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Another Problem in Possible World Semantics

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Abstract

In "A Problem in Possible-World Semantics," David Kaplan presented a consistent and intelligible modal principle that cannot be validated by any possible world frame (in the terminology of modal logic, any neighborhood frame). However, Kaplan's problem is tempered by the fact that his principle is stated in a language with propositional quantification, so possible world semantics for the basic modal language without propositional quantifiers is not directly affected, and the fact that on careful inspection his principle does not target the *world* part of possible world semantics—the *atomicity* of the algebra of propositions—but rather the idea of propositional quantification over a *complete* Boolean algebra of propositions. By contrast, in this paper we present a simple and intelligible modal principle, without propositional quantifiers, that cannot be validated by any possible world frame precisely because of their assumption of atomicity (i.e., the principle also cannot be validated by any atomic Boolean algebra expansion). It follows from a theorem of David Lewis that our logic is as simple as possible in terms of modal nesting depth (two). We prove the consistency of the logic using a generalization of possible world semantics known as *possibility semantics*. We also prove the completeness of the logic (and two other relevant logics) with respect to possibility semantics. Finally, we observe that the logic we identify naturally arises in the study of Peano Arithmetic.

Keywords: modal logic, Kripke incompleteness, Kripke inconsistency, atomic inconsistency, possibility semantics, algebraic semantics, Kaplan's paradox

1 Introduction

In his paper "A Problem in Possible-World Semantics" [17], written for a festschrift for Ruth Barcan Marcus, David Kaplan argued that there is "a problem in the conceptual/mathematical foundation of possible-world semantics (PWS) which threatens its use as a correct basis for doing the model theory of intensional languages" (p. 41). The problem is that certain consistent and

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intelligible modal principles cannot be true in any possible world model. Kaplan's example is the following principle, stating that for any proposition p, it is possible that the property expressed by Q holds of p and only p (up to necessary equivalence of propositions):

$$\forall p \Diamond \forall q (Qq \leftrightarrow \Box(p \leftrightarrow q)). \tag{A}$$

For what sentential operators Q does (A) hold? As Kaplan writes:

> Perhaps, for every proposition, it is possible that it and only it is *Queried* [That is, it is asked whether it is the case that $p \ldots$]. Or Perhaps not. It shouldn't really matter. There may be no operator expressible in English which satisfies (A). Still, *logic* shouldn't rule it out. (p. 43)

Yet standard possible world semantics rules out (A). For if propositions are in one-to-one correspondence with sets of possible worlds,[3] and $\Diamond\varphi$ (resp. $\Box\varphi$) is true if and only if φ is true at some (resp. all) worlds, then the truth of (A) requires that for every set P of worlds, there is a world w_P where the Q-property holds only of P. In other words, the truth of (A) requires an injective function sending every set of worlds to a world, contradicting Cantor's theorem.

Kaplan's paradox, as it has come to be called, has been much discussed (see, e.g., [19,20,27,1,24]). From our perspective, it has at least two weaknesses as a problem *for possible world semantics.* First, as (A) involves quantification over propositions in the object language, Kaplan's paradox does not pose a direct problem for possible world semantics for modal languages without propositional quantifiers. Second, even if we want propositional quantification, on careful inspection (A) does not in fact target the *world* part of possible world semantics.

To see why not, let us first consider a general algebraic semantics for propositional modal logic with propositional quantifiers as in, e.g., [12,3,4]. We expand a Boolean algebra B with a unary operation f on B. A valuation v assigns to each propositional variable an element of B as its semantic value. Semantic values are then assigned recursively to all formulas of the language with respect to v using operations on B associated with the sentential connectives. Boolean connectives are interpreted using the corresponding Boolean operations in B; the sentential operator Q is interpreted using the operation f; and $\Diamond$ (resp. $\Box$) is interpreted using the operation that sends a to $\top$ if $a \neq \bot$, and otherwise sends a to $\bot$ (resp. the operation that sends a to $\top$ if $a = \top$, and otherwise sends a to $\bot$). Finally, the most natural way to interpret the propositional quantifiers is to assume that B is a *complete* Boolean algebra and then to take the semantic value of $\forall p\varphi$ with respect to v to be the meet in B of all the semantic values of φ with respect to every valuation that differs from v at most in the semantic value it assigns to p.

This algebraic semantics does not make the crucial "world" assumption of

[3] *General frame* semantics, in the terminology of modal logic (see, e.g., [2, § 5.5]), is not committed to the view that every set of worlds corresponds to a proposition, so it does not fall under what we call "standard possible world semantics" here.

possible world semantics—that the algebra of propositions is *atomic*—and yet on this algebraic semantics, the semantic value of (A) must still be $\bot$.[4] Thus, (A) targets the idea of propositional quantification over a *complete* Boolean algebra of propositions. Over an incomplete Boolean algebra of propositions, there is a way of interpreting (A) as true—see [14, § 4].

In this paper, we present another problem in possible world semantics, which is not subject to the two criticisms of Kaplan's problem above. After a brief review of possible world semantics in Section 2, in Section 3 we present a simple and intelligible modal principle, without propositional quantifiers, that cannot be validated by any possible world frame precisely because of their assumption of atomicity, i.e., the principle also cannot be validated by any atomic Boolean algebra expansion. It follows from a theorem of David Lewis [18] that our logic is as simple as possible in terms of modal nesting depth (two). Using a generalization of possible world semantics known as *possibility semantics*, reviewed in Section 4, we prove the consistency of the logic in Section 5. We also prove the completeness of the logic (and two other relevant logics) with respect to possibility semantics, via completeness with respect to algebraic semantics in Section 6. In Section 7, we observe that the logic we identify naturally arises in the study of Peano Arithmetic. Finally, we conclude in Section 8 with some open questions for future research.

2 Possible World Semantics

We are interested in semantics for the following bimodal language.

Definition 2.1 Let $\mathcal{L}$ be the language generated by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid Q\varphi,$$

where p belongs to a countably infinite set Prop of propositional variables. We treat the other connectives $\vee$, $\to$, and $\leftrightarrow$ as abbreviations as usual, and we define $\Diamond\varphi := \neg\Box\neg\varphi$.

According to possible world semantics, propositions (what sentences express) are in one-to-one correspondence with sets of possible worlds, as propositions are in one-to-one corresopndence with truth conditions and truth condi-

[4] For if not, then noting that the semantic value of a formula of the form $\Diamond\varphi$ is either $\bot$ or $\top$, no matter what the valuation of p is the semantic value of $\forall q(Qq \leftrightarrow \Box(p \leftrightarrow q))$ must not be $\bot$. Given that the semantic value of $\Box(p \leftrightarrow q)$ is either $\top$ or $\bot$, and it is the former iff the valuations of p and q are the same, we see that the semantic value of $Qq \leftrightarrow \Box(p \leftrightarrow q)$ is either just the semantic value of Qq in case that p and q have the same value, or it is the complement of the semantic value of Qq in case that p and q have different values. Taking the meet of them as we vary the value of q, if the value of p is $b \in B$, then the value of $\forall q(Qq \leftrightarrow \Box(p \leftrightarrow q))$ is $h(b) := f(b) \wedge \bigwedge_{b' \in B\setminus\{b\}} \neg f(b')$, and as we said, $h(b) > \bot$. However, it is also easy to see that for any $b_1 \neq b_2$ in B, $h(b_1) \wedge h(b_2) \leq f(b_1) \wedge \neg f(b_2) \wedge f(b_2) = \bot$, and given that both $h(b_1)$ and $h(b_2)$ are not $\bot$, $h(b_1) \neq h(b_2)$. Thus, we have found an antichain $C = \{h(b) \mid b \in B\}$ in B, whose cardinality is the same as the cardinality of B. But this is impossible: by the completeness of B, any subset of C has a join in B, and for any two different subsets, they have different joins, rendering the cardinality of B to be $2^{|C|} > |C|$.

tions are satisfied or not satisfied at possible worlds. On this view, *neighborhood models* [22,25,23] give us the most general way to model propositional operators, operators that do not distinguish different syntactic ways of expressing the same proposition. We review the definitions in the current bimodal setting.

Definition 2.2 A *neighborhood frame* is a tuple $\mathfrak{F} = \langle W, N_\Box, N_Q \rangle$ where:

(i) W is a nonempty set,

(ii) $N_\Box : W \to \wp(\wp(W))$ and $N_Q : W \to \wp(\wp(W))$.

A *model* based on $\mathfrak{F}$ is a pair $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ where $V : \mathsf{Prop} \to \wp(W)$.

Definition 2.3 Given a model $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ based on $\mathfrak{F} = \langle W, N_\Box, N_Q \rangle$, $w \in W$, and formula φ, we define $\mathcal{M}, w \vDash \varphi$ as follows:

(i) $\mathcal{M}, w \vDash p$ iff $w \in V(p)$;

(ii) $\mathcal{M}, w \vDash \neg\varphi$ iff $\mathcal{M}, w \nvDash \varphi$;

(iii) $\mathcal{M}, w \vDash (\varphi \wedge \psi)$ iff $\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$;

(iv) $\mathcal{M}, w \vDash \Box\varphi$ iff $\{v \in W \mid \mathcal{M}, v \vDash \varphi\} \in N_\Box(w)$;

(v) $\mathcal{M}, w \vDash Q\varphi$ iff $\{v \in W \mid \mathcal{M}, v \vDash \varphi\} \in N_Q(w)$.

Moreover, for each formula φ, let $[\![\varphi]\!]^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \vDash \varphi\}$.

Definition 2.4 A neighborhood frame $\mathfrak{F} = \langle W, N_\Box, N_Q \rangle$ *validates* a formula φ ($\mathfrak{F} \vDash \varphi$) iff for any model $\mathcal{M}$ based on $\mathfrak{F}$ and $w \in W$, $\mathcal{M}, w \vDash \varphi$.

On the logical side, we start with the definition of congruential modal logics, which can be seen as the broadest class of extensions of classical logic with propositional operators (under the assumption that formulas are logically equivalent iff they express the same proposition). For any frame $\mathfrak{F}$, $\{\varphi \in \mathcal{L} \mid \mathfrak{F} \vDash \varphi\}$ is such a logic.

Definition 2.5 A *congruential modal logic* for $\mathcal{L}$ is a set L of formulas containing all propositional tautologies and closed under modus ponens, uniform substitution, and the congruence rule for each $O \in \{\Box, Q\}$: if $\varphi \leftrightarrow \psi \in \mathsf{L}$, then $O\varphi \leftrightarrow O\psi \in \mathsf{L}$. L is *inconsistent* if $\mathsf{L} = \mathcal{L}$. For any $\Gamma \subseteq \mathcal{L}$, let $\mathsf{Cong}(\Gamma)$ be the smallest congruential modal logic extending Γ. For any congruential modal logic L and $\varphi \in \mathcal{L}$, define $\mathsf{L} + \varphi$ to be $\mathsf{Cong}(\mathsf{L} \cup \{\varphi\})$.

3 The Split Principle

Let S be the smallest congruential modal logic containing $\Box\top$ and

$$p \to \big(\Diamond(p \wedge Qp) \wedge \Diamond(p \wedge \neg Qp)\big). \qquad \text{(Split)}$$

Suppose $\Box$ is the knowledge modality of an agent a. Then intuitively (Split) says that if p is true, then it is compatible with a's knowledge that p is true while property Q holds of p, and it is also compatible with a's knowledge that p is true while property Q does not hold of p. For example, if we interpret Qp as Kaplan suggested, as p *is queried*, then (Split) says that if p is true, then it is compatible with a's knowledge that p is true while p is queried by some agent, and it is also compatible with a's knowledge that p is true while p is not queried by some agent. We do not think that semantics should forbid an epistemic logic for reasoning about a's knowledge in which (Split) is a theorem. [5] (Later we will see an arithmetic interpretation validating (Split) in which $\Diamond$ has an "epistemic" reading as *consistency in Peano Arithmetic*; and before then we will see an interpretation involving future contingents after Theorem 5.2.) And yet, it is forbidden by possible world semantics:

Theorem 3.1 *No neighborhood frame validates* S.

In fact, no atomic Boolean algebra expansion validates S, but for readers more familiar with possible world semantics we first give the proof in terms of neighborhood frames (see Proposition 6.5 for the algebraic analogue).

Proof. Suppose $\mathfrak{F} = \langle W, N_\Box, N_Q \rangle$ validates S. Define a model $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ such that for some $w \in W$, $V(p) = \{w\}$, so $\mathcal{M}, w \vDash p$. Then since $\mathfrak{F}$ validates (Split), we have $\mathcal{M}, w \vDash \Diamond(p \wedge Qp) \wedge \Diamond(p \wedge \neg Qp)$, i.e., $[\![\neg(p \wedge Qp)]\!]^{\mathcal{M}} \notin N_\Box(w)$ and $[\![\neg(p \wedge \neg Qp)]\!]^{\mathcal{M}} \notin N_\Box(w)$. Since $V(p)$ is a singleton set, either $[\![p \wedge Qp]\!]^{\mathcal{M}} = \varnothing$ or $[\![p \wedge \neg Qp]\!]^{\mathcal{M}} = \varnothing$, so $[\![\neg(p \wedge Qp)]\!]^{\mathcal{M}} = W$ or $[\![\neg(p \wedge \neg Qp)]\!]^{\mathcal{M}} = W$. Combining the previous two steps, we have $W \notin N_\Box(w)$, which contradicts the fact that $\mathfrak{F}$ validates $\Box\top$. $\Box$

Syntactically, this logic is inconsistent with some additional principles for Q that are common in the study of specific propositional operators such as necessity and knowledge. However, these principles should not be imposed on arbitrary propositional operators (and they are even dubious for a certain notion of *querying*).

Proposition 3.2

(i) $\mathsf{S} + (Q(p \wedge q) \to Qp)$ *is inconsistent. In other words, the* Q *operator in* S *cannot be monotone.*

(ii) $\mathsf{S} + (Q(p \vee q) \to Qp)$ *is inconsistent. In other words, the* Q *operator in* S *cannot be antitone.*

(iii) *In* S, *the following two rules are derivable:*

$$\frac{\varphi \to Q\varphi}{\neg\varphi}, \qquad \frac{\varphi \to \neg Q\varphi}{\neg\varphi}.$$

(iv) *If we expand the language with propositional quantifiers and consider* $\mathsf{S}\Pi$, *the congruential extension of* S *with the standard axioms and rules for propositional quantifiers (see [3] for the axioms and rules), then* $\exists p Qp$ *and hence* $\Box\exists p Qp$ *are derivable.*

[5] One objection, suggested by a referee, is to consider p being the proposition *nothing is queried*. Then it is not plausible that it is consistent with a's knowledge that $p \wedge Qp$. Indeed, (Split) only makes sense in an epistemic logic for reasoning about the knowledge of an agent a who knows that some proposition is queried (see Proposition 3.2.iv below). Once again, however, semantics should not forbid such an epistemic logic with (Split) as a theorem.

Proof. In $\mathsf{S} + (Q(p \wedge q) \to Qp)$, we have the following derivation:

1	$\vdash Q(p \wedge \neg Qp) \to Qp$	[Monotonicity]
2	$\vdash ((p \wedge \neg Qp) \wedge Q(p \wedge \neg Qp)) \leftrightarrow \bot$	[Boolean reasoning]
3	$\vdash \Diamond((p \wedge \neg Qp) \wedge Q(p \wedge \neg Qp)) \leftrightarrow \bot$	[Congruence and $\Box\top$]
4	$\vdash (p \wedge \neg Qp) \to \Diamond((p \wedge \neg Qp) \wedge Q(p \wedge \neg Qp))$	[(SPLIT), Boolean reasoning]
5	$\vdash (p \wedge \neg Qp) \leftrightarrow \bot$	[From 3 and 4]
6	$\vdash \Diamond(p \wedge \neg Qp) \leftrightarrow \bot$	[Congruence and $\Box\top$]
7	$\vdash p \to \Diamond(p \wedge \neg Qp)$	[(SPLIT) and Boolean reasoning]
8	$\vdash p \leftrightarrow \bot$	[Boolean reasoning]

Clearly, then, $\mathsf{S} + (Q(p \wedge q) \to Qp)$ is inconsistent. For $\mathsf{S} + (Q(p \vee q) \to Qp)$, we have the following derivation:

1	$\vdash p \leftrightarrow ((p \wedge Qp) \vee p)$	[Boolean reasoning]
2	$\vdash Qp \leftrightarrow Q((p \wedge Qp) \vee p)$	[Congruence]
3	$\vdash Qp \to Q(p \wedge Qp)$	[Boolean reasoning and Antitonicity]
4	$\vdash ((p \wedge Qp) \wedge \neg Q(p \wedge Qp)) \leftrightarrow \bot$	[Boolean reasoning]
5	$\vdash \Diamond((p \wedge Qp) \wedge \neg Q(p \wedge Qp)) \leftrightarrow \bot$	[Congruence and $\Box\top$]
6	$\vdash (p \wedge Qp) \to \Diamond((p \wedge Qp) \wedge \neg Q(p \wedge Qp))$	[(SPLIT) and Boolean reasoning]
7	$\vdash (p \wedge Qp) \leftrightarrow \bot$	[From 5 and 6]
8	$\vdash \Diamond(p \wedge Qp) \leftrightarrow \bot$	[Congruence and $\Box\top$]
9	$\vdash p \to \Diamond(p \wedge Qp)$	[(SPLIT) and Boolean reasoning]
10	$\vdash p \leftrightarrow \bot$	[Boolean reasoning]

For the two rules, note that if $\vdash \varphi \to Q\varphi$, then $\vdash (\varphi \wedge \neg Q\varphi) \leftrightarrow \bot$. Then by the congruence of $\Diamond$ and $\Box\top$, $\vdash \Diamond(\varphi \wedge Q\varphi) \leftrightarrow \bot$. Using (SPLIT), we have $\vdash \varphi \to \Diamond(\varphi \wedge Q\varphi)$. Thus $\vdash \neg\varphi$. The derivation for the other rule is similar, using $\vdash \varphi \to \Diamond(\varphi \wedge \neg Q\varphi)$.

Finally, we derive $\exists pQp$:

1	$\vdash \neg\exists pQp \to \Diamond(\neg\exists pQp \wedge Q\neg\exists pQp) \wedge \Diamond(\neg\exists p\neg Qp \wedge \neg Q\neg\exists pQp)$	[(SPLIT)]
2	$\vdash Q\neg\exists pQp \to \exists pQp$	[Π-principles]
3	$\vdash (\neg\exists pQp \wedge Q\neg\exists pQp) \to (\neg\exists pQp \wedge \exists pQp)$	[Boolean reasoning]
4	$\vdash (\neg\exists pQp \wedge Q\neg\exists pQp) \leftrightarrow \bot$	[Boolean reasoning]
5	$\vdash \Diamond(\neg\exists pQp \wedge Q\neg\exists pQp) \leftrightarrow \bot$	[Congruence and $\Box\top$]
6	$\vdash \neg\exists pQp \to \bot$	[From 1 and 5]
7	$\vdash \exists pQp$	[Boolean reasoning]

Since we have the congruence rule and $\Box\top$, we can necessitate $\exists pQp$ and then obtain $\Box\exists pQp$. $\Box$

Remark 3.3 A referee informed us of a paper by Hansson and Gärdenfors [10] in which four bimodal axioms are identified that are (i) valid in an atomless Boolean algebra expanded with two operations for interpreting the two modalities but (ii) not valid on any neighborhood frame. The congruential logic axiomatized by these four axioms is strictly stronger than S (but weaker than the logic EST below). We will go beyond Hansson and Gärdenfors by proving the soundness and completeness of our neighborhood-inconsistent logic S—and the logics ES and EST below—with respect to *complete* Boolean algebra expansions, as well as by providing an arithmetic interpretation of EST.

4 Possibility Semantics

Below we will prove that S is consistent using a generalization of possible world semantics known as *possibility semantics* [16,11,13]. A key feature of possibility semantics is that it does not require the algebra of propositions to be atomic. The basic ideas are that (i) formulas are evaluated at partial *possibilities*, ordered by a refinement relation $\sqsubseteq$, so that $x \sqsubseteq y$ ("x refines y") implies that x settles as true (resp. false) every formula that y settles as true (resp. false) and possibly more; (ii) a formula is true (resp. false) at a possibility iff there is no refinement of the possibility that makes the formula false (resp. true); and (iii) a possibility settling a formula as false is equivalent to settling its negation as true (so it suffices to keep track of just the relation $\Vdash$ of settling true), and a possibility settling a conjunction as true is equivalent to settling both conjuncts as true. As for the modal operators, we interpret them using the neighborhood version of possibility semantics from [11, Remark 2.42] and [13] defined below.

Given a partially ordered set $\langle S, \sqsubseteq\rangle$, let $\mathcal{RO}(S, \sqsubseteq)$ be the collection of all $X \subseteq S$ that are *regular downsets* of $\langle S, \sqsubseteq\rangle$:

(i) for every $x \in X$, ${\downarrow}x := \{x' \in S \mid x' \sqsubseteq x\} \subseteq X$ ("persistence");

(ii) for every $x \notin X$, $\exists x' \sqsubseteq x\ \forall x'' \sqsubseteq x'\ x'' \notin X$ ("refinability").

In possibility semantics, *propositions* are regular downsets in a poset of possibilities. Below we define the analogue of neighborhood frames in possibility semantics, which differ from neighborhood frames in possible world semantics by (i) replacing the set W of worlds with a poset $\langle S, \sqsubseteq\rangle$ or possibilities and (ii) putting conditions on the neighborhood functions such that for any operator O and proposition $X \in \mathcal{RO}(S, \sqsubseteq)$, the set $\{x \in S \mid X \in N_O(x)\}$ of possibilities in which "$O(X)$ is true" is also a proposition in $\mathcal{RO}(S, \sqsubseteq)$.

Definition 4.1 A *neighborhood possibility frame* is a tuple $\mathfrak{F} = \langle S, \sqsubseteq, N_\Box, N_Q\rangle$ where:

(i) $\langle S, \sqsubseteq\rangle$ is a partially ordered set;

(ii) $N_\Box : S \to \wp(\mathcal{RO}(S, \sqsubseteq))$ and $N_Q : S \to \wp(\mathcal{RO}(S, \sqsubseteq))$ are such that for $O \in \{\Box, Q\}$:
  (a) if $X \in N_O(x)$ and $x' \sqsubseteq x$, then $X \in N_O(x')$ ("persistence");
  (b) if $X \notin N_O(x)$, then $\exists x' \sqsubseteq x\ \forall x'' \sqsubseteq x'\ X \notin N_O(x'')$ ("refinability").

A *model* based on $\mathfrak{F}$ is a pair $\mathcal{M} = \langle \mathfrak{F}, V\rangle$ where $V : \mathsf{Prop} \to \mathcal{RO}(S, \sqsubseteq)$.

Definition 4.2 Given a model $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ based on $\mathfrak{F} = \langle S, \sqsubseteq, N_\Box, N_Q \rangle$, $x \in S$, and formula φ, we define $\mathcal{M}, x \Vdash \varphi$ as follows:

(i) $\mathcal{M}, x \Vdash p$ iff $x \in V(p)$;

(ii) $\mathcal{M}, x \Vdash \neg\varphi$ iff for all $x' \sqsubseteq x$, $\mathcal{M}, x' \nVdash \varphi$

(iii) $\mathcal{M}, x \Vdash (\varphi \wedge \psi)$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$;

(iv) $\mathcal{M}, x \Vdash \Box\varphi$ iff $\{y \in S \mid \mathcal{M}, y \Vdash \varphi\} \in N_\Box(x)$;

(v) $\mathcal{M}, x \Vdash Q\varphi$ iff $\{y \in S \mid \mathcal{M}, y \Vdash \varphi\} \in N_Q(x)$.

Moreover, for each formula φ, let $[\![\varphi]\!]^{\mathcal{M}} = \{x \in S \mid \mathcal{M}, x \Vdash \varphi\}$.

Lemma 4.3 *For any formula φ and model $\mathcal{M}$ based on a neighborhood possibility frame $\mathfrak{F} = \langle S, \sqsubseteq, N_\Box, N_Q \rangle$, $[\![\varphi]\!]^{\mathcal{M}} \in \mathcal{RO}(S, \sqsubseteq)$.*

Definition 4.4 A neighborhood possibility frame $\mathfrak{F} = \langle S, \sqsubseteq, N_\Box, N_Q \rangle$ *validates* a formula φ iff for any model $\mathcal{M}$ based on $\mathfrak{F}$ and $x \in S$, $\mathcal{M}, x \Vdash \varphi$.

Proposition 4.5 *Given a model $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ based on $\mathfrak{F} = \langle S, \sqsubseteq, N_\Box, N_Q \rangle$, $x \in S$, and formulas φ and ψ:*

- $\mathcal{M}, x \Vdash (\varphi \vee \psi)$ *iff* $\forall x' \sqsubseteq x\ \exists x'' \sqsubseteq x'$*:* $\mathcal{M}, x'' \Vdash \varphi$ *or* $\mathcal{M}, x'' \Vdash \psi$*;*
- $\mathcal{M}, x \Vdash (\varphi \to \psi)$ *iff* $\forall x' \sqsubseteq x$*, if* $\mathcal{M}, x' \Vdash \varphi$*, then* $\mathcal{M}, x' \Vdash \psi$*;*
- $\mathcal{M}, x \Vdash (\varphi \leftrightarrow \psi)$ *iff* $\forall x' \sqsubseteq x$*,* $\mathcal{M}, x' \Vdash \varphi$ *iff* $\mathcal{M}, x' \Vdash \psi$*.*

5 Consistency

Our goal in this section is to show that S is consistent by constructing a possibility frame validating it. For this, we first extend S so that we can treat $\Box$ in the simplest way possible and focus on the behaviour of Q.

Definition 5.1 Let ES be the smallest congruential modal logic extending S with the following axioms:

$$\Box p \to p,\ p \to \Box\Diamond p,\ \Box p \to \Box\Box p,\ \Box(p \leftrightarrow q) \to \Box(Qp \leftrightarrow Qq).$$

Let EST be the smallest congruential modal logic extending ES by the T axiom for Q: $Qp \to p$.

Note that in ES, the first three extra axioms make $\Box$ an S5 box. The last extra axiom $\Box(p \leftrightarrow q) \to \Box(Qp \leftrightarrow Qq)$ intuitively says that if two propositions are indistinguishable by $\Box$, then their Q'ed versions are also indistinguishable by $\Box$. The reason we can further add the T axiom for Q and retain consistency [6] is, roughly speaking, that what $Qp \wedge \neg p$ means is not essential to the validity of (SPLIT). More precisely, letting $Q^*\varphi$ abbreviate $(Q\varphi \wedge \varphi)$, note that (SPLIT) is in a congruential modal logic if and only if

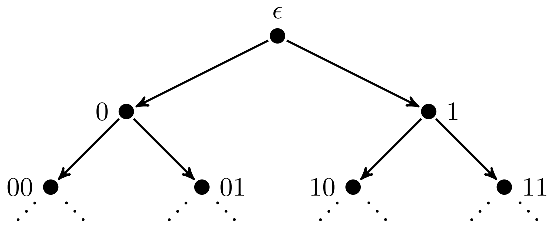
$$p \to (\Diamond(p \wedge Q^*p) \wedge \Diamond(p \wedge \neg Q^*p))$$

[6] We make no claim that the T axiom should hold for a particular operator Q such as *Queried*, but the stronger the logic we prove to be consistent, the stronger our result.

is also in the logic, since simply by Boolean reasoning, $p \wedge Q^*p$ is provably equivalent to $p \wedge Qp$, and $p \wedge \neg Q^*p$ is provably equivalent to $p \wedge \neg Qp$. Clearly $Q^*p \to p$ is in any congruential modal logic. Thus, Q^*p is in a sense the essential part of Qp that makes (SPLIT) valid, and $Qp \wedge \neg p$ is not relevant to the splitting of p by Q. Now we show that not only is S consistent, but in fact the stronger logic EST is consistent.

Theorem 5.2 *The logic* EST *is consistent.*

Proof. Consider the full infinite binary tree $2^{<\omega}$:



For $x \in 2^{<\omega}$, let $\mathsf{Par}(x)$ be the parent of x in the tree and $x0$ and $x1$ the two extensions of x by 0 and 1, respectively. In general, when y is an initial segment of x, we write $x \sqsubseteq y$ (refinements are lower down). To facilitate the definition of N_Q, for any $P \in \mathcal{RO}(2^{<\omega}) := \mathcal{RO}(2^{<\omega}, \sqsubseteq)$ and any $x \in 2^{<\omega}$, if $x \in P$, let $\mathsf{Firstin}(x, P)$ be the shortest initial segment of x that is in P, and otherwise let it be undefined. Since P is a downset, $\mathsf{Firstin}(x, P)$ is also the only y such that $x \sqsubseteq y$, $y \in P$, and $\mathsf{Par}(y) \notin P$. Moreover, $P = \bigcup_{x \in P} {\downarrow}\mathsf{Firstin}(x, P)$.

Now define N_Q by the following clause: for any $P \in \mathcal{RO}(2^{<\omega})$ and $x \in w^{<\omega}$,

$$P \in N_Q(x) \text{ iff } x \in P \text{ and } x \sqsubseteq \mathsf{Firstin}(x, P)0. \tag{1}$$

We can also define N_Q inductively as follows:

$$\begin{aligned} N_Q(\epsilon) &= \varnothing; \\ N_Q(x0) &= N_Q(x) \cup \{P \in \mathcal{RO}(2^{<\omega}) \mid x \in P,\ \mathsf{Par}(x) \notin P\}; \\ N_Q(x1) &= N_Q(x); \end{aligned}$$

but this definition is slightly harder to work with. We invite readers to verify that the inductive definition is equivalent to the definition by (1).

To show that this definition will give us a possibility frame, we claim that for any $P \in \mathcal{RO}(2^{<\omega})$, $Q(P) := \{x \in 2^{<\omega} \mid P \in N_Q(x)\} \in \mathcal{RO}(2^{<\omega})$. Pick any $P \in \mathcal{RO}(2^{<\omega})$. Now we show the two requirements for $Q(P) \in \mathcal{RO}(2^{<\omega})$.

- Suppose that $P \in N_Q(x)$ and $x' \sqsubseteq x$. By (1), $x \in P$ and $x \sqsubseteq \mathsf{Firstin}(x, P)0$. Since P is a downset, $x' \in P$. By the definition of $\mathsf{Firstin}$, clearly $\mathsf{Firstin}(x', P) = \mathsf{Firstin}(x, P)$. Hence $x' \sqsubseteq x \sqsubseteq \mathsf{Firstin}(x, P)0 = \mathsf{Firstin}(x', P)0$. Thus, $P \in N_Q(x')$. This shows that $Q(P)$ is a downset.
- Suppose that $x \notin Q(P)$, that is, $P \notin N_Q(x)$. Now we want to find a $x' \sqsubseteq x$

such that ${\downarrow}x' \cap Q(P) = \varnothing$. If $x \notin P$, then given that $P \in \mathcal{RO}(2^{<\omega})$, pick x' such that $x' \sqsubseteq x$ and ${\downarrow}x' \cap P = \varnothing$. Clearly, by the first conjunct of (1), $Q(P) \subseteq P$, and so ${\downarrow}x' \cap Q(P) = \varnothing$. Hence we are left with the case where $x \in P$. In this case, since $P \notin N_Q(x)$, it must be that $x \not\sqsubseteq \mathsf{Firstin}(x, P)0$. But then, for any $x' \sqsubseteq x$, $\mathsf{Firstin}(x', P) = \mathsf{Firstin}(x, P)$, and hence $x' \not\sqsubseteq \mathsf{Firstin}(x', P)0$ (note that we are in a tree here, and there can be only one path from $\mathsf{Firstin}(x, P)$ to x' through x). Thus, every $x' \sqsubseteq x$ fails the second conjunct of (1), and ${\downarrow}x \cap Q(P) = \varnothing$. This concludes the case where $x \in P$. Note that the above proof establishes the following:

$$\text{whenever } x \in P \text{ yet } x \notin Q(P),\ {\downarrow}x \cap Q(P) = \varnothing. \tag{2}$$

This will be useful when we show that (Split) is valid.

Now define $N_\Box$ such that for every $x \in 2^{<\omega}$, $N_\Box(x) = \{2^{<\omega}\}$. Then it is easy to see that for any $P \in \mathcal{RO}(2^{<\omega})$,

$$\Box(P) := \{x \in 2^{<\omega} \mid P \in N_\Box(x)\} = \begin{cases} 2^{<\omega} & \text{if } P = 2^{<\omega} \\ \varnothing & \text{otherwise.} \end{cases}$$

Clearly, either way, $\Box(P) \in \mathcal{RO}(2^{<\omega})$. Hence, $\mathfrak{T} := \langle 2^{<\omega}, \sqsubseteq, N_\Box, N_Q\rangle$ is a possibility frame. It is routine to verify that $\mathfrak{T}$ validates $\Box p \to p$, $p \to \Box\Diamond p$, and $\Box p \to \Box\Box p$. They all amount to discussing two cases: $V(P) = 2^{<\omega}$ and $V(P) \neq 2^{<\omega}$. It is also not hard to verify $\Box(p \leftrightarrow q) \leftrightarrow \Box(Qp \leftrightarrow Qq)$. The cases to discuss here are $V(p) = V(q)$ and $V(p) \neq V(q)$. In the former case, for all $x \in 2^{<\omega}$, $\langle \mathfrak{T}, V\rangle, x \Vdash p \leftrightarrow q$ and $\langle \mathfrak{T}, V\rangle, x \Vdash Qp \leftrightarrow Qq$. Hence the same goes for $\Box(p \leftrightarrow q)$ and $\Box(Qp \leftrightarrow Qq)$. In the case that $V(p) \neq V(q)$, $\langle \mathfrak{T}, V\rangle, \epsilon \not\Vdash p \leftrightarrow q$. Then trivially for any $x \in 2^{<\omega}$, $\langle \mathfrak{T}, V\rangle, x \Vdash \Box(p \leftrightarrow q) \to \Box(Qp \leftrightarrow Qq)$.

Now consider (Split) $= p \to (\Diamond(p \wedge Qp) \wedge \Diamond(p \wedge \neg Qp))$. To see that this is valid, first note that $\langle \mathfrak{T}, V\rangle, x \Vdash \Diamond\varphi$ iff there exists $x' \in \mathfrak{T}$ such that $\langle \mathfrak{T}, V\rangle, x' \Vdash \varphi$. Now suppose that $\langle \mathfrak{T}, V\rangle, x \Vdash p$. This means that $x \in V(p)$. Now consider $y = \mathsf{Firstin}(x, V(p))$. Clearly, by definition, $y0 \in Q(V(p))$ and hence $\langle \mathfrak{T}, V\rangle, y0 \Vdash Qp$. Now consider $y1$. Clearly, $y1 \in V(p)$ since $V(p)$ is a downset. But $y1 \notin Q(V(p))$ since $y1 \not\sqsubseteq y0 = \mathsf{Firstin}(y1, V(p))0$. Hence ${\downarrow}y1 \cap Q(V(p)) = \varnothing$ by (2). Thus, $\langle \mathfrak{T}, V\rangle, y1 \Vdash \neg Qp$. By the semantics of $\Diamond$ and $\wedge$ then, $\langle \mathfrak{T}, V\rangle, x \vDash (\Diamond(p \wedge Qp) \wedge \Diamond(p \wedge \neg Qp))$. Since V and x are arbitrary, we have shown that (Split) is valid on $\mathfrak{T}$. $\Box$

The possibility frame $\mathfrak{T}$ in the above proof can be given a natural interpretation. The partially ordered set $\langle 2^{<\omega}, \sqsubseteq\rangle$ naturally models the finitary outcomes of an infinite sequence of coin flips (say that 0 represents heads and 1 represents tails), and a crucial property is that every possibility can be further extended into two incompatible possibilities. This matches our intuitive understanding of a world with future contingencies such as random coin flips: at any time, there is at least one more coin to be flipped, and either outcome is possible.

Then our formal definition of N_Q clearly makes $Q\varphi$ express the following:

φ is now true, and the first coin flipped after φ became true landed heads up. We can also avoid temporal talk and instead speak of truth-making: φ, and the coin after the one that (exactly) makes φ true lands heads up. On this reading of Q, (Split) says that if φ is true, then it is possible that φ is true and the coin after the one that makes φ true lands heads up, and it is also possible that φ is true and the coin after the one that makes φ true lands tails up.

In addition to consistency, we will prove the following completeness theorem.

Theorem 5.3 *The logic* EST (*resp.* ES, S) *is the logic of all neighborhood possibility frames that validate* EST (*resp.* ES, S). *In other words,* EST*,* ES*, and* S *are* possibility complete*.*

This will be a corollary of the completeness theorem in the next section based on algebraic semantics.

6 Completeness

In this section, we consider algebraic semantics that generalizes possible world semantics and possibility semantics. This will help us understand exactly what it takes to validate S, ES, and EST and show that they are possibility complete.

Definition 6.1 A *Boolean algebra expansion* (BAE) $\mathcal{B}$ is a triple $\langle B, \Box, Q\rangle$ where B is a Boolean algebra and $\Box$, Q are two unary functions on B. We define $\Diamond$ and other derived operations on $\mathcal{B}$ as usual. For convenience, we omit the parentheses for the argument of unary functions as appropriate.

A valuation V on $\mathcal{B}$ is function $V : \mathsf{Prop} \to \mathcal{B}$. Then the semantics for $\mathcal{L}$ is defined by extending V to $\widehat{V} : \mathcal{L} \to \mathcal{B}$ homomorphically:

- $\widehat{V}(p) = V(p)$ for $p \in \mathsf{Prop}$;
- $\widehat{V}(\neg\varphi) = \neg\widehat{V}(\varphi)$; $\widehat{V}(\varphi \wedge \psi) = \widehat{V}(\varphi) \wedge \widehat{V}(\psi)$;
- $\widehat{V}(\Box\varphi) = \Box\widehat{V}(\varphi)$; $\widehat{V}(Q\varphi) = Q\widehat{V}(\varphi)$.

To highlight the algebra whose operations are used when obtaining $\widehat{V}$ from V, especially when V may be regarded as a valuation on two different BAEs, we may write $\widehat{V}^{\mathcal{B}}$. We say that φ is *valid* on $\mathcal{B}$ if for all valuation V on $\mathcal{B}$, $\widehat{V}(\varphi) = \top$, where $\top$ is the top element of $\mathcal{B}$.

Considering the structure of the underlying Boolean algebra, we call a BAE *complete* or a $\mathcal{C}$-BAE (resp. *atomic*, an $\mathcal{A}$-BAE) if its Boolean algebra part is a complete (resp. atomic) Boolean algebra. Then $\mathcal{CA}$-BAEs are complete and atomic BAEs. On the logical side, for $X \in \{\mathcal{C}, \mathcal{A}, \mathcal{CA}\}$, we say a set of formulas Γ is X-consistent iff there is an X-BAE validating Γ, and we say that it is X-complete iff it is the logic of the class of X-BAEs validating it (cf. [21]).

From the algebraic perspective, neighborhood frames correspond to $\mathcal{CA}$-BAEs while neighborhood possibility frames corresponds to $\mathcal{C}$-BAEs. We spell this out for possibility frames, the key fact being that the regular downsets of $\langle S, \sqsubseteq\rangle$—which are just the regular open sets in the topology on S whose opens are downsets of $\langle S, \sqsubseteq\rangle$—form a complete Boolean algebra (see, e.g., [9, § 4]).

For proofs and further discussion of the following facts relating neighborhood possibility frames and BAEs, see [13].

Proposition 6.2 *For any possibility frame $\mathcal{F} = \langle S, \sqsubseteq, N_\Box, N_Q\rangle$, let $\mathcal{F}^{\mathsf{b}} = \langle \mathcal{RO}(S, \sqsubseteq), \Box, Q\rangle$ where:*

- *$\mathcal{RO}(S, \sqsubseteq)$ is the complete Boolean algebra of regular downsets of $\langle S, \sqsubseteq\rangle$;*
- *$O(P) = \{x \in S \mid P \in N_O(x)\}$ for $O \in \{\Box, Q\}$.*

Then $\mathcal{F}^{\mathsf{b}}$ is a $\mathcal{C}$-BAE, and any valuation $V : \mathsf{Prop} \to \mathcal{RO}(S, \sqsubseteq)$ on $\mathcal{F}$ is also a valuation on $\mathcal{F}^{\mathsf{b}}$ and vice versa. Moreover, by a simple induction, for any φ, $[\![\varphi]\!]^{\langle \mathcal{F}, V\rangle} = \widehat{V}(\varphi)$. Hence $\mathcal{F}$ validates φ iff $\mathcal{F}^{\mathsf{b}}$ validates φ.

Proposition 6.3 *For any complete Boolean algebra B, let $B_\bot$ be the result of deleting $\bot$ from B and $\leq_\bot$ the result of restricting $\leq$, the lattice ordering of B, to $B_\bot$. Then $\mathcal{RO}(B_\bot, \leq_\bot)$ is isomorphic to B through the least upper bound* lub *operation. (Note that $\mathsf{lub}(\varnothing) = \bot$.)*

Thus, for any $\mathcal{C}$-BAE $\mathcal{B} = \langle B, \Box, Q\rangle$, define $\mathcal{B}_{\mathsf{u}} = \langle B_\bot, \leq_\bot, N_\Box, N_Q\rangle$ where $N_O(b) = \{P \in \mathcal{RO}(B_\bot, \leq_\bot) \mid b \leq O(\mathsf{lub}(P))\}$ for $O \in \{\Box, Q\}$. Then $\mathcal{B}_{\mathsf{u}}$ is a possibility frame, and $(\mathcal{B}_{\mathsf{u}})^{\mathsf{b}}$ is isomorphic to $\mathcal{B}$, again through lub. *Hence $\mathcal{B}$ validates a formula φ iff $\mathcal{B}_{\mathsf{u}}$ validates φ.*

A simple corollary of these two propositions is that a congruential modal logic is possibility complete iff it is $\mathcal{C}$-complete. Hence to show that S, ES, and EST are possibility complete, we show first that they are $\mathcal{C}$-complete. To this end, we begin by translating the two defining axioms of S into their conditions for being valid on BAEs.

Proposition 6.4 *A BAE $\mathcal{B} = \langle B, \Box, Q\rangle$ validates S iff the following hold:*

(i) *$\Diamond\bot = \bot$ and*

(ii) *for any $b \in \mathcal{B}$, $\Diamond(b \wedge Qb) \geq b$ and $\Diamond(b \wedge \neg Qb) \geq b$.*

A simple corollary is the following (cf. the more complicated $\mathcal{A}$-inconsistent *normal* polymodal logic in [28]).

Proposition 6.5 *If a BAE validates S, then it is atomless. Hence S is $\mathcal{A}$-inconsistent.*

Proof. Suppose a BAE $\mathcal{B}$ validates S. Pick any $b \in \mathcal{B}$ such that $b \neq \bot$. Then consider $b_1 = b \wedge Qb$ and $b_2 = b \wedge \neg Qb$. Since $\mathcal{B}$ validates S, by the previous proposition, we have (i) and (ii). By (ii), $\Diamond b_1 \geq b$ and $\Diamond b_2 \geq b$. Hence neither b_1 nor b_2 is $\bot$ since $\Diamond\bot = \bot$ by (i). But clearly $b_1 \vee b_2 = b$. Hence neither of them is b as otherwise the other is $\bot$. Thus, $\bot < b_1 < b$, so b is not an atom.$\Box$

Now we are already able to show that S is $\mathcal{C}$-complete.

Theorem 6.6 *S is the logic of the $\mathcal{C}$-BAEs validating it. Indeed, letting $\mathcal{H} = \langle H, \Box, Q\rangle$ be the Lindenbaum algebra of S and H^+ the MacNeille completion of the Boolean algebra H, there is a way to extend $\Box$ and Q to $\Box^+$ and Q^+ on H^+ such that $\mathcal{H}$ is a subalgebra of $\mathcal{H}^+ = \langle H^+, \Box^+, Q^+\rangle$ (so that $\mathcal{H}^+$ refutes all formulas not in S) and $\mathcal{H}^+$ still validates S.*

Proof. Let $\mathcal{H} = \langle H, \Box, Q \rangle$ be the Lindenbaum algebra of S. By standard algebraic logical theory, $\mathcal{H}$ validates S, and for every $\varphi \notin \mathsf{S}$, there is a valuation V_φ on $\mathcal{H}$ such that $\widehat{V_\varphi}(\varphi) \neq \top$. Since $\mathcal{H}$ validates S, by Proposition 6.5, H is atomless. Now let H^+ be the MacNeille completion of H, which is the unique (up to isomorphism) complete Boolean algebra with H being a dense subalgebra of it (in the sense that for every $b \in H^+$ such that $\bot < b$, there is a $b' \in H$ such that $\bot < b' \leq b$). (See Chap. 25 of [7] for more.) Clearly then H^+ is also atomless. Now we extend $\Box$ and Q to H^+. First, note that for any $b \in H^+ \setminus H$, there exist $b_1, b_2 \in H^+ \setminus H$ such that $b = b_1 \vee b_2$. To find such b_1 and b_2, first by density pick an $a \in H$ such that $\bot < a < b$ (note that $b \notin H$ and hence $\bot < b$). Then $b' = b \wedge \neg a$ must not be in H since otherwise $b = a \vee b'$ would also be in H. Now that H is atomless, pick $a_1, a_2 \in H \setminus \{\bot\}$ such that $a = a_1 \vee a_2$. Then let $b_1 = a_1 \vee b'$ and $b_2 = a_2 \vee b'$. Clearly $b = b_1 \vee b_2$. To see that $b_1 \notin H$, note that if it is in H, then $b' = b_1 \wedge \neg a_1$ must also be in H, contradicting that $b' \notin H$. The same reasoning applies to b_2. To fix the construction of b_1 and b_2, we can first fix an enumeration of H, which is countable, and then pick a and a_1, a_2 by going through this enumeration.

Now we define $\Box^+$ and Q^+ by the following:

$$\Box^+ b = \begin{cases} \Box b & \text{if } b \in H \\ \bot & \text{if } b \in H^+ \setminus H, \end{cases} \qquad Q^+ b = \begin{cases} Qb & \text{if } b \in H \\ b_1 & \text{if } b \in H^+ \setminus H. \end{cases}$$

Then it is easy to see by the construction of b_1 and b_2 that for every $b \in H^+ \setminus H$, $b \wedge Q^+ b$ (which is just b_1) and $b \wedge \neg Q^+ b$ (which is just b_2) are also in $H^+ \setminus H$. Also, $\Diamond^+ b := \neg \Box^+ \neg b = \top$ for all $b \in H^+ \setminus H$ since $b \notin H$ iff $\neg b \notin H$. Hence for any $b \in H^+ \setminus H$, $\Diamond^+(b \wedge Q^+ b) = \Diamond^+(b \wedge \neg Q^+ b) = \top \geq b$. Thus, by a simple discussion by cases, Proposition 6.4 applies, and $\mathcal{H}^+$ validates S. By construction, $\mathcal{H}$ is a subalgebra of $\mathcal{H}^+$. So $\mathcal{H}^+$ does not validate any formula not in S since $\mathcal{H}$ does not. Therefore, S is the logic of $\mathcal{H}^+$, a $\mathcal{C}$-BAE. $\Box$

The above strategy by MacNeille completion applies almost identically to ES and EST except that we need to focus on simple S5 algebras, those BAEs such that the $\Box$ operator essentially tests whether a proposition is $\top$ or not, so that the $\Box^+$ defined in the above proof does not destroy the validity of the $\mathsf{S5}$ axioms. To this end, we first need the following definitions.

Definition 6.7 Let $\mathcal{B} = \langle B, \Box, Q \rangle$ be a BAE. Then:

- $\mathcal{B}$ is *simple S5* if for any $b \in B$, if $b = \bot$ then $\Diamond b = \bot$, and otherwise $\Diamond b = \top$;
- $\mathcal{B}$ is *splitting* if for any $b \in B$, if $b \neq \bot$, then $b \wedge Qb \neq \bot$ and $b \wedge \neg Qb \neq \bot$;
- $\mathcal{B}$ is *deflationary* if $Qb \leq b$ for all $B \in \mathcal{B}$;
- $\mathcal{B}$ is *properly deflationary* if it is both splitting and deflationary; note that this is equivalent to: $\bot < Qb < b$ for all $b \in \mathcal{B} \setminus \{\bot\}$ and $Q\bot = \bot$.

Proposition 6.8 *A simple S5 BAE validates* ES (*resp.* EST) *iff it is also splitting* (*resp. properly deflationary*).

Proof. Let $\mathcal{B}$ be a simple S5 BAE. Then automatically $\mathcal{B}$ validates the $\mathsf{S5}$ axioms for $\Box$ and also the axiom $\Box(p \leftrightarrow q) \to \Box(Qp \leftrightarrow Qq)$, since for any valuation V on $\mathcal{B}$, $\widehat{V}(\Box(p \leftrightarrow q))$ is either $\top$ or $\bot$. If it is $\bot$, the axiom is trivially evaluated to $\top$. If it is $\top$, then $\widehat{V}(p \leftrightarrow q) = \top$, and hence $V(p) = V(q)$. Then $\widehat{V}(Qp) = \widehat{V}(Qq)$ and hence $\widehat{V}(\Box(Qp \leftrightarrow Qq))$ is also $\top$.

For (Split), it is enough to see that $\widehat{V}(\Diamond(p \wedge Qp))$ (resp. $\widehat{V}(\Diamond(p \wedge \neg Qp))$) is either $\top$ or $\bot$, and it is the former iff $V(p) \wedge QV(p) \neq \bot$ (resp. $V(p) \wedge \neg QV(p) \neq \bot$). Then the validity of (Split) translates to the condition that $\mathcal{B}$ is splitting by a simple discussion of whether $V(p) = \bot$.

For the axiom $Qp \to p$, clearly it is valid iff $\mathcal{B}$ is deflationary. $\Box$

Theorem 6.9 ES *is complete with respect to the class of all simple S5 splitting $\mathcal{C}$-BAEs.* EST *is complete with respect to the class of all simple S5 properly deflationary $\mathcal{C}$-BAEs.*

Proof. Let L be either ES or EST. Then take an arbitrary $\delta \notin \mathsf{L}$. We need to find a simple S5 splitting $\mathcal{C}$-BAE that refutes δ, and in the case that $\mathsf{L} = \mathsf{EST}$, the algebra should also be deflationary.

Consider the Lindenbaum algebra $\mathcal{H}$ of L, with $[\cdot]$ the function that sends formulas to their equivalence classes under the provable equivalence relation in L. Since $\delta \notin \mathsf{ES}$, $[\delta] \neq \top_{\mathcal{H}}$. Let $\mathcal{U}$ be an ultrafilter of the Boolean algebra base of $\mathcal{H}$ that does not contain $[\delta]$. Now define $\sim$ on $\mathcal{H}$ by $[\varphi] \sim [\psi]$ iff $\Box(\varphi \leftrightarrow \psi) \in \mathcal{U}$. This is well defined because if both $\varphi \leftrightarrow \varphi'$ and $\psi \leftrightarrow \psi'$ are in $\mathsf{ES} \subseteq \mathsf{L}$, then $\Box(\varphi \leftrightarrow \psi) \leftrightarrow \Box(\varphi' \leftrightarrow \psi')$ is also in $\mathsf{ES} \subseteq \mathsf{L}$. More importantly, $\sim$ is a congruence relation because in L we have the following theorems, with the last being a defining axiom:

- $(\Box(\varphi \leftrightarrow \psi) \wedge \Box(\varphi' \leftrightarrow \psi')) \to \Box((\varphi \wedge \varphi') \leftrightarrow (\psi \wedge \psi'))$;
- $\Box(\varphi \leftrightarrow \psi) \to \Box(\neg\varphi \leftrightarrow \neg\psi)$;
- $\Box(\varphi \leftrightarrow \psi) \to \Box(\Box\varphi \leftrightarrow \Box\psi)$;
- $\Box(\varphi \leftrightarrow \psi) \to \Box(Q\varphi \leftrightarrow Q\psi)$.

Hence we can take the quotient $\mathcal{S} = \mathcal{H}/{\sim}$. Let π be the quotient map for $\sim$, and let V be the composition of π after $[\cdot]$. Now we make three claims:

(i) $\mathcal{S}$ validates L. It is a standard exercise to show that $\mathcal{H}$ validates L. Since $\mathcal{S}$ is a quotient of $\mathcal{H}$, $\mathcal{S}$ also validates L.

(ii) $\mathcal{S}$ is a simple S5 algebra. For this, we just need to show that if $b \in \mathcal{S}$ is not $\top_{\mathcal{S}}$, then $\Box_{\mathcal{S}} b = \bot_{\mathcal{S}}$. This is again standard using the $\mathsf{S5}$ axioms.

(iii) $V|_{\mathsf{Prop}}$ is a valuation on $\mathcal{S}$, $V = \widehat{V|_{\mathsf{Prop}}}$, and $V(\delta) \neq \top_{\mathcal{S}}$.

By Proposition 6.5 and 6.8, we know then that $\mathcal{S}$ is atomless and splitting. Thus $\mathcal{S}$ is a simple S5 splitting algebra that refutes δ by V, and moreover if $\mathsf{L} = \mathsf{EST}$, $\mathcal{S}$ is also deflationary. Thus, all that is left to do is to complete $\mathcal{S}$ while preserving the three properties: being simple S5, splitting, and deflationary (if $\mathcal{S}$ is deflationary). For this, write $\mathcal{S} = \langle S, \Box, Q \rangle$, and let S^+ be the MacNeille completion S. Then pick a function $j : S^+ \to S^+$ such that for every nonbottom $b \in S^+$, $\bot < j(b) < b$. Such a j exists since $\mathcal{S}$ and hence S^+ are atomless. In fact, since S is dense in S^+ by the construction of MacNeille completion, $j(b)$ can be picked in S according to an enumeration of S (note that S is countable). Then define $\mathcal{S}^+ = \langle S^+, \Box^+, Q^+\rangle$ by

$$\Box^+ b = \begin{cases} \top & \text{if } b = \top \\ \bot & \text{otherwise,} \end{cases} \qquad Q^+ b = \begin{cases} Qb & \text{if } b \in S \\ j(b) & \text{if } b \in S^+ \setminus S. \end{cases}$$

Then clearly:

- $\mathcal{S}$ embeds into $\mathcal{S}^+$ by the identity map, and hence V_{Prop} is also a valuation on $\mathcal{S}^+$ and $V = \widehat{V|_{\mathsf{Prop}}}$;
- $\mathcal{S}^+$ is a simple S5 splitting algebra since $b \wedge j(b), b \wedge \neg j(b) > \bot$;
- if $\mathcal{S}$ is deflationary, meaning that $Qb \leq b$ for all $b \in S$, then $\mathcal{S}^+$ is also deflationary, since $j(b) \leq b$ for all $b \in S^+ \setminus S$ as well;
- $\mathcal{S}^+$ is complete.

Hence δ is refuted by V on $\mathcal{S}^+$, a simple S5 splitting $\mathcal{C}$-BAE that is deflationary if $\mathcal{L} = \mathsf{EST}$. $\Box$

An important observation about the two proofs of the $\mathcal{C}$-completeness of S, ES, and EST is that the refuting $\mathcal{C}$-BAEs we constructed are very special: their Boolean reducts are all (isomorphic to) the MacNeille completion of the countable atomless Boolean algebra, since the Lindenbaum algebra of S and the quotients of the Lindenbaum algebra of ES and EST are all countable (since the language we started with is countable) and atomless (since they all validate S). Let us call this special complete Boolean algebra B_{mca}. Then we can say that S, ES, and EST are not just $\mathcal{C}$-complete, but also B_{mca}-complete. A corollary of this is that these three logics are not just possibility complete but also complete with respect to possibility frames based on the full infinite binary tree $2^{<\omega}$.

To see this, we observe that just as $\mathcal{C}$-completeness and possibility completeness are equivalent by Propositions 6.2 and 6.3, B_{mca}-completeness and $2^{<\omega}$-completeness are also equivalent. The following proposition is the core of this new equivalence.

Proposition 6.10 *$\mathcal{RO}(2^{<\omega})$ is (isomorphic to) the MacNeille completion of the countable atomless Boolean algebra.*

Proof. Given the defining property of MacNeille completion, it is enough to see that there is a dense subalgebra of $\mathcal{RO}(2^{<\omega})$ that is countable and atomless. The subalgebra generated by principal downsets (i.e., downsets of the form $\{x \in 2^{<\omega} \mid x \sqsubseteq s\}$ for $s \in 2^{\omega}$) is such a subalgebra. $\Box$

With the above proposition, we can state the analogues of Proposition 6.2 and Proposition 6.3.

Proposition 6.11 *For any possibility frame $\mathcal{F} = \langle 2^{<\omega}, N_\Box, N_Q\rangle$ based on $2^{<\omega}$, $\mathcal{F}^{\mathsf{b}}$ is of the form $\langle B_{mca}, \Box, Q\rangle$, a BAE based on B_{mca}.*

Proposition 6.12 *For any BAE $\langle B_{mca}, \Box, Q\rangle$ based on B_{mca}, define neighborhood functions $N_\Box$ and N_Q on $2^{<\omega}$ by the following clause where σ is any isomorphism from $\mathcal{RO}(2^{<\omega})$ to B_{mca}: for any $O \in \{\Box, Q\}$, $s \in 2^{<\omega}$, and $X \in \mathcal{RO}(2^{<\omega})$, $X \in N_O(s)$ iff $s \in \sigma^{-1}(O(\sigma(X)))$. Then, $(\langle 2^{<\omega}, N_\Box, N_Q\rangle)^{\mathsf{b}} \cong \langle B_{mca}, \Box, Q\rangle$ with σ being the isomorphism.*

Thus, a logic is complete with respect to neighborhood possibility frames based on $2^{<\omega}$ iff it is complete with respect to BAEs based on B_{mca}. This completes the proof of the following strengthening of Theorem 5.3.

Theorem 6.13 *The logic* S *(resp.* ES, EST*) is the logic of all neighborhood possibility frames based on $2^{<\omega}$ that validate* S *(resp.* ES, EST*).*

Now that we have seen that S, which is defined by two very simple axioms, is consistent and $\mathcal{C}$-complete yet $\mathcal{A}$-inconsistent, we briefly comment on whether we may have a logic that is also consistent and $\mathcal{A}$-inconsistent but is defined by even simpler axioms. Recall that given a set Γ of formulas, $\mathsf{Cong}(\Gamma)$ is the smallest congruential modal logic containing Γ. Now let $\mathsf{BAE}(\Gamma)$ be the class of BAEs validating Γ. Then the following theorem is due to Lewis [18].

Theorem 6.14 (Lewis) *For every set Γ of formulas of modal depth at most 1,* $\mathsf{Cong}(\Gamma)$ *is complete with respect to all finite BAEs in* $\mathsf{BAE}(\Gamma)$*. Since finite BAEs are all complete and atomic,* $\mathsf{Cong}(\Gamma)$ *is $\mathcal{CA}$-complete.*

Hence, $\{\Box\top, (\textsc{Split})\}$ is optimal in terms of modal depth: depth 2. We can also show that it is optimal in terms of the number of propositional variables used: just 1. For this, let the language $\mathcal{L}$ now include the propositional constant $\top \notin \mathsf{Prop}$ such that for any valuation V, $\widehat{V}(\top) = \top$ on any BAE. Then we have the following simple theorem.

Theorem 6.15 *If $\Gamma \subseteq \mathcal{L}$ contains only formulas that do not use any propositional variable in* Prop*, then* $\mathsf{Cong}(\Gamma)$ *is $\mathcal{CA}$-complete.*

Proof. Let Γ be a set of variable-free formulas and $\mathsf{L} = \mathsf{Cong}(\Gamma)$. L is trivially $\mathcal{CA}$-complete if it is inconsistent. Hence we assume that it is consistent. Consider $\mathcal{H} = \langle H, \langle \nabla_i\rangle_{i\le n}\rangle$, the Lindenbaum algebra of L (here we do not assume that $\mathcal{L}$ has only $\Box$ and Q as modalities). Let $\mathcal{H}^+$ be the BAE where its Boolean base H^+ is the canonical extension of H, and its operations ∇_i^+ are defined by

$$\nabla_i^+(\boldsymbol{a}) = \begin{cases} \nabla_{\mathcal{L}}(\boldsymbol{a}) & \text{if } \boldsymbol{a} \in H^{\text{arity}(\nabla_i)} \\ \top & \text{otherwise.} \end{cases}$$

Then let V_1 be the constantly $\top$ valuation V_1 on $\mathcal{H}^+$, which is also a valuation on $\mathcal{H}$. Since by construction $\mathcal{H}$ is a subalgebra of $\mathcal{H}^+$, $\widehat{V_1}^{\mathcal{H}^+} = \widehat{V_1}^{\mathcal{H}}$. In particular, for any $\varphi \in \Gamma$, $\widehat{V_1}^{\mathcal{H}^+}(\varphi) = \widehat{V_1}^{\mathcal{H}}(\varphi) = \top$ since φ is valid on $\mathcal{H}$. But a variable-free formula is valid iff it is evaluated to $\top$ in any valuation. So all formulas in Σ are still valid on $\mathcal{H}^+$. Since $\mathcal{H}$ is a subalgebra of $\mathcal{H}^+$, formulas that are invalid in $\mathcal{H}$ are still invalid in $\mathcal{H}^+$. Hence the validities of $\mathcal{H}^+$ are precisely $\mathsf{Cong}(\Gamma)$. Since H^+ is a canonical extension, $\mathcal{H}^+$ is a $\mathcal{CA}$-BAE. Hence $\mathsf{Cong}(\Gamma)$ is $\mathcal{CA}$-complete. $\square$

However, $\{\Box\top, (\textsc{Split})\}$ is not optimal in terms of the number of modal operators used. Peter Fritz in his presentation [6] of his paper [5] defined the unimodal logic $\mathsf{Uni3}$, the smallest congruential modal logic containing the following axioms:

$$(\Box\top \wedge p) \leftrightarrow \Box(\Box\top \to (p \wedge \Box(\Box\top \wedge p))) \quad (\textsc{Uni3Ax1})$$
$$(\Box\top \wedge p) \leftrightarrow \Box(\Box\top \to (p \wedge \neg\Box(\Box\top \wedge p))) \quad (\textsc{Uni3Ax2})$$
$$\Box\Box\Box\top \quad (\textsc{Uni3Ax3})$$
$$\neg\Box\bot. \quad (\textsc{Uni3Ax4})$$

It can be shown that $\mathsf{Uni3}$ is consistent yet $\mathcal{A}$-inconsistent. Hence, an open problem here is whether there is a consistent yet $\mathcal{A}$-inconsistent logic that can be axiomatized by using only 1 modal operator, 1 (or more) propositional variables, and modal depth 2. It is also not known whether $\mathsf{Uni3}$ is $\mathcal{C}$-complete.

7 Split in Peano Arithmetic

In this section, we show how EST arises naturally in the study of Peano Arithmetic and in particular the problem of uniform density [26]. Following [26], define the following sequence of subtheories of PA:

$$\mathsf{AR}_0 = \mathrm{I}\Delta_0 + \mathsf{Exp}, \quad \mathsf{AR}_{n+1} = \mathrm{I}\Sigma_{n+1}.$$

Recall that Exp is the formula stating the totality of the exponential function defined by a Δ_0 formula (see [8, p. 299]), $\mathrm{I}\Delta_0$ is Peano Arithmetic with the induction schema applied only to Δ_0 formulas, and $\mathrm{I}\Sigma_{n+1}$ is Peano Arithemetic with the induction schema only applied to Σ_{n+1} formulas. For each n, AR_{n+1} extends AR_n, with their union being the usual PA. AR_0 is also known as *elementary arithmetic* (EA). These theories are uniformly recursively axiomatized. Hence there is a formula with two free variables, $\mathsf{Prov}(x, y)$, such that $\mathsf{Prov}(n, \ulcorner\varphi\urcorner)$ expresses "φ is provable in AR_n" in PA. For convenience, let $\mathsf{Pr}_x\varphi$ stand for $\mathsf{Prov}(x, \ulcorner\varphi\urcorner)$. Then define $\mathsf{Q}\varphi$ to be

$$\varphi \wedge \forall x(\mathsf{Pr}_x(\varphi \to \mathsf{Pr}_x(\varphi \to \bot)) \to \mathsf{Pr}_x(\varphi \to \bot)).$$

If we write $\mathsf{Con}_n\varphi$ for "φ is consistent in AR_n", then $\mathsf{Q}\varphi$ can be equivalently defined as

$$\varphi \wedge \forall x(\mathsf{Con}_x\varphi \to \mathsf{Con}_x(\varphi \wedge \mathsf{Con}_x\varphi)).$$

For example, $\mathsf{Q}\top$ is equivalent to the formula $\forall x(\mathsf{Con}_x\top \to \mathsf{Con}_x\mathsf{Con}_x\top)$, which intuitively says that for every n, if AR_n is consistent, then it is consistent in AR_n that the system AR_n is consistent. While it sounds trivial to us, PA is not able to prove or disprove this $\mathsf{Q}\top$. The following two lemmas are shown in [26] (note that their notation is C_φ instead of $\mathsf{Q}\varphi$).

Lemma 7.1 ([26], Lemma 3.4) *For any φ and ψ, if $\mathsf{PA} \vdash \varphi \leftrightarrow \psi$, then $\mathsf{PA} \vdash \mathsf{Q}\varphi \leftrightarrow \mathsf{Q}\psi$.*

Lemma 7.2 ([26], Lemma 3.5) *If φ is consistent in PA, then both $\varphi \wedge \mathsf{Q}\varphi$ and $\varphi \wedge \neg\mathsf{Q}\varphi$ are consistent in PA.*

Then it follows immediately from Proposition 6.8 that the logic of this arithmetic Q is at least EST.

Theorem 7.3 *Let H be the Lindenbaum algebra of PA. Let $\Box$ be defined on H by $\Box[\varphi] = [\top]$ if $\mathsf{PA} \vdash \varphi$ and $\Box[\varphi] = [\bot]$ otherwise. Define Q on H by $Q[\varphi] = [\mathsf{Q}\varphi]$. Then $\langle H, \Box, Q\rangle$ validates EST.*

Thus, (Split) is not only consistent and intelligible but even has a natural arithmetic interpretation.

8 Conclusion

As with other results showing that certain modal logics are incomplete with respect to possible world semantics but complete with respect to more general semantics (see [15] and references therein), we take the results of this paper to be more positive than negative, as they lead to interesting new questions for the foundations of modal logic. We conclude by mentioning a few questions.

First, on the more philosophical side, we would like to identify more modal operators for which (Split) is intuitively valid. We think that the study of truth-makers or counterfactuals is the most promising path. A related question: as we have shown in Theorem 5.2, monotonicity is inconsistent with S, but what other principles are inconsistent with S? Answering this question will help us narrow down possible interpretations of the $\Box$ and Q operators validating S.

On the more technical side, a first question is whether there are congruential extensions of S (or ES or EST) that are not $\mathcal{C}$-complete. This is essentially a test of how widely applicable our method of MacNeille completion is in proving completeness with respect to $\mathcal{C}$-BAEs. A further question is which extensions of S (or ES or EST) are *tree complete*, that is, complete with respect to a class of possibility frames whose underlying posets of possibilities are trees or even finitely branching trees. We have seen in Theorem 6.13 that the three logics, S, ES, and EST, are all tree complete (indeed, $2^{<\omega}$-complete). But the general picture for congruential logics extending these logics is not clear. It may also be interesting to see how we can axiomatize the logic of the possibility frame based on $2^{<\omega}$ defined in the proof of Theorem 5.2.

Finally, as we mentioned in Section 6, it remains to be seen whether there is a consistent but $\mathcal{A}$-inconsistent congruential modal logic axiomatized using 1 modality, 1 propositional variable, and modal nesting depth 2.

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