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Ribbon Concordances and Representation Varieites

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Ribbon Concordances and Representation Varieties

by

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Abstract

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The central motivation of this dissertation is a question of Gordon asking if an infinite descending chain of ribbon concordances $K_0 \geq K_1 \geq \ldots$ is eventually constant. Restricting to the case when these knots are hyperbolic, we approach this problem by studying relations between the $\mathrm{SL}_2\mathbb{C}$ representation varieties of ribbon concordant knots. We first rule out one potential approach by giving examples of ribbon concordances that do not induce surjections on representation varieties or character varieties. Then we provide two sufficient conditions on the K_i for Gordon's question to have a positive answer. The first condition is if the K_i satisfy a conjecture of Chinburg, Reid, and Stover. Using the theory of deformations of cone manifolds, we show prove this conjecture in the case that a knot admits a Euclidean cone structure with cone angle $\alpha \leq \pi$. The second condition is when a faithful representation and a reducible representation lie on the same component of the chararcter variety. This result is shown by making use of homology with local coefficients. In particular, these conditions show that any descending chain of 2-bridge hyperbolic knots is eventually constant.

Contents

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Chapter 1

Introduction

The aim of this dissertation is to understand the relation between $\mathrm{SL}_2\mathbb{C}$ representation varieties and ribbon concordances of knots. We begin by introducing both of these main concepts.

Representation varieties of homomorphisms from discrete groups to Lie groups are interesting for a wide assortment of reasons, with many geometric and topological applications. A major reason for their importance is the fact that $\mathrm{Hom}(\pi_1(X), G)$ for a Lie group G parametrizes the space of flat principal G-bundles over X marked with a fixed base point b and an identification of the fiber over b with G[36, Proposition 3.6.15]. Furthermore, removing the dependence on a choice of identification of fiber amounts to taking the quotient $\mathrm{Hom}(\pi_1(X), G)/G$.

Flat SU(2) bundles are of central importance in gauge theory, but particularly important to 3-dimensional topology are flat $\mathrm{SL}_2\mathbb{C}$ and $\mathrm{PSL}_2\mathbb{C}$ bundles. Thurston showed that a (G, X) geometric structure (G, X) on a manifold M induces a flat principal G bundle, and that deforming this bundle gives a corresponding deformation of the geometric structure. In fact, in many cases the space of (G, X) structures is locally homeomorphic to $\mathrm{Hom}(\pi_1(X), G)/G$ [17]. This makes $\mathrm{Hom}(\pi_1(X), \mathrm{PSL}_2\mathbb{C})$ and the space of lifts $\mathrm{Hom}(\pi_1(X), \mathrm{SL}_2\mathbb{C})$ very important to geometry. For example, Thurston's hyperbolic Dehn surgery theorem [35] and the proof of geometrization of orbifolds [7] both rely on understanding the local behavior of deformations of hyperbolic structures by understanding the $\mathrm{PSL}_2\mathbb{C}$ character varieties. Also, much of this dissertation relies on work by Porti about understanding what the existence of spherical and euclidean structures on a knot complement reveals about $\mathrm{Hom}(\pi_1(S^3 \backslash K), \mathrm{SL}_2\mathbb{C})$

Outside of geometric structures representation varieties have relevance as knot invariants. The gauge theory aspects of SU(2) representations were used by Kronheimer and Mrowka [23] to find non-cyclic representations of knot groups to SU(2) for any nontrivial knot. Since $\mathrm{SU}(2) \subset \mathrm{SL}_2\mathbb{C}$, this immediately implies both the SU(2) and $\mathrm{SL}_2\mathbb{C}$ representation variety can detect the unknot.

The equations defining a representation variety $\mathrm{Hom}(\Gamma, G)$ are elegantly simple to describe, needing only a presentation of discrete group Γ. However, being such a powerful invariant, it is inevitably difficult to work with concretely: it's very computationally intensive to determine if two varieties are isomorphic. The A-polynomial of a knot $A_K(L, M)$ by [6] is one way to capture much of the information of the representation variety in a simpler object. Roughly, it records which representations in $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{SL}_2\mathbb{C})$ arise from restricting representations of the knot complement to the peripheral elements μ, λ. In fact, Kronheimer and Mrowka's work mentioned earlier can be used to show that the A-polynomial also detects the unknot [12],[2].

The study of the central object of this dissertation, ribbon concordances, was initiated by Gordon [18] in 1981. Defined as a concordance between two knots K_0 and K_1 with only critical points of degree 0 and 1, this operation, is denoted $K_0 \leq K_1$ and amounts to taking K_0 and attaching a collection of disjoint unknots to it via band sum. Intuitively the knot K_1 is more complex than K_0 but formalizing and proving this idea has proven to be highly non-trivial and a fruitful area of research.

There are many knot invariants found to become more complicated when going from K_0 to K_1. In Gordon's original paper he shows in a sense that the knot group of K_1 is more complicated than K_0, since $\pi_1(S^3\backslash K_0)$ is shown to be a subset of a quotient of $\pi_1(S^3\backslash K_0)$.

Other knot invariants which become more complicated include:

1. The Alexander polynomial [15]
2. Heegard Floer homology, Khovanov homology, and other flavors of gauge-theoretic invariants [39],[24], [10]
3. Character varieties and their tangent spaces for G compact [10]

The last item merits more discussion; much of this dissertation revolves around $G = \mathrm{SL}_2\mathbb{C}$, a non-compact Lie group. While Daemi et al. found that representation varieties for G compact should increase in dimension, no such result is known for $G = \mathrm{SL}_2\mathbb{C}$. Daemi's work relied on surjective of maps between the character varieties, but that is shown in Section 3.2 to not always hold true outside G compact.

Despite all of these invariants behaving nicely with respect to ribbon concordances, it's still a mysterious operation. Only within the past few years did Agol [1] prove that ribbon concordance is a partial order using the aforementioned surjectivity of representation varieties for G compact.

Together with Agol's paper, one of the main inspirations for this dissertation is the following open question from Gordon's original paper: If $K_0 \geq K_1 \geq \ldots$ is an infinite sequence of ribbon concordances, is there some n for which $K_n = K_m$ for all $m \geq n$?

The results of this dissertation all revolve around trying to prove this in one way or another for the specific case of hyperbolic knots. Theorem 3.3.3 is a partial proof of a conjecture of Chinburg et al. [4] which would imply Gordon's conjecture for hyperbolic knots. Meanwhile, Theorem 4.2.10 implies a proof in the case that for each K_i, the component of the character variety $\mathrm{Hom}(\pi_1(S^3\backslash K_i), \mathrm{SL}_2\mathbb{C})$ containing the discrete faithful representation also contains a reducible representation.

Chapter 2

Basic Definitions and Properties

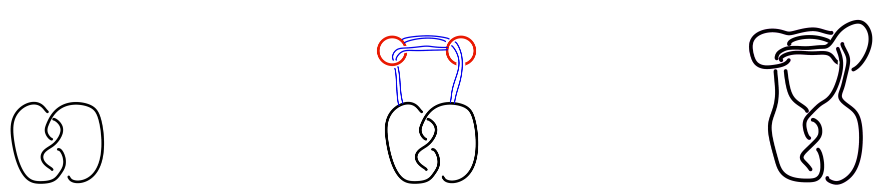
2.1 Ribbon Concordances

The following definition and properties are from Gordon's original paper defining ribbon concordances [18].

Definition 2.1.1. A *ribbon concordance* C between two knots K_1,K_0, denoted $K_1 \geq_C K_0$ is an annular concordance C smoothly and properly embedded in $S^3 \times [0,1]$ between K_0 at $t = 0$ and K_1 at $t = 1$ such that the projection $p : C \to [0,1]$ is a smooth Morse function of C with critical points of index only 0 or 1.

We denote $S^3 \backslash K_i$ as Y_i, denote $(S^3 \times [0,1]) \backslash C$ as W, and the inclusion maps as $\iota_i : Y_i \to W$.

Diagramatically, a ribbon concordance going upwards appears starting with a diagram of K_0, adding k trivial unlinked components, and attaching k bands so that the resulting diagram is a diagram of K_1. In reverse, this looks like attaching k bands to K_1 so that the resulting diagram is K_0 and k unlinked unknots, then removing these extra components.



K_0 $\qquad$ K_0 with unknots and bands $\qquad$ K_1

Figure 2.1

Corresponding to these diagrams moving upwards or downwards are associated handle decompositions: W can be constructed from $Y_0 \times [0,1]$ by attaching a 1-handle corresponding to each added unknot component and attaching a 2-handle corresponding to each band attachment. W can also be constructed from $Y_1 \times [0, 1]$ by attaching a 2-handle corresponding to each band attached, then a 3-handle corresponding to each removed unknot component.

By attaching 1 and 2-handles to $Y_0 \times [0, 1]$ to get W, we can explicitly write

$$\pi_1(W) = (\pi_1(Y_0) * \langle b_1, \ldots, b_k \rangle)/\langle s_1, \ldots s_k \rangle$$

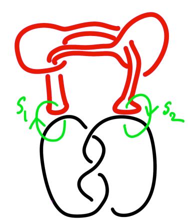


Figure 2.2: Finding the relators of Figure 2.1

The presentation of $\pi_1(W)$ can be derived from the ribbon concordance diagram: the new generators b_i correspond to the meridians of the k unknot components added, and each relator s_i comes from a loop following the band attaching the ith unknot component to K_0, containing the two arcs which are being banded together. A priori a band could attach two unknot components to each other, but we can isotope C to slide the bands over each other so they all attach to K_0.

Since the ith band will connect the ith unknot to K_0, s_i will be of the form $\mu_i m_i^{-1}$ where m_i is a meridian of the ith band and μ_i is a meridian of K_0. Since Y_0 is a knot complement it has Wirtinger presentation $\pi_1(Y_0) = \langle a_1, \ldots, a_n | r_1, \ldots r_{n-1} \rangle$ with the generators a_i corresponding to meridians. Then since all meridians on the same link component are conjugate, for any choice of $j \leq n$ there are words $w_i \in \langle a_1 \ldots a_n, b_1, \ldots, b_k \rangle$ such that $s_i = w_i^{-1} b_i w_i a_j^{-1}$. In the end we have

$$\pi_1(W) = \langle a_1, \ldots, a_n, b_1, \ldots, b_k | r_1, \ldots, r_{n-1}, s_1, \ldots, s_k \rangle$$

Proposition 2.1.2 ([18])**.** *The maps $\iota_i : Y_i \hookrightarrow W$ induce isomorphisms on H_*, an inclusion $\pi_1(Y_0) \hookrightarrow \pi_1(W)$ and a surjection $\pi_1(Y_1) \twoheadrightarrow \pi_1(W)$.*

We now study representation varieties of knots and how they interact with ribbon concordances

Definition 2.1.3. For any word $r \in F_n = \langle a_1, \ldots, a_n \rangle$ and complex linear algebraic group G, define the word map $w_r : G^n \to G$ such that w_r takes in $(g_1, \ldots, g_n) \in G^n$, replaces each a_i in r with the corresponding g_i, and evaluates the product. For any finitely generated $\gamma = \langle a_1, \ldots, a_n | r_1, \ldots \rangle$ define the representation variety $R_G(\Gamma)$ to be $\{\vec{g} \in G^n | w_{r_1}(\vec{g}) = \text{Id}, \ldots\}$.

Because the w_{r_i} are algebraic morphisms, $R_G(\Gamma)$ has the structure of an algebraic set.

Remark 2.1.4. We can also consider a real linear algebraic group G, in which case $R_G(\Gamma)$ has the structure of a real algebraic variety.

An element $\vec{g} \in R_G(\Gamma)$ defines a homomorphism $\rho : \Gamma \to G$ by sending $a_i \mapsto g_i$ and by the definition of $R_G(\Gamma)$ this assignment will send any $r_i \mapsto \text{Id}$. Similarly, $\rho : \Gamma \to G$ gives a point $(\rho(a_1), \dots, \rho(a_n)) \in R_G(\Gamma)$. These operations are inverses, thus $R_G(\Gamma)$ is the set $\text{Hom}(\Gamma, G)$ with an algebraic structure.

The algebraic structure of $R_G(\Gamma)$ is known to be invariant under change of presentation and hence is an invariant of Γ [26]. Furthermore, R_G is a contravariant functor from the category of groups to the category of algebraic varieties: any $f : \Gamma_1 \to \Gamma_2$ induces a morphism $R_G(f) : R_G(\Gamma_2) \to R_G(\Gamma_1)$ sending $\rho \mapsto \rho \circ f$.

G acts on $R_G(\Gamma)$ by conjugation: for any $\rho \in \text{Hom}(\Gamma, G)$ and $g \in G$, $g\rho g^{-1}$ is also in $\text{Hom}(\Gamma, G)$. Taking the GIT quotient by this action gives another space which is an invariant of Γ.

Definition 2.1.5. The character variety $X_G(\Gamma)$ is the GIT quotient $R_G(\Gamma)//G$.

It follows from geometric invariant theory that in the case of G a reductive group, $X_G(\Gamma)$ is an affine variety[38]. In particular when $G = \text{SL}_2\mathbb{C}$, the point a representation $\rho \in R_{\text{SL}_2\mathbb{C}}(\Gamma)$ maps to is determined by its character $\chi_\rho = \text{tr} \circ \rho$, hence the name character variety. The algebraic structure can be explicitly described: the polynomial functions on $X_{\text{SL}_2\mathbb{C}}(\Gamma)$ can be generated by the functions $\tau_{\gamma_i}(\rho) = \chi_\rho(\gamma_i)$ for a finite set of $\gamma_i \in \Gamma$ [34]. This means $X_{\text{SL}_2\mathbb{C}}(\Gamma)$ is exactly the set of characters of representations $\rho : \Gamma \to \text{SL}_2\mathbb{C}$, together with an algebraic structure. Like R_G, X_G is also a contravariant functor. In the case that the elements of X_G are characters of representations χ_ρ, then for any homomorphism $f : \Gamma_1 \to \Gamma_2$, $X_G(f)$ is given as $\chi_\rho \mapsto \chi_{\rho \circ f}$.

Remark 2.1.6. An important note is that $X_{\text{SL}_2\mathbb{C}}(\Gamma)$ is not simply the orbit space for the action of $\text{SL}_2\mathbb{C}$ on $R_{\text{SL}_2\mathbb{C}}(\Gamma)$: while it is true for irreducible representations that $\chi_\rho = \chi_{\rho'}$ if and only if ρ and ρ' are conjugate [9, Proposition 1.5.2], for reducible representations this is not the case. In particular any non-abelian reducible representation ρ has a corresponding abelian representation ρ' such that $\chi_\rho = \chi_{\rho'}$. Up to conjugation, ρ is an upper triangular representation, and ρ' is the exact same representation except with all top right entries set to 0.

We are interested in the representation varieties of knots and the morphisms induced on them by ribbon concordances. Fix the notation $R_G(Y) = R_G(\pi_1(Y))$ and $X_G(Y) = X_G(\pi(Y))$ for Y a topological space, and furthermore $R_G(K) = R_G(S^3 \backslash K)$ and $X_G(K) = X_G(S^3 \backslash K)$ for K a knot. We do the same notation for ribbon concordances C: $R_G(C) = R_G(S^3 \times [0,1] \backslash C)$ and $X_G(C) = X_G(S^3 \times [0,1] \backslash C)$. For $f : A \to B$ a continuous map of topological spaces we denote $R_G(f) = R_G(f_*)$ and $X_G(f) = X_G(f_*)$ where f_* is the induced map on fundamental groups.

For a ribbon concordance $K_1 \geq_C K_0$, $\iota_{1*} : \pi_1(Y_1) \to \pi_1(W)$ being surjective means $R_G(W)$ can be cut out from $R_G(Y_1)$ by asserting the additional relators given by $\ker \iota_1$.

Therefore $R_G(\iota_1) : R_G(W) \to R_G(Y_1)$ is the inclusion map of the closed algebraic set $R_G(W) \subset R_G(Y_1)$.

One would hope the injective map ι_0 would induce a surjection in $R_G(\iota_0)$, but the story is more complicated. When only considering varieties of representations to compact groups like $SU(2)$ and $SO(n)$, $R_G(\iota_0)$ is indeed surjective [10, Proposition 2.1]. It's natural to ask if the same holds true for $\mathrm{SL}_2\mathbb{C}$ representation varieties. To approach this question, we introduce the following definition.

Definition 2.1.7. Given presentations $\pi_1(Y_0) = \langle a_1, \ldots, a_n | r_1, \ldots, r_{n-1} \rangle$ and $\pi_1(W) = \langle a_1, \ldots, a_n, b_1, \ldots, b_k | r_1, \ldots, r_{n-1}, s_1, \ldots, s_k \rangle$, define $F_{W,Y_0} = \langle a_1, \ldots, a_n, b_1, \ldots, b_k | s_1, \ldots, s_k \rangle$ and define ι_F as the map $F_n \to F_{W,Y_0}$ sending the i-th generator of F_n to a_i.

ι_F is an inclusion since F_n is residually finite so the argument of [18, Lemma 3.1] applies. Together these maps give the commutative diagram

$$\begin{array}{ccc} F_{W,Y_0} & \twoheadrightarrow & \pi_1(W) \\ {\scriptstyle \iota_F}\big\uparrow & & \big\uparrow{\scriptstyle \iota_0} \\ F_n & \twoheadrightarrow & \pi_1(Y_0) \end{array}$$

$\pi_1(W)$ is a quotient of F_{W,Y_0}, which means $R_G(W)$ is a closed subvariety of $R_G(F_{W,Y_0})$. In fact, $R_G(W)$ is exactly $R_G(\iota_F)^{-1}(R_G(Y_0))$. In other words, the following diagram is a pullback diagram.

$$\begin{array}{ccc} R_G(F_{W,Y_0}) & \hookleftarrow & R_G(W) \\ {\scriptstyle R_G(\iota_F)}\big\downarrow & & \big\downarrow{\scriptstyle R_G(\iota_0)} \\ G^n & \hookleftarrow & R_G(Y_0) \end{array}$$

Note that F_{W,Y_0} is not an invariant of the pair (W, Y_0); it depends on the presentations of $\pi_1(W)$ and $\pi_1(Y_0)$.

The local behaviour of $R_G(F_{W,Y_0})$ can tell us a lot about the image of the map $R_G(\iota_0)$. For instance by Lemma 2.1.8 $R_G(F_{W,Y_0})$ satisfies the implicit function theorem at $\vec{\mathrm{Id}} \in G^{n+k}$. That is to say, the last k coordinates of $R_G(F_{W,Y_0}) \subset G^{n+k}$ can be written as analytic function of the first n coordinates in a neighborhood of $\vec{\mathrm{Id}} \in G^n$ using the standard topology on G^n.

Because $R_G(\iota_F)$ is an algebraic morphism and is surjective on a standard open neighborhood of $\vec{\mathrm{Id}}$, if G is an irreducible algebraic variety such as $\mathrm{SL}_2\mathbb{C}$ then $R_G(\iota_F) : R_G(F_{W,Y_0}) \to G^n$ is a dominant morphism. In a sense this means for "most" possible $R_G(Y_0) \subset G^n$, $R(W) \to R(Y_0)$ will be dominant. Note that this doesn't guarantee $R_G(\iota_0)$ is surjective or even dominant. The next section gives an explicit example of when $R_G(\iota_0)$ is not surjective.

For the statement of Lemma 2.1.8 given a fixed $\vec{g} \in G^n$ we need to define a representation $\rho_{\vec{g}}$ of the free group $F_n = \langle a_1, \ldots, a_n \rangle$ such that $\rho_{\vec{g}} : F_n \to G$ sends $a_i \mapsto g_i$. This is the same construction used to give the equivalence between $G^n = R_G(F_n)$ and $\mathrm{Hom}(F_n, G)$. We also fix the notation that $\mathrm{Ad} : G \to \mathrm{End}\,\mathfrak{g}$ sends $g \mapsto (v \mapsto gvg^{-1})$, and define the corresponding ring homomorphisms $\rho_{\vec{g}} : \mathbb{Z}[F_n] \to \mathbb{Z}[G]$ and $\mathrm{Ad} : \mathbb{Z}[G] \to \mathrm{End}\,\mathfrak{g}$ by extending our original functions linearly.

Lemma 2.1.8. *For a word x in the free group $F_n = \langle a_1, \ldots, a_n \rangle$, the partial derivative of w_x at a point $\vec{h} \in G^n$ is given by*

$$\frac{\partial w_x}{\partial g_i}\Big|_{\vec{h}} = \mathrm{Ad}(\rho_{\vec{h}}(\frac{\partial x}{\partial a_i})) \in \mathrm{End}\,\mathfrak{g}$$

where $\frac{\partial x}{\partial a_i} \in \mathbb{Z}[F_n]$ is the Fox derivative.

Proof. Here all the tangent spaces of the respective lie groups are associated with their lie algebras by right multiplication. Writing out $x = a_{j_1}^{\epsilon_1} \ldots a_{j_m}^{\epsilon_m}$ where $\epsilon_j = \pm 1$ gives us $w_x(\vec{g}) = g_{j_1}^{\epsilon_1} \ldots g_{j_m}^{\epsilon_m}$. We wish to perturb g_i by $v \in \mathfrak{g}$ and see how it changes $w_x(\vec{g})$. This means replacing each g_i by $e^v g_i$, and seeing how it perturbs the whole product. By the product rule, we can individually perturb each instance of g_i and sum up the all the different induced perturbations of w_x. For each k such that $j_k = i$ and $\epsilon_k = 1$ we can write $w_x(\vec{g})$ as $z_k g_i z_k'$, where $z_k = g_{j_1}^{\epsilon_1} \ldots g_{j_{k-1}}^{\epsilon_{k-1}}$ and $z_k' = g_{j_{k+1}}^{\epsilon_{k+1}} \ldots g_{j_m}^{\epsilon_m}$. Then perturbing the g_i in the middle by v gives

$$z_k e^v g_i z_k' = e^{\mathrm{Ad}(z_k)(v)} z_k g_i z_k' = e^{\mathrm{Ad}(z_k)(v)} w_x$$

Similarly, if instead $\epsilon_k = -1$ then the product is

$$z_k g_i^{-1} e^{-v} z_k' = e^{-\mathrm{Ad}(z_k g_i^{-1})(v)} z_k g_i^{-1} z_k' = e^{-\mathrm{Ad}(z_k g_i^{-1})(v)} w_x$$

We can see that the word v is conjugated by is exactly the corresponding summand from computing the Fox derivative $\frac{\partial x}{\partial a_i}$, except with each a_i replaced by the corresponding g_i which is exactly the action of $\rho_{\vec{g}}$. Since the total vector w_x is perturbed by is

$$\sum_{j_k=1,\epsilon_k=1} \mathrm{Ad}(z_k)(v) + \sum_{j_k=1,\epsilon_k=1} -\mathrm{Ad}(z_k g_i^{-1})(v)$$

we get all the terms arising from the Fox derivative $\frac{\partial x}{\partial g_i}$, and the result follows. □

We remark that the idea of using Fox derivatives to understand tangent spaces of representation varieties appears in [27].

Corollary 2.1.9. *If F_{W,Y_0} has presentation $F_{W,Y_0} = \langle a_1, \ldots, a_n, b_1, \ldots, b_k | s_1, \ldots, s_k \rangle$, then on a standard open set of $R_G(F_{W,Y_0}) \subset G^{n+k}$ around the trivial representation $\vec{\mathrm{Id}}$ the coordinates $g_1, \ldots, g_n$ are analytic functions of the coordinates $g_{n+1}, \ldots, g_{n+k}$.*

Proof. Define the function $\vec{w} : G^{n+k} \to G^k$ sending $\vec{g} \mapsto (w_{s_1}, \ldots, w_{s_k})$. Then $R_G(F_{W,Y_0})$ is defined as $\vec{w}^{-1}(\vec{\mathrm{Id}})$, and the proof is complete if we can show $\vec{w}$ satisfies the hypotheses of the implicit function theorem at $\vec{g} = \vec{\mathrm{Id}}$. This means we need to compute the Jacobian matrix $J_{i,j} = \frac{\partial w_{s_i}}{\partial g_{n+j}}$. By Lemma 2.1.8 we shall use the Fox derivatives $\frac{\partial s_i}{\partial b_j}$. Recall that the s_i are of the form $s_i = w_i^{-1} b_i w_i a_{k_i}$, so that

$$\frac{\partial s_i}{\partial b_j} = -w_i^{-1} \frac{\partial w_i}{\partial b_j} + w_i^{-1} \delta_{i,j} + w_i^{-1} b_i \frac{\partial w_i}{\partial b_j} + w_i^{-1} b_i w_i \frac{\partial a_{k_i}}{\partial b_j}$$

The rightmost term in the sum is 0, meanwhile all the coefficients become Id when evaluating at the identity representation, so Lemma 2.1.8 tells us after cancellation that

$$\left. \frac{\partial w_{s_i}}{\partial g_{m+j}} \right|_{\vec{\mathrm{Id}}} = \mathrm{Ad}(\delta_{i,j} \,\mathrm{Id}) = \delta_{i,j} \,\mathrm{Id}$$

It follows that $\vec{w}$ satisifies the hypotheses of the implicit function theorem, and in fact the analytic implicit function theorem, since $\vec{w}$ is a polynomial function. □

Chapter 3

Chains of Ribbon Concordances

3.1 Surjectivity of $R_G(\iota_0)$

In the original ribbon concordance paper, Gordon asked the following:

Conjecture 3.1.1 ([18])**.** *If $K_1 \geq K_2 \ldots$, does there exist some m such that $K_n = K_m$ for all $n \geq m$?*

Agol's proof [1] of the partial ordering property of ribbon concordances can be viewed as a proof of this property for the repeating sequence $K_1 \geq K_2 \geq K_1 \geq \ldots$. A natural question is if it is possible to extend these techniques to an arbitrary descending chain of ribbon concordances.

To begin, we need the following part of the proof of [1, Theorem 1.2]

Lemma 3.1.2 ([1])**.** *If $K_1 \geq K_0$ is a ribbon concordance such that the induced surjection $\pi_1(Y_1) \twoheadrightarrow \pi_1(W)$ is an isomorphism, then $K_1 = K_0$*

Proposition 3.1.3. *Suppose $K_0 \geq_{C_0} K_1 \ldots$ is an infinite sequence of ribbon concordances. For any fixed $n > 0$ there is an m such that for all $l \geq m, R_{\mathrm{SO}(n)}(K_l) = R_{\mathrm{SO}(n)}(W_l)$*

Proof. Define $S_{i,0} = R_{\mathrm{SO}(n)}(W_i) \subset R_{\mathrm{SO}(n)}(K_i)$ and denote $\phi_i : R_{\mathrm{SO}(n)}(W_i) \twoheadrightarrow R_{\mathrm{SO}(n)}(K_{i+1})$. Then we can inductively define $S_{i,j+1} = \phi_i^{-1}(S_{i+1,j})$ and it follows from induction that $S_{i,j+1} \subset S_{i,j}$. We also know $S_{i,1} = S_{i,0}$ if and only if $R_{\mathrm{SO}(n)}(W_{i+1}) = R_{\mathrm{SO}(n)}(K_{i+1})$, and for $j > 0$ we know that $S_{i,j+1} = S_{i,j}$ if and only if $S_{i+1,j} = S_{i+1,j-1}$.

Bringing this all together, this shows that $S_{i,j+1} = S_{i,j}$ if and only if $R_{\mathrm{SO}(n)}(W_{i+j+1}) = R_{\mathrm{SO}(n)}(K_{i+j+1})$. Thus a sequence of ribbon concordances with infinitely many i such that $R_{\mathrm{SO}(n)}(W_i) \neq R_{\mathrm{SO}(n)}(K_i)$ would induce an infinite descending sequence of algebraic sets, violating the Noetherian property of $R_{\mathrm{SO}(n)}(K_0)$. □

The exact same technique can be used to show this for SU(2) representation varieties, and for character varieties instead of representation varieties.

Corollary 3.1.4. *Suppose $K_0 \geq_{C_0} K_1 \ldots$ is an infinite sequence of ribbon concordances. There must exist an m such that for all $l \geq m, X_{\mathrm{SU}(2)}(K_l) = X_{\mathrm{SU}(2)}(W_l)$*

The issue with using using Proposition 3.1.3 to prove Gordon's question is that value of m can depend on the value of n, and a priori m can increase unboundedly as n grows. This means for a given chain of ribbon concordances there might not be any j for which all ribbon concordances after j induce isomorphisms on all $R_{SO(n)}$, which is needed to apply the methods of [1].

The appeal of working with hyperbolic knots is that knowing of $R_{\mathrm{SL}_2\mathbb{C}}(K_i) = R_{\mathrm{SL}_2\mathbb{C}}(C_i)$ is enough. More precisely,

Lemma 3.1.5. *Given a ribbon concordance $K_1 \geq K_0$, if $R_{\mathrm{SL}_2\mathbb{C}}(W) \subset R_{\mathrm{SL}_2\mathbb{C}}(Y_1)$ contains a faithful representation of $\pi_1(Y_1)$ then the induced surjection $\iota_* : \pi_1(Y_1) \twoheadrightarrow \pi_1(W)$ is an isomorphism.*

Proof. If $\rho \in R_{\mathrm{SL}_2\mathbb{C}}(W)$ is a faithful representation of Y_1 then it is a representation $\rho : \pi_1(Y_1) \to \mathrm{SL}_2\mathbb{C}$ for which all elements of $\ker \iota_{1*}$ are sent to Id. But since ρ is faithful, $\ker \iota_{1*}$ is trivial. □

However, since $\mathrm{SL}_2\mathbb{C}$ is not compact we cannot guarantee the map $R_{\mathrm{SL}_2\mathbb{C}}(W) \to R_{\mathrm{SL}_2\mathbb{C}}(Y_0)$ is a surjection. In fact, Example 3.2.2 gives a non-surjective example. This means the technique of Proposition 3.1.3 cannot be applied to a descending sequence of hyperbolic knots.

Despite this difficulty, if a conjecture of Chinburg, Reid, and Stover [4] were true, we would still be able to prove Conjecture 3.1.1. To begin, we need to define the canonical component of a hyperbolic knot.

Definition 3.1.6. For a hyperbolic knot K, a canonical component $\mathcal{C}$ of $X(S^3 \backslash K)$ is defined as an irreducible component of $X(S^3 \backslash K)$ such that $p_X(\mathcal{C})$ contains the character of a discrete faithful representation ρ_0.

Recall that an irreducible subset of an algebraic set is a Zariski closed subset that is not the union of two nonempty Zariski closed subsets. An irreducible component is an irreducible set that is not a proper subset of any other irreducible set. Any algebraic subset of affine space can be decomposed as a finite union of irreducible components.

Conjecture 3.1.7 ([4])**.** *For any hyperbolic knot K, the canonical component $\mathcal{C} \subset X_K$ contains a real curve of characters of $SU(2)$ representations.*

Proposition 3.1.8. *Assuming that every canonical component contains a curve of* SU(2) *representations, Conjecture 3.1.1 is true for hyperbolic knots.*

Proof. By Proposition 3.1.3, there is some m for which $X_{\mathrm{SU}(2)}(Y_l) = X_{\mathrm{SU}(2)}(W_l)$ for all $l \geq m$. Let $\mathcal{C}$ be a canonical component of $X_{\mathrm{SU}(2)}(Y_l)$. By an argument of Thurston [9, Proposition 3.2.1], $\mathcal{C}$ is one-dimensional and by hypothesis contains a real curve of characters of SU(2) representations. This real curve is infinte and therefore Zariski dense in $\mathcal{C}$. Since

$X_{\mathrm{SL}_2\mathbb{C}}(W_l)$ is a Zariski closed subset of $X_{\mathrm{SL}_2\mathbb{C}}(Y_l)$ which must contain all characters of SU(2) representations, $X_{\mathrm{SL}_2\mathbb{C}}(W_l)$ therefore contains all of $\mathcal{C}$. It therefore contains the character of a faithful representation of $\pi_1(Y_l)$, so $K_l = K_{l+1} = \dots$ □

3.2 Non-surjectivity of $R(\iota_0)$

Remark 3.2.1. For the rest of this thesis $R(\Gamma)$ and $X(\Gamma)$ will refer to $R_{\mathrm{SL}_2\mathbb{C}}(\Gamma)$ and $X_{\mathrm{SL}_2\mathbb{C}}(\Gamma)$.

Consider the presentation of the trefoil knot $K_0 = 3_1$ $\pi_1(Y_0) = \langle a, b | abab^{-1}a^{-1}b^{-1}\rangle$ and a ribbon concordance corresponding to the presentation

$$\pi_1(W) = \langle a, b, h_1, h_2 | abab^{-1}a^{-1}b^{-1}, h_2h_1h_2^{-1}a^{-1}, h_1^{-1}h_2h_1b^{-1}\rangle$$

Note that this ribbon concordance is exactly the one in fig. 2.1, and its relators follow from 2.2.

Example 3.2.2. The representation $\rho_0 \in R(K_0)$ given by

$$\rho_0(a) = \begin{pmatrix} \zeta_{12} & 0 \\ 0 & \zeta_{12}^{-1} \end{pmatrix}, \rho_0(b) = \begin{pmatrix} \zeta_{12} & 1 \\ 0 & \zeta_{12}^{-1} \end{pmatrix}$$

is not in the image of $R(\iota_0) : R(W) \to R(Y_0)$

Proof. If one wished, this could be proven by plugging the relevant equations into a computer algebra system. We will instead proceed by a method which can be generalized more easily.

We can understand the map $R(\iota_0)$ by understanding the map $R(\iota_F) : R(\Gamma_{W,Y_0}) \to R(F_2)$, since $R(\Gamma_{W,Y_0})|_W = R(\iota_0)$. Using the presentations for $\pi_1(W)$ and $\pi_1(Y_0)$ given above, $\Gamma_{W,Y_0} = \langle a, b, h_1, h_2 | h_2h_1h_2^{-1}a^{-1}, h_1^{-1}h_2h_1b^{-1}\rangle$, which simplifies to $\langle h_1, h_2\rangle$. Then $\iota_F : \langle a, b\rangle \to \langle h_1, h_2\rangle$ sends $a \mapsto h_2h_1h_2^{-1}$ and $b \mapsto h_1^{-1}h_2h_1$.

To keep dimensions as low as possible, let's analyze the character variety map

$$X(\langle h_1, h_2\rangle) \to X(\langle a, b\rangle)$$

As originally shown by Fricke and Klein [16], $X(\langle a, b\rangle) \cong \mathbb{C}[\tau_a, \tau_b, \tau_{ab^{-1}}]$ and $X(\langle h_1, h_2\rangle) \cong \mathbb{C}[\tau_{h_1}, \tau_{h_2}, \tau_{h_1h_2^{-1}}]$.

Because a and b are conjugate in $\pi_1(Y_0)$, τ_a and τ_b will be equal on all characters in $X(Y_0)$ so $X(Y_0)$ lies inside the subvariety $V \subset \mathbb{C}[\tau_a, \tau_b, \tau_{ab^{-1}}]$ cut out by the equation $\tau_a - \tau_b = 0$. This means $X(Y_0) \subset V \cong \mathbb{C}[\tau_a, \tau_{ab^{-1}}]$.

The function $\tau_a - \tau_b$ pulls back by $X(\iota_F)$ to become $\tau_{h_2h_1h_2^{-1}} - \tau_{h_1^{-1}h_2^{-1}h_1} = \tau_{h_1} - \tau_{h_2}$, therefore $V' = X(\iota_F)^{-1}(V)$ is cut out by $\tau_{h_1} - \tau_{h_2} = 0$. Since $V' \cong \mathbb{C}[\tau_{h_1}, \tau_{h_1h_2^{-1}}]$, we have reduced the problem to studying the map $X(\iota_F)|_{V'} : \mathbb{C}[\tau_{h_1}, \tau_{h_1h_2^{-1}}] \to \mathbb{C}[\tau_a, \tau_{ab^{-1}}]$.

Computing this map means writing the pullbacks

$$\tau_a \mapsto \tau_{h_2 h_1 h_2^{-1}},\ \tau_{ab^{-1}} \mapsto \tau_{h_2 h_1 h_2^{-1} h_1^{-1} h_2^{-1} h_1}$$

in terms of τ_{h_1} and $\tau_{h_1 h_2}$. .This is possible to do using the standard trace relations for matrices in $\mathrm{SL}_2\mathbb{C}$

1. $\mathrm{tr}(MN) = \mathrm{tr}(NM)$
2. $\mathrm{tr}(M)\,\mathrm{tr}(N) = \mathrm{tr}(MN) + \mathrm{tr}(MN^{-1})$
3. $\mathrm{tr}(M^{-1}) = \mathrm{tr}(M)$

For brevity, label $x = \tau_{h_1} = \tau_{h_2}$, label $y = \tau_{h_1 h_2^{-1}}$, and label $p = \tau_{h_1 h_2}$. Immediately by conjugation we get τ_a pulls back to x. Meanwhile,

$$\begin{aligned}
\tau_{h_2 h_1 h_2^{-1} h_1^{-1} h_2^{-1} h_1} &= \tau_{h_1 h_2 h_1 h_2^{-1} h_1^{-1} h_2^{-1}} \\
&= \tau_{h_1 h_2 h_1} \tau_{h_2^{-1} h_1^{-1} h_2^{-1}} - \tau_{(h_1 h_2)^3} \\
&= \tau_{h_1 h_2 h_1} \tau_{h_2 h_1 h_2} - (\tau_{(h_1 h_2)^2} p - p) \\
&= (px - x)(px - x) - ((p^2 - 2)p - p) \\
&= (p-1)^2 x^2 - (p^3 - 3p)
\end{aligned}$$

Since $p = \tau_{h_1 h_2} = x^2 - y$, we can substitute this in for p and simplify, yielding

$$\tau_{h_2 h_1 h_2^{-1} h_1^{-1} h_2^{-1} h_1} = (y-2)(x^2 - y - 1)^2 + 2$$

which means that $X(\iota_F)|_{V'}(x, y) = (x, (y-2)(x^2 - y - 1)^2 + 2)$.

We can use this to find which characters (x, y) can map to the character of our chosen representation $\chi_{\rho_0} = (\zeta_{12} + \zeta_{12}^{-1}, 2) = (\sqrt{3}, 2)$ by solving $(x, (y-2)(x^2 - y - 1)^2 + 2) = (\sqrt{3}, 2)$. This system of equations immediately yields $x^2 = 3$, so after substitution we must solve for y such that $(y-2)(3-y-1) = 0$, which only occurs when $y = 2$. This means any such $\rho \in R(\langle h_1, h_2 \rangle)$ such that $R(\iota_0)(\rho) = \rho_0$ must have $\mathrm{tr}\,\rho(h_1) = \mathrm{tr}\,\rho(h_2) = \sqrt{3}$ and $\mathrm{tr}\,\rho(h_1 h_2^{-1}) = 2$

The curve $y = 2$ is exactly the curve of characters of reducible representations, and by the proof of [9, Proposition 1.5.5] any representation ρ having the character of a reducible representation must also be reducible.

Therefore $\rho \in R(\iota_0)^{-1}(\rho_0)$ must fix a subspace W of $\mathbb{C}^2$. $\rho_0(a) = \rho(h_1 h_1 h_1^{-1})$ and $\rho_0(b) = \rho(h_2^{-1} h_2 h_1)$ would then also have to fix W, implying that the only subspace ρ can fix is $\mathbb{C}[\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)]$. Thus ρ must be of the form

$$\rho(h_1) = \begin{pmatrix} \zeta_{12} & \lambda \\ 0 & \zeta_{12}^{-1} \end{pmatrix}, \rho(h_2) = \begin{pmatrix} \zeta_{12} & \mu \\ 0 & \zeta_{12}^{-1} \end{pmatrix}$$

If this is the case then

$$\rho(h_2h_1h_2^{-1}) = \begin{pmatrix} \zeta_{12} & \lambda\zeta_{12}^2 + \mu(1-\zeta_{12}^2) \\ 0 & \zeta_{12}^{-1} \end{pmatrix}, \rho(h_1^{-1}h_2h_1) = \begin{pmatrix} \zeta_{12} & \lambda(1-\zeta_{12}^{-2}) + \mu\zeta_{12}^{-2} \\ 0 & \zeta_{12}^{-1} \end{pmatrix}$$

Since $\zeta_{12}^2 - 1 + \zeta_{12}^{-2} = 0$, these two matrices are the same. Hence there is no ρ mapping to ρ_0.

□

Despite $R(\iota_0)$ not being surjective, $X(\iota_0)$ still is. This is because the map $X(\iota_F)|_{V'}(x, y) = (x, (y-2)(x^2-y-1)^2+2)$ is clearly surjective: solving $(x, (y-2)(x^2-y-1)^2+2) = (a, b)$ amounts to solving $(y-2)(a^2-y-1)+2 = b$, and the left hand side is a non-constant polynomial.

Example 3.2.3. The ribbon concordance of the previous example has $R(\iota_0)$ a dominant map.

Proof. To see this, note that for any irreducible $\rho \in R(Y_0)$, χ_ρ being in the image of $X(\iota_0)$ means that there is some $\rho' \in R(Y_0)$ with $\chi_{\rho'} = \chi_\rho$ and ρ' in the image of $R(\iota_0)$. Because χ_ρ is the character of an irreducible representation, ρ and ρ' having the same character means they are conjugate representations. Since $R(\iota_0)$ is equivariant with respect to conjugation by $\mathrm{SL}_2\mathbb{C}$, ρ is also in the image of $R(\iota_0)$.

All abelian representations are also in the image of $R(\iota_0)$ since both $\pi_1(W)$ and $\pi_1(Y_0)$ have abelianization $\mathbb{Z}$, and ι_{0*} induces an isomorphism on these abelianizations. Thus the only representations which might not be in the image of $R(\iota_0)$ are the metabelian representations, i.e. the nonabelian reducible representations.

[3] and [11] found that a metabelian representation ρ of the knot group of K exists with $\rho(\mu)$ having eigenvalues λ, λ^{-1} exists if and only if λ^2 is a root of the Alexander polynomial $\Delta_K(t)$. In our case, $\Delta_{3_1}(t) = 1 - t + t^2$ is the 6th cyclotomic polynomial.

[20, Theorem 1.1] shows that at these metabelian representations ρ_λ, if λ^2 is a simple root of $\Delta_K(t)$ then ρ_λ belongs to an irreducible component $V \subset R(Y_0)$ containing irreducible representations. All the roots of $\Delta_{3_1}(t)$ are simple, meaning any metabelian representation lies on such a V. Since the set of irreducible representations is open and every irreducible representation is in the image of $R(\iota_0)$, V is contained in the closure of the image of $R(\iota_0)$, and $R(\iota_0)$ is therefore dominant.

□

Despite $X(\iota_0)$ being surjective, we can modify our previous example to create another example where the induced map on character varieties $X(\iota_0) : X(W) \to X(K_0)$ is not surjective.

Example 3.2.4. Taking the connect sum of the previous example with the figure-eight knot gives a ribbon concordance where $X(\iota_0) : X(W) \to X(K_0)$ is not surjective.

Proof. The knot 4_1 has knot group $\langle c, d | wcw^{-1}d^{-1} \rangle$ where $w = cd^{-1}c^{-1}d$. The knot group of $K_0 = 3_1 \# 4_1$ is then

$$\pi_1(Y_0) = \left(\pi_1(S^3 \backslash 3_1) * \pi_1(S^3 \backslash 4_1)\right) / \langle bc^{-1} \rangle$$

The corresponding representation variety $R(Y_0)$ is the subspace of

$$R(\pi_1(S^3 \backslash 3_1)) \times R(\pi_1(S^3 \backslash 4_1))$$

corresponding to points (ρ_1, ρ_2) such that $\rho_1(b) = \rho_2(c)$. Any point in $R(Y_0)$ with $\rho_1 = \rho_0$ the representation used in the previous example cannot be in the image of $R(W)$. We can find ρ_2 such that

$$\rho_2(c) = \rho_0(b) = \begin{pmatrix} \zeta_{12} & 1 \\ 0 & \zeta_{12}^{-1} \end{pmatrix}$$

by analyzing the character variety of 4_1.

Since the figure eight knot is 2-bridge, just like in the previous example the character variety lies in $\mathbb{C}[\tau_c, \tau_{cd^{-1}}]$. Labeling $x = \tau_c, y = \tau_{cd^{-1}}$, [30] computes the character variety to be cut out by the equation $(y-2)(y^2-(x^2-1)y+x^2-1) = 0$. Asserting that $x = \zeta_{12}+\zeta_{12}^{-1} = \sqrt{3}$, the equation becomes $(y-2)(y^2-2y+2)$, which has roots $2, 1+i$ and $1-i$. This means $(\sqrt{3}, 1+i)$ is a point of the character variety of 4_1, and since all the reducible representations satisfy $y = 1$, this means there exists an irreducible representation $\rho_{irr} \in R(S^3 \backslash 4_1)$ with $\operatorname{tr} \rho_{irr}(c) = \sqrt{3}$. After conjugating by an appropriate matrix, we can assume

$$\rho_{irr}(c) = \begin{pmatrix} \zeta_{12} & 1 \\ 0 & \zeta_{12}^{-1} \end{pmatrix}$$

This means $(\rho_0, \rho_{irr}) \in R(Y_0)$ and must be irreducible since ρ_{irr} is irreducible. By Remark 2.1.6, any $(\rho_1, \rho_2) \in R(Y_0)$ with the same character is conjugate to (ρ_0, ρ_{irr}) and therefore also is not in the image of $R(W)$. This means $\chi_{(\rho_0, \rho_{irr})}$ cannot be in the image of $X(W)$.

□

3.3 SU(2) representations of Euclidean knots

In this section we will find a real curve of characters of $SU(2)$ representations on the canonical component of hyperbolic knots which admit a Euclidean cone structure, partially resolving Conjecture 3.1.7.

To discuss this result, we recall the definition of a cone manifold used in [7].

Definition 3.3.1. A 3-dimensional cone manifold M is a 3-manifold admitting a PL triangulation and a metric on M such that each simplex is isometric to a geodesic simplex with constant curvature K.

Such a cone manifold will be locally isometric to the space of constant curvature K everywhere except a singular locus Σ_M, which is a subgraph of the 1-skeleton of the triangulation of M. Each edge of Σ_M can be labeled with its cone angle, which the sum of the dihedral angles of the simplices meeting at the edge. This cone angle can only change at vertices of Σ_M where more than two edges of Σ_M meet.

We will make use of the fact that $M\backslash\Sigma_M$ has a (possibly incomplete) geometric structure and thus a holonomy representation $\pi_1(M\backslash\Sigma_M) \to \text{Isom}(X_K)$ where X_K is $\mathbb{H}^3, \mathbb{E}^3$ or S^3 depending on the curvature K.

In the case of a hyperbolic cone structure, we have a representation to $\text{PSL}_2\mathbb{C}$. By the isomorphism $\text{PSL}_2\mathbb{C} \cong \text{SO}_3\,\mathbb{C}$, $\text{PSL}_2\mathbb{C}$ is a complex linear algebraic group and thus we can define a representation variety $R_{\text{PSL}_2\mathbb{C}}(\Gamma)$[19, Section 2.2].

The double cover $p : SL_2\mathbb{C} \to PSL_2\mathbb{C}$ means there are corresponding maps $p_R : R(\Gamma) \to R_{PSL_2\mathbb{C}}(\Gamma)$ and $p_X : X(\Gamma) \to X_{PSL_2\mathbb{C}}(\Gamma)$. In the case that Γ is a knot group, [19, Remark 4.3, Example 4.6] notes that $X(\Gamma) \to X_{PSL_2\mathbb{C}}(\Gamma)$ is a branched covering map. In particular this implies that the image of any irreducible component in $X(\Gamma)$ is an irreducible component of $X_{PSL_2\mathbb{C}}(\Gamma)$.

Any two $\rho, \rho' \in R(\Gamma)$ with $p_R(\rho) = p_R(\rho')$ must differ by a homomorphism $\phi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ and the group of such homomorphisms $H^1(\Gamma, \mathbb{Z}/2\mathbb{Z})$ acts on $R(\Gamma)$ and $X(\Gamma)$, acting transitively on any fiber of p_R or p_X over a point. In the case that Γ is a knot group, $H^1(\Gamma, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, and [19, Proposition 4.2] shows that $X_{PSL_2\mathbb{C}}(\Gamma)$ is isomorphic to the GIT quotient $X(\Gamma)//(\mathbb{Z}/2\mathbb{Z})$.

We require one more preliminary fact before proving Theorem 3.3.3

Lemma 3.3.2. *If any canonical component contains a curve of* $\text{SU}(2)$ *representation, then all canonical components do.*

Proof. The group $\text{SU}(2)$ is closed under complex conjugation and under multiplication by -1. Therefore the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on $X(S^3\backslash K)$ will send the character of an $SU(2)$ representation to another character of an $SU(2)$ representation. □

Theorem 3.3.3. *Let K be a hyperbolic knot in S^3. If S^3 admits a Euclidean cone structure with cone angle $\alpha \leq \pi$ then any canonical component of $X(K)$ contains a real curve of characters of $SU(2)$ representations.*

Proof. The holonomy of a Euclidean cone structure gives a representation $\pi_1(S^3\backslash K) \to \text{Isom}^+(\mathbb{E}^3)$ to the group of orientation-preserving isometries of Euclidean space. By [8, Proposition 2.1] we can lift such a representation to the double cover $\widetilde{\text{Isom}}^+(\mathbb{E}^3)$. This group has the structure of a semidirect product $\mathbb{R}^3 \rtimes SU(2)$, with rotational component map $\text{ROT} : \widetilde{\text{Isom}}^+(\mathbb{E}^3) \to SU(2)$. Define $\rho_E : \pi_1(S^3\backslash K) \to \text{SU}(2)$ as the composition of ROT with a lift of the holonomy.

[31, Lemma 6.2] says that if the Euclidean structure on (S^3, K) has cone angle not a multiple of 2π and χ_{ρ_E} is a smooth point of $X(S^3\backslash K)$, then a real curve of characters of representations into $\text{SU}(2)$ passes through χ_{ρ_E}.

Since the cone angle $\alpha \leq \pi$ then χ_{ρ_E} is guaranteed to be a smooth point by [33, Proposition 4.3]. Using this result requires an additional condition that the cone manifold is not "almost product", but this is true by [33, Lemma 2.1], [31, Lemma 9.1], and the fact that K being hyperbolic means it cannot be a torus knot.

We now need to show that χ_{ρ_E} lies on a canonical component. In the proof of [31, Theorem A], Porti finds a family of $\chi_t \in X(\pi_1(S^3 \backslash K)), t \in (0, \epsilon)$ of characters of lifted holonomies (mapping to $\mathrm{SL}_2\mathbb{C}$) of hyperbolic cone structures on (S^3, K) which approach χ_{ρ_E} as $t \to 0$ and whose cone angles approach α from below.

There must be some some χ_τ close enough to χ_{ρ_E} that it lies on the same irreducible component of $X(S^3 \backslash K)$, and for such a χ_τ, the main theorem of [22] gives a continuous path of characters of holonomies of hyperbolic cone structures in $X_{\mathrm{PSL}_2\mathbb{C}}(S^3 \backslash K)$ with decreasing cone angle from $p_X(\chi_\tau)$ to χ_{ρ_0}, the character corresponding to a discrete faithful representation. The local rigidity of Hodgson-Kerchoff [21, Theorem 4.7] implies that all the characters this path passes through are smooth points, and hence this path never leaves its original irreducible component. This shows that $p_X(\chi_{\rho_E})$ and χ_{ρ_0} lie on the same component of $X_{\mathrm{PSL}_2\mathbb{C}}(S^3 \backslash K)$.

Since $X(S^3 \backslash K)$ is a branched cover of $X_{\mathrm{PSL}_2\mathbb{C}}(S^3 \backslash K)$, any irreducible component $V \subset X(S^3 \backslash K)$ containing χ_τ must project by p_X to cover an irreducible component

$$V' \subset X_{\mathrm{PSL}_2\mathbb{C}}(S^3 \backslash K)$$

containing $p_X(\chi_\tau)$. Since $p_X(\chi_\tau)$ is a smooth point, there is only one such V', and it contains χ_{ρ_0}.

Therefore the irreducible component of $X(S^3 \backslash K)$ containing χ_{ρ_E} and χ_τ also contains a lift of χ_{ρ_0} and by Lemma 3.3.2 this means every canonical component contains a real curve of SU(2) representations. □

Candidates for such knots are hyperbolic knots whose branched double cover does not admit a hyperbolic structure. In particular these are the 2-bridge and Montesinos knots. The double cover of S^3 branched over a Montesinos knot is Seifert fibered, but not necessarily Euclidean[28]. 2-bridge knots, however do admit a Euclidean cone structure.

Corollary 3.3.4. *Any hyperbolic 2-bridge knot admits a real curve of characters of $SU(2)$ representations on its canonical component. Consequently, any infinite descending chain of hyperbolic 2-bridge knots $K_0 \geq K_1 \geq \dots$ must be eventually constant.*

Proof. By the introductory discussion of [32], any hyperbolic 2-bridge knot admits admits a Euclidean cone structure with cone angle $\alpha < \pi$, so we can apply Theorem 3.3.3. Because these knots satisfy Conjecture 3.1.7, by Proposition 3.1.8 any infinite descending sequence will be eventually constant. □

Chapter 4

Twisted Homology of Ribbon Concordances

4.1 Homology with local coefficients

The following definition of cellular (co)homology with local coefficients is from [37]

For X a CW complex with universal cover $\tilde{X}$, we can lift the CW structure to $\tilde{X}$. Then the cellular chain complex $C_*(\tilde{X})$ has a left $\pi_1(X)$ action arising from the deck transformations of $\tilde{X}$, making $C_*(\tilde{X})$ a left $\mathbb{Z}[\pi_1(X)]$ module.

A basis can be found for $C_*(\tilde{X})$ by choosing a lift of each cell in X, and this can be done in such a way that the boundary map $\partial : C_2(\tilde{X}) \to C_1(\tilde{X})$ can be computed via Fox calculus. More specifically, if X has a 0-skeleton consisting of a single point, then the $1-$cells $\{a_1, \ldots\}$ of X form a generating set of $\pi_1(X)$. Meanwhile for the $2-$cells $\{b_1, \ldots, \}$ of X, each b_i has a corresponding word r_i in the free group $\langle a_1, \ldots \rangle$ which gives its attaching map. It's possible to choose lifts $\tilde{a}_1, \ldots$ and $\tilde{b}_1, \ldots$ in $\tilde{X}$such that

$$\partial \tilde{b}_i = \sum_j \frac{\partial r_i}{\partial a_j} \tilde{a}_j$$

where $\frac{\partial r_i}{\partial a_j} \in \mathbb{Z}[\pi_1(X)]$ is the Fox derivative.

Given a left or right $\mathbb{Z}[\pi_1(X)]$ module M, we seek to take $M \otimes C_*(\tilde{X})$ and $\mathrm{Hom}(C_*(\tilde{X}), M)$ as our chain and cochain complexes to compute homology and cohomology respectively. However, $M \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\tilde{X})$ is only defined if M is a right $\mathbb{Z}[\pi_1(X)]$ module and similarly $\mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(C_*(\tilde{X}), M)$ is only defined if M is a left $\mathbb{Z}[\pi_1(X)]$ module. The solution is to define $\overline{M}$ to be the same abelian group as M, but with inverted action by $\pi_1(x)$, i.e. $m \cdot_{new} g := g^{-1} \cdot_{old} m$.

Definition 4.1.1. Given a chain complex A_* admitting a left-$\mathbb{Z}[\pi_1(X)]$ action, define

$$C_*(A_*; M) := \overline{M} \otimes_{\mathbb{Z}[\pi_1(X)]} A_*, C^*(A_*; M) := \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(A_*, M)$$

. Alternatively, if M is a right $\mathbb{Z}[\pi_1(X)]$ module $C_*(A_*; M)$ and $C^*(A_*; M)$ can be defined in the same way with $\overline{M}$ appearing in the cochain definition instead.

This allows us to define homology with twisted coefficients $H_*(X; M)$ by setting $A_* = C_*(\tilde{X})$. Likewise, we can define relative homology with twisted coefficients for a CW pair (X, Y) by setting $A_* = C_*(\tilde{X})/C_*(p^{-1}(Y_0))$, where $p : \tilde{X} \to X$ is the universal covering map.

4.2 Relative homology in local coefficients and ribbon concordances

In this section we will study the relative homology groups of a ribbon concordance in order to find points where $R(W) \to R(Y_0)$ is locally surjective.

We wish to understand when the conditions for the implicit function theorem are satisfied. Given a representation $\rho : \Gamma \to \mathrm{SL}_2\mathbb{C}$, we define $\mathrm{Ad}_\rho\, \mathfrak{sl}_2\mathbb{C}$ as the left $\mathbb{Z}[\Gamma]$ module defined by the action $\gamma \cdot v = \mathrm{Ad}_{\rho(\gamma)}(v)$.

Proposition 4.2.1. *Given a representation $\rho \in R(W)$, the cochain map*

$$\partial^* : C^1(W, Y_0; \mathrm{Ad}_\rho\, \mathfrak{sl}_2\mathbb{C}) \to C^2(W, Y_0; \mathrm{Ad}_\rho\, \mathfrak{sl}_2\mathbb{C})$$

has a matrix presentation given by the expression in Lemma 2.1.8

$$\frac{\partial w_{s_i}}{\partial g_j}\Big|_{\vec{h}} \text{ for } 1 \le i \le k, n+1 \le j \le n+k$$

where $\vec{h}$ is the point in $R(W) \subset (\mathrm{SL}_2\mathbb{C})^{n+k}$ corresponding to ρ.

Proof. W can be built from $Y_0 \times [0,1]$ by attaching k 1-handles and k 2-handles: a 1-handle for each new generator h_i and a 2-handle for each new relator s_i in the presentation

$$\pi_1(W) = \langle a_1, \ldots, a_n, b_1, \ldots, b_k | r_1, \ldots, r_{n-1}, s_1, \ldots, s_k \rangle$$

This construction lets us write (W, Y_0) as a CW pair, and this can be used to compute the relative (co)homology in twisted coefficients $H_*(W, Y_0)$ by taking the universal cover $p : \tilde{W} \to W$ and writing the cellular chain complex $C_*(\tilde{W})/C_*(p^{-1}(Y_0))$ as a left-$\mathbb{Z}[\pi_1(W)]$-module.

What results is a chain complex of the form

$$0 \longrightarrow \bigoplus_k \mathbb{Z}[\pi_1(W)] \xrightarrow{\partial} \bigoplus_k \mathbb{Z}[\pi_1(W)] \longrightarrow 0$$

The basis elements $\{\tilde{b}_1, \ldots, \tilde{b}_k\}$ of C_1 are lifts of the 1-cells attached to Y_0 corresponding to the new generators b_i. Likewise, the basis elements $\{\tilde{s}_1, \ldots, \tilde{s}_k\}$ of C_2 correspond to lifts of the 2-cells attached according to the new relators s_i in the presentation of $\pi_1(W)$.

Then the map ∂ is given by $\partial\tilde{s}_i = \sum_{j=1}^{k} \frac{\partial s_i}{\partial b_j}\tilde{b}_j$ where $\frac{\partial s_i}{\partial b_j} \in \mathbb{Z}[\pi_1(W)]$, thus the matrix for ∂ corresponding to our chosen bases is

$$(\partial)_{ij} = \frac{\partial s_j}{\partial b_i}$$

To find the twisted cohomology, we construct the cochain complex

$$C^i = \mathrm{Hom}_{\mathbb{Z}[\pi_1(W)]}(C_i, \mathfrak{sl}_2\mathbb{C})$$

where $\mathfrak{sl}_2\mathbb{C}$ is a left $\mathbb{Z}[\pi_1(W)]$ module via the Ad_ρ action $u \mapsto \rho(\gamma)u\rho(\gamma)^{-1}$, giving the chain complex

$$0 \longleftarrow \mathbb{C}^{3k} \xleftarrow[\partial^*]{} \mathbb{C}^{3k} \longleftarrow 0$$

For an $f_{ij} \in C^1$ sending $\tilde{b}_i$ to e_j a basis vector of $\mathfrak{sl}_2\mathbb{C}$ and sending all other $\tilde{b}_m$ to 0, and for any $1 \leq l \leq k$ we have

$$(\partial^* f)(\tilde{s}_l) = (f \circ \partial)(\tilde{s}_l) = f\left(\frac{\partial s_l}{\partial b_i}\tilde{b}_j\right) = \mathrm{Ad}_\rho(\frac{\partial s_l}{\partial \tilde{b}_i})(e_j)$$

This is exactly the expression in Lemma 2.1.8, hence the result follows. □

Corollary 4.2.2. $H^1(W, Y_0; \mathrm{Ad}_\rho\, \mathfrak{sl}_2\mathbb{C}) = 0$ *if and only if* $R(F_{W,Y_0}) \subset (\mathrm{SL}_2\mathbb{C})^{n+k}$ *satisfies the hypotheses of the implicit function theorem at* ρ *to be written implicitly in terms of the last* k *entries of* $(\mathrm{SL}_2\mathbb{C})^{n+k}$.

Proof. By the previous proposition, ∂^* is full rank and $H^1(W, Y_0; \mathrm{Ad}_\rho\, \mathfrak{sl}_2\mathbb{C}) = 0$ exactly when the Jacobian matrix used in the implicit function theorem is full rank. □

Note that while $R(F_{W,Y_0})$ depends on group presentations, $H^1(W, Y_0; \mathrm{Ad}_\rho)$ does not. This fact lends itself to the following definition:

Definition 4.2.3. Define a representation $\rho \in R(W)$ to be an *implicit point* of $R(W)$ when $H^1(W, Y_0; \mathrm{Ad}_\rho\, \mathfrak{sl}_2\mathbb{C}) = 0$. By the implicit function theorem, $R(\iota_0)$ is locally a homeomorphism near ρ in the standard topology.

We now restrict to the case of $\rho \in R(W)$ being a diagonal representation. Any diagonal representation ρ is determined by $\rho(\mu) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, the matrix the meridian μ is sent to.

Since conjugating a 2×2 matrix by this diagonal matrix does nothing to the top left and bottom right entries and multiplies the other two entries by λ^2 or λ^{-2}, the adjoint representation $\mathrm{Ad}_\rho\, \mathfrak{sl}_2\mathbb{C}$ decomposes as the direct sum $\mathbb{C} \oplus \mathbb{C}_{\lambda^2} \oplus \mathbb{C}_{\lambda^{-2}}$, where the $\mathbb{C}$ with no subscript denotes the trivial representation on $\mathbb{C}$ and $\mathbb{C}_{\lambda^2}$ denotes the representation on $\mathbb{C}$ on which the meridian acts on the left by multiplication by λ^2.

Since cohomology in local coefficients splits along direct sums, if we want to understand $H^1(W, Y_0; \mathrm{Ad}_\rho \mathfrak{sl}_2\mathbb{C})$ we need to understand for which values of x is $H^1(W, Y_0; \mathbb{C}_x) \neq 0$. We would like to consider homology instead of cohomology for the purposes of using the Alexander polynomial. Given the earlier discussion, switching between homology and cohomology can be confusing for the left-module $\mathbb{C}_x$, since the left action is also a right action, so we denote $\mathbb{C}'_x$ as $\mathbb{C}_x$ with the exact same action treated as a right action. Thus $\overline{\mathbb{C}_x} = \mathbb{C}'_{1/x}$

Lemma 4.2.4. *For any* $x \in \mathbb{C}^*$, $H^1(W, Y_0; \mathbb{C}_x) \cong H_1(W, Y_0; \mathbb{C}_{1/x})$

Proof. This follows from the fact that the matrices of $\partial^* : C^1(W, Y_0; \mathbb{C}_x) \to C^2(W, Y_0; \mathbb{C}_x)$ and $\mathrm{Id} \otimes \partial : C_2(W, Y_0; \mathbb{C}_{1/x}) \to C_1(W, Y_0; \mathbb{C}_{1/x})$ as maps of $\mathbb{C}$-vector spaces are adjoint. To see this, let $\phi : \pi_1(W) \to \mathbb{Z}$ be the homomorphism sending the meridian $\mu \mapsto 1$, and for any $x \in \mathbb{C}^*$ define $x^\phi : \mathbb{Z}[\pi_1(W)] \to \mathbb{C}$ as the ring homomorphism which sends any $g \in \pi_1(W)$ to $x^{\phi(g)}$.

Then our usual $\mathbb{Z}[\pi_1(W)]$-bases $\{\tilde{s_1}, \dots, \tilde{s_k}\}$ of $C_2(\tilde{W})/C_2(p^{-1}(Y_0))$ and $\{\tilde{b_1}, \dots, \tilde{b_k}\}$ of $C_1(\tilde{W})/C_1(p^{-1}(Y_0))$ induce $\mathbb{C}$-bases $\{1 \otimes \tilde{s_1}, \dots, 1 \otimes \tilde{s_k}\}$ on $C_2(W, Y_0; \mathbb{C}_{1/x})$, $\{1 \otimes \tilde{b_1}, \dots, 1 \otimes \tilde{b_k}\}$ on $C_1(W, Y_0; \mathbb{C}_{1/x})$, $\{u_1, \dots, u_k\}$ on $C^2(W, Y_0; \mathbb{C}_x)$, and $\{v_1, \dots, v_k\}$ on $C^1(W, Y_0; \mathbb{C}_x)$.

As before,

$$\partial^*(v_i) = \sum_j x^\phi(\frac{\partial s_j}{\partial h_i}) u_j$$

Meanwhile, since $C_i(W, Y_0; \mathbb{C}_{1/x}) = \overline{\mathbb{C}_{1/x}} \otimes (C_2(\tilde{W})/C_2(p^{-1}(Y_0))) = \mathbb{C}'_x \otimes (C_2(\tilde{W})/C_2(p^{-1}(Y_0)))$ we have

$$(\mathrm{Id} \otimes \partial)(1 \otimes \tilde{s_i}) = 1 \otimes (\sum_j \frac{\partial s_i}{\partial b_j} \tilde{b_j}) = \sum_j \left(x^\phi(\frac{\partial s_i}{\partial b_j}) \otimes \tilde{b_j} \right)$$

From these two computations it follows that the corresponding matrices are adjoint, therefore $\operatorname{rank} \partial^* = \operatorname{rank} \mathrm{Id} \otimes \partial$ and thus $\dim H^1(W, Y_0; \mathbb{C}_x) = \dim H_1(W, Y_0; \mathbb{C}_{1/x}) = k - \operatorname{rank} \partial^*$ □

The matrix $M(x)$ defined by $M_{ij}(x) = x^\phi(\frac{\partial s_i}{\partial b_j})$ represents the linear transformation $\mathrm{Id} \otimes \partial : C_2(W, Y_0; \mathbb{C}_{1/x}) \to C_1(W, Y_0; \mathbb{C}_{1/x})$ and is therefore a presentation matrix for the $\mathbb{C}$-module $H_1(W, Y_0; \mathbb{C}_{1/x})$. Since $M_{ij}(x)$ is square, $H_1(W, Y_0; \mathbb{C}_{1/x}) = 0$ if and only if $\det M(x) \neq 0$.

We can study the values of x for which this occurs by replacing x with the formal variable t, in which case $M(t)$ is a matrix with entries in $\mathbb{Z}[t, t^{-1}]$. In fact, $M(t)$ itself is a presentation matrix for $H_1(W, Y_0; \mathbb{Z}[t, t^{-1}]) = 0$, where $\mathbb{Z}[t, t^{-1}]$ is a $(\mathbb{Z}[t, t^{-1}], \mathbb{Z}[\pi_1(W)])$ bimodule, with left and right actions given by $q(t) \cdot p(t) \cdot g = q(t)p(t)t^{\phi(g)}$.

We now recall the definition of the Alexander polynomial.

Definition 4.2.5. Given a finitely generated $\mathbb{Z}[t, t^{-1}]$ module A, the Alexander polynomial $\Delta(t)$ is the gcd of the first Fitting ideal of the torsion submodule TA. Denote $\Delta_{W,Y_0}(t)$ to be the Alexander polynomial of $H_1(W, Y_0; \mathbb{Z}[t, t^{-1}])$, and similarly denote $\Delta_{Y_i}(t)$ as Alexander polynomial of $H_1(Y_i; \mathbb{Z}[t, t^{-1}])$.

Note that this definition only defines $\Delta_W(t)$ up to multiplication by a unit of $\mathbb{Z}[t, t^{-1}]$. We need the following collection of facts from [14, Proposition 3.2] and [5, Propositions 2.10, 2.11]

Lemma 4.2.6. *1. Since $\mathbb{Q}(t)$ is a flat module over $\mathbb{Z}[t, t^{-1}]$, for any chain complex $\mathcal{C}$ of $\mathbb{Z}[t, t^{-1}]$ modules the following sequence is exact*

$$0 \longrightarrow TH_i(\mathcal{C}) \longrightarrow H_i(\mathcal{C}) \longrightarrow H_i(\mathcal{C} \otimes \mathbb{Q}(t))$$

2. $H_i(W, Y_0; \mathbb{Q}(t)) = 0$ for all i, so all $H_i(W, Y_0; \mathbb{Z}[t, t^{-1}])$ are torsion.

Proposition 4.2.7. *The zeroes of $\Delta_{W,Y_0}(t)$ for $t \in \mathbb{C}^*$ are exactly the values of x for which $H_1(W, Y_0; \mathbb{C}_{1/x}) \neq 0$*

Proof. By Lemma 4.2.6, $H_1(W, Y_0; \mathbb{Z}[t, t^{-1}])$ is torsion. Therefore $M(t)$ is a presentation matrix for the torsion part $TH_1(W, Y_0; \mathbb{Z}[t, t^{-1}])$, and as a square matrix, its determinant is $\Delta_{W,Y_0}(t)$. As stated prior, $H_1(W, Y_0; \mathbb{C}_{1/x}) = 0$ if and only if $\det M(x) \neq 0$. □

Corollary 4.2.8. *A diagonal representation $\rho \in R(W)$ sending $\mu \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$ is an implicit point if and only if $\Delta_{W,Y_0}(x^2) \neq 0 \neq \Delta_{W,Y_0}(x^{-2})$*

Proof. Since ρ is diagonal,

$$H^1(W, Y_0; \mathrm{Ad}_\rho\, \mathfrak{sl}_2\mathbb{C}) \cong H^1(W, Y_0; \mathbb{C}) \oplus H^1(W, Y_0; \mathbb{C}_{x^{-2}}) \oplus H^1(W, Y_0; \mathbb{C}_{x^2})$$

. The first summand is just singular homology and is always 0, while the next two are 0 if and only if $\Delta_{W,Y_0}(x^2) \neq 0 \neq \Delta_{W,Y_0}(x^{-2})$. □

Proposition 4.2.9. $\Delta_{W,Y_0}(t)\Delta_{Y_0}(t) = \Delta_W(t)$

Proof. The long exact sequence for relative homology of the pair (W, Y_0) yields

$$H_2(W, Y_0; \mathbb{Z}[t, t^{-1}]) \longrightarrow H_1(Y_0; \mathbb{Z}[t, t^{-1}]) \longrightarrow H_1(W; \mathbb{Z}[t, t^{-1}]) \longrightarrow$$

$$\longrightarrow H_1(W, Y_0; \mathbb{Z}[t, t^{-1}]) \longrightarrow H_0(Y_0; \mathbb{Z}[t, t^{-1}]) \longrightarrow H_0(W; \mathbb{Z}[t, t^{-1}])$$

The first map is the zero map since the second map is an injection by [14, Proposition 3.4], and the last map can be seen to be an isomorphism between $H_0(W; \mathbb{Z}[t, t^{-1}]) = H_0(\overline{W}) \cong \mathbb{Z}$ and $H_0(Y_0; \mathbb{Z}[t, t^{-1}]) = H_0(\overline{Y}_0) \cong \mathbb{Z}$ where W and $\overline{Y}_0$ are infinite cyclic covers. Thus, the second to last map must be the zero map and the 2nd, 3rd, and 4th terms lie in a short exact sequence.

$$0 \longrightarrow TH_1(Y_0;\mathbb{Z}[t,t^{-1}]) \longrightarrow TH_1(W;\mathbb{Z}[t,t^{-1}]) \longrightarrow TH_1(W,Y_0;\mathbb{Z}[t,t^{-1}]) \longrightarrow 0$$

[5, Proposition 2.11] shows that for $X = Y_0, Y_1$, or W, we have $H_1(X;\mathbb{Q}(t)) = 0$ and thus $H_1(X;\mathbb{Z}[t,t^{-1}])$ is torsion. This fact together with Lemma 4.2.6, means that our short exact sequence is in fact a short exact sequence of torsion modules.

By [25, Proposition 5], the Alexander polynomials of these torsion modules satisfy $\Delta_W(t) = \Delta_{W,Y_0}(t) \cdot \Delta_{Y_0}(t)$ □

Note that [14] already established that $\Delta_{Y_0}(t)|\Delta_W(t)$ and $\Delta_W(t)|\Delta_{Y_1}(t)$, Proposition 4.2.9 simply gives more information about the quotient $\Delta_W(t)/\Delta_{Y_0}(t)$.

All of the results of this section come together to give the following result,

Theorem 4.2.10. *If $\Delta_W(t) = \Delta_{Y_0}(t)$ then any diagonal representation $\rho \in R(W)$ is an implicit point. In particular, this occurs when $\Delta_{Y_0}(t) = \Delta_{Y_1}(t)$.*

Proof. By Proposition 4.2.9 we know $\Delta_{W,Y_0} = 1$. Then by Corollary 4.2.8, all diagonal representations are implicit points □

4.3 Sequences of ribbon concordances

Let's now analyze how the results of the previous section can be used to study infinite sequences of ribbon concordances. Let $\mathcal{C}_{diag}(X) \subset R(X)$ for $X = Y_i$ or W be the curve of diagonal representations, parameterized by $\rho(\mu)_{1,1}$ the upper left entry of $\rho(\mu)$. Note that $\mathcal{C}_{diag}(Y_1) = \mathcal{C}_{diag}(W)$ and $R(\iota_0)$ maps $\mathcal{C}_{diag}(W)$ isomorphically to $\mathcal{C}_{diag}(Y_0)$, and both maps preserve $\rho(\mu)_{1,1}$.

Proposition 4.3.1. *If an irreducible component V of $R(Y_0)$ intersects $\mathcal{C}_{diag}(Y_0)$ at $\rho(\mu)_{1,1} = x$ and $\Delta_{W,Y_0}(x), \Delta_{W,Y_0}(x^{-1}) \neq 0$ then $R(W)$ contains an irreducible component V' which intersects $\mathcal{C}_{diag}(W)$ at $\rho(\mu)_{1,1} = x$ and maps dominantly to V.*

Proof. Let $\rho \in \mathcal{C}_{diag}(Y_0)$ be the point of intersection with V, and let $\rho' \in \mathcal{C}_{diag}(W)$ be its lift. By our assumption that $\Delta_{W,Y_0}(x), \Delta_{W,Y_0}(x^{-1}) \neq 0$, ρ' is an implicit point. This means some open subset U of V in the standard topology contains ρ and lifts to $U' \subset R(W)$ containing ρ'. The irreducible component V' of $R(W)$ containing U' then contains U in its image, and since U is Zariski dense in V, V' maps dominantly to V. □

Since V' is locally homeomorphic to V, $\dim V' = \dim V$. If we define $S_{diag}(X)$ for $X = Y_i$ or W to be the set of irreducible components of $R(X)$ which intersect $S_{diag}(X)$, and we define $\dim S_{diag}(X)$ as the multiset of dimensions of components of $S_{diag}(X)$, then we have the following corollary.

Corollary 4.3.2. *If $\Delta_{W,Y_0}(t) = 1$ then $\dim S_{diag}(Y_1) \geq \dim S_{diag}(W) \geq \dim S_{diag}(Y_0)$ where $\geq$ is the lexicographic ordering.*

Proof. This follow from the previous proposition and the fact that $S_{diag}(W) \subset S_{diag}(Y_1)$. □

Proposition 4.3.3. *Given an infinite descending sequence of ribbon concordances $K_0 \geq_{C_0} K_1 \ldots$, there is some m for which if $i \geq m$ then $S_{diag}(K_i) = S_{diag}(C_i)$.*

Proof. Since $\Delta_{K_{l+1}} | \Delta_{K_l}$ for all l, there is some n for which $\Delta_{K_{i+1}} = \Delta_{K_i}$ for all $i \geq n$. Then $\Delta_{C_i, K_{i+1}}(t) = 1$, giving an infinite descending chain of multisets $\dim S_{diag}(K_n) \geq \dim S_{diag}(C_n) \geq \dim S_{diag}(K_{n+1}) \geq \ldots$. Such a sequence must eventually become constant, meaning there is some m for which $j \geq m$ means $\dim S_{diag}(K_j) = \dim S_{diag}(C_j)$. But since $S_{diag}(C_j)$ is a closed subset of $S_{diag}(K_j)$, this means $S_{diag}(C_j) = S_{diag}(K_j)$. □

In other words, for any infinite descending chain of ribbon concordances, eventually $R(C_i)$ will contain all the components of $R(K_i)$ which contain diagonal representations.

If we would like similar results on character varieties, we should note that the character map χ sends $\mathcal{C}_{diag}$ to X_{red}, the set of characters of reducible representations. We need to understand the behavior of components of the character variety which intersect X_{red}. For any irreducible component V of the character variety, there is a corresponding irreducible component V' of the representation variety whose image under χ is V. If V intersects X_{red} then V' contains a reducible representation. In fact, it must contain a diagonal representation by the following lemma.

Lemma 4.3.4. *If an irreducible component V of a representation variety $R(\Gamma)$ contains a reducible representation, it must also contain a diagonal representation.*

Proof. Any automorphism of $R(\Gamma)$ must map irreducible components to irreducible components. Therefore the conjugation action of $\mathrm{SL}_2\mathbb{C}$ on $R(\Gamma)$ induces a homomorphism $\mathrm{SL}_2\mathbb{C}$ to $\mathrm{Sym}(S)$, the group of permutations of the finite set S of irreducible components. However, $SL_2\mathbb{C}$ is a connected lie group and has no nontrivial continuous homomorphism to a discrete group. Therefore for any $\rho \in R(G)$ belonging to an irreducible component V and for any $M \in SL_2\mathbb{C}$, it follows that $M\rho M^{-1}$ must also belong to V.

If $\rho \in V$ is reducible then it is conjugate to an upper triangular representation ρ'. Let $M(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$, so that $M(t)\rho' M(t)^{-1}$ approaches a diagonal representation as $t \to 0$. Since V is a closed set, it must contain this limiting diagonal representation.

□

Together with Proposition 4.3.3, this implies the following corollary

Corollary 4.3.5. *Given an infinite descending sequence of ribbon concordances $K_0 \geq_{C_0} K_1 \ldots$, there is some m for which if $i \geq m$ then for all irreducible components of $V \subset X(K_i)$ intersecting $X_{red}(K_i)$, it is also true that $V \subset X(C_i)$.*

Combining this result with Lemma 3.1.5 yields

Corollary 4.3.6. *For any infinite descending sequence of concordances $K_1 \geq K_2 \ldots$, if each K_i is hyperbolic with canonical component intersecting the curve of abelian characters then there is some n for which $K_m = K_n$ if $m \geq n$.*

Any two-bridge knot almost certaintly satisfies this condition on the canonical component: as shown in Section 3.2, the character varietes of two-bridge knots are plane curves, so therefore the canonical curve generically interescts the curve of characters of reducible reprentations. In particular, the figre-eight knot satisfies this. As for non two-bridge knots, the same logic should suggest a canonical component is unlikely to contain a reducible character, but the author is unaware of any enumeration of hyperbolic knots which do or do not contain reducible characters.

Chapter 5

Future Work

Recall the pullback diagram

$$\begin{array}{ccc} R(F_{W,Y_0}) & \hookleftarrow & R(W) \\ {\scriptstyle R(\iota_F)}\downarrow & & \downarrow{\scriptstyle R(\iota_0)} \\ (\mathrm{SL}_2\mathbb{C})^n & \hookleftarrow & R(Y_0) \end{array}$$

and the fact that the left map is dominant by Corollary 2.1.9.

The image $A = R(\iota_F)(R_{W,Y_0})$ of a polynomial map must be a union of locally closed sets in the Zariski topology by Chevally's theorem, which means the complement A^c is as well. Because $R(\iota_F)$ is dominant, A^c must be a union of open subsets of closed subsets of $(\mathrm{SL}_2\mathbb{C})^n$ with codimension at least 1. Generically, the components of $\pi_1(Y_0)$ should intersect A^c transversely, which would imply $R(\iota_0)$ is dominant. However, there is no obvious guarantee that this is the case for all possible ribbon concordances.

Question 5.0.1. *Is $R(\iota_0)$ dominant on each component of $R(Y_0)$?*

For example, despite having $R(\iota_0)$ not surjective, all the examples of Section 3.2 have $R(\iota_0)$ dominant,

Let's analyze a specific case that could arise if $R(\iota_0)$ is not dominant.

Lemma 5.0.2. *If S is an irreducible component of $R(F_{W,Y_0})$ such that $p_S := R(\iota_F)|_S$ is dominant, and if V is an irreducible component of $R(Y_0)$ such that $p_S(S) \cap V$ is not Zariski dense but is nonempty, then for any $\rho \in p_S(S) \cap V$, $p_S^{-1}(\rho)$ has dimension at least 1.*

Proof. The generalized Principal Ideal Theorem [13, Theorem 0.2] says that the codimension of $p_S^{-1}(V)$ as a subset of S is less than or equal to the codimension of V as a subset of $(\mathrm{SL}_2\mathbb{C})^n$. Since $\dim S \geq \dim(\mathrm{SL}_2\mathbb{C})^n$ by the dominance of p_S, this implies $\dim p_S^{-1}(V) \geq \dim V$. However, since the closure of $p_S(S) \cap V$ is a proper closed subset of V by hypothesis, then $\dim p_S(S) \cap V < \dim p_S^{-1}(V)$.

This implies any point in $\rho \in p_S(S) \cap V$ has fiber $p_S^{-1}(\rho)$ of dimension at least 1. $\square$

For such an $\rho \in R(Y_0)$, there must exist a curve $C_\rho \subset p_S^{-1}(\rho) \subset R(W)$. If ρ is irreducible, then any $\rho' \in C_\rho$ is also irreducible. The characters of $C\rho$ must then project to a curve in $X(W)$, because if not, they would all have the same character, implying they are all conjugate. However, since $R(\iota_0)$ is equivariant, thiis would give a family of $g \in \mathrm{SL}_2\mathbb{C}$ fixing ρ. However, the only $g \in \mathrm{SL}_2\mathbb{C}$ fixing an irreducible representation are $\pm\,\mathrm{Id}$ [30, Proposition 1.1.3].

Having a curve of characters in $X(W)$ lying over a fixed character $\chi_\rho \in X(Y_0)$ has interesting implications which should be explored further: By the machinery of [9], such a curve induces a nontrivial splitting of $\pi_1(W)$ as a graph of groups such that $\pi_1(Y_0)$ lies in one of the vertex groups. Thus pulls back to a splitting of $\pi_1(Y_1)$, and would induce a corresponding system of incompressible surfaces in Y_1. It would be interesting to understand what such a system of surfaces says about the topology of the ribbon concordance.

The second possible outcome under the hypotheses of Lemma 5.0.2 is that ρ is reducible. But section 4.3 shows that ρ must be an element of a component containing a diagonal representation, and that if Y_0 is late enough in the sequence of ribbon concordances, such components are mapped to dominantly since all diagonal representations are implicit points.

Finally, the most mysterious case is what happens if $p_S(S) \cap V = \emptyset$. One possible approach is to compactify S in the style of [29], and see what it means to map ideal points of S to V. This approach seems less powerful than getting a curve in $X(W)$ mapping to ρ, since it's not obvious how an action of R_{Y,W_0} on an $\mathbb{R}-$tree will be useful.

Another potential area of research is with Azumaya algebras. In [4, Theorem 1.8], Chinburg, Reid, and Stover give number-theoretic conditions for the canonical component $\mathcal{C}$ of a hyperbolic knot to admit a real curve of characters SU(2) representations. However, one of their hypotheses is a condition on the Alexander polynomial $\Delta_K(t)$ to ensure that the reducible characters of $\mathcal{C}$ are smooth points. In the context of using this result to study an infinite descending chain of ribbon concordances, Corollary 4.3.6 applies exactly in the case that infinitely many of the knots have canonical components with reducible representations. The remaining question is therefore

Question 5.0.3. *What can be said about an infinite descending sequence of hyperbolic knots $K_0 \geq K_1 \ldots$ which all satisfy the hypotheses of [4, Theorem 1.8]?*

Finally, the fact Theorem 3.3.3 doesn't necessarily apply to all hyperbolic Montesinos knots begs further investigation. The general idea of Theorem 3.3.3 is to find a family of hyperbolic cone manifolds on S^3 with singular locus K and cone angle $\alpha \in [0, \alpha_0)$ and take their holonomies to get a path in the canonical component $\mathcal{C}$. When α_0 corresponded to the cone angle of a Euclidean cone manifold, this gave a path that ended on a curve of SU(2) representations.

Cooper, Hodgson, and Kerchoff's proof of the orbifold theorem [7] studies exactly this setup, and concludes that if the hyperbolic cone manifolds terminate at $\alpha_0 < \pi$, then there is a Euclidean cone manifold with cone angle α_0. If $\alpha_0 = \pi$ however, a much wider range of geometries can occur.

Because the branched double cover of a Montesinos knot is Seifert fibered [28], it admits 1 of 6 possible geometries which then induces a geometry on S^3 with singular locus K and cone angle π.

Question 5.0.4. *What can we say about the canonical component $\mathcal{C}$ of a hyperbolic Montesinos knot based on its geometric cone structure at cone angle $\alpha = \pi$?*

The recurring theme of this dissertation is that canonical components containing SU(2) representations or reducible representations behave well with respect to infinite chains of ribbon concordances. It would be interesting to see which of these can be inferred from the various geometric cone structures.

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