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Publication Date

1979-07-01

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July 1979

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Prepared for the U. S. Department of Energy
under Contract W-7405-ENG-48

LBL-8916 C.2

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HAMILTONIAN THEORY OF PONDEROMOTIVE EFFECTS OF AN ELECTRO-
MAGNETIC WAVE IN A NONUNIFORM MAGNETIC FIELD*

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ABSTRACT

Noncanonical variables are used to construct a Hamiltonian for guiding-center motion in a nonuniform magnetostatic field. Lie methods are then used to obtain the ponderomotive (quasistatic) Hamiltonian for the perturbation by a wave of arbitrary polarization and wavenumber, with spatially modulated amplitude and wavevector. The result is applied to the problem of r.f. stoppering in mirror confinement.

*Work supported by the Office of Fusion Energy of the U. S. Department of Energy under contract No. W-7405-ENG-48.

This Letter presents a novel and powerful method for studying guiding center (g.c.) motion within a Hamiltonian framework. Although an obvious application is the exploration of g.c. effects to higher order in the small parameter $\epsilon \equiv \text{gyroradius/magnetic scale length}$, there are important new applications to be made even at lowest order. One may introduce perturbations and study their effects; this is a very fruitful approach because of the analytical simplification which arises in a Hamiltonian treatment. An important example of such a perturbation is a small-amplitude (parameterized by λ) electromagnetic (e.m.) wave; it is this problem which we treat in the second part of this Letter.

This work differs in significant ways from previous formulations¹⁻⁵. The most outstanding difference is that our g.c. Hamiltonian employs noncanonical variables in phase space, and cartesian, instead of field-line, coordinates in physical space. In addition, we use the Darboux algorithm⁶ to construct a semicanonical coordinate system in order to prepare the Hamiltonian for a perturbation analysis by Lie methods.⁷⁻⁸

For the six-dimensional phase space of a charged particle in a weakly inhomogeneous magnetostatic field $\vec{B}(\vec{x})$, we choose the following "guiding center variables": $\vec{X}$, g.c. position; P , g.c. parallel momentum; μ , magnetic moment; and θ , gyrophase. These are related to the particle position $\vec{x}$ and velocity $\vec{u}$ as follows (with $m = c = e = 1$):

$$\vec{X} = \vec{x} - \epsilon [\hat{b}(\vec{x}) \times \vec{u}] B^{-1}(\vec{x}) + O(\epsilon^2); \quad (1)$$

$$P = \hat{b}(\vec{x}) \cdot \vec{u} + O(\epsilon); \quad \mu = \frac{1}{2} [\hat{b}(\vec{x}) \times \vec{u}]^2 B^{-1}(\vec{x}) + O(\epsilon),$$

where $\hat{b}(\vec{x})$ is the field of unit vectors parallel to the magnetic field lines. The gyrophase θ is measured relative to a set of local perpendicular unit vectors. The expressions (1) are regarded as transformations from the particle variables $(\vec{x}, \vec{u})$, to the g.c. variables $(\vec{X}, P; \theta, \mu)$; higher order terms are available but not given here.

The guiding center variables are noncanonical, i.e., they do not fall into conjugate (q, p) pairs. To use noncanonical variables in Hamiltonian dynamics, one must know their fundamental Poisson brackets (P.B.):

$$\begin{aligned} \{\vec{X}, \theta\} &= \{\vec{X}, \mu\} = \{P, \theta\} = \{P, \mu\} = 0 ; \\ \{\theta, \mu\} &= 1/\epsilon; \quad \{\vec{X}, \vec{X}\} = \hat{b}(\vec{X}) \times \hat{I} / B^*(\vec{X}); \\ \{\vec{X}, P\} &= \hat{b}(\vec{X}) + [\epsilon P / B^*(\vec{X})] \hat{b}(\vec{X}) \times (\hat{b}(\vec{X}) \cdot \nabla) \hat{b}(\vec{X}), \end{aligned} \quad (2)$$

where $B^* \equiv B + \epsilon P \hat{b} \cdot \nabla \times \hat{b}$ is the modified⁹ magnetic field. These formulas (2) are exact, i.e., they are not power series in ϵ . From an abstract point of view, the fundamental P.B. are the components of a certain (Poisson) tensor⁶ on phase space; its reciprocal is the symplectic 2-form, whose components are the Lagrange brackets. The exploitation of the properties of this tensor constitutes an important novel element of this theory.

The P.B. of any two phase space functions f and g can be expressed in terms of the fundamental P.B.:

$$\begin{aligned} \{f, g\} &= \left(\frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \mu} - \frac{\partial f}{\partial \mu} \frac{\partial g}{\partial \theta} \right) \{\theta, \mu\} + \frac{\partial f}{\partial \vec{X}} \cdot \{\vec{X}, P\} \frac{\partial g}{\partial P} \\ &+ \frac{\partial f}{\partial P} \{P, \vec{X}\} \cdot \frac{\partial g}{\partial \vec{X}} + \frac{\partial f}{\partial \vec{X}} \cdot \{\vec{X}, \vec{X}\} \cdot \frac{\partial g}{\partial \vec{X}} \end{aligned} \quad (3)$$

This formula is important because the convective derivative of any function can be expressed as its P.B. with the g.c. Hamiltonian:

$$H(\vec{X}, P; \mu) = \mu B(\vec{X}) + \frac{1}{2} P^2 + O(\epsilon^2). \quad (4)$$

Note that the $O(\epsilon)$ term vanishes. Although the Hamiltonian (4) can be written down on the basis of intuition, in this theory it is rigorously derived within the framework of a systematic ordering scheme; an important bonus is the derivation of higher order terms. The familiar classic drifts are easily obtained from (2)-(4). The Hamiltonian is formally independent of θ to all orders, implying that μ is invariant to all orders.

Although the transformation (1), connecting particle and g.c. variables, is familiar, it was nevertheless not chosen on that basis. Instead, it is based on an underlying transformation theory, founded on the work of Kruskal¹⁰ on nearly periodic systems and on a theorem of Darboux¹¹; it is composed of two transformations, each of which is systematic to any order in ϵ . The first is called the Darboux transformation; it is closely connected with the constructive proof of Darboux's theorem, which concerns the transformation properties of symplectic forms. Applications of this theorem determines the fundamental P.B. (2). The Darboux transformation produces a semicanonical coordinate system in phase space, in which the variables (θ, μ) are canonically decoupled from the other four variables $(\vec{X}, P)$. Further application of the Darboux algorithm will produce additional pairs of canonically conjugate variables, related to the bounce motion and the drift surface. The second transformation comprising (1) is an averaging transformation, whose purpose is to eliminate the θ -dependence of the Hamiltonian. This is effected by Lie transforms, a novel feature of which is its extension to noncanonical variables.

The g.c. formalism thus provides us with an "unperturbed problem" to which is added an e.m. wave as a perturbation; to this problem we now turn. In plasma physics, radiation-induced processes account for many significant phenomena, such as parametric excitation, suppression of instability, profile modification in laser fusion, r.f. end-stoppering in magnetic mirrors. Physically, these processes involve the concept of ponderomotive force, the nonlinear force arising from the beating of two high frequency waves, or of one wave with itself. A consequence is that particles are expelled from regions of high radiation intensity (or attracted, in the case of a negative ponderomotive potential). Since the e.m. perturbations are usually localized, and the static fields are nonuniform, the need arises for a local description; this is accomplished by the Hamiltonian techniques just described. We shall derive the ponderomotive Hamiltonian for a particle in an e.m. wave of fixed frequency, arbitrary polarization, and with spatial modulation of amplitude and wavevector; the magnetostatic field is nonuniform, as before. We then apply the result to the containment of particles in a mirror field.

The canonical Hamiltonian is $H(\vec{q}, \vec{p}; t) = \frac{1}{2}[\vec{p} - \vec{A}_0(\vec{q}) - \vec{A}_1(\vec{q}, t)]^2 + \phi_0(\vec{q})$, where $\vec{A}_0(\vec{q})$ and $\phi_0(\vec{q})$ are the static vector and scalar potentials, and $\vec{A}_1(\vec{q}, t)$ is the e.m. wave in the radiation gauge. We make the following change of variables: $\vec{x} = \vec{q}$ and $\vec{u} = \vec{p} - \vec{A}_0(\vec{q}) = \vec{v} + \vec{A}_1(\vec{q}, t)$, where $\vec{v}$ is the particle velocity; this allows us to exhibit the perturbation in the Hamiltonian explicitly, which is the natural way of doing perturbation theory, instead of in the P.B. The Hamiltonian then reads $H = H_0 + H_1 + H_2$, where $H_0 = \frac{1}{2}u^2 + \phi_0(\vec{x})$, $H_1 = -\vec{u} \cdot \vec{A}_1(\vec{x}, t)$, $H_2 = \frac{1}{2}[\vec{A}_1(\vec{x}, t)]^2$, where $\vec{A}_1$ is expressed in eikonal form:

$\vec{A}_1(\vec{x}, t) = \lambda \vec{A}(\vec{x}) \exp i[\psi(\vec{x}) - \omega t] + \text{c.c.}$, $\vec{k}(\vec{x}) \equiv \nabla \psi(\vec{x})$. We now make the Darboux transformation [(1) and (2)], to express the Hamiltonian in terms of g.c. variables. We must assume that the scale length of the static fields $(\vec{B}, \phi_0)$ and of the wave modulation $(\vec{A}, \vec{k})$ greatly exceeds gyroradius. In g.c. variables, we obtain:

$$H_0 = \mu B(\vec{X}) + \frac{1}{2} p^2 + \phi_0(\vec{X}) + O(\varepsilon^2), \quad H_1 = \sum_{\ell=-\infty}^{+\infty} (H_\ell + H_\ell^*),$$

where $\vec{H}_\ell(\vec{X}, P; \theta, \mu; t) \equiv [-\vec{u} \cdot \vec{A}_1(\vec{x}, t)]_\ell$, the Fourier component (in θ) of the first order (in λ) perturbation, is

$$H_\ell = \frac{i\lambda}{\omega} \left(\frac{\ell \Omega_J}{k_\perp} \hat{k}_\perp + \frac{2i\Omega\mu}{k_\perp} \frac{\partial J_\ell}{\partial \mu} \hat{b} \times \hat{k} + P J_\ell \hat{b} \right) \cdot \vec{E}(\vec{X}) \exp i[-\omega t + \psi(\vec{X}) + \ell(\theta + \frac{\pi}{2}) + O(\varepsilon)] + O(\varepsilon); \quad (5)$$

$$H_2 = \lambda^2 |\vec{E}(\vec{X})|^2 / \omega^2 + \text{terms in } \exp(-2i\omega t).$$

The quantities $\Omega \equiv B$, ϕ_0 , $\vec{E}$, $\vec{k}_\perp$, $\hat{b}$ are all to be evaluated at g.c. position $\vec{X}$; the argument of the Bessel function J_ℓ is $k_\perp(\vec{X}) [2\mu/\Omega(\vec{X})]^{1/2}$.

We generate a near-identity transformation of semicanonical variables, by employing Lie techniques to eliminate the term H_1 , linear in $\vec{E}(\vec{X})$ at frequency ω , and to obtain a new Hamiltonian K , containing a static expression quadratic in $\vec{E}(\vec{X})$. The zeroth order Hamiltonian is unchanged:

$K_0 = H_0$. The first-order Lie generating function w_1 is calculated from⁸
 $\partial w_1 / \partial t + \{w_1, H_0\} = K_1 - H_1$, where we choose $K_1 = 0$, and assume $\omega \sim O(\Omega)$, i.e., we neglect drift effects:

$$w_1(\vec{X}, P; \theta, \mu; t) = \lambda \varepsilon \sum_{\ell=-\infty}^{+\infty} \frac{H_\ell(\vec{X}, P; \theta, \mu; t)}{i(\omega - \ell\Omega(\vec{X}) - k_\parallel(\vec{X})P)} + \text{c.c.} + O(\varepsilon^2).$$

The second order Hamiltonian $K_2(\vec{X}, P; \mu)$ is calculated from

$\partial w_2 / \partial t + \{w_2, H_0\} = 2(K_2 - H_2) - \{w_1, H_1\}$. We choose the next generating function w_2 to eliminate oscillatory terms at 2ω . The electromagnetic

ponderomotive Hamiltonian in a nonuniform magnetic field is then found to be

$$K_2(\vec{X}, P; \mu) = \lambda^2 \frac{|\vec{E}(\vec{X})|^2}{\omega^2} + \sum_{\ell=-\infty}^{+\infty} \left(\ell \frac{\partial}{\partial \mu} + k_{\parallel}(\vec{X}) \frac{\partial}{\partial P} \right) \frac{|H_{\ell}|^2(\vec{X}, P; \mu)}{\omega - \ell \Omega(\vec{X}) - k_{\parallel}(\vec{X}) P} + O(\epsilon). \quad (6)$$

The equations of motion valid locally are then derived from the new g.c. Hamiltonian $K = K_0 + K_2 + O(\epsilon \lambda)$. (i) G.c. velocity: $d\vec{X}/dt = \hat{b}(P + \partial K_2/\partial P) + O(\epsilon)$, showing that the g.c. parallel velocity and momentum differ by $\partial K_2/\partial P$, in the presence of the wave. Integrating $\partial K_2/\partial P$ over the g.c. distribution, we obtain the plasma contribution to the parallel wave-momentum density. The $O(\epsilon)$ terms include the perpendicular drifts, classic plus ponderomotive. (ii) Parallel force on the g.c.: $dP/dt = -\hat{b} \cdot \nabla(\mu \Omega + \phi_0 + K_2)$; it exhibits mirroring due to magnetic, electric, and ponderomotive forces, respectively. (iii) The generalized magnetic moment μ is an adiabatic invariant, not an exact invariant, because the Hamiltonian still depends on the gyrophase to order $O(\lambda^3)$. However, this θ -dependence can be transformed away to higher order and hence the new magnetic moment becomes invariant to order $O(\lambda^3)$. This process can be repeated ad infinitum, such that the generalized magnetic moment is invariant to all orders in the e.m. perturbation. The relation between the conserved magnetic moment and the unperturbed one is

$$\mu = \frac{u_{\perp}^2}{2B(\vec{X})} - \sum_{\ell=-\infty}^{+\infty} \frac{\ell H_{\ell}(\vec{X}, P; \theta, \mu, t) + \text{c.c.}}{\omega - \ell \Omega(\vec{X}) - k_{\parallel}(\vec{X}) P} + \frac{\partial}{\partial \mu} \sum_{\ell=-\infty}^{+\infty} \frac{|H_{\ell}|^2(\vec{X}, P; \mu)}{(\omega - \ell \Omega(\vec{X}) - k_{\parallel}(\vec{X}) P)^2} \quad (7)$$

plus oscillatory terms of order $O(\lambda^2)$. Recall that particle velocity is $\vec{v} = \vec{u} - \vec{A}_1(\vec{x}, t)$. (iv) Nonlinear gyrofrequency: $d\theta/dt = \Omega(\vec{X}) + \partial K_2/\partial \mu$; the gyrofrequency-shift due to the e.m. wave is $\partial K_2/\partial \mu$.

The expression for the ponderomotive Hamiltonian permits us to analyze the containment of particles in a mirror field due to r.f.

stoppering. Using conservation of bounce-action J_b and drift-flux Φ , we find the displacement in the turning point, due to the ponderomotive Hamiltonian: $\Delta X_0(\mu, J_b, \Phi) = [K_2(\vec{X}_0, P=0; \mu) - \langle K_2 \rangle(\mu, J_b, \Phi)] / \dot{P}(\mu, J_b, \Phi)$, where $\vec{X}_0$ is the unperturbed turning point, $\langle K_2 \rangle$ is the average over the drift surface, and $\dot{P}$ is the unperturbed restoring force at the turning point. We note that a wave near the mirror throat shifts the turning point inwards, while a wave in the interior displaces it outwards, unless the ponderomotive potential is negative.¹² For ions with $\mu \rightarrow 0$, the ponderomotive Hamiltonian can produce confinement, eliminating the loss cone. The fractional shifts in bounce and drift surfaces, due to the applied r.f. field, are $\Delta \omega_b / \omega_b(\mu, J_b, \Phi) = \partial \langle K_2 \rangle [\mu, J_b(K_0, \mu, \Phi), \Phi] / \partial K_0$ and $\Delta \omega_d / \omega_d(\mu, J_b, \Phi) = \partial \langle K_2 \rangle [\mu, J_b, \Phi(K_0, \mu, J_b)] / \partial K_0$. We note that the ponderomotive Hamiltonian gives the relevant physical quantities without the need to calculate the ponderomotive force explicitly.

We can straightforwardly reduce the expression for the ponderomotive Hamiltonian (6) for various limiting conditions. (i) For example, if $k_\perp(\vec{X})\rho \ll 1$, $|\omega \pm \Omega(\vec{X})| \gg k_\parallel(\vec{X})P$, and $E_\parallel(\vec{X}) \ll |\vec{E}_\perp(\vec{X})|$, then $K_2 = |\vec{E}(\vec{X})|^2 [\omega^2 - \Omega^2(\vec{X})]^{-1}$. We must not be too close to the resonance because there the perturbation expansion breaks down. If in addition we assume $\omega \ll \Omega(\vec{X})$ we obtain the Alfvén confinement¹². (ii) In the electrostatic limit $\vec{E}(\vec{X}) = -ik(\vec{X})\phi(\vec{X})$,

$$K_2 = |\phi(\vec{X})|^2 \sum_{\ell=-\infty}^{+\infty} \left\{ \frac{k_\parallel^2(\vec{X}) J_\ell^2}{[\omega - \ell\Omega(\vec{X}) - k_\parallel(\vec{X})P]^2} + \frac{\ell \frac{\partial}{\partial \mu} J_\ell^2}{\omega - \ell\Omega(\vec{X}) - k_\parallel(\vec{X})P} \right\}. \quad (8)$$

In forthcoming communications we plan to apply this formalism to study plasma kinetic theory and nonlinear wave interaction.

Acknowledgments. We are grateful to H. L. Berk for asking us the right questions. This work was supported by the U. S. Department of Energy, Office of Fusion Energy under contract No. W-7405-ENG-48.

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This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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