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OPTIMIZATION OF DC SQUID VOLTMEETER AND MAGNETOMETER CIRCUITS

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ABSTRACT

We calculate the signal-to noise ratio in a dc SQUID system as a function of source impedance taking into account the effects of current and voltage noise sources in the SQUID. The optimization of both tuned and untuned voltmeters and magnetometers is discussed and typical sensitivities are predicted using calculated noise spectra. The calculations are based on an ideal symmetric dc SQUID with $\beta = 2LI_0/\Phi_0 = 1$ and moderate noise rounding ($\Gamma = 2\pi k_B T/I_0 \Phi_0 = 0.05$), where Φ_0 is the flux quantum, T is the temperature, L is the SQUID inductance, and I_0 is the critical current of each junction. The optimum noise temperatures of tuned and untuned voltmeters are found to be $2.8(\omega L/R)T$ and $8(\omega L/R)T(1 + 1.5\alpha^2 + 0.7\alpha^4)^{1/2}/\alpha^2$ respectively, where $\omega/2\pi$ is the signal frequency, assumed to be much less

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than the Josephson frequency, and α is the coupling coefficient between the SQUID and the voltmeter circuit. It is found that tuned and untuned magnetometers can be characterized by optimum effective signal energies given by $(16 k_B T_L B / \alpha^2 R) [1 + (1 + 1.5 \alpha^2 + 0.7 \alpha^4)^{1/2} + 0.75 \alpha^2]$ and $2 k_B T_i R_i B / \omega^2 L_p$ respectively, where B is the bandwidth, R_i is the resistance representing the losses in the tuned circuit at temperature T_i , and L_p is the inductance of the pick-up coil.

1. INTRODUCTION

Several authors¹⁻⁷ have developed sensitivity criteria for dc and rf SQUIDS used as voltmeters or magnetometers. In most applications, the SQUID is coupled to a superconducting input coil of inductance L_i with a mutual inductance M_i . A widely used figure of merit for this configuration is the energy resolution per Hz referred to the input coil, $S_\Phi / (2M_i^2/L_i)$, where S_Φ is the spectral density of the total noise level at the SQUID output expressed as an equivalent magnetic flux noise applied to the SQUID. When a source circuit is connected to the input coil, the transformed source impedance may change the SQUID inductance, causing changes in the SQUID noise level and gain which in turn affect S_Φ . The energy sensitivity computed for an isolated SQUID is only useful when the input circuit does not significantly affect the operation of the SQUID. This will always be the case if the input coil, L_i , is connected in series with a voltage source with resistance $R_i \gg \omega L_i$, or with a pick-up coil of inductance $L_p \gg L_i$. These requirements are, however, not always satisfied by useful input circuits. In particular, it has recently been demonstrated that the sensitivity of SQUID systems can be increased by resonating the input circuit at the signal frequency.^{8,9} With such input circuits, the impedance of the SQUID can be significantly changed, and the noise level cannot be characterized by an energy sensitivity.

In general, the effect of the input circuit on the total output noise of a superconducting amplifier can be treated by associating two independent noise sources with the amplifier.¹⁰ It is physically

reasonable to characterize an isolated SQUID by means of a voltage noise at the output, and a circulating current noise in the SQUID loop. A reasonable approximation for the signal-to-noise ratio at the output of a SQUID coupled to a resonant input circuit can then be made by identifying these voltage and current noise levels as two noise sources associated with a noiseless linear amplifier, and computing the total signal and noise voltages at the output in the usual fashion. A detailed model of this operation of rf-biased SQUIDS, including independent noise sources, has been developed by Ehnholm,¹¹ and compared with the performance of working devices.^{12,13} Various resonant input rf SQUID circuits have also been analyzed in detail.^{9,11}

Computer calculations of the voltage noise at the output of an isolated tunnel junction dc SQUID over a wide range of device parameters have been presented in an earlier paper.¹⁴ These calculations have recently been extended¹⁵ to yield the current noise under conditions which optimize the energy resolution. In the present paper, these results are used to predict the signal-to-noise ratio which can be obtained in tuned and untuned magnetometers and voltmeter circuits employing ideal dc SQUIDS. In order to obtain results which are representative of real devices, parameters appropriate to a typical optimized cylindrical tunnel junction dc SQUID¹⁶ have been adopted. The SQUID is assumed to have a self-inductance L of 1 nH, a shunt resistance per junction R of 1 Ω , and to be maintained at 4.2 K. If the input flux Φ_a is close to $(2n+1)\Phi_0/4$, where n is an integer and Φ_0 is the flux quantum, changes in the output voltage, $\Delta\bar{V}$, of the current-biased SQUID are related to changes in Φ_a by a small-signal transfer function

$V_\phi = \partial \bar{V} / \partial \phi_a$. At the optimum¹⁴ bias current, $I_B \approx 1.6 I_0$, where I_0 is the critical current per junction, the Josephson frequency, f_J , is approximately equal to $R/10 L \approx 10^8$ Hz, and V_ϕ is approximately equal to $R/L \approx 10^9$ S⁻¹. Throughout this paper it will be assumed that the signal input frequency is several orders of magnitude smaller than f_J .

The spectral density $S_V(f)$ of the noise voltage, $V_n(t)$, appearing across the isolated SQUID may be expressed in terms of a reduced spectrum, γ_V , defined by

$$S_V(f) = 2\gamma_V k_B T R, \quad (1)$$

where the parallel shunt resistance of the SQUID is $R/2$. The noise current $J_N(t)$ circulating in the SQUID has a spectral density

$$S_J(f) = 2\gamma_J k_B T / R. \quad (2)$$

Any correlation between $V_N(t)$ and $J_N(t)$ may be represented in general by a cross spectrum $S_{VJ}(f)$ and a complex correlation coefficient γ_{VJ} given by

$$S_{VJ}(f) = 2\gamma_{VJ} k_B T. \quad (3)$$

For the SQUID parameters discussed above, the reduced spectral densities given by digital simulation^{14,15} are $\gamma_V \approx 8$, and $\gamma_J \approx 5.5$. The correlation is found to be real and given by $\gamma_{VJ} \approx 6$. These values are appropriate for signal frequencies well below f_J . Existing SQUIDS show values of γ_V which are larger by a factor of approximately 2 than those calculated.¹⁷ The parameters γ_J and γ_{VJ} have not been measured.

At sufficiently low temperatures and sufficiently high values of R/L we expect the Johnson noise in the resistive shunts to become small

compared with the shot noise in the tunnel junctions. This shot noise sets an ultimate limit on the SQUID resolution, with the reasonable assumption that the noise of the preamplifier connected to the SQUID can be made negligible. It is obviously of interest to estimate the various figures of merit in the shot noise limit. Unfortunately, no exact calculation for the shot noise in a current-biased shunted Josephson junction is presently available. As an order-of-magnitude estimate we assume that, in the shot noise limit for bias currents near $2I_0$, the spectral densities S_V , S_J , and S_{VJ} can be obtained by replacing $2k_B T$ with $eI_0 R$ in Eqs. (1) to (3) respectively,¹⁴ and keeping the ratios γ_J/γ_V and γ_{VJ}/γ_V constant.

In almost all applications, negative feedback is applied to the SQUID or the input circuit to obtain a linear overall response. The theory of such feedback circuits has been discussed elsewhere^{3,5,8} and will not be further treated here. It is important to note, however, that correctly applied negative feedback does not change the signal-to-noise ratio in a system although it can be used to change the overall frequency response.

In Section 2 of this paper, we develop a model for a dc SQUID coupled to a generalized resonant input circuit, and derive the signal-to-noise ratio of the system. In Sections 3 and 4 we use this result to calculate figures of merit for voltmeters and magnetometers.

2. MODEL FOR DC SQUID COUPLED TO INPUT CIRCUIT

Figure 1 shows a dc SQUID of self-inductance L inductively coupled to an input circuit consisting of a voltage source, $E_i(t)$, in series with an impedance Z_i . The SQUID input coil has self inductance L_i and mutual inductance M_i to the SQUID, so that the coupling coefficient α is given by $M_i = \alpha(LL_i)^{1/2}$. A small input current, $I_i(t)$, in the input circuit generates a flux, $M_i I_i(t)$, in the SQUID, producing a change $V(t) = V_\phi M_i I_i(t)$ in the output voltage across the SQUID. It will be assumed that the parameter V_ϕ is independent of the impedance presented to the SQUID by the input circuit.

An equivalent circuit for this configuration is shown in Fig. 2. The important device parameters are those relating small changes in the input variables I_i and Φ_i , to changes in the output variables I and V . Here, Φ_i is the input flux defined through the relation $V_i = j\omega\Phi_i$, where V_i is the voltage across the input coil. The relationships between ΔI_i , $\Delta\Phi_i$, ΔI , and ΔV may be conveniently expressed in the form¹²

$$\begin{pmatrix} \Delta\Phi_i \\ \Delta V \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} \Delta I_i \\ \Delta I \end{pmatrix} \quad (4)$$

The t_{ij} can be evaluated approximately as follows. The coefficient $t_{11} = (\partial\Phi_i/\partial I_i)_I$ is the effective inductance of the input coil, modified by the coupling to the SQUID. A straightforward circuit analysis shows that

$$\begin{aligned} t_{11} &\approx L_i [1 - \alpha^2 / (1 - 4j R_D / \omega L)] \\ &\approx L_i - j\alpha^2 \omega L L_i / 4R_D \end{aligned} \quad (5a)$$

at frequencies $\omega \ll R_D/L$. In Eq. (5a), $2R_D \approx R$ is the dynamic resistance of each junction. The coefficient $t_{12} = (\partial\phi_i/\partial I)_I$ is exceedingly small in the dc SQUID, that is, a change in the output current has a negligible effect on the input circuit.¹⁶ Thus, unlike the rf SQUID,¹² the dc SQUID is a non-reciprocal device, and

$$t_{12} \approx 0. \quad (5b)$$

We take $\bar{V}$ to be the time-averaged voltage at the SQUID output. The coefficient $t_{21} = (\partial\bar{V}/\partial I_i)_I$ is the forward transfer impedance Z_f :

$$t_{21} = Z_f = M_i V_\phi, \quad (5c)$$

while $t_{22} = (\partial\bar{V}/\partial I)_{\phi_i}$ is the dynamic resistance of the SQUID:

$$t_{22} = R_D \approx R/2, \quad (5d)$$

for frequencies much less than f_J . In practice, the output signal is chopped by means of a modulation signal,¹⁶ and matched to a room temperature amplifier by a resonant transformer at 4.2 K, producing an output voltage change $\Delta V_o = A\Delta V$, where A is the overall voltage gain of the amplifier and transformer. Under these conditions, the amplifier noise can be made negligible.

To characterize the noise sources shown in Fig. 2, it is convenient to define the following parameters:

$$Z_i = R_i + jX_i \quad \text{is the source impedance.}$$

$$Z_T \approx Z_i + j\omega L_i + \omega^2 M_i^2 / 4R_D \quad \text{is the total input impedance,}$$

where $\omega^2 M_i^2 / 4R_D$ is the contribution of the SQUID in the limit $\omega L \ll R_D$.

$E_N(t) = M_i dJ_N(t)/dt$ is the equivalent noise voltage source induced in the input circuit by $J_N(t)$. $E_N(t)$ has a spectral density $S_E(f) = \omega^2 M_i^2 S_J(f)$.

$V'_N(t)$ is the voltage noise across the SQUID in the presence of the input circuit, with spectral density $S'_V(f) = 2\gamma'_V k_B T_R$. $V'_N(t)$ is, in general, different from $V_N(t)$, the open input circuit value.

$S_A(f) = 4k_B T_A R_D \approx 2k_B T_A R$ is the spectral density of the equivalent voltage noise of the amplifier connected to SQUID, where T_A is the effective noise temperature. The amplifier is assumed not to load the circuit, and the amplifier current noise is assumed negligible.

We now derive the signal-to-noise ratio for the circuit. The signal $E_i(t)$ produces a voltage at the amplifier output with a mean square value

$$\langle V_o^2 \rangle = A^2 Z_f^2 \langle E_i^2 \rangle / |Z_T|^2 = A^2 M_i^2 V_\phi^2 \langle E_i^2 \rangle / |Z_T|^2 \quad (6)$$

To calculate the noise at the SQUID output, we see from the equivalent circuit that we have made the following approximation:

$$V'_N(t) = V_N(t) + M_i (dJ_N/dt) Z_f / Z_T \quad (7)$$

Calculating the spectral density of $V'_N(t)$ from Eq. (7), and comparing it with $S'_V(f) = \gamma'_V 2k_B T_R$, we find

$$\begin{aligned} \gamma'_V &= \gamma_V + 2\gamma_{IJ} \alpha^2 \omega L L_i V_\phi (X_i + \omega L_i) / |Z_T|^2 R \\ &+ \gamma_J \alpha^4 \omega^2 L^2 L_i^2 V_\phi^2 / |Z_T|^2 R^2 \end{aligned} \quad (8)$$

where it has been assumed that, as demonstrated by the digital model, γ_{VJ} is a real quantity. The noise at the amplifier output in terms of parameters for the isolated SQUID has a spectral density

$$S_N(f) = A^2 2k_B R \left[\gamma_V T + T_A + 2\gamma_{VJ} T \alpha^2 \omega L L_i V_\phi (X_i + \omega L_i) / |Z_T|^2 R \right. \\ \left. + \gamma_J T \alpha^4 \omega^2 L^2 L_i^2 V_\phi^2 / |Z_T|^2 R^2 \right] \quad (9)$$

Dividing Eq. (6) by Eq. (9), we find the signal-to-noise ratio at a frequency f

$$\frac{\langle V_o^2 \rangle}{S_N(f) B} = \frac{\alpha^2 L L_i V_\phi^2 \langle E_i^2 \rangle}{2k_B |Z_T|^2 R B} \left[\gamma_V T + T_A + 2\gamma_{VJ} T \alpha^2 \omega L L_i (X_i + \omega L_i) V_\phi / |Z_T|^2 R \right. \\ \left. + \gamma_J T \alpha^4 \omega^2 L^2 L_i^2 V_\phi^2 / |Z_T|^2 R^2 \right]^{-1} \quad (10)$$

where B is the bandwidth.

In the following sections, we apply this result to various input circuits.

3. VOLTMETERS

It is convenient to write Eq. (10) in terms of a noise temperature, T_N , defined by setting $\langle V_o^2 \rangle / S_N(f) B = 1$ with $\langle E_i^2 \rangle = 4k_B T_N R_i B$. The value of B must be small enough to ensure uniformity of the signal-to-noise ratio within the chosen bandwidth. We find

$$T_N = \frac{\omega^2 R L_i}{2\alpha^2 L R_i V_\phi^2} \left[\frac{(\gamma_V T + T_A) |Z_T|^2}{(\omega L_i)^2} + \frac{2\alpha^2 \gamma_{VJ} V_\phi L T}{R} \left(1 + \frac{X_i}{\omega L_i} \right) \right. \\ \left. + \frac{\alpha^4 \gamma_J L^2 V_\phi^2 T}{R^2} \right] \quad (11)$$

3.1. Resistive Source with Tuned Input

As an example, we assume that the source is resistive, and that the input is tuned with a capacitor, C_i , so that $Z_i = R_i - j/\omega C_i$ (see Fig. 3a). We further assume that the losses in L_i and C_i are negligible compared with the dissipation in R_i and that $T_A \ll \gamma_V T$, as is the case for a SQUID operated in the He^4 temperature range since $T_A \approx 1 \text{ K}$ and $\gamma_V \approx 8$. Equation (11) becomes

$$T_N = \frac{\omega^2 R_i L_i T}{2\alpha^2 L R_i V_\phi^2} \left\{ \gamma_V \left[\left(\frac{R_i}{\omega L_i} + \frac{\alpha^2 \omega L}{4R_D} \right)^2 + \left(1 - \frac{1}{\omega^2 L_i C_i} \right)^2 \right] + \frac{2\alpha^2 \gamma_{VJ} L V_\phi}{R} \left(1 - \frac{1}{\omega^2 L_i C_i} \right) + \frac{\alpha^4 \gamma_J L^2 V_\phi^2}{R^2} \right\}. \quad (12)$$

We optimize T_N with respect to C_i by setting $\partial T_N / \partial C_i = 0$, to find

$$\frac{1}{\omega C_i^{(\text{opt})}} = \omega L_i \left(1 + \frac{\alpha^2 \gamma_{VJ} L V_\phi}{R \gamma_V} \right), \quad (13)$$

and

$$T_N^{(\text{opt } C_i)} = \frac{\omega^2 R_i L_i T}{2\alpha^2 L R_i V_\phi^2} \left[\gamma_V \left(\frac{R_i}{\omega L_i} + \frac{\alpha^2 \omega L}{4R_D} \right)^2 + \frac{\alpha^4 L^2 V_\phi^2}{\gamma_V R^2} (\gamma_J \gamma_V - \gamma_{VJ}^2) \right]. \quad (14)$$

We now optimize Eq. (11) with respect to R_i by setting $\partial T_N / \partial R_i = 0$.

The optimization is independent of C_i and gives

$$R_i^{(\text{opt})} = \alpha^2 \omega L_i \left[\left(\frac{\omega L}{4R_D} \right)^2 + \frac{L^2 V_\phi^2}{R^2 \gamma_V^2} (\gamma_V \gamma_J - \gamma_{VJ}^2) \right]^{1/2}, \quad (15)$$

and

$$T_N^{(\text{opt } C_i, R_i)} = \frac{\gamma_V \omega RT}{\alpha^2 L V_\phi^2} \left[\frac{R_i^{(\text{opt})}}{\omega L_i} + \frac{\alpha^2 \omega L}{4 R_D} \right] \quad (16)$$

Equations (15) and (16) imply that mutually correlated components of $V_N(t)$ and $J_N(t)$ do not contribute to the optimized noise level. The non-zero of γ_{VJ} reduces the value of $R_i^{(\text{opt})}$, and thus lowers $T_N^{(\text{opt } C_i, R_i)}$ below the value $T_N \approx T(\omega^2 \gamma_V \gamma_J / V_\phi^2)^{1/2}$ that would be expected in the absence of correlations in the limit $\omega \ll R_D/L$. The value of $T_N^{(\text{opt } C_i, R_i)}$ is independent of the value of the input coupling coefficient α , but it will be found that the bandwidth over which it is maintained is not.

It is important to realize that the optimized value of C_i in Eq. (13) is for a fixed frequency, so that the optimum noise temperature given by Eq. (16) applies to narrow-band measurements close to this frequency. To obtain the optimum performance at a different frequency, one must re-optimize T_N . In practice, one is forced to optimize C_i for one particular frequency, but may need to know the performance of the given system at other frequencies. A useful way of demonstrating this behavior is to fix C_i for a particular frequency, $\omega_0 = [L_i C_i (1 + \alpha^2 \gamma_{VJ} / \gamma_V)]^{-1/2}$ (from Eq. (13), with $V_\phi = R/L$), and to regard R_i as a parameter. In Fig. 4 we have plotted the dimensionless reduced noise temperature $T_N^{(\text{opt } C_i)} V_\phi / \omega_0 T$ (from Eq. (14) in the limit $\omega \ll R_D/L$) vs. reduced frequency ω / ω_0 for various values of $Q = \omega_0 L_i / R_i$, using the SQUID parameters used in the Introduction. The value of Q that optimizes the noise temperature at ω_0 is, from Eq. (15), $Q^{(\text{opt})} = \omega_0 L_i / R_i^{(\text{opt})} =$

$\gamma_V/\alpha^2(\gamma_V\gamma_J - \gamma_{VJ}^2)^{1/2} \approx 5.6$ for a typical value $\alpha^2 = 0.5$. For larger values of R_i (lower Q), the optimum noise temperature is higher but the bandwidth is larger than for $R_i^{(opt)}$, while for smaller values of R_i (higher Q), the optimum noise temperature is again higher while the bandwidth is smaller than for $R_i^{(opt)}$.

We can now make a numerical estimate of the optimized noise temperature using the values of γ_V , γ_J , and γ_{VJ} quoted above. In most practical low-frequency applications, the term $\omega L/4R_D$ in Eqs. (15) and (16) is negligible, since it is of order $\omega/\omega_J \ll 1$. We find $1/\omega C_i^{(opt)} \approx (1 + 0.75\alpha^2)\omega L_i$, $R_i^{(opt)} \approx 0.35 \alpha^2 \omega L_i$, and $T_N^{(opt, C_i, R_i)} \approx 2.8(\omega L/R)T \approx 1.8(\omega/\omega_J)T$, for temperatures close to 4.2 K. If we take $L_i = 10^{-4}$ H, $\omega/2\pi = 10^4$ Hz, $T = 4.2$ K, and $\alpha = 0.5$, we find $C_i^{(opt)} \approx 1.8 \mu F$, $R_i^{(opt)} \approx 1.1 \Omega$, $T_N^{(opt, C_i, R_i)} \approx 0.74$ mK, and $Q^{(opt)} = \omega L_i/R_i^{(opt)} \approx 5.6$.

If the operating frequency is much lower than 10^4 Hz, the value of the capacitance may become inconveniently large with $L_i = 10^{-4}$ H. In practice it may be possible to use larger than usual values of L_i since the resulting decrease in coupling coefficient does not degrade the noise temperature on resonance. A larger bandwidth might be obtained however by the use of a transformer between the resonance source circuit and the SQUID. The analysis of this configuration is a straightforward extension of that given above.

The optimized noise temperature of the dc SQUID, $1.8(\omega/\omega_J)T$, is only a factor of 1.8 greater than expected for an ideal parametric up-converter with an input frequency of ω_J . This result suggests that the operation of the dc SQUID is analogous to that of a parametric converter pumped

at the Josephson frequency followed by a homodyne detector of noise temperature 1.8 T.

Finally, we can obtain an estimate in the shot noise limit by replacing $2k_B T$ with $eI_0 R$. We find $T_{NS}^{(opt) C_i, R_i} \sim 2.2 \hbar \omega / k_B$, where we have set $\beta = 1$.

3.2. Resistive Source Without Tuning

The untuned voltmeter (see Fig. 3b) is the circuit most often used. We set $1/\omega C_i = 0$ in Eq. (12), and optimize with respect to R_i to obtain

$$R_i^{(opt)} = \omega L_i \left[1 + \left(\frac{\alpha^2 \omega L}{4R_D} \right)^2 + \frac{2\alpha^2 \gamma_{VJ} V_\phi L}{\gamma_V R} + \frac{\alpha^4 \gamma_J V_\phi^2 R^2}{\gamma_V L^2} \right]^{\frac{1}{2}}. \quad (17)$$

The optimized noise temperature is

$$T_N^{(opt) R_i} = \frac{\gamma_V \omega R T}{\alpha^2 L V_\phi^2} \left[\frac{R_i^{(opt)}}{\omega L_i} + \frac{\alpha^2 \omega L}{4R_D} \right]. \quad (18)$$

In the limit $\omega/\omega_J \ll 1$, for the parameters given in the Introduction, we find $R_i^{(opt)} \approx \omega L_i (1 + 1.5\alpha^2 + 0.7\alpha^4)^{\frac{1}{2}}$, and $T_N^{(opt) R_i} \approx 8(\omega L/R)T(1 + 1.5\alpha^2 + 0.7\alpha^4)^{\frac{1}{2}}/\alpha^2$. Notice that the optimized noise temperature diverges as α becomes small, whereas it is independent of α for the tuned voltmeter. If we take $\alpha^2 = 0.5$, $\omega/2\pi = 10^4$ Hz, $L_i = 10^{-4}$ H, and $T = 4.2$ K, we find $R_i^{(opt)} \approx 9 \Omega$ and $T_N^{(opt) R_i} \approx 5.9$ mK, a factor of about 8 greater than for the tuned case. However, for the untuned case, it is straightforward to work at low frequencies, and in this limit the noise temperature is so far below the bath temperature that there is probably little need for a tuned circuit.

The value for T_N given by Eq. (18) applies only at the frequency chosen for the optimization of R_i . Once R_i has been fixed, the frequency dependence of T_N can easily be seen from Eq. (12). When $\omega \ll R_i/L_i$, $T_N \approx \gamma_V R_i L_i T / 2\alpha^2 R L_i$, and is independent of frequency. When $\omega \gg R_i/L_i$, $T_N \approx (\omega^2 L L_i T / 2\alpha^2 R R_i) (\gamma_V + 2\alpha^2 \gamma_{VJ} + \alpha^4 \gamma_J)$, and thus increases as ω^2 . Apart from numerical factors involving the γ 's, these two limiting results have the same form as those found by previous authors.^{18,3,5}

The shot noise limit for the optimized case is $T_{NS}^{(\text{opt } R_i)} \sim (2\pi\hbar\omega/k_B \alpha^2) (1 + 1.5\alpha^2 + 0.7\alpha^4)^{1/2} \approx 17 \hbar\omega/k_B$ for $\alpha^2 = 0.5$.

4. MAGNETOMETERS

We consider the case of a magnetometer consisting of a superconducting pick-up coil of inductance L_p connected in series with the coupling coil. In typical SQUID applications, the pickup coil is very much smaller than the wavelength corresponding to the frequency of the signal, and its radiation resistance is therefore completely negligible. For such a source, the concept of a noise temperature referred to the source is inappropriate, and it is more useful to calculate the sensitivity in terms of the mean square signal voltage $\langle E_i^2 \rangle$ induced in the pickup coil. It is convenient to focus on an effective signal energy $U_o = \langle E_i^2 \rangle / 2\omega^2 L_p$. For a pickup coil of cross sectional area S in a perpendicular magnetic field of rms amplitude $\tilde{B}$ at frequency ω , the effective signal energy is given by

$$\langle E_i^2 \rangle / 2\omega^2 L_p = \tilde{B}^2 S^{3/2} / (2\xi \mu_o) \quad , \quad (19)$$

where ξ is a dimensionless factor of order unity determined by the

shape of the coil. For a given shape, Eq. (19) shows that U_o is proportional to the coil volume. A similar treatment may be applied to gradient pick-up coils, and systems for measuring other physical quantities.

4.1. Untuned Magnetometers

The sensitivity is found by substituting the value of $\langle E_i^2 \rangle$ given by Eq. (19) into Eq. (10) with $Z_T = \omega L_i [(\alpha^2 \omega L / 4R_D) + j(1 + L_p/L_i)]$. Setting $\langle V_o^2 \rangle / S_N(f)B = 1$, we obtain an expression for the signal energy required for unity signal-to-noise ratio:

$$U_o = \frac{k_B T R L_i B}{\alpha^2 V_\phi^2 L L_p} \left\{ \gamma_V \left[\left(\frac{\alpha^2 \omega L}{4R_D} \right)^2 + \left(1 + \frac{L_p}{L_i} \right)^2 \right] + \frac{2\gamma_{VJ} \alpha^2 L (1 + L_p/L_i) V_\phi}{R} + \frac{\alpha^4 \gamma_J L^2 V_\phi^2}{R^2} \right\} \quad (20)$$

We now minimize U_o with respect to L_i , assuming L_p and α^2 to be fixed, and obtain

$$L_i^{(opt)} = L_p \left[1 + \frac{2\alpha^2 \gamma_{VJ} L V_\phi}{\gamma_V R} + \frac{\alpha^4 \gamma_J L^2 V_\phi^2}{\gamma_V R^2} + \left(\frac{\alpha^2 \omega L}{4R_D} \right)^2 \right]^{-1/2}, \quad (21)$$

and

$$U_o^{(opt L_i)} = \frac{2\gamma_V k_B T R B}{\alpha^2 L V_\phi^2} \left(1 + \frac{L_p}{L_i^{(opt)}} + \frac{\alpha^2 \gamma_{VJ} L V_\phi}{\gamma_V R} \right). \quad (22)$$

In the limit $\omega/\omega_J \ll 1$, for the parameters given in the Introduction we find $L_i^{(opt)} \approx L_p (1 + 1.5\alpha^2 + 0.7\alpha^4)^{-1/2}$, and $U_o^{(opt L_i)} \approx (16k_B T R B / \alpha^2 R) [1 + (1 + 1.5\alpha^2 + 0.7\alpha^4)^{1/2} + 0.75\alpha^2]$. These two results are independent of frequency, and $U_o^{(opt L_i)}$ is independent of the separate values of L_p

and $L_i^{(\text{opt})}$ depending only on their ratio. $U_o^{(\text{opt } L_i)}$ diverges as $\alpha \rightarrow 0$, as does the optimized noise temperature in the case of the untuned voltmeter. For $\alpha^2 = 0.5$ and the usual values of the SQUID parameters, we find $L_i^{(\text{opt})}/L_p \approx 0.72$, and $U_o^{(\text{opt } L_i)} \approx 9 k_B T(B/f_J) \approx 5 \times 10^{-30} \text{ (B/1 Hz)J}$.

It is clear from these results that since only the ratio of L_i and L_p appears in Eq. (22), the magnetic field sensitivity is improved by increasing $S^{3/2}/\xi$, keeping L_p fixed. Since one can, in principle, make the coil volume arbitrarily large, the ultimate sensitivity appears to be fundamentally limited only by dewar size assuming that spurious noise sources can be made negligible. For example, with a solenoidal pick-up coil of length 0.2 m and radius 0.05 m, we find $\tilde{B}^{(\text{opt } L_i)} \approx 2 \times 10^{-16} \text{ (B/1 Hz)}^{1/2} \text{ T}$.

In the shot noise limit, the estimated signal energy becomes $U_{OS}^{(\text{opt } L_i)} \approx 11 \text{ hB}$, giving, for the example above, a magnetic field sensitivity $\tilde{B}_S^{(\text{opt } L_i)} \approx 7 \times 10^{-18} \text{ (B/1 Hz)}^{1/2} \text{ T}$.

4.2. Tuned Magnetometer

We now consider the case in which the input circuit is tuned by a capacitance C_i in series with L_i and L_p , as shown in Fig.3(d). We introduce a series resistance R_i at T_i to represent the dissipation in the circuit. This arrangement differs from the tuned voltmeter circuit only by the addition of the inductance L_p . The total noise at the amplifier output has a spectral density $S_{NT}(f)$ given by

$$S_{NT}(f) = 4k_B (T_i + T_N^*) R_i A^2 Z_f^2 / |Z_T|^2 \quad (23)$$

$S_{NT}(f)$ includes a contribution $4k_B T_i R_i A^2 Z_f^2 / |Z_T|^2$ from R_i which is assumed to be at temperature T_i , and a contribution $4k_B T_N^* R_i A^2 Z_f^2 / |Z_T|^2$ from the SQUID noise, where it can be shown by analysis similar to that in Section 3.1 that

$$T_N^* = \frac{\omega^2 R L_i T}{2\alpha^2 L R_i V_\phi^2} \left\{ \gamma_V \left[\left(\frac{R_i}{\omega L_i} + \frac{\alpha^2 \omega L}{4R_D} \right)^2 + \left(1 + \frac{L_p}{L_i} - \frac{1}{\omega^2 L_i C_i} \right)^2 \right] + \frac{2\alpha^2 \gamma_{VJ} L V_\phi}{R} \left(1 + \frac{L_p}{L_i} - \frac{1}{\omega^2 L_i C_i} \right) + \frac{\alpha^4 \gamma_J L^2 V_\phi^2}{R^2} \right\} . \quad (24)$$

Equation (24) differs from Eq. (12) only by the inclusion of the terms in L_p .

The mean square signal at the output of the amplifier is $\langle V_o^2 \rangle = 2\omega^2 U_o L_p A^2 Z_f^2 / |Z_T|^2$. Setting $\langle V_o^2 \rangle / S_{NT}(f)B = 1$, we find that the effective signal energy required for detection in a narrow bandwidth B is given by

$$U_o = \frac{2k_B R_i B}{\omega^2 L_p} (T_i + T_N^*) . \quad (25)$$

At this point one may proceed to optimize U_o . However, for typical systems where $\omega \ll \omega_J$, T_N^* is always negligible compared with T_i , as was found for the tuned voltmeter in section 3.1. Thus, we can write immediately

$$U_o = \frac{2k_B T_i R_i B}{\omega^2 L_p} = \frac{2k_B T_i B (1 + L_i/L_p)}{Q\omega} , \quad (26)$$

where $Q = \omega(L_i + L_p)/R_i$. Equation (26) is thus independent of the SQUID

parameters and of α . Equation (26) may be found immediately by comparing the mean square Johnson noise current in the input circuit, $4k_B T_i R_i B / |Z_T|^2$, with the mean square signal current, $2\omega^2 U_o L_p / |Z_T|^2$. The response of the circuit peaks at the resonant frequency, $\omega = [(L_i + L_p) C_i]^{-1/2}$, and has a bandwidth $\omega/Q = R_i / (L_i + L_p)$. However, Eq. (26) remains true provided $T_N^* \ll T_i$, that is, over a bandwidth that is usually very much wider than ω/Q . The overall frequency response may thus be broadened considerably without degrading the energy resolution by the application of negative feedback from the output of the amplifier chain to the input circuit.⁹

It is of interest to compare the narrow-band energy sensitivity given by Eq. (26) with that for the optimized untuned circuit with $\alpha = 0.5$, $U_o^{(opt L_i)} \approx 9 k_B T (B/f_J)$. The tuned circuit has a higher sensitivity provided that

$$\frac{U_o(\text{tuned})}{U_o^{(opt L_i)}(\text{untuned})} \approx \frac{R_i \omega_J}{9\pi\omega^2 L_p} \approx \frac{(1 + L_i/L_p)\omega_J}{28 Q\omega} < 1. \quad (27)$$

If we take $\omega_J/\omega = 10^4$ ($\omega/2\pi \approx 10^4$ Hz) and assume $L_i \approx L_p$, Eq. (27) requires $Q > 700$, and a bandwidth $\omega/2\pi Q \approx 14$ Hz. Thus, to obtain an order of magnitude improvement in energy sensitivity over the untuned magnetometer, the tuned magnetometer would have a bandwidth of ~ 1 Hz in the absence of feedback. For most applications, it would therefore be highly desirable to apply feedback to increase the bandwidth. It is also evident that the ratio in Eq. (27) decreases as $1/\omega$ at constant Q . Thus, the tuned magnetometer becomes increasingly attractive compared with the untuned magnetometer as the frequency is increased. Note,

however, that for large Q input circuits operated at high frequency, the SQUID noise temperature T_N^* may be comparable to or even exceed the input resistance temperature T_i . For example, for $T = T_i$, $Q = 1000$, $L_i = L_p$, and $\omega_0 = 2 \times 10^6 \text{ s}^{-1}$, we have $T_N^*(\omega_0) = T_i$, and for $\omega_1 \neq \omega_0$, $T_N^*(\omega) > T_i$. In Fig. 5 we plot the reduced noise temperature $T_N^*(\omega) (R/\omega_0 L)/T$ versus the reduced frequency (ω/ω_0) for $Q_T = \omega_0 (L_i + L_p)/R_i = 1000$ for various values of $r = L_i/(L_i + L_p)$. Again, we have used the approximation $V_\phi = R/L$.

6. CONCLUSIONS

We have described the optimization of SQUID voltmeters and magnetometers taking into account voltage and current noise sources characteristic of ideal dc SQUIDS. Since the performance of real devices is quite close to ideal, we expect that the results will be broadly applicable in practice with small corrections to the values of γ_V , γ_J , and γ_{VJ} .

At frequencies below a few kHz we find that voltmeters may be characterized by a noise temperature, T_N , which is so much smaller than the ambient temperature that the voltage measurement is always limited by Johnson noise in the input circuit. In this frequency range, there seems little need to use the tuned voltmeter, especially as the large values of capacitance and inductance involved would make these elements rather cumbersome. However at higher frequencies, $T_N \propto \omega$, and the noise temperature of the untuned voltmeter may become comparable with the ambient temperature. In this limit the lower noise temperature offered by the tuned voltmeter may be significant.

In the same way, the untuned magnetometer is preferable to the tuned magnetometer at low frequencies. However at frequencies above a few hundred Hz the tuned magnetometer offers a clear improvement in sensitivity, provided that feedback is used properly to improve the frequency response. It seems likely that the tuned magnetometer will become widely used in future applications where high sensitivity in a relatively restricted bandwidth is required.

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REFERENCES

1. V. R. Radhakrishnan and V. L. Newhouse, J. Appl. Phys. 42, 129 (1971).
 2. J. Clarke, Proc. IEEE 61, 8 (1973).
 3. A. Davidson, R. S. Newbower, and M. R. Beasley, Rev. Sci. Instru. 45, 838 (1974).
 4. J. H. Claasen, J. Appl. Phys. 46, 2268 (1975).
 5. J. Clarke, Superconductor Applications: SQUIDS and Machines, edited by B. B. Schwartz and S. Foner, p. 67 (Plenum, 1977).
 6. V. V. Danilov, K. K. Likharev, O. V. Sniguiriev, and E. S. Soldatov, IEEE Trans. Magn., MAG-13, 240 (1977).
 7. A. V. Gusev and V. N. Rudenko, Zh. Eksp. Teor. Fiz. 74, 819 (1978) (Trans: Sov. Phys. JETP 47, 428 (1978)).
 8. F. Bordoni, P. Carelli, I. Modena, and G. L. Romani, J. de Physique 39, C6-1213 (1978).
 9. M. B. Simmonds, W. A. Fertig, and R. P. Giffard, IEEE Trans. Mag. MAG-15, 478 (1979).
 10. W. S. Goree and V. W. Hesterman, in Applied Superconductivity, Vol. 1, edited by V. L. Newhouse (Academic Press, New York, 1975).
 11. G. Ehnholm, J. Low Temp. Phys. 29, 1 (1977).
 12. R. P. Giffard and J. N. Hollenhorst, Appl. Phys. Lett. 32, 767 (1978).
 13. G. Ehnholm, S. T. Islander, P. Östman, and B. Rantala, J. de Physique 39, C6-1206 (1978).
 14. C. D. Tesche and J. Clarke, J. Low Temp. Phys. 29, 301 (1977).
 15. C. D. Tesche and J. Clarke, J. Low Temp. Phys. __, ____ (1979).
 16. J. Clarke, W. M. Goubau, and M. B. Ketchen, J. Low Temp. Phys. 25, 99 (1976).
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REFERENCES (con'td)

17. M. B. Ketchen, J. Clarke, and W. M. Goubau, in Future Trends in Superconductive Electronics, edited by B. Deaver, C. Falco, J. Harris, and S. Wolf, AIP Conference Proceedings No 44, p.22 (AIP, 1978).
18. R. A. Webb, R. P. Giffard, and J. C. Wheatley, JLTP 13, 383 (1973).

Figure Captions

Fig. 1. dc SQUID coupled to generalized input circuit.

Fig. 2. Equivalent circuit for Fig. 1 showing signal and noise voltages. The area within the box represents the SQUID.

Fig. 3. (a) Tuned voltmeter, (b) untuned voltmeter,
(c) untuned magnetometer, (d) tuned magnetometer.

Fig. 4. Dimensionless noise temperature versus reduced frequency ω/ω_0 for tuned voltmeter with $\gamma_V = 8$, $\gamma_J = 5.5$, $\gamma_{VJ} = 6$, and $\alpha^2 = 0.5$.

Fig. 5. Dimensionless noise temperature versus reduced frequency ω/ω_0 for the tuned magnetometer with $\gamma_V = 8$, $\gamma_J = 5.5$, $\gamma_{VJ} = 6$, and $\alpha^2 = 0.5$ for several values of $r = L_i/(L_i + L_p)$.

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Figure 1

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Figure 2

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Figure 3

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Figure 4

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Figure 5

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