma 4.3.10. $E[a] = a$ *for all* $a \in \mathcal{T}$.

Proof. Let $a \in \mathcal{T}$, $b, c \in \mathcal{A}_0$, then there exists an $N \in \mathbb{N}$ such that $a \in \overline{\mathcal{A}_{N+1}}^{w^*}$ and $b, c \in \mathcal{A}_{[N]}$. We can approximate a in WOT by a bounded sequence $(a_k)_{k\in\mathbb{N}} \subset \mathcal{A}_{N+1}$ in WOT. According to the definition of E and the exchangeability, we have

$$\begin{aligned} \langle E[a]\hat{c}, \hat{b}\rangle &= \phi(b^* E[a]c) \\ &= \lim_k \phi(b^* E[a_k]c) \\ &= \lim_k \lim_n \phi(b^* \alpha^n(a_k)c) \\ &= \lim_k \phi(b^* a_k c) \\ &= \phi(b^* ac) = \langle a\hat{c}, \hat{b}\rangle. \end{aligned}$$

The equation is true for all $b, c \in \mathcal{A}_0$, so $E[a] = a$. □

To check the bimodule property of E, we need to show that the quality of 4.3.7 holds for all $x \in \mathcal{A}$:

Lemma 4.3.11. *For all $a, b, x \in \mathcal{A}$, we have*

$$\phi(aE[x]b) = \phi(E[a]xE[b]).$$

Proof. By the Kaplansky's density theorem, there exist two bounded sequences $\{a_n \in \mathcal{A}_0 | \|a_n\| \leq \|a\|, n \in \mathbb{N}\}$ and $\{b_n \in \mathcal{A}_0 | \|b_n\| \leq \|b\|, n \in \mathbb{N}\}$ which converge to a and b in WOT, respectively. Since ϕ and E are normal, we have

$$\begin{aligned} \phi(aE[x]b) &= \lim_n \phi(a_n E[x]b) \\ &= \lim_n \lim_m \phi(a_n E[x]b_m) \\ &= \lim_n \lim_m \phi(E[a_n]xE[b_m]) \\ &= \lim_n \phi(E[a_n]xE[b]) \\ &= \phi(E[a]xE[b]). \end{aligned}$$

□

Lemma 4.3.12. *$E[ax] = aE[x]$ for all $a \in \mathcal{T}$ and $x \in \mathcal{A}$.*

Proof. For all $b, c \in \mathcal{A}_0$, by lemma 5.2.12 and Lemma 4.3.10, we have

$$\begin{aligned} \langle E[ax]\hat{b}, \hat{c}\rangle &= \phi(c^* E[ax]b) \\ &= \phi(E[c^*]axE[b]) \\ &= \phi((E[c^*]a)xE[b]). \end{aligned}$$

since $E[c^*]a \in \mathcal{T}$, $E[E[c^*]a] = E[c^*]a$, then

$$\begin{aligned} \phi((E[c^*]a)xE[b]) &= \phi(E[E[c^*]a]xE[b]) \\ &= \phi(E[c^*]aE[x]b) \\ &= \phi(E[c^*]E[aE[x]]b) \\ &= \phi(E[E[c^*]](aE[x])E[b]) \\ &= \phi(E[c^*](aE[x])E[b]) \\ &= \phi(c^*E[aE[x]]b) \\ &= \phi(c^*aE[x]b) \\ &= \langle aE[x]\hat{b}, \hat{c}\rangle. \end{aligned}$$

Since b, c are arbitrary, we get our desired conclusion □

Lemma 4.3.13.

$$E[x_{i_1}^{k_1} \cdots x_{i_s}^{k_s} \cdots x_{i_t}^{k_t} \cdots x_{i_n}^{k_n}] = E[x_{i_1}^{k_1} \cdots \alpha^N(x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}) \cdots x_{i_n}^{k_n}]$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $N \geq \max\{i_1, ..., i_n\}$, k_j*'s are positive integers.*

Proof. Given $a, b \in \mathcal{A}_0$, then there exists an M such that $a, b \in \mathcal{A}_{[M]}$. Then, we have

$$\begin{aligned} & \langle E[x_{i_1}^{k_1} \cdots x_{i_s}^{k_s} \cdots x_{i_t}^{k_t} \cdots x_{i_n}^{k_n}]\hat{a}, \hat{b}\rangle \\ =& \lim_{l\to\infty} \langle \alpha^l(x_{i_1}^{k_1} \cdots x_{i_s}^{k_s} \cdots x_{i_t}^{k_t} \cdots x_{i_n}^{k_n})\hat{a}, \hat{b}\rangle \\ =& \langle \alpha^M(x_{i_1}^{k_1} \cdots x_{i_s}^{k_s} \cdots x_{i_t}^{k_t} \cdots x_{i_n}^{k_n})\hat{a}, \hat{b}\rangle \\ =& \langle x_{i_1+M}^{k_1} \cdots x_{i_s+M}^{k_s} \cdots x_{i_t+M}^{k_t} \cdots x_{i_n+M}^{k_n}\hat{a}, \hat{b}\rangle, \end{aligned}$$

by lemma 4.3.6 and $i_1 + M \neq \cdots \neq i_{s-1} + M \neq i_s + M + N \neq i_{s+1} + M + N \neq \cdots \neq i_t + M + N \neq i_{t+1} + M \neq \cdots i_n + M$,

$$\begin{aligned} & \langle x_{i_1+M}^{k_1} \cdots x_{i_s+M}^{k_s} \cdots x_{i_t+M}^{k_t} \cdots x_{i_n+M}^{k_n}\hat{a}, \hat{b}\rangle \\ =& \langle x_{i_1+M}^{k_1} \cdots x_{i_s+M+N}^{k_s} \cdots x_{i_t+M+N}^{k_t} \cdots x_{i_n+M}^{k_n}\hat{a}, \hat{b}\rangle \\ =& \langle \alpha^M(x_{i_1}^{k_1} \cdots \alpha^N(x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}) \cdots x_{i_n}^{k_n})\hat{a}, \hat{b}\rangle \\ =& \lim_{l\to\infty} \langle \alpha^l(x_{i_1}^{k_1} \cdots \alpha^N(x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}) \cdots x_{i_n}^{k_n})\hat{a}, \hat{b}\rangle \\ =& \langle E[x_{i_1}^{k_1} \cdots \alpha^N(x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}) \cdots x_{i_n}^{k_n}]\hat{a}, \hat{b}\rangle. \end{aligned}$$

Because $\{\hat{a} | a \in \mathcal{A}_0\}$ is dense in $\mathcal{H}$, the proof is complete. □

Corollary 4.3.14.

$$E[x_{i_1}^{k_1} \cdots x_{i_s}^{k_s} \cdots x_{i_t}^{k_t} \cdots x_{i_n}^{k_n}] = E[x_{i_1}^{k_1} \cdots E[x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}] \cdots x_{i_n}^{k_n}],$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$.

Proof. Let $N = \max\{i_1, ..., i_n\}$. Since $E[x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}] = w^* - \lim_{l\to\infty} \alpha^l(x_{i_s}^{k_s} \cdots x_{i_t}^{k_t})$, we have

$$E[x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}] = w^* - \lim_{l\to\infty} \frac{1}{l} \sum_{s=1}^{l} \alpha^{N+l}(x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}).$$

Then, by lemma 4.3.13,

$$\begin{aligned} & E[x_{i_1}^{k_1} \cdots x_{i_s}^{k_s} \cdots x_{i_t}^{k_t} \cdots x_{i_n}^{k_n}] \\ = \ & \tfrac{1}{l} \sum_{s=1}^{l} E[x_{i_1}^{k_1} \cdots \alpha^{N+l}(x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}) \cdots x_{i_n}^{k_n}] \\ = \ & E[x_{i_1}^{k_1} \cdots [w^* - \lim_{l\to\infty} \tfrac{1}{l} \sum_{s=1}^{l} \alpha^{N+l}(x_{i_s}^{k_s} \cdots x_{i_t}^{k_t})] \cdots x_{i_n}^{k_n}] \\ = \ & E[x_{i_1}^{k_1} \cdots E[x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}] \cdots x_{i_n}^{k_n}]. \end{aligned}$$

The last two equations follow the normality of E and

$$x_{i_1}^{k_1} \cdots [\frac{1}{l} \sum_{s=1}^{l} \alpha^{N+l}(x_{i_s}^{k_s} \cdots x_{i_t}^{k_t})] \cdots x_{i_n}^{k_n} \to x_{i_1}^{k_1} \cdots E[x_{i_s}^{k_s} \cdots x_{i_t}^{k_t}] \cdots x_{i_n}^{k_n}$$

in WOT. □

Lemma 4.3.15.

$$E[b_1 x_{i_1}^{k_1} b_2 \cdots b_s x_{i_s}^{k_s} \cdots b_t x_{i_t}^{k_t} \cdots b_n x_{i_n}^{k_n}] = E[b_1 x_{i_1}^{k_1} b_2 \cdots E[b_s x_{i_s}^{k_s} \cdots b_t x_{i_t}^{k_t}] \cdots b_n x_{i_n}^{k_n}],$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $k_1, ...k_n$ are positive integers, $b_1, ...b_n \in \mathcal{A}_{N+1}$ where $N =$ $\max\{i_1, ..., i_n\}$.

Proof. By the linearity of E, we can assume that b_i's are "monomials", i.e. $b_j = x_{i_{j,1}} \cdots x_{i_{j,r_j}}$ where $i_{j,j'}$'s are greater than N. Then,

$$b_1 x_{i_1}^{k_1} b_2 \cdots b_s x_{i_s}^{k_s} \cdots b_t x_{i_t}^{k_t} \cdots b_n x_{i_n}^{k_n} = b_1 x_{i_1}^{k_1} b_2 \cdots x_{i_{s,1}} \cdots x_{i_{s,r_s}} x_{i_s}^{k_s} \cdots x_{i_{t,1}} \cdots x_{i_{t,r_t}} x_{i_t}^{k_t} \cdots b_n x_{i_n}^{k_n},$$

$i_{s,1} \geq N+1 > i_{s-1}$ and $i_{t,r_t} \geq N+1 > i_{t+1}$. Therefore, by lemma 5.2.8,

$$\begin{aligned} & E[b_1 x_{i_1}^{k_1} b_2 \cdots x_{i_{s,1}} \cdots x_{i_{s,r_s}} x_{i_s}^{k_s} \cdots x_{i_{t,1}} \cdots x_{i_{t,r_t}} x_{i_t}^{k_t} \cdots b_n x_{i_n}^{k_n}] \\ = \ & E[b_1 x_{i_1}^{k_1} b_2 \cdots E[x_{i_{s,1}} \cdots x_{i_{s,r_s}} x_{i_s}^{k_s} \cdots x_{i_{t,1}} \cdots x_{i_{t,r_t}} x_{i_t}^{k_t}] \cdots b_n x_{i_n}^{k_n}] \\ = \ & E[b_1 x_{i_1}^{k_1} b_2 \cdots E[b_s x_{i_s}^{k_s} \cdots b_t x_{i_t}^{k_t}] \cdots b_n x_{i_n}^{k_n}]. \end{aligned}$$

□

Proposition 4.3.16. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be a sequence of of selfadjoint random variables in $\mathcal{A}$ whose joint distribution is invariant of under the boolean permutations. Let E be the conditional expectation onto the non-unital tail algebra $\mathcal{T}$ of the sequence. Then, E has the following factorization property: for all $n, k \in \mathbb{N}$, polynomials $p_1, ..., p_n \in \mathcal{T}\langle X_1, ..., X_k\rangle_0$ and $i_1, ..., i_n \in \{1, ..., k\}$ such that $i_1 \neq i_2 \neq \cdots \neq i_n$, we have*

$$E[p_1(x_{i_1}) \cdots p_l(x_{i_m}) \cdots p_n(x_{i_n})] = E[p_1(x_{i_1}) \cdots E[p_l(x_{i_m})] \cdots p_n(x_{i_n})].$$

Proof. It suffices to prove the statement in the case: $p_1, ..., p_n$ are $\mathcal{T}$-monomials but none of them is an element of $\mathcal{T}$. Assume that

$$p_i(X) = b_{i,0}X^{t_{i,1}}b_{i,1}X^{t_{i,2}}b_{i,2}\cdots X^{t_{i,k_i}},$$

where $b_{i,j} \in \mathcal{T}$ and $t'_{i,j}s$ are positive integers. Let $N = \max\{i_1, ..., i_n\}$, then $b_{i,j} \in \mathcal{T} \subset \overline{\mathcal{A}_{N+1}}^{w^*}$. By the Kaplansky theorem, for every $b_{i,j}$, there exists a bounded sequence $\{b_{l,i,j}\}_{l\in\mathbb{N}}$ such that $b_{l,i,j}$ converges to $b_{i,j}$ in strong operator topology (SOT). Let $p_{n,i}(X) = b_{n,i,0}X^{t_{i,1}}b_{n,i,1}X^{t_{i,2}}b_{n,i,2}\cdots X$ then $p_{l,k}(x_{i_k})$ converges to $p_k(x_{i_k})$ in SOT. By the normality of E, we have

$$E[p_1(x_{i_1})\cdots p_l(x_{i_m})\cdots p_n(x_{i_n})] = w^* - \lim_{l\to\infty} E[p_{l,1}(x_{i_1})\cdots p_{l,m}(x_{i_m})\cdots p_{l,n}(x_{i_n})].$$

By lemma 4.3.15, we have

$$E[p_{l,1}(x_{i_1})\cdots p_{l,m}(x_{i_m})\cdots p_{l,m}(x_{i_n})] = E[p_{l,1}(x_{i_1})\cdots E[p_{l,m}(x_{i_m})]\cdots p_{l,n}(x_{i_n})].$$

It follows that $E[p_{l,m}(x_{i_m})]$ converges to $E[p_m(x_{i_m})]$ in WOT. Therefore,

$$p_{l,1}(x_{i_1})\cdots E[p_{l,m}(x_{i_m})]\cdots p_{l,n}(x_{i_n})$$

converges to $p_1(x_{i_1})\cdots E[p_m(x_{i_m})]\cdots p_m(x_{i_n})$ in WOT. Now, we have

$$\begin{aligned} & E[p_1(x_{i_1})\cdots p_l(x_{i_m})\cdots p_n(x_{i_n})] \\ = & w^* - \lim_{l\to\infty} E[p_{l,1}(x_{i_1})\cdots p_{l,m}(x_{i_m})\cdots p_{l,n}(x_{i_n})] \\ = & w^* - \lim_{l\to\infty} E[p_{l,1}(x_{i_1})\cdots E[p_{l,m}(x_{i_m})]\cdots p_{l,n}(x_{i_n})] \\ = & E[p_1(x_{i_1})\cdots E[p_m(x_{i_m})]\cdots p_n(x_{i_n})], \end{aligned}$$

the last equality follows E's WOT continuity. □

4.4 Main theorem and examples

In this section, we will complete the proof of the main theorems. In the first subsection, we assume that the unit of the original algebra is contained in the WOT-closure of the non-unital algebra generated by random variables. The proof in this case is a summary of the results in section 4.3 and section 4.2. In the second subsection, we assume that the unit of the original algebra is not contained in the WOT-closure of the non-unital algebra generated by random variables. We will show that a probability space in this situation is the unitalization of a case in subsection 4.4.1. Then, we prove the theorem by using results in section 4.2. In the last subsection, we show that boolean exchangeable sequences in a probability space with a faithful state is trivial, i.e. all the random variables are equal to each other.

4.4.1 Non-unital tail algebra case

Theorem 4.4.1. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be an infinite sequence of selfadjoint random variables. Suppose $\mathcal{A}$ is the WOT closure of the non-unital algebra generated by $(x_i)_{i\in\mathbb{N}}$ and ϕ is non-degenerated. Then the following statements are equivalent:*

a) The joint distribution of $(x_i)_{i\in\mathbb{N}}$ satisfies the invariance conditions associated with the linear coactions of the quantum semigroups $\mathcal{B}_s(n)$'s.

b) The sequence $(x_i)_{i\in\mathbb{N}}$ is identically distributed and boolean independent with respect to a ϕ-preserving normal conditional expectation E onto the non-unital tail algebra $\mathcal{T}$ of the sequence $(x_i)_{i\in\mathbb{N}}$

Proof. $a) \Rightarrow b)$: By choosing $m = 1$ in proposition 5.3.1, we have

$$\begin{aligned} & E[p_1(x_{i_1})\cdots p_2(x_{i_2})\cdots p_n(x_{i_n})] \\ = \; & E[E[p_1(x_{i_1})]p_2(x_{i_2})\cdots p_n(x_{i_n})] \\ = \; & E[p_1(x_{i_1})]E[p_2(x_{i_2})\cdots p_n(x_{i_n})] \\ & \cdots \\ = \; & E[p_1(x_{i_1})]E[p_2(x_{i_2})]\cdots E[p_n(x_{i_n})], \end{aligned}$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $p_1, ..., p_n \in \mathcal{T}\langle X\rangle_0$
$b) \Rightarrow a)$ is a special case of theorem 6.2.12 □

The non-unital tail algebra case example in the previous section is a special case of this theorem. The range of the conditional expectation in example is a one dimensional algebra.

4.4.2 Unital tail algebra case

Let $(\mathcal{A}, \phi)$ be a W^*-probability space with a non-degenerated normal state ϕ and $(x_i)_{i\in\mathbb{N}}$ be a sequence of selfadjoint random variables. Suppose $\mathcal{A}$ is the WOT closure of the unital algebra generated $(x_i)_{i\in\mathbb{N}}$ and ϕ is non-degenerated. Again, we denote by $\mathcal{A}_0$ the non-unital algebra generated by $(x_i)_{i\in\mathbb{N}}$. Let $I_\mathcal{A}$ be the unit of $\mathcal{A}$, we assume that $I_\mathcal{A}$ is not contained in $\overline{\mathcal{A}_0}^{w^*}$ and denote by I_1 the unit of $\overline{\mathcal{A}_0}^{w^*}$. Then,

$$I_2 = I_\mathcal{A} - I_1 \neq 0$$

and

$$\mathcal{A} = \mathbb{C}I_2 \oplus \overline{\mathcal{A}_0}^{w^*}.$$

For all $x \in \overline{\mathcal{A}_0}^{w^*}$, we have

$$I_2 x = (I_\mathcal{A} - I_1)x = 0.$$

Let $a \in \overline{\mathcal{A}_0}^{w^*}$ such that $\phi(xay) = 0$ for all $x, y \in \overline{\mathcal{A}_0}^{w^*}$. For $\bar{x}, \bar{y} \in \mathcal{A}$, there exist two constants $c_1, c_2 \in \mathbb{C}$ and $x, y \in \overline{\mathcal{A}_0}^{w^*}$ such that $x = c_1 I_2 + x$ and $y = c_2 I_2 + y$, then

$$\phi(\bar{x}a\bar{y}) = \phi(xab) = 0,$$

Since our $\bar{x}, \bar{y}$ are chosen arbitrarily, we have $a = 0$. Therefore, $(\overline{\mathcal{A}_0}^{w^*}, \frac{1}{\phi(I_1)}\phi)$ is a W^*-probability space with a non-degenerated normal state. Let $\mathcal{A}_{tail}$ be the unital tail algebra of $(x_i)_{i\in\mathbb{N}}$ in $(\mathcal{A}, \phi)$ and $\mathcal{T}$ be the non-unital tail algebra of $(x_i)_{i\in\mathbb{N}}$ in $(\overline{\mathcal{A}_0}^{w^*}, \frac{1}{\phi(I_1)}\phi)$. Then, we have

$$\mathcal{A}_{tail} = \bigcap_{n=1}^{\infty} vN\{x_k | k \geq n\} = \bigcap_{n=1}^{\infty} (W^*\{x_k | k \geq n\} + \mathbb{C}I_{\mathcal{A}}) = \mathcal{T} + \mathbb{C}I_{\mathcal{A}}.$$

Since $\overline{\mathcal{A}_0}^{w^*}$ is a two-sided ideal of $\mathcal{A}$, for all $\bar{x} \in \mathcal{A}_{tail}$ we have $\bar{x} = aI_{\mathcal{A}} + x$ for some $x \in \overline{\mathcal{A}_0}^{w^*}$ and $a \in \mathbb{C}$. By theorem 7.1, there is a ϕ-preserving normal conditional expectation E from $\overline{\mathcal{A}_0}^{w^*}$ onto $\mathcal{T}$. As we proceeded in section 4.1, we can extend this conditional expectation E to an conditional expectation $\bar{E}$ which is from the unitalization of $\overline{\mathcal{A}_0}^{w^*}$ to the unitalization of $\mathcal{T}$. The unitalizations of the two algebras are isomorphic to $\mathcal{A}$ and $\mathcal{A}_{tail}$, respectively. We have

Lemma 4.4.2. *The conditional expectation $\bar{E}$ is ϕ-preserving and normal.*

Proof. The normality is obvious, we just check the ϕ-preserving condition here. Let $\bar{x} = aI_{\mathcal{A}} + x \in \mathcal{A}$ for some $x \in \overline{\mathcal{A}_0}^{w^*}$ and $a \in \mathbb{C}$, we have

$$\phi(E[\bar{x}] = \phi(E[aI_{\mathcal{A}} + x]) = \phi(aI_{\mathcal{A}} + E[x]) = a + \phi(E[x]) = a + \phi(x).$$

The last equality holds because E is a $\frac{1}{\phi(I_1)}\phi$-preserving conditional expectation on $(\overline{\mathcal{A}_0}^{w^*}, \frac{1}{\phi(I_1)}\phi)$. □

Together with proposition 6.2.12, we have the following theorem:

Theorem 4.4.3. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be a sequence of selfadjoint random variables. Suppose the unit $I_{\mathcal{A}}$ of $\mathcal{A}$ is not contained in the WOT closure of the non-unital algebra generated by $(x_i)_{i\in\mathbb{N}}$ and ϕ is non-degenerated. Then the following statements are equivalent:*

a) The joint distribution of $(x_i)_{i\in\mathbb{N}}$ is boolean exchangeable.

b) The sequence $(x_i)_{i\in\mathbb{N}}$ is identically distributed and boolean independent with respect to a ϕ-preserving normal conditional expectation $\bar{E}$ onto the unital tail algebra $\mathcal{A}_{tail}$ of the $(x_i)_{i\in\mathbb{N}}$.

4.4.3 On W^*-probability spaces with faithful states

Now, we turn to consider boolean exchangeable sequences of random variables in a W^*-probability space with a faithful state:

Theorem 4.4.4. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be a sequence of selfadjoint random variables such that $\mathcal{A}$ is generated by $(x_i)_{i\in\mathbb{N}}$ and ϕ is faithful. Then the following statements are equivalent:*

a) The joint distribution of $(x_i)_{i\in\mathbb{N}}$ is boolean exchangeable.

b) $x_i = x_j$ for all $i, j \in \mathbb{N}$

Proof. $b) \Rightarrow a)$: If $x_i = x_j$ for all $i, j \in \mathbb{N}$, given a monomial $p = X_{i_1} \cdots X_{i_k} \in \mathbb{C}\langle X_1, ..., X_n\rangle$, then

$$\begin{aligned}
\mu_{x_1,\ldots x_n}(X_{i_1}\cdots X_{i_k})\mathbf{P} &= \phi(x_{i_1}\cdots x_{i_k})\mathbf{P} \\
&= \phi(x_1^k)\mathbf{P} \\
&= \sum_{j_1,\ldots,j_k=1}^{n} \phi(x_1^k)\pi(\mathbf{P}u_{j_1,i_1}\cdots u_{j_k,i_k}\mathbf{P}) \\
&= \sum_{j_1,\ldots,j_k=1}^{n} \phi(x_{j_1}\cdots x_{j_k})\mathbf{P}u_{j_1,i_1}\cdots j_k, i_k\mathbf{P} \\
&= \mu_{x_1,\ldots x_n} \otimes id_{\mathcal{B}_s(n)}(\text{Ł}p).
\end{aligned}$$

$b) \Rightarrow a)$: It is sufficient to show that $x_1 = x_2$. By theorem 7.1 and 7.3, there exists a ϕ-preserving conditional expectation E maps $\mathcal{A}$ to its unital or non-unital tail algebra such that $(x_i)_{i\in\mathbb{N}}$ is identically distributed and boolean independent with respect to E. For $k \in \mathbb{N}$ and $k > 2$, we have

$$\begin{aligned}
&\phi((x_1 - x_2)x_k((x_1 - x_2)x_k)^*) \\
=\ &\phi((x_1 - x_2)x_k^2(x_1 - x_2)) \\
=\ &\phi(E[(x_1 - x_2)x_k^2(x_1 - x_2)]) \\
=\ &\phi(E[x_1 - x_2]E[x_k^2]E[x_1 - x_2]) \\
=\ &0.
\end{aligned}$$

Since ϕ is faithful, we get

$$(x_1 - x_2)x_k = 0$$

for all $k > 2$. Let $\mathcal{A}_k$ be the WOT closure of the non-unital algebra generated by $\{x_n | n > k\}$, then we have

$$(x_1 - x_2)x = 0$$

for all $x \in \mathcal{A}_k$. Notice that $(x_i)_{i\in\mathbb{N}}$ is exchangeable, by the construction of proposition 4.2 in [23] , there exists a normal ϕ-preserving homomorphism $\alpha : \mathcal{A}_n \to \mathcal{A}_{n+1}$ such that $\alpha(x_i) = x_{i+1}$. For each $n \in \mathbb{N}$, we denote by I_n the unit of $\mathcal{A}_n$. Then, $\alpha(I_n) = I_{n+1}$ and $I_nI_{n+1} = I_{n+1}$, since I_{n+1} is a projection in $\mathcal{A}_n$. Then, we have

$$\phi((I_n - I_{n+1})^2) = \phi(I_n - I_{n+1}) = \phi(I_n) - \phi(\alpha(I_n)) = 0,$$

which implies that $I_n = I_{n+1}$. It follows that

$$I_0 = I_1 = I_2.$$

Therefore,

$$0 = (x_1 - x_2)I_2 = (x_1 - x_2)I_0 = x_1 - x_2.$$

□

Especially, we have the following:

Corollary 4.4.5. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be a sequence of boolean independent and identically distributed random variables. Then, there exists a real number a and a projection $P \in \mathcal{A}$, such that $x_i = aP$ for all $i \in \mathbb{N}$.*

Proof. The joint distribution of $(x_i)_{i\in\mathbb{N}}$ is boolean exchangeable, since $(x_i)_{i\in\mathbb{N}}$ are boolean exchangeable and identically distributed. According to Theorem 4.4.4, all these random variables $(x_i)_{i\in\mathbb{N}}$ are equal to each other. Let $x = x_1$, then

$$\phi(x^n) = \phi(\prod_{i=1}^{n} x_i) = \prod_{i=1}^{n} \phi(x_i) = \phi(x)^n.$$

Therefore, $\phi((x^2 - \phi(x)x)^2) = 0$. It implies that $x^2 = \phi(x)x$. If $\phi(x) = 0$, then $\phi(x^2) = 0$. In this case, $a = 0$, P can be any projection in $\mathcal{A}$. If $\phi(x) \neq 0$, then $\frac{1}{\phi(x)}x$ is a projection. In this case, $a = \phi(x)$, $P = \frac{1}{\phi(x)}x$. □

4.5 Two more kinds of probabilistic symmetries

In this section, we study two more kinds of probabilistic symmetries. Since $\mathbb{C}\langle X_1, ..., X_n\rangle$ is an algebra, it would be natural to define coactions of the quantum semigroups $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, ..., X_n\rangle$ as algebraic homomorphisms but not only linear maps. In the following, we will define algebraic coactions of the quantum semigroups $\mathcal{B}_s(n)$'s and $B_s(n)$'s on $\mathbb{C}\langle X_1, ..., X_n\rangle$. The invariance conditions for the joint distribution will be defined as we did in previous sections.

In our situation, the algebraic coaction $Ł'_n : \mathbb{C}\langle X_1, ..., X_n\rangle \to \mathbb{C}\langle X_1, ..., X_n\rangle \otimes \mathcal{B}_s(n)$ is given by the following formulas:

$$Ł'_n(1) = 1 \otimes I, \;\; Ł'_n(X_i) = \sum_{k=1}^{n} X_k \otimes \mathbf{P}u_{k,i}\mathbf{P}.$$

Since $\mathbb{C}\langle X_1, ..., X_n\rangle$ is freely generated by n variables $X_1, ..., X_n$, the homomorphism is well defined. Then, we would have

$$Ł_n(X_{i_1} \cdots X_{i_k}) = \sum_{j_1,\dots j_k=1}^{n} X_{j_1} \cdots X_{j_k} \otimes \mathbf{P}u_{j_1,i_1}\mathbf{P} \cdots \mathbf{P}u_{j_n,i_n}\mathbf{P}$$

and

$$(Ł'_n \otimes id_{B_s(n)})Ł'_n = (id_{\mathbb{C}_n} \otimes \Delta)Ł'_n.$$

We call $Ł'_n$ the algebraic coaction of $\mathcal{B}_s(n)$ on $\mathbb{C}\langle X_1, ..., X_n\rangle$. The invariance condition associated with $Ł'_n$ would be the defined as following:

Definition 4.5.1. *In a probability space $(\mathcal{A}, \phi)$, let $(x_i)_{i=1,...,n}$ be random variables in $\mathcal{A}$. We say that the joint distribution of $(x_i)_{i=1,...,n}$ is invariant under the algebraic coaction $Ł'_n$ of $\mathcal{B}_s(n)$ if*

$$\mu_{x_1,...,x_n}(p)\boldsymbol{P} = \mu_{x_1,...,x_n} \otimes id_{\mathcal{B}_s(n)}(Ł'_n(p)),$$

for all $p \in \mathbb{C}\langle X_1, ..., X_n\rangle$, where $\mu_{x_1,...,x_n}$ is the joint distribution of $(x_i)_{i=1,...,n}$ with respect to ϕ.

We will see that this invariance condition is so strong that we can study it in finitely generated probability spaces.

Proposition 4.5.2. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space with a non-degenerated state ϕ. Suppose $\mathcal{A}$ is the WOT closure of the unital algebra generated by selfadjoint random variables $(x_i)_{i=1,...,n}$. Then, the joint distribution of $(x_i)_{i=1,...,n}$ is invariant under the algebraic coaction $Ł'_n$ of $\mathcal{B}_s(n)$ is equivalent to $x_1 = x_2 = \cdots = x_n$.*

Proof. Suppose $x_1 = x_2 = \cdots = x_n$. Let $p = X_{i_1} \cdots X_{i_m} \in \mathbb{C}\langle X_1, ..., X_n\rangle$, then we have

$$\begin{aligned}
& \mu_{x_1,...,x_n} \otimes id_{\mathcal{B}_s(n)}(Ł'_n(X_{i_1} \cdots X_{i_m})) \\
= & \mu_{x_1,...,x_n} \otimes id_{\mathcal{B}_s(n)}(\sum_{j_1,...j_m=1}^{n} X_{j_1} \cdots X_{j_m} \otimes \mathbf{P}u_{j_1,i_1}\mathbf{P}u_{j_2,i_2}\mathbf{P} \cdots \mathbf{P}u_{j_m,i_m}\mathbf{P}) \\
= & \sum_{j_1,...j_m=1}^{n} \phi(x_{j_1} \cdots x_{j_m})\mathbf{P}u_{j_1,i_1}\mathbf{P}u_{j_2,i_2}\mathbf{P} \cdots \mathbf{P}u_{j_m,i_m}\mathbf{P} \\
= & \sum_{j_1,...j_m=1}^{n} \phi(x_1^m)\mathbf{P}u_{j_1,i_1}\mathbf{P}u_{j_2,i_2}\mathbf{P} \cdots \mathbf{P}u_{j_m,i_m}\mathbf{P} \\
= & \phi(x_1^m)\mathbf{P} \\
= & \mu_{x_1,...,x_n}(X_{i_1} \cdots X_{i_m})\mathbf{P}.
\end{aligned}$$

Since p is arbitrary, we proved $\Leftarrow$.
Suppose the joint distribution of $(x_i)_{i=1,...,n}$ is invariant under the algebraic coaction $Ł'_n$. Let $\{v_1, ..., v_{2n}\}$ be orthonormal basis of the standard $2n$-dimensional Hilbert space $\mathbb{C}^{2n}$ and denote $v_k = v_{k+2n}$ for all $k \in \mathbb{Z}$. Let

$$P_{i,j} = P_{v_{2(i-j)+1}+v_{2(j-i)+2}}$$

and

$$P = P_{v_1+v_2+\cdots+v_{2n}},$$

where P_v is the orthogonal projection onto the one dimensional subspace generated by the vector v . By lemma 3.2.3, we have a representation π of $\mathcal{B}_s(n)$ on $\mathbb{C}^{2n}$ defined by the following formulas:

$$\pi(\mathbf{P}) = P, \quad \pi(\mathbf{P}u_{i_1,j_1} \cdots u_{i_k,j_k}\mathbf{P}) = PP_{i_1,j_1} \cdots P_{i_k,j_k}P$$

for all $i_1, j_1, ..., i_k, j_k \in \{1, ..., n\}$ and $k \in \mathbb{N}$. In particular, we have

$$\pi(\mathbf{P}u_{i,j}\mathbf{P}) = PP_{i,j}P = \frac{1}{n}P.$$

Let π act on the invariance condition, we get

$$\begin{aligned}
\phi(x_{i_1}\cdots x_{i_k})P &= \pi(\mu_{x_1,...x_n}(X_{i_1}\cdots X_{i_k})\mathbf{P}) \\
&= \pi(\sum_{j_1,...,j_k=1}^{n} \mu_{x_1,...x_n}\otimes id_{\mathcal{B}_s(n)}(X_{j_1}\cdots X_{j_k}\otimes \mathbf{P}u_{j_1,i_1}\mathbf{P}\cdots \mathbf{P}u_{j_k,i_k}\mathbf{P})) \\
&= \sum_{j_1,...,j_k=1}^{n} \phi(x_{j_1}\cdots x_{j_k})\pi(\mathbf{P}u_{j_1,i_1}\mathbf{P}\cdots \mathbf{P}u_{j_k,i_k}\mathbf{P})) \\
&= \sum_{j_1,...,j_k=1}^{n} \phi(x_{j_1}\cdots x_{j_k})\tfrac{1}{n^k}P,
\end{aligned}$$

for all $i_1, ..., i_k \in \{1, ...n\}$. It implies that

$$\phi(x_{i_1}\cdots x_{i_k}) = \frac{1}{n^k}\sum_{j_1,...,j_k=1}^{n}\phi(x_{j_1}\cdots x_{j_k}).$$

Therefore, two mixed moments are the same if their degrees are the same. Given two monomials $a = x_{s_1}\cdots x_{s_{k_1}}$ and $b = x_{t_1}\cdots x_{t_{k_2}}$, then

$$\phi(a(x_i-x_j)(x_i-x_j)^*b) = \phi(a(x_i-x_j)^2b) = \phi(ax_ix_ib)-\phi(ax_ix_jb)-\phi(ax_ix_jb)+\phi(ax_ix_ib) = 0.$$

The last equation holds because all those monomials have the same degree. By the linearity of ϕ, we have

$$\phi(a(x_i - x_j)(x_i - x_j)^*b) = 0,$$

for all $a, b \in \mathcal{A}_{[n]}$, where $\mathcal{A}_{[n]}$ is the unital algebra generated by $x_1, ..., x_n$. Therefore,

$$x_i = x_j,$$

for all $i \neq j$. □

In the end of this section, we define an algebraic coaction

$$L_n : \mathbb{C}\langle X_1, ..., X_n\rangle \to \mathbb{C}\langle X_1, ..., X_n\rangle \otimes B_s(n)$$

of $B_s(n)$ on $\mathbb{C}\langle X_1, ..., X_n\rangle$ via the following formulas:

$$L_n(1) = 1\otimes I, \quad L_n(X_i) = \sum_{k=1}^{n} X_k \otimes u_{k,i},$$

where 1 is the identity of $\mathbb{C}\langle X_1, ..., X_n\rangle$ and I is the identity of $B_s(n)$. According to the definition of L_n, we have

$$L_n(X_{i_1} \cdots X_{i_k}) = \sum_{j_1,\ldots j_k=1}^{n} X_{j_1} \cdots X_{j_k} \otimes u_{j_1,i_1} \cdots u_{j_k,i_k}.$$

One can easily check

$$(L_n \otimes id_{B_s(n)})L_n = (id_{\mathbb{C}_n} \otimes \Delta)L_n,$$

where $id_{B_s(n)}$ and $id_{\mathbb{C}_n}$ are identity maps on $B_s(n)$ and $\mathbb{C}\langle X_1, ..., X_n\rangle$ respectively. The invariance condition associated with L_n is

Definition 4.5.3. *Let $(\mathcal{A}, \phi)$ be a probability space and $(x_i)_{i=1,..,n}$ be random variables in $\mathcal{A}$. The joint distribution of $(x_i)_{i=1,...,n}$ is invariant under the coaction L_n if for all $p \in \mathbb{C}\langle X_1, ..., X_n\rangle$, we have*

$$\mu_{x_1,...,x_n}(p)I = \mu_{x_1,...,x_n} \otimes id(L_n(p)).$$

Then, we have

Proposition 4.5.4. *Let $(\mathcal{A}, \phi)$ be a probability space and $(x_i)_{i=1,...,n}$ be random variables in $\mathcal{A}$. If the joint distribution of $(x_i)_{i=1,...,n}$ is invariant under the coaction L_n, then $\phi(x_{i_1}x_{i_2} \cdots x_{i_k}) = 0$ for all $i_1, ..., i_k \in \{1, ..., n\}$ and $k \in \mathbb{N}$.*

Proof. Take a trivial representation π of $B_s(n)$ on a 1-dimensional space V defined by the following formulas:

$$\pi(I) = 1, \text{ and } \pi(u_{i,j}) = \pi(\mathbf{P}) = 0,$$

where 1 is the identity of V. By the universality of $B_s(n)$, π is well defined. According to the invariance condition, we have

$$\pi(\mu_{x_1,...,x_n}(p)I) = \mu_{x_1,...,x_n} \otimes \pi(L_n(p)),$$

for all $p \in \mathbb{C}\langle X_1, ..., X_n\rangle$. Let $p = X_{i_1} \cdots X_{i_k}$, we get

$$\phi(x_{i_1}x_{i_2} \cdots x_{i_k})1 = \sum_{j_1,\ldots j_k=1}^{n} \phi(x_{j_1} \cdots x_{j_k})\pi(u_{j_1,i_1} \cdots u_{j_k,i_k}) = 0,$$

which completes the proof. □

Chapter 5

Extended De Finetti type theorems in noncommutative probability

5.1 Monotonically equivalent sequences

In order to study monotone spreadability, we need to find some relations between mixed moments of monotonically spreadable sequences of random variables. In this section, we introduce will an equivalent relation, which has a deep relation with monotone spreadability, on finite sequences of ordered indices.

Definition 5.1.1. Given two pairs of integers (a, b), (c, d) , we say these two pairs have the same order if $a - b, c - d$ are both positive or negative or 0.

For example, $(1, 2)$ and $(3, 5)$ have the same order but $(1, 2)$ and $(5, 3)$ do not have the same order.

Definition 5.1.2. Let $\mathbb{Z}$ be the set of integers with the natural order"$>$"and $\mathbb{Z}^L = \mathbb{Z}\times\cdots\times\mathbb{Z}$ be the set of finite sequences of length L. We define a partial relation $\sim_m$ on $\mathbb{Z}^L$: Given two sequences of indices $\mathcal{I} = \{i_1, ..., i_L\}, \mathcal{J} = \{j_1, ..., j_L\} \in \mathbb{Z}^L$. If for all $1 \leq l_1 < l_2 \leq L$ such that $i_{l_3} > \max\{i_{l_1}, i_{l_2}\}$ for all $l_1 < l_3 < l_2$, (i_{l_1}, i_{l_2}) and (j_{l_1}, j_{l_2}) have the same order, then we denote $\mathcal{I} \sim_m \mathcal{J}$.

Example: $(5, 3, 4) \sim_m (5, 3, 5)$ but $(5, 6, 4) \not\sim_m (5, 6, 5)$. It follows the definition that (i_l, i_{l+1}) and (j_l, j_{l+1}) have the same order for all $1 \leq l < L$ if $\mathcal{I} \sim_m \mathcal{J}$.

Remark 5.1.3. In general, the relation can be defined on any ordered set but not only $\mathbb{Z}$. We will show this partial relation is actually an equivalence relation.

To show that $\sim_m$ is an equivalent relation, we need to show that the relation $\sim_m$ is reflexive, symmetric and transitive.:

(Reflexivity) First, reflexivity is obvious, because a pair (i_{l_1}, i_{l_2}) always has the same order with itself.

Lemma 5.1.4. *(Symmetry) Let $\mathcal{I} = \{i_1, ..., i_L\}, \mathcal{J} = \{j_1, ..., j_L\} \in \mathbb{Z}^L$ such that $\mathcal{I} \sim_m \mathcal{J}$. Then, we have $\mathcal{J} \sim_m \mathcal{I}$.*

Proof. Suppose that $\mathcal{J} \not\sim_m \mathcal{I}$. Then, there exist two natural numbers $1 \leq l_1 < l_2 \leq L$ such that

$$j_{l_3} > \max\{j_{l_1}, j_{l_2}\}$$

for all $l_1 < l_3 < l_2$, but (j_{l_1}, j_{l_2}) and (i_{l_1}, i_{l_2}) do not have the same order. Fix l_1, we choose the smallest l_2 which satisfies the above property. Notice that $\mathcal{I} \sim_m \mathcal{J}$, (j_{l_1}, j_{l_1+1}) and (i_{l_1}, i_{l_1+1}) have the same order, then

$$l_2 \neq l_1 + 1.$$

According to our assumption, we have

$$j_{l'_3} > \max\{j_{l_1}, j_{l_2}\}$$

for $l_1 < l'_3 < l_2$.
Suppose that there exists an l''_3 between l_1 and l_2 such that

$$i_{l''_3} \leq \max\{i_{l_1}, i_{l_2}\}.$$

Without loss of generality, we assume that

$$i_{l_1} \geq i_{l_2},$$

then

$$i_{l''_3} \leq i_{l_1}.$$

Again, among these l''_3, we choose the smallest one. Then, we have $i_l > i_{l_1} \geq i_{l''_3}$ for

$$l_1 < l < l''_3.$$

Since $\mathcal{I} \sim_m \mathcal{J}$, $(i_{l_1}, i_{l''_3})$ and $(j_{l_1}, j_{l''_3})$ must have the same order, but $i_{l_1} \geq i_{l''_3}$ and $i_{l_1} < j_{l''_3}$. It contradicts the existence of our l''_3. Hence, $i_{l'_3} > \max\{i_{l_1}, i_{l_2}\}$ for all $l_1 < l'_3 < l_2$. It follows that (i_{l_1}, i_{l_2}) and (j_{l_1}, j_{l_2}) have the same order. But, it contradicts our original assumption. Therefore, $\mathcal{J} \sim_m \mathcal{I}$.

□

Lemma 5.1.5. *Given two sequences $\mathcal{I} = \{i_1, ..., i_L\}, \mathcal{J} = \{j_1, ..., j_L\} \in \mathbb{Z}^L$ such that $\mathcal{I} \sim_m \mathcal{J}$. Let $1 \leq l_1 < l_2 \leq L$ such that $i_{l_3} > \max\{i_{l_1}, i_{l_2}\}$ for all $l_1 < l_3 < l_2$. Then, we have*

$$j_{l_3} > \max\{j_{l_1}, j_{l_2}\}$$

for all $l_1 < l_3 < l_2$.

Proof. If the statement is false, then there exists l_3 between l_1 and l_2 such that

$$j_{l_3} \leq \max\{j_{l_1}, j_{l_2}\}.$$

Suppose $j_{l_1} \geq j_{l_2}$, then

$$j_{l_3} \leq j_{l_1}.$$

Among all these l_3, we take the smallest one. Then, we have

$$j_{l_4} > \max\{j_{l_1}, j_{l_3}\}$$

for all $l_1 < l_4 < l_3$. By Lemma 5.1.4, $\mathcal{J} \sim_m \mathcal{I}$ since $\mathcal{I} \sim_m \mathcal{J}$. Therefore, (j_{l_1}, j_{l_3}) and (i_{l_1}, i_{l_3}) must have the same order which means

$$i_{l_1} \geq i_{l_3}.$$

This is a contradiction. If we assume that $j_{l_1} < j_{l_2}$, then we just need to consider the largest one among those l_3 and we will get the same contradiction. The proof is complete. □

Lemma 5.1.6. *(Transitivity)Given three sequences $\mathcal{I} = \{i_1, ..., i_L\}, \mathcal{J} = \{j_1, ..., j_L\}, \mathcal{Q} = \{q_1, ...q_L\} \in \mathbb{Z}^L$, such that $\mathcal{I} \sim_m \mathcal{J}$ and $\mathcal{J} \sim_m \mathcal{Q}$, then $\mathcal{I} \sim_m \mathcal{J}$.*

Proof. Given $1 \leq l_1 < l_2 \leq L$ such that

$$i_{l_3} > \max\{i_{l_1}, i_{l_2}\}$$

for all $l_1 < l_3 < l_2$. By Lemma 5.1.5, we have

$$j_{l_3} > \max\{j_{l_1}, j_{l_2}\}$$

for all $l_1 < l_3 < l_2$. It follows the definition that (i_{l_1}, i_{l_2}), (j_{l_1}, j_{l_2}) have the same order and (j_{l_1}, j_{l_2}), (q_{l_1}, q_{l_2}) have the same order. Therefore, (i_{l_1}, i_{l_2}), (q_{l_1}, q_{l_2}) have the same order. Since l_1, l_2 are arbitrary, the proof is complete. □

By now, we have shown that the relation $\sim_m$ is reflexive, symmetric and transitive.

Proposition 5.1.7. *$\sim_m$ is an equivalence relation on $\mathbb{Z}^L$.*

As we mentioned before, $\mathbb{Z}$ can be replaced by any ordered set I. When there is no confusion, we always use $\sim_m$ to denote the monotone equivalence relation on I^L for ordered set I and positive integers L. For example, I can be $[n] = \{1, ..., n\}$.

Definition 5.1.8. Let $\mathcal{I} = (i_1, ..., i_L)$ be a sequence of ordered indices. An ordered subsequence $(i_{l'_1}, ..., i_{l'_2})$ of $\mathcal{I}$ is called an interval if the sequence contains all the elements $i_{l'_3}$ whose position l'_3 is between l'_1 and l'_2. An interval $(i_{l'_1}, ..., i_{l'_2})$ of $\mathcal{I}$ is called a crest if $i_{l'_1} = i_{l'_1+1} \cdots = i_{l'_2} > \max\{i_{l'_1-1}, i_{l'_2+1}\}$. In addition , we always assume that $i_0 < i_1$ and $i_L > i_{L+1}$ even though i_0, i_{L+1} are not in $\mathcal{I}$.

Example: $(1,2,3,4)$ has one crest of length 1, namely (4). $(1,2,1,3,4,4,3,5)$ has 3 crests $(2),(4,4),(5)$ and (2) is the first peak of the sequence. $(1,1,1,1,1)$ has one crest $(1,1,1,1,1)$ which is the sequence itself.

Lemma 5.1.9. *Given $\mathcal{I}=(i_1,...,i_L)\in\mathbb{Z}^L$, $\mathcal{I}$ has at least one crest.*

Proof. Since $\mathcal{I}$ consists of finite elements, it has a maximal one, i.e. i_l such that $i_l \geq i_{l'}$ for $1\leq l' \leq L$. It is obvious that i_l must be contained in an interval $(i_{l_1'},...,i_{l_2'})$ such that

$$i_{l_1'} = i_{l_1'+1}\cdots = i_{l_2'} = i_{l'}$$

and

$$i_{l'} > \max\{i_{l_1'-1}, i_{l_2'+1}\}.$$

Therefore, $\mathcal{I}$ contains a crest. □

Lemma 5.1.10. *Given two index sequences $\mathcal{I},\mathcal{J}\in\mathbb{Z}^L$ such that $\mathcal{I}\sim_m\mathcal{J}$. If $(i_{l_1'},...,i_{l_2'})$ is a crest of $\mathcal{I}$, then $(j_{l_1'},...,j_{l_2'})$ is a crest of $\mathcal{J}$.*

Proof. Since $\mathcal{I}\sim_m\mathcal{J}$, all consecutive pairs (i_l,i_{l+1}) and (j_l,j_{l+1}) have the same order. According to the definition, we have

$$i_{l_1'-1} < i_{l_1'} = i_{l_1'+1}\cdots = i_{l_2'} > j_{l_2'+1}.$$

If follows that

$$j_{l_1'-1} < j_{l_1'} = j_{l_1'+1}\cdots = j_{l_2'} > j_{l_2'+1},$$

thus $(j_{l_1'},...,j_{l_2'})$ is a crest of $\mathcal{J}$.

□

Now, we will introduce some $\sim_m$ preserving operations on index sequences. The first operation is to remove a crest from a sequence. Let $(i_{l_1'},...,i_{l_2'})$ be an interval of $\mathcal{I}=(i_1,...,i_L)$, we denote by $\mathcal{I}\setminus(i_{l_1'},...,i_{l_2'})$ the new sequence $(i_1,...,i_{l_1'-1},i_{l_2'+1},...,i_L)$. We denote by the empty set $\emptyset=\mathcal{I}\setminus\mathcal{I}$ and we assume $\emptyset\sim_m\emptyset$.

Lemma 5.1.11. *Let $\mathcal{I}=(i_1,....,i_L),\mathcal{J}=(j_1,...,j_L)\in\mathbb{Z}^L$ such that $\mathcal{I}\sim_m\mathcal{J}$. If $(i_{l_1'},...,i_{l_2'})$ is a crest of $\mathcal{I}$ and $(j_{l_1'},...,j_{l_2'})$ is a crest of $\mathcal{J}$. Then,*

$$\mathcal{I}\setminus(i_{l_1'},...,i_{l_2'}) \sim_m \mathcal{J}\setminus(j_{l_1'},...,j_{l_2'}).$$

Proof. If $\mathcal{I}\setminus(i_{l_1'},...,i_{l_2'})$ is empty, then $\mathcal{J}\setminus(j_{l_1'},...,j_{l_2'})$ must be empty because the lengths of $\mathcal{I}$, $\mathcal{J}$ are the same. The statement is true in this situation. If $\mathcal{I}\setminus(i_{l_1'},...,i_{l_2'})$ is non empty, then $\mathcal{I}$ can be written as

$$(i_1,...,i_{l_1'},...,i_{l_2'},...,i_L)$$

and

$$\mathcal{I}\setminus(i_{l_1'},...,i_{l_2'}) = (i_1,...,i_{l_1'-1},i_{l_2'+1},...,i_L) = (i_1',...,i_{l_1'-1}',i_{l_1'}',...,i_{L-l_2'+l_1'-1}')$$

and

$$\mathcal{J} \setminus (j_{l'_1}, ..., j_{l'_2}) = (j_1, ..., j_{l'_1-1}, j_{l'_2+1}, ..., j_L) = (j'_1, ..., j'_{l'_1-1}, j'_{l'_1}, ..., j'_{L-l'_2+l'_1-1}).$$

For any indices $1 \leq l_1 < l_2 < L - l'_2 + l'_1 - 1$ such that $i_{l_3} > \max\{i'_{l_1}, i'_{l_2}\}$ for all $l_1 < l_3 < l_2$:
If $l_1, l_2 \leq l'_1 - 1$ or $l_1, l_2 \geq l'_1$, then $(i'_{l_1}, ..., i'_{l_2})$ is an interval of $\mathcal{I}$. Since $\mathcal{I} \sim_m \mathcal{J}$, (i'_{l_1}, i'_{l_2}) and (j'_{l_1}, j'_{l_2}) have the same order .
If $l_1 < l'_1 \leq l_2$, then $i'_{l_2} = i_{l_2+l'_2-l'_1+1}$. We have

$$i_{l_3} > i_{l'_1-1} \geq \max\{i'_{l_1}, i'_{l_2}\}$$

for all $l'_1 \leq l_3 \leq l'_2$. It follows that

$$i_{l_3} > \max\{i_{l_1}, i_{l_2}\}$$

for all $l_1 < l_3 < l_2 + l'_2 - l'_1 + 1$. Therefore, $(i_{l_1}, i_{l_2+l'_2-l'_1+1})$ and $(j_{l_1}, j_{l_2+l'_2-l'_1+1})$ have the same order which shows that (i'_{l_1}, i'_{l_2}) and (j'_{l_1}, j'_{l_2}) have the same order.
The proof is complete. □

The same as the previous proof, by checking the definition of $\sim_m$, we have

Lemma 5.1.12. *Let $\mathcal{I} = (i_1, ..., i_L) \in \mathbb{Z}^L$ and $(i_{l'_1}, ..., i_{l'_2})$ is a crest of $\mathcal{I}$, then we have*

$$\mathcal{I} = (i_1, ...i_L) \sim_m (i_1, ..., i_{l'_1-1}, i_{l'_1} + K, ..., i_{l'_2} + K, i_{l'_2+1}, ..., i_l)$$

for any integer K such that $i_{l'_1} + K > \max\{i_{l'_1-1}, i_{l'_2+1}\}$.

Now, we turn to study some relations between $M_i(n, k)$ and $\sim_m$:

Proposition 5.1.13. *Given two sequences $\mathcal{I} = \{i_1, ..., i_L\} \in [k]^L, \mathcal{J} = \{j_1, ..., j_L\} \in [n]^L$, let $\{u_{i,j}\}_{i=1,...,n;j=1,...,k}$ be the set of standard generators of $M_i(n, k)$, then we have*

$$\sum_{(q_1,...,q_L)\sim_m \mathcal{J}} u_{q_1,i_1} \cdots u_{q_L,i_L} \boldsymbol{P} = \begin{cases} \boldsymbol{P} & \text{if } \mathcal{J} \sim_m \mathcal{I} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will prove the proposition by induction.
When $L = 1$, the statement is obviously true.
Suppose the statement is true for all $L \leq L'$. Let us consider the case $L = L' + 1$. Let $(i_{l'_1}, ..., i_{l'_2})$ be a crest of $\mathcal{I}$:
Case 1: If $(j_{l'_1}, ..., j_{l'_2})$ is not a crest of $\mathcal{J}$, then $\mathcal{I} \not\sim_m \mathcal{J}$ and one of the following cases happens:

1. There exists an index $j_{l'_3}$ of $\mathcal{J}$ such that $j_{l'_3} \neq j_{l'_3+1}$ for some $l'_1 \leq l'_3 < l'_2$.

2. $j_{l'_1} \leq j_{l'_1-1}$.

3. $j_{l'_2} \leq j_{l'_2+1}$.

But, for all $\mathcal{Q} = (q_1, ..., q_L) \sim_m \mathcal{J}$, we have:

1. $(q_{l_3'}, q_{l_3'-1})$ and $(j_{l_3'}, j_{l_3'-1})$ have the same order.
2. $(q_{l_1'}, q_{l_1'-1})$ and $(j_{l_1'}, j_{l_1'-1})$ have the same order.
3. $(q_{l_2'}, q_{l_2'+1})$ and $(j_{l_2'}, j_{l_2'+1})$ have the same order.

Therefore, we have at least one of the followings:

1. $q_{l_3'} \neq q_{l_3'-1}$ and $i_{l_3'} = i_{l_3'-1}$ for some $l_1' \leq l_3' < l_2'$.
2. $q_{l_1'} \leq q_{l_1'-1}$ and $i_{l_1'} > i_{l_1'-1}$.
3. $q_{l_2'} \leq q_{l_2'+1}$ and $i_{l_2'} > i_{l_2'+1}$.

According to the definition of $M_i(n,k)$, we have one of the following equations:

1. $u_{q_{l_3'}, i_{l_3'}} u_{q_{l_3'+1}, i_{l_3'+1}} = 0$ for some $l_1' \leq l_3' < l_2'$.
2. $u_{q_{l_1'-1}, i_{l_1'-1}} u_{q_{l_1'}, i_{l_1'}} = 0$.
3. $u_{q_{l_2'}, i_{l_2'}} u_{q_{l_2'+1}, i_{l_2'+1}} = 0$.

In this case, we always have

$$\sum_{(q_1,...,q_L)\sim_m \mathcal{J}} u_{q_1,i_1} \cdots u_{q_L,i_L} \mathbf{P} = 0.$$

Case 2: If $(j_{l_1'}, ..., j_{l_2'})$ is a crest of $\mathcal{J}$, then $(q_{l_1'}, ..., q_{l_2'})$ is a crest of $\mathcal{Q}$. Therefore,

$$u_{q_{l_1'}, i_{l_1'}} \cdots u_{q_{l_2'}, i_{l_2'}} = u_{q_{l_1'}, i_{l_1'}}.$$

By Lemma 5.1.12, if we fix the indices of $\mathcal{Q} \setminus (q_{l_1'}, ..., q_{l_2'})$, then $q_{l_1'}, ..., q_{l_2'}$ can be any integers such that $q_{l_1'} = ... = q_{l_2'}$ and $\max\{q_{l_1'-1}, q_{l_2'+1})\} < q_{l_1'} \leq n$. Therefore, we have

$$\begin{aligned}
&\sum_{\max\{q_{l_1'-1}, q_{l_2'+1})\} < q_{l_1'} \leq n} u_{q_{l_1'-1}, i_{l_1'-1}} u_{q_{l_1'}, i_{l_1'}} u_{q_{l_2'+1}, i_{l_2'+1}} \\
= &\sum_{1 \leq q_{l_1'} \leq n} u_{q_{l_1'-1}, i_{l_1'-1}} u_{q_{l_1'}, i_{l_1'}} u_{q_{l_2'+1}, i_{l_2'+1}} \\
= &\; u_{q_{l_1'-1}, i_{l_1'-1}} u_{q_{l_2'+1}, i_{l_2'+1}}.
\end{aligned}$$

The first equality holds because the extra terms are 0. The second equality uses the monotone universal condition of $M_i(n,k)$. Let $L'' = L - l_2' + l_1' + 1 \leq L'$, then $\mathcal{J} \setminus (j_{l_1'}, ..., j_{l_2'}) \in [n]^{L''}$ By

Lemma 5.1.10, $\mathcal{Q} \setminus (q_{l'_1}, ..., q_{l'_2}) \sim_m \mathcal{J} \setminus (j_{l'_1}, ..., j_{l'_2})$. If we denote by $(i'_1, ..., i'_{L''})$ the sequence $\mathcal{I} \setminus (i_{l'_1}, ..., i_{l'_2})$, then we have

$$\begin{aligned} & \sum_{(q_1,...,q_L)\sim_m \mathcal{J}} u_{q_1,i_1} \cdots u_{q_L,i_L} \mathbf{P} \\ = & \sum_{(q'_1,...,q'_{L''})\sim_m \mathcal{J}\setminus(j_{l'_1},...,j_{l'_2})} u_{q'_1,i'_1} \cdots u_{q'_{L''},i'_{L''}} \mathbf{P} \\ = & \begin{cases} \mathbf{P} & \text{if } \mathcal{J} \setminus (j_{l'_1}, ..., j_{l'_2}) \sim_m \mathcal{I} \setminus (i_{l'_1}, ..., i_{l'_2}) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The last equality comes from the assumption of our induction. By Lemma 5.1.10 and Lemma 5.1.11, $\mathcal{J} \setminus (j_{l'_1}, ..., j_{l'_2}) \sim_m \mathcal{I} \setminus (i_{l'_1}, ..., i_{l'_2})$ iff $\mathcal{J} \sim_m \mathcal{I}$. The proof is complete. $\square$

Now, we turn to show that operator valued monotone finite sequences of random variables are monotonically spreadable.

Definition 5.1.14. Let $\mathcal{I} = (i_1, ..., i_L)$ be a sequence of ordered indices and $a = \min\{i_1, ..., i_L\}$. We call the set $\S(\mathcal{I}) = \{l | i_l = a\}$ the positions of the smallest elements of $\mathcal{I}$. An interval of $(i_{l'_1}, ..., i_{l'_2})$ is called a hill of $\mathcal{I}$ if $i_{l'_1-1} = i_{l'_2+1} = a$ and $i'_{l_3} \neq a$ for all $l'_1 \leq l'_3 \leq l'_2$. Here, we assume that $i_0 = i_{L+1} = a$ for convenience.

Example: $(1, 2, 3, 4, 1, 2, 1)$ has two hills $(2, 3, 4)$ and (2). $(1, 2, 1, 3, 4,)$ has two hills (2) and $(3, 4)$. $(1, 1, 1, 1, 1)$ has no hill.

Lemma 5.1.15. *Given two sequences $\mathcal{I} = \{i_1, ..., i_L\}, \mathcal{J} = \{j_1, ..., j_L\} \in [n]^L$ such that $\mathcal{I} \sim_m \mathcal{J}$, then $\S(\mathcal{I}) = \S(\mathcal{J})$. Let $(i_{l'_1}, ..., i_{l'_2})$ be a hill of $\mathcal{I}$, then*

$$(i_{l'_1}, ..., i_{l'_2}) \sim_m (j_{l'_1}, ..., j_{l'_2}).$$

Proof. We just need to check the elements of $\mathcal{J}$ one by one. Suppose

$$\S(\mathcal{I}) = \{l''_1 < \cdots < l''_{k'}\},$$

where k' is the cardinality of $\S(\mathcal{I})$. Let $b = \min\{j_1, ...j_L\}$, we need to show that $j_{l''_1} = \cdots = j_{l''_{k'}} = b$ and $j_l > b$ for all $l \notin \S(\mathcal{I})$.
Given an integer $1 \leq p < k'$, we have

$$i_l > a = i_{l''_p} = i_{l''_{p+1}}$$

for all $l''_p < l < l''_{p+1}$. According to the definition of $\sim_m$ and Lemma 5.1.5, we have

$$j_{l''_p} = j_{l''_{p+1}}$$

and

$$j_l > \max\{j_{l''_p}, j_{l''_{p+1}}\}$$

for all $l''_p < l < l''_{p+1}$. The left is to check the elements j_l with $l < l''_1$ or $l > l''_{k'}$. If there exists and $l < l''_1$ such that $j_l \leq j_{l''_1}$, we chose the greatest such l. Then, we have

$$j_{l'} > \max\{j_l, j_{l''_1}\}$$

for all $l < l' < l''_1$. Therefore, we have

$$i_l \leq i_{l''_1}$$

which is a contradiction. It implies that

$$j_l > j_{l''_1}$$

for all $l < l''_1$. the same we have

$$j_l > j_{l''_1}$$

for all $l > l''_k$. Therefore, $j_{l''_1} = \cdots = j_{l''_{k'}} = \min\{j_1, ..., j_L\}$. The last statement of this Lemma is obvious from the definition of $\sim_m$. □

Given $\mathcal{I} = \{i_1, ..., i_L\} \in \mathbb{Z}^L$, we will denote by $x_{\mathcal{I}} = x_{i_1} x_{i_2} \cdots x_{j_L}$ for short.

Proposition 5.1.16. *Let $(\mathcal{A}, \mathcal{B}, E)$ be an operator valued probability space, and $(x_i)_{i=1,...,n}$ be a sequence of random variables in $\mathcal{A}$. If $(x_i)_{i=1,...,n}$ are identically distributed and monotonically independent. Then, for indices sequences $\mathcal{I} = \{i_1, ..., i_L\}, \mathcal{J} = \{j_1, ..., j_L\} \in [n]^L$ such that $\mathcal{I} \sim_m \mathcal{J}$, $L \in \mathbb{N}$, we have*

$$E[x_{\mathcal{I}}] = E[x_{\mathcal{J}}].$$

Proof. When $L = 1$, the statement is true since the sequence is identically distributed. Suppose the statement is true for all $L \leq L' \in \mathbb{N}$. Let us consider the case $L = L' + 1$: If $\mathcal{I}$ has no hill, then $i_1 = \cdots = i_L$ which implies $j_1 = \cdots = j_L$. The statement is true, since the sequence is identically distributed.
Suppose $\mathcal{I}$ has hills $\mathcal{I}_1, ..., \mathcal{I}_l$ and $a = \min\{i_1, ..., i_L\}$. Then, $x_{\mathcal{I}}$ can be written as

$$x_a^{n_1} x_{\mathcal{I}_1} x_a^{n_2} x_{\mathcal{I}_2} \cdots x_a^{n_l} x_{\mathcal{I}_l} x_a^{n_{l+1}},$$

where $n_2, ..., n_l \in \mathbb{N}$ and $n_1, n_{l+1} \in \mathbb{N} \cup \{0\}$. Since x_i's are monotonically independent, we have

$$E[x_{\mathcal{I}}] = E[x_a^{n_1} E[x_{\mathcal{I}_1}] x_a^{n_2} E[x_{\mathcal{I}_2}] \cdots x_a^{n_l} E[x_{\mathcal{I}_l}] x_a^{n_{l+1}}].$$

Let $b = \min\{j_1, ..., j_L\}$, by Lemma 5.1.15, $\mathcal{J}$ has hills $\mathcal{J}_1, ..., \mathcal{J}_l$ whose positions of elements correspond to the positions of elements of $\mathcal{I}_1, ..., \mathcal{I}_l$ and $\mathcal{J}_{l'} \sim_m \mathcal{J}_{l'}$ for all $1 \leq l' \leq k'$. Therefore, we have

$$\begin{aligned} E[x_{\mathcal{J}}] &= E[x_b^{n_1} E[x_{\mathcal{J}_1}] x_b^{n_2} E[x_{\mathcal{J}_2}] \cdots x_b^{n_l} E[x_{\mathcal{J}_l}] x_b^{n_{l+1}}] \\ &= E[x_b^{n_1} E[x_{\mathcal{I}_1}] x_b^{n_2} E[x_{\mathcal{I}_2}] \cdots x_b^{n_l} E[x_{\mathcal{I}_l}] x_b^{n_{l+1}}] \\ &= E[x_a^{n_1} E[x_{\mathcal{I}_1}] x_a^{n_2} E[x_{\mathcal{I}_2}] \cdots x_a^{n_l} E[x_{\mathcal{I}_l}] x_a^{n_{l+1}}] \\ &= E[x_{\mathcal{I}}], \end{aligned}$$

where the second equality follows the induction and the third equality holds because x_a and x_b are identically distributed. The proof is complete. □

Proposition 5.1.17. *Let $(\mathcal{A}, \mathcal{B}, E)$ be an operator valued probability space, and $(x_i)_{i=1,...,n}$ be a sequence of random variables in $\mathcal{A}$. If $(x_i)_{i=1,...,n}$ are identically distributed and monotonically independent with respect to E. Let ϕ be a state on $\mathcal{A}$ such that $\phi(\cdot) = \phi(E[\cdot])$. Then, $(x_i)_{i=1,...,n}$ is monotonically spreadable with respect to ϕ.*

Proof. For fixed natural numbers $n, k \in \mathbb{N}$, let $(u_{i,j})_{i=1,...,n;j=1,...,k}$ be standard generators of $M_i(n,k)$. Let $\mathcal{J} = (j_1, ..., j_L) \in [k]^L$ and denote $x_{j_1} \cdots x_{j_L}$ by $x_{\mathcal{J}}$. We denote the equivalent class of $[n]^L$ associated with $\sim_m$ by $\overline{[n^L]}$. For each $\mathcal{I} \in [n]^L$, we denote $u_{i_1,j_1} \cdots u_{i_L,j_L}$ by $u_{\mathcal{I},\mathcal{J}}$. Then, by proposition 5.1.13, we have

$$\begin{aligned}
& \sum_{\mathcal{I}\in[n]^L} \phi(x_{\mathcal{I}})\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} \\
= & \sum_{\mathcal{I}\in[n]^L} \phi(E[x_{\mathcal{I}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} \\
= & \sum_{\bar{Q}\in\overline{[n]^L}} \sum_{\mathcal{I}\in\bar{Q}} \phi(E[x_{\mathcal{I}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} \\
= & \sum_{\mathcal{J}\notin\bar{Q}\in\overline{[n]^L}} \sum_{\mathcal{I}\in\bar{Q}} \phi(E[x_{\mathcal{I}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} + \sum_{\mathcal{J}\in\bar{Q}\in\overline{[n]^L}} \sum_{\mathcal{I}\in\bar{Q}} \phi(E[x_{\mathcal{I}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} \\
= & \sum_{\mathcal{J}\notin\bar{Q}\in\overline{[n]^L}} \sum_{\mathcal{I}\in\bar{Q}} \phi(E[x_{\mathcal{Q}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} + \sum_{\mathcal{I}\sim_m\mathcal{J}} \phi(E[x_{\mathcal{J}}])\mathbf{P}u_{\mathcal{I},\mathcal{J}}\mathbf{P} \\
= & \ 0 + \phi(E[x_{\mathcal{J}}])\mathbf{P} \\
= & \ \phi(x_{\mathcal{J}})\mathbf{P}.
\end{aligned}$$

Since n, k are arbitrary, the proof is complete.

□

5.2 Tail algebras

In the previous work on distributional symmetries, infinite sequences of objects are indexed by natural numbers. For this kind of infinite sequences, the conditional expectations in de Finetti type theorems are defined via the limit of unilateral shifts. It was shown in [22] that unilateral shift is an isometry from $\mathcal{A}$ into itself if $(\mathcal{A}, \phi)$ is a W^*-probability space generated by a spreadable sequence of random variables and ϕ is faithful. Therefore, a normal conditional expectation defined via the limit of unilateral shifts exists under a very weak condition, i.e. the sequence of random variables just need to be spreadable. However, our works are in a more general situation that the state ϕ is not necessarily faithful. In our framework, we will provide an example in which the sequence is monotonically spreadable but the unilateral shift is not an isometry. Therefore, we can not get an extended de Finetti type theorem for monotone independence in the usual way. Therefore, we will turn to consider bilateral sequences of random variables. Here, we begin with an interesting example :

5.2.1 Unbounded spreadable sequences

Example: Let $\mathcal{H}$ be the standard 2-dimensional Hilbert space with orthonormal basis

$$\{v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}.$$

Let $p, A, x \in B(\mathcal{H})$ be operators on $\mathcal{H}$ with the following matrix forms:

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathscr{H} = \bigotimes_{n=1}^{\infty} \mathcal{H}$ the infinite tensor product of $\mathcal{H}$. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of selfadjoint operators in $B(\mathscr{H})$ defined as follows:

$$x_i = \bigotimes_{n=1}^{i-1} A \otimes x \otimes \bigotimes_{m=1}^{\infty} p$$

Let ϕ be the vector state $\langle \cdot v, v\rangle$ on $\mathcal{H}$ and $\Phi = \bigotimes_{n=1}^{\infty} \phi$ be a state on $B(\mathscr{H})$. It is obvious that $\Phi(x_i^n) = \phi(x^n)$ for for i. Therefore, the sequence $(x_i)_{i\in\mathbb{N}}$ is identically distributed. For any $x, y \in B(\mathcal{H})$, an elementary computation shows

$$\phi(xpy) = \phi(x)\phi(y).$$

For convenience, we denote by $A^{\otimes i-1} = \bigotimes_{n=1}^{i-1} A$ and $P^{\otimes\infty} = \bigotimes_{n=1}^{\infty} P$. Also, we denote $x_{i_1} \cdots x_{i_L} = x_{\mathcal{I}}$ for $\mathcal{I} = (i_1, ..., i_L) \in \mathbb{N}^L$. We will show that the sequence $\{x_i\}_{i\in\mathbb{N}}$ is $M_i(n,k)$-spreadable with respect to Φ.

Lemma 5.2.1. *For indices sequences $\mathcal{I} = (i_1, ..., i_L), \mathcal{J} = (j_1, ..., j_L) \in [n]^L$ such that $\mathcal{I} \sim_m \mathcal{J}$ and $L \in \mathbb{N}$, we have*

$$\Phi(x_{\mathcal{I}}) = \Phi(x_{\mathcal{J}}).$$

Proof. When $L = 1$, the statement is true since the sequence is identically distributed. Suppose the statement is true for all $L \le L'$. Let us consider the case $L = L' + 1$. If $\mathcal{I}$ has no hill, then $i_1 = \cdots = i_L$ which implies $j_1 = \cdots = j_L$. The statement is true for this case, because the sequence is identically distributed. Also, we denote by $x_i^{(n)}$ the n-the component of x_i. Then,

$$x_i^{(n)} = \begin{cases} a & \text{if } n < i \\ x & \text{if } n = i \\ p & \text{if } n > i \end{cases}$$

and $x_{\mathcal{I}}^{(n)} = x_{i_1}^{(n)} x_{i_2}^{(n)} \cdots x_{i_L}^{(n)}$.
According to the definition of Φ, we have that

$$\Phi(x_{i_1}x_{i_2}\cdots x_{j_L}) = \prod_{n=1}^{\infty}\phi(\prod_{l=1}^{L} x_i^{(n)}).$$

Suppose that $\mathcal{I}$ has hills $\mathcal{I}_1, ..., \mathcal{I}_l$ and $a = \min\{i_1, ..., i_L\}$, then $x_{\mathcal{I}}$ can be written as

$$x_a^{n_1} x_{\mathcal{I}_1} x_a^{n_2} x_{\mathcal{I}_2} \cdots x_a^{n_l} x_{\mathcal{I}_l} x_a^{n_{l+1}}$$

and

$$\phi(\prod_{l=1}^{L} x_i^{(n)}) = \begin{cases} 1 & \text{if } n < a \\ \phi(x^{n_1}A^{|\mathcal{I}_1|}x^{n_2}A^{|\mathcal{I}_2|}\cdots x^{n_l}A^{|\mathcal{I}_l|}x^{n_{l+1}}) & \text{if } n = a \\ \phi(px_{\mathcal{I}_1}^{(n)}px_{\mathcal{I}_2}^{(n)}p\cdots px_{\mathcal{I}_l}^{(n)}p) & \text{if } n > a. \end{cases}$$

It follows that

$$\phi(\prod_{l=1}^{L} x_i^{(n)}) = \prod_{n\geq \min\{\mathcal{I}\}}^{\infty}\phi(\prod_{l=1}^{L} x_i^{(n)}).$$

Because

$$\phi(px_{\mathcal{I}_1}^{(n)}px_{\mathcal{I}_2}^{(n)}p\cdots px_{\mathcal{I}_l}^{(n)}p) = \phi(x_{\mathcal{I}_1}^{(n)})\phi(x_{\mathcal{I}_2}^{(n)})\cdots\phi(x_{\mathcal{I}_l}^{(n)}),$$

we have

$$\begin{aligned} & \Phi(x_{i_1}x_{i_2}\cdots x_{j_L}) \\ = \ & \phi(x^{n_1}A^{|\mathcal{I}_1|}x^{n_2}A^{|\mathcal{I}_2|}\cdots x^{n_l}A^{|\mathcal{I}_l|}x^{n_{l+1}})\prod_{n>a}^{\infty}\phi(px_{\mathcal{I}_1}^{(n)}px_{\mathcal{I}_2}^{(n)}p\cdots px_{\mathcal{I}_l}^{(n)}p). \\ = \ & \phi(x^{n_1}A^{|\mathcal{I}_1|}x^{n_2}A^{|\mathcal{I}_2|}\cdots x^{n_l}A^{|\mathcal{I}_l|}x^{n_{l+1}})\prod_{n>a}^{\infty}\phi(x_{\mathcal{I}_1}^{(n)})\phi(x_{\mathcal{I}_2}^{(n)})\cdots\phi(x_{\mathcal{I}_l}^{(n)}) \\ = \ & \phi(x^{n_1}A^{|\mathcal{I}_1|}x^{n_2}A^{|\mathcal{I}_2|}\cdots x^{n_l}A^{|\mathcal{I}_l|}x^{n_{l+1}})\Phi(x_{\mathcal{I}_1})\Phi(x_{\mathcal{I}_2})\cdots\Phi(x_{\mathcal{I}_l}). \end{aligned}$$

Let $b = \min\{j_1, ..., j_L\}$, by Lemma 5.1.15, $\mathcal{J}$ has hills $\mathcal{J}_1, ..., \mathcal{J}_l$ whose positions of elements correspond to the positions of elements of $\mathcal{I}_1, ..., \mathcal{I}_l$ and $\mathcal{J}_{l'} \sim_m \mathcal{J}_{l'}$ for all $1 \leq l' \leq k'$. Therefore, we have

$$\begin{aligned} \Phi(x_{\mathcal{J}}) &= \Phi(x_{i_1}x_{i_2}\cdots x_{i_L}) \\ &= \phi(x^{n_1}A^{|\mathcal{J}_1|}x^{n_2}A^{|\mathcal{J}_2|}\cdots x^{n_l}A^{|\mathcal{J}_l|}x^{n_{l+1}})\Phi(x_{\mathcal{J}_1})\Phi(x_{\mathcal{J}_2})\cdots\Phi(x_{\mathcal{J}_l}) \\ &= \phi(x^{n_1}A^{|\mathcal{I}_1|}x^{n_2}A^{|\mathcal{I}_2|}\cdots x^{n_l}A^{|\mathcal{I}_l|}x^{n_{l+1}})\Phi(x_{\mathcal{I}_1})\Phi(x_{\mathcal{I}_2})\cdots\Phi(x_{\mathcal{I}_l}) \\ &= \Phi(x_{\mathcal{I}}). \end{aligned}$$

where the second equality follows the induction, the fact that $\mathcal{J}_k \sim_m \mathcal{I}_k$ and $|\mathcal{J}_k| = |\mathcal{I}_k|$ for all $1 \leq k \leq l$. □

Proposition 5.2.2. *The joint distribution of* $(x_i)_{i\in\mathbb{N}}$ *with respect to* Φ *is monotonically spreadable.*

Proof. Fixed $n > k \in \mathbb{N}$, let $\{u_{i,j}\}_{i=1,...,n;j=1,...,k}$ be the set of standard generators of $M_i(n,k)$. For all $\mathcal{I} = (i_1, ..., i_L) \in [k]^L$, we denote by $\overline{[n]^L}$ the $\sim_m$ equivalence class of $[n]^L$, then we have

$$\begin{aligned}
& \mathbf{P}\mu_{x_1,...,x_n} \otimes (id_{M_i(n,k)})(\alpha_{n,k}^{(m)}(X_{\mathcal{I}}))\mathbf{P} \\
= & \sum_{\mathcal{J}\in[n]^L} \mu_{x_1,...,x_n}(X_{\mathcal{J}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= & \sum_{\bar{\mathcal{Q}}\in\overline{[n]^L}} \sum_{\mathcal{J}\in\bar{\mathcal{Q}}} \mu_{x_1,...,x_n}(X_{\mathcal{J}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= & \sum_{\mathcal{I}\notin\bar{\mathcal{Q}}\in\overline{[n]^L}} \sum_{\mathcal{J}\in\bar{\mathcal{Q}}} \mu_{x_1,...,x_n}(X_{\mathcal{J}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} + \sum_{\mathcal{J}\sim_m\mathcal{I}} \mu_{x_1,...,x_n}(X_{\mathcal{J}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= & \sum_{\mathcal{I}\notin\bar{\mathcal{Q}}\in\overline{[n]^L}} \sum_{\mathcal{J}\in\bar{\mathcal{Q}}} \mu_{x_1,...,x_n}(X_{\mathcal{Q}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} + \sum_{\mathcal{J}\sim_m\mathcal{I}} \mu_{x_1,...,x_n}(X_{\mathcal{I}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= & \sum_{\mathcal{I}\notin\bar{\mathcal{Q}}\in\overline{[n]^L}} \mu_{x_1,...,x_n}(X_{\mathcal{Q}}) \sum_{\mathcal{J}\in\bar{\mathcal{Q}}} \mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} + \sum_{\mathcal{J}\sim_m\mathcal{I}} \mu_{x_1,...,x_n}(X_{\mathcal{I}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= & \sum_{\mathcal{I}\notin\bar{\mathcal{Q}}\in\overline{[n]^L}} \mu_{x_1,...,x_n}(X_{\mathcal{Q}}) \cdot 0 + \sum_{\mathcal{J}\sim_m\mathcal{I}} \mu_{x_1,...,x_n}(X_{\mathcal{I}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= & \sum_{\mathcal{J}\sim_m\mathcal{I}} \mu_{x_1,...,x_n}(X_{\mathcal{I}})\mathbf{P}u_{\mathcal{J},\mathcal{I}}\mathbf{P} \\
= & \mu_{x_1,...,x_n}(X_{\mathcal{I}})\mathbf{P}.
\end{aligned}$$

The proof is complete. □

By direct computations, we have

$$\prod_{i=1}^{n} x_{n+1-i} v^{\otimes\infty} = w^{\otimes n} \otimes v^{\otimes\infty}$$

and

$$x_{n+1} w^{\otimes n} \otimes v^{\otimes\infty} = 2^n w^{\otimes n+1} \otimes v^{\otimes\infty}. \tag{5.1}$$

Let $(\mathcal{H}', \pi', \xi')$ be the GNS representation of the von Neumann algebra generated by $(x_i)_{i=1,...,\infty}$ associated with Φ. We have

$$\|\pi'(x_{n+1})\| \leq \|x_{n+1}\| = 2^n,$$

but equation 5.1 shows that $\|\pi'(x_{n+1})\| \geq 2^n$. Therefore, $\|\pi'(x_{n+1})\| = 2^n$.

Let $\mathcal{A}$ be the von Neumann algebra generated by $\pi(x_i)$'s. Then, there is no bounded endomorphism α on $\mathcal{A}$ such that $\alpha(x_i) = x_{i+1}$.

5.2.2 Tail algebras of bilateral sequences of random variables

We have shown that, in a W^*-probability space with a non-degenerated normal state, the unilateral shift of a spreadable unilateral sequence of random variables may not be extended to a bounded endomorphism. Therefore, in general, we can not define a normal condition expectation by taking the limit of unilateral shifts of variables. In $(\mathcal{A}, \phi)$, a W^*-probability space with a faithful state, the norm of a selfadjoint random variable $x \in \mathcal{A}$ is controlled by the moments of X, i.e.

$$\|x\| = \lim_{n\to\infty} \phi(|x|^n)^{\frac{1}{n}}.$$

But, in non-degenerated W^*-probability spaces, the norm of a random variable depends on all mixed moments which involve it. As a kind of partial distributional symmetries, spreadability can not provide relations between all mixed moments which makes a spreadable sequence can be unbounded. To create a well-defined conditional expectation, we consider spreadable sequences of random variables indexed by $\mathbb{Z}$ but not $\mathbb{N}$. As a consequence, we will have two choices to take limits on defining normal conditional expectations and tail algebras. Before studying tail algebras of bilateral sequences, we introduce some necessary notations and assumptions here.

Let $(\mathcal{A}, \phi)$ be a W^*-probability space generated by a spreadable bilateral sequence of bounded random variables $(x_i)_{i\in\mathbb{Z}}$ and ϕ is a non-degenerated normal state. We assume that the unit of $\mathcal{A}$ is contained in the WOT-closure of the non-unital algebra generated by $(x_i)_{i\in\mathbb{Z}}$. Let $(\mathcal{H}, \pi, \xi)$ be the GNS representation of $\mathcal{A}$ associated with ϕ. Then, $\{\pi(P(x_i|i \in \mathbb{Z}))\xi | P \in \mathbb{C}\langle X_i | i \in \mathbb{Z}\rangle\}$ is dense in $\mathcal{H}$. For convenience, we will denote $\pi(y)\xi$ by $\hat{y}$ for all $y \in \mathcal{A}$. When there is no confusion, we will write y short for $\pi(y)$. We denote by A_{k+} the non-unital algebra generated by $(x_i)_{i\geq k}$ and A_{k-} the non-unital algebra generated by $(x_i)_{i\leq k}$. Let $\mathcal{A}_k^+$ and $\mathcal{A}_k^-$ be the WOT-closure of A_{k+} and A_{k-}, respectively.

Definition 5.2.3. Let $(\mathcal{A}, \phi)$ be a no-degenerated noncommutative W^*-probability space, $(x_i)_{i\in\mathbb{Z}}$ be a bilateral sequence of bounded random variables in $\mathcal{A}$ such that $\mathcal{A}$ is the WOT closure of the non-unital algebra generated by $(x_i)_{i\in\mathbb{Z}}$. The positive tail algebra $\mathcal{A}_{tail}^+$ of $(x_i)_{i\in\mathbb{Z}}$ is defined as following:

$$\mathcal{A}_{tail}^+ = \bigcap_{k>0} \mathcal{A}_k^+.$$

In the opposite direction, we define the negative tail algebra $\mathcal{A}_{tail}^-$ of $(x_i)_{i\in\mathbb{Z}}$ as following:

$$\mathcal{A}_{tail}^- = \bigcap_{k<0} \mathcal{A}_k^-.$$

Remark 5.2.4. *In general, the positive tail algebra and the negative tail algebra are different.*

Even though our framework looks quit different from the framework in [22], we can show that there exists a normal bounded shift of the sequence in a similar way. For completeness, we provide the details here.

Lemma 5.2.5. *There exists a unitary map $U : \mathcal{H} \to \mathcal{H}$ such that $U(P(x_i|i \in \mathbb{Z}))\xi = P(x_{i+1}|i \in \mathbb{Z})\xi$*

Proof. Since $(x_i)_{i\in\mathbb{Z}}$ is spreadable, we have

$$\phi((P(x_i|i \in \mathbb{Z}))^* P(x_i|i \in \mathbb{Z})) = \phi((P(x_{i+1}|i \in \mathbb{Z}))^* P(x_{i+1}|i \in \mathbb{Z})).$$

It implies that

$$U(P(x_i|i \in \mathbb{Z})\xi) = P(x_{i+1}|i \in \mathbb{Z})\xi$$

is a well defined isometry on $\{\pi(P(x_i|i \in \mathbb{Z}))\xi|P \in \mathbb{C}\langle X_i|i \in \mathbb{Z}\rangle\}$. Since $\{\pi(P(x_i|i \in \mathbb{Z}))\xi|P \in \mathbb{C}\langle X_i|i \in \mathbb{Z}\rangle\}$ is dense in $\mathcal{H}$, U can be extended to the whole space $\mathcal{H}$. It is obvious that $\{\pi(P(x_i|i \in \mathbb{Z}))\xi|P \in \mathbb{C}\langle X_i|i \in \mathbb{Z}\rangle\}$ is contained in the range of U. Therefore, the extension of U is a unitary map on $\mathcal{H}$. □

Now, we can define an automorphism α on $\mathcal{A}$ by the following formula:

$$\alpha(y) = UyU^{-1}.$$

Lemma 5.2.6. *α is the bilateral shift of $(x_i)_{i\in\mathbb{Z}}$, i.e.*

$$\alpha(x_k) = x_{k+1}$$

for all $k \in \mathbb{Z}$.

Proof. For all $y = P(x_i|i \in \mathbb{Z})\xi$, we have

$$\alpha(x_k)y = Ux_kU^{-1}P(x_i|i \in \mathbb{Z})\xi = Ux_kP(x_{i-1}|i \in \mathbb{Z})\xi = x_{k+1}P(x_i|i \in \mathbb{Z})\xi.$$

By the density of $\{\pi(P(x_i|i \in \mathbb{Z}))\xi|P \in \mathbb{C}\langle X_i|i \in \mathbb{Z}\rangle\}$, we have $\alpha(x_k) = x_{k+1}$. The proof is complete. □

Since α is a normal automorphism of $\mathcal{A}$, we have

Corollary 5.2.7. *For all $k \in \mathbb{Z}$, we have $\alpha(\mathcal{A}_k^+) = \mathcal{A}_{k+1}^+$.*

Lemma 5.2.8. *Fix $n \in \mathbb{Z}$. Let $y_1, y_2 \in A_{n-}$. Then, we have*

$$\langle \alpha^l(a)\hat{y}_1, \hat{y}_2\rangle = \langle a\hat{y}_1, \hat{y}_2\rangle,$$

where $l \in \mathbb{N}$ and $a \in \mathcal{A}_{n+1}^+$.

Proof. It is sufficient to prove the statement under the assumption that $l = 1$. Since $a \in \mathcal{A}_{n+1}^+$, by Kaplansky's theorem, there exists a sequence $(a_m)_{m\in\mathbb{N}} \subset A_{(n+1)+}$ such that $\|a_m\| \leq \|a\|$ for all m and a_m converges to a in WOT. Then, by the spreadability of $(x_i)_{i\in\mathbb{Z}}$, we have

$$\langle \alpha(a)\hat{y}_1, \hat{y}_2\rangle = \lim_{m\to\infty} \langle \alpha(a_m)\hat{y}_1, \hat{y}_2\rangle = \lim_{m\to\infty} \phi(y_2^* a_m \hat{y}_1) = \langle a\hat{y}_1, \hat{y}_2\rangle$$

□

In the following context, we fix $k \in \mathbb{Z}$.

Lemma 5.2.9. *For all $a \in \mathcal{A}_k^+$, we have that*

$$E^+[a] = WOT - \lim_{l\to\infty} \alpha^l(a)$$

exists. Moreover, $E^+[a] \in \mathcal{A}_{tail}^+$

Proof. For all $y_1, y_2 \in \{\pi(P(x_i|i \in \mathbb{Z}))\xi | P \in \mathbb{C}\langle X_i|i \in \mathbb{Z}\rangle\}$, there exits $n \in \mathbb{Z}$ such that $y_1, y_2 \in \mathcal{A}_{n-}$. For all $l > n-k$, we have $\alpha^l(a) \in \mathcal{A}_{(n+1)+}$. By Lemma 5.2.8, we have

$$\langle \alpha^{n+1-k}(a)y_1, y_2\rangle = \langle \alpha^{n+2-k}(a)y_1, y_2\rangle = \cdots .$$

Therefore,

$$\lim_{l\to\infty}\langle \alpha^l(a)y_1, y_2\rangle = \langle \alpha^{n+1-k}(a)y_1, y_2\rangle.$$

$\alpha^l(a)$ converges pointwisely to an element $E^+[a]$. Since for all $n > 0$, we have $\alpha^l(a) \in \mathcal{A}_n^+$ for all $l > n-k+1$. It follows that $WOT - \lim_{l\to\infty} \alpha^l(a) \in \mathcal{A}_n^+$ for all n. Hence, $E^+[a] \in \mathcal{A}_{tail}^+$. □

Proposition 5.2.10. *E^+ is normal on $\mathcal{A}_k^+$ for all $k \in \mathbb{Z}$.*

Proof. Let $(a_m)_{m\in\mathbb{N}} \subset \mathcal{A}_k^+$ be a bounded sequence which converges to 0 in WOT. For all $y_1, y_2 \in \{\pi(P(x_i|i \in \mathbb{Z}))\xi | P \in \mathbb{C}\langle X_i|i \in \mathbb{Z}\rangle\}$, there exits $n \in \mathbb{Z}$ such that $y_1, y_2 \in \mathcal{A}_{n-}$. Then, we have

$$\lim_{m\to\infty}\langle E^+[a_m]y_1, y_2\rangle = \lim_{m\to\infty}\langle \alpha^{n+1-k}(a_m)y_1, y_2\rangle = 0.$$

The last equality holds because α^l is normal for all $l \in \mathbb{N}$. The proof is complete. □

Remark 5.2.11. *E^+ is defined on $\bigcup_{k\in\mathbb{Z}} \mathcal{A}_k^+$ but not on $\mathcal{A}$. In general, we can not extend E^+ to the whole algebra $\mathcal{A}$.*

Lemma 5.2.12. *$E^+[a] = a$ for all $a \in \mathcal{A}_{tail}^+$.*

Proof. For all $\hat{y}_1, \hat{y}_2 \in \{\pi(P(x_i|i \in \mathbb{Z}))\xi | P \in \mathbb{C}\langle X_i|i \in \mathbb{Z}\rangle\}$, there exits $n \in \mathbb{Z}$ such that $y_1, y_2 \in A_{n-}$. Since $a \in \mathcal{A}_{tail}^+ \subset \mathcal{A}_{n+1}^+$, by Kaplansky's theorem, there exists a sequence of $(a_m)_{m\in\mathbb{N}} \subset A_{(n+1)+}$ such that $a_m \to a$ in WOT and $\|a_m\| \le \|a\|$ for all m. Then, we have

$$\langle a\hat{y}_1, \hat{y}_2\rangle = \lim_{m\to\infty}\langle a\hat{y}_1, \hat{y}_2\rangle = \lim_{m\to\infty}\langle \alpha(a_m)\hat{y}_1, \hat{y}_2\rangle = \langle \alpha(a)\hat{y}_1, \hat{y}_2\rangle.$$

Since y_1, y_2 are arbitrary, we have $a = \alpha(a)$. □

Remark 5.2.13. *One should be careful that $\mathcal{A}_{tail}^+$ could be a proper subset of the fixed points set of α.*

Lemma 5.2.14.

$$E^+[a_1ba_2] = a_1E^+[b]a_2$$

for all $b \in \mathcal{A}_k^+$, $a_1, a_2 \in \mathcal{A}_{tail}^+$.

Proof. By Lemma 5.2.12, we have

$$E^+[a_1ba_2] = \lim_{l\to\infty}\alpha^l(a_1ba_2) = \lim_{l\to\infty}\alpha^l(a_1)\alpha^l(b)\alpha^l(a_2) = \lim_{l\to\infty}a_1\alpha^l(b)a_2 = a_1E^+[b]a_2.$$

□

5.3 Conditional expectations of bilateral monotonically spreadable sequence

In this section, we assume that the joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable.

Lemma 5.3.1. *Fix $n > k \in \mathbb{N}$, let $(u_{i,j})_{i=1,...,n;\, j=1,...,k}$ be the standard generators of $M_i(n,k)$. Then, we have*

$$\phi(a_1x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m}a_2)\boldsymbol{P} = \sum_{j_1,...,j_m=1}^{n}\phi(a_1x_{j_1}^{l_1}b_1x_{j_2}^{l_2}b_2\cdots b_{m-1}x_{j_m}^{l_m}a_2)\boldsymbol{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\boldsymbol{P},$$

where $1 \leq i_1,...i_m \leq k$, $b_1,...,b_{m-1} \in A_{(n+1)+}$ and $a_1,a_2 \in A_{0-}$.

Proof. Without loss of generality, we assume that there exist $n_1,n_2 \in \mathbb{N}$ such that

$$a_1,a_2 \in A_{[-n_1+1,0]}$$

and

$$b_1,...,b_{m-1} \in A_{[n+1,n_2+k]}.$$

Since the map is linear, we just need to consider the case that a_1,a_2 and $b_1,...,b_{m-1}$ are products of $(x_i)_{i\in\mathbb{Z}}$. Let

$$a_1 = x_{s_{1,1}}\cdots x_{s_{1,t_1}}$$

and

$$a_2 = x_{s_{2,1}}\cdots x_{s_{2,t_2}}$$

for some $t_1,t_2 \in \mathbb{N}$ and $-n_1+1 \leq s_{c,d} \leq 0$. Let

$$b_i = x_{r_{i,1}}\cdots x_{r_{i,t'_i}}$$

for $t'_1,...,t'_{m-1} \in \mathbb{N}\cup\{0\}$ and $n+1 \leq r_{c,d} \leq k+n_2$. Then, $(x_{-n_1+1},...,x_{n+n_2})$ is a sequence of length $n+n_1+n_2$, we denote it by $(y_1,...,y_{n+n_1+n_2})$. Let $n' = n+n_1+n_2$ and $k' = k+n_1+n_2$. By our assumption, $a_1x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m}a_2$ is in the algebra generated by $(y_1,....,y_{k'})$. Let $(u'_{i,j})_{i=1,...,n';\, j=1,...,k'}$ be the standard generators of $M_i(n',k')$ and $\mathbf{P}'$ be the invariant projection. Let π be the C^*-homomorphism in Lemma 3.3.13 and id be the identity on $\mathbb{C}\langle X_1,....,X_{n'}\rangle$. Since $1 \leq s_{c,d}+n_1 \leq n_1$, we have

$$id\otimes\pi(\alpha_{n',k'}^{(m)}(X_{s_{i,1}+n_1}\cdots X_{s_{i,t_1}}+n_1)) = X_{s_{i,1}+n_1}\cdots X_{s_{i,t_1}+n_1}\otimes\mathbf{P}.$$

Since $n_1+n+1 \leq r_{c,d}+n_1 \leq n_1+n_2+k$, we have

$$id\otimes\pi(\alpha_{n',k'}^{(m)}(X_{r_{i,1}+n_1}\cdots X_{r_{1,t'_1}}+n_1)) = X_{r_{i,1}+n_1+n-k}\cdots X_{r_{i,t'_i}+n_1+n-k}\otimes I,$$

where I is the identity of $M_i(n,k)$. According to our assumption, we have $1 \leq i_t \leq k$ for $t = 1, ..., m$. Then

$$id \otimes \pi(\alpha^{(m)}_{n',k'}(X^{l_t}_{i_t+n_1}) = \sum_{j_t=1}^{n} X^{l_t}_{j_t+n_1} \otimes u_{j_t,i_t}.$$

According to the monotone spreadability of $(y_1, ..., y_{n'})$ and Lemma 3.3.13, we have

$$\begin{aligned}
& \phi(a_1 x^{l_1}_{i_1} b_1 x^{l_2}_{i_2} b_2 \cdots b_{m-1} x^{l_m}_{i_m} a_2)\mathbf{P} \\
= \ & \mu_{y_1,...,y_{k'}}(X_{s_{1,1}+n_1} \cdots X_{s_{1,t_1}+n_1} X^{l_1}_{i_1+n_1} \cdots X^{l_m}_{i_m+n_1} X_{s_{1,1}+n_1} \cdots X_{s_{2,t_2}+n_1})\pi(\mathbf{P'}) \\
= \ & \mathbf{P}\mu_{y_1,...,y_{n'}} \otimes \pi(\alpha^{(m)}_{n',k'}(X_{s_{1,1}+n_1} \cdots X_{s_{1,t_1}+n_1} X^{l_1}_{i_1+n_1} \cdots X^{l_m}_{i_m+n_1} X_{s_{1,1}+n_1} \cdots X_{s_{2,t_2}+n_1}))\mathbf{P} \\
= \ & \sum_{j_1,...,j_m=1}^{n} \mu_{y_1,...,y_{n'}}(X_{s_{1,1}+n_1} \cdots X_{s_{1,t_1}+n_1} X^{l_1}_{j_1+n_1} X_{r_{1,1}+n_1+n-k} \cdots \\
& X_{r_{m-1,t'_{m-1}+n_1}+n-k} X^{l_m+n_1}_{j_m} X_{s_{1,1}+n_1} \cdots X_{s_{2,t_2}})\mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P}.
\end{aligned}$$

Notice that $(y_1, ..., y_{n'})$ is spreadable and $n+1 \leq r_{,}$, the above equation becomes

$$\begin{aligned}
& \phi(a_1 x^{l_1}_{i_1} b_1 x^{l_2}_{i_2} b_2 \cdots b_{m-1} x^{l_m}_{i_m} a_2)\mathbf{P} \\
= \ & \sum_{j_1,...,j_m=1}^{n} \mu_{y_1,...,y_{n'}}(X_{s_{1,1}+n_1} \cdots X_{s_{1,t_1}+n_1} X^{l_1}_{j_1+n_1} X_{r_{1,1}+n_1} \cdots \\
& X_{r_{m-1,t'_{m-1}+n_1}} X^{l_m}_{j_m+n_1} X_{s_{1,1}+n_1} \cdots X_{s_{2,t_2}})\mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P} \\
= \ & \sum_{j_1,...,j_m=1}^{n} \phi(x_{s_{1,1}} \cdots x_{s_{1,t_1}} x^{l_1}_{j_1} x_{r_{1,1}} \cdots x_{r_{m-1,t'_{m-1}}} x^{l_m}_{j_m} x_{s_{1,1}} \cdots x_{s_{2,t_2}}) \\
& \mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P} \\
= \ & \sum_{j_1,...,j_m=1}^{n} \phi(a_1 x^{l_1}_{j_1} b_1 x^{l_2}_{j_2} b_2 \cdots b_{m-1} x^{l_m}_{j_m} a_2)\mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P}.
\end{aligned}$$

The proof is complete. □

Lemma 5.3.2. *Fix $n > k \in \mathbb{N}$, let $(u_{i,j})_{i=1,...,n;\, j=1,...,k}$ be the standard generators of $M_i(n,k)$. Then, we have*

$$E^+[x^{l_1}_{i_1} b_1 x^{l_2}_{i_2} b_2 \cdots b_{m-1} x^{l_m}_{i_m}] \otimes \boldsymbol{P} = \sum_{j_1,...,j_m=1}^{n} E^+[x^{l_1}_{j_1} b_1 x^{l_2}_{j_2} b_2 \cdots b_{m-1} x^{l_m}_{j_m}] \otimes \boldsymbol{P} u_{j_1,i_1} \cdots u_{j_m,i_m} \boldsymbol{P},$$

where $1 \leq i_1, ... i_m \leq k$, $b_1, ..., b_{m-1} \in A_{(n+1)+}$.

Proof. It is necessary to check the two sides of the equation equal to each other pointwisely, i.e.

$$\phi(a_1 E^+[x^{l_1}_{i_1} b_1 x^{l_2}_{i_2} b_2 \cdots b_{m-1} x^{l_m}_{i_m}] a_2)\mathbf{P} = \sum_{j_1,...,j_m=1}^{n} \phi(a_1 E^+[x^{l_1}_{j_1} b_1 x^{l_2}_{j_2} b_2 \cdots b_{m-1} x^{l_m}_{j_m}] a_2)\mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m}\mathbf{P} \tag{5.2}$$

for all $a_1, a_2 \in A_{[-\infty,\infty]}$. Given $a_1, a_2 \in A_{[-\infty,\infty]}$, then there exists $M \in \mathbb{N}$ such that $a_1, a_2 \in A_{M-}$. Then,

$$\alpha^{-m}(a_1), \alpha^{-m}(a_2) \in A_{0-}$$

for all $m > M$. By Lemma 5.3.1, we have

$$\begin{aligned} &\phi(\alpha^{-m}(a_1)x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m}\alpha^{-m}(a_2))\mathbf{P} \\ = &\sum_{j_1,\dots,j_m=1}^{n}\phi(\alpha^{-m}(a_1)x_{j_1}^{l_1}b_1x_{j_2}^{l_2}b_2\cdots b_{m-1}x_{j_m}^{l_m}\alpha^{-m}(a_2))\mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P}. \end{aligned}$$

Therefore, for all $m > M$,we have

$$\begin{aligned} &\phi(a_1\alpha^m(x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m})a_2)\mathbf{P} \\ = &\sum_{j_1,\dots,j_m=1}^{n}\phi(a_1\alpha^m(x_{j_1}^{l_1}b_1x_{j_2}^{l_2}b_2\cdots b_{m-1}x_{j_m}^{l_m})a_2)\mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P}. \end{aligned}$$

Let m go to $+\infty$, we get equation 6.2.
The proof is complete since a_1, a_2 are arbitrary. □

Proposition 5.3.3. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space, $(x_i)_{i\in\mathbb{Z}}$ a sequence of selfadjoint random variables in $\mathcal{A}$, E^+ be the conditional expectation onto the positive tail algebra $\mathcal{A}_{tail}^+$. Assume that the joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable, then the same is true for the joint distribution with respect to E^+, i.e. for fixed $n > k \in \mathbb{N}$ and $(u_{i,j})_{i=1,\dots,n;\,j=1,\dots,k}$ the standard generators of $M_i(n,k)$, we have that*

$$E^+[x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m}]\otimes \boldsymbol{P} = \sum_{j_1,\dots,j_m=1}^{n} E^+[x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_2\cdots b_{m-1}x_{i_m}^{l_m}]\otimes \boldsymbol{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\boldsymbol{P},$$

$1 \le i_1, ..., i_m \le k$, $l_1, ..., l_m \in \mathbb{N}$ and $b_1, ..., b_n \in \mathcal{A}_{tail}^+$.

Proof. Since $b_1, ..., b_{m-1} \in \mathcal{A}_{tail}^+ \in \mathcal{A}_n^+$, by Kaplansky's theorem, there exists sequences

$$\{b_{s,t}\}_{s=1,\dots m-1;t\in\mathbb{N}} \subset A_{n+}$$

such that $\|b_{s,t}\| \le \|b_s\|$ and $\lim_{n\to\infty} b_{s,t} = b_s$ in SOT for each $s = 1, ..., m-1$. Therefore,

$$SOT - \lim_{t_1\to\infty} x_{i_1}^{l_1}b_{1,t_1}x_{i_2}^{l_2}b_{2,t_2}\cdots b_{m-1,t_m}x_{i_m}^{l_m} = x_{i_1}^{l_1}b_1x_{i_2}^{l_2}b_{2,t_2}\cdots b_{m-1,t_m}x_{i_m}^{l_m}.$$

By Lemma 5.3.2, we have

$$E^+[x_{i_1}^{l_1}b_{1,t_1}x_{i_2}^{l_2}b_{2,t_2}\cdots b_{m-1,t_m}x_{i_m}^{l_m}]\otimes\mathbf{P} = \sum_{j_1,\dots,j_m=1}^{n} E^+[x_{j_1}^{l_1}b_{1,t_1}x_{j_2}^{l_2}b_{2,t_2}\cdots b_{m-1,t_{m-1}}x_{j_m}^{l_m}]\otimes\mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P}$$

Let t_1 go to $+\infty$, by normality of E^+, we have

$$E^+[x_{i_1}^{l_1} b_1 x_{i_2}^{l_2} b_{2,t_2} \cdots b_{m-1,t_m} x_{i_m}^{l_m}] \otimes \mathbf{P} = \sum_{j_1,\dots,j_m=1}^{n} E^+[x_{j_1}^{l_1} b_1 x_{j_2}^{l_2} b_{2,t_2} \cdots b_{m-1,t_{m-1}} x_{j_m}^{l_m}] \otimes \mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m} \mathbf{P}$$

Again, take $t_2, ..., t_{m-1}$ to $+\infty$, we have

$$E^+[x_{i_1}^{l_1} b_1 x_{i_2}^{l_2} b_2 \cdots b_{m-1} x_{i_m}^{l_m}] \otimes \mathbf{P} = \sum_{j_1,\dots,j_m=1}^{n} E^+[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{m-1} x_{j_m}] \otimes \mathbf{P} u_{j_1,i_1} \cdots u_{j_m,i_m} \mathbf{P} \tag{5.3}$$

□

If $i_s = i_{s+1}$ for some s, according to the universal conditions of $M_i(n,k)$, the terms on the right hand side are not vanished only if $j_s = j_{s+1}$. Therefore, we can shorten the product on the right hand side of (5.3) if $i_s = i_{s+1}$ for some s. We have

Proposition 5.3.4. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space, $(x_i)_{i\in\mathbb{Z}}$ a sequence of selfadjoint random variables in $\mathcal{A}$, E^+ be the conditional expectation onto the positive tail algebra $\mathcal{A}^+_{tail}$. Assume that the joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable, for fixed $n > k \in \mathbb{N}$ and $(u_{i,j})_{i=1,\dots,n;\, j=1,\dots,k}$ the standard generators of $M_i(n,k)$, we have that*

$$E^+[p_1(x_{i_1}) \cdots p_m(x_{i_m})] \otimes \boldsymbol{P} = \sum_{j_1,\dots,j_m=1}^{n} E^+[p_1(x_{j_1}) \cdots p_m(x_{j_m})] \otimes \boldsymbol{P} u_{j_1,i_1} \cdots u_{j_m,i_m} \boldsymbol{P},$$

whenever $1 \le i_1, ..., i_m \le k$, $i_1 \neq \cdots \neq i_m$ and $p_1, ..., p_m \in \mathcal{A}^+_{tail}\langle X\rangle_0$.

Lemma 5.3.5. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space, $(x_i)_{i\in\mathbb{Z}}$ a sequence of selfadjoint random variables in $\mathcal{A}$, E^+ be the conditional expectation onto the positive tail algebra $\mathcal{A}^+_{tail}$. Assume that the joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable, then*

$$E^+[p_1(x_{i_1}) \cdots p_s(x_{i_s}) \cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1}) \cdots E^+[p_s(x_{i_s})] \cdots p_m(x_{i_m})]$$

whenever $i_s > i_t$ for all $t \neq s$, $i_1 \neq \cdots \neq i_m$ and $p_1, ..., p_m \in \mathcal{A}^+_{tail}\langle X\rangle_0$.

Proof. Since $(x_i)_{i\in\mathbb{Z}}$ is spreadable, by Lemma 5.2.9, we have that

$$\alpha(p_t(x_{i_t})) = p_t(\alpha(x_{i_t}))$$

and

$$E^+[\alpha^{k'}(a)] = E^+[a]$$

for all $a \in \bigcup_{n'\in\mathbb{Z}} \mathcal{A}^+_{n'}$ and $k' \in \mathbb{Z}$.

Therefore, it is sufficient to prove the statement under the assumption that $i_1, ..., i_m > 0$.

Let $i_s = k$, $(u_{i,j})_{i=1,...,n+1;\, j=1,...,k}$ the standard generators of $M_i(n+k,k)$. By proposition 5.3.4, we have

$$E^+[p_1(x_{i_1})\cdots p_m(x_{i_m})]\otimes \mathbf{P} = \sum_{j_1,...,j_m=1}^{n+k} E^+[p_1(x_{j_1})\cdots p_m(x_{j_m})]\otimes \mathbf{P}u_{j_1,i_1}\cdots u_{j_m,i_m}\mathbf{P}.$$

Let $l_1 = \cdots = l_{k-1} = 1$ and $l_k = n+1$. By proposition 3.3.12, we have

$$E^+[p_1(x_{i_1})\cdots p_s(x_{i_s})\cdots p_m(x_{i_m})]\otimes \mathbf{P} = \frac{1}{n+1}\sum_{j_s=k}^{n+k} E^+[p_1(x_{i_1})\cdots p_s(x_{j_s})\cdots p_m(x_{i_m})]\otimes \mathbf{P}.$$

Since n is arbitrary, and E^+ is normal on $\mathcal{A}_0^+$, we have

$$\begin{aligned}
& E^+[p_1(x_{i_1})\cdots p_s(x_{i_s})\cdots p_m(x_{i_m})] \\
= \ & \tfrac{1}{n+1}\sum_{j_s=k}^{n+k} E^+[p_1(x_{i_1})\cdots p_s(x_{j_s})\cdots p_m(x_{i_m})] \\
= \ & \mathrm{WOT}-\lim_{n\to\infty} E^+[p_1(x_{i_1})\cdots(\tfrac{1}{n+1}\sum_{j_s=k}^{n+k} p_s(x_{j_s}))\cdots p_m(x_{i_m})] \\
= \ & \mathrm{WOT}-\lim_{n\to\infty} E^+[p_1(x_{i_1})\cdots(\tfrac{1}{n+1}\sum_{t=0}^{n} \alpha^t(p_s(x_{i_s}))\cdots p_m(x_{i_m})] \\
= \ & \mathrm{WOT}-\lim_{n\to\infty} E^+[p_1(x_{i_1})\cdots E^+[p_s(x_{i_s})]\cdots p_m(x_{i_m})].
\end{aligned}$$

The proof is complete. □

Now, we turn to consider the case that the maximal index is not unique.

Proposition 5.3.6. *Let $(\mathcal{A},\phi)$ be a W^*-probability space, $(x_i)_{i\in\mathbb{Z}}$ a sequence of selfadjoint random variables in $\mathcal{A}$, E^+ be the conditional expectation onto the positive tail algebra $\mathcal{A}^+_{tail}$. Assume that the joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable, then*

$$E^+[p_1(x_{i_1})\cdots p_s(x_{i_s})\cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1})\cdots E^+[p_s(x_{i_s})]\cdots p_m(x_{i_m})]$$

whenever $i_s = \max\{i_1,...,i_n\}$ for all $t\neq s$, $i_1\neq\cdots\neq i_m$ and $p_1,...,p_m\in\mathcal{A}^+_{tail}\langle X\rangle_0$.

Proof. Again, we can assume that $i_1,...,i_t > 0$ and $\max\{i_1,...,i_m\} = k$. Suppose the number k appears t times in the sequence, which are $\{i_{l_j}\}_j = 1,...,t$ such that $i_{l_j} = k$ and $l_1 < l_2 < \cdots < l_t$. Fix n,k and consider $M_i(n+k,k)$, by proposition 5.3.4 and proposition 3.3.12, we have

$$
\begin{aligned}
& E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{i_{l_1}})\cdots p_{l_2}(x_{i_{l_2}})\cdots p_m(x_{i_m})]\otimes P \\
= & \sum_{j_{l_1},j_{l_2},\ldots j_{l_t}=k}^{k+n} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\otimes PP_{j_{l_1},k}PP_{j_{l_2},k}P\cdots u_{j_{l_t},k}P \\
= & \tfrac{1}{(n+1)^t}\sum_{j_{l_1},j_{l_2},\ldots j_{l_t}=k}^{k+n} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]]\otimes P \\
= & \tfrac{1}{(n+1)^t}(\sum_{j_{l_s}\neq j_{l_r}\text{ if } s\neq r}^{N} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\otimes P \\
& + \sum_{j_{l_s}=j_{l_t}\text{ for some } s\neq t}^{N} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\otimes P).
\end{aligned}
$$

In the first part of the sum, apply proposition 5.3.5 on indices $j_{l_1},...j_{l_{l_t}}$ recursively, it follows that

$$E^+[p_1(x_{i_1})\cdots p_s(x_{j_{l_1}})\cdots p_s(x_{j_{l_2}})\cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1})\cdots E[p_{l_1}(x_{j_{l_1}})]\cdots E[p_{l_2}(x_{j_{l_2}})]\cdots p_m(x_{i_m})].$$

Since $E[p_s(x_{j_{l_1}})] = E[p_s(x_k)]$ for all $j_{l_1},...,j_{l_t}$,

$$E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1})\cdots E[p_{l_1}(x_k)]\cdots E[p_{l_2}(x_k)]\cdots p_m(x_{i_m})].$$

Then, we have

$$
\begin{aligned}
& \tfrac{1}{(n+1)^t}(\sum_{j_{l_s}\neq j_{l_r}\text{ if } s\neq r}^{N} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\otimes P \\
= & \frac{\prod_{s=0}^{t-1}(n+1-s)}{n+1^t} E^+[p_1(x_{i_1})\cdots E[p_{l_1}(x_k)]\cdots E[p_{l_2}(x_k)]\cdots p_m(x_{i_m})]\otimes P,
\end{aligned}
$$

which converges to $E[p_s(x_k)]\cdots E[p_s(x_k)]\cdots p_m(x_{i_m})]\otimes P$ in norm as n goes to $+\infty$.

To the second part of the sum, we have

$$
\begin{aligned}
& \|E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\| \\
\leq & \|p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})\| \\
\leq & \|p_1(x_{i_1})\|\cdots\|p_{l_1}(x_{j_{l_1}})\|\cdots\|p_{l_2}(x_{j_{l_2}})\|\cdots\|p_m(x_{i_m})\| \\
\leq & \|p_1(x_1)\|\cdots\|p_{l_1}(x_1)\|\cdots\|p_{l_2}(x_1)\|\cdots\|p_m(x_1)\|,
\end{aligned}
$$

which is finite. Therefore,

$$
\begin{aligned}
& \|\sum_{j_{l_s}=j_{l_t}\text{ for some } s\neq t}^{N} E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{j_{l_1}})\cdots p_{l_2}(x_{j_{l_2}})\cdots p_m(x_{i_m})]\| \\
\leq & (1-\frac{\prod_{s=0}^{t-1}(n+1-s)}{(n+1)^t})\|p_1(x_1)\|\cdots\|p_{l_1}(x_1)\|\cdots\|p_{l_2}(x_1)\|\cdots\|p_m(x_1)\|
\end{aligned}
$$

goes to 0 as n goes to $+\infty$.

Therefore, we have

$$E^+[p_1(x_{i_1})\cdots p_{l_1}(x_{i_{l_1}})\cdots p_{l_2}(x_{i_{l_2}})\cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1})\cdots E[p_{l_1}(x_k)]\cdots E[p_{l_2}(x_k)]\cdots p_m(x_{i_m})].$$

The same we can show that

$$\begin{aligned} & E^+[p_1(x_{i_1})\cdots p_{l_1}(x_k)\cdots E^+[p_s(x_{i_s})]\cdots p_{l_2}(x_k)\cdots p_m(x_{i_m})] \\ = \quad & E^+[p_1(x_{i_1})\cdots E[p_{l_1}(x_k)]\cdots E[p_{l_2}(x_k)]\cdots p_m(x_{i_m})], \end{aligned}$$

which implies

$$E^+[p_1(x_{i_1})\cdots p_s(x_{i_s})\cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1})\cdots E^+[p_s(x_{i_s})]\cdots p_m(x_{i_m})].$$

□

5.4 de Finetti type theorem for monotone spreadability

5.4.1 Proof of main theorem 1

Now, we turn to prove our main theorem for monotone independence:

Theorem 5.4.1. *Let $(\mathcal{A},\phi)$ be a non-degenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ be a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$. Let $\mathcal{A}_k^+$ be the WOT closure of the non-unital algebra generated by $\{x_i | i \geq k\}$. Then the following are equivalent:*

- *a) The joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable.*
- *b) For all $k \in \mathbb{Z}$, there exits a ϕ-preserving conditional expectation $E_k : \mathcal{A}_k^+ \to \mathcal{A}_{tail}^+$ such that the sequence $(x_i)_{i\geq k}$ is identically distributed and monotone with respect E_k. Moreover, $E_k|_{\mathcal{A}_{k'}} = E_{k'}$ when $k \geq k'$.*

Proof. "*b) ⇒ a)* "follows corollary 5.1.17

We will prove "*a) ⇒ b)* "by induction. Since the sequence is spreadable, it is suffices to prove $a) \Rightarrow b)$ for $k = 1$:
By the results in the previous two sections, there exists a conditional expectation E_k : $\mathcal{A}_k^+ \to \mathcal{A}_{tail}^+$ such that the sequence $(x_i)_{i\geq k}$ is identically distributed with respect to E_k and $E_k|_{\mathcal{A}_{k'}} = E_{k'}$ when $k \geq k'$. Actually, E_k is the restriction of E^+ on $\mathcal{A}_k^+$. Since the sequence is spreadable, we just need to show that the sequence $(x_i)_{i\in\mathbb{N}}$ is monotonically independent with respect to E_1, i.e.

$$E^+[p_1(x_{i_1})\cdots p_s(x_{i_s})\cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1})\cdots E^+[p_s(x_{i_s})]\cdots p_m(x_{i_m})] \qquad (5.4)$$

$i_{s-1} < i_s > i_{s+1}$, $i_1 \neq \cdots \neq i_m$, $i_1, ..., i_m \in \mathbb{N}$ and $p_1, ..., p_m \in \mathcal{A}^+_{tail}\langle X\rangle$.
Now, we prove this equality by induction on the maximal index of $\{i_1, ..., i_m\}$:
When $\max\{i_1, ..., i_m\} = 1$, then equality is true because $i_s = 1$ and the length of the sequence $(i_1, ..., i_m)$ can only be 1.
Suppose the equality holds for $\max\{i_1, ..., i_m\} = n$. When $\max\{i_1, ..., i_m\} = n + 1$, we have two cases:
Case 1: $i_s = n + 1$**.** In this case the equality follows proposition 5.3.6.

Case 2: $i_s \leq n$**.** Suppose the number $n + 1$ appears t times in the sequence, which are $\{i_{l_j}\}_j = 1, ..., t$ such that $i_{l_j} = k$ and $l_1 < l_2 < \cdots < l_t$. Since $i_{s-1} < i_s > i_{s+1}$, $i_{s-1}, i_s, i_{s+1} \neq n + 1$. By proposition 5.3.6, we have:

$$\begin{aligned} & E^+[p_1(x_{i_1}) \cdots p_{l_1}(x_{i_{l_1}}) \cdots p_{s-1}(x_{i_{s-1}}) p_s(x_{i_s}) p_{s+1}(x_{i_{s+1}}) \cdots p_{l_t}(x_{i_{l_t}}) \cdots p_m(x_{i_m})] \\ = \; & E^+[p_1(x_{i_1}) \cdots E^+[p_{l_1}(x_{i_{l_1}})] \cdots p_{s-1}(x_{i_{s-1}}) p_s(x_{i_s}) p_{s+1}(x_{i_{s+1}}) \cdots E^+[p_{l_t}(x_{i_{l_t}})] \cdots p_m(x_{i_m}))]. \end{aligned}$$

Notice that

$$p_1(x_{i_1}) \cdots E^+[p_{l_1}(x_{i_{l_1}})] \cdots p_{s-1}(x_{i_{s-1}}) p_s(x_{i_s}) p_{s+1}(x_{i_{s+1}}) \cdots E^+[p_{l_t}(x_{i_{l_t}})] \cdots p_m(x_{i_m}) \in \mathcal{A}^+_{tail}\langle X_1, ..., X_n\rangle,$$

by induction, we have

$$\begin{aligned} & E^+[p_1(x_{i_1}) \cdots E^+[p_{l_1}(x_{i_{l_1}})] \cdots p_{s-1}(x_{i_{s-1}}) p_s(x_{i_s}) p_{s+1}(x_{i_{s+1}}) \cdots E^+[p_{l_t}(x_{i_{l_t}}) \cdots p_m(x_{i_m})] \\ = \; & E^+[p_1(x_{i_1}) \cdots E^+[p_{l_1}(x_{i_{l_1}})] \cdots p_{s-1}(x_{i_{s-1}}) E^+[p_s(x_{i_s})] p_{s+1}(x_{i_{s+1}}) \cdots E^+[p_{l_t}(x_{i_{l_t}})] \cdots p_m(x_{i_m})] \\ = \; & E^+[p_1(x_{i_1}) \cdots p_{l_1}(x_{i_{l_1}}) \cdots p_{s-1}(x_{i_{s-1}}) E^+[p_s(x_{i_s})] p_{s+1}(x_{i_{s+1}}) \cdots p_{l_t}(x_{i_{l_t}}) \cdots p_m(x_{i_m})]. \end{aligned}$$

The last equality follows proposition 5.3.6. This our desired conclusion.

□

5.4.2 Conditional expectation E^-

We do not know whether we can extend E^+ to the whole space $\mathcal{A}$. But, the conditional expectation E^- can be extended to the whole algebra $\mathcal{A}$ if the bilateral sequence $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable. Given $a, b, c \in A_{[-\infty,\infty]}$, then there exists $L \in \mathbb{N}$ such that $a, b, c \in A_{[-L,L]}$. Therefore, $\alpha^{-3L}(c) \in \mathcal{A}_{[-4L,-3L]}$. Since $(x_{-4L}, x_{-4L+1}, ...)$ is monotonically with respect to E^+, we have

$$\begin{aligned} & \phi(aE^-[b]c) \\ = \; & \lim_{n\to\infty} \phi(a\alpha^{-n}(b)c) \\ = \; & \lim_{n\to\infty, n>4L} \phi(a\alpha^{-n}(b)c) \\ = \; & \lim_{n\to\infty, n>4L} \phi(E^+[a\alpha^{-n}(b)c]) \\ = \; & \lim_{n\to\infty, n>4L} \phi(E^+[E^+[a]\alpha^{-n}(b)E^+[c]]) \\ = \; & \lim_{n\to\infty} \phi(E^+[a]\alpha^{-n}(b)E^+[c]) \\ = \; & \lim_{n\to\infty} \phi(E^+[a]E^-[b]E^+[c]). \end{aligned}$$

Since $\mathcal{A}$ is generated by countablely many operators, by Kaplansky's density theorem, for all $y \in \mathcal{A}$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset A_{[-\infty,\infty]}$ such that $\|y_n\| \le \|y\|$ for all n and y_n converges to y in WOT. Then, for all $a, c \in A_{[-\infty,\infty]}$ we have

$$\lim_{n\to\infty} \phi(aE^-[y_n]c) = \lim_{n\to\infty} \phi(E^+[a]y_nE^+[c]) = \phi(E^+[a]yE^+[c])$$

Therefore, $E^-[y_n]$ converges to an element y' pointwisely. Moreover, y' depends only on y. If we define $E^-[y] = y'$, then we have

Proposition 5.4.2. *Let $(\mathcal{A}, \phi)$ be a non-degenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ be a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$. If $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable, then the negative conditional expectation E^- can be extend to the whole algebra $\mathcal{A}$ such that*

$$\phi(aE^-[y]b) = \phi(E^+[a]yE^+[c])$$

for all $y \in \mathcal{A}$ and $a, c \in A_{[-\infty,\infty]}$. Moreover, the extension is normal.

5.5 de Finetti type theorem for boolean spreadability

In this section, we assume that $(\mathcal{A}, \phi)$ is a W^*-probability space with a non-degenerated normal state and $\mathcal{A}$ is generated by a bilateral sequence of random variables $(x_i)_{i\in\mathbb{Z}}$ and $(x_i)_{i\in\mathbb{Z}}$ are boolean spreadable.

Lemma 5.5.1. *Let $y_i = x_{-i}$ for all $i \in \mathbb{Z}$, then $(y_i)_{i\in\mathbb{Z}}$ is also boolean spreadable.*

Proof. By proposition 3.3.22, it suffices to show that $(y_i)_{i=1,\dots,n}$ is boolean spreadable for all $n \in \mathbb{N}$. Given a natural number $k < n$, assume the standard generators of $B_i(n,k)$ are $\{u_{i,j}\}_{i=1,\dots,n;j=1,\dots,k}$ and invariant projection $\mathbf{P}$.
Consider the matrix $\{u'_{i,j}\}_{i=1,\dots,n;j=1,\dots,k}$ such that $u'_{i,j} = u_{n+1-i,k+1-j}$.. it is obvious that the entries of the matrix are are orthogonal projections and

$$\sum_{i=1}^{n} u'_{i,j}\mathbf{P} = \sum_{i=1}^{n} u_{i,k+1-j}\mathbf{P} = \mathbf{P}.$$

Given $j, j', i, i' \in \mathbb{N}$ such that $1 \le j < j' \le k$ and $1 \le i \le i' \le n$. Then, we have $n+1-i \le n+1-i'$ and $k+1-j < k+1-j'$. Therefore,

$$u'_{i,j}u'_{i',j'} = u_{n+1-i,k+1-j}u_{n+1-i',k+1-j'} = 0.$$

It implies that $\{u'_{i,j}\}_{i=1,\dots,n;j=1,\dots,k}$ and $\mathbf{P}$ satisfy the universal conditions of $B_i(n,k)$. It follows that there exists a unital C^*-homomorphism $\Phi : B_i(n,k) \to B_i(n,k)$ such that:

$$\Phi(u_{i,i}) = u'_{i,j}, \text{ and } \Phi(\mathbf{P}) = \mathbf{P}.$$

Let $z_i = x_{i-n-1}$ for $i = 1, ..., n$. Since $(x_i)_{i\in\mathbb{Z}}$ are boolean spreadable, $(z_i)_{i=1,...,n}$ is boolean spreadable. Therefore, for $i_1, ..., i_L \in [k]$, we have

$$\begin{aligned}
& \phi(y_{i_1} \cdots y_{i_L})\mathbf{P} \\
= \ & \phi(y_{n-k+i_1} \cdots y_{n-k+i_L})\mathbf{P} \\
= \ & \phi(x_{-n+k-i_1} \cdots x_{n-k-i_L})\mathbf{P} \\
= \ & \Phi(\phi(z_{k+1-i_1} \cdots z_{k+1-i_L})\mathbf{P}) \\
= \ & \Phi(\sum_{j_1,...,j_L=1}^{n} \phi(z_{j_1} \cdots z_{j_L})\mathbf{P}u_{j_1,k+1-i_1} \cdots u_{j_L,k+1-i_L}\mathbf{P}) \\
= \ & \sum_{j_1,...,j_L=1}^{n} \phi(z_{j_1} \cdots z_{j_L})\mathbf{P}u_{n+1-j_1,i_1} \cdots u_{n+1-j_L,i_L}\mathbf{P} \\
= \ & \sum_{j_1,...,j_L=1}^{n} \phi(x_{j_1-n-1} \cdots x_{j_L-n-1})\mathbf{P}u_{n+1-j_1,i_1} \cdots u_{n+1-j_L,i_L}\mathbf{P} \\
= \ & \sum_{j_1,...,j_L=1}^{n} \phi(y_{n+1-j_1} \cdots y_{n+1-j_L})\mathbf{P}u_{n+1-j_1,i_1} \cdots u_{n+1-j_L,i_L}\mathbf{P} \\
= \ & \sum_{j_1,...,j_L=1}^{n} \phi(y_{j_1} \cdots y_{j_L})\mathbf{P}u_{j_1,i_1} \cdots u_{j_L,i_L}\mathbf{P}
\end{aligned}$$

which completes the proof. □

Proposition 5.5.2. *$(\mathcal{A}, \phi)$ is a W^*-probability space with a non-degenerated normal state and $\mathcal{A}$ is generated by a bilateral sequence of random variables $(x_i)_{i\in\mathbb{Z}}$ and $(x_i)_{i\in\mathbb{Z}}$ are boolean spreadable. Then, E^- and E^+ can be extend to the whole algebra $\mathcal{A}$. Moreover, $E^- = E^+$*

Proof. Since $(x_i)_{i\in\mathbb{Z}}$ is boolean spreadable, $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable. By proposition 5.4.2, E^- can be extended to the whole algebra. By Lemma 5.5.1, $(x_{-i})_{i\in\mathbb{Z}}$ is also boolean spreadable and its negative-conditional expectation is exactly the positive conditional expectation of $(x_i)_{i\in\mathbb{Z}}$. Therefore, E^+ can also be extended the whole algebra $\mathcal{A}$ normally. Give $a, b, c \in A_{[-\infty,\infty]}$, by Lemma 5.4.2, we have

$$\begin{aligned}
\phi(aE^-[b]c) &= \phi(E^+[a]bE^+[c]) \\
&= \phi(E^+[E^+[a]bE^+[c]]) \\
&= \phi(E^+[a]E^+[b]E^+[c]) \\
&= \lim_{n\to\infty} \phi(\alpha^n(a)E^+[b]E^+[c]) \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \phi(\alpha^n(a)E^+[b]\alpha^m(c)).
\end{aligned}$$

Notice that, for fixed n, m,

$$\phi(\alpha^n(a)E^+[b]\alpha^m(c)) = \phi(\alpha^n(a)\alpha^L(b)\alpha^m(c))$$

for $L \in \mathbb{N}$ which is large enough. Since $(x_{-i})_{i\in\mathbb{Z}}$ is monotonically spreadable, by theorem **??**,

$(x_{-i})_{i\in\mathbb{Z}}$ is monotonically independent with respect to E^-. Therefore, we have

$$\begin{aligned}
& \phi(\alpha^n(a)E^+[b]\alpha^m(c)) \\
= & \phi(\alpha^n(a)\alpha^L(b)\alpha^m(c)) \\
= & \phi(E^-[\alpha^n(a)\alpha^L(b)\alpha^m(c)]) \\
= & \phi(E^-[\alpha^n(a)]E^-[\alpha^L(b)]E^-[\alpha^m(c)]) \\
= & \phi(E^-[a]E^-[b]E^-[c]) \\
= & \phi(E^-[E^-[a]bE^-[c]]) \\
= & \phi(E^-[a]bE^-[c]) \\
= & \phi(aE^+[b]c)
\end{aligned}$$

and

$$\begin{aligned}
\phi(aE^-[b]c) &= \phi(E^+[a]bE^+[c]) \\
&= \lim_{n\to\infty}\lim_{m\to\infty}\phi(\alpha^n(a)E^+[b]\alpha^m(c)) \\
&= \lim_{n\to\infty}\lim_{m\to\infty}\phi(aE^+[b]c) \\
&= \phi(aE^+[b]c).
\end{aligned}$$

It implies that $E^+[b] = E^-[b]$ for all $b \in A_{[-\infty,\infty]}$. Since $\mathcal{A}$ is the WOT closure of $A_{[-\infty,\infty]}$, the proof is complete. □

Corollary 5.5.3. *$(\mathcal{A}, \phi)$ is a W^*-probability space with a non-degenerated normal state and $\mathcal{A}$ is generated by a bilateral sequence of random variables $(x_i)_{i\in\mathbb{Z}}$ and $(x_i)_{i\in\mathbb{Z}}$ are boolean spreadable. Then, the positive tail algebra and the negative tail algebra of $(x_i)_{i\in\mathbb{Z}}$ are the same.*

Now, we are ready to prove theorem **??**

Theorem 5.5.4. *Let $(\mathcal{A}, \phi)$ be a non degenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ be a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$ as a von Neumann algebra. Then the following are equivalent:*

a) The joint distribution of $(x_i)_{i\in\mathbb{N}}$ is boolean spreadable.

b) The sequence $(x_i)_{i\in\mathbb{Z}}$ is identically distributed and boolean independent with respect to the $\phi-$preserving conditional expectation E^+ onto the non unital positive tail algebra of the $(x_i)_{i\in\mathbb{Z}}$

Proof. "$b) \Rightarrow a)$". If the sequence $(x_i)_{i\in\mathbb{Z}}$ is identically distributed and boolean independent with respect to a ϕ-preserving conditional expectation E , then sequence $(x_i)_{i\in\mathbb{Z}}$ is boolean exchangeable by theorem 7.1 in [25]. According the diagram in section 3.4, $(x_i)_{i\in\mathbb{Z}}$ is boolean spreadable.

"$a) \Rightarrow b)$". By Lemma5.5.2, $(x_i)_{i\in\mathbb{Z}}$ is monotone with respect to E^+, $(x_{-i})_{i\in\mathbb{Z}}$ is monotone with respect to E^- and $E^+ = E^-$. Therefore,

$$\begin{aligned}
& E^+[p_1(x_{i_1})\cdots p_m(x_{i_m})] = E^+[p_1(x_{i_1})]E^+[p_2(x_{i_2})\cdots p_m(x_{i_m})] = \cdots \\
= & E^+[p_1(x_{i_1})]E^+[p_2(x_{i_2})]\cdot\cdots\cdot E^+[p_m(x_{i_m})]
\end{aligned}$$

whenever $i_1 \neq \cdots \neq i_m$ and $p_1, ..., p_m \in \mathcal{A}^+_{tail}\langle X \rangle$. The proof is complete.

$\square$

Chapter 6

General De Finetti type theorems in noncommutative probability

6.1 Noncommutative symmetries

In this section, we will recall distributional symmetries for classic independence are free independence from [5].

Definition 6.1.1. An orthogonal Hopf algebra is a unital C^*-algebra A generated by n^2 selfadjoint elements $\{u_{i,j}|i,j=1,...,n\}$, such that the following hold:

1. The inverse of $u=(u_{i,j})_{i,j=1,....n}\in M_n(A)$ is the transpose $u^t=(u_{j,i})_{i,j=1,...n}$, i.e. $\sum\limits_{k=1}^{n}u_{i,k}u_{j,k}=\sum\limits_{k=1}^{n}u_{k,i}u_{k,j}=\delta_{i,j}1_A$.
2. $\Delta(u_{i,j})=\sum\limits_{k=1}^{n}u_{i,k}\otimes u_{k,j}$ determines a C^*-unital homomorphism $\Delta: A\to A\otimes_{min}A$.
3. $\epsilon(u_{i,j})=\delta_{i,j}$ defines a homomorphism $\epsilon: A\to\mathbb{C}$.
4. $S(u_{i,j})=u_{j,i}$ defines a homomorphism $S: A\to A^{op}$.

This definition adapted from the fundamental work of Woronowicz[43]. Following the notion of Wang's free quantum groups in [40, 41], one can define universal algebras A generated by n^2 noncommutative variables $\{u_{i,j}\}_{i,j=1,...,n}$ which satisfy some relations R. Moreover, for suitable choices of R, we will get Hopf algebras in the sense of Woronowicz[43].

In [5], Banica and Speicher found the following conditions which can be used to construct Hopf orthogonal algebras:

Definition 6.1.2. *A matrix $u=(u_{i,j})_{i,j=1,...,n}\in M_n(A)$ over a C^*-algebra A is called:*

- *Orthogonal, if all entries of u are selfadjoint, and $uu^t=u^tu=1_n$,*

- *Magic, if it is orthogonal, and its entries are projections.*
- *Cubic, if it is orthogonal, and* $u_{i,j}u_{i,k} = u_{j,i}u_{k,i} = 0$*, for* $j \neq k$*.*
- *Bistochastic, if it is orthogonal, and* $\sum_{i=1}^{n} u_{i,j} = \sum_{j=1}^{n} u_{k,i} = 1_A$*, for all* j, k*.*
- *Magic',if it is cubic, with the same sum on rows and columns.*
- *Bistochastic',if it is orthogonal, with the same sum on rows and columns*

The universal algebras associated with the above conditions are defined as follows:

Definition 6.1.3. *$A_g(n)$ with $g = o, s, h, b, s, s', b'$ is the universal C^*-algebra generated by the entries of an $n \times n$ matrix which is respectively orthogonal, magic, cubic, bistochastic, magic' and bistochastic'. $C_g(n)$ with $g = o, s, h, b, s, s', b'$ is the universal commutative C^*-algebra generated by the entries of an $n \times n$ matrix which is respectively orthogonal, magic, cubic, bistochastic, magic' and bistochastic'.*

Especially, for each n, $A_s(n)$ and $A_o(n)$ are Wang's quantum permutation group and quantum orthogonal group introduced in [41, 40]. $C_g(n)$ can be considered as the abelianization of $A_g(n)$ for all $g = o, s, h, b, s, s', b'$. It should be mentioned here that there are 7 easy quantum groups in total, see [42].

According to the definitions, we have the following diagram:

$$\begin{array}{ccccc} A_o(n) & \longrightarrow & A_{b'}(n) & \longrightarrow & A_b(n) \\ \downarrow & & \downarrow & & \downarrow \\ A_h(n) & \longrightarrow & A_{s'}(n) & \longrightarrow & A_s(n) \end{array}$$

and

$$\begin{array}{ccccc} C_o(n) & \longrightarrow & C_{b'}(n) & \longrightarrow & C_b(n) \\ \downarrow & & \downarrow & & \downarrow \\ C_h(n) & \longrightarrow & C_{s'}(n) & \longrightarrow & C_s(n) \end{array}$$

and

$$A_g(n) \to C_g(n),$$

for $g = o, s, h, b, s, s', b'$. Here, the arrow means that there exists a morphism of orthogonal Hopf algebras $(A, u) \to (B, v)$ which is a C^*-homomorphism from$A$ to B such that $u_{i,j} \to v_{i,j}$. In other words, $(A, u) \to (B, v)$ implies that B is a quotient C^*-algebra of A. We will use $B \subseteq A$ for $(A, u) \to (B, v)$.

Proposition 6.1.4. *Let $E(n)$ be an orthogonal Hopf algebra generated by n^2 selfadjoint elements $\{u_{i,j}\}_{i,j=1,\ldots,n}$, then*

1. *If $E(n) \nsubseteq A_h(n)$, then there exists a j such that* $\sum_{k=1}^{n} u_{k,j}^4 \neq 1_{E(n)}$.

2. *If $E(n) \nsubseteq A_b(n)$, then there exists a j such that* $\sum_{k=1}^{n} u_{k,j} \neq 1_{E(n)}$.

Proof. 1. Suppose $\sum_{k=1}^{n} u_{k,i}^4 = 1_{E(n)}$, for all i. Since $\sum_{k=1}^{n} u_{k,i}^2 = 1_{E(n)}$ and $u_{k,i}^4 \leq u_{k,i}^2$, we have

$$u_{k,i}^4 = u_{k,i}^2.$$

$(u_{i,j}^2)_{i,j=1,\ldots,n}$ is a matrix of orthogonal projections with sum 1 on rows and columns. Therefore,

$$u_{i,j}^2 u_{i,k}^2 = u_{j,i}^2 u_{k,i}^2 = 0$$

for $j \neq k$. Since $u_{i,j}$ and $u_{i,k}$ are selfadjoint, we have

$$u_{i,j} u_{i,k} = u_{j,i} u_{k,i} = 0$$

which implies that $E(n)$ is a quotient algebra of $A_h(n)$. It is a contradiction.

2. Suppose $\sum_{k=1}^{n} u_{k,i} = 1_{E(n)}$, for all i. Then, for each i, we have

$$\sum_{l=1}^{n} u_{i,l} = \sum_{l=1}^{n} \sum_{k=1}^{n} u_{i,l} u_{k,l} = \sum_{k=1}^{n} \sum_{l=1}^{n} u_{i,l} u_{k,l} = \sum_{k=1}^{n} \delta_{i,k} 1_{E(n)} = 1_{E(n).}$$

Therefore, $E(n)$ is a quotient algebra of $A_b(n)$ which leads to a contradiction. □

Proposition 6.1.5. *Let $E(n)$ be an orthogonal Hopf algebra generated by n^2 selfadjoint elements $\{u_{i,j}\}_{i,j=1,\ldots,n}$ such that $A_s(n) \subseteq E(n) \subseteq A_o(n)$. Then, the following hold:*

1. *If $E(n) \subseteq A_h(n)$ and $E(n) \subseteq A_b(n)$, then $E(n) = A_s(n)$.*

2. *If $E(n) \nsubseteq A_h(n)$ and $E(n) \subseteq A_b(n)$, then $\exists\, i'$ such that*

$$\sum_{k=1}^{n} u_{k,i'}^m \neq 1,$$

for all $m > 2$.

3. *If $E(n) \nsubseteq A_b(n)$ and $E(n) \subseteq A_h(n)$, then $\exists\, i'$ such that*

$$\sum_{k=1}^{n} u_{k,i'}^m \neq 1,$$

for all odd numbers m.

4. If $E(n) \nsubseteq A_h(n)$ and $E(n) \nsubseteq A_h(n)$, then $\exists\, i_1', i_2$ such that

$$\sum_{k=1}^{n} u_{k,i_1'}^m \neq 1,$$

for all $m > 2$, and

$$\sum_{k=1}^{n} u_{k,i_2'} \neq 1.$$

Proof. It is obvious that $\|u_{i,j}\| \leq 1$ for all $i, j = 1, ...n$.

1. By assumption, we have

$$\sum_{k=1}^{n} u_{i,k} = 1_{E(n)}$$

and

$$u_{i,j}u_{i,k} = 0$$

for $j \neq k$. Therefore,

$$u_{i,j} = u_{i,j}\sum_{k=1}^{n} u_{i,k} = u_{i,j}^2$$

for all i, j. It implies that $E(n)$ is a quotient algebra of $A_s(n)$, so $E(n) = A_s(n)$.

2. By Proposition 6.1.5, there exists i' such that

$$\sum_{k=1}^{n} u_{k,i'}^4 \neq 1.$$

Therefore, there exists k' such that

$$u_{k',i'}^4 < u_{k',i'}^2$$

which implies that the spectrum of $u_{k',i'}$ contains a number a such that $-1 < a < 1$. Therefore,

$$u_{k',i'}^m < u_{k',i'}^2$$

for all natural number $m > 2$. Hence, we have

$$\sum_{k=1}^{n} u_{k,i'}^m < 1_{E(n)},$$

for $m > 2$.

3. According to Proposition 6.1.5, there exists i' such that

$$\sum_{k=1}^{n} u_{k,i'} \neq 1.$$

Therefore, there exists k' such that $u_{k',i'}$ is not an orthogonal projection which implies that

$$u_{k',i'}^{2m+1} < u_{k',i'}^{2m}.$$

Thus, we have

$$\sum_{k=1}^{n} u_{k,i'}^{2m+1} < \sum_{k=1}^{n} u_{k,i'}^{2m} = \sum_{k=1}^{n} u_{k,i'}^{m} = 1_{E(n)},$$

4. Combine Case 2 and 3, the proof is complete. □

Following the proof above, we have

Corollary 6.1.6. *Let $E(n)$ be an orthogonal Hopf algebra generated by n^2 selfadjoint elements $\{u_{i,j}\}_{i,j=1,\ldots,n}$ such that $C_s(n) \subseteq E(n) \subseteq C_o(n)$. Then, the following hold:*

1. *If $E(n) \subseteq A_h(n)$ and $E(n) \subseteq A_b(n)$, then $E(n) = A_s(n)$.*

2. *If $E(n) \nsubseteq A_h(n)$ and $E(n) \subseteq A_b(n)$, then $\exists\, i'$ such that $\sum_{k=1}^{n} u_{k,i'}^{m} \neq 1$, for all $m > 2$.*

3. *If $E(n) \nsubseteq A_b(n)$ and $E(n) \subseteq A_h(n)$, then $\exists\, i'$ such that $\sum_{k=1}^{n} u_{k,i'}^{m} \neq 1$, for all odd numbers m.*

4. *If $E(n) \nsubseteq A_h(n)$ and $E(n) \nsubseteq A_h(n)$, then $\exists\, i_1', i_2$ such that $\sum_{k=1}^{n} u_{k,i_1'}^{m} \neq 1$, for all $m > 2$, and $\sum_{k=1}^{n} u_{k,i_2'} \neq 1$.*

Now, we turn to define noncommutative distributional symmetries by maps of quantum family of Sołtan[32]:

Definition 6.1.7. Let (A, Δ) be a quantum group and $\mathcal{V}$ be a unital algebra. By a (right) coaction of the quantum group A on $\mathcal{V}$, we mean a unital homomorphism $\alpha : \mathcal{V} \to \mathcal{V} \otimes \mathcal{A}$ such that

$$(\alpha \otimes id_A)\alpha = (id \otimes \Delta)\alpha.$$

Definition 6.1.8. Given an orthogonal Hopf algebra $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,\ldots n}$, we have a natural coaction α_n of $E(n)$ on $\mathbb{C}\langle X_1, ..., X_n\rangle$ such that

$$\alpha_n : \mathbb{C}\langle X_1, ..., X_n\rangle \to \mathbb{C}\langle X_1, ..., X_n\rangle \otimes E(n)$$

is an algebraic homomorphism defined via $\alpha_n(X_i) = \sum_{k=1}^{n} X_k \otimes u_{k,i}$ for all $i = 1, ..., n$.

Definition 6.1.9. Given a probability space $(\mathcal{A}, \phi)$, a sequence of random variables $(x_1, ..., x_n)$ of $\mathcal{A}$ and an orthogonal Hopf algebra $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...n}$. We say that the joint distribution $\mu_{x_1,...,x_n}$ of $x_1, ..., x_n$ is $E(n)$-invariant if

$$\mu_{x_1,...,x_n}(p)1_{E(n)} = \mu_{x_1,...,x_n} \otimes id_{E(n)}(\alpha_n(p)),$$

for all $p \in \mathbb{C}\langle X_1, ..., X_n\rangle$.

Remark 6.1.10. Noncommutative distributional symmetries, which are associated with $E(n)$ such that $A_s \subseteq E(n) \subseteq A_o(n)(C_s \subseteq E(n) \subseteq C_o(n))$, will be used to characterize free(classical) type de Finetti theorems.

Proposition 6.1.11. Given a probability space $(\mathcal{A}, \phi)$ and a sequence of random variables $(x_1, ..., x_n)$ of $\mathcal{A}$. $E(n)$ and $F(n)$ are two orthogonal Hopf algebras such that $E_1(n) \subseteq E_2(n)$. Then, $(x_1, ..., x_n)$ is $E_1(n)$-invariant if $E_2(n)$-invariant.

Proof. Let $\{u^{(l)}_{i,j}\}_{i,j=1,...,n}$ be generators of $E_l(n)$ for $l = 1, 2$. Since $E_1(n) \subseteq E_2(n)$, there exists a C^*-homomorphism $\Phi : E_2(n) \to E_1(n)$ such that

$$\Phi(u^{(2)}_{i,j}) = u^{(1)}_{i,j}$$

for all i, j. $(x_1, ..., x_n)$ is $E_2(n)$-invariant is equivalent to that

$$\mu_{x_1,...,x_n}(X_{\mathbf{i}})1_{E_2(n)} = \sum_{\mathbf{j}\in[n]^k} \mu_{x_1,...,x_n}(X_{\mathbf{j}}) \otimes u^{(2)}_{\mathbf{i},\mathbf{j}},$$

for all monomials $X_{i_1} \cdots X_{i_k} \in \mathbb{C}\langle X_1, ..., X_n\rangle$. Apply Φ on both sides of the above equation, we get

$$\mu_{x_1,...,x_n}(X_{\mathbf{i}})1_{E_1(n)} = \sum_{\mathbf{j}\in[n]^k} \mu_{x_1,...,x_n}(X_{\mathbf{j}}) \otimes u^{(1)}_{\mathbf{i},\mathbf{j}},$$

which implies that $(x_1, ..., x_n)$ is $E_1(n)$-invariant. □

Given an orthogonal Hopf algebra $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...,n}$. Then , for $k \in \mathbb{N}$, $E(n)$ can be considered as an orthogonal Hopf algebra $E(n, k)$ generated by $\{v_{i,j}\}_{i,j=1,...,n+k}$ such that

$$v_{i,j} = \begin{cases} u_{i,j} & \text{if } i, j \leq n \\ \delta_{i,j}1_{E(n)} & \text{otherwise} \end{cases}$$

We will call $E(n, k)$ the k-th extension of $E(n)$. To study de Finettil theorems for all orthogonal Hopf algebras $E(n)$, we need to extend $E(n)$-invariance condition on n random variables to infinitely many random variables.

Definition 6.1.12. Given a probability space $(\mathcal{A}, \phi)$, a sequence of random variables $(x_i)_{i\in\mathbb{N}}$ of $\mathcal{A}$ and an orthogonal Hopf algebra $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...n}$. We say that the joint distribution μ of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$-invariant if the joint distribution of $(x_1, ..., x_{n+k})$ is $E(n, k)$-invariant for all $k \in \mathbb{N}$.

6.2 Quantum semigroups in analogue of easy quantum groups

Inspired by the previous work in [25], we will define distributional symmetries for boolean independent random variables via quantum semigroups. We briefly recall quantum semigroups' definition here: For any C^*-algebras A and B, the set of morphisms $\mathrm{Mor}(A, B)$ consists of all C^*-algebra homomorphisms acting from A to $M(B)$, where $M(B)$ is the multiplier algebra of B, such that $\phi(A)B$ is dense in B. If A and B are unital C^*-algebras, then all unital C^*-homomorphisms from A to B are in Mor(A,B). In [32],

Definition 6.2.1. By a quantum semigroup we mean a C^*-algebra $\mathcal{A}$ endowed with an additional structure described by a morphism $\Delta \in Mor(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ such that

$$(\Delta \otimes id_{\mathcal{A}})\Delta = (id_{\mathcal{A}} \otimes \Delta)\Delta.$$

The quantum semigroups for boolean independence are unital universal C^*-algebras generated by an orthogonal projection $\mathbf{P}$ and entries of $n \times n$ matrices which satisfying certain relation R related to $\mathbf{P}$:

Definition 6.2.2. Let $u = (u_{i,j})_{i,j=1,\ldots,n} \in M_n(\mathcal{A})$ be an $n \times n$ matrix over a C^*-algebra $\mathcal{A}$ and $\mathbf{P}$ be an orthogonal projection in $\mathcal{A}$. The pair $(u, \mathbf{P})$ is called:

1. $\mathbf{P}$-orthogonal, if all entries of u are selfadjoint, and $uu^t(1_n \otimes \mathbf{P}) = u^t u(1_n \otimes \mathbf{P}) = 1_n \otimes \mathbf{P}$ i.e. $\sum\limits_{k=1}^{n} u_{i,k}u_{j,k}\mathbf{P} = \sum\limits_{k=1}^{n} u_{k,i}u_{k,j}\mathbf{P} = \delta_{i,j}\mathbf{P}$.
2. $\mathbf{P}$-magic, if it is $\mathbf{P}$-orthogonal, and the entries of u are projections.
3. $\mathbf{P}$-cubic, if it is $\mathbf{P}$-orthogonal, and $u_{i,j}u_{i,k}\mathbf{P} = u_{j,i}u_{j,k}\mathbf{P} = 0$, for $j \neq k$.
4. $\mathbf{P}$-bistochastic, if it is $\mathbf{P}$-orthogonal, and $\sum\limits_{j=1}^{n} u_{i,j}\mathbf{P} = \sum\limits_{j=1}^{n} u_{k,i}\mathbf{P} = \mathbf{P}$, for all j, k.
5. $\mathbf{P}$-', if $\sum\limits_{j=1}^{n} u_{i,j}\mathbf{P} = \sum\limits_{j=1}^{n} u_{k,i}\mathbf{P}$, for all j, k.
6. $\mathbf{P}$-magic', if it is $\mathbf{P}$-cubic and $\mathbf{P}$-'.
7. $\mathbf{P}$-bistochastic',if it is $\mathbf{P}$-orthogonal and $\mathbf{P}$-'.

Unlike universal conditions for quantum groups, these conditions cannot define universal C^*-algebras since they cannot ensure that $u_{i,j}$'s are bounded. Therefore, we need an additional condition to control the norms of $u'_{i,j}$s. We say $(u_{i,j})_{i,=1,\ldots n}$ is norm ≤ 1 if the norm $\|(u_{i,j})_{i,j=1,\ldots,n}\|$ of the matrix is ≤ 1

Definition 6.2.3. $B_g(n)$ with $g = o, s, h, b, s, s', b'$ is the unital universal C^*-algebra generated by the entries of an $n \times n$ norm ≤ 1 matrix $(u_{i,j})_{i,=1,\dots n}$ and an orthogonal projection $\mathbf{P}$ which is respectively $\mathbf{P}$-orthogonal, $\mathbf{P}$-magic, $\mathbf{P}$-cubic, $\mathbf{P}$-bistochastic, $\mathbf{P}$-magic' and $\mathbf{P}$-bistochastic'.

On the C^*-algebra $B_g(n)$ with $g = o, s, h, b, s, s', b'$, we can always define a unital C^*-homomorphism

$$\Delta : B_g(n) \to B_g(n) \otimes B_g(n)$$

by the following formulas:

$$\Delta u_{i,j} = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j}$$

and

$$\Delta\ \mathbf{P} = \ \mathbf{P} \otimes\ \mathbf{P}, \quad \Delta I = I \otimes I.$$

To show the coproduct is well defined, we need the show that the $(\Delta u_{i,j})_{i,j=1,\dots,n}$ and $\mathbf{P} \otimes \mathbf{P}$ satisfy the universal conditions as $(u_{i,j})_{i,j=1,\dots,n}$ and $\mathbf{P}$ do:
Norm condition: If $\|(u_{i,j})_{i,j=1,\dots n}\| \leq 1$, we have

$$\|(\Delta u_{i,j})_{i,j=1,\dots n}\| = \|(\sum_{k=1}^{n} u_{i,k} \otimes u_{k,j})_{i,j=1,\dots n}\| = \|(u_{i,j} \otimes 1_n)_{i,j=1,\dots n}(1_n \otimes u_{i,j})_{i,j=1,\dots n}\| \leq \|(u_{i,j})_{i,j=1,\dots n}\|^2 \leq 1.$$

P-orthogonal: If $\sum_{k=1}^{n} u_{i,k}u_{j,k}\mathbf{P} = \sum_{k=1}^{n} u_{k,i}u_{k,j}\mathbf{P} = \delta_{i,j}\mathbf{P}$, then

$$\begin{aligned}
& \sum_{k=1}^{n} \Delta u_{i,k} \Delta u_{j,k} \Delta \mathbf{P} \\
= \ & \sum_{k=1}^{n} (\sum_{l=1}^{n} u_{i,l} \otimes u_{l,k})(\sum_{m=1}^{n} u_{j,m} \otimes u_{m,k})(\mathbf{P} \otimes \mathbf{P}) \\
= \ & \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} u_{i,l}u_{j,m}\mathbf{P} \otimes u_{l,k}u_{m,k}\mathbf{P} \\
= \ & \sum_{l=1}^{n} \sum_{m=1}^{n} u_{i,l}u_{j,m}\mathbf{P} \otimes \delta_{m,l}\mathbf{P} \\
= \ & \sum_{l=1}^{n} u_{i,l}u_{j,l}\mathbf{P} \otimes \mathbf{P} \\
= \ & \delta_{i,j}\mathbf{P} \otimes \mathbf{P}.
\end{aligned}$$

The same we have $\sum_{k=1}^{n} \Delta u_{k,i} \Delta u_{k,j} \Delta \mathbf{P} = \delta_{i,j}\mathbf{P} \otimes \mathbf{P}$.

P-cubic: Since $u_{i,j}u_{i,k}\mathbf{P} = u_{j,i}u_{j,k}\mathbf{P} = 0$, for $j \neq k$, we have

$$\begin{aligned} & \Delta u_{i,j}\Delta u_{i,k}\Delta\mathbf{P} \\ = & \sum_{l,m=1}^{n} u_{i,l}u_{i,m}\mathbf{P} \otimes u_{l,j}u_{m,k}\mathbf{P} \\ = & \sum_{l=1}^{n} u_{i,l}u_{i,l}\mathbf{P} \otimes u_{l,j}u_{l,k}\mathbf{P} \\ = & \ 0, \end{aligned}$$

whenever $j \neq k$. Then same, we have

$$\Delta u_{j,i}\Delta u_{j,k}\Delta\mathbf{P} = 0,$$

whenever $j \neq k$.

P-bistochastic: If $\sum_{j=1}^{n} u_{i,j}\mathbf{P} = \sum_{j=1}^{n} u_{j,i}\mathbf{P} = \mathbf{P}$, for all $j = 1, ..., n$.

$$\begin{aligned} & \sum_{j=1}^{n} \Delta u_{i,j}\Delta\mathbf{P} \\ = & \sum_{j=1}^{n}\sum_{k=1}^{n} u_{i,k}\mathbf{P} \otimes u_{k,j}\mathbf{P} \\ = & \sum_{j=1}^{n} u_{i,j}\mathbf{P} \otimes \mathbf{P} \\ = & \ \mathbf{P} \otimes \mathbf{P}. \end{aligned}$$

The same we will have $\sum_{j=1}^{n} \Delta u_{j,i}\Delta\mathbf{P} = \mathbf{P} \otimes \mathbf{P}$, for all j.

P' -condition: Let $r = \sum_{j=1}^{n} u_{i,j}\mathbf{P} = \sum_{j=1}^{n} u_{j,i}\mathbf{P}$, for $j \neq k$.

$$\begin{aligned} & \sum_{j=1}^{n} \Delta u_{i,j}\Delta\mathbf{P} \\ = & \sum_{j,l=1}^{n} u_{i,l}\mathbf{P} \otimes u_{l,j}\mathbf{P} \\ = & \sum_{l=1}^{n} u_{i,l}\mathbf{P} \otimes r \\ = & \ r \otimes r, \end{aligned}$$

for all j. The same we will have $\sum_{j=1}^{n} \Delta u_{j,i}\Delta\mathbf{P} = r \otimes r$ for all j.

Therefore, Δ is a well defined C^*-homomorphism and $(B_g(n), \Delta)$ with $g = o, s, h, b, s, s', b'$ are quantum semigroups. As the relation for easy quantum groups, we have the following diagram for boolean quantum semigroups:

$$\begin{array}{ccccc} B_o(n) & \longrightarrow & B_{b'}(n) & \longrightarrow & B_b(n) \\ \downarrow & & \downarrow & & \downarrow \\ B_h(n) & \longrightarrow & B_{s'}(n) & \longrightarrow & B_s(n) \end{array}$$

We can see that easy quantum groups could be quotient algebras of these easy quantum semigroups with requirement of $\mathbf{P} = 1$, i.e.

$$C_g(n) \subseteq A_g(n) \subseteq B_g(n)$$

for $g = o, s, h, b, s, s', b'$.
The algebras $\mathcal{B}_g(n)$ generated by the generators of $B_g(n)$ with $g = o, s, h, b$ are quotient algebras of Hayase's Hopf algebras $C(G_n^{I_2}), C(G_n^{I}), C(G_n^{I_h}), C(G_n^{I_b})$ in [17], respectively. Actually, $B_g(n)$ with $g = o, s, h, b$ satisfy Hayase's universal conditions for $C(G_n^{I_2}), C(G_n^{I}), C(G_n^{I_h}), C(G_n^{I_b})$. To check the some vanishing conditions, we need the following notation for convenience: Given $\pi_1 \in I(k_1)$ and $\pi_2 \in I(k_2)$, $\pi = \pi_1\pi_2 \in I(k_1 + k_2)$ denotes the concatenation of π_1 and π_2. Given $\mathbf{j}_1 = (j_1, ..., j_{k_1}) \in [n]^{k_1}$ and $\mathbf{j}_2 = (j'_1, ..., j'_{k_2}) \in [n]^{k_2}$, $\mathbf{j} = \mathbf{j}_1\, \mathbf{j}_2 = (j_1, ..., j_{k_1}, j'_1, ..., j'_{k_2}) \in [n]^{k_1+k_2}$.

According to Definition 2.2.1, it is obvious that

Lemma 6.2.4. Let $\pi \in I(k_1 + k_2)$ such that $\pi = \pi_1\pi_2$ for some $\pi_1 \in I(k_1)$ and $\pi_2 \in P(k_2)$. Let $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$ such that $\mathbf{j}_1 \in [n]^{k_1}$ and $\mathbf{j}_2 \in [n]^{k_2}$. Then, $\pi \leq ker\ \mathbf{j}$ iff $\pi_i \leq \ker\ \mathbf{j}_i$ for $i = 1, 2$.

Therefore ,we have the following:

Lemma 6.2.5. Given $\pi_1 \in I(k_1)$, $\pi_2 \in P(k_2)$ and $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$ such that $\mathbf{j}_1 \in [n]^{k_1}$ and $\mathbf{j}_2 \in [n]^{k_2}$. If

$$\sum_{\mathbf{i}_i \in [n^{k_i}], \pi_i \leq \ker\ \mathbf{i}_i} u_{\ \mathbf{i}_i,\ \mathbf{j}_i} \mathbf{P} = \begin{cases} \mathbf{P} & \text{if } \pi \leq \ker\ \mathbf{j}_i \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2$. Then, we have

$$\sum_{\mathbf{i} \in [n^{k_1+k_2}], \pi_1\pi_2 \leq \ker\ \mathbf{i}} u_{\ \mathbf{i},\ \mathbf{j}} \mathbf{P} = \begin{cases} \mathbf{P} & \text{if } \pi \leq \ker\ \mathbf{j} \\ 0 & \text{otherwise} \end{cases}$$

Proof. By a direct computation, we have:

$$\sum_{\substack{\mathbf{i} \in [n^{k_1+k_2}] \\ \pi_1\pi_2 \leq \ker\ \mathbf{i}}} u_{\ \mathbf{i},\ \mathbf{j}} \mathbf{P} = \sum_{\substack{\mathbf{i}_1 \in [n^{k_1}] \\ \pi_1 \leq \ker\ \mathbf{i}_1}} \sum_{\substack{\mathbf{i}_2 \in [n^{k_2}] \\ \pi_2 \leq \ker\ \mathbf{i}_2}} u_{\ \mathbf{i}_1,\ \mathbf{j}_1} u_{\ \mathbf{i}_2,\ \mathbf{j}_2} \mathbf{P} = \begin{cases} \sum\limits_{\substack{\mathbf{i}_1 \in [n^{k_1}] \\ \pi_1 \leq \ker\ \mathbf{i}_1}} u_{\ \mathbf{i}_1,\ \mathbf{j}_1} \mathbf{P} & \text{if } \pi_1 \leq \ker\ \mathbf{j}_2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$\sum_{\substack{\mathbf{i}\in[n^{k_1+k_2}]\\ \pi_1\pi_2\leq \ker\ \mathbf{i}}} u_{\ \mathbf{i},\ \mathbf{j}}\mathbf{P}\begin{cases} =\mathbf{P} & \pi_1\leq \ker\ \mathbf{j}_1 \text{ and } \pi_2\leq \ker\ \mathbf{j}_2 \\ 0 & \text{otherwise}\end{cases},$$

which completes the proof. □

Now, we can turn to check a vanishing condition:

Lemma 6.2.6. Let $u_{i,j}$'s and $\mathbf{P}$ be the standard generators of $B_o(n)$,$B_s(n)$,$B_h(n)$,$B_b(n)$. Then, we have

$$\sum_{\mathbf{i}\in[n^k],\pi\leq \ker\ \mathbf{i}} u_{\ \mathbf{i},\ \mathbf{j}}\mathbf{P}=\begin{cases}\mathbf{P} & \text{if } \pi\leq \ker\ \mathbf{j}\\ 0 & \text{otherwise}\end{cases}$$

for $\pi\in I_2(k), I(k), I_h(k), I_b(k)$, respectively.

Proof. 1. For $B_o(n)$, $k=2$. The identity holds by the definition of $B_o(n)$. Since all partitions in $I_2(n)$ are concatenations of pair partitions by Lemma 6.2.5, the identity is true.

2. For $B_b(n)$, the identity holds by the definition of $B_b(n)$ when π is a single partition or a partition. Since all partitions in $I_b(n)$ are concatenations of single partitions and pair partitions, by Lemma 6.2.5, the identity is true.

3. For $B_h(n)$ we just need to check $\pi=1_{2m}\in I_h(2m)$ for all $m\in\mathbb{N}$. It follows that

$$\sum_{\substack{\mathbf{i}\in[n]^{2m}\\ \pi\leq\ker\ \mathbf{i}}} u_{\ \mathbf{i},\ \mathbf{j}}\mathbf{P}=\sum_{i=1}^{n} u_{i,j_1}\cdots u_{i,j_{2m}}\mathbf{P}.$$

It equals zero if $j_l\neq j_{l+1}$ for some l, otherwise

$$\sum_{i=1}^{n} u_{i,j_1}\cdots u_{i,j_{2m}}\mathbf{P}=\sum_{i=1}^{n} u_{i,j_1}^{2m}\mathbf{P}=\sum_{i=1}^{n} u_{i,j_1}^{2m-2}\sum_{l=1}^{n} u_{l,j_1}^{2}\mathbf{P}=\sum_{i=1}^{n} u_{i,j_1}^{2m-2}\mathbf{P}=\cdots=\sum_{l=1}^{n} u_{l,j_1}^{2}\mathbf{P}=\mathbf{P}.$$

Since all partitions in $I_b(n)$ are concatenations of blocks of even length, by Lemma 6.2.5, the identity is true.

4. For $B_s(n)$ we just need to check $\pi=1_m\in I(m)$, for all $m\in\mathbb{N}$. It follows that

$$\sum_{\substack{\mathbf{i}\in[n]^{m}\\ \pi\leq\ker\ \mathbf{i}}} u_{\ \mathbf{i},\ \mathbf{j}}\mathbf{P}=\sum_{i=1}^{n} u_{i,j_1}\cdots u_{i,j_{m}}\mathbf{P}.$$

It equals zero if $j_l\neq j_{l+1}$ for some l, otherwise

$$\sum_{i=1}^{n} u_{i,j_1}\cdots u_{i,j_{m}}\mathbf{P}=\sum_{i=1}^{n} u_{i,j_1}^{m}\mathbf{P}=\sum_{i=1}^{n} u_{i,j_1}^{m-1}\sum_{l=1}^{n} u_{l,j_1}\mathbf{P}=\sum_{i=1}^{n} u_{i,j_1}^{m-1}\mathbf{P}=\cdots=\sum_{l=1}^{n} u_{l,j_1}\mathbf{P}=\mathbf{P}.$$

Since all partitions in $I_b(n)$ are concatenations of blocks of arbitrary length, by Lemma 6.2.5, the identity is true. □

Now, we define noncommutative distributional symmetries for boolean independence in general:

Definition 6.2.7. An orthogonal boolean quantum semigroup is a unital C^*-algebra A generated by n^2 selfadjoint elements $\{u_{i.j}|i,j=1,...,n\}$ and an orthogonal projection $\mathbf{P}$, such that the following hold:

1. $u=(u_{i,j})_{i,j=1,...,n}\in M_n(A)$ is norm≤ 1 and (u,$\mathbf{P}$) is $\mathbf{P}$-orthogonal.

2. $\Delta(u_{i,j}) = \sum\limits_{k=1}^{n} u_{i,k}\otimes u_{k,j}$ and $\Delta\ \mathbf{P} = \ \mathbf{P}\otimes\ \mathbf{P}, \Delta I = I\otimes I$ determine a C^*-unital homomorphism $\Delta: A\to A\otimes_{min} A$.

Definition 6.2.8. Let (A,Δ) be a quantum semigroup and $\mathcal{V}$ be a unital algebra. By a right coaction of the quantum semigroup A on $\mathcal{V}$, we mean a unital homomorphism $\alpha:\mathcal{V}\to\mathcal{V}\otimes\mathcal{A}$ such that

$$(\alpha\otimes id_A)\alpha = (id\otimes\Delta)\alpha.$$

Definition 6.2.9. Given an orthogonal boolean quantum semigroup $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...n}$ and $\mathbf{P}$, we have a natural coaction α_n of $E(n)$ on $\mathbb{C}\langle X_1,...,X_n\rangle$ such that

$$\alpha_n:\mathbb{C}\langle X_1,...,X_n\rangle\to\mathbb{C}\langle X_1,...,X_n\rangle\otimes E(n)$$

is an algebraic homomorphism defined via $\alpha_n(X_i)=\sum_{k=1}^n X_k\otimes u_{k,i}$ for all i.

Definition 6.2.10. Given a probability space $(\mathcal{A},\phi)$, a sequence of random variables $(x_1,...,x_n)$ of $\mathcal{A}$ and an orthogonal boolean quantum semigroup $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...n}$ and $\mathbf{P}$. We say that the joint distribution $\mu_{x_1,...,x_n}$ of $x_1,...,x_n$ is $E(n)$-invariant if

$$\mu_{x_1,...,x_n}(p)\mathbf{P} = \mu_{x_1,...,x_n}\otimes id_{E(n)}(\alpha_n(p))\mathbf{P},$$

for all $p\in\mathbb{C}\langle X_1,...,X_n\rangle$.

The same as matrix quantum groups, we can define $E(n)$-invariance condition for infinite sequences. Given an orthogonal boolean quantum semigroup $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...,n}$ and $\mathbf{P}$ then , for $k\in\mathbb{N}$, $E(n)$ can be considered as an orthogonal boolean quantum semigroup $E(n,k)$ generated by $\{v_{i,j}\}_{i,j=1,...,n+k}$ and $\mathbf{P}'$ such that

$$v_{i,j}=\begin{cases} u_{i,j} & \text{if } i,j\leq n\\ \delta_{i,j}1_{E(n)} & \text{otherwise}\end{cases}$$

and $\mathbf{P}'=\mathbf{P}$. We will call $E(n,k)$ the k-th extension of $E(n)$.

Definition 6.2.11. *Given a probability space $(\mathcal{A}, \phi)$, a sequence of random variables $(x_i)_{i\in\mathbb{N}} \in \mathcal{A}$ and an orthogonal Hopf algebra $E(n)$ generated by $\{u_{i,j}\}_{i,j=1,...n}$. We say that the joint distribution μ of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$-invariant if the joint distribution of $(x_1, ..., x_{n+k})$ is $E(n,k)$-invariant for all $k \in \mathbb{N}$.*

Proposition 6.2.12. *Let $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ be an operator valued probability space and $\{x_i\}_{i=1,...,n}$ be a sequence of random variables in $\mathcal{A}$. Let ϕ be a linear functional on $\mathcal{A}$ such that $\phi(\cdot) = \phi(E[\cdot])$. Then, in probability space $(\mathcal{A}, \phi)$, we have*

- *If $\{x_i\}_{i=1,...,n}$ is identically distributed and boolean independent with respect to E, then the sequence is B_s-invariant.*
- *If $\{x_i\}_{i=1,...,n}$ is identically symmetric distributed and boolean independent with respect to E, then the sequence is B_h-invariant.*
- *If $\{x_i\}_{i=1,...,n}$ has identically shifted Bernoulli distribution and is boolean independent with respect to E, then the sequence is B_b-invariant.*
- *If $\{x_i\}_{i=1,...,n}$ has identically centered Bernoulli distribution and boolean independent with respect to E, then the sequence is B_o-invariant.*

Proof. Suppose that the joint distribution of $\{x_i\}_{i=1,...,n}$ satisfies one of the conditions specified in the statement of the proposition, and let $D(k)$ be the partition family associated to the corresponding quantum semigroups. Let $X_{\mathbf{j}} = X_{j_1} \cdots X_{j_k}$, by Lemma 6.2.6 and 2.2.9, we have

$$
\begin{aligned}
\mu_{x_1,\dots,x_n}(\alpha_n(X_{\mathbf{j}}))\mathbf{P} &= \sum_{\mathbf{i}\in[n]^k} \mu_{x_1,\dots,x_n}(X_{\mathbf{i}})u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\mathbf{i}\in[n]^k} \phi(x_{\mathbf{i}})u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\mathbf{i}\in[n]^k} \phi(E[x_{\mathbf{i}}])u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\mathbf{i}\in[n]^k} \sum_{\pi\in D(k)} \phi(b_E^{(\pi)}(x_{\mathbf{i}}))u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\pi\in D(k)} \sum_{\mathbf{i}\in[n]^k} \phi(b_E^{(\pi)}(x_{\mathbf{i}}))u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\pi\in D(k)} \sum_{\substack{\mathbf{i}\in[n]^k \\ \pi\leq\ker\mathbf{i}}} \phi(b_E^{(\pi)}(x_{\mathbf{i}}))u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\pi\in D(k)} \sum_{\substack{\mathbf{i}\in[n]^k \\ \pi\leq\ker\mathbf{i}}} \phi(b_E^{(\pi)}(x_1,\dots,x_1))u_{\mathbf{i},\mathbf{j}}\mathbf{P} \\
&= \sum_{\substack{\pi\in D(k) \\ \pi\leq\ker\mathbf{j}}} \phi(b_E^{(\pi)}(x_1,\dots,x_1))\mathbf{P} \\
&= \sum_{\substack{\pi\in D(k) \\ \pi\leq\ker\mathbf{j}}} \phi(b_E^{(\pi)}(x_{\mathbf{j}}))\mathbf{P} \\
&= \phi(E[x_{\mathbf{j}}])\mathbf{P} \\
&= \phi(x_{\mathbf{j}})p \\
&= \mu_{x_1,\dots,x_n}(X_{\mathbf{j}})\mathbf{P},
\end{aligned}
$$

which completes the proof. □

6.3 Main result

In this section, we will prove our main theorem. Then, we will present an application of our main theorem to easy groups $C_{s'}(n)$, easy quantum groups $C_{b'}(n)$, $A_{s'}(n)$, $A_{b'}(n)$, $A_{b\#}(n)$ and boolean quantum semigroups $B_{s'}(n)$, $B_{b'}(n)$.

Theorem 6.3.1. Let $(\mathcal{A},\phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be a sequence of random variables which generate $\mathcal{A}$

- Classical case:
  Suppose that $\mathcal{A}$ is commutative. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal Hopf algebras such that $C_s(n)\subseteq E(n)\subseteq C_o(n)$ for each $n\in\mathbb{N}$. If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$-invariant, then there are a W^*-subalgebra $1\subseteq\mathcal{B}\subseteq\mathcal{A}$ and a ϕ-preserving conditional expectation $E:\mathcal{A}\to\mathcal{B}$ such that

  1. If $E(n)=C_s(n)$ for all n, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and identically distributed with respect to E.

2. If $C_s(n) \subseteq E(n) \subseteq C_h(n)$ for all n and there exists a k such that $E(k) \neq C_s(k)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have identically symmetric distribution with respect to E.
3. If $C_s(n) \subseteq E(n) \subseteq C_b(n)$ for all n and there exists a k such that $E(k) \neq C_s(k)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have identically shifted-Gaussian distribution with respect to E.
4. If there exist k_1, k_2 such that $E(k_1) \nsubseteq C_h(k_1)$ and $E(k_2) \nsubseteq C_b(k_2)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have centered Gaussian distribution with respect to E.

- Free case:
  Suppose ϕ is faithful. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal Hopf algebras such that $A_s(n) \subseteq E(n) \subseteq A_o(n)$ for each n. If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$-invariant, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that

  1. If $E(n) = A_s(n)$ for all n, then $(x_i)_{i\in\mathbb{N}}$ are freely independent and identically distributed with respect to E.
  2. If $A_s(n) \subseteq E(n) \subseteq A_h(n)$ for all n and there exists a k such that $E(k) \neq A_s(k)$, then $(x_i)_{i\in\mathbb{N}}$ are freely independent and have identically symmetric distribution with respect to E.
  3. If $A_s(n) \subseteq E(n) \subseteq A_b(n)$ for all n and there exists a k such that $E(k) \neq A_s(k)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have identically shifted-semicircular distribution with respect to E.
  4. If there exist k_1, k_2 such that $E(k_1) \nsubseteq A_h(k_1)$ and $E(k_2) \nsubseteq A_b(k_2)$, then $(x_i)_{i\in\mathbb{N}}$ are freely independent and have centered semicircular distribution with respect to E.

- boolean case:
  If ϕ is non-degenerated. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal boolean quantum semigroups such that $B_s(n) \subseteq E(n) \subseteq B_o(n)$ for each n. If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$-invariant, then there are a W^*-subalgebra(not necessarily contains the unit of $\mathcal{A}$) $\mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that

  1. If $E(n) = B_s(n)$ for all n, then $(x_i)_{i\in\mathbb{N}}$ are boolean independent and identically distributed with respect to E.
  2. If $B_s(n) \subseteq E(n) \subseteq B_h(n)$ for all n and there exists a k such that $E(k)$ has a quotient algebra $E'(k)$ that $A_s(k) \subsetneq E'(k) \subseteq A_n(n)$, then $(x_i)_{i\in\mathbb{N}}$ are boolean independent and have identically symmetric distribution with respect to E.
  3. If $B_s(n) \subseteq E(n) \subseteq B_b(n)$ for all n and there exists a k such that $E(k)$ has a quotient algebra $E'(k)$ that $A_s(k) \subsetneq E'(k) \subseteq A_b(n)$, then $(x_i)_{i\in\mathbb{N}}$ are boolean independent and have identically shifted-Bernoulli distribution with respect to E.

4. If there exist k_1, k_2 such that $E(k_1)$ and $E(k_2)$ have quotient algebras $E'(k_1) \subseteq A_o(k_1)$ and $E'(k_2) \subseteq A_o(k_2)$ such that $E(k_1) \nsubseteq A_h(k_1)$ and $E'(k_2) \nsubseteq A_b(k_2)$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have centered Bernoulli distribution with respect to E.

The proof of free case is the most typical, we list it below:

Free case: In a W^*-probability space $(\mathcal{A}, \phi)$ such that ϕ is faithful. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal Hopf algebras such that $A_s(n) \subseteq E(n) \subseteq A_o(n)$ for each n. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of random variables which generate A. Suppose that the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$-invariant for all n. By Proposition 6.1.11, $(x_i)_{i\in\mathbb{N}}$ are $A_s(n)$-invariant for all n. By Köstler and Speicher[23], there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ that $(x_i)_{i\in\mathbb{N}}$ are freely independent and identically distributed with respect to E. It proves the statement 1 for free case. In addition, by Proposition 4.3 in [23] and Definition 6.1.12, the coaction invariant condition for ϕ can be extended to the conditional expectation E, i.e.

$$E[b_0 x_{i_1} b_1 \cdots b_{k-1} x_{i_k} b_k] \otimes 1_{E(n)} = \sum_{j_1,\dots,j_k=1}^{n} E[b_0 x_{j_1} b_1 \cdots b_{k-1} x_{j_k} b_k] \otimes u_{j_1,i_1} \cdots u_{j_k,i_k}$$

for $i_1, ..., i_k \leq n$, where $u_{i,j}$'s are generators of $E(n)$.

2. Suppose that $A_s(n) \subseteq E(n) \subseteq A_b(n)$ for all n and there exists a k such that $E(k) \neq A_s(k)$. Let $\{u_{i,j}\}_{i,j=1,\dots,k}$ be generators of $E(k)$. By proposition 6.1.5, $\exists\, i'$ such that

$$\sum_{l=1}^{k} u_{l,i'}^{m} \neq 1$$

for all $m > 2$.

Without loss of generality, we assume that $i' = 1$. In order to finish the proof, we need to show that $\kappa_l(x_1 b_1, ..., x_1 b_l) = 0$ for all $l \geq 3$, where $b_1, ..., b_l \in B$. We prove this by induction on l. First, we have that

$$\begin{aligned}
& E[x_1 b_1 \cdots x_1 b_l] \otimes 1_{E(n)} \\
= & \sum_{\mathbf{i}\in[k]^l} E[x_{i_1} b_1 \cdots x_{i_l} b_l] \otimes u_{\mathbf{i},1} \\
= & \sum_{\mathbf{i}\in[k]^l} \sum_{\pi\in NC(l)} \kappa_\pi(x_{i_1} b_1, ..., x_{i_l} b_l) \otimes u_{\mathbf{i},1} \\
= & \sum_{\pi\in NC_b(l)} \sum_{\mathbf{i}\in[k]^l} \kappa_\pi(x_{i_1} b_1, ..., x_{i_l} b_l) \otimes u_{\mathbf{i},1} + \sum_{\pi\in NC(l)\setminus NC_b(l)} \sum_{\mathbf{i}\in[k]^l} \kappa_\pi(x_{i_1} b_1, ..., x_{i_l} b_l) \otimes u_{\mathbf{i},1} \\
= & \sum_{\pi\in NC_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} \kappa_\pi(x_{i_1} b_1, ..., x_{i_l} b_l) \otimes u_{\mathbf{i},1} + \sum_{\pi\in NC(l)\setminus NC_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} \kappa_\pi(x_{i_1} b_1, ..., x_{i_l} b_l) \otimes u_{\mathbf{i},1} \\
= & \sum_{\pi\in NC_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} \kappa_\pi(x_1 b_1, ..., x_1 b_l) \otimes u_{\mathbf{i},1} + \sum_{\pi\in NC(l)\setminus NC_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} \kappa_\pi(x_1 b_1, ..., x_1 b_l) \otimes u_{\mathbf{i},1} \\
= & \sum_{\pi\in NC_b(l)} \kappa_\pi(x_1 b_1, ..., x_1 b_l) \otimes 1_{E(n)} + \sum_{\pi\in NC(l)\setminus NC_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} \kappa_\pi(x_1 b_1, ..., x_1 b_l) \otimes u_{\mathbf{i},1}.
\end{aligned}$$

The first term of the last equality follows that $E(n)$ is a quotient algebra of $A_b(n)$. On the other hand

$$E[x_1b_1,...,x_1b_l]\otimes 1_{E(n)} = \sum_{\pi\in NC_b(k)} \kappa_\pi(x_1b_1,...,x_1b_l)\otimes 1_{E(n)} + \sum_{\pi\in NC(l)\setminus NC_b(l)} \kappa_\pi(x_1b_1,...,x_1b_l)\otimes 1_{E(n)}.$$

Therefore,

$$\sum_{\pi\in NC(l)\setminus NC_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker\ \mathbf{i}}} \kappa_\pi(x_1b_1,...,x_1b_l)\otimes u_{\ \mathbf{i},1} = \sum_{\pi\in NC(l)\setminus NC_b(l)} \kappa_\pi(x_1b_1,...,x_1b_l)\otimes 1_{E(n)} \quad (6.1)$$

When $l=3$, we have $NC(3)\setminus NC_b(3) = \{1_3\}$, then

$$\sum_{\substack{\mathbf{i}\in[n]^k \\ \pi\leq \ker 1_3}} \kappa_{1_3}(x_1b_1,...,x_1b_3)\otimes u_{\ \mathbf{i},1} = \kappa_{1_3}(x_1b_1,...,x_1b_3)\otimes 1_{E(n)},$$

which is

$$\kappa_{1_3}(x_1b_1,...,x_1b_k)\otimes(\sum_{l=1}^{k} u_{l,1}^3 - 1_{E(n)}) = 0.$$

Therefore, $\kappa_{1_3}(x_1b_1,...,x_1b_3)=0$. Suppose $\kappa_{1_l}(x_1b_1,...,x_1b_l)=0$ for $3\leq l\leq m$, then for $\pi\in NC(m+1)$, $\kappa_\pi(x_{i_1}b_1,...,x_1b_{m+1})=0$ if π contains a block whose size is between 3 and m. Each partition $\pi\in NC(m+1)\setminus NC_b(m+1)$ contains at least one block whose size is greater than 2. Therefore, for $\pi\in NC(m+1)\setminus NC_b(m+1)$, $\kappa_\pi(x_1b_1,...,x_1b_k)=0$ if $\pi\neq 1_{m+1}$. Hence, equation 6.1 becomes

$$\kappa_{1_{m+1}}(x_1b_1,...,x_1b_{m+1})\otimes(\sum_{l=1}^{k} u_{l,1}^{m+1} - 1_{E(n)}) = 0$$

which implies

$$\kappa_{1_{m+1}}(x_1b_1,...,x_1b_{m+1}) = 0,$$

for all $b_1,...,b_{m+1}\in\mathcal{B}$. The proof is complete.

3. Suppose that $A_s(n)\subseteq E(n)\subseteq A_h(n)$ for all n and there exists a k such that $E(k)\neq A_s(k)$. Let $\{u_{i,j}\}_{i,j=1,...,k}$ be generators of $E(k)$. By proposition 6.1.5, $\exists\, i'$ such that

$$\sum_{l=1}^{k} u_{l,i'}^m \neq 1$$

for all odd numbers m.

Without loss of generality, we assume that $i'=1$. We need to show that $\kappa_k(x_1b_1,....,x_1b_l) = 0$ for all add numbers k where $b_1,....,b_l\in B$. Agian, we prove this by induction on l.

We have that

$$\begin{aligned} & E[x_1b_1\cdots x_1b_l]\otimes 1_{E(n)} \\ = & \sum_{\pi\in NC_h(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\ \mathbf{i}}}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes u_{\ \mathbf{i},1}+\sum_{\pi\in NC(l)\setminus NC_h(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\ \mathbf{i}}}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes u_{\ \mathbf{i},1} \\ = & \sum_{\pi\in NC_h(l)}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes 1_{E(n)}+\sum_{\pi\in NC(l)\setminus NC_h(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\ \mathbf{i}}}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes u_{\ \mathbf{i},1}\end{aligned}$$

The first term of the last equality follows that $E(n)$ is a quotient algebra of $A_h(n)$. On the other hand, we have

$$E[x_1b_1,...,x_1b_l]\otimes 1_{E(n)}=\sum_{\pi\in NC_b(l)}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes 1_{E(n)}+\sum_{\pi\in NC(l)\setminus NC_h(l)}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes 1_{E(n)}.$$

Therefore,

$$\sum_{\pi\in NC(l)\setminus NC_b(l)}\sum_{\substack{\mathbf{i}\in[k]^l\\ \pi\leq\ker\ \mathbf{i}}}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes u_{\ \mathbf{i},1}=\sum_{\pi\in NC(l)\setminus NC_b(l)}\kappa_\pi(x_1b_1,...,x_1b_l)\otimes 1_{E(n)}\quad(6.2)$$

When $l=1$, we have $NC(1)\setminus NC_h(1)=\{1_1\}$, then

$$\kappa^{(1)}(x_1b_1)\otimes(\sum_{l=1}^{k}u_{l,1}-1_{E(n)})=0.$$

Therefore, $\kappa_{1_1}(x_1b_1)=0$. Suppose $\kappa_{1_l}(x_1b_1,...,x_1b_l)=0$ for odd numbers $l\leq 2m$, then for $\pi\in NC(2m+1)$, $\kappa_\pi(x_{i_1}b_1,...,x_1b_{2m+1})=0$ if π contains a block whose size is an odd number less than $2m$. Each partition $\pi\in NC(2m+1)\setminus NC_b(2m+1)$ contains at least one block whose size is odd. Therefore, for $\pi\in NC(2m+1)\setminus NC_b(2m+1)$, $\kappa_\pi(x_1b_1,...,x_1b_{2m+1})=0$ if $\pi\neq 1_{2m+1}$. Hence, equation 6.2 becomes

$$\kappa_{1_{2m+1}}(x_1b_1,...,x_1b_{2m+1})\otimes(\sum_{l=1}^{k}u_{l,1}^{2m+1}-1_{E(n)})=0$$

which implies

$$\kappa_{1_{m+1}}(x_1b_1,...,x_1b_{m+1})=0,$$

for all $b_1,...,b_{m+1}\in\mathcal{B}$. The proof is complete.

4. If there exist k_1,k_2 such that $E(k_1)\not\subseteq A_h(k_1)$ and $E(k_2)\not\subseteq A_b(k_2)$, by Case 3 and 4, the only non-vanishing cumulants are pair partition cumulants. The proof is done.

Classical Case: The proof is almost the same as free case, we just need to replace noncrossing partitions by all partitions.

boolean Case: The proof is a little different. Some properties of boolean conditional expectation are discussed in [25], [17]. As it is shown in [25], for boolean de Finetti theorem, we need to consider random variables in W^*-probability space with a non-degenerated state $(\mathcal{A}, \phi)$. Assume that $\mathcal{A}$ is generated by a sequence of random variables $(x_i)_{i\in\mathbb{N}}$. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal boolean quantum groups such that $B_s(n) \subseteq E(n) \subseteq B_o(n)$ for each n. If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$-invariant, then the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $B_s(n)$ invariant for all n. By the main results in [25], there are a W^*-subalgebra(not necessarily contain the unit of $\mathcal{A}$) $\mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $(x_i)_{i\in\mathbb{N}}$ are boolean independent and identically distributed with respect to E. In this part of proof, we will assume that $\mathcal{B}$ does not contain $1_{\mathcal{A}}$. It should be pointed out that the case that $\mathcal{B}$ contains the unit of $\mathcal{A}$ is always a unitalization of the case that $\mathcal{B}$ does not contain $1_{\mathcal{A}}$. Under our assumption, the tail algebra

$$\mathcal{B} = \bigcap_{n=1}^{\infty} W^*\{x_k | k \geq n\},$$

where $W^*\{x_k|k \geq n\}$ is the WOT closure of the non-unital algebra generated by $\{x_k|k \geq n\}$. We call $\mathcal{B}$ the non-unital tail algebra of $\{x_i\}_{i\in\mathbb{N}}$. Unlike the proof of free and classical case, the coaction invariant condition for ϕ can be extended to the conditional expectation E directly. Actually, we have a stronger statement.

Proposition 6.3.2. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be an infinite sequence of selfadjoint random variables which generate $\mathcal{A}$ as a von Neumann algebra and the unit of $\mathcal{A}$ is contained in the WOT closure of the non-unital algebra generated by $(x_i)_{i\in\mathbb{N}}$. Let $E(n)$ be a sequence of boolean orthogonal quantum semigroups such that $B_s(n) \subseteq E(n) \subseteq B_o(n)$. If $(x_i)_{i\in\mathbb{N}}$ is $E(n)$-invariant for all n, then there exists a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is the non-unital tail algebra of $\{x_i\}_{i\in\mathbb{N}}$, such that $(x_i)_{i\in\mathbb{N}}$ is boolean independent with respect to E. Let $\mathcal{A}_n$ be the non-unital algebra generated by $\{x_i\}_{i\in\mathbb{N}}$. We have that*

$$E[a_1 b a_2] = E[a_1] b E[a_2],$$

where $a_1, a_2 \in \mathcal{A}_n$ for some n and $b \in \mathcal{B}$. Let $\{u_{i,j}\}_{i,j=1,\dots,n}$ be generators of $E(n)$. We will have that

$$E[x_{i_1} \cdots x_{i_k}] \otimes \boldsymbol{P} = \sum_{j_1,\dots,j_k=1}^{n} E[x_{j_1} \cdots x_{j_k}] \otimes u_{j_1,i_1} \cdots u_{j_k,i_k} \boldsymbol{P}$$

for $i_1, ..., i_k \leq n$.

Proof. The existence of E is prove in [25]. We will just need to prove the last two equations. Given $a_1, a_2 \in \mathcal{A}_n$ for some n and $b \in \mathcal{B}$, by assumption, b is contained in W^*-closure of the non-unital algebra generated by $\{x_i | i > n\}$. By Kaplansky theorem, $\exists$ a sequence of bounded elements y_i such that y_i is contained in the non-unital algebra generated by $\{x_i | i > n\}$ such that y_i converges to b in strong operator topology. Therefore, by normality of E, we have

$$E[a_1 b a_2] = \lim_{i\to\infty} E[a_1 y_i a_2] = \lim_{i\to\infty} E[a_1] E[y_i] E[a_2] = E[a_1] b E[a_2],$$

where the second equality follows the fact that $(x_i)_{i\in\mathbb{N}}$ are boolean independent with respect to E. The second equation can be checked pointwisely. Let $a_1, a_2 \in \mathcal{A}_m$ for some m. In [25], we showed that there exists a normal homomorphism $\alpha : \mathcal{A} \to \mathcal{A}$ such that $\alpha(x_i) = x_{i+1}$ for all $i \in \mathbb{N}$. By the proof of Lemma 6.7 in [25] and the assumption that $\{x_i\}_{i\in\mathbb{N}}$ is $E(n)$-invariant, we have

$$\begin{aligned}
& \phi(a_1 E[x_{i_1}\cdots x_{i_k}]a_2)\otimes \mathbf{P} \\
= & \lim_{l\to\infty, l>m} \phi(a_1\alpha^l(x_{i_1}\cdots x_{i_k})a_2)\otimes \mathbf{P} \\
= & \lim_{l\to\infty, l>m} \phi(\alpha^n(a_1)x_{i_1}\cdots x_{i_k}\alpha^n(a_2))\otimes \mathbf{P} \\
= & \lim_{l\to\infty, l>m} (\phi(\alpha^n(a_1)\sum_{j_1,\dots,j_k=1}^{n} x_{j_1}\cdots x_{j_k}\alpha^n(a_2))\otimes u_{j_1,i_1}\cdots u_{j_k,i_k}\mathbf{P} \\
= & \lim_{l\to\infty, l>m} \phi(a_1\alpha^l(\sum_{j_1,\dots,j_k=1}^{n} x_{j_1}\cdots x_{j_k})a_2)\otimes u_{j_1,i_1}\cdots u_{j_k,i_k}\mathbf{P} \\
= & \sum_{j_1,\dots,j_k=1}^{n} \phi(a_1 E[x_{j_1}\cdots x_{j_k}]a_2)\otimes u_{j_1,i_1}\cdots u_{j_k,i_k}\mathbf{P}
\end{aligned}$$

Since a_1, a_2 are arbitrarily from the sense set $\bigcup_{n\to\infty} \mathcal{A}_n$ of $\mathcal{A}$, the proof is done. □

Now, we turn to finish the proof of our main theorem for boolean case:

1. This is just the boolean de Finetti theorem in [25].
2. As the free case, we need to show that $b_E^{(l)}(x_1b_1, ..., x_1b_l) = 0$ for all $l \geq 3$ where $b_1, ..., b_l \in B \cup \{\mathbb{C}1_{\mathcal{A}}\}$. By proposition 6.3.2, we have

$$\begin{aligned}
& E[x_{\mathbf{i}_1} b_1 x_{\mathbf{i}_2}\cdots b_{n-1}x_{\mathbf{i}_m}] \\
= & E[x_{\mathbf{i}_1}]b_1E[x_{\mathbf{i}_2}]\cdots b_{n-1}E[x_{\mathbf{i}_m}] \\
= & \sum_{\pi_1\in I(k_1)} b_E^{(\pi_1)}(x_{i_1^{(1)}}, ...x_{i_{k_1}^{(1)}})b_1 \sum_{\pi_2\in I(k_2)} b_E^{(\pi_2)}(x_{i_1^{(2)}}, ...x_{i_{k_2}^{(2)}})\cdots b_{n-1} \sum_{\pi_m\in I(k_m)} b_E^{(\pi_m)}(x_{i_1^{(m)}}, ...x_{i_{k_m}^{(m)}}) \\
= & \sum_{\pi\in I(k_1)\times I(k_2)\times\cdots\times I(k_m)} b_E^{(\pi)}(x_{i_1^{(1)}}, ...x_{i_{k_1}^{(1)}}, b_1x_{i_1^{(2)}}, ...x_{i_{k_2}^{(2)}}, \cdots, b_{n-1}x_{i_1^{(m)}}, ...x_{i_{k_m}^{(m)}})
\end{aligned}$$

where $\mathbf{i}_l = (i_1^{(l)}, ..., i_{k_l}^{(l)}) \in [n]^{k_l}$ for all $l = 1, ..., m$ for some n and $b_1, ..., b_m \in \mathcal{B}$. Therefore, to finish the prove, we just need to show that $b_E^{(k)}(x_1, ..., x_1) = 0$ for all $l \geq 3$. The rest of the poof is almost the same as the free case:

Let $\{u_{i,j}\}_{i,j=1,...,k}$'s and $\mathbf{P}$ be generators of $E(k)$. First, by Proposition 6.3.2, we have

$$\begin{aligned}
& E[x_1 \cdots x_1] \otimes \mathbf{P} \\
= & \sum_{\mathbf{i}\in[k]^l} E[x_{\mathbf{i}}] \otimes u_{\mathbf{i},1}\mathbf{P} \\
= & \sum_{\mathbf{i}\in[k]^l} \sum_{\pi\in I(l)} b_E^{(\pi)}(x_{\mathbf{i}}) \otimes u_{\mathbf{i},1} \\
= & \sum_{\pi\in I_b(l)} \sum_{\mathbf{i}\in[k]^l} b_E^{(\pi)}(x_{i_1},...,x_{i_l}) \otimes u_{\mathbf{i},1}\mathbf{P} + \sum_{\pi\in I(l)\setminus I_b(l)} \sum_{\mathbf{i}\in[k]^l} b_E^{(\pi)}(x_{i_1},...,x_{i_l}) \otimes u_{\mathbf{i},1}\mathbf{P} \\
= & \sum_{\pi\in I_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} b_E^{(\pi)}(x_{i_1},...,x_{i_l}) \otimes u_{\mathbf{i},1}\mathbf{P} + \sum_{\pi\in I(l)\setminus I_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} b_E^{(\pi)}(x_{i_1},...,x_{i_l}) \otimes u_{\mathbf{i},1}\mathbf{P} \\
= & \sum_{\pi\in I_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} b_E^{(\pi)}(x_1,...,x_1) \otimes u_{\mathbf{i},1}\mathbf{P} + \sum_{\pi\in I(l)\setminus I_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} b_E^{(\pi)}(x_1,...,x_1) \otimes u_{\mathbf{i},1}\mathbf{P} \\
= & \sum_{\pi\in I_b(l)} b_E^{(\pi)}(x_1b_1,...,x_1b_l) \otimes \mathbf{P} + \sum_{\pi\in I(l)\setminus I_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} b_E^{(\pi)}(x_1,...,x_1) \otimes u_{\mathbf{i},1}\mathbf{P}.
\end{aligned}$$

The first term of the last equality follows that $E(n)$ is a quotient algebra of $B_b(n)$. On the other hand

$$E[x_1,...,x_1] \otimes \mathbf{P} = \sum_{\pi\in I_b(k)} b_E^{(\pi)}(x_1,...,x_1) \otimes \mathbf{P} + \sum_{\pi\in I(l)\setminus I_b(l)} b_E^{(\pi)}(x_1,...,x_1) \otimes \mathbf{P}.$$

Therefore,

$$\sum_{\pi\in I(l)\setminus I_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} b_E^{(\pi)}(x_1,...,x_1) \otimes u_{\mathbf{i},1}\mathbf{P} = \sum_{\pi\in I(l)\setminus I_b(l)} b_E^{(\pi)}(x_1,...,x_1) \otimes \mathbf{P}. \tag{6.3}$$

By assumption, $E(k)$ has a quotient algebra $E'(k)$ that $A_s(k) \subsetneq E'(k) \subseteq A_n(n)$. Let $\{u'_{i,j}\}'s$ be the generators of $E'(k)$. Then, there exists a C^*-homomorphism $\Psi : E(k) \to E'(k)$ such that

$$\Psi(u_{i,j}) = u'_{i,j} \text{ for all } i,j = 1,...,k, \text{ and } \Psi(\mathbf{P}) = 1_{E'(k)}.$$

Without loss of generality, by proposition 6.1.5, we can assume that

$$\sum_{l=1}^{k} u_{l,1}^m \neq 1$$

for all $m > 2$. Let $id \otimes \Psi$ acts on equation 6.4. Then, we get

$$\sum_{\pi\in I(l)\setminus I_b(l)} \sum_{\substack{\mathbf{i}\in[k]^l \\ \pi\leq \ker \mathbf{i}}} b_E^{(\pi)}(x_1,...,x_1) \otimes u'_{\mathbf{i},1} = \sum_{\pi\in I(l)\setminus I_b(l)} b_E^{(\pi)}(x_1,...,x_1) \otimes 1_{E'(k)}. \tag{6.4}$$

When $l = 3$, we have $I(3) \setminus I_b(3) = \{1_3\}$, then

$$\sum_{\substack{\mathbf{i}\in[n]^k \\ \pi\leq \ker 1_3}} b_E^{(3)}(x_1, ..., x_1) \otimes u'_{\mathbf{i},1} = b_E^{(3)}(x_1, ..., x_1) \otimes 1_{E'(k)},$$

which is

$$\kappa_{1_3}(x_1, ..., x_1) \otimes (\sum_{l=1}^{k} u'^3_{l,1} - 1_{E'(k)}) = 0.$$

Therefore, $b_E^{(3)}(x_1, ..., x_1) = 0$.
Suppose $b_E^{(l)}(x_1 b_1, ..., x_1 b_l) = 0$ for $3 \leq l \leq m$. Then, for $\pi \in I(m+1)$, $b_E^{(\pi)}(x_1, ..., x_1) = 0$ if π contains a block whose size is between 3 and m. Each partition $\pi \in I(m+1) \setminus I_b(m+1)$ contains at least one block whose size is greater than 2. Therefore, for $\pi \in I(m+1) \setminus I_b(m+1)$, $b_E^{(\pi)}(x_1, ..., x_1) = 0$ if $\pi \neq 1_{m+1}$. Hence, equation 6.1 becomes

$$b_E^{(m+1)}(x_1, ..., x_1) \otimes (\sum_{l=1}^{k} u'^{m+1}_{l,1} - 1_{E'(k)}) = 0$$

which implies

$$b_E^{(m+1)}(x_1, ..., x_1) = 0.$$

The proof is complete.

The same, compare to Case 3 and Case 4 in free case, by applying the method in boolean Case 2, we have Case 3 and Case 4 for boolean independence are also true.

Remark 6.3.3. *According to the proof, we can replace the condition $*_s(n) \subseteq E(n)$ for all n by $*_s(n) \subseteq E(n)$ for infinitely many n which ensures fundamental de Finetti theorems hold, where $*$ could be A, B, C. Our general de Finetti theorem for boolean independence is not complete since we know very little about classification of boolean quantum semigroups.*

According the diagrams in Section 6.1 and 6.2, we have the following:

1. $C_s(n) \subseteq C_{s'}(n) \subseteq C_b(n)$ for all n, and $C_s(n) \neq C_{s'}(n)$ for $n > 3$.

2. $C_{b'}(n) \nsubseteq C_h(n), C_b(n)$ for $n > 3$.

3. $A_s n \subseteq A_{s'}(n) \subseteq A_b(n)$ for all n, and $A_s(n) \neq A_{s'}(n)$ for $n > 3$.

4. $A_{b'}(n), A_{b\#}(n) \nsubseteq A_h(n), A_b(n)$ for $n > 3$.

5. $B_s(n) \subseteq B_{s'}(n) \subseteq B_b(n)$ for all n, and $B_s(n) \neq B_{s'}(n)$ for $n > 3$. Moreover $A_{s'}(n)$ is a quotient algebra of $B_{s'(n)}$

6. $A_{b'}$ is a quotient algebra of $B_{b'}(n)$ and $A_{b'}(n) \nsubseteq A_h(n), A_b(n)$ for $n > 3$.

By Theorem 6.3.1, we get the following:

Corollary 6.3.4. *Let $(\mathcal{A}, \phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be a sequence of random variables which generate $\mathcal{A}$.*

- *Classical case:*
  Suppose that $\mathcal{A}$ is commutative and ϕ is faithful. We have
  1. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $C_{s'}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ $(x_i)_{i\in\mathbb{N}}$ such that $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have identically symmetric distribution with respect to E.*
  2. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $C_{b'}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ $(x_i)_{i\in\mathbb{N}}$ such that $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and have centered Gaussian distribution with respect to E.*

- *Free case:*
  Suppose ϕ is faithful. there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that
  1. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $A_{s'}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $(x_i)_{i\in\mathbb{N}}$ are freely independent and have identically symmetric distribution with respect to E.*
  2. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $A_{b'}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $(x_i)_{i\in\mathbb{N}}$ are freely independent and have centered semicircular distribution with respect to E.*
  3. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $A_{b\#}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $(x_i)_{i\in\mathbb{N}}$ are freely independent and have centered semicircular distribution with respect to E.*

- *boolean case:*
  If ϕ is non-degenerated. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal boolean quantum semigroups such that $B_s(n) \subseteq E(n) \subseteq B_o(n)$ for each n. If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$-invariant, then there are a W^*-subalgebra(not necessarily contain the unit of $\mathcal{A}$) $\mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that
  1. *If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $B_{s'}(n)$ invariant for all $n \in \mathbb{N}$, then there are a W^*-subalgebra(not necessarily contain the unit of $\mathcal{A}$) $\mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $(x_i)_{i\in\mathbb{N}}$ are boolean independent and have identically symmetric distribution with respect to E.*

2. *If the joint distribution of* $(x_i)_{i\in\mathbb{N}}$ *is* $B_{b'}(n)$ *invariant for all* $n \in \mathbb{N}$*, then there are a* W^**-subalgebra(not necessarily contain the unit of* $\mathcal{A}$*)* $\mathcal{B} \subseteq \mathcal{A}$ *and a* ϕ*-preserving conditional expectation* $E : \mathcal{A} \to \mathcal{B}$ *such that* $(x_i)_{i\in\mathbb{N}}$ *are conditionally independent and have centered Bernoulli distribution with respect to* E*.*

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