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Measurability Properties on Small Cardinals

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Measurability Properties on Small Cardinals

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Monroe Blake Eskew

Dissertation Committee:
Professor Martin Zeman, Chair
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2014

DEDICATION

To Courtney

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I am indebted to Martin Zeman and Matthew Foreman, who taught me set theory.

CURRICULUM VITAE

Monroe Blake Eskew

EDUCATION

- **University of California, Irvine**
  - MS Mathematics, 2009.
  - PhD Mathematics, 2014.
- **Rice University**
  - BA Mathematics and Philosophy, May 2005. Cum Laude.

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- Recipient of JSPS Short-term Postdoctoral Fellowship to work with Masahiro Shioya at the University of Tsukuba, October 2014–September 2015.
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- GAANN Fellowship, 2008–2010. $60,000. (Graudate Assistantships for Areas of National Need, a competitive fellowship funded by the US Department of Education.)
- National Merit Scholar, 2001.

PAPERS

- *Coherent forests* (submitted).
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CONFERENCE TALKS

- "Applications of the anonymous collapse." Logic in Southern California Meeting. UCI. May 10, 2014.
- "Applications of the anonymous collapse." 15th Annual Graduate Student Conference in Logic. University of Wisconsin, Madison. April 26, 2014.
- "Ulam's measure problem, saturated ideals, and cardinal arithmetic." Joint Mathematics Meeting. Baltimore, MD. January 16, 2014.
- "Measurability properties on small cardinals." Logic in Southern California Meeting. UCLA. June 1, 2013.

- "Measurability properties on small cardinals." New York Graduate Student Conference in Logic. CUNY Graduate Center. April 19, 2013.
- "Dense ideals and small ultrapowers." 13th Annual Graduate Student Conference in Logic. University of Notre Dame. April 28, 2012.
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TEACHING EXPERIENCE

- Served as a Teaching Assistant for the following courses at UCI. Duties included leading discussion section, holding office hours, writing and administering examinations, and providing feedback to student work.
  - Calculus (x6)
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SERVICE AND LEADERSHIP EXPERIENCE

- Volunteer for "Math Fair: at the square root of fun," hosted by the Mind Research Institute, August 2014.
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- Associated Graduate Students Council Member, 2011-2012.
- Verano Residents Council, 2012-2014. The five members of this council serve as liason between the Verano Place graduate housing community office and the approximately 1250 residents of the community. Duties include: making decisions on community improvements, organizing community events, advocating for residents and resolving their concerns.
- Gave introductory talk for undergraduates, "Definability and infinity," to the UCI Anteater Mathematics Club and the Chapman University MathCS Seminar. November, 2011.

ABSTRACT OF THE DISSERTATION

Measurability Properties on Small Cardinals

By

Monroe Blake Eskew

Doctor of Philosophy in Mathematics

University of California, Irvine, 2014

Professor Martin Zeman, Chair

Ulam proved that there cannot exist a probability measure on the reals for which every set is measurable and gets either measure zero or one. He asked how large a collection of partial 0–1 valued measures is required so that every set of reals is measurable in one of them. Alaoglu and Erdős proved that if the continuum hypothesis holds, then countably many measures is not enough, and Ulam asked if $\aleph_1$ many can suffice. This question was shown to be independent of ZFC by Prikry and Woodin. Here, we examine the analogous questions on successor cardinals above $\aleph_1$ and on spaces of the form $\mathcal{P}_\kappa(\lambda)$. We generalize Woodin's consistency results to these contexts, producing models of ideals of minimal density on various spaces starting from models of almost-huge cardinals. We show some interactions between these ideals, cardinal arithmetic, and square principles. Then we show that certain characterizations of a positive answer to Ulam's question, namely the existence of dense ideals and nonregular ideals, are equivalent on $\aleph_1$ but not for higher cardinals. Some tension appears in separating these properties while preserving the GCH, but we show this is possible using structures we call "coherent forests," about which we show several results of independent interest. The main result is that if almost-huge cardinals are consistent, then ZFC+GCH does not prove that the existence of dense and nonregular ideals is equivalent for successor cardinals above $\aleph_1$. Our methods also lead to a new result on the individual consistency but collective inconsistency of some types of generic large cardinals.

Introduction

In his 1902 thesis [27], Lebesgue considered the *measure problem*: Can every subset X of the real numbers be assigned a nonnegative measure $\mu(X)$, in a way conforming to geometric criteria, and also satisfying countable additivity? Towards a positive solution, he developed what we now call the Lebesgue measure. A few years later in 1905, Vitali showed that Lebesgue did not succeed in assigning a measure to *every* set of reals, and that a positive solution was not achievable. In light of this, Banach and Kuratowski in 1929 [2] proposed loosening the geometric criteria, requiring only that the measure of an interval $[a, b]$ is $|a-b|$. They proved that this was also impossible if Cantor's Continuum Hypothesis (CH) holds.

In the following year, the Banach-Kuratowski result was strengthened by Ulam [37], who also considered a version of the question in which every subset is given measure zero or one. Ulam proved that, regardless of the cardinality of the continuum, a measure with such properties could only be defined on a space whose size is "inaccessible" compared to the real line. Ulam thus asked, for a set S of *accessible* cardinality such as $\aleph_1$, $\aleph_2$, $\mathbb{R}$, $2^{\mathbb{R}}$, etc., what is the size of the smallest collection of countably additive two-valued partial measures, each of which gives measure zero to single points and measure one to S, such that every subset of S is measurable with respect to one of these measures? Strengthening his result that one is not enough, he proved that finitely many do not suffice either. Considering measures on $\aleph_1$, Alaoglu and Erdős proved that countably many is still too few [10]. So Ulam asked, is $\aleph_1$ many measures enough [11]?

As it turns out, all of these questions touched upon the logical independence phenomena discovered by Gödel [15]. Many of them could not be settled by the Zermelo-Fraenkel axioms with Choice (ZFC), but establishing this required the use of principles that substantially transcend ZFC. Following the groundwork laid by Gödel, set theory gradually established a linear hierarchy of principles known as Large Cardinal Axioms, that empirically seem to be able to gauge the strength of any axiomatic system. System A is said to be stronger than system B when the consistency of B can be derived from the assumption that A is consistent. Establishing the independence of Ulam's questions required traveling far up this hierarchy.

Many set theorists have viewed the large cardinal axioms as natural extensions of ZFC. Originally, Gödel and others had hoped that these axioms would be able to settle Hilbert's First Problem, whether CH is true [16]. After the development of the method *forcing* by Cohen [7], and elaborations by Lévy, Solovay and others [28], these hopes were dashed. It was found that while these axioms have much to say about the consistency of various theories, they have relatively little direct influence on propositions about ordinary mathematical objects like the real line. However, as advanced by Foreman [13], there is a more general class of principles known as Generic Large Cardinals that fit under the same broad conceptual framework as the traditional large cardinals, and these more general principles have a much stronger influence on ordinary mathematical objects. Certain answers to Ulam's question about families of partial measures end up fitting into this category.

Ulam asked how many countably additive two-valued partial measures it takes to collectively measure all subsets of $\aleph_1$. A generalized version of this question is to take a set Z and ask how large a family of two-valued partial measures is required to collectively measure all subsets of Z with some additional requirements on the family, such as stipulating that they are all κ-additive for some cardinal κ, or that they satisfy other structural properties like normality and fineness. To any measure there is an associated *ideal* of measure zero sets, and when the measure is κ-additive, we say the ideal is *κ-complete*. Sets not in a given ideal I are called *I-positive*. An ideal I on a set Z is called *κ-dense* when there is a collection $\{A_\alpha : \alpha < \kappa\}$ of I-positive sets such that for every I-positive $B \subseteq Z$, there is some $\alpha < \kappa$ such that $A_\alpha \setminus B \in I$; in other words A_α is contained in B except for a negligible part. It is not hard to see that the existence of a κ-complete, κ-dense ideal on Z is equivalent to the existence of a family of partial two-valued κ-additive measures $\{\mu_\alpha : \alpha < \kappa\}$ such that every $A \subseteq Z$ is either measure zero for all μ_α or measure one for some μ_α, a strengthening of Ulam's requirement. If an ideal is both κ-complete and κ-dense, then passing to a forcing extension reveals properties of κ closely resembling the definitional properties of the traditional large cardinals, hence the phrase "generic large cardinal."

In this work, we establish the consistency of small cardinals possessing various kinds of generic largeness properties and explore the interrelationships between these properties and some more standard propositions of infinitary combinatorics. Our consistency results start from traditional large cardinal assumptions that lie between almost-huge and huge.

Chapter 1 lays out the necessary preliminaries about ideals, forcing, and elementary embeddings. Many proofs are deferred to well-known textbooks.

Chapter 2 shows how to obtain models of normal and fine, κ-complete λ-dense ideals ideals on $\mathcal{P}_\kappa(\lambda)$ where κ is a successor cardinal, giving positive answers to many generalizations of Ulam's problem. For $\kappa = \aleph_1$, these ideals have in some sense the maximal saturation property, but for higher successor cardinals, there are more structural possibilities. A key to this construction is a certain "universal" boolean algebra we dub "the anonymous collapse." Its flexibility enables saturated ideals on $\mathcal{P}_\kappa(\lambda)$ for a fixed successor κ and many values of λ simultaneously. It also has several interesting applications without the use of hypotheses near the strength of almost-huge cardinals. For example, we can use it to produce many models with the same cardinals and same reals, but very different higher-order combinatorial properties of the continuum, a phenomenon belied by the phrase "cardinal *invariants* of the continuum."

Chapter 3 explores some consequences of the existence of these generic large cardinals for cardinal arithmetic, square principles, and an old conjecture from model theory about the size of ultrapowers. We answer two open questions posed by Foreman in [13]. These explorations lead to an interesting limitation regarding successors of singular cardinals, showing the optimality of some aspects of the consistency results.

Chapter 4 focuses on several properties related to a positive solution to Ulam's question, which Taylor [33] proved equivalent relative to $\aleph_1$. We explore the extent to which the arguments for Taylor's theorem generalize to higher cardinals. The key notion is that of "regularity" of an ideal, and we show that under the Generalized Continuum Hypothesis (GCH), most degrees of regularity are equivalent. We also show that under GCH, a positive solution to the generalized Ulam problem via normal ideals is equivalent to the existence of a dense ideal, implying that the generalized Ulam problem has a negative answer at successors of singular cardinals under GCH.

Chapter 5 gives consistency results relative to generic large cardinals that were proved consistent relative to almost-huge cardinals in Chapter 2. The modular nature of this chapter means that it can piggyback on possible future results that may reduce upper bounds on the consistency strength of dense ideals above $\aleph_1$. We start with a generalization of Foreman's Duality Theorem, fixing a minor error in [14]. Using this, we separate the existence of dense ideals from nonregular ideals above $\aleph_1$, showing that Taylor's theorem is indeed specific to $\aleph_1$. Although the equivalence of a positive answer to the generalized Ulam problem and dense ideals is open, we show they can never be separated via this technique. We also use the Duality Theorem to show that strong forms of generic supercompactness are compatible with square holding globally, in contrast to traditional supercompactness. Finally, we apply these techniques to show that some types of generic large cardinals cannot coexist in one model of set theory, strengthening a result of Woodin.

There is an apparent tension between the technique used to separate density and nonregularity, and the preservation of GCH. Chapter 6 is aimed at resolving this problem, showing ultimately that if almost-huge cardinals are consistent, then ZFC+GCH does not prove a generalization of Taylor's theorem to cardinals above $\aleph_1$. We arrive at this through an investigation of structures dubbed "coherent forests," given their connection to the trees of infinitary combinatorics. The notions of being Aronszajn and Suslin carry over from trees to forests, and we explore several ways of obtaining large Aronszajn and Suslin forests. We show that large coherent Aronszajn forests can be constructed within ZFC and use the P-ideal dichotomy to show the optimality of some of these results. Then we give three ways of forcing large coherent Suslin forests. The first is a modification of Jech's method of forcing by local approximations [19], and the second generalizes the well-known argument of Todorčević that a Cohen real adds a Suslin tree [35]. The third method uses a guessing principle, which we show consistent from a Mahlo cardinal, that plays a similar role to diamond in the construction of Suslin trees. This allows a large Suslin forest to be created by a relatively small forcing. This feature leads to models with the right kind of dense ideals and large Suslin algebras existing simultaneously, allowing the techniques of Chapter 5 to be applied to achieve the main result.

Chapter 1

Preliminaries

We start by reviewing some essential facts about ideals, forcing, and elementary embeddings. Many of these results are folklore, and when proofs are omitted, they may be found in [13], [21], [22], or [25].

1.1 Basic combinatorics of ideals

Let Z be any set. An ideal I on Z is a collection of subsets of Z closed under taking subsets and pairwise unions. If κ is a cardinal, I is called κ-complete if it is also closed under unions of size less than κ. "Countably complete" is taken as synonymous with "ω_1-complete." I is called nonprincipal if $\{z\} \in I$ for all $z \in Z$, and proper if $Z \notin I$. Hereafter we will assume all our ideals are nonprincipal and proper.

Let $X = \bigcup Z$. I is called fine if for all $x \in X$, $\{z : x \notin z\} \in I$. I is called normal if for any sequence $\langle A_x : x \in X \rangle \subseteq I$, the "diagonal union" $\{z : \exists x(x \in z \in A_x)\}$ is in I. It is well-known that I is normal iff for any $A \in \mathcal{P}(Z) \setminus I$ and any function f on A such that $f(z) \in z$ for all $z \in A$, there is an x such that $f^{-1}(x) \notin I$.

To fix notation, let $I^* = \{Z \setminus A : A \in I\}$ (the I-measure one sets), $I^+ = \mathcal{P}(Z) \setminus I$ (the I-positive sets), $\hat{x} = \{z : x \in z\}$, and denote diagnoal unions by $\nabla_{x \in X} A_x$. Note that $\nabla_{x \in X} A_x = \bigcup_{x \in X} \hat{x} \cap A_x$.

Proposition 1.1. *Let I be a normal and fine ideal on $Z \subseteq \mathcal{P}(X)$. Let κ be a cardinal and let $\{x_\alpha : \alpha < \kappa\}$ be distinct elements of X. Then I is κ-complete iff for all $\beta < \kappa$, $\bigcap_{\alpha<\beta} \hat{x}_\alpha \in I^*$.*

Proof. If I is κ-complete, then by fineness $\hat{x}_\alpha \in I^*$ for any α, so $\bigcap_{\alpha<\beta} \hat{x}_\alpha \in I^*$ for any $\beta < \kappa$. For the other direction, suppose that $\{A_\alpha : \alpha < \beta < \kappa\} \subseteq I$, but $A = \bigcup_{\alpha<\beta} A_\alpha \in I^+$. Then by hypothesis, $B = A \cap (\bigcap_{\alpha<\beta} \hat{x}_\alpha) \in I^+$. Let $f : B \to X$ be defined by $f(z) = x_\alpha$, where α is the least ordinal such that $z \in A_\alpha$. By normality, some $A_\alpha \in I^+$, a contradiction. □

The following basic fact seems to have been previously overlooked–see, for example, the hypotheses of several theorems in [13] and [14].

Proposition 1.2. *All normal and fine ideals are countably complete.*

Proof. Let I be a normal and fine ideal on $Z \subseteq \mathcal{P}(X)$. By the above, it suffices to find an infinite set $\{x_n : n < \omega\} \subseteq X$ such that $\bigcap \hat{x}_n \in I^*$. Since I is proper and nonprincipal, X is infinite. We show that any infinite set of distinct elements of X suffices.

Let $\{x_n : n < \omega\}$ be distinct elements of X. Suppose the contrary, that $B = \{z : \{x_n : n < \omega\} \nsubseteq z\} \in I^+$. By fineness, $B \cap \hat{x}_0 \in I^+$. For each $z \in B \cap \hat{x}_0$, let n_z be the largest integer such that $\{x_0, ..., x_{n_z}\} \subseteq z$. Let $f : B \cap \hat{x}_0 \to X$ be defined by $f(z) = x_{n_z}$. By normality, there is an n such that $C = f^{-1}(x_n) \in I^+$. Then for all $z \in C$, $x_{n+1} \notin z$. This contradicts fineness. □

Proposition 1.3. *If I is a normal, fine, κ-complete ideal on $Z \subseteq \mathcal{P}_\kappa(\kappa)$, then $\kappa \in I^*$.*

Proof. Suppose $A = \{z \in Z : z \text{ is not an ordinal}\} \in I^+$. Let $f : A \to \kappa$ be such that $f(z)$ is the least $\alpha \in z$ such that $\alpha \nsubseteq z$. Then for some α, $f^{-1}(\alpha) \in I^+$. However, $\{z : \alpha \subseteq z\} \in I^*$ by fineness and κ-completeness. □

Lemma 1.4. *If $\kappa = \mu^+$ and I is a κ-complete, fine ideal on $Z = \mathcal{P}_\kappa(X)$, then every $A \in I^+$ can be split into $|X|$ many disjoint I-positive sets.*

Proof. We use a generalization of Ulam matrices. For each $z \in Z$, let $f_z : z \to \mu$ be an injection. For $\alpha < \mu$, $x \in X$, let $M_x^\alpha = \{z \in \hat{x} : f_z(x) = \alpha\}$. For $x \neq y$, $M_x^\alpha \cap M_y^\alpha = \emptyset$. For each $x \in X$, $\hat{x} = \bigcup_{\alpha<\mu} M_x^\alpha$. If $A \in I^+$, then by κ-completeness, $\forall x \exists \alpha (A \cap M_x^\alpha \in I^+)$. If $|X|$ is regular, then there is some $\alpha < \mu$ such that $A \cap M_x^\alpha \in I^+$ for $|X|$ many $x \in X$.

Otherwise, for each $\beta < |X|$, there is some $\alpha < \mu$ such that $|\{x \in X : A \cap M_x^\alpha \in I^+\}| \geq \beta$, since $X = \bigcup_{\alpha<\mu}\{x : A \cap M_x^\alpha \in I^+\}$. Pick some sequence $\langle \beta_i : i < \mathrm{cf}(|X|)\rangle$ converging to $|X|$, pick some $\alpha < \mu$ such that $|\{x \in X : A \cap M_x^\alpha \in I^+\}| \geq \mathrm{cf}(|X|)$, and enumerate the first $\mathrm{cf}(|X|)$ elements as $\langle x_i : i < \mathrm{cf}(|X|)\rangle$. For each x_i, apply the above argument to pick some splitting of $A \cap M_{x_i}^\alpha$ into β_i many disjoint I-positive sets. This gives a splitting of A into $|X|$ many disjoint I-positive sets. □

Alaoglu and Erdős set the stage for Ulam's question by providing an important limitation. Taylor [34] generalized their result to show that for any collection $\{I_\alpha : \alpha < \mu\}$ of μ^+-complete ideals on μ^+, there is an $A \subseteq \mu^+$ that is nonmeasurable for each I_α. A similar result holds for normal and fine ideals on $\mathcal{P}_\kappa(\lambda)$:

Theorem 1.5. *Suppose $\{I_\alpha : \alpha < \eta < \lambda\}$ is a collection of normal and fine ideals on $Z \subseteq \mathcal{P}(\lambda)$ such for each $\alpha < \eta$ and each $A \in I_\alpha^+$, A can be split into η^+ many disjoint I_α-positive sets. Then there is a sequence of disjoint sets $\{A_\alpha : \alpha < \eta\}$ such that each $A_\alpha \in I_\alpha^+$ and a set $B \subseteq Z$ that is nonmeasurable for all I_α.*

Proof. For each $\alpha < \eta$, let $\{A_\beta^\alpha : \beta < \eta^+\}$ be a collection of disjoint I_α-positive sets. Define

$f(\alpha)$ to be the least $\gamma \leq \alpha$ such that $A^\gamma_\beta \in I^+_\alpha$ for η^+ many β. Let $\delta < \eta^+$ be such that $(\forall \alpha < \eta)(\forall \gamma < f(\alpha))(\forall \beta \geq \delta) A^\gamma_\beta \in I_\alpha$. Recursively construct a one-to-one sequence $\langle B_\alpha : \alpha < \eta \rangle$ such that $B_\alpha = A^{f(\alpha)}_\beta \in I^+_\alpha$ for some $\beta \geq \delta$. Then $B_\beta \in I_\alpha$ when $f(\beta) < f(\alpha)$, and $B_\beta \cap B_\alpha = \emptyset$ when $f(\beta) = f(\alpha)$. For each $\alpha < \eta$, let $C_\alpha = B_\alpha \cap \hat{\alpha} \setminus \bigcup_{f(\beta)<f(\alpha)} (B_\beta \cap \hat{\beta})$. $C_\alpha \in I^+_\alpha$ by normality, and $C_\alpha \cap C_\beta = \emptyset$ when $\alpha \neq \beta$. Now split each C_α into two disjoint I_α-positive sets, C^0_α and C^1_α. For $i < 2$, let $D_i = \bigcup_{\alpha<\eta} C^i_\alpha$. D_0 and D_1 are disjoint and I_α-positive for all α, hence nonmeasurable for all I_α. □

Therefore, if $\kappa = \mu^+$ and $\lambda \geq \kappa$, the best we could hope for is to measure every subset of $\mathcal{P}_\kappa(\lambda)$ with λ many κ-complete, normal and fine ideals. Using large cardinals, we will show that this is possible in many cases.

1.2 Forcing

A partial order $\mathbb{P}$ is said to be *separative* when $p \not\leq q \Rightarrow (\exists r \leq p) r \perp q$. Every partial order $\mathbb{P}$ has a canonically associated equivalence relation $\sim_s$ and a separative quotient $\mathbb{P}_s$, which is isomorphic to $\mathbb{P}$ if $\mathbb{P}$ is already separative. In most cases we will assume our partial orders are separative. For every separative partial order $\mathbb{P}$, there is a canonical complete boolean algebra $\mathcal{B}(\mathbb{P})$ with a dense set isomorphic to $\mathbb{P}$.

A map $e : \mathbb{P} \to \mathbb{Q}$ is an *embedding* when it preserves order and incompatibility. An embedding is said to be *regular* when it preserves the maximality of antichains. A order-preserving map $\pi : \mathbb{Q} \to \mathbb{P}$ is called a *projection* when $\pi(1_\mathbb{Q}) = 1_\mathbb{P}$, and $p \leq \pi(q) \Rightarrow (\exists q' \leq q) \pi(q') \leq p$.

Lemma 1.6. *Suppose $\mathbb{P}$ and $\mathbb{Q}$ are partial orders.*

(1) G is a generic filter for $\mathbb{P}$ iff $\{[p]_s : p \in G\}$ is a generic filter for $\mathbb{P}_s$.

(2) $e : \mathbb{P} \to \mathbb{Q}$ *is a regular embedding iff for all* $q \in \mathbb{Q}$*, there is* $p \in \mathbb{Q}$ *such that for all* $r \leq p$*,* $e(r)$ *is compatible with* q*.*

(3) The following are equivalent:

(a) There is a regular embedding $e : \mathbb{P}_s \to \mathcal{B}(\mathbb{Q}_s)$*.*

(b) There is a projection $\pi : \mathbb{Q}_s \to \mathcal{B}(\mathbb{P}_s)$*.*

(c) There is a $\mathbb{Q}$*-name* $\dot{g}$ *for a* $\mathbb{P}$*-generic filter such that for all* $p \in \mathbb{P}$*, there is* $q \in \mathbb{Q}$ *such that* $q \Vdash p \in \dot{g}$*.*

(4) Suppose $e : \mathbb{P} \to \mathbb{Q}$ *is a regular embedding. If* G *is a filter on* $\mathbb{P}$ *let* $\mathbb{Q}/G = \{q : \neg\exists p \in G(e(p) \perp q)\}$*. The following are equivalent:*

(a) H *is* $\mathbb{Q}$*-generic over* V*.*

(b) $G = e^{-1}[H]$ *is* $\mathbb{P}$*-generic over* V*, and* H *is* $\mathbb{Q}/G$*-generic over* $V[G]$*.*

Lemma 1.7. *Suppose* $\mathbb{P}$ *and* $\mathbb{Q}$ *are partial orders.* $\mathcal{B}(\mathbb{P}_s) \cong \mathcal{B}(\mathbb{Q}_s)$ *iff the following holds. Letting* $\dot{G}, \dot{H}$ *be the canonical names for the generic filters for* $\mathbb{P}, \mathbb{Q}$ *respectively, there is a* $\mathbb{P}$*-name for a function* $\dot{f}_0$ *and a* $\mathbb{Q}$*-name for a function* $\dot{f}_1$ *such that:*

(1) $\Vdash_{\mathbb{P}} \dot{f}_0(\dot{G})$ *is a* $\mathbb{Q}$*-generic filter,*

(2) $\Vdash_{\mathbb{Q}} \dot{f}_1(\dot{H})$ *is a* $\mathbb{P}$*-generic filter,*

(3) $\Vdash_{\mathbb{P}} \dot{G} = \dot{f}_1^{\dot{f}_0(\dot{G})}(\dot{f}_0(\dot{G}))$*, and* $\Vdash_{\mathbb{Q}} \dot{H} = \dot{f}_0^{\dot{f}_1(\dot{H})}(\dot{f}_1(\dot{H}))$*.*

Proof. If $\iota : \mathcal{B}(\mathbb{P}_s) \cong \mathcal{B}(\mathbb{Q}_s)$ is an isomorphism, then we can let $\dot{f}_0$ be a $\mathbb{P}$-name for $\iota[G]$, and f_1 be a $\mathbb{Q}$-name for $\iota^{-1}[H]$.

Suppose $\mathbb{P}$ and $\mathbb{Q}$ are complete boolean algebras and $\dot{f}_0, \dot{f}_1$ are as hypothesized. Let $\iota : \mathbb{P} \to \mathbb{Q}$ be given by $p \mapsto ||\check{p} \in \dot{f}_1(\dot{H})||$. First note that ι clearly preserves order and incompatibility.

The kernel of ι is trivial, since for any nonzero $p \in \mathbb{P}$, if we take G generic with $p \in G$, then $H = f_0(G)$ is $\mathbb{Q}$-generic, and $p \in G = f_1(H)$.

It suffices to show that the range of ι is dense. Let $q \in \mathbb{Q}$ be arbitrary. First we claim there is $p \in \mathbb{P}$ that forces $q \in f_0(G)$. If H is generic with $q \in H$, then let $G = f_1(H)$ and let $p \in G$ force $q \in f_0(G) = H$. Now we claim for such p, $\iota(p) \leq q$. For whenever H is generic with $\iota(p) \in H$, $p \in f_1(H) = G$ by the definition of ι, and so by the property of p, $q \in f_0(G) = H$. □

For a broader notion of "forcing equivalence," the best that can be said in general is the following:

Lemma 1.8. *Suppose $\mathbb{P}$ and $\mathbb{Q}$ are partial orders.*

(1) If $e : \mathbb{P} \to \mathbb{Q}$ is a regular embedding, and any $\mathbb{Q}$-generic H yields $V[H] = V[e^{-1}[H]]$, then there is a predense set $A \subseteq \mathcal{B}(\mathbb{Q}_s)$ such that $\mathcal{B}(\mathbb{P}_s) \cong \mathcal{B}(\mathbb{Q}_s) \restriction a$ for all $a \in A$.

(2) $\mathbb{P}$ and $\mathbb{Q}$ yield the same generic extensions iff for a dense set of $p \in \mathbb{P}$, there is $q \in \mathbb{Q}$ such that $\mathcal{B}(\mathbb{P}_s) \restriction p \cong \mathcal{B}(\mathbb{Q}_s) \restriction q$ and vice versa.

Proof. For both claims, we will assume $\mathbb{P}$ and $\mathbb{Q}$ are complete boolean algebras.

For (1), if $V[H] = V[e^{-1}[H]]$ for any $\mathbb{Q}$-generic H, then $\mathbb{P}$ must force that the quotient $\mathbb{Q}/e[G]$ is atomic. Hence there is an isomorphism $\iota : \mathbb{P} * \dot{\mathbb{A}} \to \mathbb{Q}$ extending e, where $\dot{\mathbb{A}}$ is a $\mathbb{P}$-name for an atomic boolean algebra. The set of elements $(1, \dot{a})$ for $\dot{a}$ a name for an atom is predense, and $\mathbb{Q} \restriction \iota(1, \dot{a})$ is isomorphic to $\mathbb{P}$ for any name for an atom $\dot{a}$.

For (2), suppose $\mathbb{P}$ and $\mathbb{Q}$ yield the same extensions, and let $p_0 \in \mathbb{P}$ be arbitrary. Let $G \subseteq \mathbb{P}$ be generic with $p_0 \in G$. Since every $\mathbb{P}$-generic extension is a $\mathbb{Q}$-generic extension, there is a $\mathbb{P}$-name $\dot{h}$ for a $\mathbb{Q}$-generic filter such that $1 \Vdash \dot{G} \in V[\dot{h}]$. There must be some $q_0 \in \mathbb{Q}$ such that $||q \in \dot{h}|| \wedge p_0 \neq 0$ for all $q \leq q_0$; otherwise the set of q which p_0 forces cannot be in $\dot{h}$ is dense, contradicting the genericity of $\dot{h}$. The map $e : q \mapsto ||q \in \dot{h}|| \wedge p_0$ is a regular embedding of $\mathbb{Q} \restriction q_0$ into $\mathbb{P} \restriction ||q_0 \in \dot{h}|| \wedge p_0$, since for any maximal antichain $A \subseteq \mathbb{Q} \restriction q_0$ and any generic $G \subseteq \mathbb{P}$ with $||q_0 \in \dot{h}|| \wedge p_0 \in G$, there is some $q \in A$ such that $q \in \dot{h}^G$. Thus the hypotheses of (1) are satisfied, and $\mathbb{Q} \restriction q_0 \cong \mathbb{P} \restriction p_1$ for some $p_1 \leq p_0$. Switching the roles of $\mathbb{P}$ and $\mathbb{Q}$ gives the "vice versa" conclusion. The converse is trivial. □

A partial order $\mathbb{P}$ is said to be κ-distributive if for any collection of maximal antichains in $\mathbb{P}$, $\{A_\alpha : \alpha < \beta < \kappa\}$, there is a maximal antichain A such that A refines A_α for all $\alpha < \beta$. $\mathbb{P}$ is called (κ, λ)-distributive if the same holds restricted to antichains of size $\leq \lambda$. Forcing with $\mathbb{P}$ adds adds no new functions from any $\alpha < \kappa$ to λ iff $\mathcal{B}(\mathbb{P})$ is (κ, λ)-distributive.

A strictly stronger property than distributivity is strategic closure. For a partial order $\mathbb{P}$ and an ordinal α, we define a game $G_\alpha(\mathbb{P})$ with two players *Even* and *Odd*. *Even* starts by playing some element $p_0 \in \mathbb{P}$. At successor stages $\beta + 1$, the next player must play some element $p_{\beta+1} \leq p_\beta$. *Even* plays at limit stages β if possible, by playing a p_β that is $\leq p_\gamma$ for all $\gamma < \beta$. If *Even* cannot play at some stage below α, the game is over and *Odd* wins; otherwise *Even* wins. We say that $\mathbb{P}$ is α-strategically closed if for every $p \in \mathbb{P}$, *Even* has a winning strategy with first move p. Note that under this definition, every partial order is trivially ω-strategically closed.

A stronger property that κ-strategic closure is κ-closure. $\mathbb{P}$ is κ-closed when any descending chain of length less than κ has a lower bound. $\mathbb{P}$ is κ-directed closed when any directed set of size $< \kappa$ has a lower bound.

For any partial order $\mathbb{P}$, the saturation of $\mathbb{P}$, $\text{sat}(\mathbb{P})$, is the least cardinal κ such that every antichain in $\mathbb{P}$ has size less than κ. Erdős and Tarski [12] proved that $\text{sat}(\mathbb{P})$ is always regular. The density of $\mathbb{P}$, $\text{d}(\mathbb{P})$, is the least cardinality of a dense subset of $\mathbb{P}$. Clearly $\text{sat}(\mathbb{P}) \leq \text{d}(\mathbb{P})^+$ for any $\mathbb{P}$. We say $\mathbb{P}$ is κ-saturated if $\text{sat}(\mathbb{P}) \leq \kappa$, and $\mathbb{P}$ is κ-dense if $\text{d}(\mathbb{P}) \leq \kappa$. A synonym for κ-saturation is the κ chain condition (κ-c.c.).

The properties of distributivity, strategic closure, saturation, and density are robust in the sense that they are absolute between $\mathbb{P}$ and $\mathcal{B}(\mathbb{P})$ for any separative partial order $\mathbb{P}$, and often inherited by intermediate forcings:

Lemma 1.9. *Suppose $e : \mathbb{P} \to \mathbb{Q}$ is a regular embedding and κ is a cardinal.*

(1) If $\mathbb{Q}$ is κ-strategically closed, then so is $\mathbb{P}$.

(2) $\mathbb{Q}$ is κ-distributive iff $\mathbb{P}$ is κ-distributive and $\Vdash_{\mathbb{P}} \mathbb{Q}/\dot{G}$ is κ-distributive.

(3) $\mathbb{Q}$ is κ-saturated iff $\mathbb{P}$ is κ-saturated and $\Vdash_{\mathbb{P}} \mathbb{Q}/\dot{G}$ is κ-saturated.

(4) $\mathbb{Q}$ is κ-dense iff $\mathbb{P}$ is κ-dense and $\Vdash_{\mathbb{P}} \mathbb{Q}/\dot{G}$ is κ-dense.

Proof. We prove only (4). Suppose first that $\mathbb{Q}$ is κ-dense, and let $\{q_\alpha : \alpha < \kappa\}$ witness. Since $\mathcal{B}(\mathbb{Q}) \cong \mathcal{B}(\mathbb{P} * \mathbb{Q}/\dot{G}$, we can pick for each q_α some $(p_\alpha, \dot{r}_\alpha) \leq q_\alpha$. Then $\{p_\alpha : \alpha < \kappa\}$ is dense in $\mathbb{P}$. If $G \subseteq \mathbb{P}$ is generic, let $r \in \mathbb{Q}/\dot{G}$ be arbitrary. For any $p \in \mathbb{P}$, there is $(p_\alpha, \dot{r}_\alpha) \leq (p, \dot{r})$, so for some $p_\alpha \in G$, $p_\alpha \Vdash \dot{r}_\alpha \leq \dot{r}$. Now suppose $\{p_\alpha : \alpha < \kappa\}$ is dense in $\mathbb{P}$, and $\Vdash \{\dot{r}_\beta : \beta < \kappa\}$ is dense in $\mathbb{Q}/\dot{G}$. For any $q \in \mathbb{Q}$, let $(p, \dot{r}) \leq q$. By density, there is some $p_\alpha < p$ and some $\beta < \kappa$ such that $p_\alpha \Vdash \dot{r}_\beta \leq \dot{r}$. □

For any forcing $\mathbb{P}$ and any $\mathbb{P}$-name $\dot{X}$ for a set of ordinals, there is a canonically associated complete subalgebra $\mathcal{A}_{\dot{X}} \subseteq \mathcal{B}(\mathbb{P})$ that captures $\dot{X}$. It is the smallest complete subalgebra containing all elements of the form $||\check{\alpha} \in \dot{X}||$ for α an ordinal. $\mathcal{A}_{\dot{X}}$ has the property that whenever $G \subseteq \mathbb{P}$ is generic, $\dot{X}^G$ and $G \cap \mathcal{A}_{\dot{X}}$ are definable from each other using the parameters $\mathcal{B}(\mathbb{P})$ and its powerset, as computed in the ground model. In this case, we have $V[\dot{X}^G] = V[G \cap \mathcal{A}_{\dot{X}}]$.

1.3 Forcing with ideals

Proofs of the following facts can be found in [13]. If I is an ideal on Z, say $A \sim_I B$ if the symmetric difference $A \Delta B$ is in I. Let $[A]_I$ denote the equivalence class of A mod $\sim_I$. The equivalence classes form a boolean algebra under the obvious operations, which we denote by $\mathcal{P}(Z)/I$. Normality ensures a certain amount of completeness of the algebra:

Proposition 1.10. *Suppose I is a normal and fine ideal on $Z \subseteq \mathcal{P}(X)$. If $\{A_x : x \in X\} \subseteq \mathcal{P}(Z)$, then ∇A_x is the least upper bound of $\{[A_x]_I : x \in X\}$ in $\mathcal{P}(Z)/I$.*

If we force with this algebra, we get a generic ultrafilter G on Z extending I^*. We can form the ultrapower V^Z/G. If this ultrapower is well-founded for every generic G, then I is called precipitous. A combinatorial characterization of precipitousness is given by the following:

Theorem 1.11 (Jech-Prikry)**.** *I is a precipitous ideal on Z iff the following holds: For any sequence $\langle A_n : n < \omega \rangle \subseteq \mathcal{P}(I^+)$, such that for each n,*

(1) $B_n = \{[a]_I : a \in A_n\}$ is a maximal antichain in $\mathcal{P}(Z)/I$,

(2) B_{n+1} refines B_n,

there is a function f with domain ω such that for all n, $f(n) \in A_n$, and $\bigcap_{n<\omega} f(n) \neq \emptyset$.

For an ideal I, the saturation, density, distributivity, and strategic closure of I refers to that of the corresponding boolean algebra. The next proposition is immediate from Theorem 1.11:

Proposition 1.12. *If I is an ω_1-complete, ω_1-distributive ideal, then I is precipitous.*

Proposition 1.13. *Suppose I is a κ-complete precipitous ideal on Z, and I is nowhere κ^+-complete. Let G be $\mathcal{P}(Z)/I$-generic, and let $j : V \to M$ be the associated elementary embedding, where M is the transitive collapse of V^Z/G. Then the critical point of j is κ.*

Proposition 1.14. *Let I be an ideal $Z \subseteq \mathcal{P}(X)$. Then I is normal and fine iff $1 \Vdash_{\mathcal{P}(Z)/I} [id] = j[X]$.*

Proposition 1.15. *Suppose I is an ideal on $Z \subseteq \mathcal{P}(X)$. If I is κ-complete and κ^+-saturated, or if I is normal, fine, and $|X|^+$-saturated, then every antichain in $\mathcal{P}(Z)/I$ has a system of pairwise disjoint representatives.*

Proof. If I is κ-complete, and $\{A_\alpha : \alpha < \kappa\}$ is an antichain, replace each A_α with $A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$. If I is normal and fine, and $\{A_x : x \in X\}$ is an antichain, replace A_x by $A_x \cap \hat{x} \setminus \bigcup_{y \neq x} A_y \cap \hat{y}$. □

Theorem 1.16. *Suppose I is a countably complete ideal on Z, and every antichain in $\mathcal{P}(Z)/I$ has a system of pairwise disjoint representatives. Then:*

(1) I is precipitous.

(2) $\mathcal{P}(Z)/I$ is a complete boolean algebra.

(3) If G is generic over $\mathcal{P}(Z)/I$, $j : V \to M$ is the associated embedding, and $j[\lambda] \in M$, then M is closed under λ-sequences from $V[G]$.

1.4 Elementary embeddings

Lemma 1.17. *Suppose M and N are models of ZF^-, $j : M \to N$ is an elementary embedding, $\mathbb{P} \in M$ is a partial order, G is $\mathbb{P}$-generic over M, and H is $j(\mathbb{P})$-generic over N. Then j has a unique extension $\hat{j} : M[G] \to N[H]$ with $\hat{j}(G) = H$ iff $j[G] \subseteq H$.*

Proof. If $j[G] \subseteq H$, the only possible choice is to let $\hat{j}(\tau^G) = j(\tau)^H$ for all $\mathbb{P}$-names τ. If $M[G] \models \varphi(\tau_1^G, ..., \tau_n^G)$, then for some $p \in G$, $p \Vdash \varphi(\tau_1, ..., \tau_n)$. We have $j(p) \in H$ and

$j(p) \Vdash \varphi(j(\tau_1), ..., j(\tau_n))$, so $N[H] \models \varphi(j(\tau_1)^H, ..., j(\tau_n)^H)$. Conversely, if $\hat{j} : M[G] \to N[H]$ is elementary, extends j, and has $\hat{j}(G) = H$, then for all $p \in G$, $j(p) \in H$ by elementarity. $\square$

Lemma 1.18. *Suppose M, N are transitive models of ZFC with the same ordinals, and $j : M \to N$ is an elementary embedding. Then either j has a critical point, or j is the identity and $M = N$.*

Proof. Suppose j is the identity map on ordinals. Let x be a set of minimal rank in M such that $j(x) \neq x$. There is some ordinal $\kappa \in M$ such that $x = \{x_\alpha : \alpha < \kappa\}$. Then $j(x) = \{j(x_\alpha) : \alpha < \kappa\} = x$ by the minimality of x, so j is the identity. To show $M = N$, note that for all ordinals $\alpha \in M$, $j(V_\alpha^M) = V_\alpha^N = V_\alpha^M$. $\square$

Chapter 2

Dense ideals from large cardinals

Here we show that it is consistent relative to an almost-huge cardinal that there is a normal, κ-complete, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$, where κ is the successor of a regular cardinal μ, and $\lambda \geq \kappa$ is regular, for many particular choices for μ, λ. We also show that relative to a super-almost-huge cardinal, there can exist a successor cardinal κ such that for every regular $\lambda \geq \kappa$, there is a normal, κ-complete, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$. This generalizes a theorem of Woodin about the relative consistency of an $\aleph_1$-dense ideal on $\aleph_1$, and has the following additional advantages: (1) An explicit forcing extension is taken, rather than an inner model of an extension. (2) Careful constructions within a model where the axiom of choice fails, as presented in [13], are avoided.

Let us first recall the essential facts about almost-huge cardinals (see [22], Theorem 24.11). A cardinal κ is almost-huge if there is an elementary embedding $j : V \to M$ with critical point κ, such that $M^{<j(\kappa)} \subseteq M$.

Theorem 2.1. *The following are equivalent:*

(1) κ carries an almost-huge embedding j such that $j(\kappa) = \delta$.

(2) δ is inaccessible, and there is a sequence $\langle U_\alpha : \kappa \leq \alpha < \delta \rangle$ such that:

(a) each U_α is a normal, κ-complete ultrafilter on $\mathcal{P}_\kappa(\alpha)$,

(b) for $\alpha < \beta$, $U_\alpha = \{A \subseteq \mathcal{P}_\kappa(\alpha) : \{z \in \mathcal{P}_\kappa(\beta) : z \cap \alpha \in A\} \in U_\beta\}$, and

(c) for all $\alpha < \delta$ and all $f : \mathcal{P}_\kappa(\alpha) \to \kappa$ such that $\{z : f(z) \geq \mathrm{ot}(z)\} \in U_\alpha$, there is β such that $\alpha \leq \beta < \delta$ and $\{z : f(z \cap \alpha) = \mathrm{ot}(z)\} \in U_\beta$.

Furthermore, if a system as in (2) is given, the direct limit model and embedding witness the almost-hugeness of κ with target δ.

A system as in (2) will be called an almost-huge tower. Almost-huge towers capture almost-hugeness in a minimal way:

Corollary 2.2. *If κ has an almost-huge tower of height δ, and $j : V \to M$ is the embedding derived from the tower, then we have $\delta < j(\delta) < \delta^+$, and $j[\delta]$ is cofinal in $j(\delta)$.*

Proof. For each $\alpha < \delta$, let M_α be the transitive collapse of $V^{\mathcal{P}_\kappa(\alpha)}/U_\alpha$, and let $j_\alpha : V \to M_\alpha$ and $k_\alpha : M_\alpha \to M$ be the canonical embeddings, with $j = k_\alpha \circ j_\alpha$. Since δ is inaccessible, $j_\alpha(\kappa) < \delta$ and $j_\alpha(\delta) = \delta$ for each $\alpha < \delta$.

If $\gamma < j(\delta)$, then there are some $\alpha, \beta < \delta$ such that $k_\alpha(\beta) = \gamma$. Thus there are only δ ordinals below $j(\delta)$. Also, there is $\eta < \delta$ such that $j_\alpha(\eta) > \beta$, so $j(\eta) > \gamma$, and thus $j[\delta]$ is cofinal in $j(\delta)$. □

A super-almost-huge cardinal is a cardinal κ such that for all $\lambda \geq \kappa$, there is an almost huge tower of height $\geq \lambda$. The next result follows from considering the set of closure points under witnesses to property (c) in the tower characterization.

Corollary 2.3. *If κ has an almost-huge tower of height δ, and δ is Mahlo, then $V_\delta \models ZFC + \kappa$ is super-almost-huge, and for stationary many $\alpha < \delta$, $V_\alpha \models ZFC + \kappa$ is super-almost-huge.*

There is a vast gap in strength between almost-huge and huge:

Theorem 2.4. *If κ is a huge cardinal, then there is a stationary set $S \subseteq \kappa$ such that for all $\alpha < \beta$ in S, α has an almost-huge tower of height β.*

Proof. Suppose $j : V \to M$ is an elementary embedding with critical point κ, $j(\kappa) = \delta$, and $M^\delta \subseteq M$. Then κ carries an almost-huge tower $\vec{U}$ of length δ, and $\vec{U} \in M$. Let F be the ultrafilter on κ defined by $F = \{X \subseteq \kappa : \kappa \in j(X)\}$. Let $A = \{\alpha < \kappa : \alpha$ carries an almost-huge tower of height $\kappa\}$. Since $\kappa \in j(A)$, $A \in F$. Now let $c : \kappa^2 \to 2$ be defined by $c(\alpha, \beta) = 1$ if α carries and almost-huge tower of height β, and $c(\alpha, \beta) = 0$ otherwise. By Rowbottom's theorem, let $H \in F$ be homogeneous for c. We claim c takes constant value 1 on H. For if $\alpha \in A \cap H$, then $\{\alpha, \kappa\} \in [j(A \cap H)]^2$, and $j(c)(\alpha, \kappa) = 1$. □

2.1 Layering and absorption

Definition. *We will call a partial order $\mathbb{P}$ (μ, κ)-*nicely layered *when there is a collection $\mathcal{L}$ of regular suborders of $\mathbb{P}$ such that:*

(1) for all $\mathbb{Q} \in \mathcal{L}$, $\mathbb{Q}$ is μ-closed and has size $< \kappa$,

(2) for all $\mathbb{Q}_0, \mathbb{Q}_1 \in \mathcal{L}$, if $\mathbb{Q}_0 \subseteq \mathbb{Q}_1$, then $\Vdash_{\mathbb{Q}_0} \mathbb{Q}_1/\dot{G}$ is μ-closed, and

(3) for all $\mathbb{P}$-names $\dot{f}$ for a function from μ to the ordinals, and all $\mathbb{Q}_0 \in \mathcal{L}$, there is an $\mathbb{Q}_1 \in \mathcal{L}$ and an $\mathbb{Q}_1$-name $\dot{g}$ such that $\mathbb{Q}_0 \subseteq \mathbb{Q}_1$, and $\Vdash_{\mathbb{P}} \dot{f} = \dot{g}$.

*We will say $\mathbb{P}$ is (μ, κ)-*nicely layered with collapses, *(μ, κ)-NLC, when additionally for all $\alpha < \kappa$ and all $\mathbb{Q}_0 \in \mathcal{L}$, there is $\mathbb{Q}_1 \in \mathcal{L}$ such that $\mathbb{Q}_0 \subseteq \mathbb{Q}_1$, $\Vdash_{\mathbb{Q}_0} |\mathbb{Q}_1/\dot{G}| \geq |\alpha|$, and $\Vdash_{\mathbb{Q}_1} |\mathbb{Q}_1| = \mu$.*

Proposition 2.5. *If $\mathcal{L}$ witnesses that $\mathbb{P}$ is (μ, κ)-nicely layered, then $\mathbb{P}$ is κ-c.c. and $\bigcup \mathcal{L}$ is dense in $\mathbb{P}$.*

Proof. Suppose that $\{p_\alpha : \alpha < \kappa\} \subseteq \mathbb{P}$ is a maximal antichain. Let $\dot{f}$ be a name of a function with domain $\{0\}$ such that $f(0) = \alpha$ iff $p_\alpha \in G$. There cannot be a regular suborder $\mathbb{Q}$ of size $< \kappa$ and a $\mathbb{Q}$-name $\dot{g}$ that is forced to be equal to $\dot{f}$, since such a $\dot{g}$ would have $< \kappa$ possible values for its range.

Similarly, let $p \in \mathbb{P}$ be arbitrary, and let $\{p_\alpha : \alpha < \delta\}$ be a maximal antichain with $p = p_0$. Let $\dot{f}$ be a name of a function with domain $\{0\}$ such that $f(0) = \alpha$ iff $p_\alpha \in G$. If $\mathbb{Q}$ is a regular suborder and $\dot{g}$ is a $\mathbb{Q}$-name such that $\Vdash_{\mathbb{P}} \dot{f} = \dot{g}$, then there is some $q \in \mathbb{Q}$ forcing $\dot{g}(0) = 0$, so $q \leq p$. □

Lemma 2.6 (Folklore)**.** *If $\mathbb{P}$ is a μ-closed partial order such that $\Vdash_{\mathbb{P}} |\mathbb{P}| = \mu$, then $\mathcal{B}(\mathbb{P}) \cong \mathcal{B}(\mathrm{Col}(\mu, |\mathbb{P}|))$.*

Proof. Pick a $\mathbb{P}$-name $\dot{f}$ for a bijection from μ to $\dot{G}$. We build a tree $T \subseteq \mathbb{P}$ that is isomorphic to a dense subset of $\mathrm{Col}(\mu, |\mathbb{P}|)$, and show that it is dense in $\mathbb{P}$. Each level will be a maximal antichain in $\mathbb{P}$. Let the first level $T_0 = \{1_{\mathbb{P}}\}$. If levels $\{T_\beta : \beta < \alpha+1\}$ are defined, below each $p \in T_\alpha$, pick a $|\mathbb{P}|$-sized maximal antichain of conditions deciding $\dot{f}(\alpha)$, and let $T_{\alpha+1}$ be the union of these antichains. If $\{T_\beta : \beta < \lambda\}$ is defined up to a limit λ, pick for each descending chain b through the previous levels, a $|\mathbb{P}|$-sized maximal antichain of lower bounds to b, and set T_λ equal to the union of these anithchains. It is easy to check that T_λ is a maximal antichain. Let $T = \bigcup_{\alpha<\mu} T_\alpha$. To show T is dense, let $p \in \mathbb{P}$. Let $q \leq p$ be such that for some $\alpha < \mu$, $q \Vdash \dot{f}(\alpha) = p$. q is compatible with some $r \in T_{\alpha+1}$. Since r decides $\dot{f}(\alpha)$ and forces it in $\dot{G}$, $r \leq p$. □

Lemma 2.7. *Suppose $\mu < \kappa$ are regular, and $\mathbb{P}$ is (μ, κ)-NLC. If G is $\mathbb{P}$-generic over V, then there is a forcing $\mathbb{R} \in V[G]$ such that $\mathbb{R}$ adds a filter $H \subseteq \mathrm{Col}(\mu, < \kappa)$ which is generic over V and such that $(\mathrm{Ord}^\mu)^{V[G]} = (\mathrm{Ord}^\mu)^{V[H]}$.*

Proof. Let $\mathcal{L}$ witness the (μ, κ)-NLC property. First note that this implies $\alpha^{<\mu} < \kappa$ for all $\alpha < \kappa$. In $V[G]$, let $\mathbb{R}$ be the collection of filters $h \subseteq \mathrm{Col}(\mu, < \alpha)$ for $\alpha < \kappa$ which are generic over V, such that for some $\mathbb{Q} \in \mathcal{L}$, $V[h] = V[G \cap \mathbb{Q}]$. The ordering is end-extension.

Let $h \in \mathbb{R}$ with $\mathbb{Q}_0 \in \mathcal{L}$ a witness, and let and $\alpha < \kappa$ be arbitrary. Let $\alpha < \beta < \kappa$ and $\mathbb{Q}_1 \supseteq \mathbb{Q}_0$ in $\mathcal{L}$ be such that in $V[h]$, $|\mathbb{Q}_1/(G \cap \mathbb{Q}_0)| = |\beta|$, and $\mathbb{Q}_1$ collapses β to μ. By the definition and Lemma 2.6, $\mathbb{Q}_1/(G \cap \mathbb{Q}_0)$ is equivalent in $V[h]$ to $\mathrm{Col}(\mu, \beta)$, which is equivalent to the $< \mu$-support product of $\mathrm{Col}(\mu, \gamma)$ for $\alpha \leq \gamma \leq \beta$. The filter $G \cap \mathbb{Q}_1$ therefore gives a filter $h' \supseteq h$ on $\mathrm{Col}(\mu, < \beta + 1)$ that is generic over V, with $V[h'] = V[\mathbb{Q}_1 \cap G]$.

Let $h \in \mathbb{R}$ with $\mathbb{Q}_0 \in \mathcal{L}$ a witness, and let $f : \mu \to \mathrm{Ord}$ in $V[G]$ be arbitrary. By the definition of (μ, κ)-NLC, we can find some $\mathbb{Q}_1 \supseteq \mathbb{Q}_0$ in $\mathcal{L}$ such that $f \in V[G \cap \mathbb{Q}_1]$. By the previous paragraph, we may find $\mathbb{Q}_2 \supseteq \mathbb{Q}_1$ in $\mathcal{L}$ equivalent to some $\mathrm{Col}(\mu, < \alpha)$, and some filter $h' \subseteq \mathrm{Col}(\mu, < \alpha)$ generic over V, extending h, and such that $V[G \cap \mathbb{Q}_2] = V[h']$.

So if F is generic over $\mathbb{R}$, let $H = \bigcup_{h \in F} h$. By the above arguments, the rank of H is κ. Since $\mathrm{Col}(\mu, < \kappa)$ is κ-c.c., H is generic, since any maximal antichain from V intersects some $h \in F$. Also by the above arguments, any $f : \mu \to \mathrm{Ord}$ in $V[G]$ is in $V[H]$. Conversely, any $f : \mu \to \mathrm{Ord}$ in $V[H]$ lives in some $V[h]$ with $h \in \mathbb{R}$, so is in $V[G]$. □

2.1.1 The anonymous collapse

Let κ be a regular cardinal whose regularity is preserved by a forcing $\mathbb{P}$. Let $A(\mathbb{P})$ be the complete subalgebra of $\mathcal{B}(\mathbb{P} * \mathrm{Add}(\kappa))$ generated by the canonical name for the $\mathrm{Add}(\kappa)$-generic set. More precisely, if $e : \mathbb{P} * \mathrm{Add}(\kappa) \to \mathcal{B}(\mathbb{P} * \mathrm{Add}(\kappa))$ is the canonical dense embedding, $A(\mathbb{P})$ is completely generated by the elements of the form $e(\langle 1, \{\langle \dot{\alpha}, 1 \rangle\} \rangle)$.

In the case that $\alpha^{<\mu} < \kappa$ for all $\alpha < \kappa$ and $\mathbb{P} = \mathrm{Col}(\mu, < \kappa)$, denote $A(\mathbb{P})$ by $A(\mu, \kappa)$, and write $B(\mu, \kappa)$ for $\mathcal{B}(\mathrm{Col}(\mu, < \kappa) * \mathrm{Add}(\kappa))$.

Lemma 2.8. *If $\mathbb{P}$ is (μ,κ)-NLC, and $H \subseteq A(\mathbb{P})$ is generic over V, then $\mathcal{B}(\mathbb{P} * \mathrm{Add}(\kappa))/H$ is κ-distributive in $V[H]$.*

Proof. $V[H] = V[X_H]$ for some canonically associated $X \subseteq \kappa$, and by forcing with $\mathcal{B}(\mathbb{P} * \mathrm{Add}(\kappa))/H$ over $V[H]$, we recover a filter $G * X_H$ for $\mathbb{P} * \mathrm{Add}(\kappa)$, generic over V.

If $G * X$ is $\mathbb{P} * \mathrm{Add}(\kappa)$-generic over V, then X codes all subsets of μ that live in $V[G]$. By the definition of (μ,κ)-NLC, every $z \in (\mathrm{Ord}^{\mu})^{V[G]}$ occurs in some submodel of the form $V[G \cap \mathbb{Q}]$, where $\mathbb{Q}$ is isomorphic to $\mathrm{Col}(\mu,\alpha)$ for some $\alpha < \kappa$. Thus $z \in V[y]$ for some $y \subseteq \mu$ in $V[G]$, so $(\mathrm{Ord}^{\mu})^{V[X]} \supseteq (\mathrm{Ord}^{\mu})^{V[G]}$. Since $\mathrm{Add}(\kappa)$ adds no μ-sized sets of ordinals, $(\mathrm{Ord}^{\mu})^{V[G]} = (\mathrm{Ord}^{\mu})^{V[G*X]} \supseteq (\mathrm{Ord}^{\mu})^{V[X]}$. Thus $\mathcal{B}(\mathbb{P} * \mathrm{Add}(\kappa))/H$ is κ-distributive. □

Lemma 2.9. *Let V be a countable transitive model of ZFC (or just assume generic extensions are always available), and assume $\Vdash^V_{\mathbb{P}} \kappa$ is regular. If $X \subseteq \kappa$, the following are equivalent:*

(1) X is $A(\mathbb{P})$-generic over V.

(2) There is $G \subseteq \mathbb{P}$ such that G is generic over V, and X is $\mathrm{Add}(\kappa)$-generic over $V(P_0)$, where $P_0 = \mathcal{P}_\kappa(\kappa)^{V[G]}$.

Proof. If X is $A(\mathbb{P})$-generic then force with $\mathcal{B}(\mathbb{P} * \mathrm{Add}(\kappa))/H_X$ over $V[X]$, obtaining G such that $G * X$ is $\mathbb{P} * \mathrm{Add}(\kappa)$-generic over V. Then X is $\mathrm{Add}(\kappa)$-generic over $V[G]$, and since $\mathrm{Add}(\kappa)^{V[G]} = \mathrm{Add}(\kappa)^{V(P_0)}$, X is $\mathrm{Add}(\kappa)$-generic over $V(P_0)$.

Suppose $G \subseteq \mathbb{P}$ is generic over V, and X is $\mathrm{Add}(\kappa)$-generic over $V(P_0)$, but not $A(\mathbb{P})$-generic over V. Then some $p \in \mathrm{Add}(\kappa)^{V(P_0)}$ forces this with $\mathrm{dom}(p) = \alpha < \kappa$, and $X \restriction \alpha = p$. Take $Y \subseteq \kappa$ such that $Y \restriction \alpha = p$ that is $\mathrm{Add}(\kappa)$-generic over the larger model $V[G]$. Then Y is $A(\mathbb{P})$-generic over V, and $V(P_0)[Y]$ can see this, but this contradicts the property of p. So X was $A(\mathbb{P})$-generic over V. □

Theorem 2.10. *For any $\mathbb{P}$ that is is (μ, κ)-NLC, there is an isomorphism $\iota : A(\mathbb{P}) \to A(\mu, \kappa)$ such that $\iota(||\alpha \in \dot{X}||_{A(\mathbb{P})}) = ||\alpha \in \dot{X}||_{A(\mu,\kappa)}$ for all $\alpha < \kappa$.*

Proof. Let X be $A(\mathbb{P})$-generic over V. There is a κ-distributive forcing over $V[X]$ to get G such that $G * X$ is $\mathbb{P} * \mathrm{Add}(\kappa)$-generic over V. By Lemma 2.7, we can do further forcing to obtain $H \subseteq \mathrm{Col}(\mu, < \kappa)$ generic over V such that $(\mathrm{Ord}^\mu)^{V[H]} = (\mathrm{Ord}^\mu)^{V[G]}$. By Lemma 2.9, X is also $A(\mu, \kappa)$-generic over V.

Conversely, every $A(\mu, \kappa)$-generic X is $A(\mathbb{P})$-generic. For suppose X is a counterexample. Then there is some $(p, \dot{q}) \in \mathrm{Col}(\mu, < \kappa) * \mathrm{Add}(\kappa)$ such that $(p, \dot{q}) \Vdash \dot{X}$ is not $A(\mathbb{P})$-generic over V. Let Y be any $A(\mathbb{P})$-generic set, and let $P_0 = \mathcal{P}(\mu)^{V[Y]}$. By the above, Y is $A(\mu, \kappa)$-generic over V. Thus we can force over $V[Y]$ to get $H \subseteq \mathrm{Col}(\mu, < \kappa)$ such that $H * Y$ is generic over V. By the homogeneity of the Levy collapse, there is some automorphism $\pi \in V$ such that $p \in \pi[H] = H'$. By the homogeneity of Cohen forcing, there is some automorphism σ in $V(P_0)$ such that $\sigma[Y]$ is a generic Y' such that $Y' \restriction \mathrm{dom}(\dot{q}^{H'}) = \dot{q}^{H'}$. Y' is also $A(\mathbb{P})$-generic over V. However, $(p, \dot{q}) \in H' * Y'$, so we have a contradiction.

This implies that we have a canonical correspondence between $A(\mathbb{P})$- and $A(\mu, \kappa)$-generic filters, i.e. definable functions f, g such that for any generic H for $A(\mathbb{P})$, $f(H)$ is the generic for $A(\mu, \kappa)$ computed from X_H, and vice versa, and $g(f(H)) = H$. For $p \in A(\mathbb{P})$, put $\iota(p) = ||p \in g(\dot{H})||_{A(\mu,\kappa)}$. It is easy to see that ι is a complete embedding. For any $q \in A(\mu, \kappa)$, there is $p \in A(\mathbb{P})$ forces that $q \in f(\dot{H})$. Thus if H is generic for $A(\mu, \kappa)$ and $\iota(p) \in H$, then $p \in g(H)$, so $q \in f(g(H)) = H$, hence $\iota(p) \leq q$. The range of ι is dense, so it is an isomorphism. By the way we construct f and g, $\iota(||\alpha \in \dot{X}||_{A(\mathbb{P})}) = ||\alpha \in \dot{X}||_{A(\mu,\kappa)}$. □

This machinery has some interesting applications to the absoluteness of some properties of a given powerset. First, it is easy to see for regular $\mu < \kappa$ such that $\alpha^{<\mu} < \kappa$ for all $\alpha < \kappa$, $\mathrm{Col}(\mu, < \kappa) \times \mathrm{Add}(\mu, \lambda)$ is (μ, κ)-NLC for every λ. Thus if X is $A(\mu, \kappa)$-generic, then for any λ, we may further force to obtain a model which is a $(\mathrm{Col}(\mu, < \kappa) \times \mathrm{Add}(\mu, \lambda)) * \mathrm{Add}(\kappa)$-generic extension with the same Ord^μ. Taking inner models given by such $\mathrm{Col}(\mu, < \kappa) \times \mathrm{Add}(\mu, \lambda)$-generic sets, we produce many models with the same cardinals and same $\mathcal{P}(\mu)$, each assigning a different cardinal value for 2^μ. For example, if we add ω_1 Cohen reals to any model of M of ZFC, this is the same as forcing with $\mathrm{Col}(\omega, < \omega_1)$. There is for each uncountable ordinal $\alpha \in M$, a generic extension with the same reals and same cardinals, in which it appears we have added α many Cohen reals.

By using weakly compact cardinals, we can get even more dramatic examples. If κ is weakly compact, every κ-c.c. partial order captures small sets in small factors. To show this, first consider a partial order $\mathbb{P}$ of size κ. We can code $\mathbb{P}$ as $A \subseteq \kappa$, and by weak compactness, there is some transitive elementary extension $(V_\kappa, \in, A) \prec (M, \in, B)$. If $\mu < \kappa$, then any $\mathbb{P}$-name for function $f : \mu \to \mathrm{Ord}$ has an equivalent name $\tau \in V_\kappa$ by the κ-c.c. Since $A \in M$ and M sees A as a regular suborder of B, M thinks that τ is a $\mathbb{Q}$-name for some regular suborder $\mathbb{Q}$ of B. By elementarity, V_κ thinks that τ is a $\mathbb{Q}$-name for some regular $\mathbb{Q}$ of A. For $\mathbb{P}$ of arbitrary size, let τ be a $\mathbb{P}$-name of size $< \kappa$, take some regular θ such that $\mathbb{P}, \tau \in H_\theta$, and take an elementary $M \prec H_\theta$ with $\mathbb{P}, \tau \in M$ such that $|M| = \kappa$ and $M^{<\kappa} \subseteq M$. It is easy to see that $M \cap \mathbb{P}$ is a regular suborder of $\mathbb{P}$, and so the above considerations apply to show that there is some regular $\mathbb{Q} \subseteq \mathbb{P} \cap M \subseteq \mathbb{P}$ of size $< \kappa$ such that τ is a $\mathbb{Q}$-name.

Therefore, if κ is weakly compact and $\mathbb{P}$ is κ-c.c., the collection $\mathcal{L}$ of all regular suborders of $\mathbb{P}$ of size $< \kappa$ witnesses that $\mathbb{P}$ is (ω, κ)-nicely layered. If $\mathbb{P}$ also forces $\kappa = \aleph_1$, then this collection also witnesses that $\mathbb{P}$ is (ω, κ)-NLC. To check this, take any $\mathbb{Q}_0 \in \mathcal{L}$, any $\mathbb{P}$-name τ of size $< \kappa$, and $\alpha < \kappa$. Let $H \subseteq \mathbb{Q}_0$ be generic. Since κ is still weakly compact in $V[H]$, there is some regular $\mathbb{Q}_1 \subseteq \mathbb{P}/H$ of size $< \kappa$ in $V[H]$ such that the $(\mathbb{P}/H)$-name associated to τ is a $\mathbb{Q}_1$-name. Let $\beta \geq \max\{\alpha, |\mathbb{Q}_0 * \dot{\mathbb{Q}}_1|\}$. Since $\mathbb{P}/(\mathbb{Q}_0 * \dot{\mathbb{Q}}_1)$ adds a generic for $\mathrm{Col}(\omega, \beta)$, we have $\mathbb{Q}_2 \in \mathcal{L}$ extending $\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$ such that $\mathbb{Q}_2 \sim \mathrm{Col}(\omega, \beta)$.

In particular, if κ is weakly compact, then $\mathrm{Col}(\omega, < \kappa) * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is forced to be c.c.c., is (ω, κ)-NLC. Thus an extremely wide variety of forcing extensions with very different theories can be obtained, each sharing the same reals and same cardinals.

2.1.2 An unfortunate reality

Despite the universality of $A(\mu,\kappa)$, it is difficult to characterize its combinatorial structure. While it absorbs all of the small sets added by a (μ,κ)-NLC forcing, no such forcing completely embeds into it. The reader may opt to skip this section, as later results will not depend on it.

To show this, we first isolate two properties of a forcing extension that depend on two regular cardinals $\mu < \kappa$. The author is grateful to Mohammad Golshani for bringing these properties to his attention.

(1) *Levy(μ,κ):* $(\exists A \in [\kappa]^\kappa)(\forall y \in [\kappa]^\mu \cap V) y \not\subseteq A$.

(2) *Silver(μ,κ):* $(\exists A \in [\kappa]^\kappa)(\forall X \in [\kappa]^\kappa \cap V)(\exists y \in [X]^\mu \cap V) y \cap A = \emptyset$.

Note that these are both Σ_1 properties of the parameters $([\kappa]^\mu)^V$ and $([\kappa]^\kappa)^V$. For any partial order $\mathbb{P}$, and collection of dense subsets $\mathcal{D} \subseteq \mathcal{P}(\mathbb{P})$ the statement, "There is a filter $G \subseteq \mathbb{P}$ that is $\mathcal{D}$-generic," is also a Σ_1 property of $\mathbb{P}$ and $\mathcal{D}$. Now the following proposition either holds or fails for a given partial order $\mathbb{P}$ and cardinals $\mu < \kappa$:

$$(*)_{\mu,\kappa} : (\forall X \in [\mathbb{P}]^\kappa)(\exists y \in [X]^\mu) y \text{ has a lower bound in } \mathbb{P}.$$

Lemma 2.11. *If $\mathbb{P}$ is a separative partial order that satisfies $(*)_{\mu,\kappa}$, preserves the regularity of κ, and such that $\mathrm{d}(\mathbb{P} \restriction p) = \kappa$ for all $p \in \mathbb{P}$, then $\mathbb{P}$ forces Silver(μ,κ).*

Proof. Let $\{p_\alpha : \alpha < \kappa\}$ be a dense subset of $\mathbb{P}$. Inductively build a dense $D \subseteq \{p_\alpha : \alpha < \kappa\}$, putting $p_\alpha \in D$ just in case there is no $\beta < \alpha$ such that $p_\beta \in D$ and $p_\beta \leq p_\alpha$. D has the property that for all $p \in D$, $|\{q \in D : p \leq q\}| < \kappa$. Fixing a bijection $f : D \to \kappa$, we claim that if $G \subseteq \mathbb{P}$ is generic, $A = f[G]$ witnesses $Silver(\mu, \kappa)$. Note that since $\mathbb{P}$ is nowhere $< \kappa$-dense, A is an unbounded subset of κ. Now let $p \in D$ and $X = \{q_\alpha : \alpha < \kappa\} \in [D]^\kappa$ be arbitrary. There is some $B \in [\kappa]^\kappa$ such that for all $\alpha \in B$, $p \nleq q_\alpha$. For each $\alpha \in B$, choose $r_\alpha \leq p$ such that $r_\alpha \perp q_\alpha$. By $(*)$, there is some $y \in [B]^\mu$ such that $\{r_\alpha : \alpha \in y\}$ has a lower bound r. We have $r \Vdash \{q_\alpha : \alpha \in \check{y}\} \cap \dot{G} = \emptyset$. As p and X were arbitrary, $Silver(\mu, \kappa)$ is forced. □

Lemma 2.12. *If $\mathbb{P}$ is a κ-c.c. separative partial order of size κ satisfying $\neg(*)_{\mu,\kappa}$, then some $p \in \mathbb{P}$ forces $Levy(\mu, \kappa)$.*

Proof. Suppose $X \in [\mathbb{P}]^\kappa$ witnesses $\neg(*)_{\mu,\kappa}$. By the κ-c.c., there is some p such that $p \Vdash |\check{X} \cap \dot{G}| = \kappa$. If $y \in [X]^\mu$, then $1 \Vdash \check{y} \nsubseteq \dot{G}$, since otherwise some q is a lower bound to y. Hence p forces that $X \cap G$ witnesses $Levy(\mu, \kappa)$. □

Lemma 2.13. *Suppose $\mu < \kappa$, μ is regular for all $\alpha < \kappa$, $\alpha^\mu < \kappa$. There are two (μ, κ)-NLC partial orders $\mathbb{P}_0$ and $\mathbb{P}_1$ such that $\mathbb{P}_0$ forces $Levy \wedge \neg Silver$, and $\mathbb{P}_1$ forces $\neg Levy \wedge Silver$.*

Proof. Let $\mathbb{P}_0$ be the Levy collapse $\mathrm{Col}(\mu, < \kappa) = \prod_{\alpha<\kappa}^{<\mu-supp} \mathrm{Col}(\mu, \alpha)$, and let $\mathbb{P}_1 = \prod_{\alpha<\kappa}^{\mu-supp} \mathrm{Col}(\mu, \alpha)$. It is easy to see that $\mathbb{P}_1$ satisfies $(*)_{\mu,\kappa}$, while $\mathbb{P}_0$ fails this property, as witnessed by $X = \mathbb{P}_0$. Hence by the previous lemmas, $\mathbb{P}_0$ forces $Levy(\mu, \kappa)$, and $\mathbb{P}_1$ forces $Silver(\mu, \kappa)$. We must show that the respective negations are also forced.

Let $\dot{A}$ be a $\mathbb{P}_0$-name such that $1 \Vdash \dot{A} \in [\kappa]^\kappa$. Let $p \in \mathbb{P}$ be arbitrary, and let $\gamma < \kappa$ be such that $\mathrm{supp}(p) \subseteq \gamma$. Let $X_0 = \{\alpha < \kappa : p \nVdash \alpha \notin \dot{A}\}$. For each $\alpha \in X_0$, pick some $q_\alpha \leq p$ such that $q_\alpha \Vdash \alpha \in \dot{A}$. By a delta-system argument, let $X_1 \in [X_0]^\kappa$ be such that there is $r \leq p$ such that for all $\alpha \in X_1$, $q_\alpha \restriction \gamma = r$, and for $\alpha \neq \beta$ in X_1, $(\mathrm{supp}(q_\alpha) \setminus \gamma) \cap (\mathrm{supp}(q_\beta) \setminus \gamma) = \emptyset$. For any $q \leq r$ and $y \in [X_1]^\mu$, $q \nVdash \check{y} \cap \dot{A} = \emptyset$. This is because for such q, there is some $\alpha \in y$ such that $(\mathrm{supp}(q_\alpha) \setminus \gamma) \cap \mathrm{supp}(q) = \emptyset$, so q is compatible with q_α. Hence $r \Vdash (\exists X \in [\kappa]^\kappa \cap V)(\forall y \in [X]^\mu \cap V) y \cap \dot{A} \neq \emptyset$. As $\dot{A}$ and p were arbitrary, $\neg Silver(\mu, \kappa)$ is forced.

Now let $\dot{A}$ be a $\mathbb{P}_1$-name such that $1 \Vdash \dot{A} \in [\kappa]^\kappa$, and let $p \in \mathbb{P}_1$ be arbitrary. Form X_0, $\{q_\alpha : \alpha \in X_0\}$, and X_1 like above. We can take a $y \in [X_1]^\mu$ such that $\bigcup_{\alpha \in y} q_\alpha = q \in \mathbb{P}_1$. Then $q \Vdash \check{y} \subseteq \dot{A}$, so q forces $\neg Levy(\mu, \kappa)$. □

Corollary 2.14. *Suppose μ, κ, $\mathbb{P}_0$, and $\mathbb{P}_1$ are as above. Let G be $\mathbb{P}_0$-generic and H be $\mathbb{P}_1$-generic over V. Let $\mathbb{Q} \in V$ be partial order. If $\mathbb{Q}$ forces $Levy(\mu, \kappa)$, then $V[H]$ has no $\mathbb{Q}$-generic, and if $\mathbb{Q}$ forces $Silver(\mu, \kappa)$, then $V[G]$ has no $\mathbb{Q}$-generic. If $\mathbb{Q}$ is κ-c.c. and of size κ, then no κ-closed forcing extension of $V[G]$ or $V[H]$ can introduce a generic for $\mathbb{Q}$.*

Proof. Since $V[H]$ satisfies $\neg Levy$, and $Levy$ is a Σ_1 property with parameters in V, no inner model of $V[H]$ containing V can satisfy $Levy$. Likewise, no inner model of $V[G]$ containing V can satisfy $Silver$. To see that the non-existence of $\mathbb{Q}$-generics is preserved by κ-closed forcing, suppose that for some such forcing $\mathbb{R} \in V[G]$, $r \Vdash_{\mathbb{R}}^{V[G]} \dot{K}$ is $\mathbb{Q}$-generic over V. Since $\mathbb{Q}$ has size κ, we can build a descending sequence $\{r_\alpha : \alpha < \kappa\}$ below r such that for all $q \in \mathbb{Q}$, there is r_α deciding whether $q \in K$. Let $K' = \{q : (\exists \alpha < \kappa) r_\alpha \Vdash q \in \dot{K}\}$. Any maximal antichain $A \in V$ contained in $\mathbb{Q}$ has size $< \kappa$, thus some r_α completely decides $A \cap K$. Since $r_\alpha \Vdash \check{A} \cap \dot{K} \neq \emptyset$, we must have $K' \cap A \neq \emptyset$, so K' is $\mathbb{Q}$-generic over V. The argument for κ-closed forcing over $V[H]$ is the same. □

Theorem 2.15. *Suppose $\mu < \kappa$ are regular and $\alpha^\mu < \kappa$ for all $\alpha < \kappa$. No (μ, κ)-NLC forcing regularly embeds into $A(\mu, \kappa)$. Further, a generic extension by $A(\mu, \kappa)$ has no generic filters for any κ-c.c. forcing $\mathbb{Q}$ such that* $\mathrm{d}(\mathbb{Q} \upharpoonright q) \geq \kappa$ *for all $q \in \mathbb{Q}$.*

Proof. First note that we only need to consider $\mathbb{Q}$ such that $\mathrm{d}(\mathbb{Q} \upharpoonright q) = \kappa$ for all $q \in \mathbb{Q}$. For if $p \in A(\mu, \kappa)$ is such that $p \Vdash \dot{K}$ is $\mathbb{Q}$-generic, then there would be some $q \in \mathcal{B}(\mathbb{Q})$ and some $p' \leq p$ such that $\mathcal{B}(\mathbb{Q}) \upharpoonright q$ completely embeds into $A(\mu, \kappa) \upharpoonright p'$. Since $d(A(\mu, \kappa) = \kappa$, this implies $\mathcal{B}(\mathbb{Q}) \upharpoonright q \leq \kappa$.

Let $\mathbb{Q}$ be any κ-c.c. forcing such that $\mathrm{d}(\mathbb{Q} \restriction q) = \kappa$ for all $q \in \mathbb{Q}$. For any $p \in \mathbb{Q}$, if $(*)$ holds for $\mathbb{Q} \restriction p$, then $p \Vdash \mathit{Silver}$, and otherwise for some $q \leq p$, $q \Vdash \mathit{Levy}$. Thus $\Vdash_{\mathbb{Q}} \mathit{Levy} \vee \mathit{Silver}$. Suppose K is $\mathbb{Q}$-generic over V, and X is $A(\mu, \kappa)$-generic over V. There are two further forcings $\mathbb{R}_0, \mathbb{R}_1$ over $V[X]$ that respectively get filters G, H such that $V[G][X]$ is $\mathbb{P}_0 * \mathrm{Add}(\kappa)$-generic, and $V[H][X]$ is $\mathbb{P}_1 * \mathrm{Add}(\kappa)$-generic. If $V[K] \models \mathit{Levy}$, then $K \notin V[H][X]$, and if $V[K] \models \mathit{Silver}$, then $K \notin V[G][X]$. Thus $V[X]$ has no $\mathbb{Q}$-generics. □

2.2 Construction of a dense ideal

First we will define a useful strengthening of "nicely layered."

Definition. $\mathbb{P}$ *is* (μ, κ)-very nicely layered (with collapses) *when there is a sequence* $\langle \mathbb{Q}_\alpha : \alpha < \kappa \rangle = \mathcal{L}$ *such that:*

(1) $\mathcal{L}$ *witnesses that* $\mathbb{P}$ *is* (μ, κ)*-nicely layered (with collapses),*

(2) $\mathcal{L}$ *is* $\subseteq$*-increasing,*

(3) every subset of $\mathbb{P}$ *of size* $< \mu$ *with a lower bound has an infimum, and*

(4) there is a system of continuous projection maps $\pi_\alpha : \mathbb{P} \to \mathbb{Q}_\alpha$ *such that for each* α, $\pi_\alpha \restriction \mathbb{Q}_\alpha = id$*, and for* $\beta < \alpha < \kappa$, $\pi_\beta = \pi_\beta \circ \pi_\alpha$.

A typical example is the Levy collapse $\mathrm{Col}(\mu, < \kappa)$. In the general case, we will usually abbreviate the action of the projection maps $\pi_\alpha(q)$ by $q \restriction \alpha$. In applying clause (3), we will use the next proposition, proof of which is left to the reader.

Proposition 2.16. *If* $\mathbb{P}$ *is a partial order such that every descending chain of length* $< \mu$ *has an infimum, then every directed subset of size* $< \mu$ *has an infimum.*

Theorem 2.17. *Assume κ carries an almost-huge tower of height δ, and let $j : V \to M$ be given by the tower. Let $\mu < \kappa$ be regular, and let $\kappa \leq \lambda < \delta$. Let X be $A(\mu, \kappa)$-generic, and suppose $\mathbb{P} \in V[X]$ is a partial order such that:*

(1) $\langle \mathbb{Q}_\alpha : \alpha < \delta \rangle$ witnesses that $\mathbb{P}$ is (κ, δ)-very nicely layered, and

(2) for unboundedly many $\alpha < \delta$, $|\mathbb{Q}_\alpha| \geq |\alpha|$ and $\Vdash_{\mathbb{Q}_\alpha} |\mathbb{Q}_\alpha| = \lambda$.

If H is $\mathbb{P}$-generic over $V[X]$, then in $V[X][H]$, there is a normal, κ-complete, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$.

Proof. Let H_X be the $A(\mu, \kappa)$-generic filter computed from X. Let $K \times C$ be $B(\mu, \kappa)/H_X \times \mathrm{Col}(\mu, \lambda)$-generic over $V[X][H]$, and for brevity let $W = V[X][H][K][C]$. Note that $V[X][K] = V[G][X]$, where $G * X$ is some $\mathrm{Col}(\mu, {<}\kappa) * \mathrm{Add}(\kappa)$-generic filter over V. By the distributivity of $B(\mu, \kappa)/H_X$ in $V[X]$, $\mathbb{P}$ and its layers $\mathbb{Q}_\alpha$ are still κ-closed in $V[G][X]$. For $\alpha < \beta$, the relation $\Vdash_{\mathbb{Q}_\alpha}$ "$\mathbb{Q}_\beta/\mathbb{Q}_\alpha$ is κ-closed" holds in $V[G][X]$ because in $V[X]$, $B(\mu, \kappa)/H_X \times \mathbb{Q}_\alpha$ is κ-distributive. Furthermore, since no sequences of length $< \mu$ are added, the forcing given by the definition of $\mathrm{Col}(\mu, \lambda)$ is the same between V, W, and intermediate models.

The forcing to get from $V[G]$ to W is equivalent to $(\mathrm{Add}(\kappa) \times \mathrm{Col}(\mu, \lambda)) * \mathbb{P}$. Let $\mathcal{L}$ be the collection of subforcings of the form $(\mathrm{Add}(\kappa) \times \mathrm{Col}(\mu, \lambda)) * \mathbb{Q}_\alpha$ for $\alpha < \delta$. This sequence then witnesses the (μ, δ)-NLC property in $V[G]$. The closure properties are evident, and since the whole forcing has the δ-c.c., functions from μ to ordinals are indeed captured by these factors. The "with collapses" part of the definition holds because of clause (2) of the hypothesis.

Let $P_0 = \mathcal{P}(\mu)^W$, and consider the submodel $M(P_0)$. In W, $Q_0 = \mathcal{P}(\mathrm{Add}(\delta))^{M(P_0)}$ has cardinality δ. To show this, let $Y \subseteq \delta$ be $\mathrm{Add}(\delta)$-generic over W. By Theorem 2.10, Y is $A(\mu, \delta)$-generic over V, and hence over M since $(\mathrm{Col}(\mu, < \delta) * \mathrm{Add}(\delta))^M = (\mathrm{Col}(\mu, < \delta) *$

$\mathrm{Add}(\delta))^V$ by the closure of M. Since $M[Y]$ thinks $j(\delta)$ is inaccessible, $M[Y] \models |Q_0| < j(\delta)$, so $W[Y] \models |Q_0| = \delta$ since $j(\delta) < (\delta^+)^V$. Since $W \models 2^\mu = \delta$, W and $W[Y]$ have the same cardinals, so $W \models |Q_0| = \delta$. Therefore, working in W, we can inductively build a set $\hat{X} \subseteq \delta$ that is $\mathrm{Add}(\delta)$-generic over $M(P_0)$ with $\hat{X} \cap \kappa = X$. By Lemma 2.9, $\hat{X}$ is $A(\mu, \delta)$-generic over $M[G]$. A further forcing produces $G' \supseteq G$, such that $G' * \hat{X}$ is $\mathrm{Col}(\mu, {<}\delta) * \mathrm{Add}(\delta)$-generic over M, so have an elementary $\hat{j} : V[G][X] \to M[G'][\hat{X}]$ extending j. By elementarity, for the corresponding filters H_X and $H_{\hat{X}}$ on the respective algebras $A(\mu, \kappa)^V$ and $A(\mu, \delta)^M$, we have $j[H_X] \subseteq H_{\hat{X}}$. Hence we can define in W the restricted elementary embedding $\hat{j} : V[X] \to M[\hat{X}]$.

Now we wish to extend $\hat{j}$ to have domain $V[X][H]$. As in the argument for Lemma 2.8, every element of $(\mathrm{Ord}^\mu)^W$ is coded by some element of M and some $y \subseteq \mu$ coded in $\hat{X}$, so $M[\hat{X}]$ is closed under $< \delta$ sequences from W. Consequently, $H \cap \mathbb{Q}_\alpha$ and $\hat{j}[H \cap \mathbb{Q}_\alpha]$ are in $M[\hat{X}]$ for all $\alpha < \delta$. Also, $M[\hat{X}] \models$ "$\hat{j}(\mathbb{P})$ is $(\delta, j(\delta))$-very nicely layered." Each $\hat{j}[H \cap \mathbb{Q}_\alpha]$ is a directed set of size μ in $M[\hat{X}]$, so it has an infimum $m_\alpha \in \hat{j}(\mathbb{Q}_\alpha)$.

Let $\langle A_\alpha : \alpha < \delta \rangle \in W$ enumerate the maximal antichains of $\hat{j}(\mathbb{P})$ from $M[\hat{X}]$. (There are only δ many because $M[\hat{X}]$ thinks this partial order has inaccessible size $j(\delta)$ and is $j(\delta)$-c.c.) Inductively define an increasing sequence of ordinals $\langle \alpha_i \rangle_{i<\delta} \subseteq \delta$, and a corresponding decreasing sequence of conditions $\langle p_i \rangle_{i<\delta} \subseteq \hat{j}(\mathbb{P})$ as follows.

Assume as the induction hypothesis that we have defined the sequences up to i, and for all $\xi < i$ and all $\alpha < \delta$, p_ξ is compatible with m_α, and for all $\xi < i$, there is some $a \in A_\xi$ such that $p_\xi \leq a$. Let $q_i = \inf_{\xi<i} p_\xi$. This is compatible with all m_α because for all α, $\langle q_\xi \wedge m_\alpha : \xi < i \rangle$ is a descending chain in $\hat{j}(\mathbb{P})$. Let $\alpha_i \geq \sup_{\xi<i} \alpha_\xi$ be such that $A_i \subseteq \hat{j}(\mathbb{Q}_{\alpha_i})$ and $q_i \in \hat{j}(\mathbb{Q}_{\alpha_i})$. This is possible by the chain condition and because $j[\delta]$ is cofinal in $j(\delta)$. Choose $p_i \in \hat{j}(\mathbb{Q}_{\alpha_i})$ below $q_i \wedge m_{\alpha_i}$ and some $a \in A_i$. p_i is compatible with all m_α, because for any $\alpha > \alpha_i$, $m_\alpha \restriction j(\alpha_i) = m_{\alpha_i}$. This is because for any $\beta < \alpha < \delta$, $H \cap \mathbb{Q}_\beta = \{p \restriction \beta : p \in H \cap \mathbb{Q}_\alpha\}$, and the projections are continuous.

The upward closure of the sequence $\langle p_i \rangle_{i<\delta}$ is a filter $\hat{H}$ which is $\hat{j}(\mathbb{P})$-generic over $M[\hat{X}]$. For all $p \in H$, $\hat{j}(p) \in \hat{H}$ since there is some $m_\alpha \leq \hat{j}(p)$. Thus we get an extended elementary embedding $\hat{j} : V[X][H] \to M[\hat{X}][\hat{H}]$. In W, we define an ultrafilter U over $(\mathcal{P}(\mathcal{P}_\kappa \lambda))^{V[X][H]}$: let $A \in U$ iff $j[\lambda] \in \hat{j}(A)$. Note that $j[\lambda] \in \mathcal{P}_{j(\kappa)}(j(\lambda))^{M[\hat{X}][\hat{H}]}$. U is κ-complete and normal with respect to functions in $V[X][H]$. If $f : \mathcal{P}_\kappa(\lambda) \to \lambda$ is a regressive function in $V[X][H]$ on a set $A \in U$, then $\hat{j}(f)(j[\lambda]) = j(\alpha)$ for some $\alpha < \lambda$, so $\{z \in A : f(z) = \alpha\} \in U$.

Now the forcing to obtain U was $\mathbb{Q} = B(\mu, \kappa)/H_X \times \mathrm{Col}(\mu, \lambda)$, the product of a κ-dense and a λ-dense partial order. In $V[X][H]$, let $e : \mathcal{P}(\mathcal{P}_\kappa \lambda) \to \mathcal{B}(\mathbb{Q})$ be defined by $e(A) = ||\check{A} \in \dot{U}||$. Let I be the kernel of e. I is clearly a normal, κ-complete ideal. e lifts to a boolean embedding of $\mathcal{P}(\mathcal{P}_\kappa \lambda)/I$ into $\mathcal{B}(\mathbb{Q})$. Since $\mathbb{Q}$ is λ^+-c.c., I is λ^+-saturated. If $\langle [A_\alpha] : \alpha < \lambda \rangle$ is a maximal antichain in $\mathcal{P}_\kappa(\lambda)/I$, then ∇A_α is the least upper bound and is in the dual filter to I. $e(\nabla A_\alpha) = ||\nabla A_\alpha \in \dot{U}|| = 1$, and this is the least upper bound in $\mathcal{B}(\mathbb{Q})$ to $\{e(A_\alpha) : \alpha < \lambda\}$. This is because if there were a generic extension in which all $A_\alpha \notin U$, then $\nabla A_\alpha \notin U$ as well since U is normal with respect to sequences from $V[X][H]$. Therefore e is a complete embedding, and thus I is λ-dense. □

When λ is regular, it is easy to find a forcing $\mathbb{P} \in V[X]$ satisfying the hypotheses–for example $\mathrm{Col}(\lambda, {<}\delta)$. Though we do not explicitly assume λ is regular, it is actually required for the argument. Applying the theorem to the case of singular λ would require, at minimum, turning an inaccessible into the successor of a singular cardinal with a countably closed forcing. This was recently observed to be impossible by Asaf Karagila and Yair Hayut, who communicated their argument to the author in private correspondence.

Theorem 2.18 (Karagila-Hayut). *If $\delta > \kappa = \lambda^+$ and $\mathbb{P}$ is a forcing preserving all stationary subsets of $\kappa \cap \mathrm{cof}(\omega)$, then $\mathbb{P}$ cannot force that λ is singular and $\delta = \lambda^+$.*

Proof. Let $G \subseteq \mathbb{P}$ be generic. If λ is singular and δ is its successor in $V[G]$, then for some $\mu < \lambda$, $\mathrm{cf}(\kappa) = \mu$. In V, choose a collection $\{S_\alpha : \alpha < \lambda^+\}$ of disjoint stationary subsets of

$\kappa \cap \text{cof}(\omega)$. In $V[G]$ choose a club $\{\beta_i : i < \mu\} \subseteq \kappa$. For each $i < \mu$, there is at most one $\alpha < \kappa$ such that $\beta_i \in S_\alpha$, and for each $\alpha < \kappa$, there is an $i < \mu$ such that $\beta_i \in S_\alpha$. Thus there is a surjection $f : \mu \to \kappa$ in $V[G]$, and λ is not a cardinal in $V[G]$, a contradiction. $\square$

We can also characterize the exact structure of $\mathcal{P}(\mathcal{P}_\kappa \lambda)/I$, for the ideals produced via Theorem 2.17. First note the following about the ground model embedding $j : V \to M$. M is the direct limit of the coherent system of α-supercompactness embeddings $j_\alpha : V \to M_\alpha$ for $\alpha < \delta$. Every member of M_α is represented as $j_\alpha(f)(j_\alpha[\alpha])$ for some function $f \in V$ with domain $\mathcal{P}_\kappa(\alpha)$. If $k_\alpha : M_\alpha \to M$ is the factor map such that $j = k_\alpha \circ j_\alpha$, then the critical point of k_α is above α, so $k_\alpha(x) = k_\alpha[x]$ when $M_\alpha \models |x| \leq |\alpha|$. Since M is the direct limit, for any $x \in M$, there is some $\alpha < \delta$ and some $f \in V$ such that

$$x = k_\alpha([f]) = k_\alpha(j_\alpha(f)(j_\alpha[\alpha])) = j(f)(k_\alpha(j_\alpha[\alpha])) = j(f)(j[\alpha])).$$

Let $U \subseteq \mathcal{P}(\mathcal{P}_\kappa \lambda)/I$ be generic over $V[X][H]$, and let $j_U : V \to N$ be the generic ultrapower embedding. Since $e : \mathcal{P}(\mathcal{P}_\kappa \lambda)/I \to \mathcal{B}(\mathbb{Q})$ is a complete embedding, forcing with $\mathcal{B}(\mathbb{Q})/e[U]$ over $V[X][H][U]$ produces a model W as above. Notice that the definition of e and U makes $A \in U$ iff $j[\lambda] \in \hat{j}(A)$. Hence we can define an elementary embedding $k : N \to M[\hat{X}][\hat{H}]$ by $k([f]) = \hat{j}(f)(j[\lambda])$, and we have $\hat{j} = k \circ j_U$.

What is the critical point of k? Since $N \models \mu^+ = \delta$, certainly it must be at least δ. Let β be any ordinal. There is some α such that $\lambda \leq \alpha < \delta$ and some $f \in V$ such that $\beta = j(f)(j[\alpha])$. Let $b : \lambda \to \alpha$ be a bijection in $V[X][H]$. Then $\beta = j(f)(\hat{j}(b)[j[\lambda]])$. Furthermore, $j[\lambda] = k(j_U[\lambda])$. Therefore, $\beta = k(j_U(f)(j_U(b)[j_U[\lambda]]))$. Thus $\beta \in \text{ran}(k)$, and so k does not have a critical point. Therefore, $N = M[\hat{X}][\hat{H}]$. By the closure of $M[\hat{X}][\hat{H}]$, the generic $K \times C$ for $\mathbb{Q}$ is in $M[\hat{X}][\hat{H}] = N \subseteq V[X][H][U]$. So the quotient $\mathcal{B}(\mathbb{Q})/e[U]$ is trivial and $\mathcal{P}(\mathcal{P}_\kappa \lambda)/I \cong \mathcal{B}(\mathbb{Q}) \restriction q$ for some q.

2.2.1 Minimal generic supercompactness

Generalizing supercompactness, we will say cardinal κ is generically supercompact when for every $\lambda \geq \kappa$, there is a forcing $\mathbb{P}$ such that whenever $G \subseteq \mathbb{P}$ is generic, there is an elementary embedding $j : V \to M$, where M is a transitive class in $V[G]$, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $M^\lambda \cap V[G] \subseteq M$. We note that unlike in the case of non-generic supercompactness, the condition that $j[\lambda] \in M$ does not imply that M is closed under λ-sequences from $V[G]$. Whenever a supercompact κ is turned into a successor cardinal by a κ-c.c. forcing, we'll have that for all $\lambda \geq \kappa$, there is a normal, fine, precipitous ideal on $\mathcal{P}_\kappa(\lambda)$ whose generic embeddings always extend the original supercompactness embedding. But if $j : V \to M$ is an embedding coming from a normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$, then $2^{\lambda^{<\kappa}} < j(\kappa) < (2^{\lambda^{<\kappa}})^+$. If $\kappa = \mu^+$ in a generic extension $V[H]$, and a further extension gives $\hat{j} : V[H] \to M[\hat{H}] \subseteq V[H][G]$ extending j, then $M[\hat{H}]$ is not closed under λ-sequences from $V[H][G]$. This is because $|\lambda| = |j(\kappa)| = \mu$ in $V[H][G]$, while $M[\hat{H}]$ thinks $j(\kappa)$ is a cardinal.

Stronger properties of ideals on $\mathcal{P}_\kappa(\lambda)$ are needed to give genuine generic supercompactness. One such property is λ^+-saturation, which is implied by λ-density. We now sketch how to get a model in which there is a successor cardinal κ such that for all regular $\lambda \geq \kappa$, there is a normal, κ-complete, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$. Start with a super-almost-huge cardinal κ and a regular $\mu < \kappa$. The first part of the forcing is $A(\mu, \kappa)$. Then we do a proper class iteration, which we prefer to describe instead as an iteration up to an inaccessible $\delta > \kappa$ such that $V_\delta \models \kappa$ is super-almost-huge.

Let $T = \{\alpha < \delta : \kappa$ carries an almost-huge tower of height $\alpha\}$. Let C be the closure of T, and let $\langle \alpha_\beta \rangle_{\beta < \delta}$ be its continuous increasing enumeration. Over $V^{A(\mu,\kappa)}$, let $\mathbb{P}_\delta$ be the Easton-support limit of the following:

- Let $\mathbb{P}_0 = \mathrm{Col}(\kappa, < \alpha_0)$.

- If β is zero or a successor ordinal, let $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \mathrm{Col}(\alpha_\beta, < \alpha_{\beta+1})$.
- If β is a limit ordinal such that α_β is singular, let $\mathbb{P}_{\beta+1} = \mathrm{Col}(\alpha_\beta^+, < \alpha_{\beta+1})$.
- If β is a limit ordinal such that α_β is regular, let $\mathbb{P}_{\beta+1} = \mathrm{Col}(\alpha_\beta, < \alpha_{\beta+1})$.

It is routine to verify that this iteration preserves the regularity of the members of T, the successors of the singular limit points of T, and the regular limit points of T. Further, the set of non-limit-points of T becomes the set of successors of regular cardinals between κ and δ.

Let $X \subseteq \kappa$ be $A(\mu, \kappa)$-generic over V, and let $H \subseteq \mathbb{P}_\delta$ be generic over $V[X]$. Suppose $\kappa \leq \lambda < \delta$, and λ is regular in $V[X][H]$. Then there is some successor ordinal $\beta < \delta$ such that $\alpha_\beta \in T$ and $\alpha_\beta = \lambda^+$. Consider the subforcing $A(\mu, \kappa) * \mathbb{P}_\beta = (A(\mu, \kappa) * \mathbb{P}_{\beta-1}) * \mathrm{Col}(\lambda, < \alpha_\beta)$. The forcing $\mathbb{P}_\beta$ is (κ, α_β)-very nicely layered in $V[X]$.

If $j : V \to M_\beta$ is an almost-huge embedding with critical point κ and $j(\kappa) = \alpha_\beta$, then by Theorem 2.17, there is a normal, κ-complete, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$ in $V[X][H_\beta]$. Now note that the tail-end forcing $\mathbb{P}_{\beta,\delta}$ is α_β-closed. Since $\lambda^{<\kappa} = \lambda$ in $V[X][H_\beta]$, no new subsets of $\mathcal{P}_\kappa(\lambda)$ are added by the tail. The collection $\{A_\alpha : \alpha < \lambda\}$ witnessing the λ-density of I retains this property, as this is a local property of the boolean algebra $\mathcal{P}_\kappa(\lambda)/I$ and $\{A_\alpha : \alpha < \lambda\}$. Normality and completeness of I are likewise preserved.

Because of the generality of the hypotheses of Theorem 2.17, this method is quite flexible. It can done by iterating collapsing posets other than the Levy collapse, or by using products rather than iterations.

2.2.2 Dense ideals on successive cardinals?

At the time of this writing, it is unknown whether there can exist simultaneously a normal κ-dense ideal on κ and a normal κ^+-dense ideal on κ^+. The following is the current best approximation.

Suppose $\langle \kappa_n : n < \omega \rangle$ is a sequence of cardinals such that for all n, κ_n carries an almost-huge tower of height κ_{n+1}. Such a sequence will be called an *almost-huge chain*. Obviously, extending this to sequences of length longer than ω requires an extra idea; perhaps we just stack one ω-chain above another, or maybe postulate some relationship between the ω-chains. By Theorem 2.4, such chains occur quite often below a huge cardinal.

Suppose $\langle \kappa_n : 0 < n < \omega \rangle$ is an almost-huge chain, and $\mu < \kappa_1$ is regular. Consider the full-support iteration $\mathbb{P}$ of $\langle \mathbb{P}_n : n < \omega \rangle$, where $\mathbb{P}_0 = A(\mu, \kappa_1)$, and for all $n < \omega$, $\mathbb{P}_{n+1} = \mathbb{P}_n * A(\kappa_n, \kappa_{n+1})$. The stage $\mathbb{P}_1 = A(\mu, \kappa_1) * A(\kappa_1, \kappa_2)$ regularly embeds into $A(\mu, \kappa_1) * (\mathrm{Col}(\kappa_1, <\kappa_2) * \mathrm{Add}(\kappa_2))$. The first two stages here add a normal κ_1-dense ideal on κ_1 and make $\kappa_1 = \mu^+$, $\kappa_2 = \mu^{++}$. The third stage preserves this since it adds no subsets of κ_1. By Lemma 2.8, the quotient forcing $\mathbb{Q}$ to get from $V^{\mathbb{P}_1}$ to this three-stage extension is κ_2-distributive. Now the tail-end forcing $\mathbb{P}/\mathbb{P}_1$ is κ_2-strategically closed. Since $\mathbb{Q}$ does not add any plays of the relevant game of length $<\kappa_2$, $\mathbb{P}/\mathbb{P}_1$ remains κ_2-strategically closed in $V^{\mathbb{P}_1 * \mathbb{Q}}$, so forcing with it preserves the κ_1-dense ideal on κ_1. Also, $\mathbb{Q}$ remains κ_2-distributive in $V^{\mathbb{P}}$, since $\mathbb{Q} \times (\mathbb{P}/\mathbb{P}_1)$ is κ_2-distributive in $V^{\mathbb{P}_1}$. It thus remains the case in $V^{\mathbb{P}}$ that there is a κ_2-distributive forcing adding a normal κ_1-dense ideal on κ_1.

Similarly, consider $V^{\mathbb{P}_n}$ for $n > 1$. $\mathbb{P}_n = \mathbb{P}_{n-2} * (A(\kappa_{n-1}, \kappa_n) * A(\kappa_n, \kappa_{n+1}))$. Since $|\mathbb{P}_{n-2}| = \kappa_{n-1}$ (or μ for $n = 2$), κ_n retains an almost-huge tower of height κ_{n+1} in $V^{\mathbb{P}_{n-2}}$. Thus the same argument applies: In $V^{\mathbb{P}_n}$, there is a κ_{n+1}-distributive forcing adding a normal κ_n-dense ideal on κ_n, and this remains true in $V^{\mathbb{P}}$. Therefore, we obtain a model in which for all $n > 0$, there is a μ^{+n+1}-distributive forcing adding a normal μ^{+n}-dense ideal on μ^{+n}.

By repeating this with a tall enough stack of almost-huge chains, we obtain the consistency of ZFC with the statement, "For all regular cardinals κ, there is a κ^{++}-distributive forcing adding a normal κ^+-dense ideal on κ^+."

Chapter 3

Structural constraints

Saturated ideals have a strong influence over the combinatorial structure of the universe in their vicinity. Phenomena of this type may also be viewed as the universe imposing constraints on the structural properties of ideals. Below are some of the most interesting known results to this effect. Proofs can be found in [13].

(1) (Tarski) If I is a nowhere-prime ideal which is κ-complete and μ-saturated for some $\mu < \kappa$, then $2^{<\mu} \geq \kappa$.

(2) (Jech-Prikry) If $\kappa = \mu^+$, $2^\mu = \kappa$, and there is a κ-complete, κ^+-saturated ideal on κ, then $2^\kappa = \kappa^+$.

(3) (Jech-Prikry) If $\kappa = \mu^+$, and there is a κ-complete, κ^+-saturated ideal on κ, then there are no κ-Kurepa trees.

(4) (Woodin) If there is a countably complete, ω_1-dense ideal on ω_1, then there is a Suslin tree.

(5) (Woodin) If there is a countably complete, uniform, ω_1-dense ideal on ω_2, then $2^\omega = \omega_1$. (Uniform means that all sets of size $< \omega_2$ are in the ideal–equivalent to fineness.)

(6) (Shelah) If $2^\omega < 2^{\omega_1}$, then NS_{ω_1} is not ω_1-dense.

(7) (Gitik-Shelah) If I is a κ-complete, nowhere-prime ideal, then $\mathrm{d}(I) \geq \kappa$.

We note that result (2) easily generalizes to the following: If $\kappa = \mu^+$, $2^\mu = \kappa$, and there is a normal, fine, κ-complete, λ^+-saturated ideal on $\mathcal{P}_\kappa(\lambda)$, then $2^\lambda = \lambda^+$.

If no requirements are made for the ideal I and the set Z on which it lives, almost no structural constraints on quotient algebras remain. The following strengthens a folklore result, probably known to Sikorski. The argument was supplied by Don Monk in personal correspondence.

Proposition 3.1. *Let $\mathbb{B}$ be a complete boolean algebra, and let κ be a cardinal such that $2^\kappa \geq |\mathbb{B}|$. There is a uniform ideal I on κ such that $\mathcal{B} \cong \mathcal{P}(\kappa)/I$.*

Proof. Let κ, $\mathbb{B}$ be as hypothesized. By the theorem of Fichtenholz-Kantorovich and Hausdorff (see [21], Lemma 7.7), there exists a family F of 2^κ many subsets of κ such that for any $x_1, ..., x_n, y_1, ..., y_m \in F$, $x_1 \cap ... \cap x_n \cap (\kappa \setminus y_1) \cap ... \cap (\kappa \setminus y_m)$ has size κ. F generates a free algebra: closing F under finitary set operations gives a family of sets G such that any equation holding between elements of G expressed as boolean combinations of elements of F holds in all boolean algebras. If we pick any surjection $h_0 : F \to \mathbb{B}$ and extend it to $h_1 : G \to \mathbb{B}$ in the obvious way, then h_1 will be a well-defined homomorphism.

Let I_{bd} be the ideal of bounded subsets of κ. Since all elements of G are either empty or have cardinality κ, $G \cong G/I_{bd}$, so h_1 has an extension h_2 from the algebra generated by $G \cup I_{bd}$ to $\mathbb{B}$, where $h_2(x) = 0$ for all $x \in I_{bd}$. Finally, by Sikorski's extension theorem, there is a further extension to a homomorphism $h_3 : \mathcal{P}(\kappa) \to \mathbb{B}$. The kernel of h_3 is an ideal I such that $\mathcal{P}(\kappa)/I \cong \mathbb{B}$. □

3.1 Cardinal arithmetic and ideal structure

A careful examination of the proof of Woodin's theorem (5) shows that ω_2 can be replaced by any ω_n, $2 \leq n < \omega$. Aside from that, Woodin's argument is rather specific to the cardinals involved. In [13], Foreman asked (Open Question 27) whether the analogous statement holds one level up:

Question (Foreman)**.** *Does the existence of an ω_2-complete, ω_2-dense, uniform ideal on ω_3 imply that $2^{\omega_1} = \omega_2$?*

To answer this, we invoke an easy preservation lemma about ideals under small forcing. If I is an ideal, $\mathbb{P}$ is a partial order, and $G \subseteq \mathbb{P}$ is generic, then $\bar{I}$ denotes the ideal generated by I in $V[G]$, i.e. $\{X : (\exists Y \in I) X \subseteq Y\}$.

Lemma 3.2. *Suppose I is a κ-complete ideal on $Z \subseteq \mathcal{P}(X)$, $\mathbb{P}$ is partial order, and G is $\mathbb{P}$-generic.*

(1) If $\mathrm{sat}(\mathbb{P}) \leq \kappa$, then $\bar{I}$ is κ-complete in $V[G]$.

(2) If $\mathrm{d}(\mathbb{P}) < \kappa$, then $\mathrm{d}(\bar{I})^{V[G]} \leq \mathrm{d}(I)^V$.

Proof. For (1), let $\dot{s}$ be a $\mathbb{P}$-name for a sequence of elements of $\bar{I}$ of length less than κ. By κ-saturation, let $\beta < \kappa$ be such that $1 \Vdash dom(\dot{s}) \leq \beta$. For each $\alpha < \beta$, let A_α be a maximal antichain such that for $p \in A_\alpha$, $p \Vdash \dot{s}(\alpha) \subseteq \check{b}^p_\alpha$, where $b^p_\alpha \in I$. Then $B = \bigcup_{p,\alpha} b^p_\alpha \in I$, and $1 \Vdash \bigcup \dot{s} \subseteq \check{B}$.

For (2), let $D \subseteq \mathbb{P}$ be a dense set of size less than κ, and let $A \in \bar{I}^+$. Then $A = \bigcup_{d \in D \cap G} \{z : d \Vdash z \in \dot{A}\}$. By (1), there is some $d \in D$ such that $\{z : d \Vdash z \in \dot{A}\} \notin I$. This shows that $(\mathcal{P}(Z)/I)^V$ is dense in $(\mathcal{P}(Z)/\bar{I})^{V[G]}$, and the conclusion follows. □

Corollary 3.3. *If there is a κ^+-complete, κ^+-dense, uniform ideal on κ^{++}, then $2^\kappa = \kappa^+$.*

Proof. Suppose for a contradiction that $f : \mathcal{P}(\kappa) \to \kappa^{++}$ is a surjection. Let $\mathbb{P} = \mathrm{Col}(\omega, \kappa)$, and let G be $\mathbb{P}$-generic. Since $\mathrm{d}(\mathbb{P}) = \kappa$, Lemma 3.2 implies that $\bar{I}$ is κ^+-complete and κ^+-dense in $V[G]$. Furthermore, in $V[G]$, $\kappa^+ = \omega_1$ and $\kappa^{++} = \omega_2$. Thus Woodin's theorem implies that $V[G] \models \mathrm{CH}$. However, f witnesses the failure of CH, a contradiction. $\square$

Another interesting constraint can be derived from the following:

Theorem 3.4 (Shelah [30])**.** *Suppose $V \subseteq W$ are models of ZFC. If κ is a regular cardinal in V, and $\mathrm{cf}(\kappa) \neq \mathrm{cf}(|\kappa|)$ in W, then $(\kappa^+)^V$ is not a cardinal in W.*

Corollary 3.5. *If $\kappa = \mu^+$, $\lambda \geq \kappa$ is regular, and I is a normal, fine, κ-complete, λ^+-saturated ideal on $\mathcal{P}_\kappa(\lambda)$, then $\{z : \mathrm{cf}(z) = \mathrm{cf}(\mu)\} \in I^*$.*

Proof. Let G be a generic ultrafilter extending I^*. Since $\mathrm{crit}(j) = \kappa$ and λ^+ is preserved, $j(\kappa) = \lambda^+$, and $|\lambda| = \mu$ in $V[G]$. By Shelah's theorem, $\mathrm{cf}(\lambda) = \mathrm{cf}(\mu)$ in $V[G]$ and in the ultrapower M since $M^\mu \cap V[G] \subseteq M$. Since $1 \Vdash [id] = j[\lambda]$, Łoś's theorem gives $\{z : \mathrm{cf}(z) = \mathrm{cf}(\mu)\} \in I^*$. $\square$

Theorem 3.6. *Suppose $\kappa = \mu^+$, and I is a normal, fine, κ^+-saturated ideal on κ. Then $\mathcal{P}(\kappa)/I$ is $\mathrm{cf}(\mu)$-distributive iff $\mu^{<\mathrm{cf}(\mu)} = \mu$.*

Proof. Suppose $\mathcal{P}(\kappa)/I$ is $\mathrm{cf}(\mu)$-distributive, and let $\{f_\alpha : \alpha < \delta\}$ be an enumeration of $[\mu]^{<\mathrm{cf}(\mu)}$, where δ is a cardinal. If $\mu < \delta$, then for any $\mathcal{P}(\kappa)/I$-generic G, $([\mu]^{<\mathrm{cf}(\mu)})^V$ is a proper subset of $([\mu]^{<\mathrm{cf}(\mu)})^{V[G]}$, since $j[\delta] \neq j(\delta)$. This contradicts the distributivity of $\mathcal{P}(\kappa)/I$.

Since $\mathcal{P}(\kappa)/I$ is κ^+-saturated, it is $\mathrm{cf}(\mu)$-distributive iff it is $(\mathrm{cf}(\mu), \kappa)$-distributive. Let G be $\mathcal{P}(\kappa)/I$-generic and let M be the generic ultrapower. Let $\beta < \mathrm{cf}(\mu)$, and suppose $f \in V[G]$ is a function from β to κ. By Theorem 1.16, $f \in M$. By Corollary 3.5, $M \models \mathrm{cf}(\kappa) = \mathrm{cf}([id]) = \mathrm{cf}(\mu)$. Thus there is a $\gamma < \kappa$ such that $\mathrm{ran}(f) \subseteq \gamma$. Observe that $j({}^\beta\gamma) = ({}^\beta\gamma)^M = ({}^\beta\gamma)^V$, since $\mu^\beta < \kappa$. Hence $f \in V$. $\square$

3.2 Stationary reflection

A stationary subset S of a regular cardinal κ is said to reflect if there is some $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α. A collection of stationary subsets $\{S_i : i < \delta\}$ of κ is said to reflect simultaneously if there is some $\alpha < \kappa$ if $S_i \cap \alpha$ is stationary for all $i < \delta$. It is well known that if $\kappa = \mu^+$ and X is a set of regular cardinals below μ, then the statement that every stationary subset of $\{\alpha < \kappa : \mathrm{cf}(\alpha) \in X\}$ reflects contradicts $\square_\mu$, and the statement that every pair of stationary subsets of $\{\alpha < \kappa : \mathrm{cf}(\alpha) \in X\}$ reflect simultaneously contradicts the weaker principle $\square(\kappa)$.

Theorem 3.7. *Suppose there is a κ^+-complete, κ^{++}-saturated, uniform ideal on κ^{+n} for some $n \geq 2$. Then for $2 \leq m \leq n$, every collection of κ many stationary subsets of κ^{+m} contained in* $\mathrm{cof}(\leq \kappa)$ *reflects simultaneously.*

Proof. Suppose I is such an ideal and $j : V \to M \subseteq V[G]$ is a generic embedding arising from the ideal. The critical point of j is κ^+, and all cardinals above κ^+ are preserved. Since I is uniform, and there is a family of κ^{+n+1} many almost-disjoint functions from κ^{+n} to κ^{+n}, $j(\kappa^{+n}) \geq (\kappa^{+n+1})^V$. The first $n-1$ cardinals in V above κ must map onto the first $n-1$ cardinals in M above κ. But in M, there are at least $n-1$ cardinals in the interval $(\kappa, (\kappa^{+n+1})^V)$ since all cardinals above κ^+ are preserved. Thus if $j(\kappa^{+n}) > (\kappa^{+n+1})^V$, then κ^{+n+1} would be collapsed. So for $1 \leq m \leq n$, $j(\kappa^{+m}) = (\kappa^{+m+1})^V$.

Let $\{S_\alpha : \alpha < \kappa\}$ be stationary subsets of κ^{+m} concentrating on $\mathrm{cof}(\leq \kappa)$, where $2 \leq m \leq n$. By the κ^{++}-chain condition, these sets remain stationary in $V[G]$. By the above remarks, $\gamma = \sup(j[\kappa^{+m}]) < j(\kappa^{+m})$. For each α, $j \restriction S_\alpha$ is continuous since $\kappa < \mathrm{crit}(j)$. For each α, let C_α be the closure of S_α. In $V[G]$, we can define a continuous increasing function $f : C_\alpha \to \gamma$ extending $j \restriction S_\alpha$ by sending $\sup(S_\alpha \cap \beta)$ to $\sup(j[S_\alpha \cap \beta])$ when β is a limit point of S_α. This shows that $j[S_\alpha]$ is stationary in γ. Now M may not have $j[S_\alpha]$ as an element, but it satisfies that $j(S_\alpha) \cap \gamma$ is stationary in γ. Furthermore, $j(\{S_\alpha : \alpha < \kappa\}) = \{j(S_\alpha) : \alpha < \kappa\}$, and M sees that these all reflect at γ. By elementarity, the S_α have a common reflection point. □

Proposition 3.8. *Suppose μ, κ, λ are regular cardinals such that $\omega < \mu < \kappa = \mu^+ < \lambda$, and I is an ideal on $\mathcal{P}_\kappa(\lambda)$ as in Theorem 2.17. Then every collection $\{S_i : i < \mu\}$ of stationary subsets of $\lambda \cap \mathrm{cof}(\omega)$ reflects simultaneously.*

Proof. The algebra $\mathcal{P}(\mathcal{P}_\kappa(\lambda))/I$ is isomorphic $\mathcal{B}(\mathbb{P} \times \mathbb{Q})$, where $\mathbb{P}$ is κ-dense and $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ is countably closed." Forcing with $\mathbb{P} \times \mathbb{Q}$ thus preserves the stationarity of any subset of $\lambda \cap \mathrm{cof}(\omega)$. If $j : V \to M \subseteq V[G]$ is a generic embedding arising from the ideal, then since $j[\lambda] \in M$ and M thinks $j(\lambda)$ is regular, $\gamma = \sup(j[\lambda]) < j(\lambda)$. The restriction of j to each S_i is continuous, and as above we may define in $V[G]$ a continuous increasing function from the closure of S_i into γ, showing $j[S_i]$ is stationary in γ for each i. Thus $M \models (\forall i < \mu) j(S_i) \cap \gamma$ is stationary, so by elementarity, the collection reflects simultaneously. □

3.3 Nonregular ultrafilters

The computation of the cardinality of ultrapowers is an old problem of model theory. Originally, it was conjectured that if μ, κ are infinite cardinals, and U is a countably incomplete uniform ultrafilter on κ, then $|\mu^\kappa/U| = \mu^\kappa$ [6]. It was shown by Donder [9] that this conjecture holds in the core model below a measurable cardinal. A key tool in such computing the size of ultrapowers is the notion of regularity:

Definition. *An ultrafilter U on Z is called* (μ, κ)-regular *if there is a sequence $\langle A_\alpha : \alpha < \kappa \rangle \subseteq U$ such that for any $Y \subseteq \kappa$ of order type μ, $\bigcap_{\alpha \in Y} A_\alpha = \emptyset$.*

Theorem 3.9 (Keisler [23]). *Suppose U is a (μ, κ)-regular ultrafilter on Z, witnessed by $\langle A_\alpha : \alpha < \kappa \rangle$. For each $z \in Z$, let $\beta_z = \mathrm{ot}(\{\alpha : z \in A_\alpha\}) < \mu$. Then for any sequence of ordinals $\langle \gamma_z : z \in Z \rangle$, we have $|\prod \gamma_z^{\beta_z}/U| \geq |\prod \gamma_z/U|^\kappa$.*

Obviously any uniform ultrafilter on a cardinal κ is (κ,κ)-regular. Also, any fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$ is (κ,λ)-regular, as witnessed by $\langle \hat{\alpha} : \alpha < \lambda \rangle$. Much can be seen by exploiting a connection between dense ideals and nonregular ultrafilters.

Lemma 3.10 (Huberich [18])**.** *Suppose $\mathbb{B}$ is a complete boolean algebra of density κ, where κ is regular. Then there is an ultrafilter U on $\mathbb{B}$ such that whenever $X \subseteq \mathbb{B}$ and $\sum X \in U$, then there is $Y \subseteq X$ such that $|Y| < \kappa$ and $\sum Y \in U$.*

Proof. Let $D = \{d_\alpha : \alpha < \kappa\}$ be dense in $\mathbb{B}$. For any maximal antichain $A \subseteq \mathbb{B}$, let $\gamma_A > 0$ be least such that for all $\alpha < \gamma_A$, there are $\beta < \gamma_A$ and $a \in A$ such that $d_\beta \leq d_\alpha \wedge a$. Let $C_A = \{d \in D \restriction \gamma_A : (\exists a \in A) d \leq a\}$. Let $F = \{\sum C_A : A \text{ is a maximal antichain}\}$.

We claim F has the finite intersection property. Let $A_1, ..., A_n$ be maximal antichains. We may assume $\gamma_{A_1} \leq ... \leq \gamma_{A_n}$. Let $d_{\alpha_1} \leq d_0 \wedge a_1$ for some $a_1 \in A_1$, where $\alpha_1 < \gamma_{A_1}$. Let $d_{\alpha_2} \leq d_{\alpha_1} \wedge a_2$ for some $a_2 \in A_2$, where $\alpha_2 < \gamma_{A_2}$. Proceeding inductively, we get a descending chain $d_{\alpha_1} \leq ... \leq d_{\alpha_n}$, where each $d_{\alpha_i} \leq a_i$ for some $a_i \in A_i$. Thus $d_{\alpha_n} \leq \sum C_{A_1} \wedge ... \wedge \sum C_{A_n}$.

Let $U \supseteq F$ be any ultrafilter. If $\sum X \in U$, then we can find an antichain A that is maximal below $\sum X$ such that $(\forall a \in A)(\exists x \in X) a \leq x$. Extending A it to a maximal antichain A', we have $\sum C_{A'} \in F$. Since $|C_{A'}| < \kappa$, the conclusion follows. □

Lemma 3.11. *Suppose $\kappa = \mu^+$, λ is regular, and I is a normal and fine, κ-complete, λ-dense ideal on $Z \subseteq \mathcal{P}_\kappa(\lambda)$. Then any ultrafilter $U \supseteq I^*$ given by Lemma 3.10 is* $(\text{cf}(\mu) + 1, \lambda)$*-regular.*

Proof. By Corollary 3.5, $\{z : \text{cf}(z) = \text{cf}(\mu)\} \in I^*$. For such z, choose $A_z \subseteq z$ of order type $\text{cf}(\mu)$ that is cofinal in z. Let U be given by Lemma 3.10. We will inductively build a sequence of intervals $\{(x_\alpha, y_\alpha) : \alpha < \lambda\}$, each contained in λ, such that $y_\alpha < x_\beta$ when $\alpha < \beta$, and such that for all α, $\{z : A_z \cap (x_\alpha, y_\alpha) \neq \emptyset\} \in U$.

Suppose we have constructed the intervals up to β. Let $\lambda > x_\beta > \sup\{y_\alpha : \alpha < \beta\}$. For $z \in \hat{x}_\beta$, let $y_\beta(z) \in z$ be such that $A_z \cap (x_\beta, y_\beta(z)) \neq \emptyset$. Since I is normal, there is a maximal antichain A of I-positive sets such that for all $a \in A$, $y_z(\beta)$ is the same for all $z \in a$. There is some $A' \subseteq A$ of size $< \lambda$ such that $\sum A' \in U$. Let $y_\beta > x_\beta$ be such that for $z \in a \in A'$, $y_\beta(z) < y_\beta$.

Now for $\alpha < \lambda$, let $X_\alpha = \{z : A_z \cap (x_\alpha, y_\alpha) \neq \emptyset\}$. Since each A_z has ordertype $\text{cf}(\mu)$ and the intervals (x_α, y_α) are disjoint and increasing, each A_z cannot have nonempty intersection with all intervals in some sequence of length greater than $\text{cf}(\mu)$. Thus if $s \subseteq \lambda$ and $z \in \bigcap_{\alpha \in s} X_\alpha$, then $\text{ot}(s) \leq \text{cf}(\mu)$. □

Lemma 3.12. *Suppose $\kappa < \lambda$ are regular, and I is a κ-complete, λ-dense ideal on Z such that $\mathcal{P}(Z)/I$ is complete. Then there is an ultrafilter $U \supseteq I^*$ such that for all $\alpha < \kappa$, $|\alpha^Z/U| \leq 2^{<\lambda}$.*

Proof. Suppose $\alpha < \kappa$, and let D witness the λ-density of I, and let $U \supseteq I^*$ be given by Lemma 3.10. First we count certain special members of α^Z/U. Choose an antichain $A \subseteq D$ of size $< \lambda$, and choose $f : A \to \alpha$. There are $\sum_{\gamma<\lambda} \lambda^\gamma \cdot \alpha^\gamma = 2^{<\lambda}$ many choices. Using κ-completeness, let $\{B_\beta : \beta < \alpha\}$ be pairwise disjoint and such that each $[B_\beta]_I = \sum f^{-1}(\beta)$. Let $g_f : Z \to \alpha$ be defined by $g_f(z) = \beta$ if $z \in B_\beta$ and $g_f(z) = 0$ if $z \notin \bigcup_{\beta<\alpha} B_\beta$.

Now let $g : Z \to \alpha$ be arbitrary. By κ-completeness, $A = \{g^{-1}(\beta) : \beta < \alpha \text{ and } g^{-1}(\beta) \in I^+\}$ forms a maximal antichain. Let $A' \subseteq D$ be a maximal antichain refining A. There is some $A'' \subseteq A'$ of size $< \lambda$ such that $\sum A'' \in U$. Let $f : A'' \to \alpha$ be defined by $f(a) = \beta$ iff $a \leq_I g^{-1}(\beta)$. If $[B]_I = \sum A''$, then $\{z \in B : g(z) \neq g_f(z)\} \in I$, so $g =_U g_f$. □

The following contrasts with the consistency results of Chapter 2:

Theorem 3.13. *Suppose μ is a singular cardinal such that $2^{\text{cf}(\mu)} < \mu$, λ is regular, and $2^{<\lambda} < 2^\lambda$. Then there is no normal and fine, λ-dense ideal on $\mathcal{P}_{\mu^+}(\lambda)$.*

Proof. Suppose such an ideal exists, and let U be given by Lemma 3.10. By Lemma 3.12, $|\prod \mu/U| \leq 2^{<\lambda}$. But by Lemma 3.11, U is $(\mathrm{cf}(\mu)+1, \lambda)$-regular. Hence, Theorem 3.9 implies that $|\prod 2^{\mathrm{cf}(\mu)}/U| \geq 2^{\lambda}$, a contradiction. □

Corollary 3.14. *Suppose μ is a singular cardinal such that $2^{\mathrm{cf}(\mu)} < \mu$. Then for all α there is a regular $\lambda > \alpha$ such that there is no normal and fine, λ-dense ideal on $\mathcal{P}_{\mu^+}(\lambda)$.*

Proof. We will show that there is a proper class of regular cardinals λ such that $2^{<\lambda} < 2^{\lambda}$. Assume for a contradiction that this fails. Let α be arbitrary, and let $\kappa = 2^{\alpha}$. We will show by induction the impossible conclusion that $2^{\beta} = \kappa$ for all $\beta \geq \alpha$. Suppose that this holds for all $\gamma < \beta$. If β is regular, then by a general equation shown in [21], $2^{<\beta} = 2^{\beta}$ by assumption, so $2^{\beta} = \kappa$. If β is singular, then $2^{\beta} = (2^{<\beta})^{\mathrm{cf}(\beta)} = \kappa^{\mathrm{cf}(\beta)}$. For regular $\gamma < \beta$ above $\mathrm{cf}(\beta)$, $(2^{\gamma})^{\mathrm{cf}(\beta)} = \kappa^{\mathrm{cf}(\beta)} = 2^{\gamma} = \kappa$. □

Corollary 3.15. *If κ is singular such that $2^{\mathrm{cf}(\kappa)} < \kappa$, then there is no uniform, κ^+-complete, κ^+-dense ideal on κ^{+n} for $n \geq 2$.*

Proof. Assume I is a uniform, κ^+-complete, κ^+-dense ideal on κ^{+n} for some $n \geq 2$. Define $\phi : \mathcal{P}(\kappa^+) \to \mathcal{P}(\kappa^{+n})/I$ by $X \mapsto ||\kappa^+ \in j(X)||_{\mathcal{P}(\kappa^{+n})/I}$. Let $J = \ker \phi$. ϕ lifts to an embedding of $\mathcal{P}(\kappa^+)/J$ into $\mathcal{P}(\kappa^{+n})/I$. Since J is clearly normal and κ^{++}-saturated, the embedding is regular, since for a maximal antichain $\{A_\alpha : \alpha < \kappa^+\}$, $\Vdash \kappa^+ \in j(\nabla_{\alpha<\kappa^+} A_\alpha)$, so it is forced that for some $\alpha < \kappa^+$, $\phi(A_\alpha)$ is in the generic filter. Thus J is a normal κ^+-dense ideal on κ^+. We have $2^{<\kappa^+} < 2^{\kappa^+}$ by Corollary 3.3, so κ cannot be singular such that $2^{\mathrm{cf}(\kappa)} < \kappa$. □

These methods can also be used to deduce more cardinal arithmetic consequences of dense ideals. First we need a few more lemmas:

Theorem 3.16 (Kunen-Prikry [26])**.** *If κ is regular and U is a (κ^+, κ^+)-regular ultrafilter, then U is (κ, κ)-regular.*

Lemma 3.17. *Suppose $(L, <)$ is a linear order such that for all $x \in L$, $|\{y \in L : y < x\}| \leq \kappa$. Then $|L| \leq \kappa^+$.*

Corollary 3.18. *Suppose there is a κ^+-complete, κ^+-dense ideal on κ^{+n}, where $n \geq 2$. Then for $0 \leq m \leq n$, $2^{\kappa^{+m}} = \kappa^{+m+1}$.*

Proof. Let I be such an ideal, and let $U \supseteq I^*$ be given by Lemma 3.10. By Lemma 3.12, $|\kappa^{\kappa^{+n}}/U| \leq 2^\kappa$, which is κ^+ by Corollary 3.3. Note that for any cardinal μ, any ultrafilter V on a set Z, and any $g : Z \to \mu^+$, $\{[f]_V : f <_V g\}$ has cardinality at most $|\mu^Z/V|$. Thus, applying Lemma 3.17 inductively, we get that $|(\kappa^{+m})^{\kappa^{+n}}/U| \leq \kappa^{+m+1}$ for all $m < \omega$.

U is $(\kappa^{+n}, \kappa^{+n})$-regular, so by Theorem 3.16, it is $(\kappa^{+m}, \kappa^{+m})$-regular for $m \leq n$. Assume for induction that $2^{\kappa^{+r}} = \kappa^{+r+1}$ for $r < m \leq n$; note the base case $m = 1$ holds. Let $\{X_\alpha : \alpha < \kappa^{+m}\}$ witness $(\kappa^{+m}, \kappa^{+m})$-regularity, and let $\beta_z = \text{ot}(\{\alpha : z \in X_\alpha\})$. By Theorem 3.9 and the above observations, we have:

$$2^{\kappa^{+m}} \leq |\prod 2^{\beta_z}/U| \leq |\prod 2^{\kappa^{+m-1}}/U| = |\prod \kappa^{+m}/U| \leq \kappa^{+m+1}.$$

□

We note that if the hypothesis of Corollary 3.18 is consistent, then no cardinal arithmetic above κ^{+n} can be deduced from it, since any forcing which adds no subsets of κ^{+n} will preserve the relevant properties of the ideal.

By combining this technique with the results of Chapter 2, we can answer the following, which was Open Question 16 from [13]:

Question (Foreman)**.** *Is it consistent that there is a uniform ultrafilter U on ω_3 such that ω^{ω_3}/U has cardinality ω_3? Is it consistent that there is a uniform ultrafilter U on $\aleph_{\omega+1}$ such that $\omega^{\aleph_{\omega+1}}/U$ has cardinality $\aleph_{\omega+1}$? Give a characterization of the possible cardinalities of ultrapowers.*

Theorem 3.19. *Assume ZFC is consistent with a super-almost-huge cardinal. Then it is consistent that every regular uncountable cardinal κ carries a uniform ultrafilter U such that $|\omega^\kappa/U| = \kappa$.*

This follows from Chapter 2 and the next result.

Lemma 3.20. *Suppose $\kappa = \mu^+$, GCH holds at cardinals $\geq \mu$, and for all regular $\lambda \geq \kappa$, there is a normal and fine, κ-complete, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$. Then for every regular λ, there is a uniform ultrafilter U on λ such that $|\mu^\lambda/U| = \mu$.*

Proof. Let I be a normal and fine, κ-complete, λ-dense ideal on $Z = \mathcal{P}_\kappa(\lambda)$, where $\kappa = \mu^+$ and λ is regular. Let $U \supseteq I^*$ be given by Lemma 3.12, so that $|\mu^Z/U| \leq 2^{<\lambda}$. If $2^{<\lambda} = \lambda$, then $|\mu^Z/U| \leq \lambda$, and we can assume U is a uniform ultrafilter on λ with the same property. Since $2^\mu = \kappa$ and any ultrafilter extending I^* is (κ, λ)-regular, Theorem 3.9 implies that $|\kappa^Z/U| > \lambda$, and Lemma 3.17 implies that $|\kappa^Z/U| \leq |\mu^Z/U|^+$. Thus $|\mu^Z/U| = \lambda$. $\square$

The following extra conclusion can be immediately deduced in the case of $\mu < \aleph_\omega$ and $\lambda = \rho^+$, where $\text{cf}(\rho) = \omega$. Suppose $\mu = \omega_n$. Since $|\omega_{n+1}^Z/U| > \lambda$, we cannot have $|\omega_m^Z/U| < \rho$ for any m, since by Lemma 3.17, we would have $|\omega_r^Z/U| < \rho$ for all $r < \omega$. Also, U is (ω, ω)-regular, so Theorem 3.9 implies that $|\omega^Z/U| \geq |\omega^Z/U|^\omega$. Thus $|\omega_m^Z/U| = \lambda$ for all $m \leq n$.

Chapter 4

Ulam's problem and regularity of ideals

Taylor [33] generalized the notion of regularity of ultrafilters to arbitrary ideals:

Definition. *An ideal I is* (μ, κ)-regular *if for any sequence $\langle A_\alpha : \alpha < \kappa \rangle \subseteq I^+$, there is a sequence $\langle B_\alpha : \alpha < \kappa \rangle \subseteq I^+$ such that $B_\alpha \subseteq A_\alpha$ for all α, and for any $Y \subseteq \kappa$ of order type μ, $\bigcap_{\alpha \in Y} B_\alpha = \emptyset$.*

Note that if I^* is an ultrafilter, then I is (μ, κ)-regular iff I^* is (μ, κ)-regular per the definition in the previous chapter, since if $\langle X_\alpha : \alpha < \kappa \rangle$ witnesses (μ, κ)-regularity in the old sense, and we are given $\langle A_\alpha : \alpha < \kappa \rangle \subseteq I^+ = I^*$, the we can take $B_\alpha = A_\alpha \cap X_\alpha$. We will simply call a κ-complete ideal on κ regular if it is $(2, \kappa)$-regular, and a normal and fine ideal on $Z \subseteq \mathcal{P}(\lambda)$ regular if it is $(2, \lambda)$-regular. Taylor [33] proved the following connection between dense ideals, nonregular ideals, and Ulam's measure problem:

Theorem 4.1 (Taylor)**.** *The following are equivalent:*

(1) There is a countably complete ω_1-dense ideal on ω_1.

(2) There is a set $\{I_\alpha : \alpha < \omega_1\}$ of normal and fine ideals on ω_1 such that every $A \subseteq \omega_1$ is measureable in one of them.

(3) There is a countably complete nonregular ideal on ω_1.

We will investigate the extent to which Taylor's theorem generalizes to ideals on $\mathcal{P}_\kappa(\lambda)$. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) will go through in general, but when $\kappa \neq \omega_1$, the argument for (3) $\Rightarrow$ (1) will seem to require some additional assumptions. In the next chapter we will produce models of set theory that show that (3) does not imply either (1) or (2) when $\kappa \neq \omega_1$.

4.1 Generalizing Taylor's theorem

The implication (1) $\Rightarrow$ (2) is trivial. If I is a normal and fine, λ-dense ideal on Z, as witnessed by $\{A_\alpha : \alpha < \lambda\}$, then every subset of Z is measurable in $I \restriction A_\alpha$ for some $\alpha < \lambda$. Also, the implication (1) $\Rightarrow$ (3) is immediate. If $\{A_\alpha : \alpha < \lambda\}$ is dense, and $\{B_\alpha : \alpha < \lambda\}$ is a disjoint refinement into I-positive sets, then there is some $A_\alpha \subseteq_I B_0$ with $\alpha \neq 0$, which contradicts that $B_0 \cap B_\alpha = \emptyset$. The implication (2) $\Rightarrow$ (3) requires more work.

Lemma 4.2. *Suppose $\{I_\alpha : \alpha < \lambda\}$ is a collection of normal and fine ideals on $Z \subseteq \mathcal{P}(\lambda)$, each of which is nowhere λ^+-saturated. Then there is a collection $\{X_\alpha : \alpha < \lambda^+\}$ of subsets of Z such that $(\forall \alpha < \lambda^+)(\forall \beta < \lambda) X_\alpha \in I^+_\beta$, and for $\alpha < \beta < \lambda^+$, $X_\alpha \cap X_\beta$ is nonstationary.*

Proof. First note that whenever I is normal ideal on $Z \subseteq \mathcal{P}(\lambda)$, and $\{A_\alpha : \alpha < \lambda^+\}$ is an antichain, then we can refine the A_α's so that their pairwise intersections are nonstationary. For each $\alpha < \lambda^+$, let $f_\alpha : \lambda \to \alpha$ be a surjection. Let $B_\alpha = A_\alpha \setminus \bigcup_{\beta<\lambda} A_{f_\alpha(\beta)} \cap \hat{\beta}$. So when $\alpha \neq \beta$, $B_\alpha \cap B_\beta \cap \hat{\gamma} = \emptyset$ for some γ.

Let $\{I_\alpha : \alpha < \lambda\}$ be as hypothesized, and for each I_α pick some antichain $\{A^\alpha_\beta : \beta < \lambda^+\}$ such that the pairwise intersections are nonstationary. Let $h : \lambda \to \lambda$ be defined by $h(\alpha) =$ the least γ such that $\{A^\gamma_\beta : \beta < \lambda^+\}$ has λ^+ many I_α-positive sets. Let $\delta < \lambda^+$ be $\sup\{\beta : (\exists\alpha < \lambda)(\exists\gamma < h(\alpha))A^\gamma_\beta \in I^+_\alpha\}$. Recursively construct a one-to-one sequence $\langle B_\alpha : \alpha < \lambda\rangle$ such that $B_\alpha = A^{h(\alpha)}_\beta$ for some $\beta > \delta$, so $B_\alpha \in I^+_\alpha$ for all $\alpha < \lambda$, and $B_\alpha \cap B_\beta \in I_\alpha$ when $h(\beta) \leq h(\alpha)$. Put $C_\alpha = B_\alpha \cap \hat{\alpha} \setminus \bigcup_{h(\beta)\leq h(\alpha)} B_\beta \cap \hat{\beta}$, so $C_\alpha \in I^+_\alpha$ and $C_\alpha \cap C_\beta = \emptyset$ when $\alpha \neq \beta$. Now pick $\{D^\alpha_\beta : \alpha < \lambda,\ \beta < \lambda^+\}$ such that each D^α_β is an I_α-positive subset of C_α, and $D^\alpha_\beta \cap D^\alpha_\gamma$ is nonstationary when $\beta \neq \gamma$. For $\beta < \lambda^+$, let $E_\beta = \bigcup_{\alpha<\lambda} D^\alpha_\beta \cap \hat{\alpha}$. Then $E_\beta \cap E_\gamma = \bigcup_{\alpha<\lambda} D^\alpha_\beta \cap D^\alpha_\gamma \cap \hat{\alpha}$, and each E_β is I_α-positive for all α. □

Lemma 4.3. *Suppose I is a normal and fine ideal on $Z \subseteq \mathcal{P}(X)$. Then I is $|X|^+$-saturated iff every normal and fine $J \supseteq I$ is equal to $I \restriction A$ for some $A \subseteq Z$.*

Proof. Suppose I is $|X|^+$-saturated. Let $\{A_x : x \in X\}$ be a maximal antichain in $J \cap I^+$. Then $[\nabla A_x]$ is the largest element of $\mathcal{P}(Z)/I$ whose elements are in J. Thus $J = I \restriction (Z \setminus \nabla A_x)$. Now suppose I is not $|X|^+$-saturated, and let $\{A_\alpha : \alpha < \delta\}$ be a maximal antichain where $\delta \geq |X|^+$. Let J be the ideal generated by $\bigcup\{\Sigma_{\alpha\in Y}[A_\alpha] : Y \in \mathcal{P}_{|X|^+}(\delta)\}$. Then J is a normal, fine, proper ideal extending I. J cannot be equal $I \restriction A$ for some $A \in I^+$ because if so, there is some α where $A \cap A_\alpha \in I^+$. $A \cap A_\alpha \in J$ by construction, but every I-positive subset of A is $(I \restriction A)$-positive. □

Theorem 4.4. *If there is a set $\{I_\alpha : \alpha < \lambda\}$ of normal and fine ideals on $Z \subseteq \mathcal{P}(\lambda)$ such that every $A \subseteq Z$ is measurable in one of them, then there is a normal, fine, nonregular ideal on Z.*

Proof. Let $\mathcal{J} = \{I_\alpha : \alpha < \lambda\}$ be as hypothesized, and assume for a contradiction that every normal and fine ideal on Z is regular. Let $\mathcal{J}_0 = \{I \in \mathcal{J} : I \text{ is nowhere } \lambda^+\text{-saturated}\}$, and for $I \in \mathcal{J} \setminus \mathcal{J}_0$, choose A_I such that $I \restriction A_I$ is λ^+-saturated, and let $\mathcal{J}_1 = \{I \restriction A_I : I \in \mathcal{J}$

and I is somewhere λ^+-saturated$\}$. Clearly, every subset of Z is measurable by some ideal in $\mathcal{J}_0 \cup \mathcal{J}_1$.

Now let $\{A_\alpha : \alpha < \lambda^+\}$ be such that each A_α is I-positive for all $I \in \mathcal{J}_0$ and $A_\alpha \cap A_\beta$ is nonstationary for $\alpha \neq \beta$. Since at most one A_α can be I-measure one for any $I \in \mathcal{J}_1$, there is some $\alpha < \lambda^+$ such that $Z \setminus A_\alpha$ is I-positive for all $I \in \mathcal{J}_1$. Let $\mathcal{J}_2 = \{I \restriction (Z \setminus A_\alpha) : I \in \mathcal{J}_1\}$.

Let $J = \bigcap \mathcal{J}_2$, which is clearly normal. Since the J-positive sets are those that are I-positive for some $I \in \mathcal{J}_2$, J is λ^+-saturated. Thus by Lemma 4.3, each $I \in \mathcal{J}_1$ is $J \restriction B_I$ for some $B_I \in J^+$. Since we assume J is regular, there is a collection $\{C_I : I \in \mathcal{J}_2\}$ such each C_I is a J-positive subset of B_I, and $C_{I_0} \cap C_{I_1} = \emptyset$ for $I_0 \neq I_1$. Split each C_I into two disjoint positive sets D_I^0, D_I^1, and let $E_i = \bigcup_{I \in \mathcal{J}_1} D_I^i$ for $i < 2$. Split A_α into two disjoint sets F_0, F_1 that are I-positive for all $I \in \mathcal{J}_0$. Then $E_0 \cup F_0$, $E_1 \cup F_1$ are nonmeasurable for all $I \in \mathcal{J}$. □

The argument for (3) $\Rightarrow$ (1) in the case of ω_1 uses a result of Baumgartner, Hajnal, and Máté [3] about normal ideals on ω_1. The following generalizes their argument.

Lemma 4.5. *Suppose J is a normal, fine, κ^+-complete ideal on $Z \subseteq \mathcal{P}_{\kappa^+}(X)$ which is nowhere $|X|$-dense, and $\mathcal{P}(Z)/J$ is κ-strategically closed. Then J is regular.*

Proof. Let $\{M_x^\alpha : \alpha < \kappa, x \in X\}$ be a generalized Ulam matrix as in Lemma 1.4, so that for each $x \in X$, $\hat{x} = \bigcup_{\alpha < \kappa} M_x^\alpha$, and $M_x^\alpha \cap M_y^\alpha = \emptyset$ for $x \neq y$.

Now let $\{A_x : x \in X\}$ be any sequence of J-positive sets. For $\alpha < \kappa$, let $A_x^\alpha = A_x \cap M_x^\alpha$. Note that κ^+-completeness implies $\forall x \exists \alpha A_x^\alpha \in J^+$. Let $S = \{(\alpha, x) : A_x^\alpha \in J^+\}$. Using the winning strategies $\{\sigma_p : p \in \mathcal{P}(Z)/J\}$ for *Even*, where σ_p has opening move p, we will inductively build descending sequences of sets below A_x^α to create a disjoint refinement.

If $(\alpha, x) \in S$, we play a game below A_x^α, and denote representatives for *Even's* plays by $A_x^{\alpha,\beta}$ and those for *Odd's* plays immediately following by $B_x^{\alpha,\beta}$. Let $\gamma < \kappa$ and assume for induction that for $(\alpha, x) \in S$ we have chosen sequences $\langle A_x^{\alpha,\beta}\rangle_{\beta<\gamma}$ and $\langle B_x^{\alpha,\beta}\rangle_{\beta<\gamma}$ satisfying the following:

(1) $G_x^{\alpha,\gamma} = \langle [A_x^{\alpha,0}], [B_x^{\alpha,0}], ..., [A_x^{\alpha,\beta}], [B_x^{\alpha,\beta}], ...\rangle_{\beta<\gamma}$ is a sequence played according to $\sigma_{[A_x^\alpha]}$.

(2) If $\beta_1 < \beta_2 < \gamma$, then $A_x^\alpha \supseteq A_x^{\alpha,\beta_1} \supseteq B_x^{\alpha,\beta_1} \supseteq A_x^{\alpha,\beta_2}$.

(3) If $\beta < \gamma$, $\beta \neq \alpha$ and $x, y \in X$, then $B_x^{\beta,\beta} \cap B_y^{\alpha,\beta} = \emptyset$.

For each $(\alpha, x) \in S$, let $A_x^{\alpha,\gamma}$ be a representative of $\sigma_{[A_x^\alpha]}(G_x^{\alpha,\gamma})$ that is a subset of all previously chosen representatives. Since $\mathcal{P}(Z)/J$ is nowhere $|X|$-dense, for any $x \in X$, the set:

$$\{[A_x^{\gamma,\gamma} \cap A_y^{\alpha,\gamma}] : (\alpha, y) \in S\} \cap J^+$$

is not dense below $A_x^{\gamma,\gamma}$. Thus we may choose a J-positive $B_x^{\gamma,\gamma} \subseteq A_x^{\gamma,\gamma}$ such that $(A_x^{\gamma,\gamma} \cap A_y^{\alpha,\gamma}) \setminus B_x^{\gamma,\gamma} \in J^+$ for all α, y such that $A_x^{\gamma,\gamma} \cap A_y^{\alpha,\gamma} \in J^+$. For $(\alpha, y) \in S$ such that $\alpha \neq \gamma$, let $B_y^{\alpha,\gamma} = A_y^{\alpha,\gamma} \setminus \bigcup_{x\in X} B_x^{\gamma,\gamma}$.

Note that when $(\gamma, x), (\alpha, y) \in S$ and $\gamma \neq \alpha$,

$$(A_x^{\gamma,\gamma} \cap A_y^{\alpha,\gamma}) \setminus B_x^{\gamma,\gamma} = (A_x^{\gamma,\gamma} \cap A_y^{\alpha,\gamma}) \setminus \bigcup_{w\in X} B_w^{\gamma,\gamma} = A_x^{\gamma,\gamma} \cap B_y^{\alpha,\gamma},$$

since $B_w^{\gamma,\gamma} \cap A_x^{\gamma,\gamma} = \emptyset$ when $x \neq w$. If $A_x^{\gamma,\gamma} \cap A_y^{\alpha,\gamma} \in J^+$ for some x, then $B_y^{\alpha,\gamma} \in J^+$. If $(\alpha, y) \in S$ and $A_x^{\gamma,\gamma} \cap A_y^{\alpha,\gamma} \in J$ for all x, then $B_y^{\alpha,\gamma} \in J^+$ by the normality of J.

Clearly the induction hypotheses hold with respect to $\gamma + 1$. In the end, the collection $\{B_x^{\alpha,\alpha} : (\alpha, x) \in S\}$ is a pairwise disjoint refinement of $\{A_x^\alpha : (\alpha, x) \in S\}$ into J-positive sets. As $\{A_x : x \in X\}$ was arbitrary, this shows that J is regular. □

Corollary 4.6. *Let $Z \subseteq \mathcal{P}_{\omega_1}(\lambda)$ be stationary. The following are equivalent:*

(1) There is a normal, fine, λ-dense ideal on Z.

(2) There is a set $\{I_\alpha : \alpha < \lambda\}$ of normal and fine ideals on Z such that every $A \subseteq Z$ is measurable in one of them.

(3) There is a normal, fine, nonregular ideal on Z.

Proof. We've already seen (1) $\Rightarrow$ (2) $\Rightarrow$ (3). To see (3) $\Rightarrow$ (1), note that every normal and fine ideal on Z is ω-strategically closed. Thus by Lemma 4.5, any normal, fine, nonregular ideal on Z is somehwere λ-dense. $\square$

The theorem works for ideals on $\mathcal{P}_\kappa(\lambda)$ for $\kappa = \mu^+ > \omega_1$, if we also assume that every λ^+-saturated ideal on $\mathcal{P}_\kappa(\lambda)$ is μ-strategically closed. If we could reduce the hypothesis of μ-strategic closure in Lemma 4.5 to μ-distributivity, then by Proposition 3.6, we could prove under $\mu^{<\mu} = \mu$ that for normal ideals on μ^+, (1), (2), and (3) are equivalent. But later we will produce models proving such a reduction is impossible. However, under GCH, we can prove the equivalence of (1) and (2) by a different argument.

Definition. *If $\mathbb{B}$ is a boolean algebra, $D \subseteq \mathbb{B}^+$ is called* weakly dense *if $(\forall b \in \mathbb{B}^+)(\exists d \in D)(d \leq b$ or $d \leq \neg b)$.*

Lemma 4.7. *Let $Z \subseteq \mathcal{P}(\lambda)$ be stationary. The following are equivalent.*

(1) There is a collection $\{I_\alpha : \alpha < \lambda\}$ of normal and fine ideals on Z such that every subset of Z is measurable in one of them.

(2) There is a λ^+-saturated ideal I on Z such that $\mathcal{P}(Z)/I$ has a weakly dense subset of size $\leq \lambda$.

Proof. Let $\mathcal{J} = \{I_\alpha : \alpha < \lambda\}$ be such that every subset of Z is measurable in one of them. Let $\mathcal{J}_0 = \{I \in \mathcal{J} : I$ is nowhere λ^+-saturated$\}$, and choose sets A_I such that $\mathcal{J}_1 = \{I \restriction A_I : I \in \mathcal{J} \setminus \mathcal{J}_0$ and $I \restriction A_I$ is λ^+-saturated$\}$. As in the proof of Theorem 4.4, there is some $A \subseteq Z$ such A is positive for all $I \in \mathcal{J}_0$, $Z \setminus A$ is positive for all $I \in \mathcal{J}_1$, and every subset of Z is measurable by some ideal in $\mathcal{J}_0 \cup \mathcal{J}_1$. Let $J = \bigcap_{I \in \mathcal{J}_1} I \restriction (Z \setminus A)$. J is normal, fine, and λ^+-saturated. By Lemma 4.3, for all $I \in \mathcal{J}_1$, there is some $B_I \in J^+$ such that $I \restriction (Z \setminus A) = J \restriction B_I$.

We claim $\{B_I : I \in \mathcal{J}_1\}$ is weakly dense in $\mathcal{P}(Z \setminus A)/J$. Otherwise, there is a partition of $Z \setminus A$ into two J-positive sets B_0, B_1 such that for no $I \in \mathcal{J}_1$ is B_0 or B_1 in $(J \restriction B_I)^*$. We can partition A into A_0, A_1 that are I-positive for all $I \in \mathcal{J}_0$. Then $A_0 \cup B_0$ and $A_1 \cup B_1$ are nonmeasurable for all $I \in \mathcal{J}$.

Conversely, if I is an ideal on Z such that $\mathcal{P}(Z)/I$ has a weakly dense subset $\{A_\alpha : \alpha < \lambda\}$, then for every $B \subseteq Z$, there is some $\alpha < \lambda$ such that $B \in (I \restriction A_\alpha) \cup (I \restriction A_\alpha)^*$. □

Theorem 4.8 (Bozeman [4]). *Suppose $\mathbb{B}$ is a complete boolean algebra, $D \subseteq \mathbb{B}$ is weakly dense, $|D| = \kappa$, and $2^{<\kappa} = \kappa$. Then there is a $b \in \mathbb{B}$ and a set E of size κ that is dense below b.*

Corollary 4.9. *Suppose $2^{<\lambda} = \lambda$. Then there is a normal, fine, λ-dense ideal on $Z \subseteq \mathcal{P}(\lambda)$ iff there is a collection $\{I_\alpha : \alpha < \lambda\}$ of normal and fine ideals on Z such that every subset of Z is measurable in one of them.*

Corollary 4.10. *Assume GCH, κ is the successor of a singular cardinal, and $\lambda \geq \kappa$ is regular. If $\{I_\alpha : \alpha < \lambda\}$ is a set of normal and fine ideals on $\mathcal{P}_\kappa(\lambda)$, then there is a set $X \subseteq \mathcal{P}_\kappa(\lambda)$ that is not measurable in any them.*

4.2 Reduction to normality and degrees of regularity

In many cases, questions about κ-complete ideals on κ reduce to questions about normal and fine ideals on κ. We use this technique to extend some of the previous results to non-normal ideals, and then show that most degrees of regularity are equivalent under enough GCH.

On the other hand, a model constructed by Shelah shows the importance of the normality assumption in clause (2) of Taylor's theorem.

Lemma 4.11 (Baumgartner-Hajnal-Máté [3]). *If I is a κ-complete, nowhere κ^+-saturated ideal, then I is regular.*

Lemma 4.12 (Taylor [33]). *Suppose I is a κ-complete, κ^+-saturated ideal on $\kappa = \mu^+$. Then there is $A \in I^+$ and a bijection $f : \kappa \to \kappa$ such that $f_*(I \restriction A) = \{X \subseteq \kappa : f^{-1}(X) \in I \restriction A\}$ is normal.*

Lemma 4.13 (Taylor [33]). *Let I be a κ-complete ideal on Z. Suppose that the set $\{A \in I^+ : (I \restriction A)$ is (μ, κ)-regular$\}$ is dense in $\mathcal{P}(Z)/I$. Then I is (μ, κ)-regular.*

Corollary 4.14 (Taylor [33]). *If every normal and fine ideal on $\kappa = \mu^+$ is (μ, κ)-regular, then every κ-complete ideal on κ is (μ, κ)-regular.*

Proof. Suppose I is a κ-complete ideal on κ, and let $A \in I^+$. If $I \restriction A$ is nowhere κ^+-saturated, then $I \restriction A$ is $(2, \kappa)$-regular by Lemma 4.11. Otherwise, Lemma 4.12 implies there is some $B \subseteq A$ such that $I \restriction B$ is isomorphic to a normal ideal N. By hypothesis, N is (μ, κ)-regular, and this is clearly preserved by isomorphisms. Thus by Lemma 4.13, I is (μ, κ)-regular. □

Corollary 4.15. *The normality assumption can be replaced by κ-completeness in Proposition 3.6.*

Proof. Assume $\kappa = \mu^+$, $\mu^{<\mathrm{cf}(\mu)} = \mu$, and I is a κ-complete, κ^+-saturated ideal on κ. Suppose for a contradiction that there is $A \in I^+$ such that $[A] \Vdash$ "$\beta < \mathrm{cf}(\mu)$, $\dot{f} : \beta \to \mathrm{Ord}$, and $\dot{f} \notin V$." By lemma 4.12, there is a $B \subseteq A$ such that $I \restriction B$ is isomorphic to a normal ideal N on κ. By Proposition 3.6, forcing with $\mathcal{P}(\kappa)/N$ adds no new functions with domain β and range in the ordinals, so the same is true of $I \restriction B$. □

Lemma 4.16. *Let $\mu < \kappa$ be regular cardinals, and $\lambda \geq \kappa$ such that $\lambda^{<\mu} = \lambda$. If I is a normal, fine, κ-complete, nowhere κ^+-complete, λ^+-saturated ideal on $Z \subseteq \mathcal{P}(\lambda)$, and $\mathcal{P}(Z)/I$ is μ-distributive, then I is regular iff I is (μ, λ)-regular.*

Proof. Fix a bijection $f : \lambda \to [\lambda]^{<\mu}$. Let $j : V \to M$ be a generic embedding arising from forcing with $\mathcal{P}(Z)/I$. If $x \subseteq j[\lambda]$ has cardinality $< \mu$ in M, then by μ-distributivity, $j^{-1}[x] \in V$ and $j(j^{-1}[x]) = x$. $j^{-1}[x] = f(\alpha)$ for some $\alpha < \lambda$, so $x = j(f)(j(\alpha))$. Also, for any $\alpha < \lambda$, $j(f)(j(\alpha)) = j(f(\alpha)) \subseteq j[\lambda]$. Thus, by Łoś's theorem, $A = \{z : f \restriction z$ is a bijection between z and $[z]^{<\mu}\} \in I^*$.

If $\{A_\alpha : \alpha < \lambda\} \subseteq I^+$, let $\{B_\alpha : \alpha < \lambda\} \subseteq I^+$ be a refinement such that for any $X \subseteq \lambda$ of order type μ, $\bigcap_{\alpha \in X} B_\alpha = \emptyset$. We may assume each $B_\alpha \subseteq \hat{\alpha} \cap A$. For each $z \in Z$, let $s(z) = \{\alpha : z \in B_\alpha\}$. For all z, $s(z) \in [z]^{<\mu}$. For each B_α, there is a constant η_α such that $C_\alpha = \{z \in B_\alpha : s(z) = f(\eta_\alpha)\} \in I^+$.

Suppose $z \in C_{\alpha_0} \cap C_{\alpha_1}$. Then $\{\alpha_0, \alpha_1\} \subseteq s(z) = f(\eta_{\alpha_0}) = f(\eta_{\alpha_1})$. If $z' \in C_{\alpha_0}$, then $s(z') = f(\eta_{\alpha_0}) = f(\eta_{\alpha_1})$, so $z' \in C_{\alpha_1}$. Thus $C_{\alpha_0} \subseteq C_{\alpha_1}$. Switching the roles of α_0 and α_1, we conclude $C_{\alpha_0} = C_{\alpha_1}$. Therefore for all $\alpha_1 < \alpha_2 < \lambda$, either $C_{\alpha_0} \cap C_{\alpha_1} = \emptyset$ or $C_{\alpha_0} = C_{\alpha_1}$.

For $C \subseteq Z$, Let $t(C) = \{\alpha : C = C_\alpha\}$. If $\alpha \in t(C)$ and $z \in C$, then $t(C) \subseteq s(z)$, so $|t(C)| < \mu$. For each $C \in \{C_\alpha : \alpha < \kappa\}$, choose a splitting of C into disjoint I-positive sets $\{D^C_\xi : \xi < \mathrm{ot}(t(C))\}$. (This is possible by the assumptions: the (μ, λ)-regularity of I implies that I is nowhere-prime. If $I^* \cap \mathcal{P}(Y)$ were an ultrafilter on $Y \subseteq Z$, then by κ-completeness, it would not be (α, α)-regular for any $\alpha < \kappa$. μ-distributivity implies that $2^{<\mu} < \kappa$, and so I is nowhere μ-saturated.) By induction, define $E_\alpha \subseteq C_\alpha$ to be D^C_ξ, where $C = C_\alpha$, and ξ is the least ordinal such that $D^C_\xi \neq E_\gamma$ for all $\gamma < \alpha$. Then $\{E_\alpha : \alpha < \kappa\}$ is a pairwise disjoint refinement of $\{A_\alpha : \alpha < \kappa\}$. □

Theorem 4.17. *Suppose $\kappa = \mu^+$, $\mu^{<\mathrm{cf}(\mu)} = \mu$, and I is a κ-complete ideal on κ. Then I is $(\mathrm{cf}(\mu), \kappa)$-regular iff it is regular.*

Proof. Suppose I is a κ-complete, $(\mathrm{cf}(\mu), \kappa)$-regular ideal on $\kappa = \mu^+$. Let $A \in I^+$. If $I \restriction A$ is nowhere κ^+-saturated, then $I \restriction A$ is regular by Lemma 4.11. Otherwise, Lemma 4.12 implies there is some $B \subseteq A$ and some bijection f on κ such that $f_*(I \restriction A) = N$ is a κ^+-saturated normal ideal on κ. N is $(\mathrm{cf}(\mu), \kappa)$-regular, and by Proposition 3.6, $\mathcal{P}(\kappa)/N$ is $\mathrm{cf}(\mu)$-distributive. Hence, N is regular by Lemma 4.16. Thus, $\{[C] : I \restriction C \text{ is regular}\}$ is dense, so Lemma 4.13 implies that I is regular. The other direction is trivial. □

To show the necessity of assuming the ideals are normal in clause (2) of Taylor's theorem, we use the following result of Shelah:

Theorem 4.18 (Shelah [31]). *Suppose there is a cardinal κ such that $\{\alpha < \kappa : \alpha$ is supercompact$\}$ is stationary in κ. Then there is a forcing extension in which:*

1. $\mathcal{P}(\omega_1)/NS \cong \mathcal{B}(\mathrm{Col}(\omega, <\omega_2))$.
2. *The algebra $\mathcal{P}(\omega_1)/NS$ is the union of ω_1 many countably complete filters.*

Proposition 4.19. *In Shelah's model, there is no ω_1-dense, countably complete ideal on ω_1, but there is a set of countably complete ideals $\{I_\alpha : \alpha < \omega_1\}$ such that $\mathcal{P}(\omega_1) = \bigcup(I_\alpha \cup I_\alpha^*)$.*

Proof. Let $\mathcal{P}(\omega_1)/NS = \bigcup\{F_\alpha : \alpha < \omega_1\}$, where each F_α is a countably complete filter. For each α, let $I_\alpha = \{X : [\omega_1 \setminus X] \in F_\alpha\}$.

If there were a countably complete, ω_1-dense ideal on ω_1 in Shelah's model, then there would be a normal one J. In the model, NS_{ω_1} is ω_2-saturated, so by Lemma 4.3, $J = NS_{\omega_1} \restriction A$ for some stationary set A. But $\mathrm{Col}(\omega, <\omega_2)$ is nowhere ω_1-dense. □

Proposition 4.20. *Suppose NS_{ω_1} is ω_2-saturated but nowhere ω_1-dense. Let $\mathcal{J}$ be a of ω_1 many countably complete ideals on ω_1 such that every subset of ω_1 is measurable by one of them. $\mathcal{J}$ must have ω_1 many non-normal, nowhere ω_2-saturated members.*

Proof. Let $\mathcal{J}_0 = \{I \in \mathcal{J} : I_\alpha \text{ is nowhere } \omega_2\text{-saturated}\}$. Since NS_{ω_1} is ω_2-saturated, Lemma 4.3 implies that every member of $\mathcal{J}_0$ is not normal. Let $\{I_\alpha : \alpha < \omega_1\}$ enumerate $\mathcal{J} \setminus \mathcal{J}_0$, with repetitions in case this set is countable. For each I_α, use Lemma 4.12 to pick $(A_\alpha, f_\alpha, N_\alpha)$ such that $A_\alpha \in I_\alpha^+$, f_α is a bijection on ω_1, N_α is a normal ω_2-saturated ideal, and $N_\alpha = (f_\alpha)_*(I_\alpha \restriction A_\alpha)$. Let $\mathcal{J}_1 = \{I_\alpha \restriction A_\alpha : \alpha < \omega_1\}$, and let $J = \bigcap \mathcal{J}_1$.

For each α, let $J_\alpha = (f_\alpha)_*(J)$. So $J_\alpha \subseteq N_\alpha$, and both are ω_2-saturated. Let $\{[B_\beta] : \beta < \omega_1\}$ be a maximal antichain in $\mathcal{P}(\omega_1)/J_\alpha$ where each $B_\beta \in N_\alpha$. Let $C_\alpha = \nabla_{\beta<\omega_1} B_\beta \in N_\alpha$. If $X \in J_\alpha^+$ and $X \subseteq C_\alpha$, then $X \in N_\alpha$, so by maximality $X \cap B_\beta \in J_\alpha^+$ for some β. Thus, C_α is $\leq_{J_\alpha}$-maximal among all sets in N_α, so $N_\alpha = J_\alpha \restriction (\omega_1 \setminus C_\alpha)$. Since f_α is a bijection, we have for each α a set D_α such that $I_\alpha \restriction A_\alpha = J \restriction D_\alpha$.

J is nowhere ω_1-dense, so by Theorem 4.1, J is regular. Let $\{E_\alpha : \alpha < \omega_1\}$ be a disjoint refinement of $\{D_\alpha : \alpha < \omega_1\}$ into J-positive sets. Split each E_α into disjoint J-positive sets $\{E_\alpha^\beta : \beta < \omega_1\}$, and let $F_\alpha = \bigcup_{\beta<\omega_1} E_\alpha^\beta$. Then the collection $\{F_\alpha : \alpha < \omega_1\}$ is pairwise disjoint, and every member is $(J \restriction D_\gamma)$-positive for all relevant γ.

Assume now for a contradiction that $\mathcal{J}_0$ is countable. For each even ordinal α, let $G_\alpha = F_\alpha \cup F_{\alpha+1}$. At most one G_α can be in I^* for any $I \in \mathcal{J}_0$, so let β be such that $\omega_1 \setminus G_\beta$ is I-positive for all $I \in \mathcal{J}_0$. By the Alaoglu-Erdős theorem, there are disjoint $X_0, X_1 \subseteq \omega_1 \setminus G_\beta$, both I-positive for all $I \in \mathcal{J}_0$. Then $X_0 \cup F_\beta$ and $X_1 \cup F_{\beta+1}$ are disjoint sets which are positive for all $I \in \mathcal{J}$. This contradicts the assumption that every subset of ω_1 is measurable by some $I \in \mathcal{J}$. $\square$

Chapter 5

Consistency results from generic large cardinals

Our main goal in this chapter is to pull apart density and nonregularity at higher cardinals, showing that Theorem 4.1 and Corollary 4.6 are indeed specific to ω_1. Rather than creating a new construction from large cardinals, we start from generic large cardinal assumptions shown consistent in Chapter 2. A potential advantage of this approach is that if the consistency strength of dense ideals on spaces other than ω_1 is ever reduced below almost-huge cardinals, the following arguments will be applicable to those contexts as well. As a consequence of the techniques, we derive some new results concerning the mutual inconsistency of some generic large cardinals and strengthen a result of Woodin. We also show that the kind of generic supercompactness shown consistent in Chapter 2 is also consistent with $\square$ holding very often.

5.1 Foreman's Duality Theorem

Our main tool will be Foreman's Duality Theorem [14], which allows a precise characterization of the effect of forcing on the structure of precipitous ideals. We present here a slight generalization of Foreman's theorem, correcting a minor mistake in [14]. The mistake was not in the main result but in a statement regarding the extent of the applicability of its hypothesis. The correction leads to a more general theorem and an equivalence between the generic extendibility of elementary embeddings and a structural characterization of induced ideals.

Claim (Foreman)**.** *Suppose I is a precipitous ideal on Z, $\mathbb{P}$ is a partial order, and $\dot{m}$ is a $\mathcal{P}(Z)/I$-name such that $(1, \dot{m}) \Vdash_{\mathcal{P}(Z)/I * j(\mathbb{P})}$ "$j^{-1}[\hat{H}]$ is $\mathbb{P}$-generic over V," where $\hat{H}$ denotes the generic for $j(\mathbb{P})$. Then there is some $q \in \mathbb{P}$ such that the map e defined by $p \mapsto (1, \dot{m} \wedge \dot{j(p)})$ is a regular embedding of $\mathbb{P} \restriction q$ into $\mathcal{P}(Z)/I * (j(\mathbb{P} \restriction q) \restriction \dot{m})$.*

Counterexample Assume CH and there is an ω_1-dense ideal I on ω_1. Then $\mathcal{P}(\omega_1)/I \cong \mathcal{B}(\mathrm{Col}(\omega, \omega_1))$. Since $\mathrm{Col}(\omega, \omega_1) \sim \mathrm{Add}(\omega_1) \times \mathrm{Col}(\omega, \omega_1)$ under CH, forcing with $\mathcal{P}(\omega_1)/I$ adds a Cohen-generic subset $H \subseteq \omega_1$. Let G be generic for $\mathcal{P}(\omega_1)/I$ and let $j : V \to M \subseteq V[G]$ be the generic ultrapower embedding. Then $H \in M$, and H is a condition $m \in j(\mathrm{Add}(\omega_1)) = \mathrm{Add}(\omega_1)^{V[G]}$. If we take a $\mathcal{P}(\omega_1)/I$-name $\dot{m}$ for m, then the condition $(1, \dot{m}) \in \mathcal{P}(\omega_1)/I * \dot{\mathrm{Add}}(\omega_1)$ forces that $j^{-1}[\hat{H}]$ is $\mathrm{Add}(\omega_1)$-generic over V, where $\hat{H}$ denotes the generic for $\mathrm{Add}(\omega_1)^{V[G]}$.

Now the map e defined by $e(p) = (1, \dot{j(p)} \wedge \dot{m}) = (1, \check{p} \wedge \dot{m})$ is not a regular embedding of $\mathrm{Add}(\omega_1)$ into $\mathcal{P}(\omega_1)/I * \mathrm{Add}(\omega_1)$ for one simple reason. Its range is not contained in the purported codomain. For any nontrivial $p \in \mathrm{Add}(\omega_1)$, $[\omega_1]_I$ does not force that p is compatible with $\dot{m}$. In fact, the set $\{(p, A) : A \Vdash p \perp \dot{m}\}$ is dense in the product order of $\mathrm{Add}(\omega_1) \times \mathcal{P}(\omega_1)/I$. Thus Foreman's claim is incorrect.

To see this, let $\iota : \mathrm{Add}(\omega_1) \times \mathrm{Col}(\omega, \omega_1) \to \mathcal{P}(\omega_1)/I$ be a dense embedding, and let the name for the condition m be the projection of $\iota^{-1}[G]$ to the first coordinate. Let $(p, A) \in \mathrm{Add}(\omega_1) \times \mathcal{P}(\omega_1)/I$ be arbitrary, and let $\iota(p_0, q_0) \leq [A]_I$. If $p_0 \perp p$, then any $[B]_I \leq \iota(p_0, q_0)$ forces $\dot{m} \perp p$. Otherwise, let $p_1, p_2 \leq p_0 \wedge p$ be such that $p_1 \perp p_2$. Let $B \subseteq A$ be such that $[B]_I \leq \iota(p_1, q_0)$. Then $B \Vdash \check{p}_2 \perp \dot{m}$.

To fix this, we only need to redefine the map e. We should instead send p to $||j(p) \in \hat{H}||$. In some cases, this will coincide with Foreman's map, but not always.

Theorem 5.1. *Suppose I is a precipitous ideal on Z and $\mathbb{P}$ is a boolean algebra. Let $j : V \to M \subseteq V[G]$ denote a generic ultrapower embedding arising from I. Suppose $\dot{\mathbb{Q}}$ is a $\mathcal{P}(Z)/I$-name for a forcing and $\dot{H}_0$ is a name such that:*

(1) $1 \Vdash_{\mathcal{P}(Z)/I * \dot{\mathbb{Q}}} \dot{H}_0$ is $j(\mathbb{P})$-generic over M,

(2) $1 \Vdash_{\mathcal{P}(Z)/I * \dot{\mathbb{Q}}} j^{-1}[\dot{H}_0]$ is $\mathbb{P}$-generic over V, and

(3) for all $p \in \mathbb{P}$, $1 \nVdash_{\mathcal{P}(Z)/I * \dot{\mathbb{Q}}} j(p) \notin \dot{H}_0$.

In $V[G]$, let K be the ideal $\{p \in j(\mathbb{P}) : 1 \Vdash_{\mathbb{Q}} p \notin \dot{H}_0\}$. There is $\mathbb{P}$-name for an ideal J on Z and a canonical isomorphism

$$\iota : \mathcal{B}(\mathbb{P} * \mathcal{P}(\dot{Z})/J) \cong \mathcal{B}(\mathcal{P}(Z)/I * \dot{j(\mathbb{P})}/K).$$

Proof. Denote a generic for $\mathcal{P}(Z)/I * j(\mathbb{P})/K$ by $G * h$. In $V[G * h]$, let $\hat{H} = \{p \in j(\mathbb{P}) : [p]_K \in h\}$. First we claim $\hat{H}$ has properties (1),(2),(3) assumed for H_0.

(1) If $D \in M$ is open and dense in $j(\mathbb{P})$, then $\{[d]_K : d \in D \text{ and } d \notin K\}$ is dense in $j(\mathbb{P})/K$. For otherwise, there is $p \in j(\mathbb{P}) \setminus K$ such that $p \wedge d \in K$ for all $d \in D$. By the definition of K, we can force with $\mathbb{Q}$ to obtain a filter $H_0 \subseteq j(\mathbb{P})$ with $p \in H_0$. But H_0 cannot contain

any elements of D, so it is not generic over M, a contradiction. Thus if $h \subseteq j(\mathbb{P})/K$ is generic over $V[G]$, then $\hat{H}$ is $j(\mathbb{P})$-generic over M.

(2) If $A \in V$ is a maximal antichain in $\mathbb{P}$, then $\{[j(a)]_K : a \in A \text{ and } j(a) \notin K\}$ is a maximal antichain in $j(\mathbb{P})/K$. For otherwise, there is $p \in j(\mathbb{P}) \setminus K$ such that $p \wedge j(a) \in K$ for all $a \in A$. We can force with $\mathbb{Q}$ to obtain a filter $H_0 \subseteq j(\mathbb{P})$ with $p \in H_0$. But H_0 cannot contain any elements of $j[A]$, so $j^{-1}[H_0]$ is not generic over V, a contradiction.

(3) If $p \in \mathbb{P}$, then the assumption about $\dot{H}_0$ implies that $1 \nVdash_{\mathcal{P}(Z)/I} j(p) \in K$.

Let $e : \mathbb{P} \to \mathcal{B}(\mathcal{P}(Z)/I * j(\mathbb{P})/K)$ be defined by $p \mapsto ||j(p) \in \hat{H}||$. By (3), this map has trivial kernel. By elementarity, it is an order and antichain preserving map. If $A \subseteq \mathbb{P}$ is a maximal antichain, then it is forced that $j^{-1}[\hat{H}] \cap A \neq \emptyset$. Thus e is regular.

Whenever $H \subseteq \mathbb{P}$ is generic, there is a further forcing yielding a generic $G * h \subseteq \mathcal{P}(Z)/I * j(\mathbb{P})/K$ such that $j[H] \subseteq \hat{H}$. Thus there is an embedding $\hat{j} : V[H] \to M[\hat{H}]$ extending j. In $V[H]$, let $J = \{A \subseteq Z : 1 \Vdash_{(\mathcal{P}(Z)/I * j(\mathbb{P})/K)/e[H]} [id]_M \notin \hat{j}(A)\}$. In V, define a map $\iota : \mathbb{P} * \mathcal{P}(\dot{Z})/J \to \mathcal{B}(\mathcal{P}(Z)/I * j(\dot{\mathbb{P}})/K)$ by $(p, \dot{A}) \mapsto e(p) \wedge ||[id]_M \in \hat{j}(\dot{A})||$. It is easy to check that ι is order and antichain preserving.

We want to show the range of ι is dense. Let $(B, \dot{q}) \in \mathcal{P}(Z)/I * j(\dot{\mathbb{P}})/K$, and WLOG, we may assume there is some $f : Z \to V$ in V such that $B \Vdash \dot{q} = [[f]_M]_K$. By the regularity of e, let $p \in \mathbb{P}$ be such that for all $p' \leq p$, $e(p') \wedge (B, \dot{q}) \neq 0$. Let $\dot{A}$ be a $\mathbb{P}$-name such that $p \Vdash \dot{A} = \{z \in B : f(z) \in H\}$, and $\neg p \Vdash \dot{A} = Z$. $1 \Vdash_{\mathbb{P}} \dot{A} \in J^+$ because for any $p' \leq p$, we can take a generic $G * h$ such that $e(p') \wedge (B, \dot{q}) \in G * h$. Here we have $[id]_M \in j(B)$ and $[f]_M \in \hat{H}$, so $[id]_M \in \hat{j}(A)$. Furthermore, $\iota(p, \dot{A})$ forces $B \in G$ and $q \in h$, showing ι is a dense embedding. □

Proposition 5.2 (Foreman). *Suppose the ideal K in Theorem 5.1 is forced to be principal. Let $\dot{m}$ be such that $\Vdash_{\mathcal{P}(Z)/I} \dot{K} = \{p \in j(\mathbb{P}) : p \leq \neg \dot{m}\}$. Suppose f and A are such that*

$A \Vdash \dot{m} = [f]$, and $\dot{B}$ is a $\mathbb{P}$-name for $\{z \in A : f(z) \in H\}$. Let $\bar{I}$ be the ideal generated by I in $V[H]$. Then $\bar{I} \restriction B = J \restriction B$, and $A \setminus B \in J$.

Proof. Clearly $J \supseteq \bar{I}$. Suppose that $\Vdash \dot{C} \subseteq \dot{B}$ and $\dot{C} \in \bar{I}^+$, and let $p \in \mathbb{P}$ be arbitrary. WLOG $\mathbb{P}$ is a complete boolean algebra. For each $z \in Z$, let $b_z = ||z \in \dot{C}||$. In V, define $C' = \{z : p \wedge b_z \wedge f(z) \neq 0\}$. $p \Vdash \dot{C} \subseteq C'$, so $C' \in I^+$. If $G \subseteq \mathcal{P}(Z)/I$ is generic with $C' \in G$, then $j(p) \wedge b_{[id]} \wedge \dot{m} \neq 0$. Take $\hat{H} \subseteq j(\mathbb{P})$ generic over $V[G]$ with $j(p) \wedge b_{[id]} \wedge \dot{m} \in \hat{H}$. Since $b_{[id]} \Vdash^M_{j(\mathbb{P})} [id] \in \hat{j}(C)$, $p \nVdash \dot{C} \in J$ as $p \in H = j^{-1}[\hat{H}]$. Thus $J \restriction B = \bar{I} \restriction B$. Furthermore, if $H \subseteq \mathbb{P}$ is any generic, and $G * h$ is generic extending $e[H]$ with $A \in G$, then $\dot{m} = j(f)([id]) \in \hat{H}$, so $[id] \in \hat{j}(B)$. Thus $A \setminus B \in J$. □

Corollary 5.3. *Let I be a precipitous ideal on Z and $\mathbb{P}$ a partial order. Let $\bar{I}$ denote the ideal generated by I in $V^{\mathbb{P}}$. The following are equivalent:*

(1) In some $\mathcal{P}(Z)/I * j(\mathbb{P})$-generic extension $V[G * \hat{H}]$, $H = j^{-1}[\hat{H}]$ is $\mathbb{P}$-generic over V.

(2) For some conditions a, b, there is an isomorphism $\iota : \mathcal{B}(\mathbb{P} * \mathcal{P}(Z)/\bar{I}) \restriction a \cong \mathcal{B}(\mathcal{P}(Z)/I * j(\mathbb{P})) \restriction b$ such that $\iota(||p \in H||) = ||j(p) \in \hat{H}||$ for all $p \in \mathbb{P}$.

Proof. WLOG we may assume $\mathbb{P}$ is a complete boolean algebra. (1) implies that for some $(A, \dot{m}) \in \mathcal{P}(Z)/I * j(\mathbb{P})$, $(A, \dot{m}) \Vdash j^{-1}[\hat{H}]$ is $\mathbb{P}$-generic over V. By shrinking A if necessary, we may assume for some $f \in V$, $A \Vdash \dot{m} = [f]$. Let $e : \mathbb{P} \to \mathcal{B}(\mathcal{P}(A)/I * j(\mathbb{P}) \restriction \dot{m})$ be defined by $p \mapsto ||j(p) \in \hat{H}||$. We claim that for some $p \in \mathbb{P}$, $e(p') \neq 0$ for all $p' \leq p$. Otherwise, the set of p such that $e(p) = 0$ is dense, so whenever $G * \hat{H}$ is generic with $(A, \dot{m}) \in G * \hat{H}$, there is $p \in \mathbb{P}$ such that $j(p) \in \hat{H}$ and $||j(p) \in \hat{H}|| = 0$, a contradiction. If $D = \{p \in \mathbb{P} : (\forall p' \leq p) e(p') \neq 0\}$, then $p_0 = \sum D \in D$. Since $\{p \leq \neg p_0 : e(p) = 0\}$ is dense below $\neg p_0$, we must have $e(\neg p_0) = 0$. Thus $(A, \dot{m}) \Vdash j(p_0) \in \hat{H}$, and so $A \Vdash \dot{m} \leq j(\dot{p}_0)$.

The hypotheses of Theorem 5.1 are satisfied with respect to $I \restriction A$ and $\mathbb{P} \restriction p_0$, and the ideal K on $j(\mathbb{P} \restriction p_0)$ is the principal ideal generated by $\neg\dot{m}$. We get an isomorphism $\iota : \mathcal{B}((\mathbb{P} \restriction p_0) *$

$\mathcal{P}(\dot{A})/J) \cong \mathcal{P}(A)/I * (j(\mathbb{P} \restriction \dot{p}_0) \restriction m))$ such that $\iota(||p \in H||) = \iota(p, \dot{1}) = e(p) = ||j(p) \in \hat{H}||$ for all $p \in \mathbb{P} \restriction p_0$. By Proposition 5.2, there is a set $B \in V^{\mathbb{P}}$ such that we may replace $\mathcal{P}(A)/J$ on the left by $\mathcal{P}(B)/\bar{I}$. Thus (1) $\Rightarrow$ (2). For the other direction, if an isomorphism exists with those properties, then for any generic $G * \hat{H}$, $\iota^{-1}[G * \hat{H}] \cap \mathbb{P} = j^{-1}[\hat{H}]$. □

Corollary 5.4 (Foreman)**.** *If I is a κ-complete precipitous ideal on Z and $\mathbb{P}$ is κ-c.c., then there is a canonical isomorphism $\iota : \mathbb{P} * \mathcal{P}(Z)/\bar{I} \cong \mathcal{P}(Z)/I * j(\mathbb{P})$.*

Proof. If $G * \hat{H} \subseteq \mathcal{P}(Z)/I * j(\mathbb{P})$ is generic, then for any maximal antichain $A \subseteq \mathbb{P}$ in V, $j[A] = j(A)$, and $M \models j(A)$ is a maximal antichain in $j(\mathbb{P})$. Thus $j^{-1}[\hat{H}]$ is $\mathbb{P}$-generic over V, and clearly for each $p \in \mathbb{P}$, we can take $\hat{H}$ with $j(p) \in \hat{H}$. Thus Theorem 5.1 implies that for some $\dot{J}, \dot{K}$, we have an isomorphism $\iota : \mathcal{B}(\mathbb{P} * \mathcal{P}(Z)/J) \to \mathcal{B}(\mathcal{P}(Z)/I * j(\mathbb{P})/K)$. In this case, K is trivial, and so Proposition 5.2 implies that $J = \bar{I}$. □

Proposition 5.5. *If $Z, I, \mathbb{P}, J, K, \iota$ are as in Theorem 5.1, then whenever $H \subseteq \mathbb{P}$ is generic, J is precipitous and has the same completeness and normality that I has in V. Also, if $\bar{G} \subseteq \mathcal{P}(Z)/J$ is generic and $G * h = \iota[H * \bar{G}]$, then if $\hat{j} : V[H] \to M[\hat{H}]$ is as above, $M[\hat{H}] = V[H]^Z/\bar{G}$ and $\hat{j}$ is the canonical ultrapower embedding.*

Proof. Suppose $H * \bar{G} \subseteq \mathbb{P} * \mathcal{P}(Z)/\bar{I}$ is generic, and let $G * h = \iota[H * \bar{G}]$ and $\hat{H} = \{p : [p]_K \in h\}$. For $A \in J^+$, $A \in \bar{G}$ iff $[id]_M \in \hat{j}(A)$. If $i : V[H] \to N = V[H]^Z/\bar{G}$ is the canonical ultrapower embedding, then there is an elementary embedding $k : N \to M[\hat{H}]$ given by $k([f]_N) = \hat{j}(f)([id]_M)$, and $\hat{j} = k \circ i$. Thus N is well-founded, so J is precipitous. If $f : Z \to \mathrm{Ord}$ is a function in V, then $k([f]_N) = j(f)([id]_M) = [f]_M$. Thus k is surjective on ordinals, so it must be the identity, and $N = M[\hat{H}]$. Since $i = \hat{j}$ and $\hat{j}$ extends j, i and j have the same critical point, so the completeness of J is the same as that of I. Finally, since $[id]_N = [id]_M$, I is normal in V iff J is normal in $V[H]$, because $j \restriction \bigcup Z = \hat{j} \restriction \bigcup Z$, and normality is equivalent to $[id] = j[\bigcup Z]$. □

Proposition 5.6. *Suppose* $\iota : \mathcal{B}(\mathbb{P} * \mathcal{P}(Z)/J) \to \mathcal{B}(\mathcal{P}(Z)/I * j(\mathbb{P})/K)$ *is as in Theorem 5.1. For any* $B \in I^+$*, if* $(p, \dot{A}) \in \mathbb{P} * \mathcal{P}(Z)/J$ *and* $p \Vdash \dot{A} = \check{B}$*, then* $\iota(p, \check{A}) = (B, \dot{1}) \wedge e(p)$*.*

Proof. By definition, $\iota(p, \dot{A}) = e(p) \wedge ||[id] \in \hat{j}(\dot{A})||$. If $G * h$ is generic with this condition, then $[id] \in \hat{j}(\dot{A}^{e^{-1}[G*h]})$, and so $[id] \in j(B)$ and $(B, \dot{1}) \wedge e(p) \in G * h$. If $(B, \dot{1}) \wedge e(p) \in G * h$, then $[id] \in j(B) = \hat{j}(\dot{A}^{e^{-1}[G*h]})$, so $\iota(p, \check{A}) \in G * h$. The equality follows. □

5.2 Preservation and destruction

We show here that density and nonregularity can be consistently separated by showing that certain forcings will preserve nonregularity while destroying density. We give two different preservation results, each of which starts from different assumptions about the ground model. They provide two different paths to our desired consistency result.

Lemma 5.7. *Suppose* μ *is regular,* $\mathbb{P}$ *is* μ*-c.c.,* I *is an ideal on* Z*, and* $p \Vdash_{\mathbb{P}}$ "$\bar{I}$ *is* (μ, κ)*-regular." Then* I *is* (μ, κ)*-regular.*

Proof. Let $\langle A_\alpha \rangle_{\alpha<\kappa} \subseteq I^+$. Let $\langle \dot{B}_\alpha \rangle_{\alpha<\kappa}$ be such that $p \Vdash_{\mathbb{P}}$ "$\langle \dot{B}_\alpha \rangle_{\alpha<\kappa} \subseteq \bar{I}^+$ is a refinement of $\langle A_\alpha \rangle_{\alpha<\kappa}$ such that for all $X \subseteq \kappa$ of size μ, $\bigcap_{\alpha \in X} \dot{B}_\alpha = \emptyset$." For each α, let $C_\alpha = \{z : (\exists q \leq p) q \Vdash z \in \dot{B}_\alpha\}$. Each $C_\alpha \in I^+$ because $p \Vdash \dot{B}_\alpha \subseteq C_\alpha$. For each $z \in Z$, let $s(z) = \{\alpha : z \in C_\alpha\}$. Since $\mathbb{P}$ is μ-.c.c., $|s(z)| < \mu$ for each z. So $\langle C_\alpha \rangle_{\alpha<\kappa}$ is the desired refinement of $\langle A_\alpha \rangle_{\alpha<\kappa}$. □

Lemma 5.8. *Suppose* I *is a normal, fine, nonregular ideal on* $Z \subseteq \mathcal{P}(\lambda)$ *that is* λ^+*-saturated but nowhere* λ*-saturated. Let* $\langle A_\alpha \rangle_{\alpha<\lambda} \subseteq I^+$ *be a sequence with no disjoint refinement into* I*-positive sets. Then there is an* I*-positive* $B \in I^+$ *such that for all* I*-positive* $C \subseteq B$*,* $|\{\alpha : C \cap A_\alpha \in I^+\}| = \lambda$*.*

Proof. Suppose towards a contradiction that $\{B : |\{\alpha : B \cap A_\alpha \in I^+\}| < \lambda\}$ is dense. Let $\{B_\alpha : \alpha < \lambda\}$ be a maximal antichain contained in this collection. We may assume the B_α's are pairwise disjoint. Let $s(\alpha) = \{\beta : B_\alpha \cap A_\beta \in I^+\}$. By Theorem 1.5, for each B_α, there is a disjoint refinement of $\{B_\alpha \cap A_\beta : \beta \in s(\alpha)\}$ into I-positive sets $\{C^\alpha_\beta : \beta \in s(\alpha)\}$, as $\{I \restriction (B_\alpha \cap A_\beta) : \beta \in s(\alpha)\}$ is a set of $< \lambda$ many nowhere λ-saturated ideals on B_α. Since $\{B_\alpha : \alpha < \lambda\}$ is a maximal antichain, for all β there is α such that $\beta \in s(\alpha)$. Let $h(\beta) =$ the least α such that $\beta \in s(\alpha)$. Then $\{C^{h(\beta)}_\beta : \beta < \lambda\}$ is a disjoint refinement of $\{A_\beta : \beta < \lambda\}$ into I-positive sets, contrary to assumption. □

Lemma 5.9. *Suppose $\mu < \lambda$ are regular cardinals, $\kappa = \mu^+$, $\mathbb{P}$ is μ-c.c., and I is a normal, fine, nonregular, κ-complete ideal on $Z \subseteq \mathcal{P}(\lambda)$. Then $\Vdash_{\mathbb{P}} \bar{I}$ is nonregular.*

Proof. Suppose $\langle A_\alpha \rangle_{\alpha<\lambda}$ witnesses that I is nonregular. Let $X_0 = \{\alpha < \lambda : I \restriction A_\alpha$ is nowhere λ^+-saturated$\}$. For each $\alpha \in X_1 = \lambda \setminus X_0$, pick $A'_\alpha \subseteq A_\alpha$ such that $I \restriction A'_\alpha$ is λ^+-saturated. Note that $I \restriction (\bigcup_{\alpha \in X_1} A'_\alpha \cap \hat{\alpha})$ is λ^+-saturated. We claim that $\{A'_\alpha : \alpha \in X_1\}$ witnesses that I is nonregular.

To see this, assume towards a contradiction that $\{A''_\alpha : \alpha \in X_1\}$ is a pairwise disjoint refinement into I-positive sets. By Lemma 4.2, there is a collection $\{B_\alpha : \alpha < \lambda^+\}$ such that $(\forall \alpha < \lambda^+)(\forall \beta \in X_0) B_\alpha \cap A_\beta \in I^+$, and $(\forall \alpha < \beta < \lambda^+) B_\alpha \cap B_\beta$ is nonstationary. Let $\gamma < \lambda^+$ be such that $C_\alpha = A''_\alpha \setminus B_\gamma \in I^+$ for all $\alpha \in X_1$. Apply Lemma 4.2 again to split B_γ into pairwise disjoint subsets $\{C_\alpha : \alpha \in X_0\}$ such that $C_\alpha \in I^+$ and $C_\alpha \subseteq A_\alpha$ for all $\alpha \in X_0$. Then $\{C_\alpha : \alpha < \lambda\}$ is a pairwise disjoint refinement of $\{A_\alpha : \alpha < \lambda\}$, contrary to assumption.

Considering $I \restriction (\bigcup_{\alpha \in X_1} A'_\alpha \cap \hat{\alpha})$ and renaming the A'_α's, we can assume WLOG that we start with a λ^+-saturated, normal, fine, κ-complete, nonregular ideal I on Z with $\langle A_\alpha \rangle_{\alpha<\lambda}$ witnessing the nonregularity. Assume for the sake of a contradiction that $p \Vdash \bar{I}$ is regular. Let $\langle \dot{B}_\alpha \rangle_{\alpha<\lambda}$ be a sequence of $\mathbb{P}$-names such that $p \Vdash \langle \dot{B}_\alpha \rangle_{\alpha<\lambda}$ is a pairwise disjoint refinement of $\langle \check{A}_\alpha \rangle_{\alpha<\lambda}$ into $\bar{I}$-positive sets. So $\{(p, \dot{B}_\alpha) : \alpha < \lambda\}$ is an antichain in $\mathbb{P} * \mathcal{P}(Z)/\bar{I}$.

Let $\iota : \mathbb{P} * \mathcal{P}(Z)/\bar{I} \to \mathcal{B}(\mathcal{P}(Z)/I * j(\mathbb{P}))$ be given by Corollary 5.4. For each $\alpha < \lambda$, choose $(C_\alpha, \dot{q}_\alpha) \leq \iota(p, \dot{B}_\alpha)$. Since

$$(C_\alpha, \dot{q}_\alpha) \leq \iota(p, \dot{B}_\alpha) \leq \iota(p, \check{A}_\alpha) \leq \iota(1, \check{A}_\alpha) = (A_\alpha, \dot{1}),$$

we have $C_\alpha \subseteq_I A_\alpha$ for each $\alpha < \lambda$. Since $\langle A_\alpha \rangle_{\alpha<\lambda}$ has no disjoint refinement into I-positive sets, neither does $\langle C_\alpha \rangle_{\alpha<\lambda}$.

Since I satisfies the hypotheses of Lemma 5.8, there is $D \in J^+$ such that for all J-positive $E \subseteq D$, $|\{\alpha : E \cap C_\alpha \in J^+\}| = \lambda$. Thus, if we take a generic $G \subseteq \mathcal{P}(Z)/I$ with $D \in G$, then a density argument shows that $\{\alpha : C_\alpha \in G\}$ is unbounded in λ. Let $M = V^Z/G$. M thinks that μ is a regular cardinal and $j(\mathbb{P})$ is μ-c.c. By the closure properties of M, these hold in $V[G]$ as well. By Theorem 3.4, $\mathrm{cf}(\lambda) \geq \mu$ in $V[G]$, so there are at least μ many C_α's in G. Since $(C_\alpha, \dot{q}_\alpha) \perp (C_\beta, \dot{q}_\beta)$ for $\alpha \neq \beta$, we must have that $\dot{q}_\alpha^G \perp \dot{q}_\beta^G$ for distinct $C_\alpha, C_\beta \in G$. However, these are all elements of a μ-c.c. partial order, so we have a contradiction. Thus $\Vdash_{\mathbb{P}} \bar{I}$ is nonregular. □

Definition. *If $\mathbb{P}$ is a partial order and $Z \subseteq \mathcal{P}(X)$, we will say $\mathbb{P}$ is* Z-absolutely κ-c.c. *when for all normal, fine, $|X|^+$ saturated ideals I on Z, $\Vdash_{\mathcal{P}(Z)/I} j(\mathbb{P})$ is $j(\kappa)$-c.c.*

Lemma 5.10. *Suppose $\kappa = \mu^+$ and $Z \subseteq \{z \in \mathcal{P}_\kappa(\lambda) : z \cap \kappa \in \kappa\}$ is stationary. Suppose $\mathbb{P}$ is Z-absolutely κ-c.c. and $\kappa \leq \mathrm{d}(\mathbb{P} \restriction p) \leq \lambda$ for all $p \in \mathbb{P}$. Then in $V^{\mathbb{P}}$, there are no normal, fine, κ-complete λ-dense ideals on Z.*

Proof. Suppose $p \Vdash \dot{J}$ is a normal, fine, κ-complete, λ^+-saturated ideal on Z. Let $I = \{X \subseteq Z : p \Vdash X \in \dot{J}\}$. It is easy to check that I is normal and fine. The map $\sigma : \mathcal{P}(Z)/I \to \mathcal{B}(\mathbb{P} \restriction p * \mathcal{P}(Z)/\dot{J})$ that sends X to $(||\check{X} \in \dot{J}^+||, [\dot{X}]_J)$ is an order-preserving and antichain-preserving map, so I is λ^+-saturated.

Let H be $\mathbb{P}$-generic over V with $p \in H$. Since $\mathbb{P}$ is κ-c.c., $\bar{I}$ remains normal. By Corollary 5.4, $\mathcal{P}^{V[H]}(Z)/\bar{I} \cong (\mathcal{P}^V(Z)/I * j(\mathbb{P}))/e[H]$. Thus $\bar{I}$ is normal, fine, and λ^+-saturated, and $\bar{I} \subseteq J$. By Lemma 4.3, there is $A \in \bar{I}^+$ such that $J = \bar{I} \restriction A$. Since $j(\kappa) = \lambda^+$, $j(\mathbb{P})$ is forced to be nowhere λ-dense, and thus $\mathcal{P}(Z)/I * j(\mathbb{P})$ is nowhere λ-dense. Since $\mathrm{d}(\mathbb{P}) \leq \lambda$, $(\mathcal{P}^V(Z)/I * j(\mathbb{P}))/e[H]$ is nowhere λ-dense. Thus J is not λ-dense. $\square$

Theorem 5.11. *If almost-huge cardinals are consistent, then ZFC does not prove the analogs of Theorem 4.1 and Corollary 4.6 with ω_1 is replaced by μ^+ where $\mu > \omega$ is regular. If $\omega < \mu < \kappa \leq \lambda < \delta$ are regular, and κ carries an almost-huge tower of height δ, then there is a forcing extension in which $\kappa = \mu^+$, there are no normal, fine, κ-complete, λ-dense ideals on a stationary $Z \subseteq \mathcal{P}_\kappa(\lambda)$, but there is a normal, fine, κ-complete, nonregular ideal on Z. If κ is super-almost-huge, then there is a forcing extension in which $\kappa = \mu^+$, and for every regular $\lambda \geq \kappa$, there is a stationary $Z_\lambda \subseteq \mathcal{P}_\kappa(\lambda)$ such that there are no dense ideals on Z_λ, but there is a nonregular ideal on Z_λ.*

Proof. Let $\mu < \kappa \leq \lambda$ be as hypothesized. By Theorem 2.17, there is a forcing extension in which $\kappa = \mu^+$ there is a normal, fine, κ-complete, λ-dense ideal I on $\mathcal{P}_\kappa(\lambda)$. In this model, let $\mathbb{P}$ be any μ-c.c. partial order such that $\kappa \leq \mathrm{d}(\mathbb{P} \restriction p) \leq \lambda$ for all $p \in \mathbb{P}$, such as $\mathrm{Add}(\omega, \kappa)$. If J is any normal, fine, κ-complete, λ^+-saturated ideal on $\mathcal{P}_\kappa(\lambda)$, and $j : V \to M \subseteq V[G]$ is a generic embedding arising from J, then $M \models j(\mathbb{P})$ is μ-c.c. Since $M^\mu \cap V[G] \subseteq M$, $j(\mathbb{P})$ is μ-c.c. in $V[G]$. Hence by Lemma 5.10, we destroy all dense ideals by forcing with $\mathbb{P}$.

By Lemma 5.9, the generated ideal $\bar{I}$ remains nonregular. Furthermore, one can check that the ideal given by Theorem 2.17 is μ-distributive, so it is actually (μ, λ)-nonregular by Lemma 4.16. So we may alternatively apply Lemma 5.7 to see that $\bar{I}$ is (μ, λ)-nonregular and thus nonregular.

If we start with a super-almost-huge κ, then in a forcing extension $\kappa = \mu^+$ and these ideals on $\mathcal{P}_\kappa(\lambda)$ exist for all regular $\lambda \geq \kappa$, so forcing with $\mathbb{P}$ has the these effects simultaneously with respect to all regular $\lambda \geq \kappa$. $\square$

At present, it is unknown whether properties (1) and (2) from Theorem 4.1 are equivalent under ZFC when ω_1 is replaced by larger cardinal. Does the existence of κ many normal and fine ideals on κ that collectively measure all subsets of κ imply the existence of a κ-complete, κ-dense ideal on κ? The best we can do now is point to the need for a different approach–our methods are demonstrably inadequate:

Proposition 5.12. *Suppose GCH, $\mu < \kappa = \mu^+ \leq \lambda$, there is a normal, fine, κ-complete, λ-dense ideal I on $Z \subseteq \mathcal{P}_\kappa(\lambda)$, and $\mathbb{P}$ is Z-absolutely κ-c.c. Then in $V^{\mathbb{P}}$, either there is a normal, fine, κ-complete, λ-dense ideal on Z, or for every set $\{I_\alpha : \alpha < \lambda\}$ of normal, fine, κ-complete ideals on Z, there is $A \subseteq Z$ that is nonmeasurable for each I_α.*

Proof. If there is $p \in \mathbb{P}$ such that $\mathrm{d}(\mathbb{P} \restriction p) < \kappa$, then $p \Vdash \bar{I}$ is λ-dense. Otherwise, $\mathrm{d}(\mathbb{P} \restriction p) \geq \kappa$ for all $p \in \mathbb{P}$. Suppose $p \Vdash \{\dot{I}_\alpha : \alpha < \lambda\}$ is a set of normal, fine, κ-complete ideals on $\check{Z}$ such that every $A \subseteq \check{Z}$ is measurable in one of them. By Lemma 4.7, there is some name $\dot{I}$ such that $p \Vdash \dot{I}$ is a normal, fine, κ-complete, λ^+-saturated ideal on $\check{Z}$ such that $\mathcal{P}(\dot{Z})/I$ has a weakly dense subset $\{\dot{A}_\alpha : \alpha < \lambda\}$. In V, let $J = \{A \subseteq Z : p \Vdash A \in \dot{I}\}$. J is normal, fine, κ-complete, and λ^+-saturated. By the hypothesis on $\mathbb{P}$, Corollary 5.4, and Lemma 4.3, there is some name $\dot{B}$ such that $p \Vdash \dot{I} = \bar{J} \restriction \dot{B}$.

Let $\theta > \lambda$ be regular such that $\mathbb{P} \in H_\theta$. Let $M \prec H_\theta$ be such that $\{\mathbb{P}, p, \dot{I}, \dot{B}, \{\dot{A}_\alpha : \alpha < \lambda\}\} \in M$, $\lambda \subseteq M$, $|M| = \lambda$, and $M^{<\kappa} \subseteq M$. If $A \subseteq \mathbb{P} \cap M$ is an antichain, then by the κ-c.c. of $\mathbb{P}$, $A \in M$. If A is not maximal in $\mathbb{P}$, then by elementarity, there is some $p \in M$ such that $A \perp p$, so A is not maximal in $\mathbb{P} \cap M$. Thus $\mathbb{P}_0 = \mathbb{P} \cap M$ is a regular suborder of $\mathbb{P}$. If $G \subseteq \mathbb{P}$ is generic, then $\dot{B}^G$ and each $\dot{A}_\alpha^G$ are in $V[G \cap \mathbb{P}_0]$. Let $\bar{J}_0$ be the ideal generated by J in $V[G \cap \mathbb{P}_0]$. If $C \in \bar{J}_0 \restriction B$, then $\{A_\alpha : \alpha < \lambda\}$ is weakly dense in $\mathcal{P}(Z)/\bar{J}_0$. This is because if $C \in \mathcal{P}(Z)^{V[G \cap \mathbb{P}_0]}$, then in $V[G]$, there is some A_α such that either $A_\alpha \leq_{\bar{J}} C$ or $A_\alpha \leq_{\bar{J}} Z \setminus C$, and this property is absolute between transitive models.

However, by Lemma 5.10, forcing with $\mathbb{P}_0$ destroys all dense ideals on Z. By the κ-c.c., and GCH in V, we have $\Vdash_{\mathbb{P}_0} 2^{<\lambda} = \lambda$. Thus by Theorem 4.8, there cannot exist a normal, fine, λ^+-saturated ideal K on Z in $V[G \cap \mathbb{P}_0]$ such that $\mathcal{P}(Z)/K$ has a weakly dense subset of size λ. This contradiction establishes the dichotomy. $\square$

In contrast to density and saturation, nonregularity of ideals can be destroyed by small forcing:

Proposition 5.13. *Suppose $\kappa = \mu^+$ and $Z \subseteq \mathcal{P}_\kappa(\lambda)$ is such that there is a normal, fine, κ-complete, nonregular ideal on Z, but there are no normal, fine, κ-complete, λ-dense ideals on Z. Then forcing with* $\mathrm{Col}(\omega, \mu)$ *destroys all nonregular ideals on Z.*

Proof. Assume towards a contradiction that $G \subseteq \mathrm{Col}(\omega, \mu)$ is generic, and $J \in V[G]$ is a normal, fine, κ-complete, nonregular ideal on Z. By the first part of the argument for Lemma 5.9, we may assume J is λ^+-saturated. Let p force these properties for $\dot{J}$, and in V, let $I = \{A \subseteq Z : p \Vdash A \in \dot{J}\}$. I is normal, fine, κ-complete, and λ^+ saturated, and properties are preserved for $\bar{I}$ in $V[G]$, so by Lemma 4.3, $J = \bar{I} \restriction A$ for some A. By Lemma 4.5, there must be some J-positive $B \subseteq A$ such that $J \restriction B = \bar{I} \restriction B$ is λ-dense. Let $p \Vdash \langle \dot{B}_\alpha \rangle_{\alpha<\lambda}$ is dense in $\bar{I} \restriction \dot{B}$.

By κ-completeness, there is some $q \leq p$, $q \in G$, such that $C = \{z : q \Vdash z \in \dot{B}\} \in I^+$. For each $r \leq q$ and $\alpha < \lambda$ let $C^r_\alpha = \{z \in C : r \Vdash z \in \dot{B}_\alpha\}$. In V, let $D \subseteq C$ be I-positive. In $V[G]$, there is some $B_\alpha \subseteq_J D$, and by κ-completeness, there is some $r \in G$ such that $C^r_\alpha \in J^+$. Thus $C^r_\alpha \in I^+$ and $C^r_\alpha \setminus D \in I$. We have that $\{C^r_\alpha : r \leq q, \alpha < \lambda, \text{and } C^r_\alpha \in I^+\}$ is dense below C, contrary to assumption. $\square$

Regardless of the existence of dense ideals, we can destroy nonregular ideals while preserving saturation:

Proposition 5.14. *Suppose $\lambda \geq \kappa = \mu^+$, $\mu^{<\mu} = \mu$, and $Z \subseteq \mathcal{P}_\kappa(\lambda)$ is stationary. Forcing with* $\text{Add}(\mu, \kappa)$ *destroys all κ-complete, normal, fine, nonregular ideals on Z, while preserving the existence of κ-complete, normal, fine, λ^+-saturated ideals on Z.*

Proof. First note that $\mu^{<\mu} = \mu$ implies $\text{Add}(\mu, \kappa)$ is κ-c.c. If $j : V \to M \subseteq V[G]$ is a generic embedding arising from a κ-complete, normal, fine, λ^+-saturated ideal on Z, then $j(\text{Add}(\mu, \kappa)) = \text{Add}(\mu, j(\kappa))^M = \text{Add}(\mu, j(\kappa))^{V[G]}$, which is $j(\kappa)$-c.c. in $V[G]$, and $j(\kappa) = (\lambda^+)^V$. Thus by Corollary 5.4, and Proposition 5.5 if I is a κ-complete, normal, fine, λ^+-saturated ideal on Z, then $\bar{I}$ has these properties in $V^{\text{Add}(\mu,\delta)}$.

On the other hand, suppose $p_0 \Vdash_{\text{Add}(\mu,\kappa)} \dot{J}$ is κ-complete, normal, fine, nonregular ideal on Z. By the first part of the proof for Lemma 5.9, we may assume p also forces $\dot{J}$ is λ^+-saturated. Let $I = \{A \subseteq Z : p_0 \Vdash A \in \dot{J}\}$. Then I is normal, fine, κ-complete, and λ^+-saturated. Whenever $H \subseteq \text{Add}(\mu, \delta)$ is generic, $\dot{J}^H = \bar{I} \restriction A$ for some A.

Let $e : \text{Add}(\mu, \kappa) \to \mathcal{B}(\mathcal{P}(Z)/I * \text{Add}(\mu, j(\kappa)))$ be the embedding from Theorem 5.1. If $H \subseteq \text{Add}(\mu, \kappa)$ is generic, then there is an isomorphism $\sigma : \mathcal{B}(\mathcal{P}(Z)/I * \text{Add}(\mu, j(\kappa))/e[H] \to \mathcal{P}^{V[H]}(Z)/\bar{I}$. Suppose $\langle A_\alpha \rangle_{\alpha<\lambda}$ is a sequence of J-positive sets. For each A_α, choose $(B_\alpha, \dot{q}_\alpha) \in \mathcal{P}^V(Z)/I * \text{Add}(\mu, j(\kappa))$ such that $\sigma(B_\alpha, \dot{q}_\alpha) \leq A_\alpha$. By the λ^+-c.c., there is some $\beta < \lambda^+$ such that $\Vdash_{\mathcal{P}^V(Z)/I} \bigcup_{\alpha<\lambda} \text{dom}(\dot{q}_\alpha) \subseteq \beta$. Since $\Vdash_{\mathcal{P}^V(Z)/I} |\lambda| = \mu$, there is name for an antichain $\langle \dot{r}_\alpha \rangle_{\alpha<\lambda}$ in $\text{Add}(\mu, j(\kappa))$ such that $\Vdash \text{dom}(\dot{r}_\alpha) \cap \beta = \emptyset$ for all $\alpha < \lambda$. Then $\langle \sigma(B_\alpha, \dot{q}_\alpha \wedge \dot{r}_\alpha) \rangle_{\alpha<\lambda}$ is a disjoint refinement of $\langle A_\alpha \rangle_{\alpha<\lambda}$ into J-positive sets. □

5.3 Compatibility with square

Solovay [32] showed that $\square_\delta$ fails when $\delta \geq \kappa$ and κ is supercompact. In contrast, we'll show $(\forall \delta \geq \kappa)\square_\delta$ is consistent with the kind generically supercompact κ constructed in Chapter 2. Ironically, a common feature between traditional and generic supercompactness can show the failure of $\square$ in the traditional case, and allow $\square$ to be forced in the generic case while preserving generic supercompactness. The key difference is that nontrivial forcings may be absorbed into the quotient algebras of the ideals in the generic case.

For a cardinal δ, let $\mathbb{S}_\delta$ be the collection of bounded approximations to a $\square_\delta$ sequence. That is, a condition is a sequence $\langle C_\alpha : \alpha \in \eta \cap \mathrm{Lim} \rangle$ such that $\eta < \delta^+$ is a successor ordinal, each C_α is a club subset of α of order type $\leq \delta$, and whenever β is a limit point of C_α, $C_\alpha \cap \beta = C_\beta$. For proof of the following lemma, we refer the reader to [8].

Lemma 5.15. *For every cardinal δ, $\mathbb{S}_\delta$ is countably closed and $(\delta+1)$-strategically closed and adds a $\square_\delta$ sequence $\langle C_\alpha : \alpha \in \delta^+ \cap \mathrm{Lim} \rangle = \bigcup G$, where $G \subseteq \mathbb{S}_\delta$ is the generic filter. For every regular $\lambda \leq \delta$, there is a $\mathbb{S}_\delta$-name for a "threading" partial order $\mathbb{T}^\lambda_\delta$ that adds a club $C \subseteq (\delta^+)^V$ of order type λ and such that whenever α is a limit point of C, $C \cap \alpha = C_\alpha$. Furthermore, $\mathbb{S}_\delta * \mathbb{T}^\lambda_\delta$ has a λ-closed dense subset of size 2^δ.*

Theorem 5.16. *Suppose κ is super-almost-huge and $\mu < \kappa$ is regular. Then there is a μ-distributive forcing extension in which $\kappa = \mu^+$, $\square_\lambda$ holds for all $\lambda \geq \kappa$, and for all regular $\lambda \geq \kappa$ there is a normal, fine, κ-complete, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$.*

Proof. By Chapter 2, we may pass to a μ-distributive forcing extension in which $\kappa = \mu^+$ and for all regular $\lambda \geq \kappa$ there is a normal, fine, κ-complete, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$, and GCH holds above μ. Over this model, force with $\mathbb{P}$, the Easton support product of $\mathbb{S}_\lambda$ where λ ranges over all cardinals $\geq \kappa$. For every cardinal λ, $\mathbb{P}$ naturally factors into $\mathbb{P}_{<\lambda} \times \mathbb{P}_{\geq\lambda}$. Note that if $\lambda \geq \kappa$, $\mathbb{P}_{\geq\lambda}$ is $(\lambda+1)$-strategically closed.

First we show that for each regular $\lambda \geq \kappa$, $\mathbb{P}_{\geq\lambda}$ is λ^+-distributive in $V^{\mathbb{P}_{<\lambda}}$. Suppose that $H_0 \times H_1$ is $(\mathbb{P}_{<\lambda} \times \mathbb{P}_{\geq\lambda})$-generic, and $f : \lambda \to \mathrm{Ord}$ is in $V[H_0][H_1]$. Then in $V[H_1]$, there is a $\mathbb{P}_{<\lambda}$-name τ for f. By GCH and the fact that we take Easton support, $|\mathbb{P}_{<\lambda}| = \lambda$, so it is λ^+-c.c. in $V[H_1]$. Thus τ may be assumed to be a subset of V of size λ. By the strategic closure of $\mathbb{P}_{\geq\lambda}$, $\tau \in V$. Thus $f = \tau^{H_0} \in V[H_0]$, establishing the claim.

Next we show that $\mathbb{P}$ preserves all regular cardinals. First note that since $\mathbb{P}$ is $(\kappa+1)$-strategically closed, $\mathbb{P}$ cannot change the cofinality of any regular δ to some $\lambda \leq \kappa$. If $\mathbb{P}$ does not preserve regular cardinals, then in some generic extension $V[G]$, there are $\lambda < \delta$ which are regular in V with $\kappa < \lambda$, such that $V[G] \models \mathrm{cf}(\delta) = \lambda$. Let $H = H_0 \times H_1$, where $H_0 \subseteq \mathbb{P}_{<\lambda}$ and $H_1 \subseteq \mathbb{P}_{\geq\lambda}$. By the λ^+-c.c. of $\mathbb{P}_{<\lambda}$, $V[H_0] \models \mathrm{cf}(\delta) > \lambda$, and by the λ^+-distributivity of $\mathbb{P}_{\geq\lambda}$ in $V[H_0]$, $V[H] \models \mathrm{cf}(\delta) > \lambda$, a contradiction. Since a square sequence is upwards absolute to models with the same cardinals and $\mathbb{S}_\lambda$ regularly embeds into $\mathbb{P}$ for all $\lambda \geq \kappa$, $\mathbb{P}$ forces $(\forall \lambda \geq \kappa)\square_\lambda$.

For each regular $\lambda \geq \kappa$, let $Z_\lambda = \mathcal{P}_\kappa(\lambda)$. We want to show that in $V^{\mathbb{P}}$, for each regular $\lambda \geq \kappa$, there is a normal, fine, λ-dense ideal on Z_λ. It suffices to show that such an ideal exists in $V^{\mathbb{P}_{<\lambda}}$, since $\mathbb{P}_{\geq\lambda}$ adds no subsets of λ, and $|Z_\lambda| = \lambda$. First note that by the strategic closure of $\mathbb{P}$, the dense ideal on κ is unaffected.

Let $\mathbb{Q}$ be the Easton support product of $\mathbb{S}_\lambda * \mathbb{T}^\mu_\lambda$, where λ ranges over all cardinals. There is a coordinate-wise regular embedding of $\mathbb{P}$ into $\mathbb{Q}$. When λ is regular, $\mathbb{Q}_{<\lambda}$ has a dense μ-closed subset of size λ. Hence it regularly embeds into $\mathcal{B}(\mathrm{Col}(\mu, \lambda))$. The dense ideal I_λ on Z_λ in V has quotient algebra isomorphic to $\mathcal{B}(\mathbb{R} \times \mathrm{Col}(\mu, \lambda))$ for some small $\mathbb{R}$, and so $\mathbb{Q}_{<\lambda}$ regularly embeds into this forcing.

If $G \subseteq \mathcal{P}(Z_\lambda)/I_\lambda$ is generic, let H be the induced generic for $\mathbb{P}_{<\lambda}$, and let $j : V \to M \subseteq V[G]$ be the ultrapower embedding. Recall that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda^+$, λ^{++} is a fixed point of j, and $j[\lambda] \in M$. First note that $j[\lambda] \setminus j(\kappa)$ is an Easton set in M. If $j(\kappa) \leq \delta \leq j(\lambda)$ and δ is regular in M, then since $\mathrm{ot}(j[\lambda] \cap \delta) \leq \lambda < \delta$, $\sup(j[\lambda] \cap \delta) < \delta$.

For each cardinal δ such that $\kappa \leq \delta < \lambda$, let $\langle C^\delta_\alpha : \alpha < \delta^+ \rangle$ be the $\square_\delta$ sequence and let t_δ be the "thread" of order type μ, both given by given by $H \restriction (\mathbb{S}_\delta * \mathbb{T}^\mu_\delta)$. By the μ-distributivity of $\mathbb{S}_\delta * \mathbb{T}^\mu_\delta$, all initial segments of t_δ are in V, and since they are small, $j(t_\delta \cap \alpha) = j[t_\delta \cap \alpha]$ for $\alpha < \delta^+$, and j is continuous at all limit points of t_δ. Let $\gamma_\delta = \sup(j[\delta^+]) < j(\delta^+)$, and in

M consider $m_\delta = \bigcup_{\alpha<\delta^+} j(\langle C^\delta_\alpha : \beta < \alpha \rangle) \cup \{(\gamma_\delta, j[t_\delta])\}$. Each m_δ is a condition in $(\mathbb{S}_{j(\delta)})^M$, and the sequence $m = \langle m_\delta : \delta \in j[\lambda] \setminus \kappa \cap \mathrm{Card}^M \rangle$ is a condition in $(\mathbb{P}_{<j(\lambda)})^M$.

M thinks $\mathbb{P}_{<j(\lambda)}$ is $j(\kappa)$-strategically closed, and this is true in $V[G]$ as well since these models share the same λ-sequences, and $j(\kappa) = \lambda^+$ in $V[G]$. Since $j(\lambda^+) < (\lambda^{++})^V$, $\mathcal{P}(\mathbb{P}_{<j(\lambda)})^M$ has cardinality λ^+ in $V[G]$. Thus we may use the winning strategy to build a filter $\hat{H} \subseteq (\mathbb{P}_{<j(\lambda)})^M$ that is generic over M, with $m \in \hat{H}$. Since m is a lower bound to $j(p)$ for all $p \in H$, we have $j[H] \subseteq \hat{H}$.

Therefore, the hypotheses of Theorem 5.1 are satisfied, with respect to $Z_\lambda, I_\lambda, \mathcal{B}(\mathbb{P}_{<\lambda})$. No further forcing over $V[G]$ is required to build $\hat{H}$, so the ideal K from Theorem 5.1 is prime. Thus we have a $\mathbb{P}_{<\lambda}$-name for a normal and fine ideal J_λ on Z_λ such that $\mathcal{B}(\mathbb{P}_{<\lambda} * \mathcal{P}(\dot{Z_\lambda})/J_\lambda) \cong \mathcal{B}(\mathcal{P}(Z_\lambda)/I_\lambda)$. Hence J_λ is λ-dense in $V[H]$. $\square$

We note that in the case $\kappa = \omega_1$, $\mathcal{P}(Z_\lambda)/J_\lambda \cong \mathcal{B}(\mathrm{Col}(\omega, \lambda))$ for all regular λ. But for higher cardinals, Proposition 3.8 shows the quotient algebras must differ from those given by Theorem 2.17. The crux is that the "threading" forcings are left over as regular suborders.

It is not possible to improve this result to get the consistency of, "For all cardinals $\lambda \geq \kappa$, $\square_\lambda$ and there is a normal, fine, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$." Burke and Matsubara [5] showed that if $\mathrm{cf}(\lambda) < \kappa$ and there is normal, fine, κ-complete, λ^+-saturated ideal on $\mathcal{P}_\kappa(\lambda)$, then every stationary subset of $\lambda^+ \cap \mathrm{cof}(<\kappa)$ reflects.

5.4 Mutual inconsistency

In contrast to the observed situation with traditional large cardinals, generic large cardinals can be individually consistent yet mutually inconsistent. Only a handful of examples of this phenomenon seem to be known; this is discussed in Section 11.2 of [13]. Here, we use our "destruction" method to bring more instances to light.

Theorem 5.17. *Let $\mu^+ = \kappa < \delta$ be regular cardinals. The following are mutually inconsistent:*

(1) There is a κ-complete, κ-dense ideal I on κ.

(2) There is a normal, fine, κ-complete, δ-saturated ideal J on $Z = [\delta]^\kappa$, such that $\mathcal{P}(Z)/J$ is δ-absolutely δ-c.c. and $\mathrm{d}(\mathcal{P}(A)/J) = \delta$ for all $A \in J^+$.

Proof. Suppose the two statements hold simultaneously. We may assume that I is a normal ideal on κ. Let G be generic for $\mathcal{P}(Z)/J$ and let $j : V \to M \subseteq V[G]$ be the associated embedding. Since J is κ-complete, $\mathrm{crit}(j) \geq \kappa$, and since $j[\delta] \in j(Z)$, $|\delta| = j(\kappa)$ in M, so $\mathrm{crit}(j) = \kappa$. Since the forcing to produce G is δ-c.c., $j(\kappa) = \delta$. Thus by elementarity, $M \models$ "There is a normal, fine, δ-dense ideal on δ." By the closure properties of M, $\mathcal{P}(\delta)^{V[G]} = \mathcal{P}(\delta)^M$, so there is such an ideal in $V[G]$. But by Lemma 5.10, forcing with $\mathcal{P}(Z)/J$ destroys all dense ideals on δ. $\square$

This theorem strengthens and generalizes a result of Woodin. As described in Section 5.6 of [13], Woodin used some results of Laver and Hajnal-Juhasz on partition properties to show that the following are mutually inconsistent:

(1) There is a countably complete, ω_1-dense ideal on ω_1.

(2) There is a normal and fine ideal J on $Z = [\omega_2]^{\omega_1}$ such that $\mathcal{P}(Z)/J \cong \mathcal{B}(\mathrm{Col}(\omega, <\omega_2))$.

(3) CH.

In Theorem 5.17, we get to eliminate CH from the list, and speak about a broader class of partial orders in statement (2). It is easy to see that $\mathrm{Col}(\omega, <\omega_2)$ has uniform density

ω_2 and is ω_2-absolutely ω_2-c.c. Furthermore, Theorem 5.17 applies to more pairs of regular cardinals $\kappa < \delta$.

In Chapter 2, we saw that statement (1) in Theorem 5.17 is consistent relative to an almost-huge cardinal. It remains open whether there can be an ideal as in statement (2) when $\delta = \kappa^+$, but known results give models of (2) with $\delta > \kappa^+$. Starting from a huge cardinal, Magidor [29] produced a forcing extension in which there is a normal, fine, ω_3-saturated ideal on $[\omega_3]^{\omega_1}$. Huberich [17] improved upon this, producing ideals on more general spaces having stronger saturation properties. He started with a model of GCH, $\mu < \kappa < \lambda < \delta$ regular cardinals, and κ carrying a huge embedding with target δ. He showed that there is a forcing extension with the same cardinals in $[0, \mu] \cup [\kappa, \lambda]$, and in which $\kappa = \mu^+$, $\delta = \lambda^+$, and there is a normal, fine, κ-complete, λ-centered ideal J on $Z = [\delta]^\kappa$. This means that $\mathcal{P}(Z)/J$ is the union of λ many filters, and it implies that $\mathcal{P}(Z)/J$ is δ-c.c. in any outer model in which δ remains a cardinal. An elementarity argument shows that any λ-centered partial order is δ-absolutely δ-c.c. Huberich showed that $\mathcal{P}(Z)/J$ is isomorphic to a complete subalgebra of a certain complete boolean algebra $\mathbb{B}$, and it is easy to check that $\mathrm{d}(\mathbb{B} \restriction b) = \delta$ for all $b \in \mathbb{B}$. However, one can show using arguments similar to those following Theorem 2.17 that $\mathcal{P}(Z)/J \cong \mathbb{B}$ in this model. Therefore, statement (2) of Theorem 5.17 is consistent relative to a huge cardinal.

If we modify statement (2) to require that the ideal J has density less than δ, there is a chance it is consistent with (1). The question of whether there can be a normal, fine, ω_1-dense ideal on $[\omega_2]^{\omega_1}$ was raised by Foreman. We note that if this is possible, it would solve the question discussed at the end of Chapter 2.

Proposition 5.18. *Suppose there is a normal, fine, κ-complete, κ-dense ideal on $[\kappa^+]^\kappa$. Then there are dense ideals on κ and κ^+.*

Proof. Let J be a κ-complete, κ-dense ideal on $Z = [\kappa^+]^\kappa$. Let $I = \{X \subseteq \kappa : 1 \Vdash_{\mathcal{P}(Z)/I} \kappa \notin$

$j(X)\}$. As in the proof of Corollary 3.15, I is normal, fine, and κ-dense. If $G \subseteq \mathcal{P}(Z)/J$ is generic, then in $V[G]$ there is a normal, fine, $(\kappa^+)^V$-dense ideal K on $(\kappa^+)^V$. In V, let $\mathbb{P} = \mathcal{P}(Z)/J * \mathcal{P}(\kappa^+)/\dot{K}$, and let $L = \{X \subseteq \kappa^+ : 1 \Vdash_{\mathbb{P}} \kappa^+ \notin k(X)\}$, where k is the elementary embedding associated to K. Also as in Corollary 3.15, L is normal, fine, κ^{++}-saturated, and there is a complete embedding of $\mathcal{P}(\kappa^+)/L$ into $\mathcal{B}(\mathbb{P})$, so L is κ^+-dense. □

Chapter 6

Coherent forests

A question is immediately raised by the argument for Theorem 5.11. Can we obtain the same consistency result simultaneous with GCH? In particular, does CH + a nonregular ideal on ω_2 imply the existence of a dense ideal on ω_2? We may assume we start with a model of GCH plus normal, fine, κ-complete, λ-dense ideals on $\mathcal{P}_\kappa(\lambda)$, where $\kappa = \mu^+$. Does there exist a μ-c.c. partial order of uniform density κ that preserves GCH below κ? Typical μ-c.c., uniformly κ-dense forcings will introduce κ many subsets of some cardinal $\nu < \mu$. If we wish to preserve $\mu^{<\mu} = \mu$, we will need a special kind of forcing.

At first glance, we may conjecture that such objects cannot exist alongside dense ideals. Perhaps the assumption of μ-strategic closure in Lemma 4.5 can be weakened to μ-distributivity. Combined with Proposition 3.6, this would imply that the existence of a κ-complete, κ-dense ideal on κ is equivalent to the existence of a κ-complete nonregular ideal on κ when $\kappa = \mu^+$, μ is regular, and $\mu^{<\mu} = \mu$. But perhaps not.

Suppose $\mu^{<\mu} = \mu$, and $\mathbb{P}$ is a μ-c.c. forcing that preserves this. We can take a set of $\mathbb{P}$-names $\{\tau_\alpha : \alpha < \mu\}$ whose realizations are forced to become $\mathcal{P}_\mu(\mu)$. By the μ-c.c., we can assume that each $|\tau_\alpha| < \mu$. If $\theta > \kappa$ is regular such that $\mathbb{P} \in H_\theta$, then we can take an elementary substructure $M \prec H_\theta$ such that $\mathbb{P}, \{\tau_\alpha : \alpha < \mu\} \in M$, $|M| = \mu$, and $M^{<\mu} \subseteq M$. Then $\mathbb{P}_0 = \mathbb{P} \cap M$ is a regular suborder of $\mathbb{P}$. We have some $\mathbb{P}_0$-name $\dot{\mathbb{Q}}$ such that $\mathbb{P} \cong \mathbb{P}_0 * \dot{\mathbb{Q}}$, and $\Vdash_{\mathbb{P}_0} \mathcal{B}(\dot{\mathbb{Q}})$ is (μ, μ)-distributive. By the μ-c.c. of $\mathbb{Q}$ in $V^{\mathbb{P}_0}$, $\mathbb{Q}$ is thus forced to be μ-distributive. Hence, $\mathcal{B}(\mathbb{Q})$ is, by definition, a μ-Suslin algebra in $V^{\mathbb{P}_0}$. Since $|\mathbb{P}_0| < \kappa$, any κ-complete, λ-dense ideal will remain so after forcing with $\mathbb{P}_0$.

Therefore, if we wish to apply the technique of Chapter 5 to destroy all normal, fine, κ-complete, λ-dense ideals on $\mathcal{P}_\kappa(\lambda)$ while preserving a nonregular one, we will need a model in which there is such a dense ideal and also a μ-Suslin algebra of uniform density κ. Such a model is produced in this chapter. We also include several related results that were obtained in the pursuit of this model.

We consider a type of structure called a forest, a generalization of a tree. Forests contain many trees, but can be much wider than a single tree. Thomas Jech had previously studied the same type of object under the name "mess" [19]. The nicer choice of terminology is due to Christoph Weiß [38]. In contrast to the work of Weiß, we will focus on forests that do not contain long branches.

Definition. *A* (κ, X, μ)-forest *is a collection of functions F satisfying:*

(1) $\{\mathrm{dom}(f) : f \in F\} = \mathcal{P}_\kappa(X)$.

(2) $(\forall f \in F) \, \mathrm{ran}(f) \subseteq \mu$.

(3) For $z \in \mathcal{P}_\kappa(X)$, let $F_z = \{f \in F : \mathrm{dom}(f) = z\}$. A forest must satisfy that for $z_0 \subseteq z_1$ in $\mathcal{P}_\kappa(X)$, $F_{z_0} = \{f \restriction z_0 : f \in F_{z_1}\}$.

Forests are full of trees. If F is a (κ, X, μ)-forest, and $S = \{x_\alpha : \alpha < \kappa\}$ is an enumeration of distinct elements of X, then $T_S = \{f \in F : (\exists \beta < \kappa) \, \mathrm{dom}(f) = \{x_\alpha : \alpha < \beta\}\}$ forms a tree of height κ under the subset ordering.

A (κ, X, μ)-forest F is called *thin* if for all $z \in \mathcal{P}_\kappa(X)$, $|F_z| < \kappa$. A collection of functions F is called κ-*coherent* if for all $f, g \in F$, $|\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(x) \neq g(x)\}| < \kappa$. If F is a (κ^+, X, μ)-forest we say it is *coherent* if it is κ-coherent. Clearly, if $\mu \leq \kappa = \kappa^{<\kappa}$, then any coherent (κ^+, X, μ)-forest is thin.

A *chain* in a forest is a subset which is linearly ordered under $\subseteq$. Two elements f, g in a forest F are said to be *compatible* when they have a common extension $h \in F$. An *antichain* in a forest is a subset of pairwise incompatible elements. We say that a (κ, X, μ)-forest F is *Aronszajn* if it contains no well-ordered chain of length κ. We say it is *Suslin* if it contains no antichain of cardinality κ. If F is a (κ, X, μ)-forest with $\mu \geq 2$, closed under finite modifications, then F is Suslin only if it is Aronszajn. This is because we can "split off" from any chain of length κ to get an antichain of size κ.

Proposition 6.1. *If F is a (κ, X, μ)-forest, then for any $z \in \mathcal{P}_\kappa(X)$, F_z is a maximal antichain.*

Proof. Let $f \in F$, $z \in \mathcal{P}_\kappa(X)$. By clause (3) of the definition of forests, there is $g \in F$ such that $f \subseteq g$ and $\mathrm{dom}(g) = \mathrm{dom}(f) \cup z$. Then $g \restriction z \in F_z$, so g is a common extension of f and something in F_z. □

The following lemma will be useful in several constructions:

Lemma 6.2. *Suppose F is a coherent (κ^+, X, μ)-forest, and F is closed under $<\kappa$ modifications. Then two functions in F have a common extension in F if and only if they agree on their common domain.*

Proof. Let $f, g \in F$ agree on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. Let $h \in F$ be such that $\mathrm{dom}(h) = \mathrm{dom}(f) \cup \mathrm{dom}(g)$. By coherence, we can change the values of h on a set of size $<\kappa$ to get $h' : \mathrm{dom}(h) \to \mu$ with $h' \restriction \mathrm{dom}(f) = f$, and $h' \restriction \mathrm{dom}(g) = g$. By the closure of F, $h' \in F$. □

6.1 Aronszajn forests

The first theorem of this section generalizes of an argument of Koszmider [24].

Lemma 6.3. *Let κ be a regular cardinal, and suppose $F = \{f_\alpha : \alpha < \kappa\}$ is a κ-coherent set of partial functions from κ to μ.*

(a) There is a function $f : \kappa \to \mu$ such that $\{f\} \cup F$ is κ-coherent.

(b) If $\mu = \kappa$ and each f_α is $<\kappa$ to 1, then there is a $<\kappa$ to 1 function $f : \kappa \to \kappa$ such that $\{f\} \cup F$ is κ-coherent.

Proof. For each α, let $D_\alpha = \mathrm{dom}(f_\alpha) \setminus \bigcup_{\beta<\alpha} \mathrm{dom}(f_\beta)$. Let $E = \kappa \setminus \bigcup_\alpha D_\alpha$. For the first claim, choose any function $g : E \to \mu$, and let

$$f(\beta) = \begin{cases} f_\alpha(\beta) & \text{if } \beta \in D_\alpha \\ g(\beta) & \text{if } \beta \in E \end{cases}$$

For any α, $\{\beta : f(\beta) \neq f_\alpha(\beta)\} = \bigcup_{\gamma<\alpha}\{\beta \in D_\gamma \cap \mathrm{dom}(f_\alpha) : f_\gamma(\beta) \neq f_\alpha(\beta)\}$. This is a union of $<\kappa$ sets of size $<\kappa$, so has size $<\kappa$.

For the second claim, choose any $<\kappa$ to 1 function $g : E \to \kappa$, and let

$$f(\beta) = \begin{cases} \max(\alpha, f_\alpha(\beta)) & \text{if } \beta \in D_\alpha \\ g(\beta) & \text{if } \beta \in E \end{cases}$$

For any α, $\{\beta : f(\beta) \neq f_\alpha(\beta)\} \subseteq \bigcup_{\gamma\leq\alpha}\{\beta \in D_\gamma : f_\gamma(\beta) < \gamma \text{ or } f_\gamma(\beta) \neq f_\alpha(\beta)\}$. By the hypotheses, this set has size $<\kappa$. For each α, $f^{-1}(\alpha) \subseteq g^{-1}(\alpha) \cup \bigcup\{f_\gamma^{-1}(\beta) : \gamma, \beta \leq \alpha\}$, so f is $<\kappa$ to 1. $\square$

Theorem 6.4. *Let κ be a regular cardinal. For every $\zeta < \kappa$, there is a coherent $(\kappa^+, \kappa^{+\zeta}, \kappa)$-forest consisting of $<\kappa$ to 1 functions.*

Proof. We will prove by induction the following stronger statement: For every $\zeta < \kappa$ and every sequence $\langle (X_\alpha, F_\alpha) : \alpha < \kappa \rangle$ such that:

(1) each $X_\alpha \subseteq \kappa^{+\zeta}$,

(2) each F_α is a $(\kappa^+, X_\alpha, \kappa)$-forest of $<\kappa$ to 1 functions,

(3) $\bigcup_\alpha F_\alpha$ is κ-coherent,

there is a coherent $(\kappa^+, \kappa^{+\zeta}, \kappa)$-forest $F \supseteq \bigcup_\alpha F_\alpha$ consisting of $<\kappa$ to 1 functions.

For $\zeta = 0$, pick a collection $\{f_\alpha : \alpha < \kappa\}$ such that for each α, $f_\alpha \in F_\alpha$, and $\mathrm{dom}(f_\alpha) = X_\alpha$. By Lemma 6.3(b), there is a $< \kappa$ to 1 function $f : \kappa \to \kappa$ that coheres with each f_α, and we can take $F = \{g : \mathrm{dom}(g) \subseteq \kappa \text{ and } |\{x : f(x) \neq g(x)\}| < \kappa\}$.

Assume $\zeta = \eta + 1$ and the statement holds for η. For each $\beta < \kappa^{+\zeta}$, let $F^\beta_\alpha = \bigcup_\alpha \{f \restriction \beta : f \in F_\alpha\}$. We will construct $F \supseteq \bigcup F_\alpha$ as the union of a $\subseteq$-increasing sequence $\langle G_\beta : \beta < \kappa^{+\zeta} \rangle$ such that for each β, G_β is a coherent $(\kappa^+, \beta, \kappa)$-forest of $<\kappa$ to 1 functions containing $\bigcup_\alpha F^\beta_\alpha$. Let $G_0 = \{\emptyset\}$. Given G_β, let $G_{\beta+1} = \{f : \mathrm{dom}(f) \subseteq (\beta + 1), \mathrm{ran}(f) \subseteq \kappa, \text{ and } f \restriction \beta \in G_\beta\}$.

Suppose β is a limit ordinal of cofinality $\leq \kappa$, and let $\langle \gamma_i : i < \delta \leq \kappa \rangle$ be cofinal in β. The collection $\bigcup_{i<\delta} G_{\gamma_i} \cup \bigcup_{\alpha<\kappa} F^\beta_\alpha$ is κ-coherent, because $(\forall \alpha < \kappa)(\forall f \in F^\beta_\alpha)(\forall i < \delta)(f \restriction \gamma_i \in F^{\gamma_i}_\alpha \subseteq G_{\gamma_i})$. Since β has cardinality $\leq \kappa^{+\eta}$, the inductive assumption implies that we can extend to a forest G_β with the desired properties.

Suppose β is a limit ordinal of cofinality $> \kappa$. Let $G_\beta = \bigcup_{\gamma<\beta} G_\gamma$. Then G_β is a forest with the desired properties because $\bigcup_{\alpha<\kappa} F^\beta_\alpha = \bigcup_{\gamma<\beta}(\bigcup_{\alpha<\kappa} F^\gamma_\alpha)$. Finally, we let $F = \bigcup_{\beta<\kappa^{+\zeta}} G_\beta$.

Now assume ζ is a limit ordinal of cofinality $< \kappa$, and the statement holds for all $\eta < \zeta$. Let $\langle \gamma_i : i < \delta = \mathrm{cf}(\zeta) \rangle$ be an increasing cofinal sequence in ζ. Like above, recursively build an increasing sequence $\langle G_i : i < \delta \rangle$ such that each G_i is a $(\kappa^+, \kappa^{+\gamma_i}, \kappa)$-forest of $< \kappa$ to 1 functions extending $\bigcup_\alpha F_\alpha^{\gamma_i}$. This is done by applying the inductive hypothesis for $\kappa^{+\gamma_i}$ to $\bigcup_\alpha F_\alpha^{\gamma_i} \cup \bigcup_{j<i} G_j$. We may also assume each G_i is closed under $< \kappa$ modifications. Simply let F be the collection of functions f such that $\mathrm{dom}(f) \subseteq \kappa^{+\zeta}$, and $(\forall i < \delta) f \restriction \gamma_i \in G_{\gamma_i}$. Clearly $F \supseteq \bigcup_\alpha F_\alpha$.

First note that if $f \in F$ were not $< \kappa$ to 1, then there would be some $i < \delta$ such that $f \restriction \kappa^{+\gamma_i}$ is not $< \kappa$ to 1, which is false. If $f, g \in F$ were to disagree at κ many points, then there would be some $i < \delta$ such that $f \restriction \kappa^{+\gamma_i}$ and $g \restriction \kappa^{+\gamma_i}$ disagree at κ many points, which is false. Second, we check that for any $z \in \mathcal{P}_{\kappa^+}(\kappa^{+\zeta})$, there is an $f \in F$ such that $\mathrm{dom}(f) = z$. We can recursively build a sequence $\langle g_i : i < \delta \rangle$ such that for all $i < j < \delta$, $g_i \in G_i$, $\mathrm{dom}(g_i) = z \cap \kappa^{+\gamma_i}$, and $g_i \subseteq g_j$. If we have built such a sequence up to $j < \delta$, then $\bigcup_{i<j} g_i \in G_j$, because for any $h \in G_j$ with domain $z \cap \kappa^{+\gamma_j}$, the set of disagreements with $\bigcup_{i<j} g_i$ has size $< \kappa$. Let $f = \bigcup_{i<\delta} g_i$. □

Koszmider showed that in the case $\kappa = \omega$, if λ is a singular cardinal of cofinality ω, and $\square_\lambda$ and $\lambda^\omega = \lambda^+$ hold, then the induction can push through λ as well. The argument generalizes almost verbatim to show for any regular κ, the induction can go forward at λ of cofinality κ, under the assumptions $\square_\lambda$ and $\lambda^\kappa = \lambda^+$. As a consequence, we get that in L, for every regular κ and every $\lambda \geq \kappa$, there is a coherent, $(\kappa^+, \lambda, \kappa)$-forest of $< \kappa$ to 1 functions.

Recall that a partial order $\mathbb{P}$ is called κ-Knaster if for any $A \subseteq \mathbb{P}$ of size κ, there is $B \subseteq A$ of size κ that consists of pairwise compatible elements.

Corollary 6.5. *For every regular cardinal κ and every $\zeta < \kappa$, there is a coherent $(\kappa^+, \kappa^{+\zeta}, \kappa)$-forest, which is Aronszajn, does not have the $2^{<\kappa}$ or the κ^+ chain condition, but is $(2^\kappa)^+$-Knaster. If ζ is finite or $2^{<\kappa} < \kappa^{+\omega}$, then the forest is $(2^{<\kappa} \cdot \kappa^+)^+$-Knaster.*

Proof. Let F be given by Theorem 6.4. We may assume F is closed under $<\kappa$ modifications. To see the failure of the $2^{<\kappa}$ chain condition, note that for any $z \subseteq \kappa^{+\zeta}$ of size κ, F_z is an antichain of size $2^{<\kappa}$.

Let $\{\alpha_\beta : \beta < \kappa^+\}$ be any enumeration of distinct ordinals in $\kappa^{+\zeta}$, and for each $\gamma < \kappa^+$, let $f_\gamma \in F$ have domain $\{\alpha_\beta : \beta < \gamma\}$. Since each $f \in F$ maps into κ, there is a $\xi < \kappa$ and a stationary subset $S_0 \subseteq \{\gamma < \kappa^+ : \text{cf}(\gamma) = \kappa\}$ such that for all $\gamma \in S_0$, $f_{\gamma+1}(\alpha_\gamma) = \xi$. Since each $f \in F$ is $<\kappa$ to 1, each set $\{\beta < \gamma : f_{\gamma+1}(\alpha_\beta) = \xi\}$ is bounded below γ when $\text{cf}(\gamma) = \kappa$. Thus there is an $\eta < \kappa^+$ and a stationary $S_1 \subseteq S_0$ such that for all $\gamma \in S_1$, $\{\beta < \gamma : f_{\gamma+1}(\alpha_\beta) = \xi\} \subseteq \eta$. Therefore, for any $\gamma_0 < \gamma_1$ in $S_1 \setminus \eta$, $f_{\gamma_0+1}(\alpha_{\gamma_0}) \neq f_{\gamma_1+1}(\alpha_{\gamma_0})$. This shows that F does not have the κ^+ chain condition.

It also shows that F is Aronszajn. For otherwise, let $\langle f_\alpha : \alpha < \kappa^+\rangle$ be a strictly increasing $\subseteq$-chain in F. Let $\{\xi_\beta : \beta < \kappa^+\} = \bigcup_\alpha \text{dom}(f_\alpha)$, and for each γ let $g_\gamma = (\bigcup_\alpha f_\alpha) \restriction \{\xi_\beta : \beta < \gamma\}$. Then $\langle g_\gamma : \gamma < \kappa^+\rangle$ is a strictly increasing chain, but by the above paragraph, it contains an antichain of size κ^+, contradiction.

To show the $(2^\kappa)^+$-Knaster property, let $\{f_\alpha : \alpha < (2^\kappa)^+\} \subseteq F$. Let $T_0 \subseteq (2^\kappa)^+$ have size $(2^\kappa)^+$ and be such that $\{\text{dom}(f_\alpha) : \alpha \in T_0\}$ forms a delta-system with root r. Let $T_1 \subseteq T_0$ have size $(2^\kappa)^+$ and be such that for a fixed g, $f_\alpha \restriction r = g$ for all $\alpha \in T_1$. The union of any two of these is in F.

For the case where $\zeta < \omega$ or $2^{<\kappa} < \kappa^{+\omega}$, let $\theta = (2^{<\kappa} \cdot \kappa^+)^+$. First note that it is easy to see by induction that for every $n < \omega$, $\mathcal{P}_{\kappa^+}(\kappa^{+n})$ has a cofinal subset of size κ^{+n}. Let $A = \{f_\alpha : \alpha < \theta\} \subseteq F$, and let $S = \bigcup_\alpha \text{dom}(f_\alpha)$.

Suppose first that $|S| < \theta$. There is an $R \subseteq \mathcal{P}_{\kappa^+}(S)$ that covers $\{\text{dom}(f_\alpha) : \alpha < \theta\}$ and has cardinality $|S|$. Therefore, by the coherence of F, there is a $G \subseteq F$ of cardinality $\leq |S| \cdot 2^{<\kappa} < \theta$ such that for all $\alpha < \theta$, there is $g \in G$ with $f_\alpha \subseteq g$. Therefore there is a $g_0 \in G$ which is a common lower bound to θ many f_α.

Now suppose that $|S| = \theta$. Since θ is regular and $\theta > \kappa^+$, we can use the delta-system argument to get an $S_0 \subseteq S$ of cardinality less than θ and a $T_0 \subseteq \theta$ of cardinality θ such that for all $\alpha_0, \alpha_1 \in T_0$, $\text{dom}(f_{\alpha_0}) \cap \text{dom}(f_{\alpha_1}) \subseteq S_0$. By the above paragraph, there is a $T_1 \subseteq T_0$ of cardinality θ such that for any $\alpha_0, \alpha_1 \in T_1$, f_{α_0} and f_{α_1} agree on their common domain contained in S_0. □

One may ask whether the condition "$< \kappa$ to 1" can be strengthened to "1 to 1" in Theorem 6.4. But this cannot always be achieved:

Proposition 6.6. *If there is a coherent $(\kappa^+, \lambda, \kappa)$-forest consisting of injective functions, then there are λ many almost disjoint subsets of κ.*

Proof. Let F be such a forest, and for each $z \in \mathcal{P}_{\kappa^+}(\lambda)$, choose $f_z \in F$ with domain z. Let S be a collection of λ many pairwise disjoint subsets of λ, each of cardinality κ. For $x \neq y$ in S, $\text{ran}(f_x)$ is almost disjoint from $\text{ran}(f_y)$. This is because the sets $A = \text{ran}(f_{x\cup y} \restriction x)$ and $B = \text{ran}(f_{x\cup y} \restriction y)$ are disjoint, and $|A \triangle \text{ran}(f_x)| < \kappa$, and $|B \triangle \text{ran}(f_y)| < \kappa$. □

A positive answer in the following special case is well-known (see [25], Chapter II, Theorem 5.9 and exercise 37):

Theorem 6.7. *Let κ be a regular cardinal. There is a κ-coherent collection of functions $\{f_\alpha : \alpha < \kappa^+\}$, such that each f_α is an injection from α to κ.*

A more general positive answer can be forced:

Theorem 6.8. *Assume κ is a regular cardinal with $2^{<\kappa} = \kappa$, and $\lambda \geq \kappa$. There is a κ-closed, κ^+-c.c. partial order that adds a coherent $(\kappa^+, \lambda, \kappa)$-forest of injective functions.*

Proof. Let $\mathbb{P}$ be the collection partial functions p that assign to $< \kappa$ many $z \subseteq \lambda$ of size $\leq \kappa$, a partial injective function from z to κ defined at $< \kappa$ many points. Let $p \leq q$ when:

(a) $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$.

(b) For all $z \in \mathrm{dom}(q)$, $p(z) \supseteq q(z)$.

(c) If $z_0, z_1 \in \mathrm{dom}(q)$, $\alpha \in z_0 \cap z_1 \setminus (\mathrm{dom}(q(z_0)) \cup \mathrm{dom}(q(z_1))$, and $\alpha \in \mathrm{dom}(p(z_0))$, then $\alpha \in \mathrm{dom}(p(z_1))$ and $p(z_0)(\alpha) = p(z_1)(\alpha)$.

It is easy to check that $\leq$ is transitive and that $\langle \mathbb{P}, \leq \rangle$ is κ-closed. To check the chain condition, let $A \subseteq \mathbb{P}$ have size κ^+. Since $\kappa^{<\kappa} = \kappa$, we can find a $B \subseteq A$ of size κ^+ such that $\{\mathrm{dom}(p) : p \in B\}$ forms a delta-system with root R. Again since $\kappa^{<\kappa} = \kappa$, there is a $C \subseteq B$ of size κ^+ and a collection of functions $\{f_z : z \in R\}$ such that $\forall p \in C$, $\forall z \in R$, $p(z) = f_z$. If $p, q \in C$, then $p \cup q$ is a common extension.

If $G \subseteq \mathbb{P}$ is generic, then for all $z \in \mathcal{P}_{\kappa^+}(\lambda)^V$, G gives an injective function $f_z : z \to \kappa$ as $\bigcup\{p(z) : z \in p \in G\}$. For $z_0, z_1 \in \mathcal{P}_{\kappa^+}(\lambda)^V$, there is some $p \in G$ such that $z_0, z_1 \in \mathrm{dom}(p)$. p forces that f_{z_0} and f_{z_1} agree outside $\mathrm{dom}(p(z_0)) \cup \mathrm{dom}(p(z_1))$. Finally, by the κ^+-c.c., $\mathcal{P}_{\kappa^+}(\lambda)^V$ is cofinal in $\mathcal{P}_{\kappa^+}(\lambda)^{V[G]}$. So we can define a $(\kappa^+, \lambda, \kappa)$-forest F as $\{f : f$ is an injection into κ, $(\exists z)\, \mathrm{dom}(f) \subseteq z \in \mathcal{P}_{\kappa^+}(\lambda)^V$, and f disagrees with f_z at $< \kappa$ many points$\}$. □

Such a forest will be Aronszajn because a chain of length κ^+ would give an injection from κ^+ to κ. Unlike the forests of Theorem 6.4, it will never have the λ chain condition.

6.2 Influence of the P-ideal dichotomy

In the previous section, we saw that coherent, Aronszajn $(\omega_1, \omega_n, \omega)$-forests can be constructed in ZFC for every natural number n. Here we show that the third coordinate is optimal, in the sense that for $n < \omega$ and $\lambda \geq \omega_1$, ZFC cannot prove the existence of a coherent, Aronszajn (ω_1, λ, n)-forest. Let us recall the relevant notions:

Definition. *An ideal* $I \subseteq \mathcal{P}(X)$ *is a* P-ideal *when* $\mathcal{P}_\omega(X) \subseteq I \subseteq \mathcal{P}_{\omega_1}(X)$*, and for any* $\{z_n : n < \omega\} \subseteq I$*, there is* $z \in I$ *such that* $z_n \setminus z$ *is finite for all* n*.*

Definition. *The* P-ideal dichotomy (PID) *is the statement that for any P-ideal* I *on a set* X*, either*

(1) there is an uncountable $Y \subseteq X$ *such that* $\mathcal{P}_{\omega_1}(Y) \subseteq I$*, or*

(2) there is a partition of X *into* $\{X_n : n < \omega\}$ *such that for all* n *and all* $z \in I$*,* $z \cap X_n$ *is finite.*

PID is a consequence of the Proper Forcing Axiom, and is also known to be consistent with ZFC+GCH relative to a supercompact cardinal [36]. The restriction of PID to ideals on sets of size ω_1 is known to be consistent without the use of large cardinals, both with and without GCH [1].

Using a coherent, Aronszajn $(\omega_1, \omega_1, \omega)$-forest F, we can obtain a coherent, Aronszajn ω_1-tree T of binary functions by taking the collection of characteristic functions of members of F whose domain is an ordinal, considering the functions as subsets of $\alpha \times \omega$ for $\alpha < \omega_1$. A cofinal branch would be a function $g : \omega_1 \times \omega \to 2$ with $g \restriction (\alpha \times \omega) \in T$ for all $\alpha < \omega_1$, and this would code an uncountable well-ordered chain in F. Further, using a regressive function argument, we can see that the closure of T under finite modifications remains Aronszajn. On the other hand, forests are more flexible. If we take such a tree T, close it under subsets to get a forest F, then it may be that there is an uncountable well-ordered chain $C \subseteq F$, but with $\mathrm{dom}(\bigcup C)$ a proper subset of $\omega_1 \times \omega$. This is what happens under PID.

Theorem 6.9. *Assume PID, and let* F *be a coherent* (ω_1, λ, n)*-forest closed under finite modifications, for some* $\lambda \geq \omega_1$*,* $n < \omega$*. Then* F *is not Aronszajn.*

Proof. First we prove this for $n = 2$. Let F be a coherent $(\omega_1, \lambda, 2)$-forest closed under finite modifications. Let I be the collection of $z \subseteq \lambda$ such that for some $f \in F$, $z \subseteq \{\alpha : f(\alpha) = 1\}$.

We claim I is a P-ideal. Let $\{z_n : n < \omega\} \subseteq I$, and for each n, choose $f_n \in F$ witnessing $z_n \in I$. Let $f \in F$ have domain $\bigcup_n \operatorname{dom}(f_n)$, and let $z = \{\alpha : f(\alpha) = 1\}$. For any n, f disagrees with f_n on a finite set, so there can only be finitely many $\alpha \in z_n \setminus z$.

Assume that alternative (1) of PID holds, and let $Y \subseteq \lambda$ be uncountable such that $\mathcal{P}_{\omega_1}(Y) \subseteq I$. Enumerate Y as $\langle y_\alpha : \alpha < \omega_1 \rangle$. For each $\alpha < \omega_1$, let f_α be the function that has $f_\alpha(y_\beta) = 1$ for $\beta < \alpha$, and is undefined elsewhere. Since F is closed under subsets, each $f_\alpha \in F$, and these form an uncountable well-ordered chain.

Assume alternative (2) of PID holds. Let $X_n \subseteq \lambda$ be uncountable such that for all $z \in I$, $X_n \cap z$ is finite. Let g have constant value 0 on X_n. If $f \in F$ and $\operatorname{dom}(f) \subseteq X_n$, then $\{\alpha : f(\alpha) = 1\}$ is finite. Thus for any countable $z \subseteq X_n$, $g \restriction z \in F$, so again we have an uncountable well-ordered chain.

Now assume the result holds for n, and let F be a coherent $(\omega_1, \lambda, n+1)$-forest. Let $r(k) = 0$ for $k < n$, and $r(n) = 1$. Consider the forest $G = \{r \circ f : f \in F\}$, and let g_0, g_1 be the functions on λ with constant value 0 and 1 respectively. By the above argument, there is some uncountable $Y \subseteq \lambda$ such that either $g_0 \restriction z \in G$ for all countable $z \subseteq Y$, or likewise for g_1. The latter case shows that F is not Aronszajn. In the former case, we have that for all countable $z \subseteq Y$, there is a function $f_z \in F$ with domain z that only takes values below n. If $H = \{g : (\exists z \in \mathcal{P}_{\omega_1}(Y)) g : z \to n \text{ and } \{\alpha : g(\alpha) \neq f_z(\alpha)\} \text{ is finite}\}$, then H is a coherent (ω_1, Y, n)-forest contained in F. By induction, H contains an uncountable well-ordered chain. □

6.3 Suslin forests

Lemma 6.10. *Let κ be a regular cardinal. All Suslin (κ, λ, μ)-forests are κ-distributive.*

Proof. Let F be a Suslin (κ, λ, μ)-forest, and let $\langle A_\alpha : \alpha < \delta < \kappa \rangle$ be a sequence of maximal antichains contained in F. By the Suslin property, each A_α has size $< \kappa$, so if $z = \bigcup_\alpha \{\mathrm{dom}(f) : f \in A_\alpha\}$, $|z| < \kappa$. By maximality, for every $\alpha < \delta$ and every $g \in F_z$, there is an $f \in A_\alpha$ such that g is compatible with f. But since $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$, this means $f \subseteq g$. Thus F_z refines each A_α. □

The boolean completion of a Suslin (κ, λ, μ)-forest is a κ-Suslin algebra, which is a complete boolean algebra with that is both κ-c.c. and κ-distributive. The cardinality of this algebra is at least λ. Therefore the existence of varieties Suslin forests is constrained by the following (see [21], Theorem 30.20):

Theorem 6.11 (Solovay)**.** *If $\mathbb{B}$ is a κ-Suslin algebra, then $|\mathbb{B}| \leq 2^\kappa$.*

Large Suslin forests can be obtained by forcing. In [19], Jech defined a class of partial orders $\mathbb{P}_\lambda$ such that under CH, $\mathbb{P}_\lambda$ is countably closed, ω_2-c.c., and adds a Suslin $(\omega_1, \lambda, 2)$-forest. However, this forest fails to be coherent. Modifying his forcing slightly, we obtain:

Theorem 6.12. *Assume κ is a regular cardinal, $2^{<\kappa} = \kappa$, and $2^\kappa = \kappa^+$. Then for all $\lambda > \kappa$, there is a κ^+-closed, κ^{++}-c.c. forcing of size $\lambda^{<\kappa}$ that adds a coherent, Suslin $(\kappa^+, \lambda, 2)$-forest.*

Proof (sketch). Let $\mathbb{P}$ be the set of all partial functions f from λ to 2 of size $\leq \kappa$, and say $f \leq g$ when $\mathrm{dom}(f) \supseteq \mathrm{dom}(g)$ and $|\{\alpha : f(\alpha) \neq g(\alpha)\}| < \kappa$. κ^+-closure follows from Lemma 6.3(a), and the κ^{++}-c.c. follows from a delta-system argument. If G is $\mathbb{P}$-generic over V, in $V[G]$ let $F = \{f : (\exists g \in G)\, \mathrm{dom}(g) = \mathrm{dom}(f) \text{ and } |\{\alpha : f(\alpha) \neq g(\alpha)\}| < \kappa\}$. Clearly F is coherent. The argument that F is Suslin in $V[G]$ proceeds as in [19]. □

By adapting an argument of Todorčević that appears in [35], we can obtain large Suslin forests in a different way:

Theorem 6.13. *Assume κ is a regular cardinal, $2^{<\kappa} = \kappa$, and there is a coherent $(\kappa^+, \lambda, \kappa)$-forest of injective functions. Then adding a Cohen subset of κ adds a coherent, Suslin $(\kappa^+, \lambda, 2)$-forest.*

Proof. Let F be a coherent $(\kappa^+, \lambda, \kappa)$-forest of injections closed under $< \kappa$ modifications to other injections. Let $g : \kappa \to 2$ be an $Add(\kappa)$ generic function over V. Consider the family $G_0 = \{g \circ f : f \in F\}$. Since $Add(\kappa)$ is κ^+-c.c., $\mathcal{P}_{\kappa^+}(\lambda)^V$ is cofinal in $\mathcal{P}_{\kappa^+}(\lambda)^{V[g]}$, so G_0 generates a forest G when we close under subsets. G inherits coherence from F. We claim G is Suslin.

First we note that G is closed under $< \kappa$ modifications. If $f \in F$, then by the argument for Proposition 6.6, $\kappa \setminus \operatorname{ran}(f)$ has size κ. By a density argument, $\{\alpha \in \kappa \setminus \operatorname{ran}(f) : g(\alpha) = i\}$ has size κ for both $i = 0, 1$. So if $g \circ f \in G$, and $x \subseteq \operatorname{dom} f$ has size $< \kappa$, we can switch values of $g \circ f$ on x by choosing distinct ordinals $\{\alpha_i : i \in x\} \subseteq \kappa \setminus \operatorname{ran}(f)$ such that $g(\alpha_i) = g(f(i)) + 1 \mod 2$. If $f' = f$ except that $f'(i) = \alpha_i$ for $i \in x$, then $f' \in V$ by κ-closure, so $g \circ f' \in G$. So by Lemma 6.2, members of G have a common extension when they agree on their common domain.

Towards a contradiction, suppose $A = \{g \circ f_\alpha : \alpha < \kappa^+\}$ is an antichain in G_0, and let $p_0 \in Add(\kappa)$ force this. Since $|Add(\kappa)| = \kappa$, there is some $p_1 \leq p_0$ such that $p_1 \Vdash \dot{g} \circ \check{f} \in \dot{A}$ for κ^+ many $f \in F$. Let $A_0 = \{f : p_1 \Vdash \dot{g} \circ \check{f} \in \dot{A}\}$, and let $Z = \bigcup\{\operatorname{dom}(f) : f \in A_0\}$.

Case 1: $|Z| \leq \kappa$. Let $h \in F$ be such that $\operatorname{dom}(h) = Z$. There are at most κ many $< \kappa$ modifications of h, so there are $f_0, f_1 \in A_0$ such that both agree with the same modification of h. But p_1 forces that $g \circ f_0$ and $g \circ f_1$ are compatible, contradiction.

Case 2: $|Z| = \kappa^+$. Let $\langle \alpha_i : i < \kappa^+ \rangle$ be an enumeration of Z. Let $\beta_0 = \sup(\operatorname{dom}(p_1)) + 1$, and for each $f \in A_0$, let $X_f = \{\alpha : f(\alpha) < \beta_0\}$. Since each f is injective, each $|X_f| < \kappa$. For each X_f, let $\langle X_f(i) : i < \beta_f \rangle$ be an enumeration of X_f that agrees in order with the above enumeration of Z.

Case 2a: There is no $i < \kappa$ such that $|\{X_f(i) : f \in A_0\}| = \kappa^+$. Then there is a $\gamma < \kappa^+$ such that for all $f \in A_0$, $\{i : \alpha_i \in X_f\} \subseteq \gamma$. Since $\kappa^{<\kappa} = \kappa$, we may choose some $A_1 \subseteq A_0$ such that for all $f \in A_1$, X_f is the same set S, and further that $f \restriction S$ is the same for all $f \in A_1$.

Let $f_0, f_1 \in A_1$, and let $D = \{\alpha \in \mathrm{dom}(f_0) \cap \mathrm{dom}(f_1) : f_0(\alpha) \neq f_1(\alpha)\}$. $|D| < \kappa$, $D \cap S = \emptyset$, and if $\alpha \in D$, then $f_0(\alpha), f_1(\alpha) \geq \beta_0$. Thus we can define a $q \leq p_1$ such that for all $\alpha \in D$, $q \circ f_0(\alpha) = q \circ f_1(\alpha) = 0$. q forces that $g \circ f_0$ and $g \circ f_1$ are compatible, contradiction.

Case 2b: There is some $i < \kappa$ such that $|\{X_f(i) : f \in A_0\}| = \kappa^+$. Let i_0 be the least such ordinal. We choose a sequence $\langle f_\alpha : \alpha < \kappa^+\rangle$. Let $f_0 \in A_0$ be arbitrary. Let f_1 be such that $X_{f_1}(i_0)$ has index in the enumeration of Z above $\{i : \alpha_i \in \mathrm{dom}(f_0)\}$. Keep going in this fashion such that for $\beta < \gamma < \kappa^+$, $X_{f_\gamma}(i_0)$ has index greater than $\sup\{i : \alpha_i \in \mathrm{dom}(f_\beta)\}$. By the minimality of i_0, there is $C \subseteq \kappa^+$ of size κ^+ and a set $S \subseteq Z$ such that for all $\alpha \in C$, $\{X_{f_\alpha}(i) : i < i_0\} = S$, and $f_\alpha \restriction S$ is the same.

Now let $\beta < \gamma$ be in C, and let $D = \{\alpha \in \mathrm{dom}(f_\beta) \cap \mathrm{dom}(f_\gamma) : f_\beta(\alpha) \neq f_\gamma(\alpha)\}$. As before, $|D| < \kappa$ and $D \cap S = \emptyset$. If $\alpha \in D$, then $f_\gamma(\alpha) \geq \beta_0$, because $X_{f_\gamma} \cap \mathrm{dom}(f_\beta) = S$. We construct $q \leq p_1$ such that for all $\alpha \in D$, $q \circ f_\gamma(\alpha) = q \circ f_\beta(\alpha)$. Let $D_0 = \{\alpha \in D : f_\beta(\alpha) \in \mathrm{dom}(p_1)\}$, and let $q_0 = p_1 \cup \{\langle f_\gamma(\alpha), p_1 \circ f_\beta(\alpha)\rangle : \alpha \in D_0\}$. We are free to do this because f_γ is injective and $f_\gamma(\alpha) \notin \mathrm{dom}(p_1)$ for $\alpha \in D$.

Note that for all $\alpha \in D$, q_0 is defined at $f_\gamma(\alpha)$, only if it is defined at $f_\beta(\alpha)$. But it may be that for some $\alpha \in D_0$ and some $\alpha' \in D \setminus D_0$, $f_\gamma(\alpha) = f_\beta(\alpha')$. Assume we have a sequence $q_0 \geq ... \geq q_n$ such that:

(1) for all $k \leq n$, $D \cap f_\gamma^{-1}[\mathrm{dom}(q_k)] \subseteq D \cap f_\beta^{-1}[\mathrm{dom}(q_k)]$,

(2) for all $k \leq n$, $q_k \circ f_\gamma \restriction (D \cap f_\gamma^{-1}[\mathrm{dom}(q_k)]) = q_k \circ f_\beta \restriction (D \cap f_\gamma^{-1}[\mathrm{dom}(q_k)])$,

(3) if $k+1 \leq n$, then $D \cap f_\gamma^{-1}[\mathrm{dom}(q_{k+1})] = D \cap f_\beta^{-1}[\mathrm{dom}(q_k)]$.

If $D \cap f_\gamma^{-1}[\mathrm{dom}(q_n)] = D \cap f_\beta^{-1}[\mathrm{dom}(q_n)]$, let $q_{n+1} = q_n$. Otherwise, let $D_{n+1} = D \cap f_\beta^{-1}[\mathrm{dom}(q_n)]$, and let $q_{n+1} = q_n \cup \{\langle f_\gamma(\alpha), q_n \circ f_\beta(\alpha)\rangle : \alpha \in D_{n+1}\}$. Clearly the induction hypotheses are preserved for $n+1$.

Put $q_\omega = \bigcup q_n$. (Note in the case $\kappa = \omega$, D is finite, so $q_\omega = q_n$ for some n.) By (1) and (3), $D \cap f_\beta^{-1}[\mathrm{dom}(q_\omega)] = D \cap f_\gamma^{-1}[\mathrm{dom}(q_\omega)]$, so call this set D_ω. Let $q = q_\omega \cup \{\langle f_\beta(\alpha), 0\rangle : \alpha \in D \setminus D_\omega\} \cup \{\langle f_\gamma(\alpha), 0\rangle : \alpha \in D \setminus D_\omega\}$. This q forces that $g \circ f_\beta$ and $g \circ f_\gamma$ are compatible, again in contradiction to the assumption about p_1. □

Corollary 6.14. *Assume κ is a regular cardinal, $2^{<\kappa} = \kappa$, and $\lambda > \kappa$. Then there is a κ-closed, κ^+-c.c. forcing that adds a coherent, Suslin $(\kappa^+, \lambda, 2)$-forest.*

Proof. Apply Theorems 6.8 and 6.13. □

Large Suslin forests can also be obtained from combinatorial principles rather than forcing. As reported by Jech [19] [20] [21], Laver proved in unpublished work that the existence of Suslin $(\omega_1, \omega_2, 2)$-forests follows from Silver's principle W and $\diamondsuit$, both of which hold in L. Unfortunately, Laver's proof seems to be lost to history. In trying to reconstruct it, we encountered technical issues that led to the development of a new combinatorial principle, which we prove consistent from a Mahlo cardinal, that can be used to construct large Suslin forests. The main appeal for us is that, unlike the above forcing constructions, it allows a Suslin $(\kappa, \kappa^+, 2)$-forest to be generically added to any model with sufficiently large cardinals using a forcing of size κ rather than κ^+.

Let us establish some notation concerning trees. Suppose T is a κ-tree and $\alpha < \kappa$. T_α is the set of nodes at level α. If b is a cofinal branch in T, $\pi_\alpha(b)$ is the node at level α in b. If $\beta < \alpha$, and $x \in T_\alpha$, $\pi_{\alpha,\beta}(x)$ is the node in T_β below x.

Definition. *$W_\kappa(\lambda)$ is the statement that there is a κ-tree T, a set of cofinal branches B, and a sequence $\langle W_\alpha : \alpha < \kappa\rangle$ with the following properties:*

(1) $|B| = \lambda$.

(2) For each α, $|W_\alpha| < \kappa$, and $W_\alpha \subseteq \mathcal{P}(T_\alpha)$.

(3) For every $z \in \mathcal{P}_\kappa(B)$, there is an $\alpha < \kappa$ such that for all $\beta \geq \alpha$, $\pi_\beta[z] \in W_\beta$.

Let T, B, $\langle W_\alpha : \alpha < \kappa\rangle$ be as above. If $z \in \mathcal{P}_\kappa(B)$, say "*$z$ is captured at α*" when for all $\beta \geq \alpha$, $\pi_\beta[z] \in W_\beta$ and $\pi_\beta \restriction z$ is injective. If $z \in W_\alpha$ and $\gamma < \alpha$, say "*z is captured at γ*" when for all β such that $\gamma \leq \beta < \alpha$, $\pi_{\alpha,\beta}[z] \in W_\beta$ and $\pi_{\alpha,\beta} \restriction z$ is injective.

Definition. *$W^*_\kappa(\lambda)$ asserts $W_\kappa(\lambda)$, and that there exists a stationary $S \subseteq \kappa$ and a sequence $\langle A_\alpha : \alpha < \kappa\rangle$ with each $A_\alpha \subseteq W^2_\alpha$, such that the following additional clauses hold:*

(4) $\kappa = \mu^+$ for a regular cardinal μ, and each W_α is a μ-complete subalgebra of $\mathcal{P}(T_\alpha)$ containing all singletons.

(5) For all $\alpha \in S$, $\{z \in W_\alpha : z$ is captured below $\alpha\}$ is closed under arbitrary $< \mu$ sized unions and taking subsets which are in W_α.

(6) If $f : \kappa \to \mathcal{P}_\kappa(B)^2$ is such that $|\bigcup_{\alpha<\kappa} f_0(\alpha) \cup f_1(\alpha)| = \kappa$, let $\langle b_\alpha : \alpha < \kappa\rangle$ enumerate the elements of $\bigcup_{\alpha<\kappa} f_0(\alpha) \cup f_1(\alpha)$. The set of $\alpha \in S$ with the following properties is stationary:

(a) $\{b_\beta : \beta < \alpha\}$ is captured at α.

(b) If $z \subseteq \{\pi_\alpha(b_\beta) : \beta < \alpha\}$ is captured below α, then $\sup\{\beta : \pi_\alpha(b_\beta) \in z\} < \alpha$.

(c) $\{\langle \pi_\alpha[f_0(\beta)], \pi_\alpha[f_1(\beta)]\rangle : \beta < \alpha\} = A_\alpha$.

It is easy to see that $W_\kappa(\lambda)$ implies $2^{<\kappa} = \kappa$, and in fact $W_\kappa(\kappa)$ is equivalent to $2^{<\kappa} = \kappa$. If $\kappa = \mu^+$ and S forms part of the witness to $W^*_\kappa(\lambda)$, then clause (4) implies $\mu^{<\mu} = \mu$, and clause (6) can be used to show $\diamondsuit_\kappa(S)$. On the other hand, it follows from the next theorem that $W^*_\kappa(\lambda)$ prescribes no value for 2^κ, besides that $\lambda \leq 2^\kappa$.

Theorem 6.15. *Suppose κ is a Mahlo cardinal and $\mu < \kappa$ is regular. If $G * H \subseteq Col(\mu, <\kappa) * Add(\kappa)$ is generic, then $V[G * H]$ satisfies $W^*_\kappa(2^\kappa)$.*

Proof. In V, let T be the complete binary tree on κ, and let B be the set of all branches. For $\alpha < \kappa$, let $G_\alpha = G \cap Col(\mu, < \alpha)$, and let $W_\alpha = \mathcal{P}(T_\alpha)^{V[G_\alpha]}$. Let $S = \{\alpha < \kappa : \alpha$ is inaccessible in $V\}$. In $V[G]$, fix enumerations $\langle s^\alpha_\beta : \beta < \mu \rangle$ of the W^2_α, and in $V[G * H]$, let $A_\alpha = \{s^\alpha_\beta : H(\alpha + \beta) = 1\}$. Let us check each condition.

(1) $(2^\kappa)^V = (2^\kappa)^{V[G*H]}$, so $V[G * H] \models |B| = 2^\kappa$.

(2) Since κ is inaccessible, each W_α is collapsed to μ.

(3) Suppose $z \in \mathcal{P}_\kappa(B)$. There is some $\alpha < \kappa$ such that $z \in V[G_\alpha]$. For $\beta \geq \alpha$, $\pi_\beta[z] \in W_\beta$.

(4) The regularity of μ is preserved, and clearly each W_α contains all singletons. Let $\langle a_\xi : \xi < \delta \rangle \subseteq W_\alpha$ with $\delta < \mu$. Each $a_\xi \in A$ is $\tau_\xi^{G_\alpha}$ for some $Col(\mu, < \alpha)$-name τ_ξ. By the μ-closure of $Col(\mu, < \kappa)$, $\langle \tau_\xi : \xi < \delta \rangle \in V$, so $\langle a_\xi : \xi < \delta \rangle \in V[G_\alpha]$.

(5) By the Mahlo property, S is stationary, and by the κ-c.c. of $Col(\mu, < \kappa)$ and κ-closure of $Add(\kappa)$, it remains stationary in $V[G * H]$. Suppose $\alpha \in S$.

   (a) Unions: Let $A \in \mathcal{P}_\mu(W_\alpha)$ have the property that all a in A are captured below α. As above, $A \in V[G_\alpha]$. Now in $V[G_\alpha]$, $\alpha = \mu^+$ and $|T_\beta| = \mu$ for $\beta < \alpha$. So if $\pi_{\alpha,\beta} \restriction a$ is injective, then $V[G_\alpha] \models |a| < \alpha$, and thus $V[G_\alpha] \models |\bigcup A| < \alpha$. For distinct $x, y \in \bigcup A$, let $\gamma_{x,y} < \alpha$ be the least γ such that $\pi_{\alpha,\gamma}(x) \neq \pi_{\alpha,\gamma}(y)$. We have $\gamma = \sup\{\gamma_{x,y} : x, y \in \bigcup A\} < \alpha$. Hence if $\gamma \leq \beta < \alpha$ and all $a \in A$ are captured at β, then $\bigcup A$ is captured at β.

(b) Subsets: Suppose $z_0 \in W_\alpha$ is captured below α, and $z_1 \in W_\alpha$ is a subset of z_0. Then $V[G_\alpha] \models |z_1| < \alpha$, so by the α-c.c. of $Col(\mu, < \alpha)$, there is some $\beta < \alpha$ such that $z_1 \in V[G_\beta]$. Thus z_1 is captured below α.

(6) First work in $V[G]$. Let $\dot{f}$ be an $Add(\kappa)$-name for a function from κ to $\mathcal{P}_\kappa(B)^2$, and let $\langle \dot{b}_\alpha : \alpha < \kappa \rangle$ be as in clause (6). Let $\dot{C}$ be a name for a club, and let $p_0 \in Add(\kappa)$ be arbitrary. Build a continuous decreasing chain of conditions below p_0, $\langle p_\alpha : \alpha < \kappa \rangle \subseteq Add(\kappa)$, and a continuous increasing chain of ordinals, $\langle \xi_\alpha : \alpha < \kappa \rangle \subseteq \kappa$, with the following properties: For all α,

- $p_{\alpha+1} \Vdash \xi_\alpha \in \dot{C}$,
- $p_{\alpha+1}$ decides $\dot{f} \restriction \mathrm{dom}(p_\alpha)$ and $\{\dot{b}_\beta : \beta < \alpha\}$,
- $\mathrm{dom}(p_{\alpha+1})$ is an ordinal $> \max\{\mathrm{dom}(p_\alpha), \xi_\alpha, \alpha\}$, and
- $\xi_{\alpha+1} > \mathrm{dom}(p_{\alpha+1})$.

Let $g : \kappa \to \mathcal{P}_\kappa(B)^2$ and $\{b_\alpha : \alpha < \kappa\}$ be the objects defined by what the chain $\langle p_\alpha : \alpha < \kappa \rangle$ decides. For each $\alpha < \kappa$, there is a predense set $E_\alpha \subseteq Col(\mu, < \kappa)$ of size $< \kappa$ such that $g(\alpha)$ and b_α are decided by elements of E_α. There is a club $D \in V$ such that $\forall \alpha \in D$, $\forall \beta < \alpha$, $E_\beta \subseteq Col(\mu, < \alpha)$. For $\alpha \in D$, $g \restriction \alpha$ and $\{b_\beta : \beta < \alpha\}$ are in $V[G_\alpha]$.

Back in $V[G]$, for $\alpha < \kappa$, let γ_α be the least $\gamma \geq \alpha$ such that $\pi_{\gamma_\alpha} \restriction \{b_\beta : \beta < \alpha\}$ is injective. If α is closed under $\beta \mapsto \gamma_\beta$, then $\gamma_\alpha = \alpha$. As S is stationary, there is $\alpha \in S \cap D$ such that $\gamma_\alpha = \alpha$, $\xi_\alpha = \alpha$, and $p_\alpha \Vdash \alpha \in \dot{C}$. We have that $\{b_\beta : \beta < \alpha\}$ is captured at α, and that $\{\langle \pi_\alpha[g_0(\beta)], \pi_\alpha[g_1(\beta)] \rangle : \beta < \alpha\} \subseteq W_\alpha^2$. Since α is inaccessible in V, if $z \subseteq \{\pi_\alpha(b_\beta) : \beta < \alpha\}$ is captured below α, then $V[G_\alpha] \models |z| < \alpha$, so $\{\beta : \pi_\alpha(b_\beta) \in z\}$ is bounded below α.

Let $q \leq p_\alpha$ be such that for $\beta < \mu$, $q(\alpha + \beta) = 1$ if $s_\alpha^\beta = \langle \pi_\alpha[g_0(\beta)], \pi_\alpha[g_1(\beta)] \rangle$, and $q(\alpha + \beta) = 0$ otherwise. Then $q \Vdash \alpha \in \dot{C} \cap S$, and that items (a), (b), and (c) in clause

(6) hold at α. As p_0 was arbitrary, clause (6) is forced.

□

Question. *Can $W^*_\kappa(\lambda)$ be forced without the use of large cardinals? Can it be forced in a cardinal-preserving way? Does $L \models$ "For all regular κ, $W^*_{\kappa^+}(\kappa^{++})$"?*

Theorem 6.16. *$W^*_\kappa(\lambda)$ implies there is a coherent, Suslin $(\kappa, \lambda, 2)$-forest.*

Proof. Let $\kappa = \mu^+$, and let T, B, $\langle W_\alpha : \alpha < \kappa \rangle$, $\langle A_\alpha : \alpha < \kappa \rangle$, and $S \subseteq \kappa$ witness $W^*_\kappa(\lambda)$. We will construct a sequence of functions $\langle f_\alpha : \alpha < \kappa \rangle$ on the nodes of T that will generate a coherent family of functions on B with the desired properties. Each f_α will have domain T_α and range contained in $\{0, 1\}$.

Let f_0 be a function from T_0 to 2. Assume we have have constructed a sequence of functions $\langle f_\beta : \beta < \alpha \rangle$, with each $f_\beta : T_\beta \to 2$, satisfying the following property:

(∗) If $r \in W_\beta$ is captured at $\gamma < \beta$, then $f_\beta \restriction r$ disagrees with $f_\gamma \circ \pi_{\beta,\gamma} \restriction r$ on a set of size $< \mu$.

Let $R_\alpha = \{r \in W_\alpha : r \text{ is caputured below } \alpha\}$. Consider the set F_α of partial functions on T_α of the form $f_\gamma \circ \pi_{\alpha,\gamma} \restriction r$ for $r \in R_\alpha$ and γ witnessing its membership in R_α. Assume $\gamma_0 < \gamma_1$ and $f_{\gamma_0} \circ \pi_{\alpha,\gamma_0} \restriction r_0$ and $f_{\gamma_1} \circ \pi_{\alpha,\gamma_1} \restriction r_1$ are in F_α. By hypothesis (∗), f_{γ_1} disagrees with $f_{\gamma_0} \circ \pi_{\gamma_1,\gamma_0}$ at less than μ many points in $\pi_{\alpha,\gamma_1}[r_0]$. Therefore, there are less than μ many points in $r_0 \cap r_1$ at which $f_{\gamma_0} \circ \pi_{\alpha,\gamma_0}$ and $f_{\gamma_1} \circ \pi_{\alpha,\gamma_1}$ disagree. So F_α is a μ-coherent family.

Assume first that $\alpha \notin S$. Using Lemma 6.3(a), let $f_\alpha : T_\alpha \to 2$ be such that $\{f_\alpha\} \cup F_\alpha$ is μ-coherent. Then (∗) holds for $\langle f_\beta : \beta \leq \alpha \rangle$.

Now assume $\alpha \in S$. Let H_α be the closure of F_α under $< \mu$ modifications. Consider H_α as a partial order with $f \leq g$ iff $f \supseteq g$. The set $A_\alpha \subseteq W^2_\alpha$ codes a set of relations from subsets of T_α to 2. If $\langle a_0, a_1\rangle \in A_\alpha$, construct a relation h by putting $\langle x, i\rangle \in h$ iff $x \in a_i$, and call the set of all such things A'_α. It may be the case that every member of A'_α is a function and a member of H_α, and that A'_α is a maximal antichain in H_α. If not, ignore all these considerations, and let f_α be as in the case $\alpha \notin S$, so that $(*)$ is preserved.

Suppose A'_α is a maximal antichain in H_α. Enumerate R_α as $\langle r_\beta : \beta < \mu\rangle$. By clauses (4) and (5) of the definition of W^*, R_α is closed under unions of size $< \mu$. H_α is also a μ-closed partial order. If $\langle h_i : i < \beta < \mu\rangle$ is a decreasing sequence, then $\bigcup_{i<\beta} \mathrm{dom}(h_i) = r \in R_\alpha$, so let γ witness this. By $(*)$, each h_i disagrees with $f_\gamma \circ \pi_{\alpha,\gamma}$ on a set of size $< \mu$, and so $\bigcup_{i<\beta} h_i$ does as well by the regularity of μ.

Setting $s_\beta = \bigcup_{\xi<\beta} r_\xi$, we have $\langle s_\beta : \beta < \mu\rangle$ is an increasing cofinal sequence in R_α. For $\beta < \mu$, let γ_β be the least $\gamma < \alpha$ that witnesses $s_\beta \in R_\alpha$. Let $\langle t_\beta : \beta < \mu\rangle$ enumerate all $< \mu$ sized subsets of T_α, such that each subset is repeated μ many times. For a partial function $f : T_\alpha \to 2$ and $\beta < \mu$, let f/t_β be f with its output values switched at the points in $\mathrm{dom}(f) \cap t_\beta$.

We will define f_α inductively as $\bigcup_{\beta<\mu} h_\beta$. Let $h_0 = \emptyset$. Assume $\langle h_i : i < \beta\rangle$ has been chosen so that:

(1) for $i < j < \beta$, $h_i \subseteq h_j$;

(2) for $i < \beta$, $\mathrm{dom}(h_i) = s_{\xi_i}$ where $\xi_i \geq i$, and $\xi_i > \xi_j$ for $j < i$;

(3) for $i < \beta$, there is $a \in A'_\alpha$ such that h_{i+1}/t_i is a common extension of h_i/t_i and a.

Given h_i, there is some $a \in A'_\alpha$ that is compatible with h_i/t_i. Let $\xi_{i+1} > \xi_i$ be such that $s_{\xi_{i+1}} \supseteq \mathrm{dom}(a) \cup s_{\xi_i}$, and let $g \in H_\alpha$ be a common extension of a and h_i/t_i with domain $s_{\xi_{i+1}}$. Let $h_{i+1} = g/t_i$. Clearly (1)–(3) are preserved at successor steps. At limit steps β, we set $h_\beta = \bigcup_{i<\beta} h_i$. This is in H_α as well by μ-closure, and the preservation of (1)–(3) is trivial.

The point is this: For every $t \in \mathcal{P}_\mu(T_\alpha)$, f_α/t extends some $a \in A'_\alpha$. For let $i < \mu$ be large enough that $s_{\xi_i} \supseteq t$ and $t_i = t$. Then by (3), h_{i+1}/t extends some $a \in A'_\alpha$, and $h_{i+1}/t = (f_\alpha/t) \restriction s_{\xi_{i+1}}$. We also check that $(*)$ is preserved at α: Every $r \in R_\alpha$ is covered by some s_{ξ_i}, and $f_\alpha \restriction s_{\xi_i} = h_i$, which coheres with $f_\gamma \circ \pi_{\alpha,\gamma} \restriction s_{\xi_i}$ when s_{ξ_i} is captured at γ.

Now we define the forest. For $z \in \mathcal{P}_\kappa(B)$, let γ_z be the least $\gamma < \kappa$ such that z is captured at γ. Let $f_z : z \to 2$ be $f_{\gamma_z} \circ \pi_{\gamma_z} \restriction z$. Let F be the closure of $\{f_z : z \in \mathcal{P}_\kappa(B)\}$ under $< \mu$ modifications. Note that by $(*)$, if $\beta \geq \gamma_z$, then $f_\beta \circ \pi_\beta \restriction z$ disagrees with f_z at $< \mu$ many points. Hence F is a coherent $(\kappa, B, 2)$-forest.

Finally, we verify the κ-c.c. First note that F satisfies the κ^+-c.c. by a delta-system argument. So assume towards a contradiction that $A = \{a_\alpha : \alpha < \kappa\}$ is a maximal antichain. Let $z_\alpha = \mathrm{dom}(a_\alpha)$, and code each a_α as $\langle z^0_\alpha, z^1_\alpha \rangle$, where $z^i_\alpha = \{b : a_\alpha(b) = i\}$. Let $\langle b_\alpha : \alpha < \kappa \rangle$ enumerate the elements of $\bigcup_{\alpha<\kappa} z_\alpha$. Define:

- $C_0 = \{\alpha < \kappa : \bigcup_{\beta<\alpha} z_\beta = \{b_\beta : \beta < \alpha\}\}$.
- $C_1 = \{\alpha < \kappa : \{a_\beta : \beta < \alpha\}$ is a maximal antichain contained in $\{f \in F : (\exists \eta < \alpha)\, \mathrm{dom}(f) \subseteq \{b_\beta : \beta < \eta\}\}\}$.
- $C_2 = \{\alpha < \kappa : (\forall \beta < \alpha) \gamma_{z^0_\beta}, \gamma_{z^1_\beta}, \gamma_{z_\beta} < \alpha\}$.

It is easy to see that C_0, C_1, and C_2 are club. By clause (6) of the definition of W^*, let $\alpha \in S \cap C_0 \cap C_1 \cap C_2$ be such that $\{b_\beta : \beta < \alpha\} = \bigcup_{\beta<\alpha} z_\beta$ is captured at α, all $z \subseteq \{\pi_\alpha(b_\beta) : \beta < \alpha\}$ captured below α have $\sup\{\beta : \pi_\alpha(b_\beta) \in z\} < \alpha$, and $A_\alpha = \{\langle \pi_\alpha[z^0_\beta], \pi_\alpha[z^1_\beta] \rangle : \beta < \alpha\}$.

We claim A'_α is a maximal antichain in H_α. For $\beta < \alpha$, z_β is captured below α since $\alpha \in C_2$, so the function coded by $\langle \pi_\alpha[z^0_\beta], \pi_\alpha[z^1_\beta] \rangle$ is in H_α. If $h \in H_\alpha$ is incompatible with every member of A'_α, then consider $z = \{b_\beta : \beta < \alpha \text{ and } \pi_\alpha(b_\beta) \in \mathrm{dom}(h)\}$, and let $f = h \circ \pi_\alpha \restriction z$. Clauses (4) and (5) imply $\pi_\alpha[z]$ is captured below α, so $\sup\{\beta : b_\beta \in z\} < \alpha$. Since

$\alpha \in C_1$, f is compatible with some a_β with $\beta < \alpha$. But a_β is coded and projected down as $\langle \pi_\alpha[z^0_\beta], \pi_\alpha[z^1_\beta] \rangle \in A_\alpha$, so h is compatible with some member of A'_α after all.

Since $\{b_\beta : \beta < \alpha\}$ is captured at α, the construction has sealed this antichain. Consider any other $f \in F$ such that $\mathrm{dom}(f) \supseteq \{b_\beta : \beta < \alpha\}$. Then $f \restriction \{b_\beta : \beta < \alpha\}$ is a $< \mu$ modification of $f_\alpha \circ \pi_\alpha \restriction \{b_\beta : \beta < \alpha\}$. By the above argument, all $< \mu$ modifications of f_α extend a member of A'_α, and so f is compatible with some a_β, $\beta < \alpha$. This contradicts the assumption that $A = \{a_\gamma : \gamma < \kappa\}$ is an antichain. □

Finally, we may answer the question of whether ZFC+GCH proves an analogue of Taylor's theorem above ω_1. Start with an almost-huge cardinal κ and a Mahlo cardinal $\mu < \kappa$. Suppose κ carries a tower of height δ, and λ is regular such that $\kappa \leq \lambda$. By Theorem 2.17, if $X * H$ is $A(\mu, \kappa) * \mathrm{Col}(\lambda, < \delta)$-generic, then in $V[X][H]$ there is a normal, fine, κ-complete, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$. Furthermore, this forcing is μ-strategically closed, and it is easy to show that μ-strategically closed forcings preserve stationary subsets of μ. Thus μ remains a Mahlo cardinal in $V[X][H]$. If $\nu < \mu$ is regular and G is $\mathrm{Col}(\nu, < \mu) * \mathrm{Add}(\mu)$-generic, then in $V[X][H][G]$ there is a coherent $(\mu, \kappa, 2)$-Suslin forest, and thus a μ-Suslin algebra of uniform density κ. Since $\mathrm{Col}(\nu, < \mu) * \mathrm{Add}(\mu)$ is μ-dense, it preserves the density of κ-complete ideals.

Since the μ-Suslin algebra is μ-distributive forcing with it preserves the equation $2^\nu = \mu$. If we force with this Suslin algebra over $V[X][H][G]$, then Lemmas 5.9 and 5.10 imply that in the generic extension, there is a normal, fine, κ-complete, nonregular ideal on $\mathcal{P}_\kappa(\lambda)$, but no λ-dense ideals. Hence we have the following consistency result:

Theorem 6.17. *If ZFC+"There is an almost-huge cardinal" is consistent, then for $m \geq n \geq 2$, ZFC+GCH does not prove the statement, "If there is a nonregular ideal on $\mathcal{P}_{\omega_n}(\omega_m)$, then there is a dense ideal on $\mathcal{P}_{\omega_n}(\omega_m)$."*

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