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**UNRAVELING GEODESIC X-RAY TRANSFORMS ON THE HEISENBERG GROUP**

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in

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**Abstract**

Unraveling Geodesic X-ray Transforms on the

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We initiate the study of X-ray tomography on sub-Riemannian manifolds, for which the Heisenberg group exhibits the simplest nontrivial example. With the language of the group Fourier Transform, we prove a quantized version of the Fourier Slice Theorem and apply this new tool to show that a sufficiently regular function on the Heisenberg group is determined by its line integrals over sub-Riemannian geodesics. We also consider the families of compatible Riemannian and Lorentzian metrics $g_\epsilon$ and $g_{i\epsilon}$, respectively, taming the sub-Riemannian metric and prove that their associated X-ray transforms are injective for all $\epsilon > 0$. In the latter case, we show in particular that a function on $g_{i\epsilon}$-spacetime is determined by integrals over spacelike geodesics. These results provide explicit examples of injective X-ray transforms in a geometry with an abundance of conjugate points and with metrics of mixed curvature.

To my parents, and grandparents.

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# Part I

# Introduction

# Chapter 1

# Inverse Problems and X-ray Transforms

## 1.1 Setting

This thesis presents results in X-ray Tomography on the Heisenberg group. In the broader field of integral geometry, one attempts to reconstruct geometric information on a manifold from line integrals over geodesics or more general integral functionals. Practical examples include X-ray Computed Tomography used in medical imaging. At center stage is the X-ray transform, which encodes data sampled from noninvasive measurements.

The geometry of the problem greatly influences the mathematical approach. If the medium is highly anisotropic—constraining the degrees of freedom—such shale rock [24], then the appropriate model is a sub-Riemannian manifold, i.e., a smooth manifold $M$ equipped with a distribution $\mathcal{D}$ and a metric $g$ defined strictly on that distribution. In [17] the authors propose a method to construct memematerial arrays where light or acoustic waves propagate according to sub-Riemannian optics and for which resonance frequencies are governed by the spectrum a hypoelliptic operator. X-ray transforms have been extensively studded on Riemannian manifolds (see [26] for an exposition on standard results and applications), but to the author's knowledge the sub-Riemannian case is wide open.

Our primary object of study is the geodesic X-ray transform associated to the sub-Riemannian geometry of the Heisenberg group, which is $\mathbb{H} := \mathbb{C} \times \mathbb{R}$ with the multiplication law

$$(x + iy, t)(u + iv, s) = \left(x + u + i(y + v), t + s + \tfrac{1}{2}(xv - yu)\right),$$

and a metric defined in Section 2.1. $\mathbb{H}$ is the local model for any 3-dimensional sub-Riemannian manifold of contact type, in the same sense that 3-dimensional Euclidean space is the local model for any 3-dimensional Riemannian manifold [30, Thm. 1]. This property positions $\mathbb{H}$ as the natural homogeneous starting point for studying the integral geometry of contact manifolds, just as Radon first inverted the X-ray transform in $\mathbb{R}^2$. X-ray transforms on symmetric spaces are also extensively studied, for example in [15, 21], and the Heisenberg group arises as the boundary at infinity of complex hyperbolic space in two dimensions [7, 51].

To a function $f \in L^1(\mathbb{H})$ we associate the function $If$, its X-ray transform, defined by

$$If(\gamma) := \int f\left(\gamma(s)\right) ds,$$

where the geodesics $\gamma$ will be cast as (projections of) integral curves of the Hamiltonian flow on $T^*\mathbb{H}$ for the degenerate fiber quadratic Hamiltonian later described (2.2). Related integral transforms on $\mathbb{H}$ have been studied, for example, by Rubin [44], and Stichartz [46], who consider integration over left translates of hyperplanes. We ask whether $If$ determines $f$.

The sub-Riemannian setting, whose general theory is poorly understood, introduces qualitatively new features to this question. For example, fibers of the unit cotangent bundle $U^*\mathbb{H}$ (defined in Section 2.1) are cylinders, and there is no unique Levi-Civita connection. Thus there is no canonical splitting of its tangent space into vertical and horizontal components like there is in the unit sphere bundle of a Riemannian manifold as described in [38] or [26]. Furthermore, since the metric on a sub-Riemannian manifold is not invertible, there are no normal coordinates.

A standard geometric obstacle to such inverse problems is presented by the presence of conjugate points. In [13], [32], and [22] the authors show that conjugate points generally inhibit stable inversion of the X-ray transform on Riemannian manifolds, with unconditional loss in two dimensions. Unfortunately, the conjugate points in the Heisenberg group are ubiquitous; the cut locus to any point passes through that point—a feature generic in sub-Riemannian geometry, where the exponential map is never a local diffeomorphism at the origin [47, p.222]. Therefore, standard tools for proving injectivity, such as Pestov energy methods, which typically require a positive-definite second fundamental form [26] do not apply without a closer look. We prove that, nonetheless, the X-ray transform on the Heisenberg group is injective.

Helgason [21] created a framework for describing a large class of such integral transforms with symmetry. Suppose $X$ and $Y$ are homogeneous spaces of a Lie group $G$ with closed subgroups $H$ and $K$ and consider the following double fibration:

$$\begin{tikzcd} & G \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ X = G/K & & Y = G/H \end{tikzcd}$$

Assuming the existence of $G$-invariant measures on $G$, $X$ and $Y$, define the abstract *Radon*

*Transform* of a suitable function $f : X \to \mathbb{R}$ by

$$Rf(x) := (\pi_* \circ p^*)\, f(x) = \int_H f(xh)dh; \quad x \in X$$

(See [20] for microlocal aspects of this picture.) The principle tasks to address are

- Characterize the range and kernel of $R$.
- If invertible, in what functions spaces is the inverse $R^{-1}$ bounded?
- Explicit inversion formulas for $R$ on suitable function spaces.
- What differential operators on $X$ are pushed forward by $R$ to differential operators on $Y$?

Inversion formulas for Radon Transforms are usually have the form

$$DR^*Rf = f$$

where $R^*$ is the adjoint of $R$ and $D$ is an operator on $X$ which is usually a function of distinguished $G$-invariant differential operators as in [3, 43]. We show that the Heisenberg X-ray transform $I$ is a Radon Transform for a natural double fibration. However, defining $I^*I$ poses technical problems—even on the domain $C_c(\mathbb{H})$—due to the singular nature of the sub-Riemannian exponential map; any neighborhood of the origin in $\mathbb{H}$ contains geodesics of arbitrary length. Instead, we focus on studying the transform $I$ directly. We observe a convenient identification of the space of geodesics with a quotient of the Heisenberg group (4.2) for which $I$ is a convolution operator. We apply the group Fourier transform on the Heisenberg group and its quotient to express $I$ essentially as a multiplication operator (Theorem 2), from which we deduce that $I$ is injective (Theorem 1).

## 1.2 Approach and Main Theorems

The Heisenberg geodesics exists for all time and are left-translates of helices and straight lines. Let $\mathcal{G}$ be the set of all Heisenberg geodesics without orientation, and $\mathcal{G}_\lambda$ the set of all geodesics having a fixed value $\lambda$, for the "charge" $\lambda$, which is a constant of motion. Left translation by any element $(z,t) \in \mathbb{H}$ is an isometry of $\mathbb{H}$ and so $\mathbb{H}$ acts on $\mathcal{G}$. This action does not change the value of $\lambda$, and is a transitive action on each leaf $\mathcal{G}_\lambda$, $\lambda \neq 0$. (It is not transitive on $\mathcal{G}_0$ since it does not change the direction of the line in the plane which the $\lambda = 0$ geodesic projects to.) These facts are verified by inspecting the exponential map in (A.19). Thus we can use $\mathbb{H}$ to parameterize $\mathcal{G}_\lambda$, $\lambda \neq 0$ by fixing a particular helix $\gamma_\lambda \in \mathcal{G}_\lambda$ and left-translating it about. We take this helix to be one whose projection is a circle of radius $|R| = 1/|\lambda|$ centered at the origin and parameterized by arclength. Thus our parameterization of that part of $\mathcal{G}$ having $\lambda \neq 0$ is

$$s \mapsto (z,t)\gamma_\lambda(s), \quad \gamma_\lambda(s) = \left(Re^{i(s/R)}, \tfrac{1}{2}sR\right) \in \mathbb{H};\ R = 1/\lambda. \tag{1.1}$$

Using this identification, we may parameterize geodesics by $(z,t,\lambda)$ as above, uniquely modulo the isotropy group $\Gamma_\lambda := \{(0, k\pi R^2) \in \mathbb{H} : k \in \mathbb{Z}\}$ stabilizing $\gamma_\lambda$, and write the X-ray transform concretely as

$$If(z,t,\lambda) := I_\lambda f(z,t) := \int_{\mathbb{R}} f\left((z,t)\gamma_\lambda(s)\right) ds, \quad f \in C_c(\mathbb{H}).$$

We ignore the case when $\lambda = 0$ where the geodesics are straight lines. Furthermore, since $\lambda < 0$ corresponds to a $\lambda > 0$ geodesic with opposite orientation, we will always take $\lambda > 0$. In Proposition 8, we prove that $I_\lambda : L^1(\mathbb{H}) \to L^1(\mathcal{G}_\lambda)$, with a natural measure on the codomain, is well-defined and bounded. In [46, p. 392], Strichartz proves indirectly that a function on the

Heisenberg group may not in general be recovered from its integrals over $\lambda = 0$ geodesics, but does not consider $\lambda \neq 0$ geodesics.

We may now state our main result:

**Theorem 1.** *The Heisenberg X-ray transform $I : L^1(\mathbb{H}) \to L^1(\mathcal{G}, d\mathcal{G})$ is injective. In particular, if $f \in L^1(\mathbb{H})$, and $I_\lambda f = 0$ for all $\lambda$ in a neighborhood of zero, then $f = 0$.*

We prove this result using harmonic analysis adapted to the group structure, modifying familiar results in Euclidean space. Consider, for example, the Radon and Mean Value Transforms on $\mathbb{R}^2$:

$$Rf(s,\theta) := \int_{\mathbb{R}} f\left(se^{i\theta} + ite^{i\theta}\right) dt, \qquad M^r f(z) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z + re^{i\theta}\right) d\theta \tag{1.2}$$

where, say, $f \in C_c(\mathbb{R}^2)$. Taking the Fourier Transforms in $s$ and $z$, respectively, yields

$$\mathcal{F}_{s\mapsto\sigma} Rf(\sigma,\theta) = \hat{f}(\sigma e^{i\theta}), \qquad\qquad \mathcal{F}_{z\mapsto\zeta} M^r f(\zeta) = J_0(r|\zeta|)\hat{f}(\zeta),$$

where $J_0$ is the zeroth-order Bessel function. These results are known as Fourier Slice Theorems, or Projection Slice Theorems [37]. They reveal that $R$, thought of as a projection onto $\{\theta\}$, becomes a restriction operator onto the "slice" $\sigma \to \sigma e^{i\theta}$ in the Fourier domain, and that $M^r$ becomes a multiplication operator by $J_0(r|\zeta|)$ when viewed in the Fourier domain. Fourier Slice Theorems exist for more general Radon transforms as well; for example, in [16, 27].

The Radon and Mean Value Transforms can interpreted as integration over straight lines or magnetic geodesics in Euclidean space. In the case of $\mathbb{H}$—which is a "flat" sub-Riemannian geometry—we prove a corresponding Fourier Slice Theorem for Heisenberg geodesics. We use the operator-valued group Fourier Transform $\mathcal{F}_{\mathbb{H}}$ associated to the Bargmann-Fock representation $\beta_h$ (defined in equation (3.1)), which has proven a useful tool, for example, by Nachman in [36] to find the fundamental solution for the wave operator for the Heisenberg Laplacian. The theory of $\mathcal{F}_{\mathbb{H}}$ is extensively developed in [14, 50]. In particular it has a Plancherel Theorem and Inversion Theorem [11, 13, 50]. In the Fourier domain of $\mathcal{F}_{\mathbb{H}}$, the Heisenberg X-ray transform is simultaneously a restriction and a multiplication operator:

**Theorem 2** (Heisenberg Fourier Slice Theorem)**.** *If $f \in L^1(\mathbb{H})$, then*

$$\left(\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}\left(I_\lambda f\right)\right)(n) = (2\pi/\lambda)\,\mathcal{J}_n \circ \left(\mathcal{F}_{\mathbb{H}}\left(f\right)\right)(n\lambda^2), \quad \forall n \in \mathbb{Z}\setminus\{0\}, \forall \lambda > 0.$$

The Heisenberg Fourier Slice Theorem is an equality of operators acting on Bargmann-Fock space (originally described in [6]),

$$\mathcal{H} := \left\{ F \in \mathrm{Hol}(\mathbb{C}) : \frac{1}{\pi}\int_{\mathbb{C}} |F(\zeta)|^2 e^{-|\zeta|^2} d\zeta < \infty \right\}. \tag{1.3}$$

$\mathcal{J}_n : \mathcal{H} \to \mathcal{H}$ is the operator

$$\mathcal{J}_n F(\zeta) = \frac{1}{2\pi i}\left(\frac{e}{|n|}\right)^{|n|/2} \oint z^{|n|-1} e^{-|n|\zeta/z} F(\zeta + z)\,dz, \tag{1.4}$$

where the contour is a circle around the origin oriented counterclockwise. Loosely speaking, the Heisenberg X-ray transform $I$ is “block-diagonalized” in $\lambda$ by the group Fourier Transform, and each block is essentially a multiple of $\mathcal{J}_n$.

The classical Fourier Slice Theorem for $R$ in (1.2) states that knowledge of $Rf$ for a fixed $\theta_0$ determines the Fourier transform $\hat{f}(\zeta)$ for all $\zeta \parallel \theta_0$. Similarly, the Heisenberg Fourier Slice Theorem says that knowledge of $I_\lambda f$ for fixed $\lambda$ determines the group Fourier transform $\mathcal{F}_{\mathbb{H}} f(h)$, up to multiplication by the operator $\mathcal{J}_n$, for all $h \in \lambda^2\mathbb{Z}^*$. Therefore, injectivity of $I$ follows once we show that

**Proposition 1.** The map $\mathcal{J}_n : \mathcal{H} \to \mathcal{H}$ is injective whenever $n$ is an odd integer.

Finally, in Section 6.1, we consider the ray transforms $I^{\epsilon}$ and $I^{i\epsilon}$ for the families of left-invariant compatible Riemannian and Lorentzian metrics $g_{\epsilon}$ and $g_{i\epsilon}$ parameterized by $\epsilon > 0$. We prove Fourier Slice Theorems (Theorem 7) for the compatible Riemannian and Lorentzian geodesics which we apply in the same way to show that the X-ray transforms $I^{\epsilon}$ and $I^{i\epsilon}$ are injective (Theorem 8). In [40], the authors prove that the X-ray transform is injective on step-2 Nilpotent Lie groups of rank greater than 3 with a left-invariant metric. However, this result assumes the existence of 2-dimensional totally geodesic surfaces, which excludes the Heisenberg group.

## 1.3 Heuristics

The heart of the proof is the following idea, which we describe informally. All of the relevant objects in this section will be defined in the sequel. We stress that this section is not entirely rigorous.

In analogy with the Peter-Weyl theorem, the Heisenberg Fourier Transform exhibits the following isomorphism of $L^2$ spaces

$$\mathcal{F}_{\mathbb{H}} : L^2(\mathbb{H}) \cong \int_{\mathbb{R}^*}^{\oplus} S_2 \, d\mu(\lambda); \quad d\mu(\lambda) = \pi^{-2}|\lambda| d\lambda$$

where $S_2 \cong \mathcal{H} \otimes \mathcal{H}^*$ is the space of Hilbert-Schmidt operators on $\mathcal{H}$. Similarly

$$\mathcal{F}_{\mathbb{H}} : L^1(\mathbb{H}) \to C\left(\mathbb{R}^*, S_{\infty}\right), \qquad \mathcal{F}_{\mathbb{H}/\Gamma} : L^1(\mathbb{H}/\Gamma) \to \bigoplus_{n \in \mathbb{Z}^*} S_{\infty}$$

where $S_\infty$ is the Banach space of compact operators on $\mathcal{H}$ and $C(\mathbb{R}^*, S_\infty)$ is the space of strongly continuous $S_\infty$-valued functions.

Given a parameterized curve $c(s)$ in the plane, its horizontal lift $\gamma(s)$ to $\mathbb{H}$ is defined by prescribing $\Theta(\frac{d}{ds}\gamma(s)) = 0$. If the curve $c(s)$ is closed, then its holonomy at a point is the change in the vertical variable of its horizontal lift to $\mathbb{H}$ after traversing one period. Since Heisenberg geodesics are horizontal lifts of magnetic geodesics in the plane to helices in $\mathbb{H}$, they are periodic modulo their holonomy. In Section 4.3 we show for fixed $\lambda = 1$ (and later proceed using a dilation argument) that the X-ray transform factors as

$$I_1 = I_1^{\mathrm{red}} \circ P,$$

where $P : L^1(\mathbb{H}) \to L^1(\mathbb{H}/\Gamma)$ is the periodization map and $I_1^{\mathrm{red}} : L^1(\mathbb{H}/\Gamma) \to L^1(\mathbb{H}/\Gamma)$ is the X-ray transform $I_1$ descended to the quotient $\mathbb{H}/\Gamma$. Intuitively, $P$ captures the periodicity of Heisenberg geodesics, and $I_1^{\mathrm{red}}$ captures their holonomy.

The following diagram commutes.

$$\begin{tikzcd}
L^1(\mathbb{H}) \arrow[r, "\mathcal{F}_{\mathbb{H}}"] \arrow[d, "P"] \arrow[dd, bend right=60, "I_1"'] & C(\mathbb{R}^*, S_\infty) \arrow[d, "\mathrm{res}"] \\
L^1(\mathbb{H}/\Gamma) \arrow[r, "\mathcal{F}_{\mathbb{H}/\Gamma}"] \arrow[d, "I_1^{\mathrm{red}}"] & \bigoplus_{n\in\mathbb{Z}^*} S_\infty \arrow[d, "2\pi\mathcal{J}_n"] \\
L^1(\mathbb{H}/\Gamma) \arrow[r, "\mathcal{F}_{\mathbb{H}/\Gamma}"] & \bigoplus_{n\in\mathbb{Z}^*} S_\infty
\end{tikzcd}$$

So on the Fourier domain, the periodization map $P$ is essentially a restriction map of continuous functions from $\mathbb{R}^*$ to $\mathbb{Z}^*$, and the reduced X-ray transform is a multiplier. (We ignore zero in this above diagram to avoid considering continuous functions on a non-Hausdorff space.) As $\lambda > 0$ varies, the restriction operator restricts to the varying set $\lambda^{-2}\mathbb{Z}^*$. Given $\lambda$ in a neighborhood of zero, we recover the Fourier Transform $\mathcal{F}_{\mathbb{H}} f$ of a function $f \in L^1(\mathbb{H})$ on the entire domain $\mathbb{R}^*$.

# Part II

# Preliminaries

# Chapter 2

# Heisenberg Geometry

## 2.1 The sub-Riemannian Structure

We define the sub-Riemannian metric on $\mathbb{H}$ by declaring the left-invariant vector fields

$$X = \partial_x - \frac{1}{2} y \partial_t \qquad\qquad Y = \partial_y + \frac{1}{2} x \partial_t \tag{2.1}$$

to be orthonormal, and the length of $T = \partial_t$ to be infinite. Then any finite length smooth path in $\mathbb{H}$ must be tangent to the nonintegrable distribution $\mathcal{D}_q := \text{Span}\{X_q, Y_q\}$, $q \in \mathbb{H}$. We call such a path *horizontal*. The length of a horizontal path equals the length of its projection to the plane by the map

$$\pi(x, y, t) = (x, y).$$

A minimizing Heisenberg geodesic is a shortest horizontal path joining two points of $\mathbb{H}$. That any two points in $\mathbb{H}$ are connected by a horizontal path is guaranteed by Chow's Theorem and the fact that $\mathcal{D}$ satisfies the Hörmander condition (i.e. $\mathcal{D}$ is bracket-generating).

The fiber quadratic Hamiltonian $H : T^*\mathbb{H} \to \mathbb{R}$ given in canonical coordinates by

$$H(x, y, t, p_x, p_y, p_t) = \frac{1}{2}\left((p_x - \tfrac{1}{2}yp_t)^2 + (p_y + \tfrac{1}{2}xp_t)^2\right); \tag{2.2}$$

generates the Heisenberg geodesics. By 'generate' we mean that any solution to Hamilton's equations for $H$ projects, via the canonical projection $T^*\mathbb{H} \to \mathbb{H}$, to a sub-Riemannian geodesic, and conversely, all Heisenberg geodesics arise this way [34, Sec 1.5]. If we want geodesics parameterized by arclength we only take solutions for which $H = 1/2$. (Thus, we define the unit cotangent bundle $U^*\mathbb{H}$ as the set of all $(q, p) \in T^*\mathbb{H}$ for which $H(q, p) = 1/2$.) These geodesics can be best understood by their projection under $\pi$ to the plane: they are circles or lines. Indeed

$$\dot{p}_t = -\frac{\partial H}{\partial t} = 0,$$

so that $\lambda := p_t$ is a constant of motion. (Note that $t$ is not a temporal variable, but rather the vertical variable in $\mathbb{H} \cong \mathbb{R}^3$.) If we interpret $\lambda$ as the charge $e$ of a particle, then $H$, viewed as a Hamiltonian on $T^*\mathbb{R}^2$, is the Hamiltonian for a particle of charge $e = \lambda$ traveling in the plane under the influence of a constant unit strength magnetic field. These solutions are well-known and easy to derive [34, p. 12] . When $H = 1/2$ they are circles of radius $R = 1/|\lambda|$ for $\lambda \neq 0$, and lines when $\lambda = 0$. In the next section, we find a concrete expression of all geodesics.

## 2.2 Left-Trivialization of the Cotangent Bundle $T^*\mathbb{H}$

In contrast with the Riemannian case, the sub-Riemannian geodesic flow has no natural realization on the tangent bundle. To avoid unnecessary choices (as mathematicians typically do), we work on $T^*\mathbb{H}$, and cast all the geometric information of interest there. To the horizontal distribution $\mathcal{D}$ defined in the previous section, associate the left-invariant contact form

$$\Theta := dt - \tfrac{1}{2}\left(xdy - ydx\right),$$

so that $\mathcal{D}_q = \ker \Theta_q$.

For any vector field $V$ on $\mathbb{H}$, define its momentum function $P_V : T^*\mathbb{H} \to \mathbb{R}$ by

$$P_V(\eta_q) := \eta_q(V_q); \quad \eta_q \in T_q^*\mathbb{H}.$$

On other words, $P_V$ identifies the vector $V_q$ at any point $q \in \mathbb{H}$ with its double dual, which maps $T_q^*\mathbb{H} \to \mathbb{R}$. The functions $P_X, P_Y, P_T : T^*\mathbb{H} \to \mathbb{R}$ are related to the canonical coordinates $(x, y, t, p_x, p_y, p_t)$ on $T^*\mathbb{H}$ by

$$\begin{cases} P_X = p_x - \tfrac{1}{2}yp_t \\ P_Y = p_y + \tfrac{1}{2}xp_t \\ P_T = p_t. \end{cases}$$

The utility of this formalism is that $(x+iy, t, P_X+iP_Y, P_T)$ are global left-invariant coordinates on the cotangent bundle $T^*\mathbb{H}$. These coordinates establish the following isomorphism,

$$T^*\mathbb{H} \cong \mathbb{H} \times \mathfrak{h}^*, \tag{2.3}$$

$$\eta_q \mapsto dL_{q^{-1}}^*(\eta_q)$$

called the *left-trivialization* of $T^*\mathbb{H}$. In particular, for any $\eta_q \in T_q^*\mathbb{H}$,

$$\eta_q = P_X(\eta_q)dx + P_Y(\eta_q)dy + P_T(\eta_q)\Theta,$$

—since$(dx, dy, \Theta)$ is the coframe dual to $(X, Y, T)$—which we simply write as

$$\eta_q = (P_X(\eta_q) + iP_Y(\eta_q), P_T(\eta_q)) \, .$$

The above identification is the meaning of the isomorphism 2.3.

The cotangent bundle $T^*\mathbb{H}$ carries a canonical symplectic structure. In the left-invariant coordinates $(x + iy, t, P_X + iP_Y, P_T)$, the Liouville form $\alpha$ is

$$\begin{aligned} \alpha &= p_x dx + p_y dy + p_t dt \\ &= P_X dx + P_Y dx + P_T \Theta. \end{aligned}$$

The canonical symplectic form $\omega = -d\alpha$ on $T^*\mathbb{H}$ is then

$$\begin{aligned} \omega &= dx \wedge dp_x + dy \wedge dp_y + dt \wedge dp_t \\ &= dx \wedge dP_X + dy \wedge dP_Y + \Theta \wedge dP_T + P_T dx \wedge dy \end{aligned}$$

since $d\Theta = dx \wedge dy$.

## 2.3 Derivation of the Geodesic Flow

We find the geodesics for the sub-Riemannian and its compatible Riemannian, and Lorentzian metrics simultaneously. We see that all these cases, the geodesics only differ by a constant motion in the vertical direction.

For $\tau \in \mathbb{R}$, consider the following perturbation of the Hamiltonian in 2.2

$$\begin{aligned} &H_\tau : T^*\mathbb{H} \to \mathbb{R} \\ &H_\tau(q, p) = \frac{1}{2}(P_X^2 + P_Y^2 + \tau P_T^2). \end{aligned}$$

The term $\frac{1}{2}\tau P_T^2$ represents the energy "penalty" incurred by integral curves of the Hamiltonian flow that leave the contact distribution $\mathcal{D}$ spanned pointwise by $X$ and $Y$.

**Remark 1.** The Hamiltonian $H_\tau$ models the following cases:

When $\tau = \epsilon^2$, it is the Hamiltonian for the Riemannian metric $g_\epsilon := dx^2 + dy^2 + \left(1/\epsilon^2\right)\Theta^2$.

When $\tau = -\epsilon^2$, it is the Hamiltonian for the Lorentzian metric $g_{i\epsilon} := dx^2 + dy^2 - \left(1/\epsilon^2\right)\Theta^2$.

When $\tau = 0$, it is the Hamiltonian for the sub-Riemannian metric $g := dx^2 + dy^2|_{\mathcal{D}}$.

The Poisson bracket $\{,\}$ of functions $f, g \in C^\infty(T^*\mathbb{H})$ is given in canonical coordinates by

$$\{f, g\} := \sum_{q \in \{x,y,t\}} \frac{\partial f}{\partial q}\frac{\partial g}{\partial p_q} - \frac{\partial f}{\partial p_q}\frac{\partial g}{\partial q}.$$

We have the following Poisson bracket relations on $T^*\mathbb{H}$:

$$\begin{aligned} \{x, P_X\} &= 1 & \qquad \{t, P_X\} &= -\tfrac{1}{2}y \\ \{y, P_Y\} &= 1 & \qquad \{t, P_Y\} &= \tfrac{1}{2}x \\ \{t, P_T\} &= 1 & \qquad \{P_X, P_Y\} &= -P_T \end{aligned}$$

with all remaining relations zero.

Hamilton's equations, written compactly as $dH = \iota_{X_H}\omega$, are equivalent to

$$\dot{f} = \{f, H_\tau\}$$

as $f \in C^\infty(T^*\mathbb{H})$ ranges over the coordinate functions of $T^*\mathbb{H}$:

$$\begin{aligned}
\dot{x} &= P_X & \dot{P}_X &= -P_T P_Y \\
\dot{y} &= P_Y & \dot{P}_Y &= P_T P_X \\
\dot{t} &= -\frac{1}{2} y P_X + \frac{1}{2} x P_Y + \tau P_T & \dot{P}_T &= 0.
\end{aligned}$$

Let $z = x + iy$. Solving these equations with initial condition

$$q_0 = (z(0), t(0)) = 0 \in \mathbb{H}; \qquad p_0 = (P_X(0) + iP_Y(0), P_T(0)) = (re^{i\theta}, \lambda) \in T_0^*\mathbb{H};$$

yields

$$\begin{aligned}
z(s) &= r\frac{ie^{i\theta}}{\lambda}(1 - e^{i\lambda s}) \\
t(s) &= \int_0^s (-\tfrac{1}{2} y P_X + \tfrac{1}{2} x P_Y + \tau P_T) ds' \\
&= \frac{1}{2}\int_0^s (x\dot{y} - \dot{x}y) ds' + \tau\lambda s \\
&= \frac{1}{2}\int_0^s \mathrm{Im}(\bar{z}\dot{z}) ds' + \tau\lambda s \\
&= r^2 \frac{\lambda s - \sin\lambda s}{2\lambda^2} + \tau\lambda s.
\end{aligned}$$

Furthermore

$$P_T = \lambda$$

$$(P_X + iP_Y) = re^{iP_T s}(P_X(0) + iP_Y(0)) = re^{i\lambda s} e^{i\theta} = re^{i(\lambda s + \theta)}.$$

So the integral curve $\gamma^\tau_{(q_0,p_0)} : \mathbb{R} \to T^*\mathbb{H}$ for the Hamiltonian flow of $H_\tau$ with initial condition $q_0 = 0$ and $p_0 = (re^{i\theta}, \lambda)$, and for which $\lambda \neq 0$, is

$$\gamma^\tau_{(q_0,p_0)}(s) = \left( r\frac{ie^{i\theta}}{\lambda}(1 - e^{i\lambda s}),\ \tau\lambda s + r^2 \frac{\lambda s - \sin(\lambda s)}{2\lambda^2};\ re^{i(\lambda s + \theta)},\ \lambda \right) \in \mathbb{H} \times \mathfrak{h}^* \cong T^*\mathbb{H}. \tag{2.4}$$

For the case in which $\lambda = 0$, just take the pointwise limit of 2.4 as $\lambda \to 0$. Since the Hamiltonian is left-invariant, we may left translate this integral curve by $(z,t) \in \mathbb{H}$ to account for initial condition $q_0 = (z,t)$.

The sub-Riemannian flow $\varphi_s : U^*\mathbb{H} \to U^*\mathbb{H}$, is then given by

$$\varphi_s\left(z,t;\, e^{i\theta},\lambda\right) = \left((z,t)\left(e^{i\theta}\frac{(e^{i\lambda s}-1)}{i\lambda},\, \frac{\lambda s - \sin(\lambda s)}{2\lambda^2}\right);\, e^{i(\lambda s+\theta)},\, \lambda\right)$$
$$\in \mathbb{H}\times\mathbb{S}^1\times\mathbb{R} \cong U^*\mathbb{H},$$

for $\lambda \neq 0$, otherwise

$$\varphi_s\left(z,t;e^{i\theta},0\right) = \left((z,t)\left(se^{i\theta},0\right);e^{i\theta},0\right).$$

The sub-Riemannian flow talks to the group structure in the following ways:

Left-Invariance:

$$\varphi_s(z,t;\theta,\lambda) = (z,t)\varphi_s(0;\theta,\lambda)$$

Roration-Invariance:

$$\mathcal{R}_\phi\varphi_s(z,t;\theta,\lambda) = \varphi_s(e^{i\phi}z,t;\theta+\phi,\lambda)$$

Here $\mathcal{R}_\phi(z,t;\theta,\lambda) := (e^{i\phi}z,t;\theta+\phi,\lambda)$.

## 2.4 Double Fibration of the Heisenberg Motion Group

In this section we observe how the X-ray transform fits into Helgason's double fibration picture.

Two covectors $(q,p)$ and $(q',p') \in U^*\mathbb{H}$ determine the same oriented geodesic if and only if $\varphi_{s_0}(q,p) = (q',p')$ for some $s_0 \in \mathbb{R}$. In this case, say $(q,p) \sim_\varphi (q',p')$. We identify the space of oriented geodesics with the quotient

$$\mathcal{G} \cong U^*\mathbb{H}/\sim_\varphi . \tag{2.5}$$

In Appendix A.3, we prove that $\mathcal{G}$ is a symplectic manifold for which of the quotient map $\pi_{\mathcal{G}} : U^*\mathbb{H} \to \mathcal{G}$ is a symplectic reduction cotangent bundle $T^*\mathbb{H}$.

Consider the following double fibration:

$$\begin{tikzcd} & U^*\mathbb{H} \arrow[dl, "\pi_{\mathbb{H}}"'] \arrow[dr, "\pi_{\mathcal{G}}"] & \\ \mathbb{H} & & \mathcal{G} \end{tikzcd}$$

Define the pullback and—with respect to the volume forms $d\mathrm{vol}_{\mathcal{G}}$ and $d\mathrm{vol}_{U^*\mathbb{H}}$—the pushforward on functions:

$$\pi_{\mathbb{H}}^* : C_c(\mathbb{H}) \to C(HM) \qquad\qquad (\pi_{\mathcal{G}})_* : C_c(U^*\mathbb{H}) \to C_c(\mathcal{G})$$

We would like to define the X-ray transform as $I = (\pi_{\mathcal{G}})_* \circ \pi_{\mathbb{H}}^*$. However, the fibers of $U^*\mathbb{H}$ are not compact and neither of the above maps are proper, so it's not clear that $(\pi_{\mathcal{G}})_* \circ \pi_{\mathbb{H}}^*$ makes sense. Instead of introducing a cuffoff function, we restrict this picture in the $\lambda$ variable as follows.

The additional ingredient that we will need is the Heisenberg Motion group $HM := \mathbb{H} \rtimes U(1)$, defined in Appendix A.1, which is (the identity component of) the isometry group for the sub-Riemannian metric on $\mathbb{H}$. The action of $HM$ preserves the Hamiltonian $H = \frac{1}{2}\left(P_X^2 + P_Y^2\right)$ and therefore it acts on $U^*\mathbb{H}$ on the left via

$$\Big((z,t);\, e^{i\theta}\Big) \cdot \Big((z_0,t_0);\, e^{i\theta_0}, \lambda\Big) = \Big((z,t)(e^{i\theta}z_0, t_0);\, e^{i(\theta+\theta_0)}, \lambda\Big)$$

Recall that our model sub-Riemannian geodesic with $\lambda \neq 0$ is

$$\gamma_\lambda(s) = \left(Re^{is/R}, sR/2\right);\ R = 1/\lambda.$$

The geodesics $\gamma_\lambda$ is an integral curve of a Hamiltonian flow $H$ with initial condition

$$p_\lambda = \left((z_0, t_0);\ e^{i\theta_0}, \lambda\right) = \left((\lambda^{-1}, 0);\ i, \lambda\right) \in U^*\mathbb{H} \cong \mathbb{H} \times \mathbb{S}^1 \times \mathbb{R}.$$

If we identify $HM$ with its orbit in $U^*\mathbb{H}$ over $p_\lambda \in U^*\mathbb{H}$,

$$\Phi_{p_\lambda} : HM \to U^*\mathbb{H} :\ \left((z,t);\ e^{i\theta}\right) \mapsto \left((z,t);\ e^{i\theta}\right) \cdot p_\lambda$$

then we obtain the following restriction of the previous double fibration:

$$\begin{tikzcd} & HM \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\pi_\lambda"] & \\ \mathbb{H} & & \mathcal{G}_\lambda \end{tikzcd}$$

The sub-Riemannian geodesics are exactly orbits in $U^*\mathbb{H}$ of certain 1-parameter subgroups of $HM$, analogous to the relationship between to magnetic geodesics in $\mathbb{R}^2$ and 1-parameter subgroups of $SE(2)$. Indeed, the sub-Riemannian flow has the form

$$\varphi_s\left(z, t;\ e^{i\theta}, \lambda\right) = \left(z, t;\ e^{i\theta}\right) e^{s(\mathbb{X}+\lambda V)} \cdot ((0,0);\ 1, \lambda) \tag{2.6}$$

(with the vector fields $\mathbb{X}$ an $V$ defined in the Appendix A.1) and one may check directly that

$$e^{s(\mathbb{X}+\lambda V)} := \left(\left(\frac{(e^{i\lambda s} - 1)}{i\lambda}, \frac{\lambda s - \sin(\lambda s)}{2\lambda^2}\right);\ e^{i\lambda s}\right)$$

is a 1-parameter subgroup of $HM$. Thus, the projected curve $s \mapsto \mathrm{pr}\left(e^{s(\mathbb{X}+\lambda V)}\right) \in \mathbb{H}$ is the geodesic with initial condition $((0,0); 1, \lambda)$, and its isotropy group in $HM$ is

$$\{e^{s(\mathbb{X}+\lambda V)} : s \in \mathbb{R}\} \subset HM.$$

Let $HM_\lambda$ be the stabilizer of $\gamma_\lambda$ for the action of $HM$. Since $p_\lambda = \big((\lambda^{-1}, 0); i\big) \cdot ((0,0); 1, \lambda)$, we have

$$HM_\lambda = AD_{(\lambda^{-1},0;i)}\{e^{s(\mathbb{X}+\lambda V)} : s \in \mathbb{R}\} = \Big\{ \Big(0, \tfrac{s}{2\lambda};\, e^{i\lambda s}\Big) : s \in \mathbb{R}\Big\}.$$

For $\Gamma_\lambda := \{(0, k\pi\lambda^{-2}) \in \mathbb{H} : k \in \mathbb{Z}\}$, consider the maps

$$\begin{tikzcd}
\mathbb{H} \arrow[r, hook, "i"] \arrow[d] & HM \arrow[r, "\Phi_{p_\lambda}"] \arrow[d] \arrow[dr, "\pi_\lambda"] & \mathrm{Orb}_{HM}(p_\lambda) \arrow[d, "\pi_{\mathcal{G}}"] \\
\mathbb{H}/\Gamma_\lambda \arrow[r, "i_\lambda"] & HM/HM_\lambda \arrow[r] & \mathcal{G}_\lambda
\end{tikzcd}$$

where $i : \mathbb{H} \hookrightarrow HM$ is the inclusion, and

$$HM/HM_\lambda \to \mathcal{G}_\lambda; \quad (z, t; e^{i\theta})HM_\lambda \mapsto (z, t; e^{i\theta})\gamma_\lambda$$

is the orbit-stabilizer isomorphism. Using the relation

$$\big((0,t); 1\big)HM_\lambda = \Big((0,0);\, e^{-2i\lambda^2 t}\Big)\, HM_\lambda; \quad t \in \mathbb{R},$$

(which is the helical symmetry $(0, t; 1)\cdot\gamma_\lambda = (0, 0; e^{-2i\lambda^2 t})\cdot\gamma_\lambda$ on the level of cosets) the map

$$i_\lambda : \mathbb{H}/\Gamma_\lambda \to HM/HM_\lambda;\ (z,t)\Gamma_\lambda \mapsto (z, t;\ 1)HM_\lambda,$$

is well-defined and is an isomorphism with inverse given by

$$HM/HM_\lambda \to \mathbb{H}/\Gamma_\lambda : (z, t;\ e^{i\theta})HM_\lambda \to (z, t - \tfrac{\theta}{2\lambda^2})\Gamma_\lambda.$$

Therefore we identity

$$\psi_\lambda : \mathbb{H}/\Gamma_\lambda \to \mathcal{G}_\lambda; \quad (z,t)\Gamma_\lambda \mapsto (z,t)\gamma_\lambda.$$

With the volume forms $d\mathcal{G}_\lambda = \lambda dx \wedge dy \wedge dt$ and $d\mathrm{vol}_{HM} := dx \wedge dy \wedge dt \wedge d\theta$ we define the following pullbacks and pushforwards on functions.

$$\mathrm{pr}^* : C_c(\mathbb{H}) \to C_c(HM) \qquad\qquad (\pi_\lambda)_* : C_c(HM) \to C_c(\mathcal{G}_\lambda)$$

Here the pushforward $(\pi_\lambda)_*$ is defined by

$$\begin{aligned}\int_{\mathcal{G}_\lambda} (\pi_\lambda)_* f g d\mathcal{G}_\lambda &= \int_{HM} f \pi_\lambda^* g d\mathrm{vol}_{HM} \\ &= \int_{\frac{HM}{HM_\lambda}} \int_{\mathbb{R}} f\left((q; e^{i\theta}) e^{s(\mathbb{X}+\lambda V)}\right) g(q; e^{i\theta}) ds \, d(q; e^{i\theta}) HM_\lambda, \text{ see 2.7} \\ &= \int_{\mathbb{H}/\Gamma_\lambda} \int_{\mathbb{R}} f\left(q\gamma_\lambda(s); e^{i(\theta+\lambda s)}\right) ds \, d(q\Gamma_\lambda)\end{aligned}$$

We may then show that the following is an equivalent definition of the X-ray transform.

**Definition 1** (The X-ray Transform)**.** For $f \in C_c(\mathbb{H})$,

$$\begin{aligned}I_\lambda f(z,t) :=& \psi_\lambda^* \circ (\pi_\lambda)_* \circ \mathrm{pr}^* f(z,t) \\ =& \int_{\mathbb{R}} \mathrm{pr}^* f\left((z,t;1) e^{s(\mathbb{X}+\lambda V)}\right) ds HM_{\gamma_\lambda} \\ =& \int_{\mathbb{R}} f\left((z,t)\gamma_\lambda(s)\right) ds\end{aligned}$$

**Theorem** ([11])**.** *If $G$ is a locally compact unimodular Lie group with left Haar measure $dx$, and $H$ is a closed subgroup with Haar measure $d\xi$, then there exists a $G$-invariant measure $d(xH)$ on $G/H$. This measure is unique up to a constant factor, and if this factor is suitably chosen we have*

$$\int_G f(x) dx = \int_{G/H} \int_H f(x\xi) d\xi d(xH); \quad f \in C_c(G). \tag{2.7}$$

Applying 2.7 with $G = HM$ and $H = HM_\lambda$ yields a version of Santaló's formula.

$$\begin{aligned}\int_{HM} f(q;\, e^{i\theta}) d\mathrm{vol}_{HM} &= \int_{\frac{HM}{HM_\lambda}} \int_{\mathbb{R}} f\left((q; e^{i\theta}) e^{s(\mathbb{X}+\lambda V)}\right) ds\, d(q; e^{i\theta}) HM_\lambda \\ &= \int_{\mathbb{H}/\Gamma_\lambda} \int_{\mathbb{R}} f\left(q\gamma_\lambda(s)\right) ds\, dq\Gamma_\lambda\end{aligned}$$

# Chapter 3

# Heisenberg Harmonic Analysis

## 3.1 Bargmann-Fock Representation

We start by giving a brief description of the representation theory of the Heisenberg group. A more detailed discussion can be found in [10]. Denote by $\mathcal{U}(\mathcal{H})$ the set of unitary operators on Bargmann-Fock space, defined in (1.3). For each $h \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, the map (motivated in Section B.6)

$$\beta_h : \mathbb{H} \to \mathcal{U}(\mathcal{H})$$

given by

$$\beta_h(z,t)F(\zeta) := e^{2iht - \sqrt{h}\zeta\bar{z} - \frac{h}{2}|z|^2} F(\zeta + \sqrt{h}z), \quad h > 0, \tag{3.1}$$

and $\beta_h(z,t) = \beta_{|h|}(\bar{z}, -t)$ for $h < 0$, is a strongly continuous unitary representation of the Heisenberg group on $\mathcal{H}$. Moreover, it is known that these representations are irreducible, and by the Stone-von Neumann Theorem, up to unitary equivalence, these are all of the irreducible unitary representations on $\mathbb{H}$ that are nontrivial on the center of $\mathbb{H}$ [10].

## 3.2 The Group Fourier Transform

We define the group Fourier Transform of an integrable function on $\mathbb{H}$. Denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$. The Heisenberg Fourier Transform of $f \in L^1(\mathbb{H})$ is the operator-valued function

$$\mathcal{F}_{\mathbb{H}}(f) : \mathbb{R}^* \to \mathcal{B}(\mathcal{H})$$
$$\mathcal{F}_{\mathbb{H}}(f)(h) := \int_{\mathbb{H}} f(q)\beta_h(q)^* dq$$

where the integral is taken in the Bochner sense [50, p. 11]. Think of $h$ as a semi-classical parameter.

**Remark 2.** Many authors define $\mathcal{F}_{\mathbb{H}}$ alternatively with the Schrödinger representations. Our definition seems more natural for studying the X-ray transform due to the simplicity of (1.4), and is equivalent by conjugation with a unitary intertwining map anyway; the choice is largely a personal preference. We also normalize the representations $\beta_h$ in such a way that they all act on the same space $\mathcal{H}$, rather than a family of spaces parameterized by $h \in \mathbb{R}^*$, as in [10].

If $f \in L^1(\mathbb{H}) \cap L^2(\mathbb{H})$, then $\mathcal{F}_{\mathbb{H}}(f)(h)$ is a Hilbert-Schmidt operator on $\mathcal{H}$ [13, 14]. Let $S_2$ denote the space of Hilbert-Schmidt operators on $\mathcal{H}$, and define the Hilbert Space $L^2(\mathbb{R}^*, S_2; d\mu) = L^2(S_2)$ via the inner product

$$\langle A, B\rangle_{L^2(S_2)} := \int_{\mathbb{R}^*} \operatorname{tr}\left(A(h)B(h)^*\right) d\mu(h), \quad d\mu = \pi^{-2}|h|dh.$$

**Theorem** (Plancherel Theorem). *If $f \in L^1(\mathbb{H}) \cap L^2(\mathbb{H})$, then $||f||_{L^2(\mathbb{H})} = ||\mathcal{F}_{\mathbb{H}} f||_{L^2(S_2)}$.*

**Theorem** (Fourier Inversion Theorem)**.** *If $f \in S(\mathbb{H})$, Schwartz space on $\mathbb{R}^3$, then*

$$f(q) = \int_{\mathbb{R}^*} \operatorname{tr}\left(\beta_h(q)\mathcal{F}_{\mathbb{H}} f(h)\right) d\mu(h), \quad q \in \mathbb{H}. \tag{3.2}$$

Thus $\mathcal{F}_{\mathbb{H}}$ extends to an isometry from $L^2(\mathbb{H})$ into $L^2(S_2)$. In fact, it is onto as well. Furthermore if $f \in L^1(\mathbb{H})$, then by convolving $f$ with an approximation of identity, we may use (3.2) to prove $\mathcal{F}_{\mathbb{H}}$ is injective on $L^1(\mathbb{H})$.

While the definition above is sufficient for our purposes, we remark that $\mathcal{F}_{\mathbb{H}}$ has been extended to much more general classes of function such as tempered distributions [4]. In [5, 48], and much more generally in [28] the authors use the group Fourier transform to develop theory of pseudo-differential operators.

## 3.3 Fourier Transform on the Reduced Heisenberg Group

For any $\tau > 0$, the set $\tau\Gamma := \{(0, k\tau\pi) \in \mathbb{H} : k \in \mathbb{Z}\}$, is a discrete subgroup of the center of $\mathbb{H}$. Since $\beta_h(z,t) = e^{2iht}\beta_h(z,0)$, the representation $\beta_h$ descends to the so-called reduced Heisenberg group $\mathbb{H}/\tau\Gamma$ if and only if $h \in \tau^{-1}\mathbb{Z}^*$. To a function $g \in L^1(\mathbb{H}/\tau\Gamma)$, we associate the *reduced Fourier Transform*, defined as

$$\mathcal{F}_{\mathbb{H}/\tau\Gamma}(g) : \mathbb{Z}^* \to \mathcal{B}(\mathcal{H})$$
$$\mathcal{F}_{\mathbb{H}/\tau\Gamma}(g)(n) := \int_{\mathbb{H}/\tau\Gamma} g(q)\beta_{n/\tau}(q)^* dq,$$

where $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$.

**Remark 3.** The reduced Fourier Transform defined above is not invertible unless we also consider the representations $(z,t) \mapsto e^{iz\cdot\xi}$; $\xi \in \mathbb{C}$, which are trivial on the center, in the definition.

(Indeed, if $\partial_t f(z,t) = 0$, then $\mathcal{F}_{\mathbb{H}/\tau\Gamma} f = 0$.) This extension is not necessary for our purposes.

For simplicity, we will take $\tau = 1$ and rescale as necessary in the sequel. Consider the following subspaces of $L^2(\mathbb{H}/\Gamma)$

$$\begin{aligned} L^2(\mathbb{C}) \cong& \{f \in L^2(\mathbb{H}/\Gamma) : f(z,t) = f(z,0),\ \forall (z,t) \in \mathbb{H}/\Gamma\} \\ {}^0L^2(\mathbb{H}/\Gamma) :=& \{f \in L^2(\mathbb{H}/\Gamma) : \int_0^\pi f(z,t)dt = 0,\ \forall z \in \mathbb{C}\}. \end{aligned} \tag{3.3}$$

**Lemma 1.** *We have the orthogonal decomposition*

$$L^2(\mathbb{H}/\Gamma) \cong L^2(\mathbb{C}) \oplus {}^0L^2(\mathbb{H}/\Gamma). \tag{3.4}$$

*Proof.* If $f \in L^2(\mathbb{H}/\Gamma)$, let $f_0(z,t) := \frac{1}{\pi}\int_0^\pi f(z,t)dt$ and $g = f - f_0$, then $f_0 \in L^2(\mathbb{C})$ and $g \in {}^0L^2(\mathbb{H}/\Gamma)$.

Furthermore, for arbitrary $f_0 \in L^2(\mathbb{C})$, and $g \in {}^0L^2(\mathbb{H}/\Gamma)$,

$$\int_{\mathbb{H}/\Gamma} f_0(z,t)\overline{g(z,t)}d(z,t)\Gamma = \int_{\mathbb{C}} f_0(z) \int_0^\pi \overline{g(z,t)}dtdz = 0.$$

$\square$

Note that the above decomposition remains true for $L^2(\mathbb{H}/\tau\Gamma)$, $\tau > 0$.

See [50, Ch. 4] for a proof of the following theorem.

**Theorem** (Plancherel Theorem for the Reduced Fourier Transform)**.** *The reduced Fourier Transform $\mathcal{F}_{\mathbb{H}/\Gamma}$ is an isometric isomorphism between ${}^0L^2(\mathbb{H}/\Gamma)$ and $L^2(\mathbb{Z}^*, S_2)$.*

The author could not find a source that explicitly states the following inversion theorem for the reduced Fourier Transform on $\mathbb{H}/\Gamma$, so we provide one here:

**Theorem 3.** *Let* $\Gamma = \{(0,\pi k) \in \mathbb{C}\times\mathbb{R} : k \in \mathbb{Z}\}$. *For* $f \in S(\mathbb{H}/\Gamma)$*, the usual Schwartz class, with* $\int_0^\pi f(z,t)dt = 0$

$$f(z,t) = \frac{1}{\pi^2}\sum_{n\in\mathbb{Z}^*} |n|\mathrm{Tr}\left(\beta_n(z,t)\mathcal{F}_{\mathbb{H}/\Gamma}(f)(n)\right). \tag{3.5}$$

*Proof Sketch.* We use the inversion theorem for the Weyl Transfom $\widetilde{W}_h g := \int_{\mathbb{C}} g(z)\beta_h(z,0)^* dz$ associated to the Bargmann-Fock representation (see in B.31 in the Appendix) which says that

$$g(z) = \frac{|\lambda|}{\pi}\mathrm{Tr}\left(\beta_\lambda(z,0)^*\widetilde{W}_\lambda(g)\right), \quad g \in S(\mathbb{C}) \tag{3.6}$$

and the fact that

$$\begin{aligned}
\mathrm{Tr}\left(\beta_n(z,t)\mathcal{F}_{\mathbb{H}/\Gamma}(f)(n)\right) &= \pi\mathrm{Tr}\left(\beta_n(z,0)\left(\widetilde{W}_n\overline{f_n}\right)^*\right)e^{2int} \\
&= \pi\mathrm{Tr}\left(\beta_n(z,0)^*\left(\widetilde{W}_n\overline{f_n}\right)\right)e^{2int} \\
&= \frac{\pi^2}{|n|}f_n(z)e^{2int}
\end{aligned}$$

where $f_n(z) := \frac{1}{\pi}\int_0^\pi f(z,t)e^{-2int}dt$. □

## 3.4 Convolution and Fourier Transform of Measures

Let $G = \mathbb{H}$ or $\mathbb{H}/\tau\Gamma$. Define

$$f * g\,(q) := \int_G f\,(p)\,g(qp^{-1})dp; \quad f,g \in L^1(G). \tag{3.7}$$

Similarly, given a Radon measure $\nu$ on $G$ and $f \in L^1(G)$, let

$$\nu * f(q) := \int_G f(qp^{-1})d\nu(p). \tag{3.8}$$

The Fourier Transform interchanges convolution and operator composition:

$$\mathcal{F}_G\left(f * g\right)(h) = \mathcal{F}_G(f)(h) \circ \mathcal{F}_G(g)(h); \quad f, g \in L^1(G). \tag{3.9}$$

If $\nu$ is a compactly supported finite Radon measure on $G$, then we may define its Fourier Transform

$$\mathcal{F}_G\left(\nu\right)(h) := \int_G \beta_h(q)^* d\nu(q). \tag{3.10}$$

Then 3.9 extends to convolution of measures and functions:

$$\mathcal{F}_G\left(\nu * f\right)(h) = \mathcal{F}_G(\nu)(h) \circ \mathcal{F}_G(f)(h). \tag{3.11}$$

For more details on group convolution and Fourier Transforms of Radon measures, see [10] or [45].

## 3.5 Lemmas on the group Fourier Transform

We now prove a few general properties of the group Fourier Transform. In the first, we observe how the Fourier Transforms behave with respect to dilations.

**Lemma 2** (Dilation Property)**.** *Fix* $\lambda > 0$.

*If* $f \in L^1(\mathbb{H})$*, then*

$$\mathcal{F}_{\mathbb{H}}\left(\delta_\lambda^* f\right)(h) = \lambda^{-4} \mathcal{F}_{\mathbb{H}}(f)(h/\lambda^2), \quad \forall h \in \mathbb{R}^*.$$

*If* $g \in L^1(\mathbb{H}/\tau\Gamma)$ *for* $\tau > 0$ *then* $\delta_\lambda^* g \in L^1(\mathbb{H}/\lambda^2\tau\Gamma)$ *and*

$$\mathcal{F}_{\mathbb{H}/(\lambda^2\tau)\Gamma}\left(\delta_\lambda^* g\right)(n) = \lambda^{-4} \mathcal{F}_{\mathbb{H}/\tau\Gamma}(g)(n), \quad \forall n \in \mathbb{Z}^*.$$

We expect the above exponent of $\lambda$ because the homogeneous dimension of the Heisenberg group is $4$.

*Proof.*

$$\mathcal{F}_{\mathbb{H}}\left(\delta_\lambda^* f\right)(h) = \int_{\mathbb{H}} f(\lambda z, \lambda^2 t)\beta_h(z,t)^* d(z,t) = \lambda^{-4}\int_{\mathbb{H}} f(z,t)\beta_h\left(\lambda^{-1}z, \lambda^{-2}t\right)^* d(z,t)$$
$$= \lambda^{-4}\int_{\mathbb{H}} f(z,t)\beta_{h/\lambda^2}(z,t)^* d(z,t) = \lambda^{-4}\mathcal{F}\left(f\right)(h/\lambda^2),$$

and the proof of for the reduced Fourier Transform is nearly identical. □

Next is a Poisson Summation formula for $\mathbb{H} \to \mathbb{H}/\Gamma$ - a consequence of the classical version. The author has not found a reference for this version, but does not believe it is new.

**Lemma 3** (Poisson Summation Formula ). *If $f \in L^1(\mathbb{H})$, then*

$$\mathcal{F}_{\mathbb{H}/\Gamma}\left(Pf\right)(n) = \mathcal{F}_{\mathbb{H}}(f)\left(n\right), \quad \forall n \in \mathbb{Z}^*. \tag{3.12}$$

*More generally, if $\tau > 0$, and $P_{\tau\Gamma}$ is the Poisson summation for for the discrete subgroup $\tau\Gamma = \{(0, k\pi\tau) : k \in \mathbb{Z}\}$, we have*

$$\mathcal{F}_{\mathbb{H}/\tau\Gamma}\left(P_{\tau\Gamma} f\right)(n) = \mathcal{F}_{\mathbb{H}}(f)\left(n/\tau\right), \quad \forall n \in \mathbb{Z}^*. \tag{3.13}$$

*Proof.* For $F, G \in \mathcal{H}$,

$$\begin{aligned}
\langle \mathcal{F}_{\mathbb{H}/\Gamma}\left(Pf\right)(n)F, G\rangle_{\mathcal{H}} &:= \int_{\mathbb{H}/\Gamma} \sum_{k\in\mathbb{Z}} f(z, t+k\pi)\langle \beta_n(z,t)^* F, G\rangle_{\mathcal{H}} d(z,t)\Gamma \\
&= \int_{\mathbb{H}/\Gamma} \sum_{k\in\mathbb{Z}} f(z, t+k\pi)\langle \beta_n(z,t+k\pi)^* F, G\rangle_{\mathcal{H}} d(z,t)\Gamma, \\
&\quad (\text{since } \beta_n(z,t) = e^{2int}\beta_n(z,0),\,) \\
&= \int_{\mathbb{H}} f(z,t)\langle \beta_n(z,t)^* F, G\rangle_{\mathcal{H}} d(z,t),
\end{aligned}$$

where the third equality follows from (4.8), and the fact that $f(z,t)\langle\beta_n(z,t)^*F, G\rangle_{\mathcal{H}} \in L^1(\mathbb{H})$ by the Cauchy-Schwartz inequality. Since $F$ and $G$ were arbitrary, (3.12) follows from the definition of the Bochner integral.

Next, observe that

$$P_{\tau\Gamma} f = \delta^*_{\sqrt{\tau}} P^1 \delta^*_{1/\sqrt{\tau}} f, \quad f \in C_c(\mathbb{H}).$$

Then

$$\begin{aligned}
\mathcal{F}_{\mathbb{H}/\tau\Gamma}\left(P_{\tau\Gamma} f\right)(n) &= \mathcal{F}_{\mathbb{H}/\tau\Gamma}\left(\delta^*_{\sqrt{\tau}} P^1 \delta^*_{1/\sqrt{\tau}} f\right)(n) \\
&= \tau^{-2} \mathcal{F}_{\mathbb{H}/\tau\Gamma}\left(P^1 \delta^*_{1/\sqrt{\tau}} f\right)(n) \\
&= \tau^{-2} \mathcal{F}_{\mathbb{H}}\left(\delta^*_{1/\sqrt{\tau}} f\right)(n) \\
&= \mathcal{F}_{\mathbb{H}}\left(f\right)(n/\tau)
\end{aligned}$$

as desired. □

## 3.6 Matrix Coefficients

Most of the discussion here follows Thangavelu, but we record the following facts to establish our notation and perspective.

We describe an important class of functions which are the cylindrical harmonics of the Heisenberg group.

**Definition 2** (Matrix Coefficients)**.** The matrix coefficients of the Bargmann-Fock representation with respect to the standard basis of monomials $\{\omega_k\}$ are defined as

$$M^h_{jk}(z,t) := \langle \beta_h(z,t)\omega_j, \omega_k\rangle_{\mathcal{H}}.$$

**Proposition 2** (Multiplicativity of Matrix Coefficients)**.** For $(z,t)$ and $(w,s)$ in $\mathbb{H}$,

$$M_{jk}^{n}\left((z,t)(w,s)\right)=\sum_{l=0}^{\infty}M_{jl}^{n}(w,s)M_{lk}^{n}(z,t);\quad \forall j,k\in\mathbb{N},\, n\in\mathbb{Z}^{*}. \tag{3.14}$$

*Proof.*

$$\begin{aligned}
M_{jk}^{n}\left((z,t)(w,s)\right) &= \langle\beta_n\left((z,t)(w,s)\right)\omega_j,\omega_k\rangle_{\mathcal{H}}\\
&= \langle\beta_n(z,t)\circ\beta_n(w,s)\omega_j,\omega_k\rangle_{\mathcal{H}}\\
&= \sum_{l=0}^{\infty}\langle\beta_n(z,t)\omega_l,\omega_k\rangle_{\mathcal{H}}\langle\beta_n(w,s)\omega_j,\omega_l\rangle_{\mathcal{H}}\\
&= \sum_{l=0}^{\infty}M_{jl}^{n}(w,s)M_{lk}^{n}(z,t).
\end{aligned}$$

□

With respect to the standard orthonormal basis $\omega_k=\zeta^k/\sqrt{k!}$, $k=0,1,...$, of $\mathcal{H}$, the matrix coefficients of the Bargmann-Fock representation, (3.1), $M_{jk}^{h}(z,t):=\langle\beta_h(z,t)\omega_j,\omega_k\rangle_{\mathcal{H}}$ are given for $h>0$ via a brute force computation by

$$M_{jk}^{h}(z,t)=\begin{cases}\sqrt{\frac{k!}{j!}}\left(+\sqrt{h}z\right)^{j-k}L_k^{(j-k)}\left(h|z|^2\right)e^{-h|z|^2/2}e^{2iht} & j\geq k\\ \sqrt{\frac{j!}{k!}}\left(-\sqrt{h}\overline{z}\right)^{k-j}L_j^{(k-j)}\left(h|z|^2\right)e^{-h|z|^2/2}e^{2iht} & j\leq k\end{cases}, \tag{3.15}$$

and $M_{jk}^{h}(z,t)=M_{jk}^{|h|}(\overline{z},-t)$ for $h<0$ (see Appendix B.7 for a conversion between Folland's [10, p. 64] and our conventions).

Here $L_j^{(\alpha)}(x)$ is the generalized Laguerre polynomial, defined recursively by

$$\begin{aligned}
L_0^{(\alpha)}(x)&=1\\
L_1^{(\alpha)}(x)&=1+\alpha-x\\
(j+1)L_{j+1}^{(\alpha)}(x)&=(2j+1+\alpha-x)\,L_j^{(\alpha)}(x)-(j+\alpha)\,L_{j-1}^{(\alpha)}(x).
\end{aligned} \tag{3.16}$$

The matrix coefficients are not in $L^2(\mathbb{H})$. This means that the infinite dimensional irreducible unitary representations of $\mathbb{H}$ are *not square-integrable representations*. Nonetheless we construct a natural family Hilbert spaces in which the matrix coefficients live.

**Definition 3.** For each $j \in \mathbb{N}$ and $h \in \mathbb{R}^*$ let $\mathcal{H}_j^h \subset C^\infty(\mathbb{H})$ be the space of smooth functions $f$ satisfying

$$\mathcal{L}f = (2j+1)|h|f \qquad\qquad -iTf = hf$$

and the condition $f(z,0) \in L^2(\mathbb{C})$.

**Proposition 3.** For each $j \in \mathbb{N}$ and $h \in \mathbb{R}^*$, the space $\mathcal{H}_j^h$ together with the inner product

$$(f,g)_{\mathcal{H}_j^h} := \frac{|h|}{\pi}\int_{\mathbb{C}} f(z,0)\overline{g(z,0)}dz, \quad f,g \in \mathcal{H}_j^h. \tag{3.17}$$

is a Hilbert space for which the set of matrix coefficients with first index fixed

$$\{M_{jk}^h : k \in \mathbb{N}\} \tag{3.18}$$

are a complete orthonormal basis.

See [51, Ch.3]

*Proof.* We only show orthogonality. Note that

$$M_{jk}^h(z,0) = \sqrt{2\pi}V_{\psi_j}(\psi_k, \sqrt{2h}z), \quad h > 0$$

(see B.29 in the Appendix) with $z$ replaced by $\overline{z}$ if $h < 0$.

Given $M^h_{jk}$, $M^h_{jk'} \in \mathcal{H}^h_j$ we have

$$\begin{aligned}
\langle M^h_{jk}, M^h_{jk'} \rangle_{\mathcal{H}^h_j} &= \frac{|h|}{\pi} \int_{\mathbb{C}} M^h_{jk}(z,0) \overline{M^h_{jk'}(z,0)} dz \\
&= 2|h| \int_{\mathbb{C}} V_{\psi_j}(\psi_k, \sqrt{2|h|}z) \overline{V_{\psi_j}(\psi_{k'}, \sqrt{2|h|}z)} dz \\
&= \int_{\mathbb{C}} V_{\psi_j}(\psi_k, z) \overline{V_{\psi_j}(\psi_{k'}, z)} dz \\
&= (\psi_j, \psi_j)\, (\psi_k, \psi_{k'}) \\
&= \delta(k - k').
\end{aligned}$$

□

## 3.7 Fourier Modes on the Reduced Heisenberg group

Let $\Gamma = \{(0, k\pi) \in \mathbb{H} : k \in \mathbb{Z}\}$. We call $\mathbb{H}/\Gamma$ the *reduced Heisenberg group*. Since $\beta_h(z,t) = e^{2iht}\beta_h(z,0)$ by definition, the Bargmann-Fock representation descends to the reduced Heisenberg group if and only if $h \in \mathbb{Z}^*$. More generally, given $\tau > 0$ the representation $\beta_h$ descends to $\mathbb{H}/\tau\Gamma$ if and only if $\tau h \in \mathbb{Z}^*$.

**Definition 4** (Normalized Matrix Coefficients on $L^2(\mathbb{H}/\Gamma)$)**.** Let

$$m^n_{jk}(z,t) := \frac{\sqrt{|n|}}{\pi} M^n_{jk}(z,t); \quad j,k \in \mathbb{N},\, n \in \mathbb{Z}^*. \tag{3.19}$$

Then by rescaling Proposition 3 the functions $\{m^n_{jk} : j,k \in \mathbb{N}, h \in \mathbb{Z}^*\}$ are orthonormal on $L^2(\mathbb{H}/\Gamma)$ with respect to the inner product

$$\langle f, g \rangle_{L^2(\mathbb{H}/\Gamma)} = \int_0^{\pi} \int_{\mathbb{C}} f(z,t)\overline{g(z,t)} dz dt. \tag{3.20}$$

The functions $m^n_{jk}$ for $n \in \mathbb{Z}^*$ and $j, k \in \mathbb{N}$ form an orthonormal basis for ${}^0L^2(\mathbb{H}/\Gamma)$.

First

**Theorem 4.** *For $f \in S(\mathbb{H}/\Gamma)$,*

$$f(z,t) = f_0(z,0) + \frac{1}{(2\pi)^2} \sum_{n \in \mathbb{Z}^*} |n| \sum_{j,k \in \mathbb{N}} \left(f, M^n_{jk}\right)_{L^2(\mathbb{H}/\Gamma)} M^n_{jk}(z,t). \tag{3.21}$$

*where $f_0(z) = \frac{1}{\pi} \int_0^\pi f(z,t)dt = 0$.*

*Proof.* Suppose $\int_0^\pi f(z,t)dt = 0$. We expand the inversion formula in Theorem 3 with respect to the orthonormal basis $\omega_k = \zeta^k/\sqrt{k!}$ on $\mathcal{H}$.

$$\begin{aligned}
\mathrm{Tr}\langle \beta_n(z,t)\mathcal{F}_{\mathbb{H}/\Gamma}(f)(n)\rangle &= \sum_{k \in \mathbb{N}} \langle \beta_n(z,t))\mathcal{F}_{\mathbb{H}/\Gamma}(f)(n)\omega_k, \omega_k\rangle_{\mathcal{H}} \\
&= \sum_{j,k \in \mathbb{N}} \langle \mathcal{F}_{\mathbb{H}/\Gamma}(f)(n)\omega_k, \omega_j\rangle \langle \beta_n(z,t)\omega_j, \omega_k\rangle \\
&= \sum_{j,k \in \mathbb{N}} \int_{\mathbb{H}/\Gamma} f(w,s)\langle \beta_n(w,s)^*\omega_k, \omega_j\rangle d(w,s) \langle \beta_n(z,t)\omega_j, \omega_k\rangle \\
&= \sum_{j,k \in \mathbb{N}} \int_{\mathbb{H}/\Gamma} f(w,s)\overline{M^n_{jk}}(w,s)d(w,s) M^n_{jk}(z,t) \\
&= \sum_{j,k \in \mathbb{N}} \langle f, M^n_{jk}\rangle_{L^2(\mathbb{H}/\Gamma)} M^n_{jk}(z,t)
\end{aligned}$$

Then 3.21 follows form Theorem 3. If $\int_0^\pi f(z,t)dt \neq 0$, then set $f_0(z) := \frac{1}{\pi}\int_0^\pi f(z,t)dt$ and apply the previous result to $g = f - f_0$. $\square$

**Corollary 1.** Let $f \in L^2(\mathbb{H}/\Gamma)$ such that $\int_0^\pi f(z,t)dt = 0$, then we have

$$f = \sum_{n \in \mathbb{Z}^*} \sum_{j,k=0}^{\infty} \langle f, m^n_{jk}\rangle_{L^2(\mathbb{H}/\Gamma)} m^n_{jk}. \tag{3.22}$$

# Part III

# Proofs of Main Results

# Chapter 4

# The Heisenberg X-ray transform

## 4.1 The space of geodesics

Recall that $\mathbb{H}$ acts transitively on $\mathcal{G}_\lambda$ on the left. Since

$$(0, \pi R^2)\gamma_\lambda(s) = \left(Re^{is/R}, \tfrac{R}{2}(s + 2\pi R)\right) = \gamma_\lambda\,(s + 2\pi R)\,;\ R = 1/\lambda, \tag{4.1}$$

the subgroup $\Gamma_\lambda := \{(0, k\pi R^2) \in \mathbb{H} : k \in \mathbb{Z}\}$ stabilizes $\mathcal{G}_\lambda$. Upon fixing $\gamma_\lambda$, we have the identification

$$\begin{aligned} \mathcal{G}_\lambda &\cong \mathbb{H}/\Gamma_\lambda \\ (z,t)\gamma_\lambda &\mapsto (z,t)\Gamma_\lambda. \end{aligned} \tag{4.2}$$

When $\lambda = 1$, we omit subscripts and write $\Gamma = \Gamma_1$.

Let $d(z,t)\Gamma_\lambda \cong dx \wedge dy \wedge dt$ be the Haar measure on $\mathbb{H}/\Gamma_\lambda$, and let $\mathcal{G}_\lambda$ inherit a multiple of the Haar measure $d\mathcal{G}_\lambda := \lambda dx \wedge dy \wedge dt$, normalized to agree with $\iota_{\partial_\lambda} d\mathrm{vol}_\mathcal{G}$, where $d\mathrm{vol}_\mathcal{G} = \lambda dx \wedge dy \wedge dt \wedge d\lambda$, in Appendix A.3. Furthermore, set $d\mathcal{G} = e^{-\lambda} d\mathrm{vol}_\mathcal{G}$, chosen to ensure boundedness in Proposition 8.

## 4.2 Symmetries of $I_\lambda$

We discuss basic infinitesimal and integrated symmetries of the X-ray transform $I_\lambda$. The results in this section are not necessary for proving injectivity of $I_\lambda$, but are informative nonetheless. Of particular interest ot the author is Theorem 5, which says that $I_\lambda$ does not commute with the left-invariant Heisenberg sub-Laplacian $\mathcal{L}_0$ (defined below) on $\mathbb{H}$ and $\mathbb{H}/\Gamma_\lambda$, but rather intertwines $\mathcal{L}_0$ with the wave operator $\Box_\lambda := \lambda^{-2}T - X^2 - Y^2$ on the $\mathbb{H}/\Gamma_\lambda$ for the Lorentzian metric $g_{i\lambda} := dx^2 + dy^2 - \lambda^{-2}\Theta^2$.

**Lemma 4** (Equivariance of Ray Transform)**.** $I_\lambda : C_c(\mathbb{H}) \mapsto C_c(\mathcal{G}_\lambda)$ *has the following symmetries:*

$$I_\lambda \left( L^*_{(w,s)} f \right)(z,t) = L^*_{(w,s)} \left( I_\lambda f \right)(z,t) \tag{4.3}$$

$$I_\lambda \left( \mathcal{R}^*_\theta f \right)(z,t) = I_\lambda f \left( e^{i\theta} z, t - \frac{\theta}{2\lambda^2} \right) \tag{4.4}$$

$$I_\lambda (S^*_{-1} f)(z,t) = I_\lambda f(\overline{z}, -t) \tag{4.5}$$

*where* $\mathcal{R}_\theta(z,t) = (e^{i\theta} z, t)$*;* $S_{-1}(z,t) = (\overline{z}, -t)$*; and* $L_{(w,s)}(z,t) = (w,s)(z,t)$.

*Proof.* For $f \in C_c(\mathbb{H})$, $\lambda > 0$, $(z,t) \in \mathbb{H}/\Gamma_\lambda$ and $(w,s) \in \mathbb{H}$

$$\begin{aligned} I_\lambda \left( L^*_{(w,s)} f \right)(z,t) &= \int_{\mathbb{R}} L^*_{(w,s)} f \left( (z,t)\gamma_\lambda(\tau) \right) d\tau = \int_{\mathbb{R}} f \left( (w,s)(z,t)\gamma_\lambda(\tau) \right) d\tau \\ &= I_\lambda f \left( (w,s)(z,t) \right) = \left( L^*_{(w,s)} I_\lambda f \right)(z,t). \end{aligned}$$

Generalizing (4.1), we see the cylindrical symmetry of $\gamma_\lambda$:

$$\mathcal{R}_\theta\gamma_\lambda(s) = (e^{i\lambda(s+\theta/\lambda)}/\lambda, s/2\lambda) = (0, -\theta/2\lambda^2)\gamma_\lambda(s+\theta/\lambda).$$

So

$$\begin{aligned}
I_\lambda\left(\mathcal{R}_\theta f\right)(z,t) :=& \int_{\mathbb{R}} f\left(\mathcal{R}_\theta\left((z,t)\gamma_\lambda(s)\right)\right) ds \\
=& \int_{\mathbb{R}} f\left(\mathcal{R}_\theta(z,t)\mathcal{R}_\theta\gamma_\lambda(s)\right) ds \\
=& \int_{\mathbb{R}} f\left((e^{i\theta}z, t - \tfrac{\theta}{2\lambda^2})\gamma_\lambda(s+\theta/\lambda)\right) ds \\
=& \int_{\mathbb{R}} f\left((e^{i\theta}z, t - \tfrac{\theta}{2\lambda^2})\gamma_\lambda(s)\right) ds \\
=& I_\lambda f(e^{i\theta}z, t - \tfrac{\theta}{2\lambda^2}).
\end{aligned}$$

Observe that $S_{-1}\gamma_\lambda(s) = S_{-1}\left(e^{i\lambda s}/\lambda, s/2\lambda\right) = \left(e^{-i\lambda s}, -s/2\lambda\right) = \gamma_\lambda(-s)$. Then

$$\begin{aligned}
I_\lambda\left(S_{-1}^* f\right)(z,t) :=& \int_{\mathbb{R}} f\left(S_{-1}\left((z,t)\gamma_\lambda(s)\right)\right) ds \\
=& \int_{\mathbb{R}} f\left(S_{-1}(z,t)S_{-1}\gamma_\lambda(s)\right) ds \\
=& \int_{\mathbb{R}} f\left((\overline{z}, -t)\gamma_\lambda(-s)\right) ds \\
=& I_\lambda f(\overline{z}, -t).
\end{aligned}$$

□

**Proposition 4.** If $f \in C_c^\infty(\mathbb{H})$ then $I_\lambda f \in C_c^\infty(\mathcal{G}_\lambda)$.

*Proof.* If we define the measure $\overline{\nu}_\lambda$ on $\mathbb{H}$ by

$$\int_{\mathbb{H}} f(z,t)\overline{\nu}_\lambda(z,t) := \int_{\mathbb{R}} f\left(\gamma_\lambda(s)^{-1}\right) ds$$

(here we use a bar because $\nu$ will be reserved for a related measure in the sequel) then for $f \in C_c(\mathbb{H})$,

$$I_\lambda f(z,t) = \overline{\nu}_\lambda * f(z,t).$$

Therefore if $f \in C_c^\infty(\mathbb{H})$ then $I_\lambda f \in C_c^\infty(\mathcal{G}_\lambda)$ since convolution of a Radon measure by a smooth function is smooth. (See [8].) $\square$

Recall the standard left invariant vector fields on the Heisenberg group

$$X = \partial_x - \tfrac{1}{2} y \partial_t \qquad Y = \partial_y + \tfrac{1}{2} x \partial_t \qquad T = \partial_t.$$

The standard right invariant vector fields on $\mathbb{H}$ are

$$\tilde{X} = \partial_x + \tfrac{1}{2} y \partial_t \qquad \tilde{Y} = \partial_y - \tfrac{1}{2} x \partial_t \qquad \tilde{T} = \partial_t.$$

We will also use the definitions above to denote the standard left and right invariant vector fields on the reduced Heisenberg group.

**Definition 5.** For $\alpha \in \mathbb{R}$ the left and right invariant Heisenberg Laplacians are defined, respectively, by

$$-\mathcal{L}_\alpha := X^2 + Y^2 + i\alpha T, \qquad -\tilde{\mathcal{L}}_\alpha := \tilde{X}^2 + \tilde{Y}^2 + i\alpha T.$$

Explicitly,

$$\begin{aligned} \mathcal{L}_\alpha &:= -\left(\partial_x - \tfrac{1}{2} y \partial_t\right)^2 - \left(\partial_y + \tfrac{1}{2} x \partial_t\right)^2 - i\alpha\partial_t \\ &= \left(\partial_x^2 + \partial_y^2\right) + \tfrac{1}{4}\left(x^2 + y^2\right)\partial_t^2 + \partial_\theta \partial_t - i\alpha\partial_t. \end{aligned}$$

and

$$\tilde{\mathcal{L}}_\alpha := -\left(\partial_x + \tfrac{1}{2}y\partial_t\right)^2 - \left(\partial_y - \tfrac{1}{2}x\partial_t\right)^2 - i\alpha\partial_t$$

$$= -\left(\partial_x^2 + \partial_y^2\right) - \tfrac{1}{4}\left(x^2+y^2\right)\partial_t^2 + \partial_\theta\partial_t - i\alpha\partial_t.$$

where $\partial_\theta := (x\partial_y - y\partial_x)$.

**Proposition 5.** If $\tilde{V}$ is a right-invariant vector field on $\mathbb{H}$ and $f \in C_c^\infty(\mathbb{H})$, then

$$\tilde{V}\left(I_\lambda f\right)(z,t) = I_\lambda\left(\tilde{V} f\right)(z,t).$$

*Proof.* This follows from the fact that $I_\lambda$ is a convolution operator. For $f \in C_c^\infty(\mathbb{H})$,

$$\tilde{V}\left(I_\lambda f\right) = \tilde{V}\left(\overline{\nu}_\lambda * f\right) = \overline{\nu}_\lambda * \tilde{V} f = I_\lambda\left(\tilde{V} f\right).$$

□

**Corollary 2.** For $f \in C_c^\infty(\mathbb{H})$,

$$I_\lambda\left(\tilde{\mathcal{L}}_\alpha f\right) = \tilde{\mathcal{L}}_\alpha\left(I_\lambda f\right).$$

Differentiating (4.4) yields the following infinitesimal symmetry:

**Proposition 6.** For $f \in C_c^\infty(\mathbb{H})$,

$$I_\lambda\left(\partial_\theta f\right) = \left(\partial_\theta - \tfrac{1}{2\lambda^2}T\right) I_\lambda(f).$$

We compute the pushforward of the left Laplacian by the X-ray transform.

**Theorem 5.** *For* $f \in C_c^\infty(\mathbb{H})$,

$$I_\lambda\left(\mathcal{L}_\alpha f\right) = \left(\mathcal{L}_\alpha + \lambda^{-2}T^2\right) I_\lambda f.$$

*Proof.* By the above remark, we have

$$\tilde{\mathcal{L}}_\alpha - \mathcal{L}_\alpha = 2\partial_\theta\partial_t$$

Then

$$I_\lambda\left((\tilde{\mathcal{L}}_\alpha - \mathcal{L}_\alpha)f\right) = 2I_\lambda\left(\partial_\theta\partial_t f\right) = \left(2\partial_\theta\partial_t - \tfrac{1}{\lambda^2}\partial_t^2\right) I_\lambda f.$$

so

$$\begin{aligned} I_\lambda\left(\mathcal{L}_\alpha f\right) &= I_\lambda\left(\tilde{\mathcal{L}}_\alpha f\right) - \left(2\partial_\theta\partial_t - \tfrac{1}{\lambda^2}\partial_t^2\right) I_\lambda f \\ &= \tilde{\mathcal{L}}_\alpha\left(I_\lambda f\right) - \left(2\partial_\theta\partial_t - \tfrac{1}{\lambda^2}\partial_t^2\right) I_\lambda f \\ &= \mathcal{L}_\alpha\left(I_\lambda f\right) + \tfrac{1}{\lambda^2}\partial_t^2\left(I_\lambda f\right) \end{aligned}$$

as desired. $\square$

**Remark 4.** It is interesting to note that for $\alpha = 0$, Proposition 5 says that the pushfoward by $I_\lambda$ of the sublaplacian $\mathcal{L}_0$ on $\mathbb{H}$ is the wave operator $\square_\lambda = \lambda^{-2}T - X^2 + Y^2$ on $\mathbb{H}/\Gamma_\lambda$.

## 4.3 Reduction to $I^{\mathbf{red}}$

The dilation map, $\delta_\lambda(z,t) := (\lambda z, \lambda^2 t)$, is also an automorphism of the Heisenberg group for $\lambda \neq 0$. Furthermore,

$$\delta_\lambda : \Gamma_\lambda \ni (0, k\pi\lambda^{-2}) \mapsto (0, k\pi) \in \Gamma,$$

so $\delta_\lambda : \mathbb{H}/\Gamma_\lambda \to \mathbb{H}/\Gamma$ is well-defined. Denote by $\delta_\lambda^*$ the pullback relation defined on functions:

$$\begin{aligned} \delta_\lambda^* &: L^1(\mathbb{H}) \to L^1(\mathbb{H}) & \delta_\lambda^* f(z,t) &= f(\lambda z, \lambda^2 t) \\ \delta_\lambda^* &: L^1(\mathbb{H}/\Gamma) \to L^1(\mathbb{H}/\Gamma_\lambda) & \delta_\lambda^* g\left((z,t)\Gamma_\lambda\right) &= g\left((\lambda z, \lambda^2 t)\Gamma\right). \end{aligned}$$

**Remark 5.** In the sequel, we write any function $g : \mathbb{H}/\Gamma_\lambda \to \mathbb{C}$ as $g(z,t)$, in place of $g\left((z,t)\Gamma_\lambda\right)$, understanding that the $t$ variable is taken mod $\pi\lambda^{-2}$.

The dilation map $\delta_\lambda$ is especially relevant because it is a conformal map for the sub-Riemannian metric (with constant conformal factor $\lambda^2$). Consequently, we have the following homogeneity of the ray transform:

**Proposition 7** (Homogeneity of $I$)**.** For $f \in C_c(\mathbb{H})$,

$$I_\lambda f(z,t) = (1/\lambda)\delta_\lambda^* \left(I_1(\delta_{1/\lambda}^* f)\right)(z,t). \tag{4.6}$$

*Proof.* Note that dilation preserves geodesics but rescales their speed:

$$\delta_{1/\lambda}\gamma_1(s) = \gamma_\lambda(s/\lambda). \tag{4.7}$$

Then

$$\begin{aligned}
\delta_\lambda^* \left(I_1(\delta_{1/\lambda}^* f)\right)(z,t) &= I_1\left(\delta_{1/\lambda}^* f\right)(\lambda z, \lambda^2 t) \\
&= \int_{\mathbb{R}} \delta_{1/\lambda}^* f\left((\lambda z, \lambda^2 t)\gamma_1(s)\right) ds \\
&= \int_{\mathbb{R}} f\left(\delta_{1/\lambda}(\lambda z, \lambda^2 t)\delta_{1/\lambda}\left(\gamma_1(s)\right)\right) ds, \qquad \text{because } \delta_\lambda \in \mathrm{Aut}(\mathbb{H}), \\
&= \int_{\mathbb{R}} f\left((z,t)\gamma_\lambda(s/\lambda)\right) ds, \qquad \text{by (4.7)}, \\
&= \lambda \int_{\mathbb{R}} f\left((z,t)\gamma_\lambda(s)\right) ds = \lambda I_\lambda f(z,t).
\end{aligned}$$

□

Next, we exploit the periodic symmetry of Heisenberg geodesics to reduce the X-ray transform to one period.

**Proposition 8.** For any $\lambda > 0$, $I_\lambda : L^1(\mathbb{H}) \to L^1(\mathcal{G}_\lambda)$ is well-defined, bounded, and factors in the following way:

$$\begin{tikzcd} L^1(\mathbb{H}) \arrow[r, "I_\lambda"] \arrow[d, "P_\lambda"'] & L^1(\mathcal{G}_\lambda) \\ L^1(\mathbb{H}/\Gamma_\lambda) \arrow[ur, "I_\lambda^{\mathrm{red}}"'] & \end{tikzcd}$$

where the maps which we call *Poisson Summation* and the *reduced X-ray transform* are given by

$$P_\lambda f\,(z,t) = \sum_{k\in\mathbb{Z}} f\left(z, t + k\pi R^2\right), \qquad I_\lambda^{\mathrm{red}} g(z,t) = \int_0^{2\pi R} g\left((z,t)\gamma_\lambda(s)\right) ds;\; R = 1/\lambda.$$

Furthermore, $I : L^1(\mathbb{H}) \to L^1(\mathcal{G}, d\mathcal{G})$ is well-defined and bounded.

*Proof.* By homogeneity (4.6), and since pullback by $\delta_\lambda$ is bounded in the above $L^1$ spaces for $\lambda \neq 0$, it suffices to prove the proposition for $\lambda = 1$. For this case, we omit subscripts and write $P$ and $I^{\mathrm{red}}$. The map

$$C_c(\mathbb{H}) \ni f \mapsto \int_{\mathbb{H}/\Gamma} Pf\,(z,t)\, d(z,t)\Gamma$$

is a left-invariant positive linear functional on $C_c(\mathbb{H})$. By uniqueness of the Haar measure on $\mathbb{H}$ (which is just the Lebesgue measure), and the Riesz-Representation theorem, $\exists c > 0$ such that

$$\int_{\mathbb{H}/\Gamma} Pf\,(z,t)\, d(z,t)\Gamma = c \int_{\mathbb{H}} f(z,t) d(z,t), \tag{4.8}$$

and one may check that $c = 1$ (see [11, Thm. 2.49] for the general statement). So in particular, $||Pf||_{L^1(\mathbb{H}/\Gamma)} \leq ||f||_{L^1(\mathbb{H})}$.

For $g \in C_c(\mathbb{H}/\Gamma)$,

$$\begin{aligned}
||I^{\text{red}}g||_{L^1(\mathcal{G}_1)} &= \int_{\mathcal{G}_1} |I^{\text{red}}g\,(z,t)\,| d\mathcal{G}_1 \\
&= \int_{\mathbb{H}/\Gamma} \left| \int_0^{2\pi} g\left((z,t)\gamma_1(s)\right) ds \right| d(z,t)\Gamma \\
&\leq \int_0^{2\pi} \int_{\mathbb{H}/\Gamma} |g\left((z,t)\gamma_1(s)\right)| d(z,t)\Gamma ds \\
&= \int_0^{2\pi} \int_{\mathbb{H}/\Gamma} |g\left((z,t)\right)| d\left((z,t)\gamma_1(s)^{-1}\right) \Gamma ds \\
&= \int_0^{2\pi} \int_{\mathbb{H}/\Gamma} |g\left((z,t)\right)| d(z,t)\Gamma ds, \qquad \text{since } \mathbb{H}/\Gamma \text{ is unimodular,} \\
&= 2\pi ||g||_{L^1(\mathbb{H}/\Gamma)}.
\end{aligned}$$

Thus $P$ and $I^{\text{red}}$ extend to $L^1$ bounded maps. Given $f \in C_c(\mathbb{H})$, since $Pf \in C_c(\mathbb{H}/\Gamma)$ and

$$\begin{aligned}
I^{\text{red}}Pf(z,t) &= \int_0^{2\pi} \sum_{k\in\mathbb{Z}} f\left((z,t+k\pi)\gamma_1(s)\right) ds \\
&= \int_0^{2\pi} \sum_{k\in\mathbb{Z}} f\left((z,t)\gamma_1(s+2\pi k)\right) ds, \quad \text{by (4.1),} \\
&= \sum_{k\in\mathbb{Z}} \int_0^{2\pi} f\left((z,t)\gamma_1(s+2\pi k)\right) ds \\
&= \sum_{k\in\mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} f\left((z,t)\gamma_1(s)\right) ds = I_1 f(z,t),
\end{aligned}$$

we have $||I_1 f||_{L^1(\mathcal{G}_1)} \leq 2\pi ||f||_{L^1(\mathbb{H})}$. The third equality follows from uniform convergence of the integrand on the interval $[0, 2\pi] \ni s$. Therefore $I_1$ extends to a bounded map from $L^1(\mathbb{H})$ to $L^1(\mathcal{G}_1)$. In particular one may check, using (4.6), that $||I_\lambda f||_{L^1(\mathcal{G}_\lambda)} = ||I_1 f||_{L^1(\mathcal{G}_1)} \leq 2\pi ||f||_{L^1(\mathbb{H})}$.

Finally, for $f \in L^1(\mathbb{H})$, we have

$$\begin{aligned}||If||_{L^1(\mathcal{G})} :=& \int_{\mathcal{G}} |If(z,t,\lambda)| d\mathcal{G} \\ =& \int_0^\infty \int_{\mathcal{G}_\lambda} |I_\lambda f(z,t)| d\mathcal{G}_\lambda e^{-\lambda} d\lambda \\ \leq& 2\pi ||f||_{L^1(\mathbb{H})} \int_0^\infty e^{-\lambda} d\lambda = 2\pi ||f||_{L^1(\mathbb{H})}.\end{aligned}$$

as desired. $\square$

**Remark 6.** The reduced X-ray transform $I^{\mathrm{red}} : L^1(\mathbb{H}/\Gamma) \to L^1(\mathcal{G}_1)$ is not injective. In fact, if

$$g\,(z,t) = z^2 e^{-|z|^2} e^{4it},$$

then $I^{\mathrm{red}} g = 0$. In Appendix 5.3.2 we give essentially a Singular Value Decomposition of $I^{\mathrm{red}}$.

## 4.4 A sub-Riemannian Santaló Formula

**Remark 7.** From these computations, we may also deduce a sub-Riemannian Santaló formula:

$$\int_{\mathcal{G}_\lambda} I_\lambda f(z,t) d\mathcal{G}_\lambda = 2\pi \int_{\mathbb{H}} f(z,t) d(z,t), \quad f \in L^1(\mathbb{H}). \tag{4.9}$$

This is an example of a Santaló formula like those proven in [41], but without the latter's restriction to the "reduced unit cotangent bundle."

# Chapter 5

# The Fourier Slice Theorem

## 5.1 Statement and Proof

We show that $I^{\mathrm{red}}$ is a convolution operator by a compactly supported measure. Thus, it should be a Fourier multiplier in a generalized sense. The next propositions make this heuristic explicit.

**Definition 6.** Define the compactly supported measure $\nu$ on $\mathbb{H}/\Gamma$ by

$$\int_{\mathbb{H}} g(z,t)\nu(z,t) := \int_0^{2\pi} g\left(\gamma_1(\theta)^{-1}\right) d\theta; \quad g \in C_c(\mathbb{H}/\Gamma).$$

**Lemma 5.** *For* $g \in L^1(\mathbb{H}/\Gamma)$,

$$I^{\mathrm{red}} g = \nu * g.$$

*Proof.*

$$\nu * g(z,t) := \int_{\mathbb{H}/\Gamma} g\left((z,t)(w,s)^{-1}\right) d\nu(w,s) = \int_0^{2\pi} g\left((z,t)\gamma_1(\theta)\right) d\theta = I^{\mathrm{red}} g(z,t).$$

□

**Proposition 9.** If $g \in L^1(\mathbb{H}/\Gamma)$, then for all $n \in \mathbb{Z}^*$,

$$\mathcal{F}_{\mathbb{H}/\Gamma}\left(I^{\text{red}}g\right)(n) = (2\pi)\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}/\Gamma}(g)(n).$$

and more generally, for $\lambda > 0$ and $g \in L^1(\mathbb{H}/\Gamma_\lambda)$,

$$\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}\left(I_\lambda^{\text{red}}g\right)(n) = (2\pi/\lambda)\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}/\Gamma_\lambda}(g)(n)$$

with $\mathcal{J}_n$ defined in (1.4).

*Proof.* Applying the reduced Fourier Transform to $I^{\text{red}}(g) = \nu * g$, we obtain

$$\mathcal{F}_{\mathbb{H}/\Gamma}\left(I^{\text{red}}\right)(n) = \mathcal{F}_{\mathbb{H}/\Gamma}(\nu)(n) \circ \mathcal{F}_{\mathbb{H}/\Gamma}(g)(n).$$

So it remains to compute

$$\begin{aligned}
\mathcal{F}_{\mathbb{H}/\Gamma}(\nu)(n) :=& \int_{\mathbb{H}/\Gamma} \beta_n(z,t)^* d\nu(z,t) \\
=& \int_0^{2\pi} \beta_n\left(\gamma_1(s)^{-1}\right)^* ds \\
=& \int_0^{2\pi} \beta_n\left(\gamma_1(s)\right) ds, \quad \text{since } \beta_n \text{ is a unitary rep,} \\
=& (2\pi)\mathcal{J}_n
\end{aligned}$$

□

where the "multiplier"

$$\mathcal{J}_n := \frac{1}{2\pi}\int_0^{2\pi} \beta_n\left(\gamma_1(s)\right) ds \tag{5.1}$$

is the same as (1.4).

**Remark 8.** $\mathcal{J}_n$ is similar to the "representation integral" considered in [25], though $s \mapsto \beta_n(\gamma_1(s))$ is not a homomorphism. Such integration of representations over geodesics also appear in [19], where the authors used the Principal Series representations of $SL(2,\mathbb{R})$ to show that the normal operator $I^*I$ associated to the X-ray transform on constant negative curvature surfaces is a nontrivial function of the Laplace-Beltrami operator.

Together with Proposition 8, these imply the Heisenberg Fourier Slice Theorem:

*Proof of Theorem 2.* Let $f \in L^1(\mathbb{H})$, $\lambda > 0$ and $n \in \mathbb{Z}^*$. By Proposition 8 and 9 , we have

$$\mathcal{F}_{\mathbb{H}/\Gamma}(I_1 f)(n) = \mathcal{F}_{\mathbb{H}/\Gamma}\left(I^{\mathrm{red}} P f\right)(n) = (2\pi)\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}/\Gamma}(Pf)(n) = (2\pi)\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}}(f)(n). \tag{5.2}$$

Exploiting homogeneity of $I$,

$$\begin{aligned}
\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}(I_\lambda f)(n) &= \lambda^{-1}\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}\left(\delta_\lambda^* I_1\left(\delta_{1/\lambda}^* f\right)\right)(n), && \text{Proposition 7}\\
&= \lambda^{-5}\mathcal{F}_{\mathbb{H}/\Gamma}\left(I_1\left(\delta_{1/\lambda}^* f\right)\right)(n), && \text{Lemma 2}\\
&= 2\pi\lambda^{-5}\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}}\left(\delta_{1/\lambda}^* f\right)(n), && \text{by (5.2),}\\
&= 2\pi\lambda^{-1}\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}}(f)(n\lambda^2), && \text{Lemma 2}
\end{aligned}$$

as desired. □

**Remark 9.** In the special case when $n = 0$ or $h = 0$, the group Fourier Transforms are qualitatively different; they are the Euclidean Fourier transform in the $z$ variable (the precise sense in which this limiting behavior occurs is formalized by Geller in [13]). In this case, the Fourier Slice theorem takes the form

$$\widetilde{(I_\lambda f)}(\lambda\zeta, 0) = (2\pi/\lambda)J_0(|\zeta|)\widehat{f}(\lambda\zeta, 0); \quad \forall \lambda > 0,\ f \in L^1(\mathbb{H}),$$

where $J_0$ is the classical Bessel function of order zero, and

$$\widehat{f}(\zeta,0)=\int_{\mathbb{C}}\int_{\mathbb{R}} f(z,t)e^{-i\zeta\cdot z}dtdz,\ f\in L^1(\mathbb{H}),$$
$$\widetilde{g}(\zeta,0)=\int_{\mathbb{C}}\int_0^{\pi\lambda^{-2}} g(z,t)e^{-i\zeta\cdot z}dtdz;\ g\in L^1(\mathbb{H}/\Gamma_\lambda).$$

## 5.2 SVD of $\mathcal{J}_n(r)$

The following mild generalization of (5.1) will be useful for subsequent computations.

**Definition 7.** For $n\in\mathbb{Z}^*$, let

$$\mathcal{J}_n(r):=\frac{1}{2\pi}\int_0^{2\pi}\beta_n\left(re^{i\theta},\theta/2\right)d\theta,\quad r>0. \tag{5.3}$$

In particular, $\mathcal{J}_n(1)=\mathcal{J}_n$, defined in 5.1.

**Lemma 6.** *For any $\tau\in\mathbb{R}^*$, $j,k\in\mathbb{N}$ and $n\in\mathbb{Z}^*$,*

$$\frac{1}{2\pi}\int_0^{2\pi}e^{in\theta}M_{jk}^{n/\tau}(e^{i\theta},0)d\theta=M_{jk}^{n/\tau}(1,0)\begin{cases}\delta(|n|+j-k), & \text{if } \tau>0\\ \\ \delta(|n|+k-j), & \text{if } \tau<0\end{cases}$$

*Proof.* Note that

$$\begin{aligned}\int_0^{2\pi}e^{in\theta}M_{jk}^{n/\tau}(e^{i\theta},0)d\theta&=\int_0^{2\pi}e^{-in\theta}M_{jk}^{n/\tau}(e^{-i\theta},0)d\theta\\&=\int_0^{2\pi}e^{-in\theta}M_{jk}^{-n/\tau}(e^{i\theta},0)d\theta.\end{aligned}$$

If $n>0$, and $\tau>0$ then

$$M_{jk}^{n/\tau}(e^{i\theta},0)=e^{i(j-k)\theta}M_{jk}^{n/\tau}(1,0)$$

and if $n > 0$ and $\tau < 0$, then

$$M_{jk}^{n/\tau}(e^{i\theta}, 0) = e^{i(k-j)\theta} M_{jk}^{n/\tau}(1, 0)$$

hence the result. □

**Proposition 10** (SVD of $\mathcal{J}_n(r)$)**.** For every $n \in \mathbb{Z}^*$ and $r > 0$, the operator $\mathcal{J}_n(r) : \mathcal{H} \to \mathcal{H}$ is bounded in the operator-norm topology. Furthermore, $\mathcal{J}_{-n}(r) = \mathcal{J}_n(r)$, and, with respect to the orthonormal basis $\{\omega_j = \zeta^j/\sqrt{j!} : j = 0, 1, 2, ...\}$ of $\mathcal{H}$, we have

$$\mathcal{J}_n(r)\omega_j = (-1)^n \sqrt{\frac{j!}{(j+n)!}} \left(nr^2\right)^{n/2} e^{-nr^2/2} L_j^{(n)}\left(nr^2\right) \omega_{j+n}, \quad \forall j \in \mathbb{N},\ n > 0. \tag{5.4}$$

*Proof.* $\mathcal{J}_n(r) : \mathcal{H} \to \mathcal{H}$ is bounded in the operator-norm topology for any $n \in \mathbb{Z}^*$ since

$$||\mathcal{J}_n(r)||_{\text{op}} \leq \frac{1}{2\pi} \int_0^{2\pi} ||\beta_n\left(re^{i\theta}, \theta/2\right)||_{\text{op}} d\theta = 1.$$

Note that, for $n \in \mathbb{Z}^*$,

$$\mathcal{J}_{-n}(r) = \frac{1}{2\pi} \int_0^{2\pi} \beta_n(re^{-i\theta}, -\theta/2) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \beta_n\left(re^{i\theta}, \theta/2\right) d\theta = \mathcal{J}_n(r).$$

For $j, k \in \mathbb{N}$,

$$\begin{aligned}
\langle \mathcal{J}_n(r)\omega_j, \omega_k \rangle_{\mathcal{H}} &= \frac{1}{2\pi} \int_0^{2\pi} \langle \beta_n\left(re^{i\theta}, \theta/2\right) \omega_j, \omega_k \rangle_{\mathcal{H}} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} M_{jk}^n\left(re^{i\theta}, \theta/2\right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} M_{jk}^n\left(re^{i\theta}, 0\right) d\theta \\
&= \delta(|n| + j - k) M_{jk}^n(r, 0), \quad \text{by Lemma 6 with } \tau = 1/r^2
\end{aligned} \tag{5.5}$$

in which case,

$$\mathcal{J}_n(r)\omega_j = M_{j,j+|n|}^n(r, 0)\omega_{j+|n|},$$

and, by 3.15

$$M^{n}_{j,j+|n|}(r,0) = (-1)^n \sqrt{\frac{j!}{(j+|n|)!}} \left(|n|r^2\right)^{|n|/2} e^{-|n|r^2/2} L_j^{(|n|)}\left(|n|r^2\right). \tag{5.6}$$

□

**Corollary 3.** Let $r > 0$ and $n \in \mathbb{Z}^*$ be fixed. Since $\mathcal{J}_n(r)$ is bounded, it is injective if and only if $L_j^{(|n|)}\left(|n|r^2\right)$ is nonzero for all $j \in \mathbb{N}$.

*Proof of Proposition 1.* Given $n \in 2\mathbb{Z}+1$, by Corollary 3, the operator $\mathcal{J}_n$ is injective if and only if the sequence $\{L_j^{(n)}(n)\}_{j=0}^{\infty}$ is nonvanishing.

Set $a_j^{(n)} = j! L_j^{(n)}(n) \in \mathbb{Z}$. Then $a_0^{(n)} = a_1^{(n)} = 1$, and by (3.16),

$$\begin{aligned} a_{j+1}^{(n)} =& (2j+1)a_j^{(n)} - j(j+n)a_{j-1}^{(n)} \\ =& a_j^{(n)} \pmod 2 \end{aligned}$$

since $n$ is odd. Therefore $a_j^{(n)} = a_0^{(n)} = 1 \pmod 2$ for all $j = 0, 1, 2, \ldots.$ In particular, $L_j^{(n)}(n) = a_j^{(n)}/j! \neq 0$ for $j \in \mathbb{N}$. Therefore $\mathcal{J}_n$ is injective whenever $n$ is an odd integer. □

**Remark 10.** We know what $\mathcal{J}_2$ is not injective since $L_2^{(2)}(2) = 0$. However, the author is not currently aware of a general statement characterizing all $(j,n) \in \mathbb{N}\times\mathbb{N}^*$ for which $L_j^{(n)}(n) = 0$. While knowing this is not essential for proving injectivity of $I$, it would provide more ways to invert $I$. This is because the space of geodesics is four dimensional, and so we only need a subset of the overdetermined data to reconstruct $f$ from $If$.

## 5.3 Applications

### 5.3.1 Proof of injectivity of the X-ray Transform $I$

We now make use of the Heisenberg Fourier Slice theorem to prove injectivity of $I$ (Theorem 1); the proof is now almost immediate.

*Proof of Theorem 1.* Suppose $I_\lambda f = 0$ for all $\lambda \in (0, \eta)$, where $\eta > 0$. By the Heisenberg Fourier Slice Theorem,

$$0 = \mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}}(f)(n\lambda^2), \quad \forall n \in \mathbb{Z}^*,\ \forall \lambda \in (0, \eta).$$

By Proposition 1

$$0 = \mathcal{F}_{\mathbb{H}}(f)(n\lambda^2), \quad \forall n \in 2\mathbb{Z}+1,\ \forall \lambda \in (0, \eta). \tag{5.7}$$

In which case

$$0 = \mathcal{F}_{\mathbb{H}}(f)(h), \quad \forall h \in \bigcup_{n \in 2\mathbb{Z}+1} n\left(0, \eta^2\right) = \mathbb{R}^*.$$

Therefore $f = 0$ by the Fourier Inversion theorem for $\mathcal{F}_{\mathbb{H}}$. $\square$

### 5.3.2 SVD of $I^{\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma)}$

While not strictly necessary for our main result, the computation in Proposition 10 also gives us the SVD of $I^{\text{red}}$ when restricted to a specific subspace.

The next result is a consequence of the fact that $I^{\text{red}}$ is convolution by a compactly supported distribution, but we proceed explicitly.

**Proposition 11.**

$$I^{\text{red}} : L^2(\mathbb{H}/\Gamma) \to L^2(\mathbb{H}/\Gamma)$$

is well-defined and bounded.

*Proof.* For $g \in C_c(\mathbb{H}/\Gamma)$ the Cauchy-Schwartz Inequality gives

$$|I^{\text{red}}g(z,t)|^2 = \left| \int_0^{2\pi} g\left((z,t)(e^{i\theta},\theta/2)\right) d\theta \right|^2 \leq 2\pi \int_0^{2\pi} \left| g\left((z,t)(e^{i\theta},\theta/2)\right) \right|^2 d\theta. \tag{5.8}$$

Then

$$\begin{aligned}
||I^{\text{red}}g||^2_{L^2(\mathbb{H}/\Gamma)} &= \int_{\mathbb{H}/\Gamma} |I^{\text{red}}g(z,t)|^2 d(z,t)\Gamma \\
&\leq (2\pi) \int_{\mathbb{H}/\Gamma} \int_0^{2\pi} |g\left((z,t)(e^{i\theta},\theta/2)\right)|^2 d\theta d(z,t)\Gamma \\
&= (2\pi) \int_0^{2\pi} \int_{\mathbb{H}/\Gamma} |g\,(z,t)\,|^2 d(z,t)\Gamma d\theta, \qquad \text{by left translation,} \\
&= (2\pi)^2 ||g||^2_{L^2(\mathbb{H}/\Gamma)},
\end{aligned}$$

so $I^{\text{red}}$ extends to a bounded function from $L^2(\mathbb{H}/\Gamma)$ to itself. $\square$

**Proposition 12.** $I^{\text{red}}$ preserves the orthogonal decomposition in Lemma 1. i.e.

$$I^{\text{red}}|_{L^2(\mathbb{C})} : L^2(\mathbb{C}) \to L^2(\mathbb{C}) \tag{5.9}$$

$$I^{\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma)} : {}^0L^2(\mathbb{H}/\Gamma) \to {}^0L^2(\mathbb{H}/\Gamma). \tag{5.10}$$

Furthermore, the restriction $I^{\text{red}}|_{L^2(\mathbb{C})}$ is essentially $2\pi$ times the Mean Value Transform $M^1$.

*Proof.* For $f \in L^2(\mathbb{C})$,

$$\begin{aligned}
I^{\text{red}}|_{L^2(\mathbb{C})} f(z,t) &= \int_0^{2\pi} f\left((z,t)(e^{i\theta},\theta/2)\right) d\theta \\
&= \int_0^{2\pi} f\left(z+e^{i\theta}, t+\theta/2+\tfrac{1}{2}\mathrm{Im}\left(\overline{z}e^{i\theta}\right)\right) d\theta \\
&= \int_0^{2\pi} f(z+e^{i\theta}) d\theta = 2\pi M^1 f(z),
\end{aligned}$$

and so $I^{\text{red}} f \in L^2(\mathbb{C})$.

For $g \in {}^0L^2(\mathbb{H}/\Gamma)$,

$$\begin{aligned}
\int_0^{\pi} I^{\text{red}} g(z,t) dt &= \int_0^{\pi}\int_0^{2\pi} g\left(z+e^{i\theta}, t+\theta/2+\tfrac{1}{2}\mathrm{Im}\left(\overline{z}e^{i\theta}\right)\right) d\theta dt \\
&= \int_0^{2\pi}\int_0^{\pi} g(z+e^{i\theta},t) dt d\theta = 0,
\end{aligned}$$

so that $I^{\text{red}} g \in {}^0L^2(\mathbb{H}/\Gamma)$. □

We know that $I^{\text{red}}|_{L^2(\mathbb{C})} = 2\pi M^1$ has a continuous spectrum (see (1.2), or Remark 9), so we restrict the reduced X-ray transform to ${}^0L^2(\mathbb{H}/\Gamma)$, where is has a discrete spectrum, and compute its Singular Value Decomposition there.

**Theorem 6** (SVD of $I^{\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma)}$)**.** *For all* $j,k \in \mathbb{N}$, $n \in \mathbb{Z}^*$,

$$I^{\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma)} m^n_{jk} = 2\pi(-1)^n \sqrt{\frac{j!}{(j+|n|)!}}\, (|n|/e)^{|n|/2}\, L_j^{(|n|)}(|n|) m^n_{j+|n|,k}$$

*with* $m^n_{jk}$ *defined in* (3.19).

*Proof.* This is really nothing more than the expansion of Proposition 9 with respect to the orthonormal basis $\{\omega_j\}_j$ of $\mathcal{H}$ and the multiplicative property of the matrix coefficients $M^n_{jk}$.

$$\begin{aligned}
I^{\mathrm{red}}|_{{}^0L^2(\mathbb{H}/\Gamma)} m^n_{jk}(z,t) &= \frac{\sqrt{|n|}}{\pi}\int_0^{2\pi} M^n_{jk}\left((z,t)(e^{i\theta},\theta/2)\right)d\theta \\
&= \frac{\sqrt{|n|}}{\pi}\sum_{l=0}^{\infty}\int_0^{2\pi} M^n_{jl}(e^{i\theta},\theta/2)M^n_{lk}(z,t)d\theta, && \text{by (3.14)}, \\
&= \frac{\sqrt{|n|}}{\pi}\sum_{l=0}^{\infty}\int_0^{2\pi} M^{|n|}_{jl}(e^{i\theta},\theta/2)\,d\theta\,M^n_{lk}(z,t) \\
&= \frac{\sqrt{|n|}}{\pi}\sum_{l=0}^{\infty} 2\pi\delta(j-l+|n|)M^{|n|}_{jl}(1,0)M^n_{lk}(z,t), && \text{by (5.5)}, \\
&= 2\pi M^n_{j,j+|n|}(1,0)m^n_{j+|n|,k}(z,t) \\
&= 2\pi\sqrt{\frac{j!}{(j+|n|)!}}\,(|n|/e)^{n/2}\,L_j^{(|n|)}(|n|)m^n_{j+|n|,k}(z,t).
\end{aligned}$$

□

For $n \in \mathbb{Z}^*$ functions $f(z,t) \in \mathcal{H}^n_j \subset C^\infty(\mathbb{H})$ are periodic in $t$ with period $\pi$. Thus, we may identify $\mathcal{H}^n_j$ with a subspace of $L^2(\mathbb{H}/\Gamma)$ in fact

$$^0L^2(\mathbb{H}/\Gamma) \cong \bigoplus_{j\in\mathbb{N};\,n\in\mathbb{Z}^*} \mathcal{H}^n_j$$

**Corollary 4.** The reduced X-ray transform $I^{\mathrm{red}}$ decomposes as a collection of weighted shifts on the spaces of matrix coefficients

$$I^{\mathrm{red}}|_{\mathcal{H}^n_j} : \mathcal{H}^n_j \to \mathcal{H}^n_{j+|n|} \tag{5.11}$$

where component each is essentially a multiple of powers of the creation operator:

$$I^{\mathrm{red}}|_{\mathcal{H}^n_j} = 2\pi(-1)^n\frac{j!}{(j+|n|)!}\,(|n|/e)^{|n|/2}\,L_j^{(|n|)}(|n|)\left(-\overline{Z}\right)^{|n|}. \tag{5.12}$$

# Chapter 6

# Compatible X-ray Transforms

## 6.1 X-ray Transform for the Riemannian metric $g_\epsilon$

We use the same machinery to prove injectivity of the X-ray transform associated to the family of left-invariant taming metric on $\mathbb{H}$

Consider the family of left-invariant Riemannian metrics for $\epsilon > 0$:

$$g_\epsilon := dx^2 + dy^2 + (1/\epsilon)^2 \, \Theta^2,$$

where $\Theta := dt - \frac{1}{2}(xdy - ydx)$ is a contact form for the Heisenberg distribution $\mathcal{D}_q :=$ ker$\Theta_q$. Geodesics of $(\mathbb{H}, g_\epsilon)$ converge uniformly to the sub-Riemannian geodesics as $\epsilon \to 0$, [7, p. 33]. The explicit expression for $g_\epsilon$ geodesics is derived in [7, Sec. 2.4.4]. We record the exponential map for $g_\epsilon$ in (A.20).

**Remark 11.** To avoid quantifying $\epsilon$ in every proposition of this section, with the exception of Theorems 7 and 8, we will assume that we have chosen a fixed $\epsilon > 0$.

Let $\mathcal{G}^\epsilon$ be the set of geodesics for $g_\epsilon$ without orientation, and $\mathcal{G}^\epsilon_\lambda$ be the subset of geodesics having charge $\lambda$ (which is still a constant of motion). Geodesics with $\lambda \neq 0$ still project to circles in the plane, and those with $\lambda = 0$ project to lines; they only differ from sub-Riemannina geodesics by an $\epsilon$-dependent velocity in the $T = \partial_t$ direction. Left translation by any element $(z, t) \in \mathbb{H}$ is a $g_\epsilon$-isometry, and so $\mathbb{H}$ acts on $\mathcal{G}^\epsilon$ by pointwise left multiplication. This action does not change the value of $\lambda$ and is transitive when $\lambda \neq 0$.

We choose a particular geodesic $\gamma^\epsilon_\lambda$ to be the one whose projection to the plane is a unit-speed circular path of radius $R = 1/|\lambda|$, and parameterize the set of $g_\epsilon$ geodesics having charge $\lambda$ by

$$s \to (z,t)\gamma_\lambda(s), \quad \gamma^\epsilon_\lambda(s) = \left( Re^{is/R}, s\frac{(R^2 + 2\epsilon^2)}{2R} \right) \in \mathbb{H}; \quad R = 1/\lambda. \tag{6.1}$$

**Remark 12.** The geodesics described by (6.11) are not arclength parameterized; indeed, $g_\epsilon(\dot{\gamma}^\epsilon_\lambda(s), \dot{\gamma}^\epsilon_\lambda(s)) = 1 + \epsilon^2\lambda^2$. Instead, we insist that their projections to the plane are unit-speed.

We define the X-ray transform associated to the taming metric $g_\epsilon$ by

$$I^\epsilon f(z, t, \lambda) := I^\epsilon_\lambda f(z, t) := \int_{\mathbb{R}} f\left((z, t)\gamma^\epsilon_\lambda(s)\right) ds, \quad f \in C_c(\mathbb{H}). \tag{6.2}$$

Note that

$$\gamma^\epsilon_\lambda(s + 2\pi R) = \gamma^\epsilon_\lambda(s)(0, \pi R^2 + 2\pi\epsilon^2). \tag{6.3}$$

Therefore the isotropy group of $\gamma^\epsilon_\lambda$ for the action of $\mathbb{H}$ by left translation on $\mathcal{G}^\epsilon_\lambda$ is

$$\Gamma^\epsilon_\lambda := \{(0, k\pi(R^2 + 2\epsilon^2)) \in \mathbb{H} : k \in \mathbb{Z}\}.$$

We have the identification

$$\mathcal{G}^{\epsilon}_{\lambda} \cong \mathbb{H}/\Gamma^{\epsilon}_{\lambda}$$

$$(z,t)\gamma^{\epsilon}_{\lambda} \mapsto (z,t)\Gamma^{\epsilon}_{\lambda}.$$

**Remark 13.** Again, when $\lambda = 1$, we omit subscripts and write $\Gamma^{\epsilon} = \Gamma^{\epsilon}_1$. We will also write $g(z,t)$, for any function $g : \mathbb{H}/\Gamma^{\epsilon}_{\lambda} \to \mathbb{C}$, in place of $g\left((z,t)\Gamma^{\epsilon}_{\lambda}\right)$.

Let $d(z,t)\Gamma^{\epsilon}_{\lambda} \cong dx \wedge dy \wedge dt$ be the Haar measure on $\mathbb{H}/\Gamma^{\epsilon}_{\lambda}$, and let $\mathcal{G}^{\epsilon}_{\lambda}$ inherit a multiple of the Haar measure $d\mathcal{G}^{\epsilon}_{\lambda} := \lambda dx \wedge dy \wedge dt$. Furthermore, let $d\mathcal{G}^{\epsilon} := \lambda e^{-\lambda} dx \wedge dy \wedge dt \wedge d\lambda$.

Note the homogeneity of geodesics with respect to dilation:

$$\delta_{1/\lambda}\gamma^{\epsilon\lambda}_1(s) = \gamma^{\epsilon}_{\lambda}(s/\lambda); \quad R = 1/\lambda. \tag{6.4}$$

## 6.2 Mapping Properties of $I^{\epsilon}$

**Proposition 13** (Homogeneity of $I^{\epsilon}$)**.** For $f \in C_c(\mathbb{H})$, we have

$$I^{\epsilon}_{\lambda}(f)(z,t) = \lambda^{-1}\delta^*_{\lambda} I^{\epsilon\lambda}_1\left(\delta^*_{1/\lambda} f\right)(z,t). \tag{6.5}$$

*Proof.* This is essentially the same proof as 4.6:

$$\begin{aligned}
\delta_\lambda^* I_1^{\epsilon\lambda}\left(\delta_{1/\lambda}^* f\right)(z,t) &= I_1^{\epsilon\lambda}\left(\delta_{1/\lambda}^* f\right)(\lambda z, \lambda^2 t) \\
&= \int_{\mathbb{R}} \delta_{1/\lambda}^* f\left((\lambda z, \lambda^2 t)\gamma_1^{\epsilon\lambda}(s)\right) ds \\
&= \int_{\mathbb{R}} f\left((z,t)\delta_{1/\lambda}\gamma_1^{\epsilon\lambda}(s)\right) ds \\
&= \int_{\mathbb{R}} f\left((z,t)\gamma_\lambda^{\epsilon}(s/\lambda)\right) ds, && \text{by (6.4)}, \\
&= \lambda \int_{\mathbb{R}} \delta_{1/\lambda}^* f\left((\lambda z, \lambda^2 t)\gamma_1^{\epsilon\lambda}(\lambda s)\right) ds \\
&= \lambda I_\lambda^\epsilon f(z,t).
\end{aligned}$$

□

Furthermore, in virtually the same way as Proposition 8, we reduce the X-ray transform $I^\epsilon$ to one period:

**Proposition 14.** For any $\lambda > 0$, $I_\lambda^\epsilon : L^1(\mathbb{H}) \to L^1(\mathcal{G}_\lambda^\epsilon)$ is well-defined, bounded, and factors in the following way:

$$\begin{tikzcd}
L^1(\mathbb{H}) \arrow[r, "I_\lambda^\epsilon"] \arrow[d, "P_\lambda^\epsilon"'] & L^1(\mathcal{G}_\lambda^\epsilon) \\
L^1(\mathbb{H}/\Gamma_\lambda^\epsilon) \arrow[ur, "I_\lambda^{\epsilon,\mathrm{red}}"'] &
\end{tikzcd}$$

where

$$P_\lambda^\epsilon f(z,t) = \sum_{k\in\mathbb{Z}} f\left(z, t + k\pi(R^2 + 2\epsilon^2)\right), \quad I_\lambda^{\epsilon,\mathrm{red}} g(z,t) := \int_0^{2\pi R} g\left((z,t)\gamma_\lambda^\epsilon(s)\right) ds;\ R = 1/\lambda. \tag{6.6}$$

Furthermore, $I^\epsilon : L^1(\mathbb{H}) \to L^1(\mathcal{G}^\epsilon, d\mathcal{G}^\epsilon)$ is well-defined and bounded.

*Proof.* By homogeneity (6.5), and since pullback by $\delta_\lambda$ is bounded in the above $L^1$ spaces for $\lambda \neq 0$, it suffices to prove the proposition for $\lambda = 1$. For this case, we omit subscripts and write $P^\epsilon$ and $I^{\epsilon,\text{red}}$.

For exactly the same reason as (4.8), $P^\epsilon$ maps $C_c(\mathbb{H})$ to $C_c(\mathbb{H}/\Gamma^\epsilon)$, and

$$\int_{\mathbb{H}/\Gamma^\epsilon} P^\epsilon f\,(z,t)\,d(z,t)\Gamma^\epsilon = \int_{\mathbb{H}} f(z,t)d(z,t). \tag{6.7}$$

So in particular, $||P^\epsilon f||_{L^1(\mathbb{H}/\Gamma^\epsilon)} \leq ||f||_{L^1(\mathbb{H})}$.

For $g \in C_c(\mathbb{H}/\Gamma^\epsilon)$,

$$\begin{aligned}
||I^{\epsilon,\text{red}} g||_{L^1(\mathcal{G}_1^\epsilon)} &= \int_{\mathcal{G}_1^\epsilon} |I^{\epsilon,\text{red}} g\,(z,t)\,| d\mathcal{G}_1^\epsilon \\
&= \int_{\mathbb{H}/\Gamma^\epsilon} \left| \int_0^{2\pi} g\left((z,t)\gamma_1^\epsilon(s)\right) ds \right| d(z,t)\Gamma^\epsilon \\
&\leq \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^\epsilon} |g\left((z,t)\gamma_1^\epsilon(s)\right)| d(z,t)\Gamma^\epsilon ds \\
&= \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^\epsilon} |g\left((z,t)\right)| d\left((z,t)\gamma_1^\epsilon(s)^{-1}\right)\Gamma^\epsilon ds \\
&= \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^\epsilon} |g\,(z,t)\,| d(z,t)\Gamma^\epsilon ds, \qquad \text{since } \mathbb{H}/\Gamma^\epsilon \text{ is unimodular,} \\
&= 2\pi ||g||_{L^1(\mathbb{H}/\Gamma^\epsilon)}.
\end{aligned}$$

Thus $P^\epsilon$ and $I^{\epsilon,\text{red}}$ extend to $L^1$ bounded maps. Given $f \in C_c(\mathbb{H})$, since $Pf \in C_c(\mathbb{H}/\Gamma^\epsilon)$ and

$$\begin{aligned}
I^{\epsilon,\text{red}} P^\epsilon f(z,t) &= \int_0^{2\pi} \sum_{k\in\mathbb{Z}} f\left((z, t + k\pi(1+2\epsilon^2))\gamma_1^\epsilon(s)\right) ds \\
&= \int_0^{2\pi} \sum_{k\in\mathbb{Z}} f\left((z,t)\gamma_1^\epsilon(s+2\pi k)\right) ds, \quad \text{by (6.13),} \\
&= \sum_{k\in\mathbb{Z}} \int_0^{2\pi} f\left((z,t)\gamma_1(s+2\pi k)\right) ds \\
&= \sum_{k\in\mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} f\left((z,t)\gamma_1(s)\right) ds = I_1^\epsilon f(z,t),
\end{aligned}$$

we have $||I_1^\epsilon f||_{L^1(\mathcal{G}_1^\epsilon)} \leq 2\pi||f||_{L^1(\mathbb{H})}$. The third equality follows from uniform convergence of the integrand on the interval $[0, 2\pi] \ni s$. Therefore $I_1^\epsilon$ extends to a bounded map from $L^1(\mathbb{H})$ to $L^1(\mathcal{G}_1^\epsilon)$. In particular one may check, using (6.5), that $||I_\lambda^\epsilon f||_{L^1(\mathcal{G}_\lambda^\epsilon)} = ||I_1^\epsilon f||_{L^1(\mathcal{G}_1^\epsilon)} \leq 2\pi||f||_{L^1(\mathbb{H})}$.

Finally we have, for $f \in L^1(\mathbb{H})$,

$$\begin{aligned}||If||_{L^1(\mathcal{G}^\epsilon)} :=& \int_{\mathcal{G}^\epsilon} |I^\epsilon f(z,t,\lambda)| d\mathcal{G}^\epsilon \\ =& \int_0^\infty \int_{\mathcal{G}_\lambda^\epsilon} |I_\lambda f(z,t)| d\mathcal{G}_\lambda^\epsilon e^{-\lambda} d\lambda \\ \leq& 2\pi||f||_{L^1(\mathbb{H})} \int_0^\infty e^{-\lambda} d\lambda = 2\pi||f||_{L^1(\mathbb{H})}.\end{aligned}$$

as desired. □

We note a Poisson summation formula for $P_\lambda^\epsilon$:

**Lemma 7.** *For $f \in L^1(\mathbb{H})$,*

$$\mathcal{F}_{\mathbb{H}/\Gamma_\lambda^\epsilon} \left(P_\lambda^\epsilon f\right)(n), = \mathcal{F}_{\mathbb{H}}\left(f\right)\left(\frac{n\lambda^2}{1+2\epsilon^2\lambda^2}\right), \quad \forall n \in \mathbb{Z}^*. \tag{6.8}$$

*Proof.* Observe that $\Gamma_\lambda^\epsilon = (\lambda^{-2} + 2\epsilon^2)\Gamma$, and apply the Poisson Summation Formula (3.13) with $\tau = \lambda^{-2} + 2\epsilon^2$.

□

Observe how the Fourier Transform respects dilations:

**Lemma 8.** *For $g \in L^1(\mathbb{H}/\Gamma^\epsilon)$, $\lambda > 0$,*

$$\mathcal{F}_{\mathbb{H}/\Gamma_\lambda^\epsilon}\left(\delta_\lambda^* g\right) = \lambda^{-4}\mathcal{F}_{\mathbb{H}/\Gamma^{\epsilon\lambda}}(g)(n), \quad \forall n \in \mathbb{Z}^*. \tag{6.9}$$

*Proof.* Observe that $\Gamma_\lambda^\epsilon = (\lambda^{-2} + 2\epsilon^2)\Gamma$, and $\Gamma^{\epsilon\lambda} = (1 + 2\epsilon^2\lambda^2)\Gamma$. Then apply Lemma 2. □

## 6.3 Fourier Slice Theorem for $I^\epsilon$

As before, $I^{\epsilon,\text{red}}$ is a convolution operator by a compactly supported measure. We compute its Fourier multiplier.

**Definition 8.** Define the measure $\nu^\epsilon$ on $\mathbb{H}/\Gamma^\epsilon$ by

$$\int_{\mathbb{H}/\Gamma^\epsilon} g(z,t)d\nu(z,t) := \int_0^{2\pi} g\left(\gamma_1^\epsilon(\theta)^{-1}\right) d\theta; \quad g \in C_c(\mathbb{H}/\Gamma).$$

**Lemma 9.** *For $g \in L^1(\mathbb{H}/\Gamma)$,*

$$I^{\epsilon,\text{red}} g = \nu^\epsilon * g.$$

*Proof.*

$$\nu^\epsilon * g(z,t) := \int_{\mathbb{H}/\Gamma^\epsilon} g\left((z,t)(w,s)^{-1}\right) d\nu^\epsilon(w,s) = \int_0^{2\pi} g\left((z,t)\gamma_1^\epsilon(\theta)\right) d\theta = I^{\text{red}} g(z,t).$$

□

**Proposition 15.** For $g \in L^1(\mathbb{H}/\Gamma^\epsilon)$,

$$\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}\left(I^{\epsilon,\text{red}} g\right)(n) = (2\pi)\mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2}}\right) \circ \mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(g)(n), \quad \forall n \in \mathbb{Z}^*.$$

*Proof.* Applying the reduced Fourier Transform to $I^{\epsilon,\text{red}}(g) = \nu^\epsilon * g$, we obtain

$$\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}\left(I^{\text{red}}\right)(n) = \mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(\nu^\epsilon)(n) \circ \mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(g)(n).$$

So it remains to compute

$$
\begin{aligned}
\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}\left(\nu^\epsilon\right)(n) :=& \int_{\mathbb{H}/\Gamma^\epsilon} \beta_{n/(1+2\epsilon^2)}(z,t)^* d\nu^\epsilon(z,t) \\
=& \int_0^{2\pi} \beta_{n/(1+2\epsilon^2)}\left(\gamma_1^\epsilon(s)^{-1}\right)^* ds \\
=& \int_0^{2\pi} \beta_{n/(1+2\epsilon^2)}\left(\gamma_1^\epsilon(s)\right) ds, \quad \text{since } \beta_n \text{ is a unitary rep,} \\
=& \int_0^{2\pi} \beta_{n/(1+2\epsilon^2)}\left(e^{is}, \frac{s}{2}(1+2\epsilon^2)\right) ds \\
=& \int_0^{2\pi} \beta_n\left(\delta_{1/\sqrt{1+2\epsilon^2}}\left(e^{is}, \frac{s}{2}(1+2\epsilon^2)\right)\right) ds \\
=& \int_0^{2\pi} \beta_n\left(\frac{1}{\sqrt{1+2\epsilon^2}}e^{is}, s/2\right) ds \\
=& (2\pi)\mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2}}\right).
\end{aligned}
$$

□

We may now prove the Heisenberg Fourier Slice Theorem for $g_\epsilon$:

**Theorem 7** ($g_\epsilon$ Heisenberg Fourier Slice Theorem)**.** *If* $f \in L^1(\mathbb{H})$, *and* $\epsilon > 0$ *then*

$$
\mathcal{F}_{\mathbb{H}/\Gamma_\lambda^\epsilon}\left(I_\lambda^\epsilon f\right)(n) = (2\pi/\lambda)\mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2\lambda^2}}\right) \circ \mathcal{F}_{\mathbb{H}}(f)\left(\frac{n\lambda^2}{1+2\epsilon^2\lambda^2}\right), \quad \forall n \in \mathbb{Z}^*,\ \forall \lambda > 0.
$$

*Proof.* Combining Proposition 14 and 15,

$$
\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}\left(I^\epsilon f\right) = 2\pi\mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2}}\right) \circ \mathcal{F}_{\mathbb{H}}(f)\left(\frac{n}{1+2\epsilon^2}\right). \tag{6.10}
$$

Now, exploiting homogeneity of $I^\epsilon$,

$$\begin{aligned}
\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon_\lambda}\left(I^\epsilon_\lambda f\right)(n) &= \lambda^{-1}\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon_\lambda}\left(\delta^*_\lambda I_1^{\epsilon\lambda}\left(\delta^*_{1/\lambda}f\right)\right)(n), && \text{by Propositoin 13,}\\
&= \lambda^{-5}\mathcal{F}_{\mathbb{H}/\Gamma^{\epsilon\lambda}}\left(I_1^{\epsilon\lambda}\left(\delta^*_{1/\lambda}f\right)\right)(n), && \text{by Lemma 8,}\\
&= 2\pi\lambda^{-5}\mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2\lambda^2}}\right)\circ\mathcal{F}_{\mathbb{H}}(\delta^*_{1/\lambda}f)\left(\frac{n}{1+2\epsilon^2\lambda^2}\right), && \text{by (6.10),}\\
&= (2\pi/\lambda)\mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2\lambda^2}}\right)\circ\mathcal{F}_{\mathbb{H}}(f)\left(\frac{n\lambda^2}{1+2\epsilon^2\lambda^2}\right), && \text{by Lemma 8.}
\end{aligned}$$

□

**Proposition 16.** Let $\epsilon > 0$ and $n \in \mathbb{Z}^*$ be fixed. Then $\mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2\lambda^2}}\right) : \mathcal{H} \to \mathcal{H}$ is injective for almost all $\lambda > 0$.

*Proof.* Set $r = \frac{1}{\sqrt{1+2\epsilon^2\lambda^2}}$. By Corollary 3, the operator $\mathcal{J}_n(r)$ is injective if and only if $nr^2$ is not a zero of $L_j^{(n)}$ for any $j \in \mathbb{N}$. Since there are only countably many such zeros, the proposition follows. □

**Theorem 8.** *For all $\epsilon > 0$, the Heisenberg taming X-ray transform $I^\epsilon : L^1(\mathbb{H}) \to L^1(\mathcal{G}^\epsilon, d\mathcal{G}^\epsilon)$ is injective. In particular, if $f \in L^1(\mathbb{H})$ and $I^\epsilon_\lambda f = 0$ for all $\lambda$ in a neighborhood of zero, then $f = 0$.*

Suppose, $I^\epsilon_\lambda(f) = 0$ for all $\lambda \in (0, \eta)$, where $\eta > 0$. Then by Theorem 7 and Proposition 16,

$$0 = \mathcal{F}_{\mathbb{H}}(f)\left(\frac{n\lambda^2}{1+2\epsilon^2\lambda^2}\right)$$

for almost all $\lambda \in (0, \eta)$, and all $n \in \mathbb{Z}^*$. Let $A$ be the set of all such $\lambda \in (0, \eta)$, and

$B = \{\lambda^2/(1+2\epsilon^2\lambda^2) : \lambda \in A\}$. Then in other words

$$0 = \mathcal{F}_{\mathbb{H}}(f)(h) \qquad\qquad \forall h \in \bigcup_{n\in\mathbb{Z}}^{\infty} nB.$$

Since $B$ has full-measure on the interval $(0, \frac{\eta^2}{(1+2\epsilon^2\eta^2)})$, we know $\mathcal{F}_{\mathbb{H}} f(h) = 0$ for almost all $h \in \mathbb{R}^*$. Therefore $f = 0$ by the Fourier Inversion Theorem.

## 6.4 X-ray Transform for the Lorentzian metric $g_\epsilon$

We repeat the computations from the previous section for the Lorentzian metric, which are almost identical. The author thinks the two sections can be unified elegantly, but there are enough differences to warrant repetition for sake of juxtaposition.

Consider the family of left-invariant Lorentzian metrics for $\epsilon > 0$:

$$g_{i\epsilon} := dx^2 + dy^2 - (1/\epsilon)^2\, \Theta^2,$$

The explicit expression for $g_{i\epsilon}$ geodesics is derived in 2.2.

**Remark 14.** Again we will assume that we have chosen a fixed $\epsilon > 0$.

Let $\mathcal{G}^{i\epsilon}$ be the set of geodesics for $g_{i\epsilon}$ without orientation, and $\mathcal{G}^{i\epsilon}_\lambda$ be the subset of geodesics having charge $\lambda$ (which is still a constant of motion). Geodesics with $\lambda \neq 0$ still project to circles in the plane, and those with $\lambda = 0$ project to lines; they only differ from sub-Riemannina geodesics by an $\epsilon$-dependent velocity in the $T = \partial_t$ direction. Left translation by any element $(z, t) \in \mathbb{H}$ is a $g_{i\epsilon}$-isometry, and so $\mathbb{H}$ acts on $\mathcal{G}^{i\epsilon}$ by pointwise left multiplication. This action does not change the value of $\lambda$ and is transitive when $\lambda \neq 0$.

We choose a particular geodesic $\gamma_\lambda^{i\epsilon}$ to be the one whose projection to the plane is a unit-speed circular path of radius $R = 1/|\lambda|$, and parameterize the set of $g_{i\epsilon}$ geodesics having charge $\lambda$ by

$$s \to (z,t)\gamma_\lambda(s), \quad \gamma_\lambda^{i\epsilon}(s) = \left(Re^{is/R}, s\frac{(R^2 - 2\epsilon^2)}{2R}\right) \in \mathbb{H}; \quad R = 1/\lambda. \tag{6.11}$$

**Remark 15.** We say a Lorentzian geodesic $\gamma$ is *spacelike* if $g_{i\epsilon}(\dot{\gamma}, \dot{\gamma}) > 0$, *timelike* if $g_{i\epsilon}(\dot{\gamma}, \dot{\gamma}) < 0$ and *lightlike* if $g_{i\epsilon}(\dot{\gamma}, \dot{\gamma}) = 0$.

Note that $g_{i\epsilon}\left(\dot{\gamma}_\lambda^{i\epsilon}, \dot{\gamma}_\lambda^{i\epsilon}\right) = 1 - \lambda^2\epsilon^2$. Therefore, $\gamma_\lambda^{i\epsilon}$ is *spacelike* if $0 < \lambda < 1/\epsilon$, *timelike* if $1/\epsilon < \lambda < \infty$, and *lightlike* if $\lambda = \epsilon$.

**Definition 9.** We define the X-ray transform associated to the taming metric $g_{i\epsilon}$ by

$$I^{i\epsilon}f(z,t,\lambda) := I_\lambda^{i\epsilon}f(z,t) := \int_{\mathbb{R}} f\left((z,t)\gamma_\lambda^{i\epsilon}(s)\right) ds, \quad f \in C_c(\mathbb{H}). \tag{6.12}$$

where $\lambda \in (0,\infty) \setminus \{1/(\sqrt{2}\epsilon)\}$.

**Remark 16.** We omit the case in which $\epsilon^2\lambda^2 = 1/2$, in which the geodesics are closed. Indeed, the X-ray transform in this case is just the Heisenberg spherical mean operator studied by Thangavelu in [49].

Note that

$$\gamma_\lambda^{i\epsilon}(s + 2\pi R) = \gamma_\lambda^{i\epsilon}(s)(0, \pi R^2 - 2\pi\epsilon^2). \tag{6.13}$$

Therefore the isotropy group of $\gamma_\lambda^{i\epsilon}$ for the action of $\mathbb{H}$ by left translation on $\mathcal{G}_\lambda^{i\epsilon}$ is

$$\Gamma_\lambda^{i\epsilon} := \{(0, k\pi(R^2 - 2\epsilon^2)) \in \mathbb{H} : k \in \mathbb{Z}\}.$$

We have the identification

$$\mathcal{G}_\lambda^{i\epsilon} \cong \mathbb{H}/\Gamma_\lambda^{i\epsilon}$$

$$(z,t)\gamma_\lambda^{i\epsilon} \mapsto (z,t)\Gamma_\lambda^{i\epsilon}.$$

**Remark 17.** Again, when $\lambda = 1$, we omit subscripts and write $\Gamma^{i\epsilon} = \Gamma_1^{i\epsilon}$. We will also write $g(z,t)$, for any function $g : \mathbb{H}/\Gamma_\lambda^{i\epsilon} \to \mathbb{C}$, in place of $g\left((z,t)\Gamma_\lambda^{i\epsilon}\right)$.

Let $d(z,t)\Gamma_\lambda^{i\epsilon} \cong dx \wedge dy \wedge dt$ be the Haar measure on $\mathbb{H}/\Gamma_\lambda^{i\epsilon}$, and let $\mathcal{G}_\lambda^{i\epsilon}$ inherit a multiple of the Haar measure $d\mathcal{G}_\lambda^{i\epsilon} := \lambda dx \wedge dy \wedge dt$. Furthermore, let $d\mathcal{G}^{i\epsilon} := \lambda e^{-\lambda} dx \wedge dy \wedge dt \wedge d\lambda$.

Note the homogeneity of geodesics with respect to dilation:

$$\delta_{1/\lambda}\gamma_1^{i\epsilon\lambda}(s) = \gamma_\lambda^{i\epsilon}(s/\lambda); \quad R = 1/\lambda. \tag{6.14}$$

## 6.5 Mapping Properties of $I^{i\epsilon}$

**Proposition 17** (Homogeneity of $I^{i\epsilon}$)**.** For $f \in C_c(\mathbb{H})$, we have

$$I_\lambda^{i\epsilon}(f)(z,t) = \lambda^{-1}\delta_\lambda^* I_1^{i\epsilon\lambda}\left(\delta_{1/\lambda}^* f\right)(z,t). \tag{6.15}$$

*Proof.* This is essentially the same proof as 4.6:

$$\begin{aligned}
\delta_\lambda^* I_1^{i\epsilon\lambda}\left(\delta_{1/\lambda}^* f\right)(z,t) &= I_1^{i\epsilon\lambda}\left(\delta_{1/\lambda}^* f\right)(\lambda z, \lambda^2 t) \\
&= \int_{\mathbb{R}} \delta_{1/\lambda}^* f\left((\lambda z, \lambda^2 t)\gamma_1^{i\epsilon\lambda}(s)\right) ds \\
&= \int_{\mathbb{R}} f\left((z,t)\delta_{1/\lambda}\gamma_1^{i\epsilon\lambda}(s)\right) ds \\
&= \int_{\mathbb{R}} f\left((z,t)\gamma_\lambda^{i\epsilon}(s/\lambda)\right) ds, && \text{by (6.14),} \\
&= \lambda \int_{\mathbb{R}} \delta_{1/\lambda}^* f\left((\lambda z, \lambda^2 t)\gamma_1^{i\epsilon\lambda}(\lambda s)\right) ds \\
&= \lambda I_\lambda^{i\epsilon} f(z,t).
\end{aligned}$$

□

Furthermore, in virtually the same way as Proposition 8, we reduce the X-ray transform $I^{i\epsilon}$ to one period:

**Proposition 18.** For any $\lambda > 0$, $I_\lambda^{i\epsilon} : L^1(\mathbb{H}) \to L^1(\mathcal{G}_\lambda^{i\epsilon})$ is well-defined, bounded. If $\lambda \neq \sqrt{2}\epsilon$, then $I_\lambda^{i\epsilon}$ factors in the following way:

$$\begin{tikzcd}
L^1(\mathbb{H}) \arrow[r, "I_\lambda^{i\epsilon}"] \arrow[d, "P_\lambda^{i\epsilon}"'] & L^1(\mathcal{G}_\lambda^{i\epsilon}) \\
L^1(\mathbb{H}/\Gamma_\lambda^{i\epsilon}) \arrow[ur, "I_\lambda^{i\epsilon,\text{red}}"'] &
\end{tikzcd}$$

where

$$P_\lambda^{i\epsilon} f\,(z,t) = \sum_{k\in\mathbb{Z}} f\left(z, t + k\pi(R^2 - 2\epsilon^2)\right), \quad I_\lambda^{i\epsilon,\text{red}} g(z,t) := \int_0^{2\pi R} g\left((z,t)\gamma_\lambda^{i\epsilon}(s)\right) ds;\ R = 1/\lambda. \tag{6.16}$$

Furthermore, $I^{i\epsilon} : L^1(\mathbb{H}) \to L^1(\mathcal{G}^{i\epsilon}, d\mathcal{G}^{i\epsilon})$ is well-defined and bounded.

*Proof.* By homogeneity (6.5), and since pullback by $\delta_\lambda$ is bounded in the above $L^1$ spaces for $\lambda \neq 0$, it suffices to prove the proposition for $\lambda = 1$. For this case, we omit subscripts and write $P^{i\epsilon}$ and $I^{i\epsilon,\text{red}}$.

For exactly the same reason as (4.8), $P^{i\epsilon}$ maps $C_c(\mathbb{H})$ to $C_c(\mathbb{H}/\Gamma^{i\epsilon})$, and

$$\int_{\mathbb{H}/\Gamma^{i\epsilon}} P^{i\epsilon} f\,(z,t)\,d(z,t)\Gamma^{i\epsilon} = \int_{\mathbb{H}} f(z,t)d(z,t). \tag{6.17}$$

So in particular, $||P^{i\epsilon}f||_{L^1(\mathbb{H}/\Gamma^{i\epsilon})} \leq ||f||_{L^1(\mathbb{H})}$.

For $g \in C_c(\mathbb{H}/\Gamma^{i\epsilon})$,

$$\begin{aligned}
||I^{i\epsilon,\text{red}}g||_{L^1(\mathcal{G}_1^{i\epsilon})} &= \int_{\mathcal{G}_1^{i\epsilon}} |I^{i\epsilon,\text{red}}g\,(z,t)\,|d\mathcal{G}_1^{i\epsilon} \\
&= \int_{\mathbb{H}/\Gamma^{i\epsilon}} \left| \int_0^{2\pi} g\left((z,t)\gamma_1^{i\epsilon}(s)\right) ds \right| d(z,t)\Gamma^{i\epsilon} \\
&\leq \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^{i\epsilon}} |g\left((z,t)\gamma_1^{i\epsilon}(s)\right)| d(z,t)\Gamma^{i\epsilon} ds \\
&= \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^{i\epsilon}} |g\left((z,t)\right)| d\left((z,t)\gamma_1^{i\epsilon}(s)^{-1}\right)\Gamma^{i\epsilon} ds \\
&= \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^{i\epsilon}} |g\,(z,t)\,| d(z,t)\Gamma^{i\epsilon} ds, \qquad \text{since } \mathbb{H}/\Gamma^{i\epsilon} \text{ is unimodular,} \\
&= 2\pi ||g||_{L^1(\mathbb{H}/\Gamma^{i\epsilon})}.
\end{aligned}$$

Thus $P^{i\epsilon}$ and $I^{i\epsilon,\text{red}}$ extend to $L^1$ bounded maps. Given $f \in C_c(\mathbb{H})$, since $Pf \in C_c(\mathbb{H}/\Gamma^{\epsilon})$ and

$$\begin{aligned}
I^{i\epsilon,\text{red}}P^{i\epsilon}f(z,t) &= \int_0^{2\pi} \sum_{k\in\mathbb{Z}} f\left((z, t + k\pi(1-2\epsilon^2))\gamma_1^{i\epsilon}(s)\right) ds \\
&= \int_0^{2\pi} \sum_{k\in\mathbb{Z}} f\left((z,t)\gamma_1^{i\epsilon}(s+2\pi k)\right) ds, \quad \text{by (6.13),} \\
&= \sum_{k\in\mathbb{Z}} \int_0^{2\pi} f\left((z,t)\gamma_1(s+2\pi k)\right) ds \\
&= \sum_{k\in\mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} f\left((z,t)\gamma_1(s)\right) ds = I_1^{i\epsilon} f(z,t),
\end{aligned}$$

we have $||I_1^{i\epsilon}f||_{L^1(\mathcal{G}_1^{i\epsilon})} \leq 2\pi||f||_{L^1(\mathbb{H})}$. The third equality follows from uniform convergence of the integrand on the interval $[0, 2\pi] \ni s$. Therefore $I_1^{i\epsilon}$ extends to a bounded map from $L^1(\mathbb{H})$ to $L^1(\mathcal{G}_1^{i\epsilon})$. In particular one may check, using (6.5), that $||I_\lambda^{i\epsilon}f||_{L^1(\mathcal{G}_\lambda^{i\epsilon})} = ||I_1^{i\epsilon}f||_{L^1(\mathcal{G}_1^{i\epsilon})} \leq 2\pi||f||_{L^1(\mathbb{H})}$.

Finally we have, for $f \in L^1(\mathbb{H})$,

$$\begin{aligned}||If||_{L^1(\mathcal{G}^{i\epsilon})} :=& \int_{\mathcal{G}^{i\epsilon}} |I^{i\epsilon}f(z,t,\lambda)|d\mathcal{G}^{i\epsilon} \\ =& \int_0^\infty \int_{\mathcal{G}_\lambda^{i\epsilon}} |I_\lambda f(z,t)| d\mathcal{G}_\lambda^{i\epsilon} e^{-\lambda} d\lambda \\ \leq& 2\pi||f||_{L^1(\mathbb{H})} \int_0^\infty e^{-\lambda} d\lambda = 2\pi||f||_{L^1(\mathbb{H})}.\end{aligned}$$

as desired. □

We note a Poisson summation formula for $P^{i\epsilon}$:

**Lemma 10.** *For $f \in L^1(\mathbb{H})$, and $\epsilon\lambda \neq 1/\sqrt{2}$,*

$$\mathcal{F}_{\mathbb{H}/\Gamma_\lambda^{i\epsilon}}\left(P_\lambda^{i\epsilon}f\right)(n), = \mathcal{F}_{\mathbb{H}}(f)\left(\frac{n\lambda^2}{|1-2\epsilon^2\lambda^2|}\right), \quad \forall n \in \mathbb{Z}^*. \tag{6.18}$$

*Proof.* Observe that $\Gamma_\lambda^{i\epsilon} = |\lambda^{-2} - 2\epsilon^2|\Gamma$ and apply the Poisson Summation Formula (3.13) with $\tau = .|\lambda^{-2} - 2\epsilon^2|$.

□

Observe how the Fourier Transform respects dilations:

**Lemma 11.** *For $g \in L^1(\mathbb{H}/\Gamma^{i\epsilon})$, $\lambda > 0$,*

$$\mathcal{F}_{\mathbb{H}/\Gamma_\lambda^\epsilon}(\delta_\lambda^* g) = \lambda^{-4}\mathcal{F}_{\mathbb{H}/\Gamma^{\epsilon\lambda}}(g)(n), \quad \forall n \in \mathbb{Z}^*. \tag{6.19}$$

*Proof.* Observe that $\Gamma_\lambda^{i\epsilon} = |\lambda^{-2} - 2\epsilon^2|\Gamma$, and $\Gamma^{i\epsilon\lambda} = |1 - 2\epsilon^2\lambda^2|\Gamma$. Then apply Lemma 2. □

## 6.6 Fourier Slice Theorem for $I^{i\epsilon}$

As before, $I^{i\epsilon,\text{red}}$ is a convolution operator by a compactly supported measure. We compute its Fourier multiplier.

**Definition 10.** Define the compactly supported measure $\nu^{i\epsilon}$ on $\mathbb{H}/\Gamma^{i\epsilon}$ by

$$\int_{\mathbb{H}/\Gamma^{i\epsilon}} g(z,t)d\nu^{i\epsilon}(z,t) := \int_0^{2\pi} g\left(\gamma_1^{i\epsilon}(\theta)^{-1}\right) d\theta; \quad g \in C^\infty(\mathbb{H}/\Gamma^{i\epsilon}).$$

**Lemma 12.** *For $g \in L^1(\mathbb{H}/\Gamma)$,*

$$I^{i\epsilon,\text{red}} g = \nu^{i\epsilon} * g.$$

*Proof.*

$$\nu^{i\epsilon} * g(z,t) := \int_0^{2\pi} g\left((z,t)(w,s)^{-1}\right) d\nu^{i\epsilon}(w,s) = \int_0^{2\pi} g\left((z,t)\gamma_1^{i\epsilon}(s)\right) ds = I^{i\epsilon,\text{red}} g(z,t).$$

$\square$

**Proposition 19.** For $g \in L^1(\mathbb{H}/\Gamma^{i\epsilon})$, if $0 < \epsilon < 1/\sqrt{2}$, then

$$\mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}\left(I^{i\epsilon,\text{red}} g\right)(n) = 2\pi \mathcal{J}_n\left(\frac{1}{\sqrt{1-2\epsilon^2}}\right) \circ \mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}(g)(n), \quad \forall n \in \mathbb{Z}^*,$$

and if $\epsilon > 1/\sqrt{2}$, then

$$\mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}\left(I^{i\epsilon,\text{red}} g\right)(n) = 2\pi(-1)^n \mathcal{J}_n\left(\frac{1}{\sqrt{2\epsilon^2-1}}\right)^* \circ \mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}(g)(n), \quad \forall n \in \mathbb{Z}^*,$$

*Proof.* Applying the reduced Fourier Transform to $I^{i\epsilon,\text{red}}(g) = \nu^{i\epsilon} * g$ for $\epsilon \neq 1/\sqrt{2}$, we obtain

$$\mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}\left(I^{i\epsilon,\text{red}} g\right)(n) = \mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}(\nu^{i\epsilon})(n) \circ \mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}(g)(n).$$

So it remains to compute

$$
\begin{aligned}
\mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}\left(\nu^{i\epsilon}\right)(n) &:= \int_{\mathbb{H}/\Gamma^{\epsilon}} \beta_{n/|1-2\epsilon^2|}(z,t)^* d\nu^{\epsilon}(z,t) \\
&= \int_0^{2\pi} \beta_{n/|1-2\epsilon^2|}\left(\gamma_1^{i\epsilon}(s)^{-1}\right)^* ds \\
&= \int_0^{2\pi} \beta_{n/|1-2\epsilon^2|}\left(\gamma_1^{i\epsilon}(s)\right) ds, \quad \text{since } \beta_{n/|1-2\epsilon^2|} \text{ is a unitary rep,} \\
&= \int_0^{2\pi} \beta_{n/|1-2\epsilon^2|}\left(e^{is}, \frac{s}{2}(1-2\epsilon^2)\right) ds.
\end{aligned}
$$

For $\epsilon < 1/\sqrt{2}$,

$$
\begin{aligned}
\mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}\left(\nu^{i\epsilon}\right)(n) &= \int_0^{2\pi} \beta_{n/(1-2\epsilon^2)}\left(e^{is}, \frac{s}{2}(1-2\epsilon^2)\right) ds \\
&= \int_0^{2\pi} \beta_n\left(\frac{1}{\sqrt{1-2\epsilon^2}}e^{is}, \tfrac{s}{2}\right) ds \\
&= \mathcal{J}_n\left(\frac{1}{\sqrt{1-2\epsilon^2}}\right).
\end{aligned}
$$

For $\epsilon > 1/\sqrt{2}$

$$
\begin{aligned}
\mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}\left(\nu^{i\epsilon}\right)(n) &= \int_0^{2\pi} \beta_{n/(2\epsilon^2-1)}\left(e^{is}, \tfrac{s}{2}(1-2\epsilon^2)\right) ds \\
&= \int_0^{2\pi} \beta_n\left(\frac{1}{\sqrt{2\epsilon^2-1}}e^{is}, -\tfrac{s}{2}\right) ds \\
&= \int_{-\pi}^{\pi} \beta_n\left(\frac{1}{\sqrt{2\epsilon^2-1}}e^{i(s+\pi)}, -\tfrac{s+\pi}{2}\right) ds \\
&= \int_{-\pi}^{\pi} \beta_n\left(\frac{-1}{\sqrt{2\epsilon^2-1}}e^{is}, -\tfrac{s}{2}\right) e^{in\pi} ds \\
&= (-1)^n \int_{-\pi}^{\pi} \beta_n\left(\frac{1}{\sqrt{2\epsilon^2-1}}e^{is}, \tfrac{s}{2}\right)^* ds \\
&= (-1)^n \int_0^{2\pi} \beta_n\left(\frac{1}{\sqrt{2\epsilon^2-1}}e^{is}, \tfrac{s}{2}\right)^* ds \\
&= (-1)^n \mathcal{J}_n\left(\frac{1}{\sqrt{2\epsilon^2-1}}\right)^*
\end{aligned}
$$

as desired. $\square$

We may now prove the Heisenberg Fourier Slice Theorem for $g_{i\epsilon}$:

**Theorem 9** ($g_{i\epsilon}$ Heisenberg Fourier Slice Theorem)**.** *If $f \in L^1(\mathbb{H})$, and $\epsilon > 0$ then for $0 < \lambda < 1/(\sqrt{2}\epsilon)$, (note that all such geodesics are spacelike)*

$$\mathcal{F}_{\mathbb{H}/\Gamma^{\epsilon}_{\lambda}}\left(I^{i\epsilon}_{\lambda} f\right)(n) = (2\pi/\lambda)\mathcal{J}_n\left(\frac{1}{\sqrt{1-2\epsilon^2\lambda^2}}\right)\circ\mathcal{F}_{\mathbb{H}}(f)\left(\frac{n\lambda^2}{1-2\epsilon^2\lambda^2}\right); \quad \forall n \in \mathbb{Z}^*, \tag{6.20}$$

*and for $1/(\sqrt{2}\epsilon) < \lambda$,*

$$\mathcal{F}_{\mathbb{H}/\Gamma^{\epsilon}_{\lambda}}\left(I^{i\epsilon}_{\lambda} f\right)(n) = (2\pi/\lambda)(-1)^n\mathcal{J}_n\left(\frac{1}{\sqrt{2\epsilon^2\lambda^2-1}}\right)^*\circ\mathcal{F}_{\mathbb{H}}(f)\left(\frac{n\lambda^2}{2\epsilon^2\lambda^2-1}\right); \quad \forall n \in \mathbb{Z}^*, \tag{6.21}$$

*Proof.* We start with $\lambda = 1$. Combining Proposition 18 and 15,

$$\begin{aligned}
\mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}\left(I^{i\epsilon} f\right) &= 2\pi\mathcal{J}_n\left(\frac{1}{\sqrt{1-2\epsilon^2}}\right)\circ\mathcal{F}_{\mathbb{H}}(f)\left(\frac{n}{1-2\epsilon^2}\right); \quad \epsilon < 1/\sqrt{2} \qquad (6.22)\\
\mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}}\left(I^{i\epsilon} f\right) &= 2\pi(-1)^n\mathcal{J}_n\left(\frac{1}{\sqrt{2\epsilon^2-1}}\right)^*\circ\mathcal{F}_{\mathbb{H}}(f)\left(\frac{n}{1-2\epsilon^2}\right); \quad \epsilon > 1/\sqrt{2}
\end{aligned}$$

Now, exploiting homogeneity of $I^{i\epsilon}$, if $\epsilon < 1/(\lambda\sqrt{2})$, then

$$\begin{aligned}
\mathcal{F}_{\mathbb{H}/\Gamma^{\epsilon}_{\lambda}}\left(I^{i\epsilon}_{\lambda} f\right)(n) &= \lambda^{-1}\mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon}_{\lambda}}\left(\delta^*_{\lambda} I^{i\epsilon\lambda}_1\left(\delta^*_{1/\lambda} f\right)\right)(n), && \text{by Propositoin 17,}\\
&= \lambda^{-5}\mathcal{F}_{\mathbb{H}/\Gamma^{i\epsilon\lambda}}\left(I^{i\epsilon\lambda}_1\left(\delta^*_{1/\lambda} f\right)\right)(n), && \text{by Lemma 11,}\\
&= 2\pi\lambda^{-5}\mathcal{J}_n\left(\frac{1}{\sqrt{1-2\epsilon^2\lambda^2}}\right)\circ\mathcal{F}_{\mathbb{H}}(\delta^*_{1/\lambda} f)\left(\frac{n}{1-2\epsilon^2\lambda^2}\right), && \text{by (6.22),}\\
&= (2\pi/\lambda)\mathcal{J}_n\left(\frac{1}{\sqrt{1-2\epsilon^2\lambda^2}}\right)\circ\mathcal{F}_{\mathbb{H}}(f)\left(\frac{n\lambda^2}{1-2\epsilon^2\lambda^2}\right), && \text{by Lemma 11.}
\end{aligned}$$

and (6.21) follows by the same homogeneity argument with $\epsilon > 1/(\lambda\sqrt{2})$. ☐

**Proposition 20.** Let $\epsilon > 0$ and $n \in \mathbb{Z}^*$ be fixed. Then $\mathcal{J}_n\left(\frac{1}{\sqrt{|1-2\epsilon^2\lambda^2|}}\right) : \mathcal{H} \to \mathcal{H}$ is injective for almost all $\lambda > 0$.

*Proof.* Set $r = \frac{1}{\sqrt{|1-2\epsilon^2\lambda^2|}}$. By Corollary 3, the operator $\mathcal{J}_n(r)$ is injective if and only if $nr^2$ is not a zero of $L_j^{(n)}$ for any $j \in \mathbb{N}$. Since there are only countably many such zeros, the proposition follows. $\square$

**Theorem 10.** *For all $\epsilon > 0$, the Heisenberg taming X-ray transform $I^{i\epsilon} : L^1(\mathbb{H}) \to L^1(\mathcal{G}^{i\epsilon}, d\mathcal{G}^{i\epsilon})$ is injective. In particular, if $f \in L^1(\mathbb{H})$ and $I^{i\epsilon}_\lambda f = 0$ for all $\lambda$ in a neighborhood of zero, then $f = 0$. Furthermore, if neighborhood if sufficiently small, the X-ray transform integrates exclusively over spacelike geodesics.*

Suppose, $I^{i\epsilon}_\lambda(f) = 0$ for all $\lambda \in (0, \eta)$, where $0 < \eta < 1/(\sqrt{2}\epsilon)$. Then by Theorem 9 and Proposition 20,

$$0 = \mathcal{F}_{\mathbb{H}}(f)\left(\frac{n\lambda^2}{1-2\epsilon^2\lambda^2}\right) \tag{6.23}$$

for almost all $\lambda \in (0, \eta)$, and all $n \in \mathbb{Z}^*$. Let $A$ be the set of all such $\lambda \in (0, \eta)$, and $B = \{\lambda^2/(1-2\epsilon^2\lambda^2) : \lambda \in A\}$. Then in other words

$$0 = \mathcal{F}_{\mathbb{H}}(f)(h) \qquad\qquad \forall h \in \bigcup_{n\in\mathbb{Z}}^{\infty} nB.$$

Since $B$ has full-measure on the interval $(0, \frac{\eta^2}{(1-2\epsilon^2\eta^2)})$, we know $\mathcal{F}_{\mathbb{H}} f(h) = 0$ for almost all $h \in \mathbb{R}^*$. Therefore $f = 0$ by the Fourier Inversion Theorem.

**Remark 18.** Note that in the Theorem 10, by taking $\eta < 1/(\sqrt{2}\epsilon) < 1/\epsilon$ we restrict to *spacelike* geodesics. Therefore only integrals over spacelike geodesics are needed to determine a function $f \in L^1(\mathbb{H})$. However, inasmuch as the $(\mathbb{H}, g_{i\epsilon})$ models a real spacetime, sampling over such geodesics is not physically realizable.

In the timelike case, when $\lambda > 1/\epsilon$, we have

$$\frac{|n|}{2\epsilon^2} \leq \left|\frac{n\lambda^2}{2\epsilon^2\lambda^2 - 1}\right| < \frac{|n|}{\epsilon^2}, \quad n \in \mathbb{Z}^*.$$

Therefore we conjecture that the kernel of Lorentzian X-ray transform restricted to timelike geodesics is the set of all $f \in L^1(\mathbb{H})$ whose Fourier Transform $\mathcal{F}_{\mathbb{H}} f(h)$ is supported on $|h| \leq (2\epsilon)^{-1}$.

## 6.7 SVD Computations

For $\tau > 0$ the space ${}^0L^2(\mathbb{H}/\tau\Gamma)$ has orthonormal Hilbert basis given by the functions

$$\tau^{-1} m_{jk}^n(z/\sqrt{\tau}, t/\tau); \quad j, k \in \mathbb{N},\, n \in \mathbb{Z}^* \tag{6.24}$$

with respect to the inner product

$$\langle f, g\rangle_{L^2(\mathbb{H}/\tau\Gamma)} = \int_0^{\tau\pi} \int_{\mathbb{C}} f(z,t)\overline{g(z,t)}dzdt$$

which is seen by rescaling (3.22).

### 6.7.1 SVD of $I^{\epsilon,\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma^\epsilon)}$

**Proposition 21.**

$$I^{\epsilon,\text{red}} : L^p(\mathbb{H}/\Gamma^\epsilon) \to L^p(\mathbb{H}/\Gamma^\epsilon)$$

with $1 \leq p \leq \infty$ is well-defined and bounded.

*Proof.* This follows from the fact that $I^{\epsilon,\text{red}} g = \nu^\epsilon * g$ and that $\nu^\epsilon$ is compactly supported. $\square$

**Proposition 22.** $I^{\epsilon,\text{red}}$ preserves the orthogonal decomposition in Lemma 1. i.e.

$$I^{\epsilon,\text{red}}|_{L^2(\mathbb{C})} : L^2(\mathbb{C}) \to L^2(\mathbb{C}) \tag{6.25}$$

$$I^{\epsilon,\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma^\epsilon)} : {}^0L^2(\mathbb{H}/\Gamma^\epsilon) \to {}^0L^2(\mathbb{H}/\Gamma^\epsilon). \tag{6.26}$$

Furthermore, the restriction $I^{\epsilon,\text{red}}|_{L^2(\mathbb{C})}$ is essentially $2\pi$ times the Mean Value Transform $M^1$.

*Proof.* For $f \in L^2(\mathbb{C})$,

$$\begin{aligned} I^{\epsilon,\text{red}}|_{L^2(\mathbb{C})} f(z,t) &= \int_0^{2\pi} f\left((z,t)(e^{i\theta}, \theta/2)\right) d\theta \\ &= \int_0^{2\pi} f\left(z + e^{i\theta}, t + \theta(1+2\epsilon^2)/2 + \tfrac{1}{2}\text{Im}\left(\overline{z}e^{i\theta}\right)\right) d\theta \\ &= \int_0^{2\pi} f(z+e^{i\theta}) d\theta = 2\pi M^1 f(z), \end{aligned}$$

and so $I^{\epsilon,\text{red}} f \in L^2(\mathbb{C})$.

For $g \in {}^0L^2(\mathbb{H}/\Gamma^\epsilon)$,

$$\begin{aligned} \int_0^{\pi} I^{\epsilon,\text{red}} g(z,t) dt &= \int_0^{\pi} \int_0^{2\pi} g\left(z + e^{i\theta}, t + \theta(1+2\epsilon^2)/2 + \tfrac{1}{2}\text{Im}\left(\overline{z}e^{i\theta}\right)\right) d\theta dt \\ &= \int_0^{2\pi} \int_0^{\pi} g(z+e^{i\theta}, t) dt d\theta = 0, \end{aligned}$$

so that $I^{\epsilon,\text{red}} g \in {}^0L^2(\mathbb{H}/\Gamma^\epsilon)$. ∎

The functions

$$m^{\epsilon,n}_{jk}(z,t) := \frac{\sqrt{|n|}}{\pi(1+2\epsilon^2)} M^{n/(1+2\epsilon^2)}_{jk}(z,t)$$

obtained by setting $\tau = (1+2\epsilon^2)$ in (6.24), are an orthonormal Hilbert basis of ${}^0L^2(\mathbb{H}/\Gamma^\epsilon)$.

**Theorem 11** (SVD of $I^{\epsilon,\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma^\epsilon)}$)**.** *For all $j,k \in \mathbb{N}$ and $n \in \mathbb{Z}^*$,*

$$I^{\epsilon,\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma^\epsilon)} m^{\epsilon,n}_{jk} = 2\pi M^{n/(1+2\epsilon^2)}_{j,j+|n|}(1,0) m^{\epsilon,n}_{j+|n|,k}$$

*where*

$$M^{n/(1+2\epsilon^2)}_{j,j+|n|}(1,0) = (-1)^n \sqrt{\frac{j!}{(j+|n|)!}} \left(\frac{|n|}{1+2\epsilon^2}\right)^{|n|/2} L_j^{(|n|)}\left(\frac{|n|}{1+2\epsilon^2}\right) e^{\frac{-|n|/2}{1+2\epsilon^2}}$$

*Proof.* By multiplicativity of the matrix coefficients,

$$M^{n/(1+2\epsilon^2)}_{jk}\left((z,t)\gamma_1^\epsilon(\theta)\right) = \sum_{l=0}^{\infty} M^{n/(1+2\epsilon^2)}_{jl}\left(\gamma_1^\epsilon(\theta)\right) M^{n/(1+2\epsilon^2)}_{lk}(z,t)$$

and

$$\begin{aligned}
\int_0^{2\pi} M^{n/(1+2\epsilon^2)}_{jl}\left(\gamma_1^\epsilon(\theta)\right) d\theta &= \int_0^{2\pi} M^{n/(1+2\epsilon^2)}_{jl}\left(e^{i\theta}, \theta(1+2\epsilon^2)/2\right) d\theta \\
&= \int_0^{2\pi} e^{in\theta} M^n_{jl}\left(e^{i\theta}, 0\right) d\theta \\
&= 2\pi\delta(|n|+j-l) M^{n/(1+2\epsilon^2)}_{jl}(1,0)\,,
\end{aligned}$$

where the last equality follows from Lemma 6 with $\tau = 1+2\epsilon^2$. Thus

$$\begin{aligned}
I^{\epsilon,\mathrm{red}} M^{n/(1+2\epsilon^2)}_{jk}(z,t) &= \int_0^{2\pi} M^{n/(1+2\epsilon^2)}_{jk}\left((z,t)\gamma_1^\epsilon(\theta)\right) d\theta \\
&= 2\pi M^{n/(1+2\epsilon^2)}_{j,j+|n|}(1,0)\, M^{n/(1+2\epsilon^2)}_{j+|n|,k}(z,t)
\end{aligned}$$

Rescaling by the normalization factor $\frac{\sqrt{|n|}}{\pi(1+2\epsilon^2)}$ yields the result.

□

For $n \in \mathbb{Z}^*$ functions $f(z,t) \in \mathcal{H}_j^{n/(1+2\epsilon^2)} \subset C^\infty(\mathbb{H})$ are periodic in $t$ with period $\pi(1+2\epsilon^2)$. Thus, we may identify $\mathcal{H}_j^{n/(1+2\epsilon^2)}$ with a subspace of $L^2(\mathbb{H}/\Gamma^\epsilon)$. In fact

$${}^0L^2(\mathbb{H}/\Gamma^\epsilon) \cong \bigoplus_{j\in\mathbb{N};\, n\in\mathbb{Z}^*} \mathcal{H}_j^{n/(1+2\epsilon^2)}.$$

This is essentially (3.22).

**Corollary 5.** $I^{\epsilon,\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma^\epsilon)}$ decomposes as a collection of weighted shifts on the spaces of matrix coefficients

$$I^{\epsilon,\text{red}}|_{\mathcal{H}_j^{n/(1+2\epsilon^2)}} : \mathcal{H}_j^{n/(1+2\epsilon^2)} \to \mathcal{H}_{j+|n|}^{n/(1+2\epsilon^2)} \tag{6.27}$$

where component each is essentially a multiple of powers of the creation operator:

$$I^{\epsilon,\text{red}}|_{\mathcal{H}_j^{n/(1+2\epsilon^2)}} = 2\pi(-1)^n \frac{j!}{(j+|n|)!} \left(\frac{|n|}{1+2\epsilon^2}\right)^{|n|/2} L_j^{(|n|)}\left(\frac{|n|}{1+2\epsilon^2}\right) e^{\frac{-|n|/2}{1+2\epsilon^2}} \left(-\overline{Z}\right)^{|n|}. \tag{6.28}$$

### 6.7.2 SVD of $I^{i\epsilon,\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma^{i\epsilon})}$

**Proposition 23.**

$$I^{i\epsilon,\text{red}} : L^p(\mathbb{H}/\Gamma^{i\epsilon}) \to L^p(\mathbb{H}/\Gamma^{i\epsilon})$$

with $1 \leq p \leq \infty$ is well-defined and bounded.

*Proof.* This follows from the fact that $I^{i\epsilon,\text{red}}g = \nu^{i\epsilon} * g$ and $\nu^{i\epsilon} \in \mathcal{E}'(\mathbb{H}/\Gamma^{i\epsilon})$. □

**Proposition 24.** $I^{i\epsilon,\text{red}}$ preserves the orthogonal decomposition in Lemma 1. i.e.

$$I^{i\epsilon,\text{red}}|_{L^2(\mathbb{C})} : L^2(\mathbb{C}) \to L^2(\mathbb{C}) \tag{6.29}$$

$$I^{i\epsilon,\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma^{i\epsilon})} : {}^0L^2(\mathbb{H}/\Gamma^{i\epsilon}) \to {}^0L^2(\mathbb{H}/\Gamma^{i\epsilon}). \tag{6.30}$$

Furthermore, the restriction $I^{i\epsilon,\text{red}}|_{L^2(\mathbb{C})}$ is essentially $2\pi$ times the Mean Value Transform $M^1$.

*Proof.* For $f \in L^2(\mathbb{C})$,

$$\begin{aligned} I^{i\epsilon,\mathrm{red}}|_{L^2(\mathbb{C})} f(z,t) &= \int_0^{2\pi} f\left((z,t)(e^{i\theta},\theta/2)\right) d\theta \\ &= \int_0^{2\pi} f\left(z+e^{i\theta}, t+\theta(1-2\epsilon^2)/2 + \tfrac{1}{2}\mathrm{Im}\left(\overline{z}e^{i\theta}\right)\right) d\theta \\ &= \int_0^{2\pi} f(z+e^{i\theta}) d\theta = 2\pi M^1 f(z), \end{aligned}$$

and so $I^{i\epsilon,\mathrm{red}} f \in L^2(\mathbb{C})$.

For $g \in {}^0L^2(\mathbb{H}/\Gamma^{i\epsilon})$,

$$\begin{aligned} \int_0^{\pi} I^{i\epsilon,\mathrm{red}} g(z,t) dt &= \int_0^{\pi} \int_0^{2\pi} g\left(z+e^{i\theta}, t+\theta(1-2\epsilon^2)/2 + \tfrac{1}{2}\mathrm{Im}\left(\overline{z}e^{i\theta}\right)\right) d\theta dt \\ &= \int_0^{2\pi} \int_0^{\pi} g(z+e^{i\theta}, t) dt d\theta = 0, \end{aligned}$$

so that $I^{i\epsilon,\mathrm{red}} g \in {}^0L^2(\mathbb{H}/\Gamma^{i\epsilon})$. □

For $\epsilon \neq 1/\sqrt{2}$, the functions

$$m_{jk}^{i\epsilon,n}(z,t) := \frac{\sqrt{|n|}}{\pi|1-2\epsilon^2|} M_{jk}^{n/(1-2\epsilon^2)}(z,t)$$

obtained by setting $\tau = (1-2\epsilon^2)$ in (6.24), are an orthonormal Hilbert basis of ${}^0L^2(\mathbb{H}/\Gamma^{i\epsilon})$.

**Theorem 12** (SVD of $I^{i\epsilon,\mathrm{red}}|_{{}^0L^2(\mathbb{H}/\Gamma^{i\epsilon})}$). *Suppose $\epsilon \neq 1/\sqrt{2}$.*

*Timelike case: if $\epsilon \in (0, \frac{1}{\sqrt{2}})$, then for all $j,k \in \mathbb{N}$ and $n \in \mathbb{Z}^*$,*

$$I^{i\epsilon,\mathrm{red}}|_{{}^0L^2(\mathbb{H}/\Gamma^{i\epsilon})} m_{jk}^{i\epsilon,n} = 2\pi M_{j,j+|n|}^{n/(1-2\epsilon^2)}(1,0)\, m_{j+|n|,k}^{i\epsilon,n}$$

*where*

$$M_{j,j+|n|}^{n/(1-2\epsilon^2)}(1,0) = (-1)^n \sqrt{\frac{j!}{(j+|n|)!}} \left(\frac{|n|}{1-2\epsilon^2}\right)^{|n|/2} L_j^{(|n|)}\left(\frac{|n|}{1-2\epsilon^2}\right) e^{\frac{-|n|/2}{(1-2\epsilon^2)}}.$$

*Spacelike case: if $\epsilon \in (\frac{1}{\sqrt{2}}, \infty)$, then for all $j, k \in \mathbb{N}$ and $n \in \mathbb{Z}^*$,*

$$I^{i\epsilon,\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma^{i\epsilon})} m_{jk}^{i\epsilon,n} = 2\pi M_{j,j-|n|}^{n/(1-2\epsilon^2)}(1,0)\, m_{j-|n|,k}^{i\epsilon,n}$$

*where*

$$M_{j,j-|n|}^{n/(1-2\epsilon^2)}(1,0) = (-1)^n \sqrt{\frac{(j-|n|)!}{j!}} \left(\frac{|n|}{2\epsilon^2-1}\right)^{|n|/2} L_j^{(|n|)}\left(\frac{|n|}{2\epsilon^2-1}\right) e^{\frac{-|n|/2}{(2\epsilon^2-1)}}$$

*and where we set $M_{j,j-|n|}^{n/(1-2\epsilon^2)}(1,0) := 0$ if $j - |n| < 0$.*

*Proof.* By multiplicativity of the matrix coefficients,

$$M_{jk}^{n/(1-2\epsilon^2)}\left((z,t)\gamma_1^{i\epsilon}(\theta)\right) = \sum_{l=0}^{\infty} M_{jl}^{n/(1-2\epsilon^2)}\left(\gamma_1^{i\epsilon}(\theta)\right) M_{lk}^{n/(1-2\epsilon^2)}(z,t).$$

If $\epsilon \neq 1/\sqrt{2}$, then

$$\begin{aligned}
\int_0^{2\pi} M_{jl}^{n/(1-2\epsilon^2)}\left(\gamma_1^{i\epsilon}(\theta)\right) d\theta &= \int_0^{2\pi} M_{jl}^{n/(1-2\epsilon^2)}\left(e^{i\theta}, (1-2\epsilon^2)\theta/2\right) d\theta \\
&= \int_0^{2\pi} e^{in\theta} M_{jl}^{n/(1-2\epsilon^2)}\left(e^{i\theta}, 0\right) d\theta \\
&= 2\pi M_{jl}^{n/(1-2\epsilon^2)}(1,0) \begin{cases} \delta(|n|+j-l), & \text{if } \epsilon \in (0, \frac{1}{\sqrt{2}}) \\ \delta(|n|+l-j), & \text{if } \epsilon \in (\frac{1}{\sqrt{2}}, \infty). \end{cases}
\end{aligned}$$

so for $\epsilon \in (0, \frac{1}{\sqrt{2}})$

$$\begin{aligned}
I^{i\epsilon,\text{red}} M_{jk}^{n/(1-2\epsilon^2)}(z,t) &= \int_0^{2\pi} M_{jk}^{n/(1-2\epsilon^2)}\left((z,t)\gamma_1^{i\epsilon}(\theta)\right) d\theta \\
&= 2\pi M_{j,j+|n|}^{n/(1-2\epsilon^2)}(1,0)\, M_{j+|n|,k}^{n/(1-2\epsilon^2)}(z,t)
\end{aligned}$$

and for $\epsilon \in (\frac{1}{\sqrt{2}}, \infty)$,

$$\begin{aligned}
I^{i\epsilon,\text{red}} M_{jk}^{n/(1-2\epsilon^2)}(z,t) &= \int_0^{2\pi} M_{jk}^{n/(1-2\epsilon^2)}\left((z,t)\gamma_1^{i\epsilon}(\theta)\right) d\theta \\
&= 2\pi M_{j,j-|n|}^{n/(1-2\epsilon^2)}(1,0)\, M_{j-|n|,k}^{n/(1-2\epsilon^2)}(z,t)
\end{aligned}$$

Rescaling by the normalization factor $\frac{\sqrt{|n|}}{\pi|1-2\epsilon^2|}$ yields the result.

For $n \in \mathbb{Z}^*$ functions $f(z,t) \in \mathcal{H}_j^{n/(1-2\epsilon^2)} \subset C^\infty(\mathbb{H})$ are periodic in $t$ with period $\pi|1-2\epsilon^2|$. Thus, we may identify $\mathcal{H}_j^{n/(1-2\epsilon^2)}$ with a subspace of $L^2(\mathbb{H}/\Gamma^{i\epsilon})$. In fact

$$^0L^2(\mathbb{H}/\Gamma^{i\epsilon}) \cong \bigoplus_{j\in\mathbb{N};\, n\in\mathbb{Z}^*} \mathcal{H}_j^{n/(1-2\epsilon^2)}.$$

**Corollary 6.** $I^{i\epsilon,\mathrm{red}}|_{^0L^2(\mathbb{H}/\Gamma^{i\epsilon})}$ decomposes as a collection of weighted shifts on the spaces of matrix coefficients:

If $\epsilon \in (0, \frac{1}{\sqrt{2}})$ then

$$I^{i\epsilon,\mathrm{red}}|_{\mathcal{H}_j^{n/(1-2\epsilon^2)}} : \mathcal{H}_j^{n/(1-2\epsilon^2)} \to \mathcal{H}_{j+|n|}^{n/(1-2\epsilon^2)} \tag{6.31}$$

where component each is essentially a multiple of powers of the creation operator:

$$I^{i\epsilon,\mathrm{red}}|_{\mathcal{H}_j^{n/(1-2\epsilon^2)}} = 2\pi(-1)^n \frac{j!}{(j+|n|)!} \left(\frac{|n|}{1-2\epsilon^2}\right)^{|n|/2} L_j^{(|n|)}\left(\frac{|n|}{1-2\epsilon^2}\right) e^{\frac{-|n|/2}{(1-2\epsilon^2)}} \left(-\overline{Z}\right)^{|n|}. \tag{6.32}$$

If $\epsilon \in (\frac{1}{\sqrt{2}}, \infty)$ then

$$I^{i\epsilon,\mathrm{red}}|_{\mathcal{H}_j^{n/(1-2\epsilon^2)}} : \mathcal{H}_j^{n/(1-2\epsilon^2)} \to \mathcal{H}_{j-|n|}^{n/(1-2\epsilon^2)} \tag{6.33}$$

where component each is essentially a multiple of powers of the annihilation operator:

$$I^{i\epsilon,\mathrm{red}}|_{\mathcal{H}_j^{n/(1-2\epsilon^2)}} = 2\pi(-1)^n \frac{(j-|n|)!}{j!} \left(\frac{|n|}{2\epsilon^2-1}\right)^{|n|/2} L_j^{(|n|)}\left(\frac{|n|}{2\epsilon^2-1}\right) e^{\frac{-|n|/2}{(2\epsilon^2-1)}} Z^{|n|}. \tag{6.34}$$

$\square$

# Part IV

# Future Work

# Chapter 7

# Next Steps Regarding $I_\lambda$

## 7.1 Stability and Smoothening Properties

It is well-known that the Euclidean X-ray transform in a disc $\mathbb{R}^n$ is a smoothening operator of order $-\frac{1}{2}$, and that the transform is stable. This amount to showing the X-ray transform in bounded above and below with respect to the right Sobolev norms. The natural question is to prove the analogous theorem for the Sololev spaces associated to the Heisenberg Laplacian. See [9][12].

## 7.2 Explicit Inversion Formulas

For $n \in 2\mathbb{Z}+1$ one might expect to choose a left inverse of $\mathcal{J}_n$ (such as $\mathcal{J}_n^{-1} := (\mathcal{J}_n^* \mathcal{J}_n)^{-1} \mathcal{J}_n$) and apply the Fourier Inversion theorem to

$$(\lambda/2\pi)\mathcal{J}_n^{-1}\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}\left(I_\lambda f\right)(n) = \mathcal{F}_{\mathbb{H}}(f)(n\lambda^2); \quad f \in S(\mathbb{H});\ n \in 2\mathbb{Z}+1$$

(with $h = n\lambda^2$) to obtain an inversion formula for $If$.

$$\frac{n^2}{\pi^3}\sum_{\sigma=\pm1}\int_0^\infty \mathrm{Tr}\left(\beta_{\sigma n\lambda^2}(q)\circ\mathcal{J}_n^{-1}\circ\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}\left(I_\lambda f\right)(\sigma n)\right)\lambda^4 d\lambda = f(q); \quad f\in S(\mathbb{H}).$$

However, the operator $\mathcal{J}_n^{-1}$ is not exactly explicit. Furthermore, since $\mathcal{J}_n$ is compact, its inverse is not bounded and therefore this is not a stable inversion formula. Fortunately, the inversion formula is completely independent of $n$. We expect that the right approach to a more explicit and stable inversion formula is to find a "generating function" for the family of operators of the form

$$\sum_{n\in\mathbb{Z}^*} a_n\mathcal{J}_n = \mathcal{K}; \quad a_n\in\mathbb{C}$$

where $\mathcal{K}:\mathcal{H}\to\mathcal{H}$ is bounded below. We have, for example, that

$$\sum_{n=1}^{\infty}\left(\frac{n}{e}\right)^{n/2}\mathcal{J}_n F(\zeta) = \frac{1}{2\pi i}\oint\frac{F(\zeta+z)}{e^{\zeta/z}-z}dz, \quad F\in\mathcal{H} \tag{7.1}$$

counterclockwise over $|z|>|\zeta|$.

## 7.3 Support Theorems

Proving Helgason-type support theorems for the X-ray transform should be a straightforward application of the Paley-Wiener theory developed in Chapter 2.4 of [50], which relates support properties of a function $f$ on the Heisenberg group to decay properties of ts Weyl transform $Wf$.

# Part V

# Appendix

# Appendix A

# Further Results in Heisenberg Geometry

## A.1 A characterization of left-invariant geodesics on $\mathbb{H}$

We show that sub-Riemannian geodesics are 1-parameter subgroups of the Heisenberg Motion group:

**Definition 11.** The Heisenberg Motion group $HM = \mathbb{H} \rtimes U(1)$ has the multiplication law

$$\left((z,t);e^{i\theta}\right)\left((w,s);e^{i\phi}\right) = \left((z,t)(e^{i\theta}w,s);e^{i(\theta+\phi)}\right). \tag{A.1}$$

Let $X = \partial_x - \frac{1}{2}y\partial_t$, $Y = \partial_y + \frac{1}{2}x\partial_t$, and $T = \partial_t$ be the standard left-invariant vector fields on $\mathbb{H}$, and $\mathcal{D}_{(z,t)} = \text{Span}_{(z,t)}\{X,Y\}$.The standard the left-invariant vector fields on $HM$

are

$$\mathbb{X} = \cos\theta X + \sin\theta Y$$

$$\mathbb{X}_\perp = -\sin\theta X + \cos\theta Y$$

$$T = \partial_t$$

$$V = \partial_\theta.$$

**Definition 12.** We say that a left-invariant nondegenerate bilinear form on $T\mathbb{H}$ is compatible with the sub-Riemannian metric $g := dx^2 + dy^2|_{\mathcal{D}}$, if its restriction to $\mathcal{D}$ agrees with $g$.

**Theorem 13.** *The compatible metrics for the sub-Riemannian metric on the Heisenberg group are exactly the family of left-invariant Riemannian metrics,*

$$g_\epsilon := dx^2 + dy^2 + (1/\epsilon^2)\Theta^2 \tag{A.2}$$

*together with the family of left-invariant Lorentzian metrics*

$$g_{i\epsilon} := dx^2 + dy^2 - (1/\epsilon^2)\Theta^2 \tag{A.3}$$

*for all $\epsilon > 0$.*

**Theorem 14.** *Every complete geodesic for $g_\epsilon$, $g_{i\epsilon}$ or $g$ on $\mathbb{H}$ is a projection via* $\mathrm{pr} : HM \to \mathbb{H}$; $(z, t; e^{i\theta}) \to (z, t)$ *of a 1-parameter subgroup of $HM$ to $\mathbb{H}$ or its left translate by $(z, t; e^{i\theta})$. In particular, if*

$$\gamma(s) = \left(z, t; e^{i\theta}\right) e^{sW} \in HM; \quad W = \mathbb{X} + cT + \lambda V;\ c, \lambda \in \mathbb{R}$$

*then the curve (defined for $\lambda = 0$ by taking the pointwise limit as $\lambda \to 0$),*

$$\mathrm{pr} \circ \gamma(s) = \left( \frac{e^{i\lambda s} - 1}{i\lambda}, cs + \frac{\lambda s - \sin \lambda s}{2\lambda^2} \right),$$

*1) is a geodesic for the Riemannian metric $g_\epsilon$ when $c = \epsilon^2 \lambda$ ,*

*2) is a geodesic for Lorentzian metric $g_{i\epsilon}$ when $c = -\epsilon^2 \lambda$,*

*3) is a geodesic for sub-Riemannian metric $g$ when $c = 0$.*

*Furthermore, the projection of every 1-parameter subgroup on $HM$ is a geodesic for the sub-Riemannian or a compatible metric.*

***Proof.*** Since the quotient

$$q : HM \to SE(2) : \left(z, t; e^{i\theta}\right) \to (z; e^{i\theta}) \tag{A.4}$$

is a homomorphism, we have

$$q_* \mathbb{X} = \cos\theta \partial_x + \sin\theta \partial_y \tag{A.5}$$

$$q_* \mathbb{X}_\perp = -\sin\theta \partial_x + \cos\theta \partial_y \tag{A.6}$$

$$q_* T = 0 \tag{A.7}$$

$$q_* V = \partial_\theta \tag{A.8}$$

If $\gamma : \mathbb{R} \to HM$ is a 1-parameter subgroup, then so is $q \circ \gamma : \mathbb{R} \to SE(2)$, i.e. $q \circ \gamma(s) = e^{sW}$ for some $W \in \mathfrak{se}(2)$ on. There exists constants $c, \lambda \in \mathbb{R}$ for which

$$\gamma'(s) = \mathbb{X} + cT + \lambda V, \quad \forall s \in \mathbb{R}. \tag{A.9}$$

Then since,

$$W = \frac{d}{ds} e^{sW}|_{s=0} = \frac{d}{ds}\left(q \circ \gamma(s)\right)|_{s=0} = dq \circ \left(\gamma'(0)\right) \tag{A.10}$$

$$= dq\left(\mathbb{X}\right) + c dq\left(T\right) + \lambda dq\left(v\right) \tag{A.11}$$

$$= q_* \mathbb{X} + \lambda q_* V \tag{A.12}$$

which is the magnetic vector field for a constant magnetic field in $\mathbb{R}^2$, we have

$$q \circ \gamma(s) = \left(\frac{e^{i\lambda s} - 1}{i\lambda}; e^{i\lambda s}\right) \in SE(2) \tag{A.13}$$

Thus

$$\gamma(s) = \left(\frac{e^{i\lambda s} - 1}{i\lambda}, t(s);\ e^{i\lambda s}\right) \in HM \tag{A.14}$$

for some undetermined function $t(s)$.

Let $\Theta = dt - \frac{1}{2}\left(xdy - ydx\right) = dt - \frac{1}{2}\mathrm{Im}\left(\overline{z}dz\right)$ be the contact form on $\mathbb{H}$ with Reeb field $T$. Then since $\Theta(T) = 1$ and vanishes on the other vector fields, we have

$$c = \Theta\left(\gamma'(s)\right). \tag{A.15}$$

Then

$$cs = \int_0^s \Theta\left(\gamma'(\tau)\right) d\tau = \int_0^s t'(\tau) d\tau - \frac{1}{2}\int_\gamma \mathrm{Im}\left(\overline{z}dz\right) \tag{A.16}$$

and since $t(0) = 0$ (because $\gamma(0)$ is the identity in $HM$),

$$t(s) = cs + \frac{1}{2}\int_\gamma \mathrm{Im}\left(\overline{z}dz\right) = cs + \frac{1}{2}\left(\frac{\lambda s - \sin\lambda s}{2\lambda^2}\right) \tag{A.17}$$

from which we obtain

$$\gamma(s) = \left(\frac{e^{i\lambda s} - 1}{i\lambda}, cs + \frac{\lambda s - \sin\lambda s}{2\lambda^2};\ e^{i\lambda s}\right). \tag{A.18}$$

When $c = \epsilon^2$, the projection of (A.18) to $\mathbb{H}$ is the equation for a Riemannian geodesic for the metric $g_\epsilon$. When $c = -\epsilon^2$ it is a Lorentzian geodesic for $g_{i\epsilon}$. When $c = 0$, it is a sub-Riemannian geodesic for $g$. $\square$

## A.2 Exponential Map for Heisenberg Geodesics

The sub-Riemannian flow maps from the unit cotangent bundle $U^*\mathbb{H} := H^{-1}(\frac{1}{2})$ to itself. We work in the left-trivialization of the unit cotangent bundle: $U^*\mathbb{H} \cong \mathbb{H} \times \mathbb{S}^1 \times \mathbb{R} \ni (z, t, e^{i\phi}, \lambda)$. The exponential map $\exp : \mathbb{R} \times U^*\mathbb{H} \to \mathbb{H}$ is given in these coordinates by

$$\exp_{(z,t)}\left(s(e^{i\phi}, \lambda)\right) = (z,t)\begin{cases}\left(e^{i\phi}\frac{(e^{i\lambda s}-1)}{i\lambda}, \frac{\lambda s - \sin(\lambda s)}{2\lambda^2}\right) & \lambda \neq 0 \\ \left(se^{i\phi}, 0\right) & \lambda = 0\end{cases}. \tag{A.19}$$

(see [34, Ch. 1]), which describes the unit-speed geodesic with initial point $(z, t)$ whose projection to the plane is a counterclockwise circle of radius $R = 1/|\lambda|$, and has initial velocity in the direction of $\phi$ if $\lambda > 0$, and $\phi + \pi$ if $\lambda < 0$. If $\lambda = 0$ this projection is a strait line in the direction $\phi$. The geodesics in (1.1) are obtained by rotations and left translation of (A.19).

The Riemannian exponential map $\exp^\epsilon$ for $g_\epsilon$ is given in the same coordinates by

$$\exp^\epsilon_{(z,t)}\left(s(e^{i\phi}, \lambda)\right) = \exp_{(z,t)}\left(s(e^{i\phi}, \lambda)\right)(0, \epsilon^2\lambda s) \tag{A.20}$$

(see [34, Thm. 11.8] for an explanation). Because we are using cylindrical coordinates in the fibers, neither of these exponential maps describe geodesics with initial condition strictly in the $\lambda$ direction. In the case of $g$, these geodesics are fixed points in $\mathbb{H}$, and in the case of $g_\epsilon$ these geodesics are integral curves of the Reeb vector field $\epsilon^2\lambda T$. In both cases, the X-ray transforms are inverted without considering these geodesics.

## A.3 The Space of Geodesics as a Symplectic Reduction of the Cotangent Bundle

**Proposition 25.** $\mathcal{G}$ is the symplectic reduction of the cotangent bundle $T^*\mathbb{H}$ by the sub-Riemannian flow.

We will use the following lemma.

**Lemma 13.** *Suppose $G$ is a Lie group acting continuously on a manifold $M$. We say this action is proper if the map $G \times M \to M \times M;\ (g, p) \to (g \cdot p, p)$ is a proper map. That $G$ acts properly on $M$ is equivalent to the following statement:*

*If $\{p_i\}$ is a sequence in $M$ and $\{g_i\}$ is a sequence $G$ such that both $\{p_i\}$ and a $\{g_i \cdot p_i\}$ converge, then a subsequence of $\{g_i\}$ converges.*

See, of example [29]. For the following theorems on symplectic reduction, see [1].

**Definition 13.** Suppose $G$ is a Lie group with a Hamiltonian action on a symplectic manifold $(M, \omega)$. A moment map $\mu : M \to \mathfrak{g}^*$ is a map for which

$$\langle d\mu_p, A\rangle = \iota_{\underline{A}_p}\omega; \quad p \in M,\ A \in \mathfrak{g} \tag{A.21}$$

$$\mu_{g\cdot p} = Ad^*_g\mu; \quad p \in M,\ g \in G. \tag{A.22}$$

Here $\langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the dual pairing. We also implicitly use the identification $T_{\mu(p)}\mathfrak{g}^* \cong \mathfrak{g}^*$ and let $d\mu_p : T_pM \to \mathfrak{g}^*$.

**Theorem.** *Suppose $G$ is a Lie group having a Hamiltonian action on a symplectic manifold*

$(M, \omega)$ *with momentum map* $\mu : M \to \mathfrak{g}^*$. *Further suppose* $\eta \in \mathfrak{g}^*$ *is fixed by the coadjoint action of G. If the action is smooth, free and proper on* $\mu^{-1}(\eta)$, *then*

1) $\mu^{-1}(\eta)/G$ *is a smooth manifold of dimension dimM* $-$ *2dimG, and* $\pi : \mu^{-1}(\eta) \to \mu^{-1}(\eta)/G$ *is a principle G-bundle.*

2) *There is a unique symplectic form* $\omega^{red}$ *on* $\mu^{-1}(\eta)/G$ *for which* $\pi^*\omega^{red} = i^*\omega$. *Here* $i : \mu^{-1}(\eta) \to M$ *is the inclusion map.*

*Proof.* $\mathbb{R}$ acts on $T^*\mathbb{H}$ via the sub-Riemannian flow $\varphi : \mathbb{R} \times T^*\mathbb{H} \to T^*\mathbb{H}$; $(s, (q, p)) \mapsto \varphi_s(q, p)$. By definition, this action is Hamiltonian for

$$H = \frac{1}{2}\left(P_X^2 + P_Y^2\right). \tag{A.23}$$

Furthermore, for trivial reasons, $H : T^*\mathbb{H} \to \mathbb{R}$ is a momentum map for the action of $\mathbb{R}$ by the sub-Riemannian flow. Indeed,

$$dH = \iota_{\mathbf{X}}\omega \tag{A.24}$$

and

$$H\left(s \cdot (q, p)\right) = H\left(\varphi_s(q, p)\right) = H(q, p) = Ad_s^* H(q, p) \tag{A.25}$$

where the last line follow from the fact the $Ad_s^* = Id_{\mathbb{R}}$. If we take $\eta = 1/2$ then $\eta$ is coadjoint fixed since the coadjoint action of $\mathbb{R}$ is trivial, and $H^{-1}(\frac{1}{2}) = U^*\mathbb{H}$.

Because $\mathbb{R}$ acts on $T^*\mathbb{H}$ by the flow of a smooth vector field, the action is smooth. Since the geodesics exist for all time and none are closed, the action is free. Suppose the action is not proper on $U^*\mathbb{H}$. Then these exists sequences $\{s_i\}$ and $\{(q_i; \theta_i, \lambda_i)\}$ in $\mathbb{R}$ and $U^*\mathbb{H}$, respectively, for which

- $\{(q_i;\, \theta_i, \lambda_i)\}$ converges as $i \to \infty$

- $\{\varphi_{s_i}(q_i;\, \theta_i, \lambda_i)\}$ converges as $i \to \infty$

- $\{s_i\}$ has no convergent subsequence.

We may assume without loss of generality that $|s_i| \to \infty$ as $i \to \infty$

By (2.3), we have

$$\varphi_{s_i}(q_i;\, \theta_i, \lambda_i) = (q_i;\, e^{i\theta_i})\varphi_{s_i}(0; 0, \lambda_i) \tag{A.26}$$

Therefore $\varphi_{s_i}(0; 0, \lambda_i)$ converges as $i \to \infty$.

Passing to a subsequence of $\{\lambda_i\}$, if one such exists, suppose $\lambda_i = 0$ for all $i = 1, 2, ....$ Then

$$\varphi_{s_i}(0; 0, \lambda_i) = (s_i, 0; 0, 0)\,, \quad i = 1, 2, ... \tag{A.27}$$

which doesn't converge since $|s_i| \to \infty$ as $i \to \infty$, a contradiction.

Otherwise, passing to a subsequence of $\{\lambda_i\}$, suppose $\lambda_i \neq 0$ for all $i = 1, 2, ...$ Then

$$\varphi_{s_i}(0; 0, \lambda_i) = \left( \frac{e^{i\lambda_i s_i} - 1}{i\lambda_i}, \frac{\lambda_i s_i - \sin \lambda_i s_i}{2\lambda_i^2}; \lambda_i s_i, \lambda_i \right), \quad i = 1, 2, ... \tag{A.28}$$

But the term $\frac{\lambda_i s_i - \sin \lambda_i s_i}{2\lambda_i^2}$ does not converges as $i \to \infty$ since $\{\lambda_i\}$ converges and $|s_i| \to \infty$ , a contradiction.

Therefore, $\mathbb{R}$ acts properly on $U^*\mathbb{H}$. By the Marsden-Weinstein-Meyer Theorem, $\mathcal{G} \cong U^*\mathbb{H}/\mathbb{R}$ is a symplectic manifold with the unique symplectic form $\omega^{\text{red}}$ for which $\pi_{\mathcal{G}}^*\omega^{\text{red}} = i_{U^*\mathbb{H}}^*\omega$. Here $i_{U^*\mathbb{H}} : U^*\mathbb{H} \to T^*\mathbb{H}$ is the inclusion map, and $\pi_{\mathcal{G}} : U^*\mathbb{H} \to \mathcal{G}$ is the quotient map.

$\square$

We compute the symplectic form on $\mathcal{G}$ in a specific coordinate chart. The coordinate function $P_T : U^*\mathbb{H} \to \mathbb{R}$; $(z, t; \theta, \lambda) \mapsto \lambda$ is continuous. Since $\lambda$ is a constant of motion, $P_T$ descends to a function on $\mathcal{G}$. Because $\pi_{\mathcal{G}} : U^*\mathbb{H} \to \mathcal{G}$ is a smooth submersion, $P_T : \mathcal{G} \to \mathbb{R}$ is continuous. Thus $\mathcal{G}_+ := P_T^{-1}(0, \infty)$ is an open submanifold of $\mathcal{G}$.

The inclusion $i_{U^*\mathbb{H}} : U^*\mathbb{H} \to T^*\mathbb{H}$ induces the relations $P_X = \cos\theta$, $P_Y = \sin\theta$, and $P_T = \lambda$, where $\theta$ and $\lambda$ are the angular and vertical variables of the cylindrical fibers in the left-trivialization $U^*\mathbb{H} \cong \mathbb{H} \times \mathbb{S}^1 \times \mathbb{R}$. We then have

$$i_{U^*\mathbb{H}}^*\omega = (\cos\theta dy - \sin\theta dx) \wedge d\theta + \lambda dx \wedge dy + \Theta \wedge d\lambda. \tag{A.29}$$

Define the maps

$$\begin{tikzcd} \mathbb{H} \times (0, \infty) \arrow[r, "\Psi"] \arrow[dr, "\psi"'] & U^*\mathbb{H} \arrow[d, "\pi_{\mathcal{G}}"] \\ & \mathcal{G} \end{tikzcd}$$

where $\Psi(z, t, \lambda) = ((z, t); 0, \lambda) \in U^*\mathbb{H}$ and $\psi = \pi_{\mathcal{G}} \circ \Psi$. Then by inspecting the expression for the geodesic flow (2.5), we see that $\Psi(z, t, \lambda) = \Psi(z', t', \lambda')$ if and only if $z = z'$, $\lambda = \lambda'$ and $t = t' \pmod{\pi/\lambda^2}$. Thus $\psi : \mathbb{H} \times (0, \infty) \to \mathcal{G}_+$ is an injective smooth submersion, and hence a local diffeomorphism. In the coordinates above, we have

$$\begin{aligned} \psi^*\omega^{\text{red}} &= (\pi_{\mathcal{G}} \circ \Psi)^*\omega^{\text{red}} = \Psi^*\pi_{\mathcal{G}}^*\omega^{\text{red}} \\ &= \Psi^*\left(\omega|_{U^*\mathbb{H}}\right) \\ &= \lambda dx \wedge dy + \Theta \wedge d\lambda. \end{aligned}$$

Therefore, if we define $d\text{vol}_{\mathcal{G}} := \frac{1}{2}\left(\omega^{\text{red}} \wedge \omega^{\text{red}}\right)$, then

$$\begin{aligned}\psi^* d\text{vol}_{\mathcal{G}} :=& \tfrac{1}{2}\phi^*\left(\omega^{\text{red}} \wedge \omega^{\text{red}}\right) = \tfrac{1}{2}\phi^*\omega^{\text{red}} \wedge \phi^*\omega^{\text{red}} \\ =& \lambda dx \wedge dy \wedge d\Theta \wedge d\lambda \\ =& \lambda dx \wedge dy \wedge dt \wedge d\lambda.\end{aligned}$$

So in the above chart for $\mathcal{G}_\times$ we make the identification

$$d\text{vol}_{\mathcal{G}} = \lambda dx \wedge dy \wedge dt \wedge d\lambda. \tag{A.30}$$

The symplectic manifold $\mathcal{G}$ is foliated by leaves $\mathcal{G}_\lambda = P_T^{-1}(\lambda)$. For $\lambda \neq 0$, the volume form on $G_\lambda$ is given by

$$d\mathcal{G}_\lambda = \iota_{\partial_\lambda} d\text{vol}_{\mathcal{G}} = \lambda dx \wedge dy \wedge dt.$$

# Appendix B

# Representation Theory

## B.1 Schrödinger Representation

Recall the definition of group multiplication on $\mathbb{H} \cong \mathbb{R}^3$,

$$(x + iy, t)(u + iv, s) = \left(x + u + i(y + v), t + s + \tfrac{1}{2}(xv - uy)\right),$$

and its left invariant vector fields

$$X = \partial_x - \tfrac{1}{2}y\partial_t \qquad Y = \partial_y + \tfrac{1}{2}x\partial_t \qquad T = \partial_t.$$

There is a unique unitary action of $\mathbb{H}$ on $L^2(\mathbb{R})$ for which respectively $x$, $y$ and $z$ act on $\varphi \in L^2(\mathbb{R})$ via

$$\tau_x\varphi(\xi) := \varphi(\xi + x) \qquad m_y\varphi(\xi) := e^{iy\xi}\varphi(\xi) \qquad itI\varphi(\xi) := it\varphi(\xi).$$

Their infintesimal actions are given by

$$\rho_*X = \partial_\xi \qquad \rho_*Y = i\xi \qquad \rho_*T = iI.$$

Define the Schrödonger representation $\rho : \mathbb{R}^3 \to U(L^2(\mathbb{R}))$,

$$\rho(x, y, t)\varphi(\xi) := e^{(x\partial_\xi + yi\xi + tiI)}\varphi(\xi), \quad \varphi \in L^2(\mathbb{R}).$$

A application of the Baker-Campbell-Hausdorff formula gives

$$e^{x\partial_\xi + yi\xi + tiI}\varphi(\xi) = e^{i(t - \frac{i}{2}xy)}\tau_x m_y \varphi(\xi) = e^{it}e^{i(xy/2 + y\xi)}\varphi(\xi + x).$$

Consider the family of automorphisms of the Heisenberg group

$$S_{-1}(x, y, t) := (x, -y, -t), \quad \delta_{\sqrt{h}}(x, y, t) := (\sqrt{h}x, \sqrt{h}y, ht),\ h > 0.$$

Then for $h > 0$,

$$\begin{aligned} \rho_h :=& \rho \circ \delta_{\sqrt{h}} \\ \rho_{-h} :=& \rho \circ \delta_{\sqrt{h}} \circ S_{-1} \end{aligned}$$

are irreducible unitary representations of $\mathbb{H}$. These representations are pairwise inequivalent, and by the Stone-Von Neuman theorem, exhaust all irreducible unitary representations for which the center of $\mathbb{H}$ acts nontrivially. Furthermore,

$$\rho_{\pm h}(x, y, t)\varphi(\xi) = e^{\pm i(ht + hxy/2 + \sqrt{h}y\xi)}\varphi(\xi + \sqrt{h}x). \tag{B.1}$$

**Theorem 15** (Stone-von Neumann)**.** *Every unitary representation of $\mathbb{H}$ is unitarily equivalent to one and only one of the following representations.*

*i)* $(x, y, t) \in \mathbb{H} \mapsto \rho_h(x, y, t)$*;* $h \in \mathbb{R} \setminus \{0\}$

*ii)* $(x, y, t) \in \mathbb{H} \mapsto e^{i(x\xi + y\eta)}$*;* $\xi, \eta \in \mathbb{R}$*.*

## B.2 Fourier and Weyl Transforms

Since we will see later that the representations in $(ii)$ are negligible in the spectral decomposition of $f \in L^2(\mathbb{H})$, our definition of the Heisenberg Fourier Transform does not include them. Here we use $\hat{f}(h)$ to denote the Fourier transform defined using the Schrödonger representations.

**Definition 14** (Fourier Transform - Schrödonger Representation)**.** For a function $f \in L^1 \cap L^2(\mathbb{H})$, it's Heisenberg Fourier Transform $\hat{f}(h)$, for $h \neq 0$, is the operator on $L^2(\mathbb{R})$ given by

$$\begin{aligned}\hat{f}(h)\{\varphi\} &:= \int_{\mathbb{H}} f(z,t)\rho_h(z,t)^*\{\varphi\}dzdt, \quad \varphi \in L^2(\mathbb{R})\\ &= \int_{\mathbb{H}} f(x+iy,t)e^{ih(-t-y\xi+xy/2)}\varphi(\xi - x)dxdydt\end{aligned}$$

which we will show is a Hilbert-Schmidt operator on $L^2(\mathbb{R})$.

We care about this transform because it's inversion formula gives a useful spectral decomposition of $f \in L^2(\mathbb{H})$. Note that $\rho_h(z,t) = e^{iht}\rho_h(z,0)$, so the interesting part of $\hat{f}(h)$ is not in the $t$ direction:

$$\begin{aligned}\hat{f}(h) &= \int_{\mathbb{C}}\int_{\mathbb{R}} f(z,t)e^{-iht}\rho_h(z,0)^*dzdt\\ &= \int_{\mathbb{C}} f^h(z)\rho_h(z)^*dz,\end{aligned}$$

where $f^h(z) = \int_{\mathbb{R}} f(z,t)e^{-iht}dt$, and $\rho_h(z) := \rho_h(z,0)$. Thus, lets momentarily consider

**Definition 15** (Weyl Transform)**.** For a nice function $g \in L^1 \cap L^2(\mathbb{C})$, its Weyl Transform

$W_h(g)$ is an operator on $L^2(\mathbb{R})$ defined for $h \neq 0$ by

$$\begin{aligned} W_h(g)\{\varphi\} &:= \int_{\mathbb{C}} g(z)\rho_h(z)\{\varphi\}dz, \quad \varphi \in L^2(\mathbb{R}) \\ &= \int_{\mathbb{R}^2} g(x+iy)e^{i(y\xi+xy/2)}\varphi(\xi+x)dxdy \\ &= \int_{\mathbb{R}^2} g(x-\xi+iy)e^{iy(x/2+\xi/2)}\varphi(x)dxdy \end{aligned}$$

Set $h = 1$ for now and denote the Weyl transform by $W(g)$. Then for $\varphi \in L^2(\mathbb{R})$

$$W(g)\varphi(\xi) = \int_{\mathbb{R}} K_g(\xi, x)\varphi(x)dx, \qquad K_g(\xi, x) = \int_{\mathbb{R}} g(x-\xi, y)e^{\frac{i}{2}y(\xi+x)}dy \tag{B.2}$$

**Lemma 14.** *$W_h$ extends to an isometric embedding of $L^2(\mathbb{C})$ into Hilbert-Schmidt operators on $L^2(\mathbb{R}, 2\pi h^{-1}dh)$.*

*Proof.* For $g \in L^1 \cap L^2(\mathbb{C})$, we have

$$\begin{aligned} ||W(g)||_{HS}^2 &:= \int_{\mathbb{R}^2} |K_g(\xi, x)|^2 d\xi dx \\ &= (2\pi)\int_{\mathbb{R}^2} |g(x, y)|^2 dxdy, \end{aligned}$$

where the last step follows from the Euclidean Plancherel Theorem. For $h \neq 0$, this generalizes to

$$||W_h(g)||_{HS}^2 = 2\pi h^{-1}||g||_{L^2(\mathbb{R})}^2. \tag{B.3}$$

$\square$

**Proposition 26.** Since $W_h(g)$ is a Hilbert-Schmidt operator, it is also trace class. We may recover $g \in S(\mathbb{C})$ via

$$tr(\rho_h(z)^* W_h(g)) = (2\pi)|h|^{-1}g(z). \tag{B.4}$$

*Proof.* First,

$$\begin{aligned}
\rho(w)^*W(g) &= \int_{\mathbb{C}} g(z)\rho_h(-w)\rho_h(z)dz \\
&= \int_{\mathbb{C}} g(z+w)e^{-\frac{i}{2}\mathrm{im}(\overline{w}z)}\rho_h(z)dz \\
&= W(g_w), \quad g_w(z) := g(z+w)e^{-\frac{i}{2}\mathrm{im}(\overline{w}z)}.
\end{aligned}$$

Then, since $K_{g_w}(\xi,\xi) = \int_{\mathbb{R}} g_w(x,0)e^{ix\xi}dx$,

$$\begin{aligned}
tr(\rho(w)^*W(g)) := &\int_{\mathbb{R}} K_{g_w}(\xi,\xi)d\xi \\
= &\int_{\mathbb{R}}\int_{\mathbb{R}} g_w(x,0)e^{ix\xi}dxd\xi \\
= &2\pi g_w(0) = 2\pi g(w)
\end{aligned}$$

□

Therefore $g \mapsto W_h(g)$ is a one to one map from $L^2(\mathbb{C})$ to the set of Hilbert-Schmidt operators on $L^2(\mathbb{R})$.

**Theorem 16** (Plancherel)**.** *If $f \in L^1 \cap L^2(\mathbb{H})$, then*

$$||f||^2_{L^2(\mathbb{H})} = (2\pi)^{-2}\int_{\mathbb{R}} ||\hat{f}(h)||^2_{H.S.}|h|dh \tag{B.5}$$

*Proof.* Note that $\hat{f}(h) = W_h\left(\overline{f^h}\right)^*$. Then

$$||\hat{f}(h)||^2_{HS} = ||W_h\overline{f^h}||^2_{HS} = (2\pi)|h|^{-1}||\overline{f^h}||^2_{L^2(\mathbb{C})} = (2\pi)|h|^{-1}||f^h||^2_{L^2(\mathbb{C})} \tag{B.6}$$

so

$$\int ||\hat{f}(h)||^2_{HS}(2\pi)^{-2}|h|dh = (2\pi)^{-1}\int_{\mathbb{R}} ||f^h(z)||^2_{L^2(\mathbb{C})}dh \tag{B.7}$$

$$= ||f||^2_{L^2(\mathbb{H})} \tag{B.8}$$

where the last step follows from the Euclidean Plancherel theorem. ☐

**Theorem 17** (Fourier Inversion - Schrödinger Version). *If $f \in L^1 \cap L^2(\mathbb{H})$, then*

$$f(z,t) = \int_{\mathbb{R}} tr[\rho_h(z,t)\hat{f}(h)](2\pi)^{-2}|h|dh \tag{B.9}$$

*Proof.* Use Proposition 26 and the fact that $\hat{f}(h) = W_h\left(\overline{f^h}\right)^*$. ☐

## B.3 Fourier-Wigner Transform

Given $\varphi, \psi \in L^2(\mathbb{R})$, consider the function

$$V_\varphi(\psi, z) = (2\pi)^{-1/2}\left(\rho(z)\varphi, \psi\right)_{L^2(\mathbb{R})}.$$

This is called the Fourier-Wigner Transform of $\varphi$ and $\psi$. It is given explicitly by

$$V_\varphi(\psi, z) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{i\xi y}\varphi(\xi + x/2)\overline{\psi(\xi - x/2)}d\xi$$

**Theorem 18.** *Let $f, g, \varphi, \psi$ be in $L^2(\mathbb{R})$. Then*

$$(V_\varphi(\psi), V_f(g)) = (\varphi, f)(g, \psi).$$

*Proof.* By the usual Plancherel Theorem in the $y$ variable

$$\begin{aligned}
||V_\varphi(\psi, z)||^2_{L^2(\mathbb{C})} &= \int_{\mathbb{R}}\int_{\mathbb{R}}(2\pi)^{-1}\left|\mathcal{F}_\xi\left(\varphi(\xi + x/2)\overline{\psi(\xi - x/2)}\right)(-y)\right|^2 dxdy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(y + x/2)\overline{\psi(y - x/2)}dxdy \\
&= ||\varphi||^2_{L^2(\mathbb{R})}||\psi||^2_{L^2(\mathbb{R})}.
\end{aligned}$$

The claim follows by polarization. ☐

## B.4 Harmonic Oscillator and Hermite Functions

The operator

$$\hat{H} = -\Delta_{\mathbb{R}} + |\xi|^2 : S(\mathbb{R}) \to L^2(\mathbb{R}) \tag{B.10}$$

is called the quantum harmonic oscillator Hamiltonian. If $\Delta_{\mathbb{H}} = X^2 + Y^2$, then

$$(\rho_h)_*(-\Delta_{\mathbb{H}}) = |h|\hat{H}, \tag{B.11}$$

which is why we care about $\hat{H}$. $L^2(\mathbb{R})$ has an orthonormal basis of eigenvectors for $\hat{H}$, given by the Hermite functions indexed by $k = 0, 1, 2, ...$:

$$\psi_k(x) := c_k(-1)^k e^{x^2/2} \frac{d^k}{dx^k}\left(e^{-x^2}\right), \qquad c_k = (2^k\sqrt{\pi}k!)^{-1/2}. \tag{B.12}$$

Their eigenvalues are

$$\hat{H}\psi_k = (2k+1)\psi_k. \tag{B.13}$$

Alternately, induction on the product rule yields

$$\psi_k(x) = c_k\left(-\partial_x + x\right)^k e^{-x^2/2} = \frac{1}{\sqrt{k!}}\left(A^*\right)^k h_0(x), \tag{B.14}$$

where

$$A^* = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right) \qquad A := \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right). \tag{B.15}$$

In fact, for $k \geq 0$,

$$Ah_{k+1} = \sqrt{k+1}h_k \qquad A^*\psi_k = \sqrt{(k+1)}h_{k+1}. \tag{B.16}$$

**Remark 19.** For $Z$, $\overline{Z}$ defined as,

$$Z := \frac{1}{2}\left(X - iY\right) \qquad \overline{Z} := \frac{1}{2}\left(X + iY\right), \tag{B.17}$$

we have

$$\rho_*\left(\sqrt{2}Z\right) = A \qquad \rho_*\left(\sqrt{2}\overline{Z}\right) = -A^* \tag{B.18}$$

These creation and annihilation operators are related to the Harmonic Oscillator in the following ways

$$A^*A = \frac{1}{2}\left(\hat{H} - I\right) \qquad AA^* = \frac{1}{2}\left(\hat{H} + I\right), \tag{B.19}$$

Furthermore

$$A\hat{H} = 2AA^*A + A \qquad \hat{H}A = 2AA^*A - A \tag{B.20}$$

$$\hat{H}A^* = 2A^*AA^* + A^* \qquad A^*\hat{H} = 2A^*AA^* - A^*. \tag{B.21}$$

Combining these, we obtain

$$[A, \hat{H}] = 2A \qquad [A^*, \hat{H}] = -2A^*. \tag{B.22}$$

which are equivalent to

$$\hat{H}A = A(\hat{H} - 2) \qquad \hat{H}A^* = A^*(\hat{H} + 2). \tag{B.23}$$

## B.5 Bargmann Transform

**Definition 16** (Bargmann Kernel)**.** Define the kernel $B : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$,

$$\begin{aligned} B(x,w) :=& \sum_{k=0}^{\infty} \psi_k(x)\omega_k(w) \\ =& \sum_{k=0}^{\infty} \frac{H_k(x)}{\sqrt{2^k \pi^{1/2} k!}} e^{-x^2/2} \frac{w^k}{\sqrt{k!}} \\ =& \frac{1}{\pi^{1/4}} \sum_{k=0}^{\infty} H_k(x) \frac{(w/\sqrt{2})^k}{k!} e^{-x^2/2} \\ =& \frac{1}{\pi^{1/4}} e^{2x(w/\sqrt{2})-(w/\sqrt{2})^2} e^{-x^2/2} \\ =& \frac{1}{\pi^{1/4}} e^{-\frac{1}{2}(x-\sqrt{2}w)^2} e^{\frac{1}{2}w^2}. \end{aligned}$$

The series converges uniformly over compact subsets of $\mathbb{C}$.

**Definition 17** (Bargmann Transform)**.** The Bargmann Transform takes functions $f \in L^2(\mathbb{R})$ to entire functions $Bf(w)$ on $\mathbb{C}$. Let

$$Bf(w) := \int_{\mathbb{R}} B(x,w) f(x) dx \tag{B.24}$$

$$= \frac{1}{\pi^{1/4}} e^{\frac{1}{2}w^2} \int_{\mathbb{R}} f(x) e^{-\frac{1}{2}(x-\sqrt{2}w)^2} dx. \tag{B.25}$$

**Remark 20.** The constant $\pi^{-1/4}$ is a result of out normalization choices for the inner products on $L^2(\mathbb{R})$ and $\mathcal{H}$.

Note that, by construction,

$$B\psi_k = \omega_k. \tag{B.26}$$

**Theorem 19.** *The Bargmann Transform is an isometric isomorphism of $L^2(\mathbb{R})$ and $\mathcal{H}$.*

*Proof.* By the dominated convergence theorem

$$Bf(w) = \sum_{k=0}^{\infty} (f, \psi_k) \omega_k(w) \tag{B.27}$$

Hence $B\psi_k = \omega_k$. Then $B$ maps the orthonormal basis $\{h_0, h_1, ...\}$ of $L^2(\mathbb{R})$ to the orthonormal basis $\{\omega_0, \omega_1, ...\}$ of $\mathcal{H}$. In particular,

$$\int_{\mathbb{C}} |Bf(w)|^2 e^{-|w|^2} dw = \sum_{k=0}^{\infty} (f, \psi_k)^2 = \int_{\mathbb{R}} |f(x)|^2 dx. \tag{B.28}$$

Therefore $B$ is an isometric isomorphism from $L^2(\mathbb{R})$ to $\mathcal{H}$. $\square$

**Theorem 20.** *$B : L^2(\mathbb{R}) \to \mathcal{H}$ intertwines $\rho_{2h}$ and $\beta_h$.*

*Proof Sketch.* Since $\rho_{2h}$ and $\beta_h$ vary by precomposition of Heisenberg automorphisms as $h$ varies, it suffices to prove the proposition for $h = 1$. One may check that $B_*(A) = (1/\sqrt{2})\partial_\zeta$ and $B_*(A^*) = (-1/\sqrt{2})\zeta$. Thus $B_* \circ (\rho_2)_*(Z) = \beta_*(Z)$ and $B_* \circ (\rho_2)_*(\overline{Z}) = \beta_*(\overline{Z})$. By integrating these relations, one may obtain

$$B \circ \rho_2(z, t) = \beta(z, t) \circ B; \quad (z, t) \in \mathbb{H}.$$

$\square$

**Corollary 7.** The matrix coefficients for the Schrödinger representation $\rho_2$ with respect to the Hermite functions $\{\psi_k\}$ are the same as the matrix coefficients of the Bargmann-Fock representation $\beta_1$ with respect to the standard basis $\{\omega_k\}$.

*Proof.*

$$
\begin{aligned}
(\rho_{2h}(z,t)\psi_j,\psi_k)_{L^2(\mathbb{R})} &= (B\rho_{2h}(z,t)\psi_j, B\psi_k)_{\mathcal{H}}\,, \quad \text{by Theorem 19}\\
&= (\beta_h(z,t)B\psi_j, B\psi_k)_{\mathcal{H}}\,, \quad \text{by Theorem 20,}\\
&= (\beta_h(z,t)\omega_j,\omega_k)_{\mathcal{H}}\,, \quad \text{by (B.26)}\\
&= M^h_{jk}(z,t).
\end{aligned}
$$

□

In particular

$$
M^1_{jk}(z,0) = \left(\rho_1(\sqrt{2}z,0)\psi_j,\psi_k\right) = \sqrt{2\pi}V_{\psi_j}\left(\psi_k,\sqrt{2}z\right) \tag{B.29}
$$

One may also alternatively define the Weyl Transform using the Bargmann-Fock representations

$$
\widetilde{W}_h(g) := \int_{\mathbb{C}} g(z)\beta_h(z,0)dz : \mathcal{H} \to \mathcal{H}.
$$

Then $\widetilde{W}$ is related to the usual Weyl Transform via

$$
\widetilde{W}_h = B \circ W_{2h} \circ B^{-1}. \tag{B.30}
$$

We then have an analogous inversion theorem for $\widetilde{W}_h$:

$$
\operatorname{tr}\left(\beta_h(z,0)^*\widetilde{W}_h g\right) = \pi|h|^{-1}g(z). \tag{B.31}
$$

## B.6 Infinitesimal Representation

Define the complex vector fields on $\mathbb{H}$

$$
Z := \frac{1}{2}\left(X - iY\right), \qquad\qquad \overline{Z} := \frac{1}{2}\left(X + iY\right)
$$

where $X$ and $Y$ are given in (2.1). Then $\beta_h : \mathbb{H} \to \mathcal{U}(\mathcal{H})$ is the unique strongly continuous unitary representation of $\mathbb{H}$ on $\mathcal{H}$ for which, on the level of Lie algebras,

$$(\beta_h)_* Z = \sqrt{h}\partial_\zeta \qquad\qquad (\beta_h)_* \overline{Z} = -\sqrt{h}\zeta. \tag{B.32}$$

Note that $\zeta$ and $\partial_\zeta$ are the creation and annihilation operators on $\mathcal{H}$. One may check that

$$\beta_h(z,t) = e^{2iht+\sqrt{h}\left(z\partial_\zeta - \overline{z}\zeta\right)}$$

is the same as (3.1).

## B.7 Alternate Conventions

Folland [10] defines the Bargmann-Fock representation on the 1-parameter family of Hilbert spaces

$$\mathcal{H}^h := \left\{ F \in \mathrm{Hol}(\mathbb{C}) : h \int_{\mathbb{C}} |u(\zeta)|^2 e^{-\pi h |\zeta|^2} d\zeta < \infty \right\}, \quad h > 0,$$

and $\mathcal{H}^h := \left\{ F : \overline{F} \in \mathcal{H}^{|h|} \right\}$ for $h < 0$.

For $h \in \mathbb{R}^*$ and $\lambda > 0$, the maps

$$\begin{aligned} S_\lambda : \mathcal{H}^h &\to \mathcal{H}^{\lambda h}; & S(F)(\zeta) :=& F(\sqrt{\lambda}\zeta) \\ c : \mathcal{H}^h &\to \mathcal{H}^{-h}; & c(F) :=& \overline{F} \end{aligned}$$

are all isometries.

Folland defines the Fock representation, for $h > 0$, as

$$\beta_h^{\mathrm{Fol}}(z,t)F(\zeta) := e^{2\pi hit - \pi h \zeta \overline{z} - \pi h |z|^2/2} F(\zeta + z), \quad F \in \mathcal{H}^h$$

and $\beta_h^{\text{Fol}}(z,t) = c \circ \beta_{|h|}^{\text{Fol}}(\overline{z}, -t) \circ c$ for $h < 0$.

Our definition is rescaled so that every $\beta_h$ acts on the same space $\mathcal{H} = \mathcal{H}^{1/\pi}$. Follands definition, $\beta_h^{\text{Fol}}$, is related to ours via

$$\beta_h^{\text{Fol}}(z,t) = S_{\pi h} \circ \beta_{\pi h}(z,t) \circ S_{\pi h}^{-1}, \quad h > 0.$$

An advantage of this convention is that as $h$ varies, $\beta_h$ varies by precomposition with automorphisms of $\mathbb{H}$:

$$\beta_h(z,t) = \beta_1(\sqrt{h}z, ht), \quad \text{for } h > 0$$

$$\beta_h(z,t) = \beta_{|h|}(\overline{z}, -t), \quad \text{for } h < 0.$$

Granted, an advantage of Folland's definition is that the Fourier transform defined with $\beta_h^{\text{Fol}}$ does "converge" to the Euclidean Fourier transform as $h \to 0$.

# Appendix C

# Special Functions

### C.0.1 Bessel Functions

The classical Bessel function of order $n$ is defined by

$$J_n(r) := \frac{1}{2\pi i^n} \int_0^{2\pi} e^{ir\cos\theta} e^{-in\theta} d\theta.$$

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