GROUPS AND FIELDS WITH NTP$_2$

ARTEM CHERNIKOV, ITAY KAPLAN AND PIERRE SIMON

Abstract. NTP$_2$ is a large class of first-order theories defined by Shelah and generalizing simple and NIP theories. Algebraic examples of NTP$_2$ structures are given by ultra-products of $p$-adics and certain valued difference fields (such as a non-standard Frobenius automorphism living on an algebraically closed valued field of characteristic 0). In this note we present some results on groups and fields definable in NTP$_2$ structures. Most importantly, we isolate a chain condition for definable normal subgroups and use it to show that any NTP$_2$ field has only finitely many Artin-Schreier extensions. We also discuss a stronger chain condition coming from imposing bounds on burden of the theory (an appropriate analogue of weight), and show that every strongly dependent valued field is Kaplansky.

1. Introduction

The class of NTP$_2$ theories (i.e. theories without the tree property of the second kind) was introduced by Shelah [She80, She90]. It generalizes both simple and NIP theories and turns out to be a good context for the study of forking and dividing, even if one is only interested in NIP theories: in [CK12, Che12, BC12] it is demonstrated that the theory of forking in simple theories [Kim96] can be viewed as a special case of the theory of forking in NTP$_2$ theories over an extension base.

What are the known algebraic examples of NTP$_2$ theories?

Fact 1.1. [Che12] Let $\bar{K} = (K, \Gamma, k, v, ac)$ be a Henselian valued field of equicharacteristic 0 in the Denef-Pas language. Assume that $k$ is NTP$_2$ (respectively, $\Gamma$ and $k$ are strong, of finite burden — see Section 4). Then $\bar{K}$ is NTP$_2$ (respectively strong, of finite burden).

Example 1.2. Let $\mathcal{U}$ be a non-principal ultra-filter on the set of prime numbers $P$. Then:

1. $\bar{K} = \prod_{p \in P} \mathbb{Q}_p/\mathcal{U}$ is NTP$_2$. This follows from Fact 1.1 because:
   - The residue field is pseudo-finite, so of burden 1 (as burden is bounded by weight in a simple theory by [Adl07]).
   - The value group is a $\mathbb{Z}$-group, thus dp-minimal, and burden equals dp-rank in NIP theories by [Adl07].

   We remark that, while $\mathbb{Q}_p$ is dp-minimal for each $p$ by [DGL11], the field $\bar{K}$ is neither simple nor NIP even in the pure ring language (as the valuation ring is definable by [Ax65]).

The first author was partially supported by the [European Community’s] Seventh Framework Programme [FP7/2007-2013] under grant agreement n° 238381.

The second author was supported by SFB 879.
(2) $\bar{K} = \prod_{p \in P} F_p((t))/\mathcal{U}$ is NTP$_2$, of finite burden, as it has the same theory as the previous example by AK65 (while each of $F_p((t))$ has TP$_2$ by Corollary 3.3).

**Fact 1.3.** [CH12] Let $\bar{K} = (K, \Gamma, k, v, ac, \sigma)$ be a $\sigma$-Henselian contractive valued difference field of equicharacteristic 0, i.e., $\sigma$ is an automorphism of the field $K$ such that for all $x \in K$ with $v(x) > 0$ we have $v(\sigma(x)) > n \cdot v(x)$ for all $n \in \omega$ (see Azg10). Assume that both $(K, \sigma)$ and $(\Gamma, \sigma)$, with the naturally induced automorphisms, are NTP$_2$. Then $\bar{K}$ is NTP$_2$.

**Example 1.4.** Let $(F_p, \Gamma, k, v, \sigma)$ be an algebraically closed valued field of characteristic $p$ with $\sigma$ interpreted as the Frobenius automorphism. Then $\prod_{p \in P} F_p/\mathcal{U}$ is NTP$_2$. This case was studied by Hrushovski Hru and later by Durhan Azg10. It follows from Hru that the reduct to the field language is a model of ACFA, hence simple but not NIP. On the other hand this theory is not simple as the valuation group is definable.

Moreover, certain valued difference fields with a value preserving automorphism are NTP$_2$. Of course, any simple or NIP field is NTP$_2$, and there are further conjectural examples of pure NTP$_2$ fields such as bounded pseudo real closed or pseudo $p$-adically closed fields (see Section 5.1).

But what does knowing that a theory is NTP$_2$ tell us about properties of algebraic structures definable in it? In this note we show some initial implications. In Section 2 we isolate a chain condition for normal subgroups uniformly definable in an NTP$_2$ theory. In Section 3 we use it to demonstrate that every field definable in an NTP$_2$ theory has only finitely many Artin-Schreier extensions, generalizing some of the results of [KSW11]. In Section 4 we impose bounds on the burden, a quantitative refinement of NTP$_2$ similar to SU-rank in simple theories, and observe that some results for type-definable groups existing in the literature actually go through with a weaker assumption of bounded burden, e.g. every strong field is perfect, and every strongly dependent valued field is Kaplansky. The final section contains a discussion around the topics of the paper: we pose several conjectures about new possible examples (and non-examples) of NTP$_2$ fields and about definable envelopes of nilpotent/soluble groups in NTP$_2$ theories. We also remark how the stabilizer theorem of Hrushovski from [Hru12] could be combined with properties of forking established in CK12 and BC12 in the NTP$_2$ context.

We would like to thank Arno Fehm for his comments on Section 5.1 and Example 5.5. We would also like to thank the anonymous referee for many useful remarks.

**Preliminaries.** Our notation is standard. As usual, we will be working in a monster model $\mathfrak{C}$ of the theory under consideration. Let $G$ be a group, and $H$ a subgroup of $G$. We write $[G : H]$ $< \infty$ to denote that the index of $H$ in $G$ is bounded, which in the case of definable groups means finite. We assume that all groups (and fields) are finitary — contained in some finite Cartesian product of the monster.

**Definition 1.5.** We recall that a formula $\varphi(x, y)$ has TP$_2$ if there are tuples $(a_{i,j})_{i,j \in \omega}$ and $k \in \omega$ such that:

- $\{\varphi(x, a_{i,j}) \mid j < \omega\}$ is $k$-inconsistent, for each $i \in \omega$,
- $\{\varphi(x, a_{i,f(i)}) \mid i < \omega\}$ is consistent for each $f : \omega \to \omega$. 


A formula is NTP₂ otherwise, and a theory is called NTP₂ if no formula has TP₂.

**Fact 1.6.** \[\text{Theorem } T \text{ is NTP}_2 \text{ if and only if every formula } \varphi(x, y) \text{ with } |x| = 1 \text{ is NTP}_2.\]

We note that every simple or NIP formula is NTP₂. See \[\text{Che12}\] for more on NTP₂ theories.

2. Chain conditions for groups with NTP₂

**Lemma 2.1.** Let \(G\) be a definable group and \((H_i)_{i \in \omega}\) a uniformly definable family of normal subgroups of \(G\), with \(H_i = \varphi(x, a_i)\). Let \(H = \bigcap_{i \in \omega} H_i\) and \(H_{\neq j} = \bigcap_{i \in \omega \setminus \{j\}} H_i\). Then there is some \(i^* \in \omega\) such that \([H_{\neq i^*} : H]\) is finite.

**Proof.** Let \((H_i)_{i \in \omega}\) be given and assume that the conclusion fails. Then for each \(i \in \omega\) we can find \((b_{i, j})_{j \in \omega}\) with \(b_{i, j} \in H_{\neq i}\) and such that \((b_{i, j}H)_{j \in \omega}\) are pairwise different cosets in \(H_{\neq i}\). We have:

- \(b_{i, j}H_i \cap b_{k, i}H_i = \emptyset\) for \(j \neq k \in \omega\) and every \(i\).
- For every \(f : \omega \to \omega\), the intersection \(\bigcap_{i \in \omega} b_{i, f(i)}H_i\) is non-empty. Indeed, fix \(f\), by compactness it is enough to check that \(\bigcap_{i \leq n} b_{i, f(i)}H_i \neq \emptyset\) for every \(n \in \omega\). Take \(b = \prod_{i \leq n} b_{i, f(i)}\) (the order of the product does not matter). As \(b_{i, f(i)}H_i \in H_j\) for all \(i \neq j\), it follows by normality that \(b \in b_{i, f(i)}H_i\) for all \(i \leq n\).

But then \(\psi(x, y, z) = \exists w \left( \varphi(w, y) \land x = z \cdot w \right)\) has TP₂ as witnessed by the array \((c_{i, j})_{i, j \in \omega}\) with \(c_{i, j} = a_i b_{i, j}\). \(\Box\)

**Problem 2.2.** Is the same result true without the normality assumption? See also Theorem 1.12.

**Corollary 2.3.** Let \(T\) be NTP₂ and suppose that \(G\) is a definable group. Then for every \(\varphi(x, y)\) there are \(k_{\varphi}, n_{\varphi} \in \omega\) such that:

- If \((\varphi(x, a_i))_{i < K}\) is a family of normal subgroups of \(G\) and \(k_{\varphi} < K\) then there is some \(i^* < K\) such that \(\left| \bigcap_{i < K, i \neq i^*} \varphi(x, a_i) : \bigcap_{i < K} \varphi(x, a_i) \right| < n_{\varphi}\).

**Proof.** Follows from Lemma 2.1 and compactness. \(\Box\)

**Theorem 2.4.** Let \(G\) be NTP₂ and \(\{\varphi(x, a) \mid a \in C\}\) be a family of normal subgroups of \(G\). Then there is some \(k \in \omega\) (depending only on \(\varphi\)) such that for every finite \(C' \subseteq C\) there is some \(C_0 \subseteq C'\) with \(|C_0| \leq k\) and such that

\[
\left| \bigcap_{a \in C_0} \varphi(x, a) : \bigcap_{a \in C'} \varphi(x, a) \right| < \infty.
\]

**Proof.** Let \(k_{\varphi}\) be as given by Corollary 2.3. If \(|C'| > k_{\varphi}\), by Corollary 2.3 we find some \(a_0 \in C'\) such that \(\left| \bigcap_{a \in C' \setminus \{a_0\}} \varphi(x, a) : \bigcap_{a \in C'} \varphi(x, a) \right| < \infty\). If \(|C' \setminus \{a_0\}| > k_{\varphi}\), by Corollary 2.3 again we find some \(a_1 \in C' \setminus \{a_0\}\) such that

\[
\left| \bigcap_{a \in C' \setminus \{a_0, a_1\}} \varphi(x, a) : \bigcap_{a \in C' \setminus \{a_0\}} \varphi(x, a) \right| < \infty.
\]
Continuing in this way we end up with $a_0, \ldots, a_m \in C'$ such that for all $i < m$,

$$\left[ \bigcap_{a \in C' \setminus \{a_0, \ldots, a_{i+1}\}} \varphi(x, a) : a \in C' \setminus \{a_0, \ldots, a_i\} \right] < \infty,$$

and, letting $C_0 = C' \setminus \{a_0, \ldots, a_m\}$, we have that $|C_0| \leq k_\varphi$. □

**Corollary 2.5.** Let $G$ be a torsion-free group with NTP$_2$ and assume that $\varphi(x, y)$ defines a divisible normal subgroup for every $y$. Then $\varphi(x, y)$ is NIP.

**Proof.** Assume that $\varphi(x, y)$ has IP and let $\bar{a} = (a_i)_{i \in \mathbb{Z}}$ be an indiscernible sequence witnessing this. Taking $H_i = \varphi(\mathcal{E}, a_i)$, $H_{\neq 0} \setminus H_0 \neq \emptyset$. Let $H = \bigcap_{i \in \mathbb{Z}} H_i$, so it is divisible (here we used the assumption that $G$ is torsion-free) as is $H_{\neq 0}$. But then $H_{\neq 0}/H$ is a divisible non-trivial group, so infinite. By indiscernibility $|H_{\neq i} : H| = \infty$ for all $i$ — contradicting Lemma 2.1. □

3. **Fields with NTP$_2$**

Let $K$ be a field of characteristic $p > 0$. Recall that a field extension $L/K$ is called an Artin-Schreier extension if $L = K(\alpha)$ for some $\alpha \in L \setminus K$ such that $\alpha^p - \alpha \in K$. $L/K$ is an Artin-Schreier extension if and only if it is Galois and cyclic of degree $p$.

**Theorem 3.1.** Let $K$ be an infinite field definable in an NTP$_2$ theory. Then it has only finitely many Artin-Schreier extensions.

**Proof.** We follow the proof of the fact that dependent fields have no Artin-Schreier extensions in [KSW11].

We may assume that $K$ is $\aleph_0$-saturated, and we put $k = K^\infty = \bigcap_{n \in \mathbb{Z}} K^n$, a type-definable perfect sub-field which is infinite by saturation (all contained in an algebraically closed $K$).

For a tuple $\bar{a} = (a_0, \ldots, a_{n-1})$, let

$$G_{\bar{a}} = \{(t, x_0, \ldots, x_{n-1}) \in K^{n+1} : t = a_i \cdot \varphi(x_i) \text{ for } i < n\},$$

where $\varphi(x) = x^p - x$ is the Artin-Schreier polynomial. We consider it as an algebraic group (a subgroup of $(K^{n+1}, +)$). As such, by [KSW11] Lemma 2.8, when the elements of $\bar{a}$ are algebraically independent it is connected. If in addition $\bar{a}$ belong to some perfect field $k$, then $G_{\bar{a}}$ is isomorphic by an algebraic isomorphism over $k$ to $(K, +)$ by [KSW11] Corollary 2.9.

By Theorem 2.4 there is some $n < \omega$, an algebraically independent $(n + 1)$-tuple $\bar{a} \in k$ and an $n$-subtuple $\bar{a}'$, such that $|\bigcap_{a \in \bar{a}} a \cdot \varphi(K) : \bigcap_{a \in \bar{a}} a \cdot \varphi(K)| < \infty$. It follows that the image of the projection map $\pi : G_{\bar{a}}(K) \to G_{\bar{a}'}(K)$ has finite index in $G_{\bar{a}'}(K)$.

We have algebraic isomorphisms $G_{\bar{a}} \to (K, +)$ and $G_{\bar{a}'} \to (K, +)$ over $k$. Hence we can find an algebraic map $\rho$ over $k$ (i.e. a polynomial) which makes the following diagram commute:

$$\begin{array}{ccc}
G_{\bar{a}} & \xrightarrow{\pi} & G_{\bar{a}'} \\
\downarrow & & \downarrow \\
(K, +) & \xrightarrow{\rho} & (K, +)
\end{array}$$
As all groups and maps are defined over $k \subseteq K$, we can restrict to $K$. We saw that $[G_\alpha : \pi (G_\alpha (K))] < \infty$ so $[K : \rho (K)] < \infty$ as well (in the group $(K, +)$). In the proof of [KSW11] Theorem 4.3, it is shown that there is some $c \in K$ such that, letting $\rho^c(x) = \rho (c \cdot x)$, $\rho^c$ has the form $a \cdot \rho (x)$ for some $a \in K^\times$. The way it is done there is by choosing any $0 \neq c \in \ker (\rho) \subseteq k$, and then since $\rho^c$ is additive with kernel $F_p$ and degree $p$ (as this is the degree of $\pi$), there exists such an $a \in k$. Since $\rho^c (K) = \rho (K)$ has finite index in $K$, so does the image of $\rho = a^{-1} \rho^c$.

By [KSW11] Remark 2.3, this index is finite if and only if the number of Artin-Schreier extensions is finite.

**Proposition 3.2.** Suppose $(K, v, \Gamma)$ is a valued field of characteristic $p > 0$ that has finitely many Artin-Schreier extensions. Then the valuation group $\Gamma$ is $p$-divisible.

**Proof.** (This proof is similar to the proof of [KSW11] Proposition 5.4.) Recall that $\varrho$ is the Artin-Schreier polynomial. By Artin-Schreier theory (this is explained in [KSW11] Remark 2.3), the index $[K : \varrho (K)]$ in the additive group $(K, +)$ is finite. Suppose $\{a_i \mid i < l\}$ are representatives for the cosets of $\varrho (K)$ in $(K, +)$. Let $\alpha \in \Gamma$ be smaller than $\alpha_0 = \min \{v (a_i) \mid i < l\} \cup \{0\}$. Suppose $v (x) = \alpha$ for $x \in K$. But then there is some $i < l$ such that $x - a_i \in \varrho (K)$, and since $v (x) = v (x - a_i)$, we may assume that $x = a_i \in \varrho (K)$, so there is some $y$ such that $y^p - y = x$. But then, $v (y) < 0$, so $v (y^p) = p \cdot v (y) < v (y)$, so

$$\alpha = v (x) = v (y^p - y) = v (y^p) = p \cdot v (y).$$

So $\alpha$ is $p$-divisible. Take any negative $\beta \in \Gamma$, then $\beta + p \cdot \alpha_0$ is $p$-divisible, so $\beta$ is also $p$-divisible. Since this is true for all negative values, $\Gamma$ is $p$-divisible.

**Corollary 3.3.** $F_p ((t))$ is not NTP$_2$.

**Proof.** Follows from Theorem 5.1 and Proposition 5.2.

4. Strong theories and bounded burden

In this section we are going to consider groups and fields whose theories satisfy quantitative refinements of NTP$_2$ in terms of a bound on its burden (similar to the bounds on the rank in simple theories).

For notational convenience we consider an extension $\operatorname{Card}^+$ of the linear order on cardinals by adding a new maximal element $\omega$ and replacing every limit cardinal $\kappa$ by two new elements $\kappa_-$ and $\kappa_+$. The standard embedding of cardinals into $\operatorname{Card}^+$ identifies $\kappa$ with $\kappa_+$. In the following, whenever we take a supremum of a set of cardinals, we will be computing it in $\operatorname{Card}^+$.

**Definition 4.1.** Let $T$ be a complete theory.

1. An inp-pattern of depth $\kappa$ consists of $(\bar{a}_\alpha, \varphi_{\alpha, (x, y_{\alpha}), k_{\alpha}})_{\alpha \in \kappa}$ with $\bar{a}_\alpha = (a_{\alpha, i})_{i \leq \omega}$ and $k_{\alpha} \in \omega$ such that:
   - $\{ \varphi_{\alpha, (x, a_{\alpha, i})} \mid i < \omega \}$ is $k_{\alpha}$-inconsistent for every $\alpha \in \kappa$.
   - $\{ \varphi_{\alpha, (x, a_{\alpha, f (\alpha)})} \mid \alpha < \kappa \}$ is consistent for every $f : \kappa \to \omega$.

2. An inp$^+$-pattern of depth $\kappa$ consists of $(\bar{a}_\alpha, b_{\alpha}, \phi_{\alpha, (x, y_{\alpha}, z_{\alpha}), \beta})_{\alpha < \kappa}$, where $\phi_{\alpha, l, \alpha} = (a_{\alpha, i})_{i \leq \omega}$, and $b_{\alpha} \subseteq \bigcup \{ \bar{a}_\beta \mid \beta < \alpha \}$, such that:
   - $(\bar{a}_\alpha)_{\alpha < \kappa}$ are mutually indiscernible.
   - $\{ \phi_{\alpha, (x, a_{\alpha, i}, b_{\alpha})} \mid i < \omega \}$ is inconsistent for every $\alpha$.
   - $\{ \phi_{\alpha, (x, a_{\alpha, 0}, b_{\alpha})} \mid \alpha < \kappa \}$ is consistent.
Proposition 4.5.

3.11, Corollary 3.12] with some easy modifications:

4.1.

Strong groups and fields.

Let \( G \) be a type-definable group and \( G_i \leq G \) type-definable normal subgroups for \( i < \omega \).

(1) If \( T \) is strong, then there is some \( i_0 \) such that \( \bigcap_{i \neq i_0} G_i : \bigcap_{i < \omega} G_i \bigcup < \infty \).

(2) If \( T \) is of finite burden, then there is some \( n \in \omega \) and \( i_0 < n \) such that \( \bigcap_{i \neq i_0, i < n} G_i : \bigcap_{i < n} G_i \bigcup < \infty \).

Proof. (1) Assume not. Then, for each \( i < \omega \), we have an indiscernible sequence \( (a_{i,j})_{j<\omega} \) (over the parameters defining all the groups) such that \( a_{i,j} \in \bigcap_{k \neq i} G_k \) and for \( j_1 < j_2 < \omega \), \( a_{i,j_1}^{-1} \cdot a_{i,j_2} \notin G_i \). By compactness there is a formula \( \psi_i(x) \) in the type defining \( G_i \) such that \( \neg \psi_i \left( a_{i,j_1}^{-1} \cdot a_{i,j_2} \right) \) holds (by indiscernibility it is the same for all \( j_1 < j_2 \)). We may assume, applying Ramsey, that the sequences
\[ \{ \langle a_{i,j} \rangle_{i < \omega} \mid i < \omega \} \]
are mutually indiscernible. Let \( \psi_i' \) be another formula in the type
defining \( G_i \) such that \( \psi_i' (x) \land \psi_j' (y) \models \psi_i (x^{-1} \cdot y) \). Let \( \varphi_i (x,y) = \psi_i' (x^{-1} \cdot y) \).

Now we check that the set \( \{ \varphi_i (c,a_{i,0}) \mid i < n \} \) is consistent for each \( n < \omega \). Let \( c = a_{0,0} \cdots a_{n-1,0} \) (the order does not really matter, but for the proof it is easier
to fix one). So \( \varphi_i (c,a_{i,0}) \) holds if and only if \( \psi_i' (a_{n-1,0}^{-1} \cdots a_{i,0}^{-1} \cdots a_{0,0}^{-1} \cdot a_{i,0}) \)
holds. But since \( G_i \) is normal, \( a_{i,0}^{-1} \cdots a_{0,0}^{-1} \cdot a_{i,0} \in G_i \), so the entire product is in
\( G_i \), so \( \varphi_i (c,a_{i,0}) \) holds. On the other hand, if for some \( c' \), \( \varphi_i (c',a_{i,0}) \wedge \varphi_i (c',a_{i,1}) \)
holds, then \( \psi_i (a_{i,0}^{-1} \cdot a_{i,1}) \) holds — contradiction. So the rows are inconsistent which
contradicts strength.

(2) Follows from the proof of (1) using Fact 4.2. \( \square \)

**Corollary 4.6.** If \( G \) is an abelian group type-definable in a strong theory and \( S \subseteq \omega \) is
an infinite set of pairwise co-prime numbers, then for almost all (i.e. for all but
finitely many) \( n \in S \), \([G : G^n] < \infty \). In particular, if \( K \) is a definable field in a
strong theory, then for almost all primes \( p \), \([K^\times : (K^\times)^p] < \infty \).

**Proof.** Let \( K \subseteq S \) be the set of \( n \in S \) such that \([G : G^n] < \infty \). If \( S \setminus K \) is infinite, we replace \( S \) with \( S \setminus K \).

For \( i \in S \), let \( G_i = G_i \) (so it is type-definable). By Proposition 4.6, there is
some \( n \) such that \( \prod_{i \neq n} G_i : \prod_{i \in S} G_i \) \( < \infty \). Now it is enough to show that
\( \prod_{i \neq n} G_i / \prod_{i \in S} G_i \cong G/G_n \). For this we show that the natural map \( \prod_{i \neq n} G_i \to
G/G_n \) is onto. To show this, we may assume by compactness that \( S \) is finite. Let
\( r = \prod S \setminus \{ n \} \), then since \( r \) and \( n \) are co-prime, there are some \( a,b \in \mathbb{Z} \) such that
\( ar + bn = 1 \) so for any \( g \in G \), \( g^{ar} \equiv g \mod G_n \), and we are done. \( \square \)

The proof of the following proposition is taken from [KP11, Proposition 2.3] so we observe that it goes through in larger generality.

**Proposition 4.7.** Any infinite strong field is perfect.

**Proof.** Let \( K \) be of characteristic \( p > 0 \), and suppose that \( K^p \neq K \). Then there are
\( b_1,b_2 \in K \) linearly independent over \( K^p \). Let \( \langle a_i : i \in \mathbb{Q} \rangle \) be an indiscernible non-constant sequence over \( b_1,b_2 \). By compactness we can find \( a \) and \( (c_i)_{i < \omega} \) from
\( K \) such that \( c_0 = a \) and \( c_i = b_1 c_{i+1} + b_2 a_i^p \). Since \( b_1,b_2 \) are linearly independent over
\( K^p \), we get that \( a_i \in \text{dcl}(b_1,b_2 a) \) for every \( i < \omega \). For each \( i < \omega \), let \( \varphi_i (y_1,b_1,b_2,a) \) be a formula defining \( a_i \). We may assume that \( \forall x_1,y_1,y_2 \exists j = 1,2 \varphi_i (y_j,b_1,b_2,x) \to
y_1 = y_2 \). So:

- The sequences \( I_i = \langle a_{j,i} \rangle_{i-1/2 < j < i+1/2} \) where \( i < \omega \) are mutually indiscernible over \( b_1,b_2 \).
- \( \{ \varphi_i (a_{j,i},b_1,b_2,x) \mid i - 1/2 < j < i + 1/2 \} \) is 2-inconsistent.
- \( \{ \varphi_i (a_{i,i},b_1,b_2,x) \mid i < \omega \} \) is consistent (realized by \( a \)).

Which contradicts strength. \( \square \)

**Definition 4.8.** A valued field \((K,v)\) of characteristic \( p > 0 \) is Kaplansky if it satisfies:

1. The valuation group \( \Gamma \) is \( p \)-divisible.
2. The residue field \( k \) is perfect, and does not admit a finite separable extension
whose degree is divisible by \( p \).

**Corollary 4.9.** Every strongly dependent valued field is Kaplansky.
Proof. Combining Proposition \[4.7\] Proposition \[3.2\] and \[KSW11\] Corollary 4.4. □

4.2. Strong\(^2\) theories. The following is just a repetition of \[KST1\] Proposition 2.5:

**Proposition 4.10.** Suppose \( T \) is strong\(^2\), then it is impossible to have a sequence of type-definable groups \( \langle G_i \mid i < \omega \rangle \) such that \( G_{i+1} \leq G_i \) and \( |G_i : G_{i+1}| = \infty \).

**Proof.** Without loss of generality, we shall assume that all groups are type-definable over \( \emptyset \). Suppose there is such a sequence \( \langle G_i \mid i < \omega \rangle \). Let \( \langle \bar{a}_i \mid i < \omega \rangle \) be mutually indiscernible, where \( \bar{a}_i = \langle a_{i,j} \mid j < \omega \rangle \), such that for \( i < \omega \), the sequence \( \langle a_{i,j} \mid j < \omega \rangle \) is a sequence from \( G_i \) (in \( \mathcal{C} \)) such that \( a_{i,j}^{-1} \cdot a_{i,j} \notin G_{i+1} \) for all \( j < j' < \omega \). We can find such an array because of our assumption and Ramsey.

For each \( i < \omega \), let \( \psi_i(x) \) be in the type defining \( G_{i+1} \) such that \( \neg \psi_i(a_{i,j}^{-1} \cdot a_{i,j}) \) for \( j' < j \). By compactness, there is a formula \( \xi_i(x) \) in the type defining \( G_{i+1} \) such that for all \( a, b \in \mathcal{C} \), if \( \xi_i(a) \land \xi_i(b) \) then \( \psi_i(a^{-1} \cdot b^{-1}) \) holds. Let \( \varphi_i(x,y,z) = \xi_j(y^{-1} \cdot z^{-1} \cdot x) \). For \( i < \omega \), let \( b_0 = a_{0,0} \cdot \ldots \cdot a_{i-1,0} \) (so \( b_0 = 1 \)).

Let us check that the set \( \{ \varphi_i(x,a_{i,0},b_i) \mid i < \omega \} \) is consistent. Let \( i_0 < \omega \), and let \( c = b_{i_0} \). Then for \( i < i_0 \), \( \varphi_i(c,a_{i,0},b_i) \) holds if and only if \( \xi_i(a_{i+1,0} \cdot \ldots \cdot a_{i-1,0}) \) holds, but the product \( a_{i+1,0} \cdot \ldots \cdot a_{i-1,0} \) is an element of \( G_{i+1} \) and \( \xi_i \) is in the type defining \( G_{i+1} \), so \( \varphi_i(c,a_{i,0},b_i) \) holds. Now, if \( \varphi_i(c',a_{i,0},b_i) \land \varphi_i(c',a_{i,0},b_i) \) holds for some \( c' \), then \( \xi_i(a_{i,0}^{-1}b_i^{-1}c') \) and \( \xi_i(a_{i,0}^{-1}b_i^{-1}c') \) hold, so also \( \varphi_i(a_{i,0}^{-1}a_{i,1}) \) holds. So the rows are inconsistent, contradicting strength\(^2\). □

We also get (exactly as \[KST1\] Proposition 2.6):

**Corollary 4.11.** Assume \( T \) is strong\(^2\). If \( G \) is a type-definable group and \( h \) is a definable homomorphism \( h : G \to G \) with finite kernel then \( h \) is almost onto \( G \), i.e., the index \( [G : h(G)] \) is bounded (i.e. \( < \infty \)). If \( G \) is definable, then the index must be finite.

Theorem \[2.4\] holds for type-definable subgroups without the normality assumption.

**Theorem 4.12.** Let \( G \) be strong\(^2\) and \( \{ \varphi(x,a) \mid a \in C \} \) be a family of definable subgroups of \( G \). Then there is some \( k \in \omega \) such that for every finite \( C' \subseteq C \) there is some \( C_0 \subseteq C' \) with \( |C_0| \leq k \) and such that

\[
\left[ \bigcap_{a \in C_0} \varphi(x,a) : \bigcap_{a \in C'} \varphi(x,a) \right] < \infty.
\]

**Proof.** The proof of Theorem \[2.4\] relied on Proposition \[2.1\]. So we only need to show that this proposition goes through. Let \( H_i = \varphi(x,a_i) \) for \( i < \omega \). Consider \( H_i = \bigcap_{j<i} H_j \). At some point \( [H_j : H_{j+1}] < \infty \). But then also \([H 
\neq j : \bigcap_{i<\omega} H_i] < \infty \). □

5. Questions, Conjectures and Further Research Directions

5.1. More pure NTP\(_2\) fields. Recall that a field is pseudo algebraically closed (or PAC) if every absolutely irreducible variety defined over it has a point in it. It is well-known \[Cht99\] that the theory of a PAC field is simple if and only if it is bounded (i.e. for any integer \( n \) it has only finitely many Galois extensions of degree
n). Moreover, if a PAC field is unbounded, then it has TP$_2$ [Cha08, Section 3.5]. On the other hand, the following fields were studied extensively:

1. Pseudo real closed (or PRC) fields: a field $F$ is PRC if every absolutely irreducible variety defined over $F$ has a rational point in every real closure of $F$, has an $F$-rational point [Pre90, Pre81, Pre85].

2. Pseudo $p$-adically closed (or PpC) fields: a field $F$ is PpC if every absolutely irreducible variety defined over $F$ that has a rational point in every $p$-adic closure of $F$, has an $F$-rational point [Kim89a, Kim89b, Efr91, HJ88].

**Conjecture 5.1.** A PRC field is NTP$_2$ if and only if it is bounded. Similarly, a PpC field is NTP$_2$ if and only if it is bounded.

We remark that if $K$ is an unbounded PRC field then it has TP$_2$. Indeed, since $K$ is PRC then $L = K(\sqrt{-1})$ is PAC (because every finite extension of a PRC field is PRC and $L$ has no real closures at all). By [FJ05, Remark 16.10.3(b)] $L$ is unbounded. And of course, $L$ is interpretable in $K$. But by the result of Chatzidakis cited above $L$ has TP$_2$, thus $K$ also has TP$_2$.

5.2. **More valued fields with NTP$_2.**$ Is there an analogue of Fact 1.1 in positive characteristic? A similar result for NIP was established in [B299, Corollaire 7.6].

**Conjecture 5.2.** Let $(K,v)$ be a valued field of characteristic $p > 0$, Kaplansky and algebraically maximal. Then $(K,v)$ is NTP$_2$ (strong) if and only if $K$ is NTP$_2$ (resp. strong).

The following is demonstrated in [KSW11, Proposition 5.3].

**Fact 5.3.** Let $(K,v)$ be an NIP valued field of characteristic $p > 0$. Then the residue field contains $\mathbb{F}_p^{alg}$ (so in particular is infinite).

Hrushovski asked if the following is true:

**Problem 5.4.** Assume that $(K,v)$ is an NTP$_2$ (Henselian) valued field of positive characteristic. Does it follow that the residue field is infinite?

We remark that the finite number of Artin-Schreier extensions alone is not sufficient to conclude that the residue field is infinite:

**Example 5.5.** (Due to Arno Fehm) Let $\Omega = (\mathbb{F}_p((t)))^{sep}$, so the restriction map $\text{Gal}(\Omega/\mathbb{F}_p((t))) \to \text{Gal}(\mathbb{F}_p^{alg}/\mathbb{F}_p)$ is onto. Let $f \in \text{Gal}(\mathbb{F}_p^{alg}/\mathbb{F}_p)$ be the Frobenius automorphism, and let $\tau \in \text{Gal}(\Omega/\mathbb{F}_p((t)))$ be such that $\tau | f = \sigma$. Let $F$ be the fixed field of $\sigma$. Then $F$ has exactly one Artin-Schreier extension (as $\text{Gal}(\Omega/F)$ is pro-cyclic and $F$ is a regular extension of $\mathbb{F}_p$). Since $F_p((t))$ is an Henselian valued field, its usual valuation extends uniquely to an Henselian valuation on $F$. Since every element of $\mathbb{F}_p^{alg}/\mathbb{F}_p$ is moved by $\sigma$, one can see that the residue field must be $\mathbb{F}_p$.

**Example 5.6.** (Due to the anonymous referee) Let $\Omega$ be the generalized power series $\mathbb{F}_p^{alg}((t^{1/k}))$ — the field of formal sums $\sum a_i t^i$ with well-ordered support where $i \in \mathbb{Q}$ and $a_i \in \mathbb{F}_p^{alg}$. This field is algebraically closed. Let $f \in \text{Aut}(\Omega)$ be the map $\sum a_i t^i \mapsto \sum a_i^q t^i$. Let $F$ be the fixed field of $f$, so $F = \mathbb{F}_p((t^{1/k}))$. Then $F$ is Henselian with residue field $\mathbb{F}_p$ and (as in Example 5.5) has exactly one Artin-Schreier extension.
5.3. **Definable envelopes.** Assume that we are given a subgroup of an NTP$_2$ group. Is it possible to find a definable subgroup which is close to the subgroup we started with and satisfies similar properties?

**Fact 5.7.**

1. [She09, Ald] If $G$ is a group definable in an NIP theory and $H$ is a subgroup which is abelian (nilpotent of class $n$; normal and soluble of derived length $n$) then there is a definable group containing $H$ which is also abelian (resp. nilpotent of class $n$; normal and soluble of derived length $n$).

2. [Mii] Let $G$ be a group definable in a simple theory and let $H$ be a subgroup of $G$.
   - (a) If $H$ is nilpotent of class $n$, then there is a definable (with parameters from $H$) nilpotent group of class at most $2n$, finitely many translates of which cover $H$. If $H$ is in addition normal, then there is a definable normal nilpotent group of class at most $3n$ containing $H$.
   - (b) If $H$ is a soluble of class $n$, then there is a definable (with parameters from $H$) soluble group of derived length at most $2n$, finitely many translates of which cover $H$. If $H$ is in addition normal, then there is a definable normal soluble group of derived length at most $3n$ containing $H$.

Thus it seems very natural to make the following conjecture.

**Conjecture 5.8.** Let $G$ be an NTP$_2$ group and assume that $H$ is a subgroup. If $H$ is nilpotent (soluble), then there is a definable nilpotent (resp. soluble) group finitely many translates of which cover $H$. If $H$ is in addition normal, then there is a definable normal nilpotent (resp. soluble) group containing $H$.

5.4. **Hrushovski’s stabilizer theorem.** Let $I$ be an ideal in the Boolean algebra of definable sets in a fixed variable $x$, with parameters from the monster model (i.e. $\emptyset \in I; \phi(x,a) \vdash \psi(x,b)$ and $\psi(x,b) \in I$ imply $\phi(x,a) \in I; \phi(x,a) \in I$ and $\psi(x,b) \in I$ imply $\phi(x,a) \lor \psi(x,b) \in I$). An ideal $I$ is invariant over a set $A$ if $\phi(x,a) \in I$ and $a \equiv_A b$ implies $\phi(x,b) \in I$. An $A$-invariant ideal is called S1 if for every sequence $(a_i)_{i \in \omega}$ indiscernible over $A$, $\phi(x,a_0) \land \phi(x,a_1) \in I$ implies $\phi(x,a_0) \in I$. A partial type $q(x)$ over $A$ is called wide (or $I$-wide) if it implies no formula in $I$.

In the following, $\tilde{G}$ is a subgroup of some definable group, generated by some definable set $X$.

**Fact 5.9.** [Hru12] Theorem 3.5] Let $M$ be a model, $\mu$ an $M$-invariant S1 ideal on definable subsets of $\tilde{G}$, invariant under (left or right) translations by elements of $\tilde{G}$. Let $q$ be a wide type over $M$ (contained in $\tilde{G}$). Assume:

(F) There exist two realizations $a, b$ of $q$ such that $\text{tp}(b/Ma)$ does not fork over $M$ and $\text{tp}(a/Mb)$ does not fork over $M$.

Then there is a wide type-definable over $M$ subgroup $S$ of $G$. We have $S = (q^{-1}q)^2$; the set $qq^{-1}q$ is a coset of $S$. Moreover, $S$ is normal in $\tilde{G}$, and $S \setminus q^{-1}q$ is contained in a union of non-wide $M$-definable sets.

In [CK12] it is proved that if $M$ is a model of an NTP$_2$ theory and $q \subseteq S(M)$, then it has a global strictly invariant extension $p \subseteq S(\emptyset)$ (meaning that $p$ is an $M$-invariant type and for every $N \supset M$ and $a \models p|_N$ we have $\text{tp}(N/Ma)$ does not fork over $M$). It thus follows that the assumption (F) is always satisfied in NTP$_2$.
theories. In [BC12 Section 2 + discussion before Proposition 3.5] it is proved that in an \( NTP_2 \) theory, the ideal of formulas forking over a model \( M \) is \( S_1 \). However, in general the ideal of forking formulas is not invariant under the action of the definable group. By [Hru12 Theorem 3.5, Remark (4)] the assumption of invariance under the action of \( G \) can be replaced by the existence of an \( f \)-generic extension of \( q \). It seems interesting to find a right version of this result generalizing the theory of stabilizers in simple theories [Pi98].

References


[Mill] Cédric Milliet. Definable envelopes in groups with simple theory. *http://hal.archives-ouvertes.fr/hal-00657716/fr/*.


