Title
Solution to Monthly Problem #11418

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Solution to Monthly Problem #11418

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Problem 11418 asks to evaluate (for complex $|a| > 1$)

$$J := \int_{-\infty}^{\infty} \frac{t^2 \text{sech}^2(t)}{a - \tanh(t)} dt.$$ 

A variable change of $x = \tanh(t)$ produces for $|a| > 1$ that

$$J = \int_{-1}^{1} \frac{\arctanh^2(x)}{a - x} dx = \frac{1}{12} \ln^3 \left( \frac{a + 1}{a - 1} \right) + \frac{\pi^2}{12} \ln \left( \frac{a + 1}{a - 1} \right).$$ (1)

The corresponding Maple 12 code that obtained this is

\begin{verbatim}
J1:=a->int(t^2*sech(t)^2/(a-tanh(t)),t=-infinity..infinity): J1(a)
assuming a>1;
\end{verbatim}

\begin{verbatim}
limit(t ln(a + 1) - t ln(a + 1 + exp(2 t) a - exp(2 t))
\end{verbatim}

\begin{verbatim}
   / (a - 1) exp(2 t)\ 1
- t polylog[2, - -----------------] + - polylog[3, - -----------------]
\ / a + 1 \ a + 1 /
2 + t ln(exp(2 t) + 1) + t polylog(2, -exp(2 t)) - - polylog(3, -exp(2 t)),
2
\end{verbatim}

\begin{verbatim}
t = infinity, left|
\end{verbatim}

\begin{verbatim}
J2:=simplify(student[changervar](x=tanh(t),J1(a),x)) assuming a>1:J2;
\end{verbatim}

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\[ - \frac{\pi}{12} \ln(a - 1) + \frac{\pi}{12} \ln(a + 1) - \frac{\ln(a - 1)}{4} \ln(a + 1) \]

The newer syntax of

\texttt{[IntegrationTools](Change(J1(a),x=tanh(t)^2))}

fails to produce the desired evaluation in (1). It returns the correct but less helpful polylogarithmic limit above.

The evaluation would appear to be valid except when \(-1 < a < 1\).

A human proof can be obtained from (1) on using the geometric series and integrating term-by-term carefully—which is much easier once the answer is known.