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# Birational Geometry of Additive Varieties

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Mathematics

by

Felipe Castellano Macías

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The Dissertation of Felipe Castellano Macías is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

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ABSTRACT OF THE DISSERTATION

Birational Geometry of Additive Varieties

by

Felipe Castellano Macías

Doctor of Philosophy in Mathematics

University of California San Diego, 2026

Professor James M$^{\text{c}}$Kernan, Chair

An additive action is an effective action of $\mathbb{G}_a^n$ on an $n$-dimensional variety with an open orbit, and an additive variety is a normal variety admitting an additive action. In the first half of this dissertation, we study additive actions on projective space, which are classified by certain local Artinian algebras, and we provide a geometric explanation for the unexpected appearance of some moduli spaces within this classification. We also describe the $\mathbb{G}_a^2$-equivariant Sarkisov program for smooth additive surfaces. In the second half, motivated by the fact that not all additive varieties are Mori dream spaces, we present some conjectures about extremal rays in the Kleiman–Mori cone of additive varieties, and we prove them in specific cases.

# Introduction

There are two fundamental operations in mathematics: addition and multiplication. In algebraic geometry, these operations are encoded in the additive group $\mathbb{G}_a$ and the multiplicative group $\mathbb{G}_m$ of a field $k$, which act naturally on the affine line $\mathbb{A}^1$ by translations and dilations, respectively. These are the only one-dimensional connected linear algebraic groups. Moreover, Chevalley's theorem (see for example [Con02], [Bri17], or [Kol26]) shows that $\mathbb{G}_a$ and $\mathbb{G}_m$, together with abelian varieties, are fundamental building blocks of any algebraic group:

**Theorem 0.1** (Chevalley)**.** *Let $G$ be a connected algebraic group over a perfect field $k$. Then $G$ is in a unique way an extension of an abelian variety $A$ by a normal linear algebraic closed subgroup $L$ of $G$, that is there exists a unique short exact sequence of algebraic groups*

$$0 \to L \to G \to A \to 0$$

*such that $L$ is a linear algebraic group and $A$ is an abelian variety. Moreover, if the characteristic of $k$ is zero, then the Levi decomposition states that $L$ is isomorphic to a semidirect product $R \ltimes U$ of a reductive group $R$ and a unipotent group $U$, and if in addition $G$ is commutative, then $L$ is isomorphic to $\mathbb{G}_m^r \times \mathbb{G}_a^s$.*

In this dissertation, we study open embeddings $G \hookrightarrow X$ of algebraic groups into projective varieties $X$, such that the natural action of $G$ on itself extends to an action of $G$ on $X$. These varieties are known as *equivariant compactifications of* $G$ (see Section 1.2 for a precise definition). If $G$ is projective, then $G$ is called an abelian variety, and we obtain that $X$ is isomorphic to $G$. In contrast, we are particularly interested in the case where $G$ is affine, which allows us to study $X$ through the geometry of the boundary divisor of the embedding, using modern methods of equivariant birational geometry. Ultimately, these equivariant compactifications provide a bridge between the affine geometry of $G$ and the projective geometry of $X$.

Now let $G$ be a connected reductive group and let $G/H$ be a homogeneous space. The theory of Luna and Vust [LV83] (see also the article by Knop [Kno91]) says that the normal partial equivariant compactifications of $G/H$ can be classified using the combinatorics of colored fans if and only if every normal equivariant compactification of $G/H$ contains only finitely many $G$-orbits. In this case, $G/H$ is called a spherical homogeneous space, and its normal partial equivariant compactifications $X$ are called spherical varieties. Because they are classified combinatorially, the set of normal equivariant compactifications of a fixed $G/H$ is discrete and does not admit continuous moduli. Spherical varieties include the class of normal equivariant compactifications of $\mathbb{G}_m^n$, which are known as toric varieties.

On the other hand, if $G$ is not reductive, it is not possible in general to describe equivariant compactifications of $G$ using combinatorics. For example, $\mathbb{P}^n$ has the structure of an equivariant compactification of $\mathbb{G}_a^n$ such that every point in the hyperplane $H$ at infinity (which is the complement of the open orbit) is fixed pointwise. Blowing up any closed subscheme of $H$ gives a new equivariant compactification of $\mathbb{G}_a^n$. This way, we obtain infinitely many nonisomorphic equivariant compactification of $\mathbb{G}_a^n$, showing that their classification is at least as complicated as the classification of closed subschemes of $\mathbb{P}^{n-1}$, so there is no hope to describe them using discrete invariants. Such an action of $\mathbb{G}_a^2$ on $\mathbb{P}^2$ can be written as

$$(a_1, a_2) \cdot [x_0 : x_1 : x_2] = [x_0 : x_1 + a_1 x_0 : x_2 + a_2 x_0].$$

Notice that the line at infinity $H = \{x_0 = 0\}$ is fixed pointwise by $\mathbb{G}_a^2$.

Furthermore, equivariant compactifications of unipotent groups may not be unique: $\mathbb{P}^2$ admits another structure of an equivariant compactification of $\mathbb{G}_a^2$ with only three orbits, which can be written as

$$(a_1, a_2) \cdot [x_0 : x_1 : x_2] = [x_0 : x_1 + a_1 x_0 : x_2 + a_1 x_1 + \left(\tfrac{1}{2}a_1^2 + a_2\right) x_0].$$

Moreover, for $n \geq 6$, results of Hassett and Tschinkel [HT99] or Knop and Lange [KL84] show that $\mathbb{P}^n$ admits uncountably many nonequivalent structures as an equivariant compactification of $\mathbb{G}_a^n$.

The main objects of study in this dissertation are normal equivariant compactifications of $\mathbb{G}_a^n$, which we call additive varieties, and effective actions of $\mathbb{G}_a^n$ with an open orbit, which we call additive actions.

## 0.1 Existing literature on additive varieties

Since the paper of Hassett and Tschinkel [HT99] studying additive actions on projective space and on other varieties of small dimension, there has been a lot of interest in additive varieties. We refer the reader to the thorough survey on additive varieties by Arzhantsev and Zaitseva [AZ22], which contains many results about classifications of additive actions on different types of varieties.

Another important motivation for additive varieties comes from results due to Chambert–Loir and Tschinkel ([CLT00a], [CLT00b], [CLT12]), who obtained arithmetic results concerning the distribution of rational points of bounded height on additive varieties defined over number fields.

It is also interesting to determine which varieties are additive: this was achieved by Derenthal and Loughran [DL10] for singular del Pezzo surfaces, by Huang and Montero [HM20] for smooth Fano threefolds, and by Fu and Montero [FM19] for smooth Fano varieties of high index. Fu and Hwang [FH20] conjecture that the class of additive smooth Fano varieties of Picard number one coincides with the class of Euler-symmetric projective varieties.

Furthermore, it seems a difficult problem to classify additive varieties admitting an additive action with finitely many orbits. A classification for surfaces with du Val singularities has recently been obtained by Perepechko [Per25]. Crowley [Cro26] has also found a nice class of additive varieties (which are subvarieties of $(\mathbb{P}^1)^n$) with finitely many orbits.

In the context of positivity questions, additive varieties also offer interesting results. The ampleness and nefness of tangent bundles for smooth projective varieties are well-understood through the results of Mori [Mor79] and the Campana–Peternell conjecture. Expanding on this, Liu [Liu23] recently showed that the tangent bundle of a smooth projective additive variety is big.

Finally, additive varieties are intimately connected to Hirzebruch's [Hir54] problems on classifying all compactifications of affine spaces (not necessarily equivariant), which seems out of reach in full generality, with progress in small dimensions by many authors.

## 0.2 Structure of this dissertation

In Chapter 1, we give precise definitions and facts relating to additive varieties, provide many examples, compare additive varieties with toric varieties, and present the tools of birational geometry that we will use.

In Chapter 2, we study additive actions on $\mathbb{P}^n$, which are in bijection with some local Artininian algebras by the Knop–Lange and Hassett–Tschinkel correspondences. As a byproduct of this classification, Hassett and Tschinkel noticed that many additive actions correspond to points in the moduli space of elliptic curves, smooth genus 5 curves, or K3 surfaces of degree 8, which prompted them to ask for a geometric explanation of this phenomenon. We propose an answer to this question, showing the these varieties arise as indeterminacy loci of birational maps between different effective $\mathbb{G}_a^n$-actions on $\mathbb{P}^n$ with an open orbit, which we call *untwists* (see Section 2.4 for a definition). More precisely, we are able to prove the following:

**Theorem 0.2** (= Corollary 2.33). *Let $X$ be an elliptic curve, smooth genus 5 curve, or K3 surface of degree 8, which corresponds to an additive action $\phi$ on $\mathbb{P}^6$, $\mathbb{P}^8$, or $\mathbb{P}^9$, respectively. Let $\iota$ denote the unique additive action on $\mathbb{P}^n$ fixing pointwise the hyperplane at infinity. Then the indeterminacy locus of any birational automorphism of $\mathbb{P}^n$ conjugating $\iota$ to $\phi$ is the projective cone over $X$ with a line as its vertex.*

The above theorem is a consequence of a more general statement about indeterminacy loci between additive actions on $\mathbb{P}^n$ (Theorem 2.32). Along the way, we also describe all possible $\mathbb{G}_a^2$-equivariant Sarkisov links between smooth additive surfaces (Theorem 2.28).

In Chapter 3, motivated by the observation that not all additive varieties are Mori dream spaces (as in the blowup of $\mathbb{P}^3$ along 9 very general points contained in a plane, described in Example 1.28), we initiate the study of extremal rays in the Kleiman–Mori cone of additive varieties. We propose some conjectures describing when additive varieties have finitely many negative extremal rays, and prove these conjectures for blowups of $\mathbb{P}^n$ along distinct points contained in a fixed hyperplane. For example, we are able to show the following:

**Theorem 0.3** (= Theorem 3.15). *Set $n \geq 3$. Let $X$ be the blowup of $\mathbb{P}^n$ along $r$ points contained in a hyperplane $\Gamma$, and let $D_0, D_1, \ldots, D_r$ be the irreducible components of the additive boundary divisor, where $D_0$ is the strict transform of $\Gamma$. Then $X$ has finitely many $(K_X + \sum_{i=0}^r (1-\varepsilon_i)D_i)$-negative extremal rays for any $(\varepsilon_0, \ldots, \varepsilon_r) \in (0,1) \times [0,1]^r$.*

# Chapter 1

# Preliminaries on equivariant compactifications

Throughout this dissertation, we work over an algebraically closed field $k$ of characteristic zero. Varieties are assumed to be irreducible. Actions of algebraic groups on varieties are assumed to be regular, unless otherwise specified. Numerical equivalence is denoted by $\equiv$.

## 1.1 Algebraic groups

An algebraic group is a group object in the category of varieties over $k$. Equivalently, an algebraic group $G$ is a variety which is also a group, such that the group operations $m(x, y) = xy$ and $i(x) = x^{-1}$ are morphisms of varieties. It follows that algebraic groups are smooth, since any algebraic group acts transitively on itself and a generic point of a variety is smooth.

An action of an algebraic group $G$ on a variety $X$ is a group action $G \times X \to X$ that is also a morphism of varieties. It is effective if the induced homomorphism $G \to \mathrm{Aut}(X)$ is injective (note that the automorphism group functor of a variety is not always representable by an algebraic group).

An abelian variety is a proper algebraic group, a posteriori projective. The group $\mathrm{GL}_n$ of all invertible $n \times n$ matrices over $k$ is an algebraic group under matrix multiplication.

It has the structure of an affine variety. A closed subgroup of an algebraic group is again an algebraic group. We say that that an algebraic group is linear if it is a closed subgroup of $\mathrm{GL}_n$. It turns out that an algebraic group is linear if and only if it is affine.

The following are examples of linear algebraic groups:

- The group $\mathbb{U}_n \subset \mathrm{GL}_n$ of upper triangular matrices with 1's along the main diagonal. We say that an algebraic group is unipotent if it is a closed subgroup of $\mathbb{U}_n$. Equivalently, an algebraic group $G \subset \mathrm{GL}_n$ is unipotent if any $g \in G$ is unipotent, that is $g - I_n$ is nilpotent.

- The additive group $\mathbb{G}_a$ is defined to be the algebraic group whose underlying variety is the affine line $\mathbb{A}^1 = \operatorname{Spec} k[x]$ and whose group operation is addition, that is $m(x, y) = x + y$ and $i(x) = -x$. Equivalently, the group operation is given by

$$\begin{aligned} k[x] &\to k[x] \otimes_k k[x] \\ x &\mapsto x \otimes 1 + 1 \otimes x. \end{aligned}$$

  A representation of $\mathbb{G}_a$ on $\mathrm{GL}_2$ is given by

$$a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

  A commutative unipotent group is isomorphic to its Lie algebra via the exponential map. Therefore, any commutative unipotent group is isomorphic to a vector group

$\mathbb{G}_a^n$, which can be embedded into $\mathrm{GL}_{n+1}$ by

$$(a_1, \ldots, a_n) \mapsto \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

- The multiplicative group $\mathbb{G}_m$ is defined to be the algebraic group whose underlying variety is the punctured affine line $\mathbb{A}^1 \setminus \{\mathrm{pt}\} = \operatorname{Spec} k[x, x^{-1}]$ and whose group operation is multiplication, that is $m(x, y) = xy$ and $i(x) = x^{-1}$. Equivalently, the group operation is given by

$$\begin{aligned} k[x, x^{-1}] &\to k[x, x^{-1}] \otimes_k k[x, x^{-1}] \\ x &\mapsto x \otimes x. \end{aligned}$$

  An algebraic torus $\mathbb{G}_m^n$ can be represented on $\mathrm{GL}_n$ by

$$(t_1, \ldots, t_n) \mapsto \begin{pmatrix} t_1 & 0 & 0 & \cdots & 0 \\ 0 & t_2 & 0 & \cdots & 0 \\ 0 & 0 & t_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_n \end{pmatrix}.$$

- The unipotent radical $R_u(G)$ of a linear algebraic group $G$ is the unique maximal connected unipotent normal closed subgroup of $G$. A connected linear algebraic group $G$ is reductive if its unipotent radical is trivial, that is if it does not have any nontrivial connected unipotent normal closed subgroups. For example, $\mathbb{G}_m^n$ is reductive: all its elements are diagonalizable, and because a matrix is unipotent if and only if all its eigenvalues are 1, it follows that the only matrix which is both diagonalizable and unipotent is the identity matrix. Similarly, $\mathrm{GL}_n$, $\mathrm{SL}_n$, and finite groups are reductive. On the other hand, $\mathbb{G}_a$ is not reductive since unipotent groups are not reductive. Also, the group of upper triangular matrices in $\mathrm{GL}_n$ is not reductive since it contains $\mathbb{U}_n$ as a closed normal subgroup.

- A connected linear algebraic group is solvable if it is isomorphic to a semidirect product $T \ltimes U$, where $T$ is an algebraic torus and $U$ is a connected unipotent group. A direct consequence of this definition is that $\mathbb{U}_n$, $\mathbb{G}_m^n$, and $\mathbb{G}_a^n$ are solvable. Also, the subgroup of $\mathrm{GL}_n$ consisting of upper triangular matrices is solvable, since it decomposes as $\mathbb{G}_m^n \ltimes \mathbb{U}_n$. In contrast, $\mathrm{GL}_n$ and $\mathrm{SL}_n$ are not solvable for $n \geq 2$.

**Theorem 1.1** (Lie–Kolchin theorem)**.** *Let $G$ be a connected solvable algebraic group. If $\rho\colon G \to \mathrm{GL}(V)$ is a representation on a nonzero finite-dimensional vector space $V$, then $\rho(G)$ has an invariant line $L \subset V$, that is $\rho(G)(L) = L$.*

A consequence of the Lie–Kolchin theorem is that any solvable group is conjugate to a subgroup of upper triangular matrices in $\mathrm{GL}_n$.

A generalization of the Lie–Kolchin theorem is the Borel fixed-point theorem.

**Theorem 1.2** (Borel fixed-point theorem)**.** *Let $G$ be a connected linear algebraic group. Then $G$ is solvable if and only if every action of $G$ on a complete variety $X$ has at least one fixed point.*

We refer the reader to [Bor91], [Hum75], [Spr09], or [DG70] for more information about linear algebraic groups.

## 1.2 Additive actions and additive varieties

**Definition 1.3.** Let $G$ be a connected algebraic group. A partial equivariant compactification of $G$ is a variety $X$ containing $G$ as a Zariski-open subset, together with an action of $G$ on $X$ extending the natural action of $G$ on itself. An equivariant compactification of $G$ is a proper equivariant partial compactification of $G$.

**Lemma 1.4.** *If $X$ is a partial equivariant compactification of $G$, then $X$ admits an effective action of G with an open orbit. If in addition G is commutative, then the converse is also true.*

*Proof.* Let $X$ be a partial equivariant compactification of $G$ and let $j\colon G \hookrightarrow X$ be the open embedding. We have that the action of $G$ of $j(G)$ is the natural action by left multiplication, which is a free and transitive action. Thus, the orbit of the identity element $e_G \in j(G) \subset X$ is $j(G)$, which is the open orbit. And since the action of $G$ on the open dense subset $j(G)$ is effective, we obtain that the action of $G$ on $X$ must be effective too.

Now assume that $G$ is commutative. We start with an effective action of $G$ on $X$ with an open orbit $U$. Choose a basepoint $x_0 \in U$ and consider the orbit map

$$\begin{aligned} \phi\colon G &\to U \\ g &\mapsto g \cdot x_0. \end{aligned}$$

Since $U$ is an orbit, the action of $G$ on $U$ is transitive, so $U \simeq G/H$ via the orbit map, where $H$ is the stabilizer of $x_0$. The kernel of the action of $G$ on $G/H$ is then the intersection of all the stabilizers of points, which is equal to $\bigcap_{g\in G} gHg^{-1}$. And since $G$ is commutative, this kernel is $H$. But the action of $G$ on $U$ is effective, so the kernel of the action of $G$ on $U$ is trivial. Hence $H = \{e_G\}$ is trivial, so we obtain the desired open embedding $\phi\colon G \hookrightarrow X$ via the orbit map. Finally, the fact that the action of $G$ on $X$ extends the natural action of $G$ on itself is a consequence of the orbit map: for $g_1, g_2 \in G$, we have that

$$g_2 \cdot \phi(g_1) = g_2 \cdot g_1 \cdot x_0 = \phi(g_2 \cdot g_1). \qquad \square$$

**Example 1.5.** A toric variety is a normal partial equivariant compactification of $\mathbb{G}_m^n$.

**Example 1.6.** An abelian variety is an equivariant compactification of itself.

We now arrive at our main objects of study.

**Definition 1.7.** An additive variety is a normal equivariant compactification of $\mathbb{G}_a^n$. An additive action is an effective action of $\mathbb{G}_a^n$ with an open orbit.

In dealing with additive actions, it will be useful to have a notion of equivalence of actions. Informally, two actions should be equivalent if they are related by a linear change of coordinates. In particular, equivalent actions will have the same number of orbits.

**Definition 1.8.** Let $\phi_i \colon G \times X \to X$ be actions of an algebraic group $G$ on a variety $X$ for $i = 1, 2$. We say that the actions $\phi_1$ and $\phi_2$ are equivalent if there exist an automorphism $\alpha$ of $G$ and an automorphism $\beta$ of $X$ such that the following diagram commutes:

$$\begin{tikzcd} G \times X \arrow[r, "{(\alpha,\beta)}"] \arrow[d, "\phi_1"'] & G \times X \arrow[d, "\phi_2"] \\ X \arrow[r, "\beta"] & X. \end{tikzcd}$$

An equivalence class of an effective action of $G$ with an open orbit on a variety $X$ will be called a $G$-structure on $X$.

If $X$ is an equivariant compactification of $G$, then $X \setminus G$ is a Weil divisor, which we will call the boundary (divisor). If $G = \mathbb{G}_a^n$, we call $X \setminus \mathbb{G}_a^n$ the additive boundary (divisor).

## 1.3 Properties of additive varieties

In this section we collect some useful facts about additive varieties.

**Theorem 1.9** ([HT99, Theorems 2.5 and 2.7])**.** *Let $X$ be a projective additive variety with additive boundary $D = D_1 \cup \cdots \cup D_r$, where the $D_i$ are its irreducible components. Then*

$$K_X = -\sum_{i=1}^{r} a_i D_i,$$

*where $a_i \geq 2$. In addition,* $\mathrm{Pic}(X)$ *is freely generated by $D_1, \ldots, D_r$, and the cone of effective Cartier divisors is given by*

$$\mathrm{Eff}(X) = \oplus_{i=1}^{r} \mathbb{R}_{\geq 0}[D_i].$$

**Lemma 1.10** ([HT99, Remark 2.6])**.** *Every effective cycle on an equivariant compactification of a connected affine algebraic group $G$ is rationally equivalent to an effective cycle supported on the boundary.*

The next theorem is called Blanchard's lemma. It implies that every step of the MMP can be made $G$-equivariant in the presence of an action of $G$. In particular, it allows us to run the $\mathbb{G}_a^n$-equivariant MMP.

**Theorem 1.11** ([HM20, Theorem 2.4])**.** *Let $G$ be a connected algebraic group acting on a variety $X$. If $f\colon X \to Y$ is a proper morphism to a variety $Y$ such that $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$, then there exists a unique action of $G$ on $Y$ such that $f$ is $G$-equivariant.*

The following is a consequence of Blanchard's lemma:

**Corollary 1.12** ([HM20, Corollary 2.5])**.** *Let $X$ be an additive variety and let $Z \subset X$ be a $\mathbb{G}_a^n$-stable subvariety. Then additive actions on $X$ are in bijection with additive actions on the blowup* $\mathrm{Bl}_Z(X)$ *of $X$ along $Z$.*

We also have that additive actions on a product on projective spaces correspond to additive actions on each of the factors, as the following result shows:

**Proposition 1.13** ([HM20, Proposition 3.7])**.** *Let $r$ be the number of additive actions on $\mathbb{P}^n$ and let $s$ be the number of additive actions on $\mathbb{P}^m$. Then the number of additive actions on $\mathbb{P}^n \times \mathbb{P}^m$ is $rs$.*

Since additive actions are regular, an additive action on $\mathbb{P}^n$ corresponds to a subgroup of $\operatorname{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}$ isomorphic to $\mathbb{G}_a^n$. The following lemma states that we can work with a representation in $\mathrm{GL}_{n+1}$ rather than in $\mathrm{PGL}_{n+1}$:

**Lemma 1.14** ([DL15, Lemma 2.3])**.** *For any $m \in \mathbb{N}$, any projective representation $\mathbb{G}_a^n \to \mathrm{PGL}_m$ admits a lift to a representation $\mathbb{G}_a^n \to \mathrm{GL}_m$.*

The following lemma describes the centralizer of any additive action:

**Lemma 1.15.** *Let $\phi\colon \mathbb{G}_a^n \to \operatorname{Aut}(X)$ be an additive action on $X$ with open orbit $U \subset X$. Then the centralizer $C_{\operatorname{Aut}(U)}(\phi(\mathbb{G}_a^n))$ of $\phi(\mathbb{G}_a^n)$ in $\operatorname{Aut}(U)$ is $\phi(\mathbb{G}_a^n)$.*

*Proof.* Let $x_0 \in U$. Since the $\mathbb{G}_a^n$-action of is free and transitive on $U$, for any $x \in U$ there exists a unique $a \in \mathbb{G}_a^n$ such that

$$x = \phi(a)(x_0)$$

in $U$. Let $C \in C_{\operatorname{Aut}(U)}(\phi(\mathbb{G}_a^n))$. Applying $C$ to both sides of this equation, we get

$$C(x) = C(\phi(a)(x_0)) = \phi(a)(C(x_0)),$$

where the second equality follows from the definition of centralizer. Again, using that the action is free and transitive, there exists a unique $b \in \mathbb{G}_a^n$ such that

$$C(x_0) = \phi(b)(x_0).$$

Combining the previous two equations, we obtain

$$\begin{aligned} C(x) &= \phi(a)(\phi(b)(x_0)) \\ &= \phi(b)(\phi(a)(x_0)) \\ &= \phi(b)(x), \end{aligned}$$

where we use that $\phi$ is a group homomorphism and $\mathbb{G}_a^n$ is commutative to derive the second equality. We have concluded that $C(x) = \phi(b)(x)$ for all $x \in U$, so it must be that $C = \phi(b)$ in $\mathrm{Aut}(U)$. □

## 1.4 Additive vs. toric

In this section, we assume that all varieties are $n$-dimensional. We will now highlight some of the differences between additive ($G = \mathbb{G}_a^n$) and toric ($G = \mathbb{G}_m^n$) varieties.

**Number of effective $G$-actions with an open orbit**

Firstly, any toric variety $T$ admits a unique $\mathbb{G}_m^n$-structure. If $T$ is projective, then $\mathrm{Aut}^0(X)$ is a linear algebraic group. Any two maximal tori are conjugate in a linear algebraic group, and any maximal torus of $\mathrm{Aut}^0(T)$ acts effectively with an open orbit. If $T$ is not projective, then more specific arguments are needed (see [Ber03, Theorem 4.1]). Therefore, any two toric varieties are isomorphic in the category of varieties if and only if they are isomorphic in the category of toric varieties.

In contrast, an additive variety may admit multiple (even infinitely many) $\mathbb{G}_a^n$-structures. For example, $\mathbb{P}^2$ admits two $\mathbb{G}_a^2$-structures, and $\mathbb{P}^6$ admits uncountably many $\mathbb{G}_a^6$-structures, as explained in Section 2.1.

The reason for this discrepancy can be explained as follows. Any effective $G$-action on a variety $X$ corresponds to a subgroup of $\mathrm{Aut}^0(X)$ isomorphic to $G$. If $X$ is a rational projective variety, then its Albanese is trivial, so $\mathrm{Aut}^0(X)$ is a linear algebraic group by Chevalley's theorem. In any linear algebraic group, any two maximal tori are conjugate, and any two maximal unipotent subgroups are conjugate. However, the group $\mathbb{G}_a^n$ in $\mathrm{Aut}^0(X)$ corresponding to an additive action may not be maximal. For example, the dimension of a maximal unipotent group in $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}$ is $\frac{n(n+1)}{2}$, while the dimension of a maximal torus in $\mathrm{Aut}(\mathbb{P}^n)$ is $n$.

**Number of orbits**

Any toric variety has finitely many $\mathbb{G}_m^n$-orbits, which correspond to cones of the fan by the orbit-cone correspondence.

On the other hand, an additive variety may have infinitely many $\mathbb{G}_a^n$-orbits. For example, the $\mathbb{G}_a^2$-action on $\mathbb{P}^2$ given by

$$(a_1, a_2) \cdot [x_0 : x_1 : x_2] = [x_0 : x_1 + a_1 x_0 : x_2 + a_2 x_0]$$

has infinitely many orbits (the line $x_0 = 0$ is fixed pointwise), while the $\mathbb{G}_a^2$-action on $\mathbb{P}^2$ given by

$$(a_1, a_2) \cdot [x_0 : x_1 : x_2] = [x_0 : x_1 + a_1 x_0 : x_2 + a_1 x_1 + \left(\tfrac{1}{2}a_1^2 + a_2\right) x_0]$$

has only three orbits (the open orbit $x_0 \neq 0$ is 2-dimensional, $x_0 = 0$ and $x_1 \neq 0$ defines a 1-dimensional orbit, and the point $x_0 = x_1 = 0$ is fixed).

**Can it be covered by $G$-invariant affine open charts?**

Sumihiro's theorem ([Sum74], [Sum75]) states that any toric variety can be covered by $\mathbb{G}_m^n$-invariant affine open subsets, so the study of toric varieties can be reduced to gluings. In fact, a toric variety $T_\Sigma$ given by a fan $\Sigma$ can be written as a union of affine open charts corresponding to the cones of the fan, that is

$$T_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma.$$

By contrast, an additive variety cannot be covered by $\mathbb{G}_a^n$-invariant affine open subsets. This is a consequence of the following theorem:

**Theorem 1.16** (Kostant–Rosenlicht theorem)**.** *Every orbit of an action of a unipotent group on an affine variety is closed.*

Therefore, if a unipotent group acts on an affine variety with an open orbit, then the action must be transitive. In particular, not every point of an additive variety has a $\mathbb{G}_a^n$-invariant affine open neighborhood, since any such neighborhood would intersect the open orbit. For example, under the $\mathbb{G}_a^2$-action on $\mathbb{P}^2$ given by

$$(a_1, a_2) \cdot [x_0 : x_1 : x_2] = [x_0 : x_1 + a_1 x_0 : x_2 + a_2 x_0],$$

any point in the line at infinity $x_0 = 0$ does not have a $\mathbb{G}_a^2$-invariant affine open neighborhood.

## 1.5 Examples

**Example 1.17.** $\mathbb{P}^n$ is an additive variety. The classification of additive actions on $\mathbb{P}^n$ is explained in Section 2.1. From this classification, there always exists an additive action on $\mathbb{P}^n$ where the hyperplane at infinity is fixed pointwise by the $\mathbb{G}_a^n$-action. Blowing up any closed subscheme of the hyperplane at infinity, one obtains infinitely many nonisomorphic additive varieties of dimension $n$.

**Example 1.18.** Smooth quadrics $Q_n \subset \mathbb{P}^{n+1}$ are additive. Indeed, we may obtain $Q_n$ by stereographic unprojection from $\mathbb{P}^n$: take a lower-dimensional smooth quadric $Q_{n-2}$

contained in a hyperplane $H \subset \mathbb{P}^n$, blow up $Q_{n-2}$, and blow down the strict transform of $H$. Note that $Q_{n-2}$ is $\mathbb{G}_a^n$-stable under the additive action on $\mathbb{P}^n$ fixing the hyperplane at infinity pointwise, so there is an induced additive action on $Q_n$. One can also show that the blowup of $Q_n$ at a point $p$ is isomorphic to the blowup of $\mathbb{P}^n$ at $Q_{n-2}$. This is summarized by Figure 1.1:

$$\begin{tikzcd}
\mathrm{Bl}_p Q_n \arrow[r, leftrightarrow, "\simeq"] \arrow[d] & \mathrm{Bl}_{Q_{n-2}}\mathbb{P}^n \arrow[d] \arrow[dl] \\
Q_n & \mathbb{P}^n \supset H \supset Q_{n-2}.
\end{tikzcd}$$

**Figure 1.1.** Obtaining $Q_n$ via stereographic unprojection from $\mathbb{P}^n$.

$Q_n$ has a unique additive action up to equivalence [Sha09, Theorem 4]. It has infinitely many orbits but only one fixed point. The additive boundary is a tangent hyperplane, which is a singular quadric.

$Q_2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and is thus toric, but $Q_n$ for $n \geq 3$ is not toric (its Picard number is 1 by the Lefschetz hyperplane theorem and the only smooth proper toric variety of Picard number 1 is projective space).

Singular quadrics are also additive, but they may admit infinitely many nonequivalent additive actions [AS11, Section 4].

It is shown in [AZ22, Proposition 2.4] that the only smooth projective additive hypersurfaces in projective space are hyperplanes and smooth quadrics.

**Example 1.19.** Grassmannians are additive [Arz07] and admit a unique additive action ([FH14], [Dev15]). For example, the Grassmannian $\mathrm{Gr}(2, 4)$ of planes in $\mathbb{C}^4$ is isomorphic to $Q_4$ via the Plücker embedding.

Let $X$ be a toric variety. Demazure [Dem70] proved that effective $\mathbb{G}_a$-actions on $X$ normalized by the acting torus are in bijection with certain vectors of the fan of $X$ known as Demazure roots. Moreover, if $X$ is complete, then any effective $\mathbb{G}_a$-action on $X$ can be normalized by the acting torus. Building on this, and based on work of Arzhantsev and Romaskevich [AR17], Dzhunusov obtained the following criterion for when a complete toric variety is additive.

**Proposition 1.20.** *([Dzh21, Proposition 1]) Let $X_\Sigma$ be a complete $n$-dimensional toric variety given by a fan $\Sigma$ in the vector space $N_\mathbb{Q}$. Then $X_\Sigma$ is additive if and only if we can order the rays of $\Sigma$ in such a way that the primitive vectors on the first $n$ rays form a basis of the lattice $N$ and the remaining rays lie in the negative orthant with respect to this basis.*

Here, the negative orthant with respect to a basis $B = \{v_1, \dots, v_n\}$ of $N_\mathbb{Q}$ is defined to be the cone of nonpositive linear combinations of elements of $B$, that is

$$\left\{ \sum_{i=1}^{n} a_i v_i \,\middle|\, v_i \in \mathbb{Q} \text{ and } a_i \leq 0 \right\} \subset N_\mathbb{Q}.$$

**Example 1.21.** The blowup of $\mathbb{P}^2$ along one or two points is toric and additive.

**Example 1.22.** The blowup of $\mathbb{P}^2$ along three toric points is toric but not additive since we cannot choose the three toric points to lie in the same line. We could also observe that there is no way to arrange the six rays of the fan so that they are all contained in the first and third quadrants.

If a toric variety $X$ is not complete, then some effective $\mathbb{G}_a$-actions may fail to be normalized by the acting torus, so we cannot use the language of Demazure roots (on which Proposition 1.20 is based) to describe all additive actions on $X$. Nevertheless, the following lemma, which we could not find in the literature, states that we can still obtain analogues of Proposition 1.20 in some simple situations for noncomplete toric varieties.

**Lemma 1.23.** *Let $X$ be an $n$-dimensional $\mathbb{Q}$-factorial toric variety with precisely $n+1$ rays $\{-e_1, \dots, -e_n, v\}$ (note that $X$ is not necessarily complete). Let $\mathbb{U}$ be a unipotent group and assume that $X$ admits an effective $\mathbb{U}$-action with open orbit such that $\sigma = \mathrm{Cone}(-e_1, \dots, -e_n)$ corresponds to the open orbit. Then $v$ lies in the first orthant.*

*Proof.* Let $v = (v_1, \ldots, v_n)$. Suppose for contradiction that $v_i < 0$ for some $i$.

The $\mathbb{U}$-boundary divisor is $D_v$. Let $I(D_v) \subset \mathcal{O}(X)$ denote the ideal of regular functions which vanish on $D_v$. We have an induced action of $\mathbb{U}$ on regular functions.

We may evaluate the order of vanishing of the character $f_i = \chi^{-e_i^*}$ along the rays of the fan:

- $\langle -e_i^*, -e_i \rangle = 1$.
- $\langle -e_i^*, -e_j \rangle = 0$ for all $j \neq i$.
- $\langle -e_i^*, v \rangle = -v_1 > 0$.

Therefore $f_i$ is a regular function on $X$ and $f_i \in I(D_v)$.

Since $D_v$ is $\mathbb{U}$-invariant, so is $I(D_v)$. Let $W \subset I(D_v)$ denote the linear span of the $\mathbb{U}$-orbit of $f_i$. By the Lie–Kolchin theorem, there exists some $0 \neq g \in W^{\mathbb{U}}$. Since $\mathbb{U}$ acts transitively on the open orbit $U_\sigma$, we must then have that $g$ restricted to $U_\sigma$ is a constant. And since $U_\sigma$ is Zariski-dense, $g$ is actually a constant. But then this contradicts that $g$ vanishes on $D_v$. $\square$

## 1.6 Birational geometry

Roughly speaking, the minimal model program (MMP) is an algorithm that takes a variety and performs a sequence of birational surgery operations, determined by the positivity of the canonical divisor, to output a simpler representative of its birational equivalence class. The input of the MMP is usually a normal projective variety $X$ and an effective $\mathbb{R}$-Weil divisor $\Delta$ on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier, which we call a log pair $(X, \Delta)$. Restricting to normal projective varieties is justified since any variety is birational to a projective one by Chow's lemma, and the normalization of a variety is a birational morphism. Furthermore, requiring $K_X + \Delta$ to be $\mathbb{R}$-Cartier allows us to pull it back along morphisms and to intersect it with 1-cycles, which is essential for studying singularities and positivity properties of $K_X + \Delta$.

In order to preserve the group action through the MMP, we should use invariant divisors.

**Definition 1.24.** An $\mathbb{R}$-Weil divisor $\Delta$ on an additive variety $X$ is additive if each prime component of $\Delta$ is invariant under the additive action on $X$.

Already in dimension three, the MMP does not preserve smoothness. Therefore, we require the log pair $(X, \Delta)$ to have mild singularities which are preserved under the MMP. To define some of the main types of singularities of the MMP, let $\pi\colon Y \to X$ be a projective birational morphism from a normal variety $Y$, and let $E_1, \ldots, E_r$ be the exceptional divisors of $\pi$. Choose $K_Y$ so that $\pi_* K_Y = K_X$. Let $\pi_*^{-1}\Delta$ denote the strict transform of $\Delta$. Since $K_Y + \pi_*^{-1}\Delta$ and $\pi^*(K_X + \Delta)$ agree outside of $\cup E_i$, we can write

$$K_Y + \pi_*^{-1}\Delta + \sum E_i \equiv \pi^*(K_X + \Delta) + \sum a_i E_i.$$

Let $E \subset Y$ be a prime divisor. We define the log discrepancy of $(X, \Delta)$ at $E$ to be

$$a_E(X, \Delta) = 1 - \mathrm{coeff}_E(\pi^*(K_X + \Delta) - K_Y).$$

Note that if $E = E_i$, then $a_E(X, \Delta) = a_i$. We say that $(X, \Delta)$ is canonical (respectively klt and log canonical) if $a_E(X, \Delta) \geq 1$ (respectively $a_E(X, \Delta) > 0$ and $a_E(X, \Delta) \geq 0$) for all prime divisors $E$ over $X$. We say that $(X, \Delta)$ is terminal if $a_E(X, \Delta) > 1$ for all prime divisors $E$ over $X$ which are exceptional over $X$.

By [BCHM10], each individual step in the MMP is known to exist. It is not known whether the algorithm always terminates, although it terminates in many special cases as well as in low dimensions.

The Kleiman–Mori cone $\overline{\mathrm{NE}}(X)$, which we now introduce, dictates the steps of the MMP. Let $X$ be a complete variety, and let $N_1(X)_{\mathbb{R}}$ denote the finite-dimensional real vector space of numerical equivalence classes of 1-cycles on $X$, on which we put the Euclidean topology. We define the cone of curves $\mathrm{NE}(X) \subset N_1(X)_{\mathbb{R}}$ to be the cone spanned by the numerical equivalence classes of all effective 1-cycles on $X$, that is

$$\mathrm{NE}(X) = \left\{ \sum a_i[C_i] \,\middle|\, C_i \text{ is an irreducible curve on } X \text{ and } a_i \geq 0 \right\}.$$

Its closure $\overline{\mathrm{NE}}(X) \subset N_1(X)_{\mathbb{R}}$ is defined to be the Kleiman–Mori cone of $X$. It is a closed convex cone.

Every numerically nontrivial $\mathbb{R}$-Cartier divisor $D$ on $X$ induces a linear functional on $N_1(X)_{\mathbb{R}}$ via the intersection pairing. The kernel of this functional is the hyperplane

$$D^{\perp} = \{\alpha \in N_1(X)_{\mathbb{R}} \,|\, (D \cdot \alpha) = 0\},$$

which naturally slices the ambient vector space $N_1(X)_{\mathbb{R}}$ into two half-spaces,

$$D_{\geq 0} = \{\alpha \in N_1(X)_{\mathbb{R}} \,|\, (D \cdot \alpha) \geq 0\}$$

and

$$D_{<0} = \{\alpha \in N_1(X)_{\mathbb{R}} \,|\, (D \cdot \alpha) < 0\}.$$

This allows us to decompose the Kleiman–Mori cone into a nonnegative part $\mathrm{NE}(X)_{D_{\geq 0}}$ and a strictly negative part $\mathrm{NE}(X)_{D_{<0}}$. If $D = K_X + \Delta$, even though the nonnegative side may be complicated, the shape of the negative side is determined by the Cone Theorem:

**Theorem 1.25** ([KM98, Theorem 3.7], [JLR26])**.** *Let $(X, \Delta)$ be a log canonical pair. Then there are countably many rational curves $C_j \subset X$ such that*

$$0 < -(K_X + \Delta) \cdot C_j \leq \dim X + 1,$$

*and*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X+\Delta)_{\geq 0}} + \sum \mathbb{R}^+[C_j].$$

We say that $R \subset \overline{\mathrm{NE}}(X)$ is a ray if $R = \mathbb{R}^+\alpha$ for some nonzero $\alpha$. A ray $R \subset \overline{\mathrm{NE}}(X)$ is extremal if it has the property that if $\beta + \gamma \in R$ for some $\beta, \gamma \in \overline{\mathrm{NE}}(X)$, then $\beta, \gamma \in R$. In the Cone Theorem, the rays $\mathbb{R}^+[C_j]$ are extremal. We say that $\overline{\mathrm{NE}}(X)$ is rational polyhedral if it is generated by a finite number of rational vectors, that is if there exist $v_1, \ldots, v_r \in N_1(X)_{\mathbb{Q}}$ such that

$$\overline{\mathrm{NE}}(X) = \left\{ \sum_{i=0}^{r} \lambda_i v_i \,\middle|\, \lambda_i \in \mathbb{R}_{\geq 0} \right\}.$$

The Cone Theorem states that $\overline{\mathrm{NE}}(X)$ is generated by its $(K_X + \Delta)$-nonnegative part together with a countable family of $(K_X + \Delta)$-negative extremal rays.

The following property of the Kleiman–Mori cone will be helpful when dealing with additive varieties:

**Lemma 1.26.** *Let $X$ be a normal projective variety and let $S$ be a prime divisor on $X$. Assume that every effective 1-cycle in $X$ is numerically equivalent to an effective 1-cycle in $S$. If the ray $\mathbb{R}^+\alpha \subset \overline{\mathrm{NE}}(X)$ is extremal, then there exists some effective 1-cycle $\beta$ in $S$ such that $\beta \equiv t\alpha$ for some $t \in \mathbb{R}^+$ and the ray $\mathbb{R}^+\beta \subset \overline{\mathrm{NE}}(S)$ is extremal.*

*Proof.* It follows from the fact that the inclusion $i \colon S \hookrightarrow X$ induces a surjective map $i_* \colon \overline{\mathrm{NE}}(S) \to \overline{\mathrm{NE}}(X)$. □

A particular class of varieties which behave well with respect to the MMP is the collection of Mori dream spaces, defined by Hu and Keel in [HK00].

**Definition 1.27.** A normal projective $\mathbb{Q}$-factorial variety $X$ is a Mori dream space if

(i) numerical equivalence coincides with linear equivalence for $\mathbb{Q}$-divisors,

(ii) the nef cone is a rational polyhedral cone generated by the classes of finitely many semi-ample line bundles, and

(iii) there are finitely $1 \leq i \leq k$ many birational maps $f_i \colon X \dashrightarrow X_i$ which are isomorphisms in codimension one, such that each $X_i$ satisfies condition (ii), and if $D$ is a movable divisor on $X$, then there is a semi-ample divisor $D_i$ on $X_i$ for some $1 \leq i \leq k$ such that $D = f_i^* D_i$.

If $X$ is Mori dream space, then one can run the $D$-MMP and this terminates for any $\mathbb{Q}$-Cartier divisor $D$ on $X$. Also, the Kleiman–Mori cone $\overline{\mathrm{NE}}(X)$ is rational polyhedral since $\overline{\mathrm{NE}}(X)$ is dual to the nef cone. Furthermore, it was shown in [BCHM10] that if $X$ is log Fano (i.e. $X$ is a normal projective variety and there exists an effective $\mathbb{Q}$-divisor $\Delta$ on $X$ such that $(X, \Delta)$ is klt and $-(K_X + \Delta)$ is ample), then $X$ is a Mori dream space.

Any toric variety is a Mori dream space. Also, smooth quadrics are always Fano and hence Mori dream spaces. The next example shows that not all additive varieties are Mori dream spaces. It appears in [Cas18].

**Example 1.28.** Let $\Gamma$ denote a plane in $\mathbb{P}^3$ and let $\pi \colon X \to \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ along very general points $p_1, \ldots, p_r \in \Gamma$. Then $X$ is an additive variety since we blew up fixed points under the $\iota$-action on $\mathbb{P}^3$. Let $S = \pi_*^{-1}\Gamma$ denote the strict transform of $\Gamma$, which is isomorphic to the blowup of $\Gamma \simeq \mathbb{P}^2$ along $p_1, \ldots, p_9$. Let $E_1, \ldots, E_9$ denote the exceptional divisors on $S$.

One may prove (see Section 3.3) that every effective 1-cycle in $X$ is equivalent to an effective 1-cycle in $S$, and moreover $\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(S)$. If $r \geq 9$, we will show that $S$ has infinitely many $(-1)$-curves, each of which spans a $K_S$-negative extremal ray of $\overline{\mathrm{NE}}(S)$, so $\overline{\mathrm{NE}}(S)$ is not rational polyhedral, thus neither is $\overline{\mathrm{NE}}(X)$. This implies that $X$ is not a Mori dream space, so it is not log Fano.

Let $C$ be a $(-1)$-curve in $S$. We may write $C = dL - \sum a_i E_i$, where $L$ is the pullback of a line, $d > 0$ is the degree of the image of the curve in $\mathbb{P}^2$, and $a_i$ is the multiplicity of the image of the curve in $\mathbb{P}^2$ at $p_i$. The fact that $C$ is a $(-1)$-curve is equivalent to

$$C^2 = -1 \quad \text{and} \quad K_S \cdot C = -1.$$

From $C^2 = -1$, we obtain that

$$d^2 - \sum a_i^2 = -1, \tag{1.1}$$

and from $K_S \cdot C = -1$ (using that $K_S = -3L + \sum E_i$) we obtain that

$$-3d + \sum a_i = -1. \tag{1.2}$$

The Cauchy–Schwartz inequality says that

$$\left( \sum_{i=1}^{r} a_i \right)^2 \leq r \sum_{i=1}^{r} a_i^2,$$

so after plugging in (1.1) and (1.2), we conclude that

$$d^2(9 - r) - 6d + 1 \leq r. \tag{1.3}$$

If $r \leq 8$, inequality (1.3) has finitely many solutions for $d$ and $a_i$, so $S$ has finitely many $(-1)$-curves. In fact $S$ is Fano, hence a Mori dream space, and $X$ is a Mori dream space too.

On the other hand, if $r \geq 9$, then there are infinitely many solutions for $d$ and $a_i$, so (after showing that each curve $C$ defined by a solution has $|C| \neq \emptyset$) $S$ has infinitely $(-1)$-curves, hence both $S$ and $X$ fail to be Mori dream spaces.

Alternatively, to show that the blowup of $\mathbb{P}^2$ at nine very general points has infinitely many $(-1)$-curves, we may first consider a special configuration. Let $p_1, \ldots, p_9$ be the nine basepoints of a general pencil of cubic curves in $\mathbb{P}^2$. Blowing up these nine points resolves the indeterminacy locus of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined by the pencil, yielding a rational surface $S$ and a morphism $\pi\colon S \to \mathbb{P}^1$. Because a general element of this linear system is a smooth cubic, the generic fiber of $\pi$ is an elliptic curve, making $\pi$ an elliptic fibration. The nine exceptional divisors $E_1, \ldots, E_9$ are sections of this elliptic fibration. We can choose one of these exceptional divisors to be the zero section, which corresponds to the identity element of the group structure on the generic fiber. The set of sections forms the Mordell–Weil group, and the remaining eight sections generate a free abelian subgroup of rank 8 (isomorphic to $\mathbb{Z}^8$) because the points are general. Using the group structure of the elliptic curve, translation by any section extends to a global automorphism of the surface $S$. Therefore, translating any $(-1)$-curve by the elements of this infinite group yields an infinite orbit of distinct $(-1)$-curves on $S$. Finally, by a deformation argument, this holds true if the points $p_1, \ldots, p_9$ are very general.

However, if instead of $K_X$ we consider the $\mathbb{R}$-divisor $K_X + \lambda S$ for $\lambda \in (0, 1)$, then the situation changes. We will explore this in Chapter 3.

Part of Chapter 1 is being prepared for submission for publication. The dissertation author was the primary investigator and author of this material.

# Chapter 2

# Additive actions on projective space

## 2.1 The Knop–Lange and Hassett–Tschinkel correspondences

All algebras are commutative associative unital $k$-algebras. By the rank of an algebra, we mean its dimension as a $k$-vector space.

*Remark* 2.1. If $R$ is an Artinian local algebra with maximal ideal $\mathfrak{m}$ and $v \in \mathfrak{m}$, then

$$\exp(v) = \sum_{i=0}^{\infty} \frac{v^i}{i!}$$

is a well-defined element of $R$ since $\mathfrak{m}$ is nilpotent.

Hassett and Tschinkel [HT99] classified additive actions on $\mathbb{P}^n$ by establishing a correspondence with local algebras of rank $n+1$. Earlier, Knop and Lange [KL84, §5] established a more general correspondence between $(\mathbb{G}_m^r \times \mathbb{G}_a^{n-r})$-structures on $\mathbb{P}^n$ and algebras of rank $n+1$ with exactly $r+1$ maximal ideals. Note that any finite-rank algebra is a product of local algebras, and any connected commutative linear algebraic group is isomorphic to $\mathbb{G}_m^r \times \mathbb{G}_a^s$.

For our purposes, we state the version of Hassett and Tschinkel:

**Theorem 2.2** ([HT99, Proposition 2.15])**.** *There is a bijection between*

1. *equivalence classes of additive actions on $\mathbb{P}^n$, and*

*2. isomorphism classes of local algebras of rank $n+1$.*

Roughly, the idea of this correspondence is as follows:

Any additive action on $\mathbb{P}^n$ corresponds to a faithful representation $\mathbb{G}_a^n \to \mathrm{GL}_{n+1}(k)$ by Lemma 1.14. Take the differential at the identity to obtain a Lie algebra representation $\mathfrak{g} \to \mathfrak{gl}_{n+1}(k)$ of the Lie algebra of $\mathbb{G}_a^n$ by commuting nilpotent elements, which in turn defines a representation $\rho\colon \mathfrak{U}(\mathfrak{g}) \to \mathrm{Mat}_{n+1}(k)$ of the universal enveloping algebra. Since $\mathbb{G}_a^n$ is commutative, its universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ coincides with the symmetric algebra $\mathfrak{S}(\mathfrak{g})$, so if we choose a basis of $\mathfrak{g}$ given by $S_1 = \frac{\partial}{\partial x_1}, \ldots, S_n = \frac{\partial}{\partial x_n}$, then there is an isomorphism of algebras $\mathfrak{U}(\mathfrak{g}) \simeq k[S_1, \ldots, S_n]$ such that $\mathfrak{g}$ is identified with the vector subspace $\langle S_1, \ldots, S_n \rangle$. Then $\operatorname{im} \rho$ will be a local algebra due to the fact that $\rho(S_i)$ are nilpotent matrices, and its rank will be $n+1$ as a consequence of the open orbit.

Conversely, given a local algebra $R$ of rank $n+1$ with $k$-basis $v_1, \ldots, v_n$ of its maximal ideal, an additive action

$$(a_1, \ldots, a_n) \cdot [x_0 : x_1 : \cdots : x_n]$$

on $\mathbb{P}(A) \simeq \mathbb{P}^n$ is determined by the product

$$\exp(a_1 v_1 + \cdots + a_n v_n)(x_0 + x_1 v_1 + \cdots + x_n v_n).$$

There are finitely many isomorphism classes of local algebras of rank $n+1$ for $n \leq 5$, and infinitely many for any $n \geq 6$. See [Poo08] or Appendix A for a list. As a consequence, we obtain the following classification of additive actions on $\mathbb{P}^n$ for $n \leq 5$:

**Table 2.1.** Total number $A_n$ of additive actions on $\mathbb{P}^n$ up to equivalence.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $A_n$ | 1 | 2 | 4 | 9 | 25 | $\infty$ |

We now proceed to give examples of additive actions on $\mathbb{P}^n$ together with their corresponding local algebras.

**Example 2.3.** $\mathbb{P}^n$ admits a unique additive action where the additive boundary hyperplane is fixed pointwise. We can write this $\mathbb{G}_a^n$-action on $\mathbb{P}^n$ as

$$(a_1, \dots, a_n) \cdot [x_0 : x_1 : \cdots : x_n] = [x_0 : x_1 + a_1 x_0 : x_2 + a_2 x_0 : \cdots : x_n + a_n x_0].$$

The additive boundary is the hyperplane $x_0 = 0$. The closure of the orbit of any 1-parameter subgroup is a line.

Its corresponding local algebra of rank $n+1$ is

$$R = k[S_1, \dots, S_n]/(S_1, \dots, S_n)^2 \simeq k[S_1, \dots, S_n]/(S_i S_j)_{1 \le i,j \le n}$$
$$\mathfrak{m} = \langle S_1, \dots, S_n \rangle,$$

from which we can recover the action by

$$\begin{aligned}(a_1, \dots, a_n) \cdot [x_0 : x_1 : \cdots : x_n] &= \exp(a_1 S_1 + \cdots + a_n S_n)(x_0 + x_1 S_1 + \cdots + x_n S_n) \\ &= (1 + a_1 S_1 + \cdots + a_n S_n)(x_0 + x_1 S_1 + \cdots + x_n S_n) \\ &= x_0 + (x_1 + a_1 x_0) S_1 + \cdots + (x_n + a_n x_0) S_n \\ &= [x_0 : x_1 + a_1 x_0 : \cdots : x_n + a_n x_0].\end{aligned}$$

**Example 2.4.** $\mathbb{P}^n$ admits a unique additive action with finitely many orbits. For $n = 2$, we can write this $\mathbb{G}_a^2$-action on $\mathbb{P}^2$ as

$$(a_1, a_2) \cdot [x_0 : x_1 : x_2] = [x_0 : x_1 + a_1 x_0 : x_2 + a_1 x_1 + \left(\tfrac{1}{2} a_1^2 + a_2\right) x_0].$$

The additive boundary is the line $x_0 = 0$, which is the disjoint union of a 1-dimensional orbit and the fixed point $[0 : 0 : 1]$. The closure of a general orbit of a general 1-parameter subgroup is a conic.

This action on $\mathbb{P}^n$ with finitely many orbits corresponds to the local algebra of rank $n+1$ given by

$$R = k[S]/(S^{n+1})$$
$$\mathfrak{m} = \langle S, S^2, \ldots, S^n \rangle.$$

For $n = 2$, the additive action on $\mathbb{P}^2$ can be recovered by

$$\begin{aligned}
(a_1, a_2) \cdot [x_0 : x_1 : x_2] &= \exp(a_1 S + a_2 S^2)(x_0 + x_1 S + x_2 S^2) \\
&= (1 + a_1 S + \left(\tfrac{1}{2}a_1^2 + a_2\right) S^2)(x_0 + x_1 S + x_2 S^2) \\
&= x_0 + (x_1 + a_1 x_0)S + \left(x_2 + a_1 x_1 + \tfrac{1}{2}a_1^2 x_0 + a_2 x_0\right) S^2 \\
&= [x_0 : x_1 + a_1 x_0 : x_2 + a_1 x_1 + \left(\tfrac{1}{2}a_1^2 + a_2\right) x_0].
\end{aligned}$$

**Definition 2.5.** We define the $\tau$-action on $\mathbb{P}^n$ to be the unique additive action on $\mathbb{P}^n$ with finitely many orbits (as in Example 2.4) up to equivalence. We also define the $\iota$-action on $\mathbb{P}^n$ for $n \geq 2$ to be the unique additive action on $\mathbb{P}^n$ which fixes a hyperplane pointwise (as in Example 2.3) up to equivalence.

*Remark* 2.6. As a mnemonic, $\tau$ stands for "toric" (since toric varieties have finitely many orbits) and $\iota$ stands for "identity" (since the $\iota$-action acts as the identity on the hyperplane at infinity).

Starting with $n = 6$, we get moduli. Let $(R, \mathfrak{m})$ be a local algebra of rank $n$. For $i \geq 1$, let $d_i = \dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1})$. The Hilbert–Samuel function of $R$ is defined to be the sequence $(d_i)_{i \geq 1}$, where we retain only the finitely many nonzero terms.

**Example 2.7.** A local algebra $(R, \mathfrak{m})$ of rank 7 and Hilbert–Samuel function $(4, 2)$ has $\mathfrak{m}^3 = 0$, so the multiplication in the algebra (i.e. the algebra structure) is completely determined by the symmetric bilinear map $\mathfrak{m}/\mathfrak{m}^2 \times \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^2$. Composing with linear functionals $\mathfrak{m}^2 \to k$, we see that a choice of algebra structure corresponds to a pair of symmetric $4 \times 4$ matrices. These in turn correspond to a pair of quadratic forms in 4 variables. Their common vanishing locus in $\mathbb{P}^3$ is a complete intersection of two quadrics, which defines in the generic case a smooth curve $C$, which has degree $4 = 2 \cdot 2$. By adjunction,

$$\omega_C = (\mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_{\mathbb{P}^3}(2))\big|_C = \mathcal{O}_C,$$

so

$$2g - 2 = \deg \omega_C = 0.$$

Therefore, the curve $C$ has genus $g = 1$, so it is an elliptic curve, whose $j$-invariant is also an invariant of $R$. Note also that every elliptic curve arises in this way. Since infinitely many $j$-invariants arise from elliptic curves, we obtain that there are infinitely many isomorphism classes of $R$, and consequently infinitely many nonequivalent additive actions on $\mathbb{P}^6$.

Alternatively, we could argue that any symmetric bilinear map $\mathfrak{m}/\mathfrak{m}^2 \times \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^2$ factors uniquely through a linear map $\mathrm{Sym}^2(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^2$, so the dimension of the space of such maps is $20 = \binom{4+2-1}{2} \cdot 2$. But an isomorphism of algebras corresponds to a change of coordinates in $\mathfrak{m}/\mathfrak{m}^2$ and $\mathfrak{m}^2$, so we need to consider maps up to the action of $\mathrm{GL}_4 \times \mathrm{GL}_2$, which has dimension $20 = 4^2 + 2^2$. However, the kernel of the action of $\mathrm{GL}_4 \times \mathrm{GL}_2$ is one-dimensional. Therefore, we obtain a family of such algebras of dimension $1 = 20 - 19$.

In general, local algebras with Hilbert–Samuel function $(d_1, d_2)$ have the following interpretation:

**Lemma 2.8** ([Poo08, Lemma 1.3])**.** *Local algebras of rank $n$ with Hilbert–Samuel function $(d_1, d_2)$ correspond to $d_2$-dimensional subspaces of the symmetric matrices forms on a $d_1$-dimensional vector space $V$, up to* $\mathrm{GL}(V)$*equivalence. Equivalently, they correspond to*

$(d_1(d_1+1)/2-d_2)$*-dimensional subspaces of the space of quadratic forms on a* $d_1$*-dimensional vector space* $W$*, up to* $\mathrm{GL}(W)$*-equivalence.*

**Example 2.9.** For local algebras of rank 8 with Hilbert–Samuel function $(5,3)$, we get the moduli space of genus 5 curves. Indeed, such an algebra $R$ corresponds to 3 quadratic forms in 5 variables. Their common vanishing locus in $\mathbb{P}^4$ corresponds in the generic case to a smooth curve $C$ of degree $8 = 2 \cdot 2 \cdot 2$. By adjunction,

$$\omega_C = (\mathcal{O}_{\mathbb{P}^3}(-5) \otimes \mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_{\mathbb{P}^3}(2))\big|_C = \mathcal{O}_C(1),$$

so

$$2g - 2 = \deg \omega_C = \deg C = 8.$$

Therefore, the curve $C$ has genus $g = 5$.

**Example 2.10.** For local algebras of rank 9 with Hilbert–Samuel function $(6,3)$, we get the moduli space of K3 surfaces of degree 8. Indeed, such an algebra $R$ corresponds to 3 quadratic forms in 6 variables. Their common vanishing locus in $\mathbb{P}^5$ corresponds in the generic case to a smooth surface $S$ of degree $8 = 2 \cdot 2 \cdot 2$. By adjunction,

$$\omega_S = (\mathcal{O}_{\mathbb{P}^3}(-6) \otimes \mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_{\mathbb{P}^3}(2))\big|_S = \mathcal{O}_S,$$

so $X$ is a K3 surface.

Motivated by these examples of additive actions on $\mathbb{P}^n$ with geometric significance, Hassett and Tschinkel asked the following question:

**Question 2.11** ([HT99])**.** *Can the appearence of these K3 surfaces and genus 5 curves be explained geometrically, in terms of birational maps between different additive actions on projective space?*

## 2.2 The $G$-Sarkisov program

By [BCHM10, Corollary 1.3.3], we may run an MMP on a nonpseudoeffective $\mathbb{Q}$-factorial klt pair to obtain a Mori fiber space. However, the outcome may not be unique: different MMPs on the same such pair may output different Mori fiber spaces. Note however that any two such Mori fiber spaces are birational, since they are both outcomes of an MMP starting from the same pair and the MMP preserves birational equivalence. The Sarkisov program, proved to hold by Hacon and M$^{\mathrm{c}}$Kernan [HM13], says that we can factor any birational map between Mori fiber spaces using only four types of elementary transformations, known as Sarkisov links.

The four different types of Sarkisov links between Mori fiber spaces $f\colon X \to S$ and $g\colon Y \to T$ are:

I

$$\begin{tikzcd} X' \arrow[r, dashed] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[d, "f"'] & T \arrow[dl] \\ S & \end{tikzcd}$$

II

$$\begin{tikzcd} X' \arrow[r, dashed] \arrow[d] & Y' \arrow[d] \\ X \arrow[d, "f"'] & Y \arrow[d, "g"] \\ S \arrow[r, phantom, "="] & T \end{tikzcd}$$

III

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "f"'] & Y' \arrow[d] \\ S \arrow[dr] & Y \arrow[d, "g"] \\ & T \end{tikzcd}$$

IV

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[d, "f"'] & & Y \arrow[d, "g"] \\ S \arrow[dr] & & T \arrow[dl] \\ & R. & \end{tikzcd}$$

**Figure 2.1.** The four types of Sarkisov links.

All horizontal maps are a sequence of flops (with respect to a suitable boundary), and the vertical maps are extremal (i.e. of relative Picard number one); in particular, maps to $X$ or $Y$ are extremal divisorial contractions.

We are interested in the setting of a $\mathbb{Q}$-factorial klt pair $(Z, D)$ where $Z$ is equipped with a regular action of a connected algebraic group $G$ and the boundary is $G$-invariant. By Blanchard's lemma, every step of an MMP which starts with $(Z, D)$ is $G$-equivariant. This was noted in [Flo20] to obtain the $G$-equivariant Sarkisov program, which is the version that we state.

To state the main theorem in the Sarkisov program in the equivariant setting, let us make a definition first of what it means to be birational under a group action.

**Definition 2.12.** Let $(Z, D)$ and $(Z', D')$ be pairs such that $Z$ and $Z'$ have a regular action of a connected algebraic group $G$, and the boundaries $D$ and $D'$ are $G$-invariant. Then $(Z, D)$ and $(Z', D')$ are $G$-birational if there exists a $G$-equivariant birational map $h\colon Z \dashrightarrow Z'$ satisfying the following: for any common resolution $p\colon W \to Z$ and $q\colon W \to Z'$ of $h$, the strict transforms of $D$ and $D'$ coincide on $W$, that is $p_*^{-1}D = q_*^{-1}D'$.

Note that $(Z, D)$ and $(Z', D')$ are $G$-birational $\mathbb{Q}$-factorial terminal pairs if and only if $(Z, D)$ and $(Z', D')$ are the output of running two MMPs from for an appropriate log pair $(X, B)$.

We are now able to state the main result of the $G$-equivariant Sarkisov program.

**Theorem 2.13** ([HM13])**.** *Two Mori fiber spaces $X \to S$ and $Y \to T$ of $\mathbb{Q}$-factorial terminal pairs are $G$-birational if and only if they are related by a finite sequence of $G$-equivariant Sarkisov links.*

*Remark* 2.14. The notions of Sarkisov link and "relation" between Sarkisov links (as in [Kal13]) can also be understood through the more general notion of "rank $r$ fibration" [BLZ21].

## 2.3 The Sarkisov program for surfaces

To see the original motivation for the Sarkisov program, let us abandon the equivariant setting for a moment and focus on surfaces.

The smooth two-dimensional rational Mori fiber spaces are $\mathbb{P}^2 \to \mathrm{pt}$ and the Hirzebruch surfaces $\mathbb{F}_r \to \mathbb{P}^1$ for $r \geq 0$, where $\mathbb{F}_0$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ is the blowup of $\mathbb{P}^2$ at a point.

The Hirzebruch surface $\mathbb{F}_r$ can be defined as $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-r))$, that is as a $\mathbb{P}^1$-bundle obtained from a rank 2 vector bundle over $\mathbb{P}^1$. It can also be constructed by starting with the line $L = \mathbb{P}^1 \times \infty$ in $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, taking the $r$ lines $L_i = [i:1] \times \mathbb{P}^1$ for $1 \leq i \leq r$ meeting $L$ at $p_i$, blowing up $p_1, \ldots, p_r$ (which separates $L$ and $L_i$), and blowing down the strict transforms of $L_1, \ldots, L_r$, which had become $(-1)$-curves. See Figure 2.2.

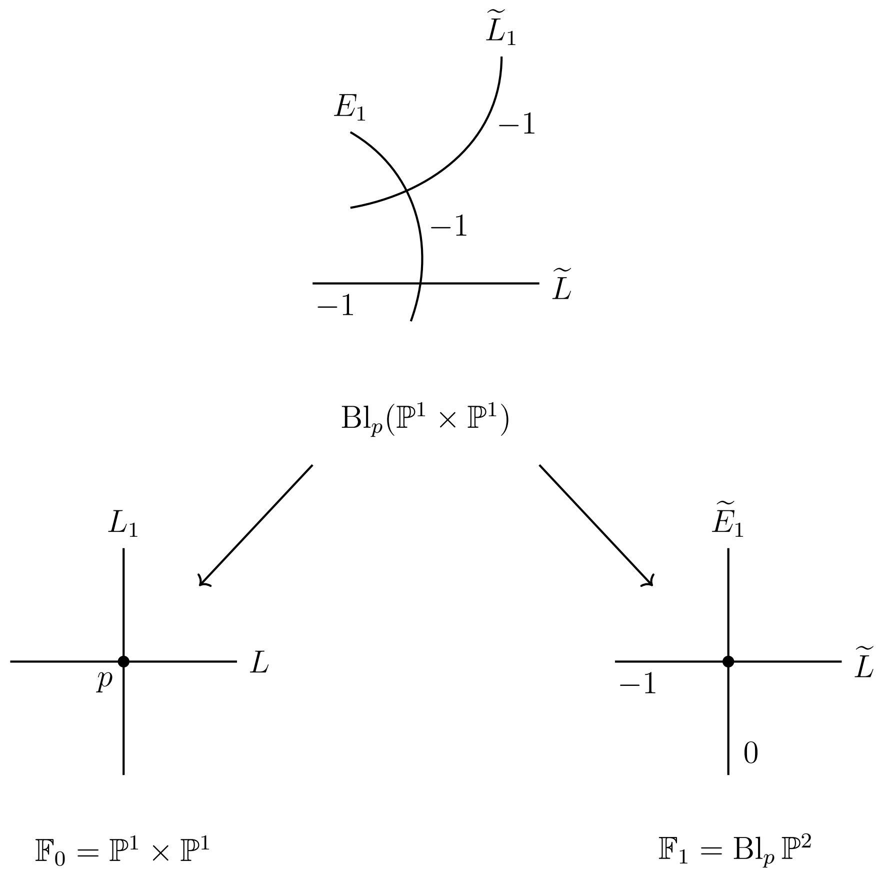


**Figure 2.2.** Construction of $\mathbb{F}_1$ from $\mathbb{F}_0$ using a Sarkisov link of type II. On $\mathbb{F}_1$, the negative section $\xi_1$ is $\widetilde{L}$ and the fiber $f$ is $\widetilde{E}_1$.

The cone of effective divisors on $\mathbb{F}_r$ is generated by two divisors: the negative section $\xi_r$ with $\xi_r^2 = -r$ (which is the strict transform of $L$ from $\mathbb{P}^1 \times \mathbb{P}^1$) and the fiber $f$ with $f^2 = 0$ (which is the strict transform of the exceptional divisor of the last blowup). We also have that $f \cdot \xi_r = 1$.

A theorem of Zariski says that any birational map between smooth surfaces is a composition of point blowups and inverses of point blowups. Therefore, we can very explicitly describe the four types of Sarkisov links between smooth rational Mori fiber spaces:

- Type I: the inverse $\mathbb{P}^2 \dashrightarrow \mathbb{F}_1$ of a point blowup.
- Type II: an elementary transformation $\mathbb{F}_r \dashrightarrow \mathbb{F}_{r\pm 1}$, defined as a blowup of a point $p \in \mathbb{F}_r$ followed by the contraction of the strict transform of the fiber through $p$. It is of type II.1 if $p$ is in the negative section $\xi_r$, in which case the elementary transformation is $\mathbb{F}_r \dashrightarrow \mathbb{F}_{r+1}$. Otherwise, it is of type II.2, in which case the elementary transformation is $\mathbb{F}_r \dashrightarrow \mathbb{F}_{r-1}$.
- Type III: a point blowup $\mathbb{F}_1 \to \mathbb{P}^2$.
- Type IV: the involution $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ which exchanges the two factors.

This description allows us to draw the Sarkisov graph in this setting (Figure 2.3), which shows all possible Sarkisov links between two-dimensional smooth rational Mori fiber spaces.

$$\begin{tikzcd}
 & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[loop above, "\mathrm{IV}"] \arrow[d, dashed, shift left, "\mathrm{II.1}"] \\
\mathbb{P}^2 \arrow[r, dashed, shift left, "\mathrm{I}"] & \mathbb{F}_1 \arrow[l, shift left, "\mathrm{III}"] \arrow[u, dashed, shift left, "\mathrm{II.2}"] \arrow[d, dashed, shift left, "\mathrm{II.1}"] \\
 & \mathbb{F}_2 \arrow[u, dashed, shift left, "\mathrm{II.2}"] \arrow[d, dashed, shift left, "\mathrm{II.1}"] \\
 & \mathbb{F}_3 \arrow[u, dashed, shift left, "\mathrm{II.2}"] \arrow[d, dashed, shift left, "\mathrm{II.1}"] \\
 & \vdots \arrow[u, dashed, shift left, "\mathrm{II.2}"]
\end{tikzcd}$$

**Figure 2.3.** The two-dimensional Sarkisov graph.

## 2.4 Untwists

**Lemma 2.15.** *Let $X$ and $Y$ be $n$-dimensional varieties equipped with additive actions $\phi$ and $\psi$, respectively. Then $X$ and $Y$ are $\mathbb{G}_a^n$-birational.*

*Proof.* Let $U \subset X$ and $V \subset Y$ be the open orbits of $\phi$ and $\psi$, respectively. For any $x \in U$, the orbit map $\phi_x \colon g \mapsto g \cdot_\phi x$ is an isomorphism between $\mathbb{G}_a^n$ and $U$ (similarly for any $y \in V$). Then a $\mathbb{G}_a^n$-equivariant birational map $X \dashrightarrow Y$ is provided by $\psi_y \circ \phi_x^{-1}$. □

*Remark* 2.16. By definition, any two $n$-dimensional additive varieties have the same action of $\mathbb{G}_a^n$ on the open orbit by translations, up to an automorphism of $\mathbb{A}^n$. In the proof above, the composition $\psi_y \circ \phi_x^{-1}$ restricts to a $\mathbb{G}_a^n$-equivariant isomorphism between the open orbits $U$ and $V$, both of which are isomorphic to $\mathbb{A}^n$.

In particular, any two additive actions on $\mathbb{P}^n$ are $\mathbb{G}_a^n$-birational, so they can be conjugated in $\mathrm{Bir}(\mathbb{P}^n)$ by some $\mathbb{G}_a^n$-equivariant birational automorphism $\sigma$ of $\mathbb{P}^n$, which can be decomposed into $\mathbb{G}_a^n$-equivariant Sarkisov links by the $\mathbb{G}_a^n$-Sarkisov program. If the two additive actions have the same open orbit $U \subset \mathbb{P}^n$, then the conjugating element $\sigma$ will belong to $\mathrm{Aut}(U) \subset \mathrm{Bir}(\mathbb{P}^n)$. Otherwise, we will need to conjugate further by some birational automorphism of $\mathbb{P}^n$ so that the open orbits coincide, thus it will not be the case that $\sigma$ is an automorphism of $\mathbb{A}^n$. Therefore, we will usually assume that all additive actions on $\mathbb{P}^n$ share the same open orbit.

**Definition 2.17.** Let $\phi_1$ and $\phi_2$ be additive actions on $X$. A $\mathbb{G}_a^n$-equivariant birational automorphism $F$ of $X$ is said to untwist $\phi_1$ into $\phi_2$ if $\phi_2$ is the conjugate of $\phi_1$ by $F$ in $\mathrm{Bir}(X)$, that is $F \circ \phi_1 \circ F^{-1} = \phi_2$.

*Remark* 2.18. The motivation for the word "untwist" is that any additive action on $\mathbb{P}^n$ different from the $\iota$-action may be considered as a "twisted" version of the $\iota$-action. For example, closures of orbits of one-parameter subgroups under the $\iota$-action are lines, while closures of orbits of one-parameter subgroups under other additive actions on $\mathbb{P}^n$ are not lines in general.

*Remark* 2.19. Let $X$ be an $n$-dimensional additive variety, and let $\phi_1$ and $\phi_2$ be additive actions on $X$ with the same open orbit $U \subset X$. We have that $F \in \mathrm{Bir}(X)$ untwists $\phi_1$ into $\phi_2$ if and only if

$$F(g \cdot_{\phi_1} p) = g \cdot_{\phi_2} F(p)$$

for all $g \in \mathbb{G}_a^n$ and all $p \in U$.

**Definition 2.20.** Let $(R, \mathfrak{m})$ be a local algebra of rank $n+1$, and let $v_1, \dots, v_n$ be a basis of $\mathfrak{m}$. Then

$$\exp(x_1 v_1 + \cdots + x_n v_n) = 1 + f_1 v_1 + \cdots + f_n v_n \in R[x_1, \dots, x_n],$$

where the $f_i$ are polynomials in the $x_i$. We say that $f_1, \dots, f_n$ are the basic polynomials of $R$.

The following theorem shows that an untwist of $\iota$ into another additive action on $\mathbb{P}^n$ with local algebra $R$ is given by the basic polynomials of $R$, and thus can be computed explicitly.

**Theorem 2.21** ([Arz24, Theorem 1])**.** *Let $\phi$ be an additive action on $\mathbb{P}^n$ with open orbit $U \subset \mathbb{P}^n$ corresponding to a local algebra $R$ of rank $n+1$. Consider the additive action $\iota$ on $\mathbb{P}^n$ with open orbit $U$. Let $f_1, \dots, f_n$ be the basic polynomials of $R$. Then $f = (f_1, \dots, f_n) \in \mathrm{Aut}(U) \subset \mathrm{Bir}(\mathbb{P}^n)$ is an untwist of $\iota$ into $\phi$.*

*Remark* 2.22. In the notation of Theorem 2.21, if $U = \{x_0 \neq 0\}$, then the untwist $f$, viewed as a birational automorphism of $\mathbb{P}^n$, is given by

$$F = [x_0^r : F_1 : \cdots : F_n],$$

where $r$ is the largest degree among the $f_i$ and

$$F_i = x_0^r f_i \left( \tfrac{x_1}{x_0}, \dots, \tfrac{x_n}{x_0} \right)$$

is the homogenization of $f_i$ of degree $r$ using $x_0$.

See Appendix A for a list of all untwists of $\iota$ into each of the other additive actions on $\mathbb{P}^n$ for $n \leq 6$.

The following lemma gives an explicit description of the set of all untwists between two additive actions, showing that untwists are unique up to translations:

**Lemma 2.23.** *Let $\phi$ and $\psi$ be additive actions on $X$ with the same open orbit $U \subset X$, and let $f \in Aut(U)$ be an untwist of $\phi$ into $\psi$. Then*

$$\mathcal{F} = \{f \circ \phi(g) \,|\, g \in \mathbb{G}_a^n\} = \{\psi(g) \circ f \,|\, g \in \mathbb{G}_a^n\} \subset \mathrm{Aut}(U)$$

*is the set of all untwists of $\phi$ into $\psi$.*

*Proof.* Let $n$ be the dimension of $X$. Let $g \in \mathbb{G}_a^n$ and let $f_1, f_2 \in \mathrm{Aut}(U)$ be untwists of $\phi$ into $\psi$. By the definition of untwist,

$$f_1 \circ \phi(g) \circ f_1^{-1} = \psi(g) \quad \text{and} \quad f_2 \circ \phi(g) \circ f_2^{-1} = \psi(g).$$

Equating the right-hand-sides,

$$f_1 \circ \phi(g) \circ f_1^{-1} = f_2 \circ \phi(g) \circ f_2^{-1},$$

and applying $f_1^{-1}$ on the right and $f_2^{-1}$ on the left,

$$\phi(g) \circ f_1^{-1} \circ f_2 = f_1^{-1} \circ f_2 \circ \phi(g).$$

Let $C = f_1^{-1} \circ f_2$. The above equation says that $C$ belongs to the centralizer $C_{\mathrm{Aut}(U)}(\phi(\mathbb{G}_a^n))$ of the $\phi$-action in $\mathrm{Aut}(U)$.

By Lemma 1.15, the centralizer $C_{\mathrm{Aut}(U)}(\phi(\mathbb{G}_a^n))$ is $\phi(\mathbb{G}_a^n)$, so $C = \phi(a)$ for some $a \in \mathbb{G}_a^n$. Putting everything together we conclude that

$$f_2 = f_1 \circ \phi(a) = \psi(a) \circ f_1. \qquad \square$$

**Corollary 2.24.** *Let $\phi$ be an additive action on $\mathbb{P}^n$ with open orbit $U \subset \mathbb{P}^n$, and consider the $\iota$-action on $\mathbb{P}^n$ with open orbit $U$. Let $f = (f_1, \dots, f_n) \in \mathrm{Aut}(U)$ be an untwist of $\iota$ into $\phi$. Then the set of all untwists of $\iota$ into $\phi$ is*

$$\{(f_1(x_1 + a_1, \dots, x_n + a_n), \dots, f_n(x_1 + a_1, \dots, x_n + a_n)) \,|\, (a_1, \dots, a_n) \in \mathbb{G}_a^n\} \subset \mathrm{Aut}(U).$$

*Proof.* By Lemma 2.23, we have that the set of all untwists of $\iota$ into $\phi$ is

$$\{f \circ \iota(a) \,|\, a \in \mathbb{G}_a^n\}.$$

The result follows from the fact that $\iota(g)(x) = (x_1 + a_1, \dots, x_n + a_n)$ for any $x = (x_1, \dots, x_n) \in U$. $\square$

Finally, the following question is very natural from the point of view of the $\mathbb{G}_a^n$-Sarkisov program:

**Problem 2.25** ([HT99])**.** Provide an explicit factorization for any $\mathbb{G}_a^n$-equivariant birational automorphism of $\mathbb{P}^n$.

Note that any $\mathbb{G}_a^n$-equivariant birational automorphism of $\mathbb{P}^n$ can be written as a composition of automorphisms of $\mathbb{P}^n$ with untwists, so the question above reduces to factorizing untwists.

*Remark* 2.26. Let $\phi$ be an additive action on $\mathbb{P}^n$ with open orbit $U \subset \mathbb{P}^n$, and consider the $\iota$-action on $\mathbb{P}^n$ with open orbit $U$. It was noted in [Arz24] that untwists of $\iota$ into $\phi$ are triangular automorphisms of $\mathbb{A}^n$, and hence tame.

Since an additive variety may admit nonequivalent additive actions, it will be convenient to use $(X, \phi)$ to denote the equivalence class of the additive action $\phi$ on $X$. However, we further emphasize that even if $\phi$ and $\phi'$ are not equivalent additive actions on an $n$-dimensional $X$, we always have that $(X, \phi)$ and $(X, \phi')$ are $\mathbb{G}_a^n$-birational by Lemma 2.15.

## 2.5 The $\mathbb{G}_a^2$-Sarkisov program for surfaces

Let $[x_0 : x_1 : x_2]$ be the coordinates in $\mathbb{P}^2$, let $(x, y)$ be the coordinates in $U = \{x_0 \neq 0\} \subset \mathbb{P}^2$, and let $(a_1, a_2)$ be an element of $\mathbb{G}_a^2$.

There are two additive actions on $\mathbb{P}^2$ up to equivalence:

$$\iota(a_1, a_2) = \begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \tau(a_1, a_2) = \begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 + \frac{1}{2}a_1^2 & a_1 & 1 \end{pmatrix}.$$

With these coordinates, the additive boundary is $x_0 = 0$ for both actions and the point $[1 : 0 : 0]$ is fixed by both actions.

The orbit of the point $[0 : 0 : 1]$ under a general one-parameter subgroup of the $\tau$-action is a conic in $U$ given by

$$x = t + \tfrac{1}{2}y^2$$

for some $t \in \mathbb{A}^1$. The orbit of the point $[0 : 0 : 1]$ under a one-parameter subgroup of the $\iota$-action is a line in $U$.

$\mathbb{P}^1 \times \mathbb{P}^1$ admits a unique additive action, induced by a $\mathbb{G}_a$-action on each factor.

There are four orbits: the 2-dimensional open orbit, two 1-dimensional orbits, and a fixed point. The two irreducible components of the additive boundary are lines which meet at the fixed point. We call this action the $\tau$-action on $\mathbb{P}^1 \times \mathbb{P}^1$.

**Proposition 2.27** ([HT99, Proposition 5.5])**.** *The Hirzebruch surfaces $\mathbb{F}_r$ with $r > 0$ each have two nonequivalent additive actions, which we call the $\iota$- and $\tau$-actions on $\mathbb{F}_r$. The additive boundary of $\mathbb{F}_r$ consists of the distinguished fiber $f$ and the negative section $\xi_r$. Both actions act on $\xi_r$ with a one-dimensional orbit. These two actions differ on how they act on $f$:*

- *$\iota$-action: fixes $f$ pointwise.*
- *$\tau$-action: acts on $f$ with a 1-dimensional orbit.*

We have four types of $\mathbb{G}_a^2$-equivariant Sarkisov links of type II:

(II.1a) $(\mathbb{F}_r, \tau) \dashrightarrow (\mathbb{F}_{r+1}, \iota)$ for $r \geq 0$: blow up the fixed point $p$ at the intersection of $f$ and $\xi_r$ (the exceptional divisor will be fixed pointwise because it intersects the strict transforms of $f$ and $\xi_r$), and contract the strict transform of $f$. We adopt the convention that the case $r = 0$ belongs to this type.

(II.1b) $(\mathbb{F}_r, \iota) \dashrightarrow (\mathbb{F}_{r+1}, \iota)$ for $r \geq 1$: blow up the fixed point $p$ at the intersection of $f$ and $\xi_r$ (the exceptional divisor will be fixed pointwise because it intersects the strict transforms of $f$ and $\xi_r$), and contract the strict transform of $f$.

(II.2a) $(\mathbb{F}_r, \iota) \dashrightarrow (\mathbb{F}_{r-1}, \tau)$ for $r \geq 1$: blow up a general point $p \in f$ not in $\xi_r$, and contract the strict transform of $f$.

(II.2b) $(\mathbb{F}_r, \iota) \dashrightarrow (\mathbb{F}_{r-1}, \iota)$ for $r \geq 2$: the inverse of (II.1b). For any point $p$ in the additive boundary of $\mathbb{P}^2$, there exists a plane conic $C$ tangent to the additive boundary at $p$. The link (II.2b) starts with the blowup of the unique fixed point which intersects the strict transform of this conic, which is also the blowup of the point to which the distinguished fiber of $(\mathbb{F}_1, \iota)$ was contracted.

Note that there is no $\mathbb{G}_a^2$-equivariant Sarkisov link from $(\mathbb{F}_r, \tau)$ to $\mathbb{F}_{r-1}$, since $f \setminus (\xi_r \cap f)$ has no fixed points.

**Theorem 2.28.** *All $\mathbb{G}_a^2$-equivariant Sarkisov links between smooth additive two-dimensional Mori fiber spaces are described in Figure 2.4.*

$$
\begin{tikzcd}
 & & (\mathbb{P}^1 \times \mathbb{P}^1, \tau) \arrow[loop above, "\mathrm{IV}"] \arrow[d, dashed, shift left, "\mathrm{II.1a}"] \\
 & (\mathbb{P}^2, \iota) \arrow[r, dashed, shift left, "\mathrm{I}"] & (\mathbb{F}_1, \iota) \arrow[l, shift left, "\mathrm{III}"] \arrow[u, dashed, shift left, "\mathrm{II.2a}"] \arrow[d, dashed, shift left, "\mathrm{II.1b}"] \\
(\mathbb{P}^2, \tau) \arrow[r, dashed, shift left, "\mathrm{I}"] & (\mathbb{F}_1, \tau) \arrow[l, shift left, "\mathrm{III}"] \arrow[r, dashed, shift left, "\mathrm{II.1a}"] & (\mathbb{F}_2, \iota) \arrow[l, dashed, shift left, "\mathrm{II.2a}"] \arrow[u, dashed, shift left, "\mathrm{II.2b}"] \arrow[d, dashed, shift left, "\mathrm{II.1b}"] \\
 & (\mathbb{F}_2, \tau) \arrow[r, dashed, shift left, "\mathrm{II.1a}"] & (\mathbb{F}_3, \iota) \arrow[l, dashed, shift left, "\mathrm{II.2a}"] \arrow[u, dashed, shift left, "\mathrm{II.2b}"] \arrow[d, dashed, shift left, "\mathrm{II.1b}"] \\
 & & \vdots \arrow[u, dashed, shift left, "\mathrm{II.2b}"]
\end{tikzcd}
$$

**Figure 2.4.** The two-dimensional $\mathbb{G}_a^2$-Sarkisov graph.

*Proof.* Start with $\mathbb{P}^1 \times \mathbb{P}^1$. The only thing we can do is blow up the unique fixed point and contract the strict transform of a fiber (type II.1a) to obtain $(\mathbb{F}_1, \iota)$. On $(\mathbb{F}_1, \iota)$, we only have three possibilities: contract the negative section (type III) to obtain $(\mathbb{P}^2, \iota)$, do an elementary transformation (type II.1b) to obtain $(\mathbb{F}_2, \iota)$, or do an elementary transformation in the other direction (type II.2a) to get back to $\mathbb{P}^1 \times \mathbb{P}^1$. On $(\mathbb{P}^2, \iota)$, we can only blow up a fixed point (type I), which takes us back to $(\mathbb{F}_1, \iota)$.

So far, the only unexplored path is the link of type II.1b from $(\mathbb{F}_1, \iota)$ to $(\mathbb{F}_2, \iota)$. On $(\mathbb{F}_2, \iota)$, we can only perform an elementary transformation up (type II.1b) to $(\mathbb{F}_3, \iota)$, down (type II.2a) to $(\mathbb{F}_1, \tau)$, or back (type II.2b) to $(\mathbb{F}_1, \iota)$. On $(\mathbb{F}_1, \tau)$ we can only contract the negative section (type III) to get $(\mathbb{P}^2, \tau)$ or perform an elementary transformation up (type II.2) back to $(\mathbb{F}_2, \iota)$. On $(\mathbb{P}^2, \tau)$, the only thing we can do is to blow up the fixed point (type I) back to $(\mathbb{F}_1, \tau)$.

On $(\mathbb{F}_3, \iota)$, we can only perform an elementary transformation (type II.1b) up to $(\mathbb{F}_4, \iota)$, down (type II.2a) to $(\mathbb{F}_2, \tau)$, or back (II.2b) to $(\mathbb{F}_2, \iota)$. There are no other links out of $(\mathbb{F}_2, \tau)$ since there are no links from $(\mathbb{F}_r, \tau)$ down to $\mathbb{F}_{r-1}$. $\square$

**Example 2.29.** Hassett and Tschinkel [HT99, 3.1(1)] note that the quadratic involution $\sigma$ of $\mathbb{P}^2$ consisting of the blowup of three infinitely near points is an untwist of $\iota$ into $\tau$. The birational automorphism $\sigma$ can be factorized as follows into a sequence of blowups and blowdowns of points fixed by the $\mathbb{G}_a^2$-action (see Figure 2.5 for an illustration):

Let $C$ be the closure of a general orbit of a general one-parameter subgroup under the $\tau$-action on $\mathbb{P}^2$, and let $L$ be the line at infinity which is the complement of the open orbit. The curve $C$ is a plane conic which is tangent to $L$ at the unique fixed point $p$. Blow up $p$ to obtain an exceptional divisor $E$, whose point of intersection with the strict transform of $L$ must be a fixed point $q$, since the intersection of invariant subvarieties must be an invariant subvariety. Now blow up $q$ to obtain an exceptional divisor $F$, which intersects the strict transforms of $L$ and $E$ transversely, so it must have two fixed points. Thus $F$ is fixed pointwise, since any $\mathbb{G}_a^2$-action with two fixed points must be trivial. Then blow up the unique point of intersection of $F$ and the strict transform of $C$, thus obtaining a third exceptional divisor $G$. Observe that $G$ is fixed pointwise since it intersects transversely the strict transforms of $C$ and $F$, and $C$ has now become a line with self-intersection 1. After this third blowup, the strict transform of $L$ became a $(-1)$-curve, so it is contractible. Now blow down the strict transforms of $L$, $F$, and $E$, in this order. At the end of this process of three blowups and three blow downs, the resulting variety is $\mathbb{P}^2$ with the $\iota$-action, where the strict transform of $G$ has become the new additive boundary (the line at infinity), and the strict transform of $C$ became the closure of the orbit of any one-parameter subgroup under the $\iota$-action (namely, a line).

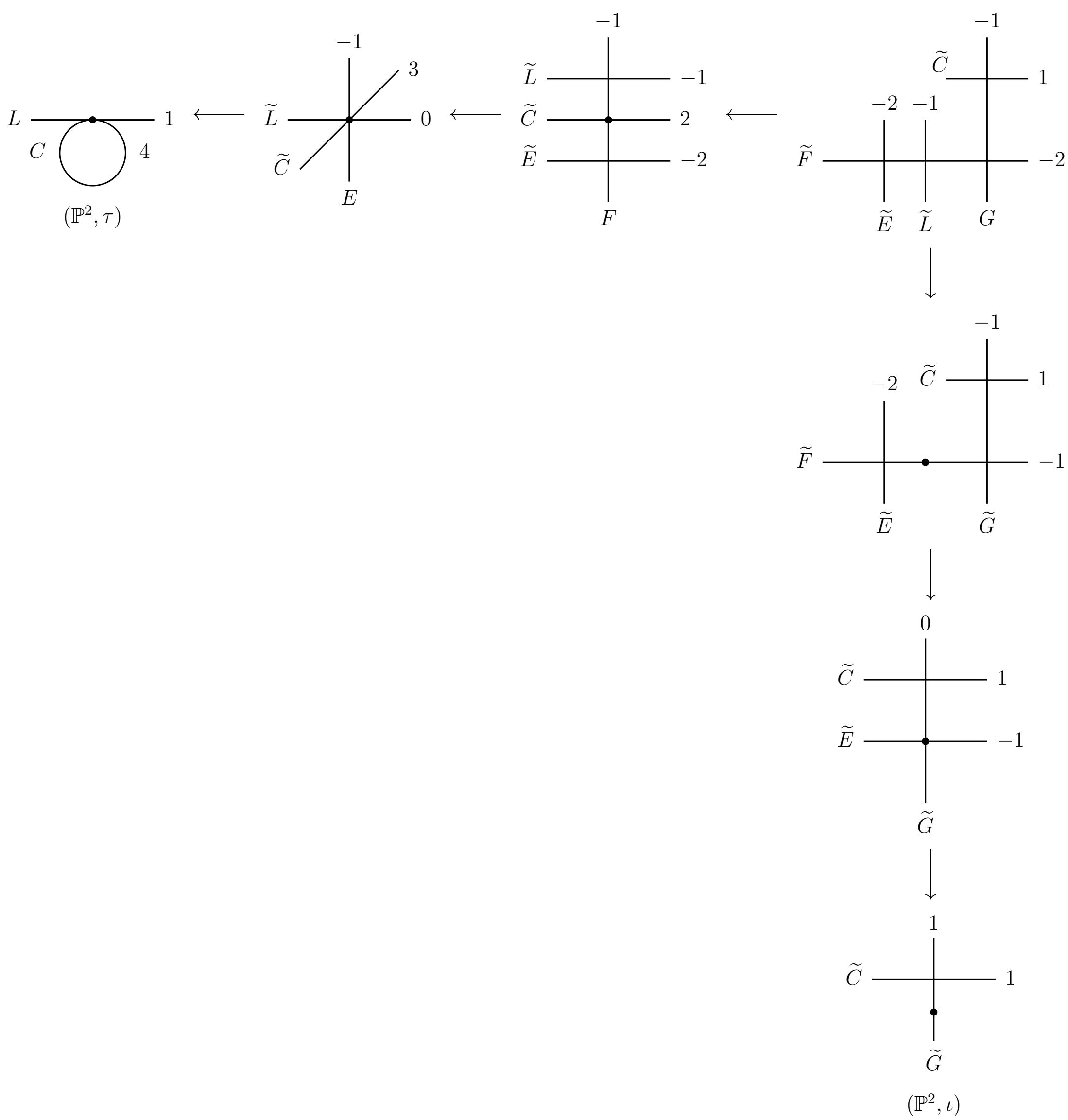


**Figure 2.5.** Untwist of $\iota$ into $\tau$ on $\mathbb{P}^2$.

By the $\mathbb{G}_a^2$-Sarkisov program, we may factor $\sigma$ into a sequence of $\mathbb{G}_a^2$-equivariant Sarkisov links. In particular, it corresponds to the following path in the $\mathbb{G}_a^2$-Sarkisov graph

(Figure 2.4) from $(\mathbb{P}^2,\tau)$ to $(\mathbb{P}^2,\iota)$:

$$\begin{tikzcd}
(\mathbb{P}^2,\tau) \arrow[r, dashed, "\mathrm{I}"] \arrow[rrrr, dashed, bend right, "\sigma"'] & (\mathbb{F}_1,\tau) \arrow[r, dashed, "\mathrm{II.1a}"] & (\mathbb{F}_2,\iota) \arrow[r, dashed, "\mathrm{II.2b}"] & (\mathbb{F}_1,\iota) \arrow[r, "\mathrm{III}"] & (\mathbb{P}^2,\iota).
\end{tikzcd}$$

We may also write down $\sigma$ in coordinates using Theorem 2.21. The local algebra of rank 3 corresponding to $\tau$ is $R = k[x]/(x^3)$ and its maximal ideal is $\mathfrak{m} = (x)$. A $k$-basis for $\mathfrak{m}$ is given by $\{x, x^2\}$. Since $\mathfrak{m}^3 = 0$, we have that

$$\begin{aligned}
\exp(x_1x + x_2x^2) &= 1 + x_1x + x_2x^2 + \tfrac{1}{2}(x_1x + x_2x^2)^2 \\
&= 1 + (x_1)x + (x_2 + \tfrac{1}{2}x_1^2)x^2.
\end{aligned}$$

Therefore, the basic polynomials of $R$ are $f_1(x_1,x_2) = x_1$ and $f_2(x_1,x_2) = (x_2 + \frac{1}{2}x_1^2)$, so we conclude that

$$\sigma = [x_0^2 : x_1x_0 : x_2x_0 + \tfrac{1}{2}x_1^2].$$

We can see that the indeterminacy locus of $\sigma$ is the point $[0:0:1]$ and its exceptional set is the line $x_0 = 0$.

The Noether–Castelnuovo theorem states that $\mathrm{Bir}(\mathbb{P}^2)$, the birational automorphism group of the plane, is generated by $\sigma$ together with automorphisms of $\mathbb{P}^2$ (it is usually stated that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathrm{PGL}_3$ together with the standard quadratic involution

$$\rho\colon [x_0 : x_1 : x_2] \mapsto \left[\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right],$$

but it is possible to factorize $\rho$ into a sequence of birational maps consisting of $\sigma$ and elements of $\mathrm{PGL}_3$).

Since $\sigma$ arises as an untwist in dimension two, it is tempting to ask if untwists in higher dimension generate $\mathrm{Bir}(\mathbb{P}^n)$ together with $\mathrm{Aut}(\mathbb{P}^n)$. However, this is not true: a result of Hudson and Pan [Pan99] states that one needs generators of unbounded degree to generate $\mathrm{Bir}(\mathbb{P}^n)$ for $n \geq 3$, while the degree of untwists is always bounded above.

## 2.6 Indeterminacy loci of untwists

In this section, we assume that all additive actions on $\mathbb{P}^n$ share the same open orbit $U \subset \mathbb{P}^n$.

The following definition will be useful for stating the main theorem of this section:

**Definition 2.30.** Let $(R, \mathfrak{m})$ be a local algebra of rank $n+1$ with Hilbert–Samuel function $(d_1, \dots, d_r)$. Let $V_1 = \mathfrak{m}/\mathfrak{m}^2$ and $V_r = \mathfrak{m}^r$. Consider the multiplication map

$$\mu_r \colon \operatorname{Sym}^r(V_1) \to V_r.$$

We define the directrix of $R$ to be the closed subscheme of $\mathbb{P}(V_1) = \mathbb{P}^{d_1-1}$ given by the ideal generated by the image of the dual map

$$\mu_r^\vee \colon V_r^\vee \to \operatorname{Sym}^r(V_1)^\vee.$$

*Remark* 2.31. Note that in Definition 2.30, the index of nilpotency of $\mathfrak{m}$ is $r+1$ (i.e. $\mathfrak{m}^s = 0$ for $s \geq r+1$ and $\mathfrak{m}^r \neq 0$). The multiplication map $\mu_r$ is determined by $d_r$ homogeneous polynomials $P_1, \dots, P_{d_r}$ in $d_1$ variables of degree $r$. Indeed, let $\{v_1, \dots, v_{d_1}\}$ and $\{w_1, \dots, w_{d_r}\}$ be $k$-bases of $V_1$ and $V_r$, respectively. An arbitrary element of $V_1$ is of the form

$$u = x_1 v_1 + \cdots + x_{d_1} v_{d_1}.$$

Then

$$\begin{aligned} u^r &= \sum c_{(a_1, \dots, a_{d_1})} \left(x_1^{a_1} \dots x_{d_1}^{a_{d_1}}\right) \left(v_1^{a_1} \dots v_{d_1}^{a_{d_1}}\right) \\ &= P_1 w_1 + \cdots + P_{d_r} w_{d_r}, \end{aligned}$$

where $c_{(a_1,\dots,a_{d_1})} \in k$. Notice that each $P_i$ is a homogeneous polynomial of degree $r$ in $x_1, \dots, x_{d_1}$ because $a_1 + \cdots + a_{d_1} = r$. The directrix $X$ is then the closed subscheme of $\mathbb{P}^{d_1-1}$ cut out by the homogeneous ideal $(P_1, \dots, P_{d_r})$.

Using Theorem 2.21 due to Arzhantsev [Arz24], we can describe the indeterminacy loci of untwists:

**Theorem 2.32.** *Let $\phi$ be an additive action on $\mathbb{P}^n$ corresponding to a local algebra $(R, \mathfrak{m})$ of rank $n+1$ with Hilbert–Samuel function $(d_1, \dots, d_r)$. Then the indeterminacy locus of any untwist of $\iota$ into $\phi$ is the projective cone over the directrix $X \subset \mathbb{P}^{d_1-1}$ of $R$ with vertex $\mathbb{P}^{n-d_1-1}$.*

*Proof.* We may assume that the open orbit $U$ shared by $\phi$ and $\iota$ is $\{x_0 \neq 0\}$. We have that the index of nilpotency of $\mathfrak{m}$ is $r+1$. We may choose a $k$-basis $\{v_1, \dots, v_n\}$ for $\mathfrak{m}$ that respects the filtration $\mathfrak{m} \supset \mathfrak{m}^2 \supset \dots \mathfrak{m}^r \supset 0$, so that $\{v_1, \dots, v_{d_1}\}$ forms a $k$-basis for $\mathfrak{m}/\mathfrak{m}^2$ and $\{v_{n-d_r+1}, \dots, v_n\}$ forms a $k$-basis for $\mathfrak{m}^r$. Let $v = x_1v_1 + \cdots + x_nv_n \in \mathfrak{m} \otimes_k k[x_1, \dots, x_n]$. We have that

$$\exp(v) = 1 + v + \frac{v^2}{2} + \cdots + \frac{v^r}{r!} = 1 + f_1v_1 + \cdots + f_nv_n,$$

where the $f_i$ are the basic polynomials of $R$. By Theorem 2.21, an untwist of $\iota$ into $\phi$ is given by $F = [x_0^r : F_1 : \cdots : F_n]$, where $F_i$ is the homogenization of $f_i$. Hence the indeterminacy locus $\mathrm{Ind}(F)$ is contained in the hyperplane at infinity $\{x_0 = 0\}$.

The only monomials of $F_i$ which are not divisible by $x_0$ are those whose dehomogenization are monomials of $f_i$ of degree exactly $r$. In turn, monomials of $f_i$ of degree exactly $r$ arise exclusively from the term $\frac{v^r}{r!}$. Because the index of nilpotency of $\mathfrak{m}$ is $r+1$, any term in $v$ belonging to $\mathfrak{m}^2$ or higher will vanish when raised to the $r$-th power. Consequently, $v^r$ depends only on the degree-1 variables $x_1, \dots, x_{d_1}$. Furthermore, the image of the map $v \mapsto v^r$ lies entirely in $\mathfrak{m}^r$. This yields two consequences. First, for $1 \leq i \leq n - d_r$, the maximum degree of any monomial in $f_i$ is strictly less than $r$. Therefore, $F_i$ is divisible by $x_0$ and vanishes entirely on $\{x_0 = 0\}$. Second, for $i > n - d_r$, the degree-$r$ homogeneous component of $f_i$ is precisely the polynomial $P_i(x_1, \ldots, x_{d_1})$ of degree $r$ defining the directrix $X$ of $R$. Thus, restricting to $\{x_0 = 0\}$ leaves $F_i(0, x_1, \ldots, x_n) = P_i(x_1, \ldots, x_{d_1})$.

It follows that $\mathrm{Ind}(F)$ in $\{x_0 = 0\} \simeq \mathbb{P}^{n-1}$ is cut out by the system of equations

$$P_{n-d_r+1}(x_1, \ldots, x_{d_1}) = \cdots = P_n(x_1, \ldots, x_{d_1}) = 0.$$

Since these equations depend only on the variables $x_1, \ldots, x_{d_1}$, they define the directrix $X \subset \mathbb{P}^{d_1-1}$. The remaining $n - d_1$ variables $x_{d_1+1}, \ldots, x_n$ are completely unconstrained, spanning a linear susbspace isomorphic to $\mathbb{P}^{n-d_1-1}$. Geometrically, this vanishing locus is exactly the projective cone over $X$ with vertex $\mathbb{P}^{n-d_1-1}$.

Finally, from the description of all untwists of $\iota$ into $\phi$ provided by Corollary 2.24, it follows that the indeterminacy locus is the same for any untwist. $\square$

As a corollary, we propose the following answer to Question 2.11:

**Corollary 2.33.** *Let $X$ be an elliptic curve, smooth genus 5 curve, or K3 surface of degree 8, which corresponds to an additive action $\phi$ on $\mathbb{P}^6$, $\mathbb{P}^8$, or $\mathbb{P}^9$, respectively. Then the indeterminacy locus of any untwist of $\iota$ into $\phi$ is the projective cone over $X$ with a line as its vertex.*

*Proof.* An elliptic curve, smooth genus 5 curve, or K3 surface of degree 8 corresponds to a local algebra $R$ with Hilbert–Samuel function $(4, 2)$, $(5, 3)$, or $(6, 3)$, respectively. Notice that the directrix of $R$ is precisely the complete intersection of the quadratic forms defining $X$, that is the directrix of $R$ is $X$. The result then follows directly from Theorem 2.32. $\square$

Part of Chapter 2 is being prepared for submission for publication. The dissertation author was the primary investigator and author of this material.

# Chapter 3

# Extremal rays on additive varieties

## 3.1 Finiteness of extremal rays

Let $X$ be a normal projective variety, and let $V$ be a finite dimensional affine subspace of the real vector space $\mathrm{WDiv}_{\mathbb{R}}(X)$ which is defined over the rationals. We have that

$$\mathcal{L}(V) = \{\Delta \in V \mid K_X + \Delta \text{ is log canonical and } \Delta \geq 0\}$$

is a rational polytope [BCHM10, Lemma 3.7.2].

**Conjecture 3.1.** *Let $X$ be a projective additive variety and let $V$ be a finite-dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ which is defined over the rationals. There exist two distinct facets $\mathcal{F}_V$ and $\mathcal{G}_V$ of the rational polytope $\mathcal{L}(V)$ such that for any compact subspace*

$$\mathcal{K} \subset \mathcal{L}(V) \setminus (\mathcal{F}_V \cup \mathcal{G}_V),$$

*there are extremal rays $R_1, \dots, R_t \subset \overline{NE}(X)$ satisfying the following property: for any additive $\Delta \in \mathcal{K}$, we have that $\overline{NE}(X)_{(K_X+\Delta)<0}$ is spanned by a subset of $\{R_1, \dots, R_t\}$.*

**Conjecture 3.2.** *Let $X$ be a projective additive variety and let $D_0, D_1, \dots, D_r$ be the irreducible components of the additive boundary divisor of $X$. Fix $\delta_1, \delta_2 \in (0,1)$ such that $\delta_1 \leq \delta_2$. Let $(\lambda_0, \lambda_1, \dots, \lambda_r) \in (\delta_1, \delta_2) \times [0,1]^r$. If $(X, \sum_{i=0}^{r} \lambda_i D_i)$ is log canonical, then $X$*

*has finitely many $(K_X + \sum_{i=0}^r \lambda_i D_i)$-negative extremal rays for some ordering of the $D_i$.*

In some sense, Conjecture 3.1 says that $X$ satisfies uniform finiteness, while Conjecture 3.2 says that $X$ satisfies pointwise finiteness. Conjectures 3.1 and 3.2 are equivalent by a compactness argument.

We also propose a weaker conjecture.

**Conjecture 3.3.** *Let $X$ be a projective additive variety and let $D_0, D_1, \ldots, D_r$ be the irreducible components of the additive boundary divisor of $X$. Let $(\lambda_0, \lambda_1, \ldots, \lambda_r) \in (0,1) \times [0,1]^r$. If $(X, \sum_{i=0}^r \lambda_i D_i)$ is log canonical, then $X$ has finitely many $(K_X + \sum_{i=0}^r \lambda_i D_i)$-negative extremal rays for some ordering of the $D_i$.*

*Remark* 3.4. Mori dream spaces, such as toric varieties, have rational polyhedral Kleiman–Mori cone, so Conjectures 3.1, 3.2, and 3.3 are automatically true for them. However, not all additive varieties are Mori dream spaces, as Example 1.28 shows, which is also our original motivation for stating these conjectures.

We are able to prove Conjecture 3.3 for blowups of $\mathbb{P}^n$ along points contained in a hyperplane, thus equivariant with respect to the $\iota$-action. Using similar methods, we believe it should be possible to extend this result to (possibly infinitely near) blowups of $\mathbb{P}^n$ along smooth subvarieties contained in a fixed hyperplane, which is again equivariant with respect to the $\iota$-action. Although not all infinitely near blowups are additive, we believe that the $\mathbb{G}_a^n$-equivariant MMP may be needed to prove these conjectures, and that is why we state these conjectures for additive varieties.

*Remark* 3.5. When working with Conjecture 3.3 and applying adjunction, it is more natural to use coefficients of the form $1 - \varepsilon_i$ instead of $\lambda_i$.

## 3.2 Intersection theory on blowups

Let $\pi \colon X_r^n \to \mathbb{P}^n$ denote the blowup of $\mathbb{P}^n$ along distinct points $p_1, \ldots, p_r \in \mathbb{P}^n$, with exceptional divisors $E_1, \ldots, E_r$. Let $H$ be the class of the pullback of a hyperplane, let $h$ be the class of the strict transform of a line not containing any of the $p_i$, and let $e_i$ be the class of a line in $E_i$. Note that rational, numerical, and homological equivalence coincide on $X_r^n$. The following lemma will be used implicitly:

**Lemma 3.6.** *We have the following relations between classes in $X_r^n$:*

$$H \cdot h = 1, \qquad H \cdot e_i = 0, \qquad E_i \cdot h = 0, \qquad E_i \cdot e_j = -\delta_{ij}.$$

*Let $C$ be a curve on $X = X_r^n$.*

1. *$C \subset E_i$ for some $i$ if and only if $[C] = a_i e_i$ for $a_i > 0$, which in turn is equivalent to $E_i \cdot C < 0$. In this case:*

$$\begin{aligned} K_X \cdot C &= \Big(-(n+1)H + (n-1)\sum E_i\Big) \cdot (a_i e_i) \\ &= -(n-1)a_i. \end{aligned}$$

2. *Otherwise, $[C] = dh - \sum a_i e_i$ with $d \geq a_i \geq 0$. The coefficient $d > 0$ is the degree of the image of the curve in $\mathbb{P}^n$ and $a_i = E_i \cdot C$ (equivalently, $a_i$ is the multiplicity of the image of the curve in $\mathbb{P}^n$ at $p_i$). In this case:*

$$\begin{aligned} K_X \cdot C &= \Big(-(n+1)H + (n-1)\sum E_i\Big) \cdot \Big(dh - \sum a_i e_i\Big) \\ &= -(n+1)d + (n-1)\sum a_i. \end{aligned}$$

*Assume that $p_1, \ldots, p_r$ are contained in a fixed hyperplane, and $S$ denotes the class of the strict transform of this hyperplane. Then $H$ is equivalent to $S + \sum E_i$. If*

$[C] = dh - \sum a_i e_i$, *then*

$$\begin{aligned} S \cdot C &= \Big(H - \sum E_i\Big) \cdot \Big(dh - \sum a_i e_i\Big) \\ &= d - \sum a_i. \end{aligned} \qquad \square$$

Although not used here, the following description of the Chow groups of a blowup might also be useful when dealing with blowups of positive-dimensional subvarieties:

**Proposition 3.7** ([Bea77, Proposition 0.1.3], [Ful98, §6.7], [EH16, §13.6])**.** *Let $W$ denote the blowup of a smooth variety $X$ along a smooth subvariety $Z$ of codimension $2+c$ with exceptional divisor $E = \mathbb{P}\mathcal{N}_{Z/X}$, so that we have the following diagram:*

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "\tau"'] & W \arrow[d, "\pi"] \\ Z \arrow[r, hook, "i"] & X. \end{tikzcd}$$

*Let $\zeta = c_1(\mathcal{O}_E(1)) \in A^1(E)$. Then we have the following isomorphism of Chow groups:*

$$\begin{aligned} A^q(X) \oplus \sum_{r=0}^{c} A^{q-1-r}(Z) &\simeq A^q(W) \\ (x, z_0, \dots, z_c) &\mapsto \pi^* x - j_* \left( \sum_{r=0}^{c} \zeta^r \cdot \tau^* z_r \right). \end{aligned}$$

*Furthermore, we have the following formulas for multiplication in $A(W)$:*

1. $\pi^*\alpha \cdot \pi^*\beta = \pi^*(\alpha \cdot \beta)$ *for* $\alpha, \beta \in A(X)$.

2. $\pi^*\alpha \cdot j_*\gamma = j_*(\gamma \cdot \tau^* i^* \alpha)$ *for* $\alpha \in A(X)$ *and* $\gamma \in A(E)$.

3. $j_*\gamma \cdot j_*\delta = -j_*(\gamma \cdot \delta \cdot \zeta)$ *for* $\gamma, \delta \in A(E)$.

## 3.3 Blowups of $\mathbb{P}^3$ along points in a plane

Let $\Gamma$ be a plane in $\mathbb{P}^3$ and choose points $p_1, \dots, p_r \in \Gamma$. Let $\pi\colon X \to \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ along $p_1, \dots, p_r$ and denote by $F_1, \dots, F_r$ the exceptional divisors on $X$. Then $X$ is an additive variety, since we blew up fixed points under the $\iota$-action. Let $S = \pi_*^{-1}\Gamma$ be the strict transform of $\Gamma$, which is isomorphic to the blowup of $\Gamma \simeq \mathbb{P}^2$ along $p_1, \dots, p_r$. Let $E_1, \dots, E_r$ be the exceptional divisors on $S$. The additive boundary divisor of $X$ is $D = \pi^*\Gamma$, which is equivalent to $S + \sum F_i$. Denote the irreducible components of $D$ by $D_0 = S, D_1 = F_1, \dots, D_r = F_r$. Let $(\varepsilon_0, \dots, \varepsilon_r) \in (0,1) \times [0,1]^r$.

We are able to prove Conjecture 3.3 in this setup:

**Theorem 3.8.** *Let $X$ be the blowup of $\mathbb{P}^3$ along $r$ points contained in a plane $\Gamma$, and let $D_0, D_1, \dots, D_r$ be the irreducible components of the additive boundary divisor, where $D_0$ is the strict transform of $\Gamma$. Then $X$ has finitely many $(K_X + \sum_{i=0}^r (1-\varepsilon_i)D_i)$-negative extremal rays for any $(\varepsilon_0, \dots, \varepsilon_r) \in (0,1) \times [0,1]^r$.*

*Remark* 3.9. If $V$ is the real vector space spanned by $D_0, D_1, \dots, D_r$, then $\mathcal{L}(V) = [0,1]^{r+1}$.

Before proceeding to the proof, we need some preparation.

**Lemma 3.10.** *Let $C$ be an effective curve in $D$. Then $C$ is equivalent to a multiple of an effective curve in $S$.*

*Proof.* Assume that $C$ is contained in some $F_i$. Then $C$ is equivalent to a multiple of a line, since $F_i \simeq \mathbb{P}^2$. The line $E_i$ is in $F_i$ and also lies in $S$. Hence $C$ is equivalent to a multiple of $E_i$, since any two lines in projective space are equivalent. □

**Lemma 3.11.** *Let $C$ be an effective curve in $X$. If $(K_X + \sum_{i=0}^r (1-\varepsilon_i)D_i) \cdot C < 0$, then $(K_X + S) \cdot C < 0$.*

*Proof.* We have

$$\left(K_X + \sum_{i=0}^{r}(1-\varepsilon_i)D_i\right)\cdot C = \Big(K_X + (1-\varepsilon_0)S + \sum(1-\varepsilon_i)F_i\Big)\cdot C$$
$$= (K_X+S)\cdot C - \varepsilon_0 S\cdot C + \sum(1-\varepsilon_i)F_i\cdot C$$
$$< 0.$$

So there are three possibilities: $(K_X+S)\cdot C<0$, $S\cdot C>0$, or $F_i\cdot C<0$ for some $i$.

First, if $F_i\cdot C<0$, then $C\subset F_i$, so we may write $C=a_iE_i$ for some $a_i>0$. We may then compute

$$(K_X+S)\cdot C = K_X\cdot C + S\cdot C = -2a_i + a_i = -a_i < 0.$$

In what follows, we may therefore assume that $F_i\cdot C\geq 0$, in which case we can write $C=dH-\sum a_iE_i$. To finish the proof, we will show that there is no such curve $C$ on $X$ satisfying both $(K_X+S)\cdot C\geq 0$ and $S\cdot C>0$, hence the only possibility is that $(K_X+S)\cdot C<0$.

So suppose that $(K_X+S)\cdot C\geq 0$ and $S\cdot C>0$. Adding these two inequalities together,

$$(K_X+S)\cdot C + S\cdot C = K_X\cdot C + 2S\cdot C > 0,$$

which translates to

$$-4d + 2\sum a_i + 2d - 2\sum a_i = -2d > 0,$$

contradicting the fact that $d>0$. □

**Lemma 3.12.** *Let $R$ be a $(K_X+\sum_{i=0}^{r}(1-\varepsilon_i)D_i)$-negative extremal ray of $\overline{\mathrm{NE}}(X)$. Then $R$ is spanned by a $(-1)$-curve on $S$.*

*Proof.* The Cone Theorem tells us that $R$ is spanned by a curve $C$. Lemmas 1.10 and 3.10

allow us to apply Lemma 1.26 and find an effective 1-cycle $\beta$ in $S$ such that $C \equiv t\beta$ and $\beta$ spans an extremal ray of $\overline{\mathrm{NE}}(S)$. By Lemma 3.11 and adjunction, we get that

$$(K_X + S) \cdot \beta = K_S \cdot \beta < 0,$$

implying that $\mathbb{R}^+\beta$ is a $K_S$-negative extremal ray of $\overline{\mathrm{NE}}(S)$. Since $S$ is a smooth rational surface with Picard number greater than two, its $K_S$-negative extremal rays are spanned by $(-1)$-curves. Thus, the extremal ray in $\overline{\mathrm{NE}}(S)$ is spanned by a $(-1)$-curve on $S$. Because $C \equiv t\beta$, this same $(-1)$-curve spans the extremal ray $R$ in $\overline{\mathrm{NE}}(X)$. □

If $r \leq 8$, then $S$ has finitely many $(-1)$-curves. If $r \geq 9$ and the points $p_i$ are very general, then $S$ has infinitely many $(-1)$-curves. Also, if all the points lie on a line, then $X$ is a Mori dream space [Ott11], so it has finitely many $(-1)$-curves.

The proof of Theorem 3.8 relies on the following fact:

**Lemma 3.13.** *The set*

$$\mathcal{A}_M = \{C \,|\, C \textit{ is a } (-1)\textit{-curve on } S \textit{ and } -S \cdot C < M\}$$

*is finite for any $M$.*

*Proof.* Let $C \in \mathcal{A}_M$. We have

$$C = dH - \sum a_i E_i \quad \text{and} \quad K_S = -3H + \sum E_i.$$

We first use the condition that $-S \cdot C < M$:

$$-S \cdot C = -d + \sum a_i < M. \tag{3.1}$$

The condition that $C$ is a $(-1)$-curve implies that $K_S \cdot C = -1$, and so

$$K_S \cdot C = -3d + \sum a_i = -1. \tag{3.2}$$

Rearranging (3.2), we obtain $\sum a_i = -1 + 3d$, which plugged into (3.1) gives us that $-d - 1 + 3d < M$, or

$$d < \frac{M+1}{2}.$$

We now show that this limits $C$ to a finite set. If $C = E_i$ for some $i$, there are only $r$ such curves. If $C \neq E_i$, then because $C$ is effective and irreducible, we must have $a_i = C \cdot E_i \geq 0$ for all $i$. Since $\sum a_i = 3d - 1$ and $d < (M+1)/2$, the integers $d$ and $a_i$ are all bounded. This implies there are only finitely many possible numerical classes for $C$ in $N_1(S)$. Finally, if two distinct irreducible $(-1)$-curves $C_1$ and $C_2$ shared the same numerical class, their intersection would be $C_1 \cdot C_2 = C_1^2 = -1 < 0$, which is impossible. Thus, there is at most one $(-1)$-curve in each numerical class. Since the number of numerical classes is finite, the set $\mathcal{A}_M$ must be finite.

Alternatively, to show that $\mathcal{A}_M$ is finite, we could measure degree with respect to an ample divisor. Choose $\delta > 0$ sufficiently small such that $A = H - \delta \sum E_i$ is ample on $S$. The degree of $C$ with respect to $A$ is

$$\deg_A(C) = A \cdot C = d - \delta \sum a_i = d - \delta(3d - 1) = d(1 - 3\delta) + \delta.$$

Since $d$ is bounded from above, the degree $\deg_A(C)$ is also strictly bounded from above. By the theory of Hilbert schemes, effective curves with bounded degree with respect to a fixed ample divisor form a bounded family. Since $C$ is a $(-1)$-curve, it is isomorphic to $\mathbb{P}^1$ with normal bundle $N_{C/S} = \mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore, $H^0(C, N_{C/S}) = 0$, meaning $C$ is rigid and does not deform. A bounded family consisting entirely of rigid objects is finite (since the Hilbert scheme pararametrizing such objects is finite type and thus Noetherian), hence $\mathcal{A}_M$ is a finite set. $\square$

*Proof of Theorem 3.8.* Let $R$ be an extremal ray of $\overline{\mathrm{NE}}(X)$ that is $(K_X + \sum_{i=0}^{r}(1-\varepsilon_i)D_i)$-negative. By Lemma 3.12, $R$ is spanned by a $(-1)$-curve $C$ on $S$. We will show that $C \in \mathcal{A}_{1/\varepsilon_0}$ (i.e. $-S \cdot C < \frac{1}{\varepsilon_0}$), which is a finite set by Lemma 3.13.

We have that $[C] = dH - \sum a_i E_i$. By applying adjunction,

$$\begin{aligned}\left(K_X + \sum_{i=0}^{r}(1-\varepsilon_i)D_i\right) \cdot C &= \left(K_X + (1-\varepsilon_0)S + \sum (1-\varepsilon_i)F_i\right) \cdot C \\ &= K_S \cdot C - \varepsilon_0 S \cdot C + \sum (1-\varepsilon_i)F_i \cdot C \\ &= -1 - \varepsilon_0 S \cdot C + \sum (1-\varepsilon_i)a_i < 0.\end{aligned}$$

Rearranging,

$$-\varepsilon_0 S \cdot C < 1 - \sum (1-\varepsilon_i)a_i \leq 1,$$

so

$$-S \cdot C < \frac{1}{\varepsilon_0}.$$ $\square$

## 3.4 Blowups of $\mathbb{P}^n$ along points in a hyperplane

A similar argument as before allows us to generalize the results of the previous section. We are not allowed to use the geometry of $(-1)$-curves anymore, but we can use the bound on the length of extremal rays given by the Cone Theorem.

To simplify some of our proofs, we introduce the following notation:

**Definition 3.14.** Let $X$ be a normal projective variety of dimension $n$ and $\Delta$ an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $R$ be a $(K_X + \Delta)$-negative extremal ray on $X$. We define $\mathcal{C}_X(R)$ to be the set of all rational curves $C \subset X$ spanning $R$ such that

$$-(\dim X + 1) \leq K_X \cdot C < 0,$$

up to numerical equivalence.

The Cone Theorem (Theorem 1.25) says that $\mathcal{C}_X(R)$ is not empty.

Set $n \geq 3$. Let $\Gamma$ be a hyperplane in $\mathbb{P}^n$ and choose points $p_1, \ldots, p_r \in \Gamma$. Let $\pi \colon X \to \mathbb{P}^n$ be the blowup of $\mathbb{P}^n$ along $p_1, \ldots, p_r$ and denote by $F_1, \ldots, F_r$ the exceptional divisors on $X$. Then $X$ is an additive variety, since we blew up fixed points under the $\iota$-action. Let $S = \pi_*^{-1}\Gamma$ be the strict transform of $\Gamma$, which is isomorphic to the blowup of $\Gamma \simeq \mathbb{P}^{n-1}$ along $p_1, \ldots, p_r$. Let $E_1, \ldots, E_r$ be the exceptional divisors on $S$. The additive boundary divisor of $X$ is $D = \pi^*\Gamma$, which is equivalent to $S + \sum F_i$. Denote the irreducible components of $D$ by $D_0 = S, D_1 = F_1, \ldots, D_r = F_r$. Let $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_r) \in (0,1) \times [0,1]^r$.

We are able to prove Conjecture 3.3 in this setup, generalizing Theorem 3.8.

**Theorem 3.15.** *Set $n \geq 3$. Let $X$ be the blowup of $\mathbb{P}^n$ along $r$ points contained in a hyperplane $\Gamma$, and let $D_0, D_1, \ldots, D_r$ be the irreducible components of the additive boundary divisor, where $D_0$ is the strict transform of $\Gamma$. Then $X$ has finitely many $(K_X + \sum_{i=0}^r (1-\varepsilon_i)D_i)$-negative extremal rays for any $(\varepsilon_0, \ldots, \varepsilon_r) \in (0,1) \times [0,1]^r$.*

*Remark* 3.16. If $V$ is the real vector space spanned by $D_0, D_1, \ldots, D_r$, then $\mathcal{L}(V) = [0,1]^{r+1}$.

We need to slightly modify some of the proofs of the previous section.

**Lemma 3.17.** *Let $C$ be an effective curve in $D$. Then $C$ is equivalent to a multiple of an effective curve in $S$.*

*Proof.* Assume that $C$ is contained in some $F_i$. Then $C$ is equivalent to a multiple of a line, as $F_i \simeq \mathbb{P}^n$. And since any two lines in projective space are equivalent, $C$ is equivalent to a multiple of a line in $E_i$. □

**Lemma 3.18.** *Let $C$ be an effective curve in $X$. If $(K_X + \sum_{i=0}^r (1-\varepsilon_i)D_i) \cdot C < 0$, then $(K_X + S) \cdot C < 0$.*

*Proof.* We have

$$\left(K_X + \sum_{i=0}^{r}(1-\varepsilon_i)D_i\right)\cdot C = \Big(K_X + (1-\varepsilon_0)S + \sum(1-\varepsilon_i)F_i\Big)\cdot C$$
$$= (K_X+S)\cdot C - \varepsilon_0 S\cdot C + \sum(1-\varepsilon_i)F_i\cdot C$$
$$< 0.$$

So there are three possibilities: $(K_X+S)\cdot C<0$, $S\cdot C>0$, or $F_i\cdot C<0$ for some $i$.

First, if $F_i\cdot C<0$, then $C\subset F_i$, so we may write $C=a_ie_i$ for some $a_i>0$. We may then compute:

$$(K_X+S)\cdot C = K_X\cdot C + S\cdot C = -(n-1)a_i - a_i = -(n-2)a_i < 0.$$

In what follows, we may therefore assume that $F_i\cdot C\geq 0$, in which case we can write $C = dh - \sum a_ie_i$. To finish the proof, we will show that there is no such curve $C$ on $X$ satisfying both $(K_X+S)\cdot C\geq 0$ and $S\cdot C>0$, hence the only possibility is that $(K_X+S)\cdot C<0$.

So suppose that $(K_X+S)\cdot C\geq 0$ and $S\cdot C>0$. We now compute:

$$(K_X+S)\cdot C = -nd + (n-2)\sum a_i \geq 0. \tag{3.3}$$

$$S\cdot C = d - \sum a_i > 0. \tag{3.4}$$

Then (3.4) gives us $\sum a_i < d$, which plugged into (3.3) yields

$$0 < -nd + (n-2)d = -2d,$$

contradicting the fact that $d>0$. □

**Lemma 3.19.** *Let $R$ be a $(K_X+\sum_{i=0}^{r}(1-\varepsilon_i)D_i)$-negative extremal ray of $\overline{\mathrm{NE}}(X)$. Then*

*$R$ is spanned by a $K_S$-negative rational curve $C$ on $S$, which also spans a $K_S$-negative extremal ray of $\overline{\mathrm{NE}}(S)$.*

*Proof.* The Cone Theorem on $X$ tells us that $R$ is spanned by a rational curve $C'$. Lemmas 1.10 and 3.17 allow us to apply Lemma 1.26 and find an effective 1-cycle $\beta$ in $S$ such that $C' \equiv t\beta$ and $\beta$ spans an extremal ray $R'$ of $\overline{\mathrm{NE}}(S)$. By Lemma 3.18 and adjunction, we get that

$$(K_X + S) \cdot \beta = K_S \cdot \beta < 0.$$

By the Cone Theorem on $S$, we may choose $\beta$ to be a $K_S$-negative rational curve $C$. Because $C' \equiv t\beta$, we have that $\beta$ spans the extremal ray $R$ in $\overline{\mathrm{NE}}(X)$. □

The proof of Theorem 3.15 relies on the following fact:

**Lemma 3.20.** *The set*

$$\mathcal{A}_M = \big\{R \,\big|\, R \text{ is a } K_S\text{-negative extremal ray of } \overline{\mathrm{NE}}(S) \text{ such that}$$
$$\text{there exists } C \in \mathcal{C}_S(R) \text{ with } -S \cdot C < M\big\}$$

*is finite for any $M$.*

*Proof.* Let $R \in \mathcal{A}_M$ and let $C \in \mathcal{C}_S(R)$ such that $-S \cdot C < M$. We have

$$C = dh - \sum a_i e_i \quad \text{and} \quad K_S = -nH + (n-2) \sum E_i.$$

We first use the condition that $-S \cdot C < M$:

$$-S \cdot C = -d + \sum a_i < M. \tag{3.5}$$

By the definition of $C$ being in $\mathcal{C}_S(R)$, we have that $-n \leq K_S \cdot C < 0$, and so

$$K_S \cdot C = -nd + (n-2)\sum a_i \geq -n. \tag{3.6}$$

Rearranging (3.5), we obtain

$$\sum a_i < M + d,$$

which plugged into (3.6) gives us that

$$-n < -nd + (n-2)(M+d),$$

or

$$d < \frac{n + M(n-2)}{2}.$$

To show this limits the numerical classes of $C$ (and thus the extremal rays) to a finite set, we measure degree with respect to an ample divisor. Choose $\delta > 0$ sufficiently small such that $A = h - \delta \sum e_i$ is ample on $S$. The degree of $C$ with respect to $A$ is

$$\deg_A(C) = A \cdot C = d - \delta \sum a_i = d - \delta(M+d) = d(1-\delta) - \delta M.$$

Since $d$ is bounded from above, the degree $\deg_A(C)$ is also strictly bounded from above. Effective curves of bounded degree with respect to a fixed ample divisor are parameterized by a Hilbert scheme of finite type. Because a scheme of finite type has only finitely many connected components, these curves can only represent finitely many distinct numerical classes. Therefore, there are only finitely many such extremal rays, and $\mathcal{A}_M$ is a finite set. $\square$

*Proof of Theorem 3.15.* Let $R$ be a $(K_X + \sum_{i=0}^{r}(1-\varepsilon_i)D_i)$-negative extremal ray of $\overline{\mathrm{NE}}(X)$. By Lemma 3.19, $R$ is spanned by a $K_S$-negative rational curve $C$ on $S$, which also spans a $K_S$-negative extremal ray of $\overline{\mathrm{NE}}(S)$. We will show that the ray spanned by $C$ in $\overline{\mathrm{NE}}(S)$ is in $\mathcal{A}_{n/\varepsilon_0}$ (i.e. $-S \cdot C < \frac{n}{\varepsilon_0}$), which is a finite set by Lemma 3.20, thus completing the proof.

The Cone Theorems on $X$ and $S$ tell us that we may choose $C$ so that $-(n+1) \leq (K_X + \sum_{i=0}^{r}(1-\varepsilon_i)D_i) \cdot C < 0$ and $-n \leq K_S \cdot C < 0$, respectively. We also have that $[C] = dh - \sum a_i e_i$. By applying adjunction,

$$
\begin{aligned}
0 &> \left(K_X + \sum_{i=0}^{r}(1-\varepsilon_i)D_i\right) \cdot C \\
&= \left(K_X + (1-\varepsilon_0)S + \sum (1-\varepsilon_i)F_i\right) \cdot C \\
&= K_S \cdot C - \varepsilon_0 S \cdot C + \sum (1-\varepsilon_i)F_i \cdot C \\
&\geq -n - \varepsilon_0 S \cdot C + \sum (1-\varepsilon_i)a_i \\
&\geq -n - \varepsilon_0 S \cdot C.
\end{aligned}
$$

Rearranging,

$$-S \cdot C < \frac{n}{\varepsilon_0}. \qquad \square$$

## 3.5 Finiteness of models

The following observation can be applied anytime there are finitely many negative extremal rays:

Assume Conjecture 3.1 is true. Let $X$ be an $n$-dimensional projective additive variety and let $\Delta$ belong to a compact subspace of $\mathcal{L}(V) \setminus (\mathcal{F}_V \cup \mathcal{G}_V)$, so that $X$ has finitely many $(K_X + \Delta)$-negative extremal rays. Consider the tree of the $\mathbb{G}_a^n$-equivariant $(K_X + \Delta)$-MMP: vertices are pairs that can be reached by this MMP, and there is an edge between two nodes if the pairs are related by a flip or divisorial contraction. König's lemma says that a finitely branching tree with no infinite paths is finite. Assuming termination of $\mathbb{G}_a^n$-equivariant flips, we obtain that there are no infinite paths. Conjecture 3.1 implies that each vertex has at most finitely many children. Therefore, under these assumptions, we conclude that $(K_X + \Delta)$ has finitely many Mori fiber spaces.

Part of Chapter 3 is being prepared for submission for publication. The dissertation author was the primary investigator and author of this material.

# Appendix A

# Untwists

The computations in this appendix were carried out using SageMath [The26]. The list of local algebras of finite rank was taken from [Poo08]. Recall that for a local Artinian algebra $(R, \mathfrak{m})$, we let $d_j = \dim_k(\mathfrak{m}^j/\mathfrak{m}^{j+1})$, and we define the Hilbert–Samuel function of $R$ to be the sequence of integers $(d_j)_{j\geq 1}$, where we retain only the finitely many nonzero terms. The untwist $F_{n,i}$ of the $\iota$-action on $\mathbb{P}^n$ into the additive action on $\mathbb{P}^n$ given by the local algebra $R_{n,i}$ is computed using Theorem 2.21. In our tables, we also include the indeterminacy locus of $F_{n,i}$. In this appendix, we assume that the open orbit of all additive actions on $\mathbb{P}^n$ is $\{x_0 \neq 0\}$.

Part of Appendix A is being prepared for submission for publication. The dissertation author was the primary investigator and author of this material.

**Table A.1.** Local algebras of rank $\leq 6$.

| $(n,i)$ | Local algebra $R_{n,i}$ of rank $n+1$ | Hilbert–Samuel function of $R_{n,i}$ |
|---|---|---|
| $(1,1)$ | $k[x]/(x^2)$ | $(1)$ |
| $(2,1)$ | $k[x]/(x^3)$ | $(1,1)$ |
| $(2,2)$ | $k[x,y]/(x,y)^2$ | $(2)$ |
| $(3,1)$ | $k[x]/(x^4)$ | $(1,1,1)$ |
| $(3,2)$ | $k[x,y]/(x^2,xy,y^3)$ | $(2,1)$ |
| $(3,3)$ | $k[x,y]/(x^2,y^2)$ | $(2,1)$ |
| $(3,4)$ | $k[x,y,z]/(x,y,z)^2$ | $(3)$ |
| $(4,1)$ | $k[x]/(x^5)$ | $(1,1,1,1)$ |
| $(4,2)$ | $k[x,y]/(x^2,xy,y^4)$ | $(2,1,1)$ |
| $(4,3)$ | $k[x,y]/(x^2+y^3,xy)$ | $(2,1,1)$ |
| $(4,4)$ | $k[x,y]/(xy,x^3,y^3)$ | $(2,2)$ |
| $(4,5)$ | $k[x,y]/(x^2,xy^2,y^3)$ | $(2,2)$ |
| $(4,6)$ | $k[x,y,z]/(x^2,y^2,xy,xz,yz,z^3)$ | $(3,1)$ |
| $(4,7)$ | $k[x,y,z]/(x^2,y^2,z^2,xy,xz)$ | $(3,1)$ |
| $(4,8)$ | $k[x,y,z]/(xy,xz,yz,x^2+y^2,x^2+z^2)$ | $(3,1)$ |
| $(4,9)$ | $k[x,y,z,w]/(x,y,z,w)^2$ | $(4)$ |
| $(5,1)$ | $k[x]/(x^6)$ | $(1,1,1,1,1)$ |
| $(5,2)$ | $k[x,y]/(x^2,xy,y^5)$ | $(2,1,1,1)$ |
| $(5,3)$ | $k[x,y]/(x^2+y^4,xy)$ | $(2,1,1,1)$ |
| $(5,4)$ | $k[x,y]/(xy,x^3,y^4)$ | $(2,2,1)$ |
| $(5,5)$ | $k[x,y]/(xy,x^3+y^3)$ | $(2,2,1)$ |
| $(5,6)$ | $k[x,y]/(x^2,xy^2,y^4)$ | $(2,2,1)$ |
| $(5,7)$ | $k[x,y]/(x^2+y^3,xy^2,y^4)$ | $(2,2,1)$ |
| $(5,8)$ | $k[x,y]/(x^2,y^3)$ | $(2,2,1)$ |
| $(5,9)$ | $k[x,y]/(x,y)^3$ | $(2,3)$ |
| $(5,10)$ | $k[x,y,z]/(x^2,xy,y^2,xz,yz,z^4)$ | $(3,1,1)$ |
| $(5,11)$ | $k[x,y,z]/(x^2,xy,y^2+z^3,xz,yz)$ | $(3,1,1)$ |
| $(5,12)$ | $k[x,y,z]/(x^2,xy+z^3,y^2,xz,yz)$ | $(3,1,1)$ |
| $(5,13)$ | $k[x,y,z]/(xy,yz,z^2,y^2-xz,x^3)$ | $(3,2)$ |
| $(5,14)$ | $k[x,y,z]/(xy,z^2,xz-yz,x^2+y^2-xz)$ | $(3,2)$ |
| $(5,15)$ | $k[x,y,z]/(x^2,xy,xz,y^2,yz^2,z^3)$ | $(3,2)$ |
| $(5,16)$ | $k[x,y,z]/(x^2,xy,xz,yz,y^3,z^3)$ | $(3,2)$ |
| $(5,17)$ | $k[x,y,z]/(xy,xz,y^2,z^2,x^3)$ | $(3,2)$ |
| $(5,18)$ | $k[x,y,z]/(xy,xz,yz,x^2+y^2-z^2)$ | $(3,2)$ |
| $(5,19)$ | $k[x,y,z]/(x^2,xy,yz,y^2-z^2)$ | $(3,2)$ |
| $(5,20)$ | $k[x,y,z]/(x^2,xy,y^2,z^2)$ | $(3,2)$ |
| $(5,21)$ | $k[x,y,z,w]/(x^2,y^2,z^2,xy,xz,xw,yz,yw,zw,w^3)$ | $(4,1)$ |
| $(5,22)$ | $k[x,y,z,w]/(x^2,y^2,z^2,w^2,xy,xz,xw,yz,yw)$ | $(4,1)$ |
| $(5,23)$ | $k[x,y,z,w]/(x^2,y^2+z^2,y^2+w^2,xy,xz,xw,yz,yw,zw)$ | $(4,1)$ |
| $(5,24)$ | $k[x,y,z,w]/(x^2,y^2,z^2,w^2,xy-zw,xz,xw,yz,yw)$ | $(4,1)$ |
| $(5,25)$ | $k[x,y,z,w,v]/(x,y,z,w,v)^2$ | $(5)$ |

**Table A.2.** Untwist $F_{n,i}$ of $\iota$ into the additive action on $\mathbb{P}^2$ given by each local algebra $R_{2,i}$ of rank 3.

| $(n,i)$ | $F_{n,i}$ | Indeterminacy locus |
|---|---|---|
| $(2,1)$ | $[x_0^2 : x_0x_1 : \frac{1}{2}x_1^2 + x_0x_2]$ | $x_0 = x_1 = 0$ |

**Table A.3.** Untwist $F_{n,i}$ of $\iota$ into the additive action on $\mathbb{P}^3$ given by each local algebra $R_{3,i}$ of rank 4.

| $(n,i)$ | $F_{n,i}$ | Indeterminacy locus |
|---|---|---|
| $(3,1)$ | $[x_0^3 : x_0^2 x_1 : \frac{1}{2}x_0 x_1^2 + x_0^2 x_2 : \frac{1}{6}x_1^3 + x_0 x_1 x_2 + x_0^2 x_3]$ | $x_0 = x_1 = 0$ |
| $(3,2)$ | $[x_0^2 : x_0 x_1 : x_0 x_2 : \frac{1}{2}x_2^2 + x_0 x_3]$ | $x_0 = x_2 = 0$ |
| $(3,3)$ | $[x_0^2 : x_0 x_1 : x_0 x_2 : x_1 x_2 + x_0 x_3]$ | $x_0 = x_1 x_2 = 0$ |

**Table A.4.** Untwist $F_{n,i}$ of $\iota$ into the additive action on $\mathbb{P}^4$ given by each local algebra $R_{4,i}$ of rank 5.

| $(n,i)$ | $F_{n,i}$ | Indeterminacy locus |
|---|---|---|
| $(4,1)$ | $[x_0^4 : x_0^3 x_1 : \frac{1}{2}x_0^2 x_1^2 + x_0^3 x_2 : \frac{1}{6}x_0 x_1^3 + x_0^2 x_1 x_2 + x_0^3 x_3$ $: \frac{1}{24}x_1^4 + \frac{1}{2}x_0 x_1^2 x_2 + \frac{1}{2}x_0^2 x_2^2 + x_0^2 x_1 x_3 + x_0^3 x_4]$ | $x_0 = x_1 = 0$ |
| $(4,2)$ | $[x_0^3 : x_0^2 x_1 : x_0^2 x_2 : \frac{1}{2}x_0 x_2^2 + x_0^2 x_3 : \frac{1}{6}x_2^3 + x_0 x_2 x_3 + x_0^2 x_4]$ | $x_0 = x_2 = 0$ |
| $(4,3)$ | $[x_0^3 : x_0^2 x_1 : x_0^2 x_2 : \frac{1}{2}x_0 x_1^2 - \frac{1}{6}x_2^3 + x_0^2 x_3 - x_0 x_2 x_4 : \frac{1}{2}x_0 x_2^2 + x_0^2 x_4]$ | $x_0 = x_2 = 0$ |
| $(4,4)$ | $[x_0^2 : x_0 x_1 : x_0 x_2 : \frac{1}{2}x_1^2 + x_0 x_3 : \frac{1}{2}x_2^2 + x_0 x_4]$ | $x_0 = x_1 = x_2 = 0$ |
| $(4,5)$ | $[x_0^2 : x_0 x_1 : x_0 x_2 : x_1 x_2 + x_0 x_3 : \frac{1}{2}x_2^2 + x_0 x_4]$ | $x_0 = x_2 = 0$ |
| $(4,6)$ | $[x_0^2 : x_0 x_1 : x_0 x_2 : x_0 x_3 : \frac{1}{2}x_3^2 + x_0 x_4]$ | $x_0 = x_3 = 0$ |
| $(4,7)$ | $[x_0^2 : x_0 x_1 : x_0 x_2 : x_0 x_3 : x_2 x_3 + x_0 x_4]$ | $x_0 = x_2 x_3 = 0$ |
| $(4,8)$ | $[x_0^2 : x_0 x_1 : x_0 x_2 : x_0 x_3 : -\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + x_0 x_4]$ | $x_0 = x_1^2 - x_2^2 - x_3^2 = 0$ |

**Table A.5.** Untwist $F_{n,i}$ of $\iota$ into the additive action on $\mathbb{P}^5$ given by each local algebra $R_{5,i}$ of rank 6.

| $(n,i)$ | $F_{n,i}$ | Indeterminacy locus |
|---|---|---|
| $(5,1)$ | $[x_0^5 : x_0^4x_1 : \frac{1}{2}x_0^3x_1^2 + x_0^4x_2 : \frac{1}{6}x_0^2x_1^3 + x_0^3x_1x_2 + x_0^4x_3$<br>$: \frac{1}{24}x_0x_1^4 + \frac{1}{2}x_0^2x_1^2x_2 + \frac{1}{2}x_0^3x_2^2 + x_0^3x_1x_3 + x_0^4x_4$<br>$: \frac{1}{120}x_1^5 + \frac{1}{6}x_0x_1^3x_2 + \frac{1}{2}x_0^2x_1x_2^2 + \frac{1}{2}x_0^2x_1^2x_3 + x_0^3x_2x_3 + x_0^3x_1x_4 + x_0^4x_5]$ | $x_0 = x_1 = 0$ |
| $(5,2)$ | $[x_0^4 : x_0^3x_1 : x_0^3x_2 : \frac{1}{2}x_0^2x_2^2 + x_0^3x_3 : \frac{1}{6}x_0x_2^3 + x_0^2x_2x_3 + x_0^3x_4$<br>$: \frac{1}{24}x_2^4 + \frac{1}{2}x_0x_2^2x_3 + \frac{1}{2}x_0^2x_3^2 + x_0^2x_2x_4 + x_0^3x_5]$ | $x_0 = x_2 = 0$ |
| $(5,3)$ | $[x_0^4 : x_0^3x_1 : x_0^3x_2 : \frac{1}{2}x_0^2x_1^2 - \frac{1}{24}x_2^4 + x_0^3x_3 - \frac{1}{2}x_0x_2^2x_4 - \frac{1}{2}x_0^2x_4^2 - x_0^2x_2x_5$<br>$: \frac{1}{2}x_0^2x_2^2 + x_0^3x_4 : \frac{1}{6}x_0x_2^3 + x_0^2x_2x_4 + x_0^3x_5]$ | $x_0 = x_2 = 0$ |
| $(5,4)$ | $[x_0^3 : x_0^2x_1 : x_0^2x_2 : \frac{1}{2}x_0x_1^2 + x_0^2x_3 : \frac{1}{2}x_0x_2^2 + x_0^2x_4 : \frac{1}{6}x_2^3 + x_0x_2x_4 + x_0^2x_5]$ | $x_0 = x_2 = 0$ |
| $(5,5)$ | $[x_0^3 : x_0^2x_1 : x_0^2x_2 : \frac{1}{2}x_0x_1^2 + x_0^2x_3 : \frac{1}{2}x_0x_2^2 + x_0^2x_4$<br>$: -\frac{1}{6}x_1^3 + \frac{1}{6}x_2^3 - x_0x_1x_3 + x_0x_2x_4 + x_0^2x_5]$ | $x_0 = x_1^3 - x_2^3 = 0$ |
| $(5,6)$ | $[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1x_2 + x_0^2x_3 : \frac{1}{2}x_0x_2^2 + x_0^2x_4 : \frac{1}{6}x_2^3 + x_0x_2x_4 + x_0^2x_5]$ | $x_0 = x_2 = 0$ |
| $(5,7)$ | $[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1x_2 + x_0^2x_3$<br>$: \frac{1}{2}x_0x_1^2 - \frac{1}{6}x_2^3 + x_0^2x_4 - x_0x_2x_5 : \frac{1}{2}x_0x_2^2 + x_0^2x_5]$ | $x_0 = x_2 = 0$ |
| $(5,8)$ | $[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1x_2 + x_0^2x_3 : \frac{1}{2}x_0x_2^2 + x_0^2x_4$<br>$: \frac{1}{2}x_1x_2^2 + x_0x_2x_3 + x_0x_1x_4 + x_0^2x_5]$ | $x_0 = x_1x_2 = 0$ |
| $(5,9)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_1x_2 + x_0x_3 : \frac{1}{2}x_1^2 + x_0x_4 : \frac{1}{2}x_2^2 + x_0x_5]$ | $x_0 = x_1 = x_2 = 0$ |
| $(5,10)$ | $[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0^2x_3 : \frac{1}{2}x_0x_3^2 + x_0^2x_4 : \frac{1}{6}x_3^3 + x_0x_3x_4 + x_0^2x_5]$ | $x_0 = x_3 = 0$ |
| $(5,11)$ | $[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0^2x_3 : \frac{1}{2}x_0x_2^2 - \frac{1}{6}x_3^3 + x_0^2x_4 - x_0x_3x_5 : \frac{1}{2}x_0x_3^2 + x_0^2x_5]$ | $x_0 = x_3 = 0$ |
| $(5,12)$ | $[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0^2x_3 : x_0x_1x_2 - \frac{1}{6}x_3^3 + x_0^2x_4 - x_0x_3x_5 : \frac{1}{2}x_0x_3^2 + x_0^2x_5]$ | $x_0 = x_3 = 0$ |
| $(5,13)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : \frac{1}{2}x_1^2 + x_0x_4 : \frac{1}{2}x_2^2 + x_1x_3 + x_0x_5]$ | $x_0 = x_1 = x_2 = 0$ |
| $(5,14)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : -\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_0x_4$<br>$: \frac{1}{2}x_1^2 + x_1x_3 + x_2x_3 + x_0x_5]$ | $x_0 = x_1 - x_2 = x_2^2 + 4x_2x_3 = 0$ |
| $(5,15)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_2x_3 + x_0x_4 : \frac{1}{2}x_3^2 + x_0x_5]$ | $x_0 = x_3 = 0$ |
| $(5,16)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : \frac{1}{2}x_2^2 + x_0x_4 : \frac{1}{2}x_3^2 + x_0x_5]$ | $x_0 = x_2 = x_3 = 0$ |
| $(5,17)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : \frac{1}{2}x_1^2 + x_0x_4 : x_2x_3 + x_0x_5]$ | $x_0 = x_1 = x_2x_3 = 0$ |
| $(5,18)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : -\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_0x_4 : \frac{1}{2}x_1^2 + \frac{1}{2}x_3^2 + x_0x_5]$ | $x_0 = x_1^2 + x_3^2 = x_2^2 + x_3^2 = 0$ |
| $(5,19)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1x_3 + x_0x_4 : \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + x_0x_5]$ | $x_0 = x_1x_2 = x_1x_3 = x_2^2 + x_3^2 = 0$ |
| $(5,20)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1x_3 + x_0x_4 : x_2x_3 + x_0x_5]$ | $x_0 = x_1x_3 = x_2x_3 = 0$ |
| $(5,21)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_0x_4 : \frac{1}{2}x_4^2 + x_0x_5]$ | $x_0 = x_4 = 0$ |
| $(5,22)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_0x_4 : x_3x_4 + x_0x_5]$ | $x_0 = x_3x_4 = 0$ |
| $(5,23)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_0x_4 : -\frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + \frac{1}{2}x_4^2 + x_0x_5]$ | $x_0 = x_2^2 - x_3^2 - x_4^2 = 0$ |
| $(5,24)$ | $[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_0x_4 : x_1x_2 + x_3x_4 + x_0x_5]$ | $x_0 = x_1x_2 + x_3x_4 = 0$ |

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