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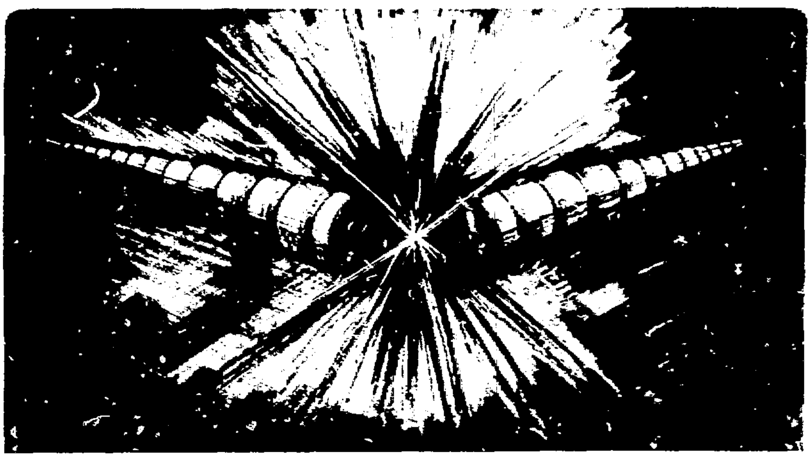

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ABSTRACT

A kinetic equation approach to stochastic cooling is presented. Equations for one and two particle distribution functions are derived from the principle of conservation of the number of ensemble systems. The violation of Liouville's theorem is expressed by certain self-interaction terms. The two-particle distribution describes Schottky noise and feedback effects and is analyzed by techniques of the Lenard-Balescu equation for plasmas. The resulting expression for the one particle distribution is of the form of a Fokker-Planck equation. The suppression of Schottky signals for arbitrary machine impedance is discussed in terms of particle correlations.

1. INTRODUCTION

The notion of mixing in stochastic cooling describes both the effects of the narrowing of Schottky signals with diminishing frequency spread and the feedback introduced by previous corrections[1] to the emittance. These issues are both manifestations of correlations in the cooled variable developing between the beam particles. In a plasma physics context, the details of two-particle correlations are described by the Lenard-Balescu equation.[2] Since it is derived from Liouville's thereom, these results are not directly applicable to stochastic cooling where the feedback system introduces dissipation. However, starting from first principles, equations can be obtained which are amenable to the techniques developed in the Lenard-Balescu analysis.

2. KINETIC EQUATIONS FOR A NON-LIOUVILLIAN SYSTEM

Consider the 2N-dimensional ensemble distribution $D(q_1, p_1, \ldots, q_n, p_n)$ describing a one dimensional system of N particles, normalized to unity integral. Conservation of the number of ensemble systems is expressed by

$$\frac{\partial D}{\partial t} + \vec{\nabla} \cdot (\vec{u}\, D) = 0 \qquad (1)$$

where $\vec{u} = (\dot{q}_1, \dot{p}_1, \ldots, \dot{q}_n, \dot{p}_n)$[3]. If the system dynamics are described by Hamilton's equations, then (1) reduces to the condition of incompressible fluid flow (Liouville's theorem), with ∇ acting on D alone. For a stochastic cooling system, (ignoring amplifier noise), the dynamics are not Hamiltonian, but are of the form

$$\frac{dp_i}{dt} = \sum_j G\,(q_i, q_j, p_j) \qquad (2)$$

$$\frac{dq_i}{dt} = Q\,(p_i) \qquad (3)$$

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As in the Liouvillian case, equation (1) may still be integrated over 2(N-1), 2(N-2), etc. variables to yield equations for one, two, etc. -particle distribution functions. There is an infinite hierarchy of equations; in particular, the two-particle equation contains the three-particle distribution. For plasmas with long range forces, the assumption is made that n-particle correlation effects vanish as the (n-1) power of the ratio of interaction energy to thermal energy. For stochastic cooling systems the corresponding ratio is of the strength of the correction to frequency spread. Decompose the two-particle distribution into

$$f_2 = f\,(q_1,\ p_1,\ t)\ f\,(q_2,\ p_2,\ t) + g\,(q_1,\ p_1,\ q_2,\ p_2,\ t) \tag{4}$$

where f is the one-particle distribution and g describes all two-particle correlations. Then, on integrating (1) over 2(N-1) and 2(N-2) variables and dropping all terms of the order of three-particle correlations, we have[5)]

$$\begin{aligned}\frac{\partial f}{\partial t} + \dot{q}\,\frac{\partial f}{\partial q} + N\,\frac{\partial f}{\partial p}\int dq'dp'G\,(q,\ q',\ p')\ f\,(q',\ p',\ t)\\ = - N\,\frac{\partial}{\partial p}\int dq'dp'G\,(q,\ q',\ p')\ g\,(q,\ p,\ q',\ p',\ t)\\ -\left[\frac{\partial}{\partial p}\,(G(q,\ q,\ p)\ f\,(q,\ p,\ t))\right]\end{aligned} \tag{5}$$

and

$$\begin{aligned}&\frac{\partial g}{\partial t}\,(q_1,\ p_1,\ q_2,\ p_2,\ t) + \dot{q}_1\,\frac{\partial g}{\partial q_1} + \dot{q}_2\frac{\partial g}{\partial q_2} + N\,\frac{\partial g}{\partial p_1}\int dq'dp'G\,(q_1,\ q',\ p')\ f\,(q',\ p',\ t)\\ &+ N\,\frac{\partial g}{\partial p_2}\int dq'dp'G\,(q_2,\ q',\ p')\ f\,(q',\ p',\ t) =\\ &- N\,\frac{\partial f}{\partial p_1}\int dq'dp'G\,(q_1,\ q',\ p')\ g\,(q_2,\ p_2,\ q',\ p',\ t)\\ &- N\,\frac{\partial f}{\partial p_2}\int dq'dp'G\,(q_2,\ q',\ p')\ g\,(q',\ p',\ q_1,\ p_1,\ t)\\ &- \frac{\partial}{\partial p_1}\left[G\,(q_1,\ q_2,\ p_2)\ f\,(q_1,\ p_1,\ t)\ f\,(q_2,\ p_2,\ t)\right]\\ &- \frac{\partial}{\partial p_2}\left[G\,(q_2,\ q_1,\ p_1)\ f\,(q_1,\ p_1,\ t)\ f\,(q_2,\ p_2,\ t)\right]\end{aligned} \tag{6}$$

The cooling of phase-space appears in the bracketed term of equation (5). To the order of the approximation no equivalent self-interaction term is found in equation (6), and it is formally identical to the Lenard-Balescu equation. The last term on the left hand side of (5) is a Vlasov-like expression which vanishes for stochastic cooling systems. The first term on the right hand side of (5) describes Schottky signal and feedback effects. For equation (6), the last two terms on the right hand side are the direct effect of other beam particles; i.e., the Schottky signals. The first two terms on the right hand side describe the suppression of both the coherent cooling rate and Schottky noise through feedback. The $\dot{q}$ terms on the left hand side effect mixing through frequency spread. Similar results follow for two dimensional systems and can be applied to transverse cooling. Amplifier noise may be added with the appearance of the usual noise term in (5) and an additional term in (6) describing feedback of the noise signal through the beam.

2. MOMENTUM COOLING

We take as our variables the azimuthal angle θ and $x = (E - E_0)$, and we model the equations of motion by

$$\frac{dx_i}{dt} = \sum_j \sum_n G_n(x_j)\, e^{in(\theta_i - \theta_j)} \tag{7}$$

where G is the system transfer function and $G_0 = 0$. We also assume that f is independent of θ, and $g(\theta_1, x_1, \theta_2, x_2, t)$ is a function of $\theta_1 - \theta_2$, x_1, and x_2; that is,

$$g(\theta_1, \theta_2, x_1, x_2, t) = \sum_{\ell=-\infty}^{\infty} g_\ell(x_1, x_2, t)\, e^{i\ell(\theta_1 - \theta_2)} \tag{8}$$

The function g_ℓ describes signal suppression in the ℓth Schottky band. Inserting (8) into (6) yields

$$\begin{aligned}\frac{\partial g}{\partial t} &+ i\ell(\omega_1 - \omega_2)\, g_\ell(x_1, x_2, t) = \\ &- \frac{\partial}{\partial x_1}\left[G_\ell(x_2)\, f(x_1, t)\, f(x_2, t)\right] - \frac{\partial}{\partial x_2}\left[G_{-\ell}(x_1)\, f(x_1, t)\, f(x_2, t)\right] \\ &- N\frac{\partial f}{\partial x_1}\int dx'\, G_\ell(x')\, g_{-\ell}(x_2, x', t) \\ &- N\frac{\partial f}{\partial x_2}\int dx'\, G_{-\ell}(x')\, g_\ell(x_1, x', t)\end{aligned} \tag{9}$$

Equation (9) may be solved by standard Laplace transform techniques developed in the Lenard-Balescu analysis[4] under under the assumption that the relaxation time for g is fast on the scale of variations of f (Schottky signal suppression is fast compared to cooling times). That is, we solve (9) with f assumed constant. Inserting the solution of (9) into (5) yields

$$\frac{\partial f}{\partial t} = -\sum_\ell \left\{\frac{\partial}{\partial x}\left[\frac{G_\ell(x) f(x,t)}{\epsilon_{-\ell}(x)}\right] - \frac{\partial}{\partial x}\left[\frac{N\pi}{|\ell|}\left|\frac{\partial x}{\partial \omega}\right|\left|\frac{G(x)}{\epsilon_{-\ell}(x)}\right|^2 \frac{\partial f}{\partial x}\, f\right]\right\} \tag{10}$$

where

$$\epsilon_{\pm|\ell|} = 1 + \frac{N}{|\ell|}\int_{\eta \to 0_+} dx'\, \frac{\frac{\partial f}{\partial x'}\, G_{\mp\ell}(x')}{\eta \pm i(\omega - \omega')} \tag{11}$$

and $\omega(x)$ is the angular revolution frequency corresponding to x.[5]

This result is of the form of a Fokker-Planck equation with one derivative acting on f alone[6, 7]. The first term on the right hand side is the coherent cooling of a particle's energy error; the ϵ_ℓ factor describes the feedback of the coherent signal through the beam. The second term contains the effects of the beam signal, including again feedback. The form of interaction (7) is directly applicable to the Palmer method of momentum cooling, where the weighting function $G_n(x)$ derives from position measurements in a transverse pickup and the electronic gain is essentially constant over a Schottky band. For the filter method, in which energy information is obtained through variation of the electronic gain with frequency, the ϵ factors are modified, with the corresponding $G_n(x)$ outside the integration in (11). If amplifier noise P is included there will be an additional term on the right hand side:

$$\sum_{\ell} \frac{\partial}{\partial x}\left[\pi\left(\frac{e\omega}{2\pi}\right)^2 \frac{P(\ell\omega)}{\left|\epsilon_\ell(\omega)\right|^2} \frac{\partial f}{\partial x}\right] \tag{12}$$

3. SCHOTTKY SIGNAL SUPPRESSION

The ℓ^{th} Schottky signal at frequency $\ell\omega(x)$ is proportional to

$$f(x) + \mathrm{Re}\int_{\text{resolution}} dx'\; g_\ell(x, x') \tag{13}$$

where the integral is over the resolution of the analyzing device. Since the correction function is highly peaked near $x = x'$, this integral is well approximated by

$$\mathrm{Re}\int dx'\; g_\ell(x, x') \approx \left(\frac{1}{\left|\epsilon_\ell\right|^2} - 1\right) f(x) \tag{14}$$

With well-centered notches and cooling, $|\epsilon_\ell| > 1$ and a negative correlation exists, corresponding to the average energy error being corrected toward zero. The resulting Schottky signal is modified to

$$\frac{f(x)}{\left|\epsilon_\ell\right|^2} \tag{15}$$

The factor $\left|\epsilon_\ell\right|^{-2}$ describes Schottky signal suppression and is a direct consequence of equation (6). This equation remains valid for space charge and wall effects (since it is independent of the cooling self-interaction term) with the appropriate impedance substituted for G. In fact, where dissipative effects of the impedance are negligible, Liouville's theorem holds, and the original derivation of Lenard and Balescu is applicable. The solution of (6), or equivalently (9) will determine associated ϵ_ℓ and g_ℓ which describe correlations due to the impedance. For space charge forces, $|\epsilon_\ell| > 1$ below transition and Schottky signals are suppressed. This effect is most pronounced in situations of small frequency spread, as has been observed in electron cooling experiments.[8]

REFERENCES

1) F. Sacherer, Stochastic Cooling Theory, CERN-ISR-TH/78-11 (1978).

2) S. Ichimaru, Basic Principles of Plasma Physics, W. A. Benjamin (1973).

3) R. Liboff, Introduction to the Theory of Kinetic Equations, John Wiley and Sons, (1969), p. 52

4) Yu. Klimontovich, Statistical Theory of Non-equilibrium Processes in a Plasma, MIT Press (1967), p. 122.

5) J. Bisognano, Kinetic Equations for Longitudinal Stochastic Cooling, LBID-149.

6) G. Carron, et al., Experiments on Stochastic Cooling in ICE, IEEE Transactions in Nucl. Sci., NS-26 3, 3456 (1979).

7) G. Lambertson, et al., Stochastic Cooling of 200 MeV Protons, this conference.

8) F. Krienen and G. Pettrucci (Private Communication).