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Morse-Bott and Equivariant Theories Using Polyfolds

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Abstract

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In this paper, we propose a general method of defining equivariant theories in symplectic geometry using polyfolds. The construction is twofold, one is for closed theories like equivariant Gromov-Witten theory, the other is for open theories like equivariant Floer cohomology.

When a compact Lie group G acts on a tame strong polyfold bundle $p : W \to Z$, we construct a quotient polyfold bundle $\overline{p} : W/G \to Z/G$ if the G-action on Z only has finite isotropy. For a general group action and if Z has no boundary, then every G-equivariant sc-Fredholm section $s : Z \to W$ induces a $H^*(BG)$ module map $s_* : H^*_G(Z) \to H^{*-\mathrm{ind}\, s}(BG)$, which can be viewed as a generalization of the integration over the zero set $s^{-1}(0)$ when equivariant transversality holds. When Z is the Gromov-Witten polyfold, s_* yields a definition equivariant Gromov-Witten invariant for any symplectic manifold. We obtain a localization theorem for s_* if there exist tubular neighborhoods around the fixed locus in the sense of polyfold. For open theories, we first obtain a construction for the Morse-Bott theories under minimal transversality requirement. We discuss the relations between different constructions of cochain complexes for Morse-Bott theory. We explain how homological perturbation theory is used in Morse-Bott cohomology, in particular, both our construction and the cascades construction can be interpreted in that way, In the presence of group actions, we construct cochain complexes for the equivariant theory. Expected properties like the independence of approximations of the classifying spaces and existence of action spectral sequences are proven. We carry out our construction for finite dimensional Morse-Bott cohomology using a generic metric and prove it recovers the regular cohomology. We outline the project of combining our construction with polyfold theory, which is expected to give a general construction for both Morse-Bott and equivariant Floer cohomology.

In the last part, we show that for any asymptotically dynamically convex contact manifold Y, the vanishing of symplectic homology $SH(W) = 0$ is a property independent of the choice of topologically simple (i.e. $c_1(W) = 0$ and $\pi_1(Y) \to \pi_1(W)$ is injective) Liouville filling W. As a consequence, we answer a question of Lazarev partially: a contact manifold Y admitting flexible fillings determines the integral cohomology of all the topologically simple Liouville fillings of Y.

Dedicated to my parents and my life partner Chenxue.

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Chapter 1

Introduction

The use of J-holomorphic curves in symplectic geometry was initiated by Gromov [52] and then by Floer [40, 41]. Roughly speaking, there two types of theories. One is the closed theory, where the moduli spaces of holomorphic curves are closed. When transversality holds, the moduli spaces have fundamental classes in the homology, which can be used to define invariants. A typical example is the symplectic Gromov-Witten theory [52, 80]. The other type is the open theory, where they are families of compact moduli spaces, but the moduli spaces have boundary and corner. Therefore the best thing one can hope is fundamental chains in the chain complexes. Although each fundamental chain depends on some auxiliary data used in the construction of the moduli spaces. One can define a global structure using all the fundamental chains, which is usually invariant in a suitable sense. Typical examples in this category are Floer type theories like Floer homology [38, 40, 41], Fukaya category [45, 98] and symplectic field theory [33]. When there exist extra group actions on the moduli spaces, one can upgrade both closed and open theories to equivariant theories, e.g. equivariant Gromov Witten invariant [47] and equivariant Floer homology [18, 17, 56].

The difficulties of defining equivariant theories for general symplectic manifolds are twofold. One is the usual transversality problem, which has been seen in non-equivariant theories. It was intensively studied by many researchers [73, 93, 23, 66] for particular settings. Moreover, there are two techniques aimed to solve such difficulty in its most general form:

1. Kuranishi type approaches [67, 45, 81, 88] deal with problem by local finite-dimensional reductions;

2. polyfold theory [65] works with infinite dimensional smooth objects by introducing new notation of differentiability and new local models.

The other challenge comes from the group action, since equivariant transversality is often obstructed. In this paper, we propose a uniform treatment of the equivariant theories for both closed and open theories in the abstract setting of polyfolds, which provides constructions for equivariant Gromov-Witten theory and equivariant Floer cohomology on general symplectic manifolds.

1.1 Quotient theorems in polyfold theory and equivariant transversality

Polyfold theory [58, 59, 60, 63, 64, 65] developed by Hofer, Wysocki and Zehnder provides an analytic framework to regularize moduli spaces of J-holomorphic curves in symplectic geometry [61, 62, 74, 102]. Roughly speaking, a polyfold is a topological space Z which contains a complicated moduli space $\mathcal{M} := s^{-1}(0)$ as zero set of a section $s : Z \to W$. Polyfolds have level structures, i.e. we have a sequence of continuous inclusions of subsets equipped with different topology $\ldots Z_{i+1} \subset Z_i \subset \ldots \subset Z_0 := Z$ and $Z_\infty := \cap_{i\in\mathbb{N}} Z_i$ is dense in Z_i for all $i \geq 0$. Then we can raise the levels up by k to get another polyfold Z^k by $(Z^k)_i := Z_{k+1}$. All the information needed to regularize $\mathcal{M}$ is encoded in Z_∞, which contains $\mathcal{M}$ and any of its regular perturbations. The level structure is required only to express in what sense the section $s : Z_\infty \to W_\infty$ is sc-smooth and sc-Fredholm. Hence forgetting the first finitely many levels is immaterial, i.e. the regularization theory for $\mathcal{M} \subset Z^k$ is independent of the shift $k \geq 0$.

The first part of this paper focuses on the construction of quotients of polyfolds, which is an analogue of the classical quotient theorem in differential topology. The quotient construction provides the infrastructure to the equivariant theories. In particular, the quotient construction allows us to carry out the Borel construction on polyfolds for equivariant theories, which is important in Part II and III. The main theorem of Part I is the following.

Theorem 1.1. *Let $p : W \to Z$ be a regular tame strong polyfold bundle (Definition 4.11), where the polyfold Z is infinite-dimensional. A compact Lie group G acts on p sc-smoothly (Definition 4.27). If the G-action on Z only has finite isotropy, then there is a G-invariant open dense set $\widehat{Z} \subset Z^2$ containing Z_∞, such that $p^{-1}(\widehat{Z})/G \to \widehat{Z}/G$ is a tame strong polyfold bundle. Moreover, the topological quotient map $\pi_G : p^{-1}(\widehat{Z}) \to p^{-1}(\widehat{Z})/G$ is covered by sc-smooth strong polyfold bundle map $\mathfrak{q} : p^{-1}(\widehat{Z}) \to p^{-1}(\widehat{Z})/G$.*

If $s : Z \to W$ is a G-equivariant proper sc-Fredholm section. Then s induces a proper sc-Fredholm section $\overline{s} : \widehat{Z}^1/G \to p^{-1}(\widehat{Z}^1)/G$ by $\mathfrak{q}^*\overline{s} = s|_{\widehat{Z}^1}$. If s is in good position to ∂Z (Definition 5.9), then so is $\overline{s}$. If s is oriented and the G-action preserves orientation, then $\overline{s}$ is orientable.

The uniqueness of polyfold structures on the quotient $p^{-1}(\widehat{Z})/G \to \widehat{Z}/G$ is addressed in Proposition 4.49. The quotient construction provides the technical foundation for the equivariant fundamental class in Theorem 1.2 and construction of equivariant Floer cohomology sketched in Chapter 17.

Using Theorem 1.1, we can study the equivariant transversality problem in polyfolds. In the case of Banach manifolds, if the group action only has finite isotropy, Cieliebak, Riera and Salamon [25] constructed an Euler class by equivariant multivalued perturbations. Since multivalued perturbation has become a part of the perturbation package in polyfold theory [65, Chapter 13], we know that if the group action only has finite isotropy then we can find equivariant multivalued transverse perturbations, see Corollary 6.1 and Corollary 6.4.

Beyond finite isotropy case, it is known that equivariant transversality is obstructed even in finite dimensional case, e.g. see [27]. In this more general case, we analyze the equivariant transversality near the fixed locus. Theorem 6.17 gives a sufficient condition which guarantees the existence of equivariant transverse perturbations near the fixed locus. In the special case of S^1-action, since the quotient polyfold can be defined outside the fixed locus, we have the following.

Corollary 6.18. *Assume $G = S^1$ and the tubular neighborhood assumption near the fixed locus without isotropy (Definition 6.9) holds. If for all weights $\lambda \in \mathbb{N}^+$ and $x \in s^{-1}(0) \cap Z^{S^1}$ we have*$\operatorname{ind} \mathrm{D}^\lambda s_x + 2 > \operatorname{ind} \mathrm{D}^{S^1}$, *then there exists a S^1-invariant neighborhood $\widehat{Z} \subset Z^3$ containing Z_∞ and an equivariant transverse sc^+-multisection perturbation on $\widehat{Z}$.*

As an example, Corollary 6.18 can be applied to Hamiltonian Floer homology with C^2 small time-independent Hamiltonians and time-independent almost complex structures. In that case, the S^1-action is given by the reparametrization in the S^1 direction of the cylinder. In particular, we can generalize Floer's proof of the weak Arnold conjecture [40] to the polyfold setting for any closed symplectic manifold.

Part I is organized as follows: Chapter 2 reviews sc-calculus. Chapter 3 constructs the quotient for M-polyfolds with free action. Chapter 4 proves the Theorem 1.1. Orientations and boundaries are discussed in Chapter 5. Chapter 6 discussed equivariant transversality when there are fixed points. Chapter 7 outline the application to Hamiltonian-Floer homology and the proof of the weak Arnold conjecture.

1.2 Equivariant fundamental class and localization

Equivariant Gromov-Witten theory was introduced by Givental in [47] for ample Kähler manifolds. When equivariant transversality holds (which is the case for ample Káhler manifolds), the moduli spaces of J-holomorphic curves are G-orbifolds. Then equivariant Gromov-Witten invariants are defined to be integrations over the moduli spaces. In this case, localization theorem for equivariant cohomology [3, 9] can be applied to compute the invariants. This method was used in [47] to verify some mirror conjectures when equivariant transversality is possible. However, equivariant transversality is often obstructed even in finite dimensional case. Therefore we can not rely on equivariant transversality for a definition equivariant symplectic Gromov-Witten theory for general symplectic manifolds.

On the other hand, what one needs for a construction of equivariant Gromov-Witten invariants is a homomorphism on the equivariant cohomology. The main result of Part II is a construction of such homomorphism for general polyfolds.

Theorem 1.2. *Let G be a compact Lie group and $p : W \to Z$ a regular tame strong polyfold strong bundle, such that Z has no boundary and is infinite dimensional. Assume G acts on p sc-smoothly and $s : Z \to W$ is a G-equivariant oriented proper sc-Fredholm section. Suppose G preserves the orientation. Then there is a $H^*(BG)$ module homomorphism for $i \geq 3$,*

$$s_* : H^*_G(Z, \tau_i) \to H^{*-k}(BG),$$

where $k := \operatorname{ind} s$. Moreover, s_* is compatible with the sequence (8.4),

$H^*_G(Z, \tau_i)$ can be understood as the equivariant cohomology of Z_∞ using the Z_i topology τ_i. The reason for $i \geq 3$ is the three levels shifted in Theorem 1.1. When s is transverse to 0, $s_*(\theta) = \int_{s^{-1}(0)} \iota^*\theta$, where $\iota : s^{-1}(0) \subset Z$ is the inclusion. Moreover, the multiplication and reduction property used in [48] holds for the equivariant fundamental class.

When Z is the Gromov-Witten polyfold constructed in [61], Z is equipped with evaluation maps $ev_1, \ldots, ev_k$ to the symplectic manifold M induced from the marked points. If compact Lie group G acts on M preserving the symplectic structure, then Z is equipped with a G-action. One can define an equivariant Gromov-Witten invariant by $I(\theta_1, \ldots, \theta_k) := s_*(ev_1^*\theta_1 \wedge \ldots \wedge ev_k^*\theta_k)$, where $\theta_i \in \mathbb{H}^*_G(M)$. This definition coincides with the equivariant Gromov-Witten invariants in [47].

In Chapter 10, we provide a localization theorem of equivariant fundamental class for torus actions under the tubular neighborhood assumption.

Theorem 1.3. *Under the tubular neighborhood assumption (Definition 6.9), then we can assume the fixed part $s|_{Z^{T^n}} : Z^{T^n} \to W^{T^n}$ is transverse. Let $\mathcal{M}^{T^n} := s|^{-1}_{Z^{T^n}}(0)$. When $n > 1$, we assume in addition Definition 10.6 holds. Then there exists an invertible element $e \in H^*(\mathcal{M}^{T^n}) \otimes \mathbb{R}(u_1, \ldots, u_n)$, such that the following localization formula holds for $i \geq 3$,*

$$s_*(\theta) = \int_{\mathcal{M}^{T^n}} \iota^*\theta \wedge e^{-1}, \quad \forall \theta \in H^*_{T^n}(Z, \tau_i),$$

where $\iota : \mathcal{M}^{T^n} \to Z$ is the inclusion.

As an example of the localization formula, we compute the equivariant curve counting in a Hirzebruch's surface with an S^1 action in Example 10.12, the non-vanishing result of the curve counting implies the equivariant transversality is obstructed in that example. We also use the localization formula to provide a proof of the Hamiltonian-Floer homology $HF^*(M)$ is isomorphic to the Morse homology $H^*(M, \Lambda)$ with Novikov field Λ coefficient in Example 10.13. This proof does not require analyzing the equivariant transversality near Morse flow lines.

Part II is organized as follows: Chapter 8 reviews the de Rham theory on polyfolds and defines equivariant cohomology for polyfolds. Theorem 1.2 is proven in Chapter 9. Chapter 10 proves the localization formula.

1.3 Morse-Bott cohomology and equivariant cohomology

In the study of open theories like Floer cohomology or Morse cohomology, the underlying polyfolds Z involved have boundary and corner. In particular, the moduli spaces $\mathcal{M} := s^{-1}(0)$ have boundary and corner even if it is regularized. When G acts Z, the construction in Theorem 1.2 does not yield a well defined map $H^*_G(Z) \to H^*(BG)$ due the appearance of boundary. Hence the invariants in the open theory is more subtle. Moreover, there is another feature of equivariant open theory, namely the setup tends to be Morse-Bott [14, 17]. For example, if one can consider a closed manifold M with a G-action, then it is possible that there is no G-invariant Morse function. However, G-invariant Morse-Bott functions exist in abundance [6]. Therefore before studying the equivariant theory, we need to set up a framework for Morse-Bott theory.

Morse-Bott functions were introduced by Bott in [11] as generalizations of Morse functions and proven to be extremely useful for studying spaces in the presence of symmetries [10, 12]. After Witten's work [105] on the construction of Morse cochain complex and Floer's generalization [38, 39, 40, 41] to various infinite dimensional settings, attention was drawn to the construction of a cochain complex for the Morse-Bott counterpart. Since there are several benefits of working in the Morse-Bott case: Morse-Bott functions usually reflect some extra symmetries of the problem; computations in Morse-Bott theory are usually simpler because of the extra symmetries; Morse-Bott theory appears in equivariant theory.

In the finite dimensional case, there are several methods one can apply to get a chain resp. cochain complex for a Morse-Bott function, we will review some of them briefly in Chapter 11:

- Austin-Braam's model [6]: The cochain complex is generated by differential forms of the critical manifolds, and the differential is defined by pullback and pushforward of differential forms through the moduli spaces of gradient flow lines.

- Fukaya's model [44]: The chain complex is generated by certain singular chain complex of the critical manifolds, and the differential is defined by pushforward and pullback of singular chains through the moduli spaces of gradient flow lines.

- Cascades [14, 43]: The cochain complex is generated by Morse cochain complexes of critical manifolds after we assign appropriate Morse functions to each critical manifold. The differential is defined by counting the "cascades".

- Latschev's model [70]: This can be viewed as a generalization of Harvey and Lawson's work [55] on Morse theory.

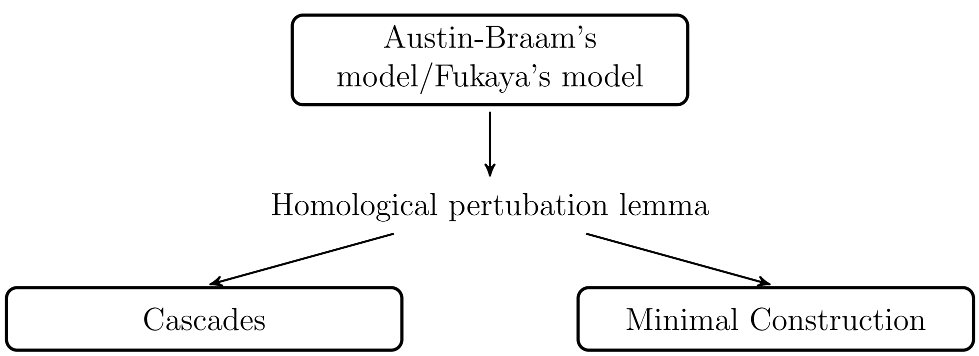
Under various conditions, the first three models can be generalized to infinite dimensional case like Floer theories. In fact, the first three models can be transplanted to the abstract setting of flow categories, where flow category was first introduced by Cohen, Jones and Segal in [26] to organize all the moduli spaces of flow lines in Morse and Floer theory. Roughly speaking, the objects of a flow category usually come from critical points and the morphisms come from gradient flow lines. In Part III, we work with flow categories under the minimal transversality requirement, i.e. the fiber products of moduli spaces are transverse[1]. The aims of Part III are:

- Provide another construction of Morse-Bott cochain complex, applicable to polyfold theory, called the minimal Morse-Bott cochain complex. It is minimal in the sense that the cochain complex is generated by the cohomology of the critical manifolds.
- Study the relations between Austin-Braam's model, Fukaya's model, cascades and our minimal construction.
- Provide equivariant counterpart of the minimal construction.

The first Theorem of Part III is the following, which is rigorously stated in Theorem 12.7

Theorem 1.4. *To an oriented flow category, we assign a minimal Morse-Bott cochain complex (BC, d_{BC}) over $\mathbb{R}$ generated by the cohomology of the critical manifolds.*

To illustrate that the Theorem 1.4 is nontrivial, we point out that: (1) when the flow category arises from a Morse-Bott function on a closed manifold, the cohomology of the minimal Morse-Bott cochain complex is the cohomology of the base manifold, see Chapter 16; (2) when the flow category arises from a Morse case, i.e. critical points are isolated, the cochain complex is the usual cochain complex with differential defined by counting rigid flow lines, see Remark 12.9. To motivate the minimal construction, we study the relations between Austin-Braam's model and cascades. It turns out both cascades and our minimal construction can be viewed as finite dimensional reductions of Austin-Braam's model using homological perturbation theory. In applications of homological perturbation theory, one needs to choose a projection and a homotopy. For cascades, the projection and homotopy are provided by Harvey and Lawson's work [55] on Morse theory. While the minimal construction is based on a more direct construction, which is closely related to the Hodge decomposition.



[1] Unlike the fiberation assumption in Austin-Braam's model, this is generic among perturbations without changing the Morse-Bott functional.

When there is a G-symmetry on the Morse-Bott theory, the cohomology theory can be enriched to a G-equivariant theory. One typical method is approximating the homotopy quotient. From this perspective, cascades construction does not work very well with equivariant theory, one of the reasons is that we have to choose nice Morse functions on the approximations. However, the minimal construction works very well with Borel's construction for equivariant cohomology. Then an equivariant cochain complex can be realized as a homotopy limit. In particular, we have the following theorem, which is rigorously stated in Theorem 15.16.

Theorem 1.5. *Let compact Lie group G acts on an oriented flow category $\mathcal{C}$ and preserves the orientations. Then there is a cochain complex (BC_G, d^G_{BC}). The homotopy type of the cochain complex is unique, i.e. independent of all the choices we made along the construction, in particular, the choice of finite dimensional approximations of the classifying space $EG \to BG$.*

Thus the only problem of applying the minimal construction is constructing a flow category, which is essentially a problem of regularizing moduli spaces. For this purpose, we will apply the polyfold theory and our minimal construction is applicable to polyfold theory. We can enrich a flow category, i.e. a system of manifolds to a system of polyfold bundles with sc-Fredholm sections, such that the boundaries and corners of the polyfolds come from transverse fiber products of polyfolds. We will refer to such system as a polyflow category. Using the perturbation theory in [65], we can find coherent perturbations such that zero sets of the sc-Fredholm sections form a flow category. In the presence of a group action, Theorem 1.5 requires G-equivariant transversality, which is often obstructed. However, the whole construction for Theorem 1.5 can be lifted to polyfold category using the quotient Theorem 1.1. We outline such construction in Chapter 17 and details will appear in our future work.

Our construction of Morse-Bott cochain complex can be generalized to other structures in Morse-Bott cases, as long as the all the relevant moduli spaces satisfy the fiber products transversality assumption. The desired algebraic relation can be derived in a similar way as the proofs presented in this paper. In particular, one can define A_∞ version of flow categories. Following a similar construction, we can get A_∞ algebras and categories out of it. We will discuss this in detail in our future work.

Part III is organized as follows: Chapter 11 discusses the motivation of the minimal construction from homological perturbation theory and interprets cascades construction as an example of the application of homological perturbation theory. Chapter 12 defines the minimal cochain complex as well as continuation maps and homotopies explicitly, and proves that they satisfy the desired properties. Chapter 13 discusses the action spectral sequence. Chapter 14 explains how the orientations used in Chapter 12 arise in Morse/Floer theory. Chapter 14 also generalizes the construction to the case with local systems. Chapter 14 also provides a more general construction which allows us to prove statements like the Gysin exact sequence. Chapter 15 is about the equivariant theory. Chapter 16 is devoted to the case of Morse-Bott functions on closed manifolds and proves our minimal construction recovers the cohomology of the base manifold. Chapter 17 outlines the project of combining our construction with polyfold theory.

1.4 Vanishing of symplectic homology

Invariants like (equivariant) Gromov-Witten invariants and (equivariant) Floer homology is very useful in the study of symplectic geometry. The last part of this paper is devoted to symplectic fillings of contact manifolds using symplectic homology, which is a special form of Floer homology. The study of the topology of fillings of a given contact manifold was initiated by Gromov and Eliashberg in the 1980s. A combination of results in [52, 30] shows that the Weinstein filling of the standard contact 3-sphere (S^3, ξ_{std}) is unique up to symplectic deformation. In general, one can ask in what case contact manifolds can determine their Liouville fillings. For the topology of the fillings, Eliashberg-Floer-McDuff [79] proved that any Liouville filling of (S^{2n-1}, ξ_{std}) is diffeomorphic to D^{2n}. Barth-Geiges-Zehmisch [7] generalized the Eliashberg-Floer-McDuff theorem to the subcritical case: all Liouville fillings of a subcritically-fillable, simply-connected contact manifold are diffeomorphic to each other. Regarding the symplectic topology of fillings, Seidel and Smith [97] proved that any Liouville filling of (S^{2n-1}, ξ_{std}) has vanishing symplectic homology, which provides evidence that the filling of (S^{2n-1}, ξ_{std}) might be unique. In this note, we more generally consider asymptotically dynamically convex contact manifolds (Definition 18.5) and their topologically simple Liouville fillings (Definition 18.2). We prove the following theorem in Section 18.2.

Theorem 1.6. *Let (Y, ξ) be an asymptotically dynamically convex contact manifold of dimension $2n-1 \geq 5$. Assume Y has a Liouville filling W that is topologically simple (i.e. $c_1(W) = 0$ and $\pi_1(Y) \to \pi_1(W)$ is injective.) and has $SH_*(W;\mathbb{Z}) = 0$. Then $SH_*(W';\mathbb{Z}) = 0$ for every topologically simple Liouville filling W'.*

Note that all the Liouville fillings of (S^{2n-1}, ξ_{std}) are diffeomorphic to D^{2n} by [79], hence all Liouville fillings of (S^{2n-1}, ξ_{std}) are topologically simple. Since (S^{2n-1}, ξ_{std}) is asymptotically dynamically convex and $SH_*(D^{2n};\mathbb{Z}) = 0$ for the standard filling (D^{2n}, ω_{std}). Thus Theorem 1.6 reproves the Seidel-Smith theorem [97].

Recently, Lazarev [71] showed that all flexible fillings of a contact manifold have the same integral cohomology. He also raised the following question.

Question 1.7 ([71, §8 Problem 1])**.** *If (Y, ξ) has a flexible filling W, do all Liouville fillings of (Y, ξ) have the same cohomology as W?*

We answer Question 1.7 partially in Corollary 18.13: All topologically simple Liouville fillings of Y, in particular, all Weinstein fillings, have the same integral cohomology as W. Moreover, they all have vanishing symplectic homology. Thus corollary 11.2 provides evidence towards the following question.

Question 1.8. *If (Y, ξ) has a flexible filling W, is the Liouville filling of Y unique?*

Theorem 1.6 also provides an obstruction to fillability by flexible Weinstein domains. We use this to prove that the Brieskorn manifolds of dimension greater or equal to 5 cannot be filled by flexible Weinstein domains. We also consider the exotic contact structures on S^{2n-1}, which were first constructed by Eliashberg [31], and were intensively studied in [46, 28, 101, 100]. Lazarev [71, Corollary 1.12] proved that for any sphere S^{2n-1} of dimension $2n-1 \geq 5$, there exist infinitely many different contact structures for the standard almost contact structure. The contact structures Lazarev constructed can be filled by flexible Weinstein domains. However, the exotic contact structures on $S^{4m+1}, S^7, S^{11}, S^{15}$ constructed in [101, 100] can not be filled with flexible Weinstein domains [71, Corollary 1.12]. So a natural question is whether we can find infinitely many contact structures on S^{2n-1} for a fixed almost contact structure, such that they can not be filled by flexible Weinstein domains. We answer this question affirmatively in theorem 18.18 for a large class of almost contact structures by a modification of Lazarev's construction.

Part I

Quotient Theorems in Polyfold Theory and Equivariant Transversality

Chapter 2

Basics of Sc-Calculus

Sc-calculus was introduced in [58] as a generalization of the classical calculus on Banach spaces. It is a weaker notion of differentiability such that reparametrization becomes smooth in sc-calculus, see Example 2.8. Moreover, the dimension jump phenomenon in the gluing analysis of J-holomorphic curves [80] can be described by sc-calculus in a smooth way, see [61, 35]. In this chapter, we refer some basic notions and properties of sc-calculus from [65] that will be used in this paper.

2.1 Sc-differentiability

Definition 2.1 ([65, Definition 1.1])**.** *A **sc-Banach space** $\mathbb{E}$ consists a decreasing sequence of Banach spaces,*

$$\mathbb{E}_0 \supset \mathbb{E}_1 \supset \dots,$$

such that the following two conditions are satisfied,

1. *the inclusion $\mathbb{E}_{i+1} \hookrightarrow \mathbb{E}_i$ is compact,*
2. *the intersection $\mathbb{E}_\infty = \cap_{i\geq 0}\mathbb{E}_i$ is dense in every $\mathbb{E}_m$.*

Given a sc-Banach space $\mathbb{E}$, we can shift the levels up by k to get another sc-Banach space $\mathbb{E}^k$, that is $(\mathbb{E}^k)_i := \mathbb{E}_{k+i}$. Note that if $\dim \mathbb{E}_i < \infty$ for some $i \in \mathbb{N}$, then the dense condition implies that $\mathbb{E}_0 = \mathbb{E}_1 = \ldots = \mathbb{R}^n$. As a consequence, we point here that sc-calculus reviewed here, M-polyfolds and polyfolds reviewed in Chapter 3 and 4 are the regular calculus, manifolds and orbifolds respectively, when the underlying sc-Banach spaces are finite dimensional.

Example 2.2. *The major examples are sequences of Sobolev spaces, e.g.$\mathbb{E} := H^0(S^1) \supset H^1(S^1) \supset \ldots$ and $\mathbb{F} := \mathrm{C}^0(S^1) \supset \mathrm{C}^1(S^1) \supset \ldots$. Then $\mathbb{E}_\infty = \mathbb{F}_\infty = \mathrm{C}^\infty(S^1)$, hence we will refer to points in the infinite level $\mathbb{E}_\infty$ as smooth points. On noncompact manifolds, to ensure the the inclusion between levels are compact, we need to consider Sobolev space* *with weights. For example,* $\mathbb{H} := H^{0,\delta_0}(\mathbb{R}) \supset H^{1,\delta_1}(\mathbb{R}) \subset \ldots$, *where* $0 = \delta_0 < \delta_1 < \ldots$ *and* $|f|^2_{H^{m,\delta_m}} := \sum_{0\le i\le m} \int e^{\delta_m |x|}(\mathrm{D}^i f(x))^2 \mathrm{d}x$.

Definition 2.3 ([65, Definition 1.2])**.** *A linear operator* $T : \mathbb{E} \to \mathbb{F}$ *is called a* ***sc-operator*** *if* $T : \mathbb{E}_0 \to \mathbb{F}_0$ *is linear,* $T(\mathbb{E}_m) \subset \mathbb{F}_m$ *and* $T : \mathbb{E}_m \to \mathbb{F}_m$ *is continuous.*

A sc-subspace $\mathbb{F} \subset \mathbb{E}$ is a sc-space $\mathbb{F}$ such that $\mathbb{F}_i \subset \mathbb{E}_i$ are closed subspace and $\mathbb{F}_i = \mathbb{F}_0 \cap \mathbb{E}_i$. A sc-subspace $\mathbb{F} \subset \mathbb{E}$ has a complement $\mathbb{G}$, if $\mathbb{G} \subset \mathbb{E}$ is a sc-subspace and on every level m we have the topological direct sum $\mathbb{E}_m = \mathbb{F}_m \oplus \mathbb{G}_m$. The following propositions asserts the existence of complement for finite dimensional subspace inside the infinite level.

Proposition 2.4 ([65, Proposition 1.1])**.** *A finite dimensional space* F *is a sc-subspace of a sc-Banach space* $\mathbb{E}$ *iff* $F \subset \mathbb{E}_\infty$*, and a finite dimensional sc-subspace always has a complement.*

An open set $U \subset \mathbb{E}$ of a sc-Banach space is open set $U \subset \mathbb{E}_0$, then it has a filtration $\{U_m := U \cap \mathbb{E}_m\}_{m\ge 0}$ and U_m is an open subset of $\mathbb{E}_m$. Moreover, we can shift levels up k to get an open set $U^k \subset \mathbb{E}^k$.

Definition 2.5 ([65, Definition 1.7])**.** *Let* $U \subset \mathbb{E}$ *and* $V \subset \mathbb{F}$ *be two open subsets, then a map* $f : U \to V$ *is* $\mathbf{sc^0}$ *or* ***sc-continuous*** *iff* $f(U_m) \subset V_m$ *and the induced maps* $f : U_m \to V_m$ *are continuous for all* m*.*

For an open set $U \subset \mathbb{E}$, the tangent space TU of U is defined to be $U^1 \oplus \mathbb{E}$, see [65, Definition 1.8]. In particular, $TU \subset T\mathbb{E} = \mathbb{E}^1 \oplus \mathbb{E}$ is an open subset. The reason of the level shift in the base is the following definition.

Definition 2.6 ([65, Definition 1.8])**.** *Let* $U \subset \mathbb{E}$ *and* $V \subset \mathbb{F}$ *be two open subsets, then a map* $f : U \to V$ *is* $\mathbf{sc^1}$ *provided the following conditions hold.*

1. f *is* sc^0.

2. *For every* $x \in U_1$*, there exists a bounded linear operator* $\mathrm{D}f_x : \mathbb{E}_0 \to \mathbb{F}_0$ *such that for* $h \in \mathbb{E}_1$ *with* $x + h \in U_1$*,*

$$\lim_{|h|_1 \to 0} \frac{|f(x+h) - f(x) - \mathrm{D}f_x h|_0}{|h|_1} = 0.$$

3. *The tangent map* $Tf : TU \to TV$ *defined by*

$$Tf(x,h) = (f(x), \mathrm{D}f_x h), \quad x \in U_1, h \in \mathbb{E}_0,$$

is sc^0.

Remark 2.7. *The map* $x \mapsto \mathrm{D}_x f$ *is usually not continuous using the norm topology for a* sc^1 *map.*

A map $f : U \to V$ is sc^k iff $Tf : TU \to TV$ is sc^{k-1}. A map f is sc^∞ if f is sc^k for all $k \geq 0$. One of the key examples of sc^∞ maps is the action by reparametrization.

Example 2.8. *Let $\mathbb{F} := \mathrm{C}^0(S^1) \supset \mathrm{C}^1(S^1) \supset \ldots$. We can define the map $f : \mathbb{R} \oplus \mathbb{F} \to \mathbb{F}$ by $(t, h) \mapsto h(t + \cdot)$. Then f is sc^∞ but not C^1 on any level, see [35, Section 2.2] for more details.*

Another important example of sc-smooth map is the following retraction, which is closely related to the gluing construction in Gromov-Witten theory and Floer theories.

Example 2.9 ([64, Example 1.22])**.** *Let the sc-Hilbert space $\mathbb{E}_m := H^m_{\delta_m}(\mathbb{R})$, where δ_m is the exponential decay weight. Choose a smooth compactly supported positive function β, such that $\int_\mathbb{R} \beta^2 = 1$. Define $r_t : \mathbb{E} \to \mathbb{E}$ to be:*

$$r_t(f) = \begin{cases} 0 & t \leq 0 \\ \int_\mathbb{R} f(x)\beta(x + e^{\frac{1}{t}})dx \cdot \beta(x + e^{\frac{1}{t}}) & t > 0 \end{cases}$$

It was checked in [64] that $\pi := (t, r_t) : \mathbb{R} \times \mathbb{E} \to \mathbb{R} \times \mathbb{E}$ is sc-smooth. Topologically, $\operatorname{im} \pi$ is the union of the half line $(-\infty, 0]$ with the open half plane $(0, \infty) \times \mathbb{R}$.

Note that π in Example 2.9 is a retraction, i.e. $\pi \circ \pi = \pi$. In contrast to the classical calculus on Banach space, where the image of a smooth retraction will be a sub-Banach manifold [20], here the image of the retraction $\operatorname{im} \pi$ does not posses a usual (Banach) manifold structure, in particular, the dimension jumps. However, $\operatorname{im} \pi$ is a smooth object in polyfold theory, see Chapter 3.

2.2 Properties of Sc-Calculus

Proposition 2.10 ([65, Proposition 1.6])**.** *Let U, V be two open subsets of sc-Banach spaces and $f : U \to V$ a sc^k map. Then $f : U^m \to V^m$ is also sc^k for all $m \geq 0$.*

Proposition 2.11 ([65, Proposition 1.7])**.** *Let U, V be two open subsets of sc-Banach spaces and $f : U \to V$ a sc^k map. Then for every $m \geq 0$, the induced maps $f : U_{m+k} \to V_m$ is of class C^k. Moreover, $f : U_{m+l} \to V_m$ is of class C^l for $0 \leq l \leq k$.*

Proposition 2.12 ([65, Proposition 1.8])**.** *Let U, V be two open subsets of sc-Banach spaces and $f : U \to V$ a map preserving the sc-structure, i.e. $f(U_m) \subset V_m$. Assume for every $m \geq 0$ and $0 \leq l \leq k$, $f : U_{m+l} \to V_m$ is of class C^{l+1}. Then f is sc^{k+1}.*

In the case of the target $\mathbb{F} = \mathbb{R}^n$, the Proposition 2.12 has the following form.

Corollary 2.13 ([65, Corollary 1.1])**.** *Let U be an open subset of a sc-Banach space and $f : U \to \mathbb{R}^n$ a map. Assume for some k and all $0 \leq l \leq k$, $f : U_l \to \mathbb{R}^n$ is of class C^{l+1}, then f is sc^k.*

One of the most important fact about the sc-calculus is that chain rule holds.

Theorem 2.14 ([65, Theorem 1.1])**.** *Let U, V, W be three open subsets of sc-Banach spaces and $f : U \to V, g : V \to W$ two sc^1 maps. Then the composition $g \circ f : U \to W$ is sc^1. Moreover, $T(g \circ f) = Tg \circ Tf$.*

The chain rule provides a foundation towards a manifold resp. orbifold theory using sc-calculus, which will be reviewed in the next chapter.

Chapter 3

Free Quotients of M-polyfolds

M-polyfolds were introduced by Hofer, Wysocki and Zehnder [65, Definition 2.8] as a generalization of manifolds. In this section we prove that the quotient of a tame M-polyfold by a free group action is still a tame M-polyfold and an equivariant sc-Fredholm section descends to a sc-Fredholm section on the quotient.

Theorem 3.1. *Let $\mathcal{X}$ be an infinite-dimensional tame M-polyfold (Definition 3.10), $p : \mathcal{Y} \to \mathcal{X}$ a tame strong M-polyfold bundle (Definition 3.12) and $s : \mathcal{X} \to \mathcal{Y}$ a proper sc-Fredholm section (Definition 3.35). Assume a compact Lie group G acts on the tame strong M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$ sc-smoothly (Definition 3.14) such that the induced action $\rho_{\mathcal{X}}$ on the base $\mathcal{X}$ is free and the section s is G-equivariant. Then the following holds.*

1. *There exists a G-invariant open set $\widehat{\mathcal{X}} \subset \mathcal{X}^2$ containing $\mathcal{X}_\infty$, such that $\overline{p} : p^{-1}(\widehat{\mathcal{X}})/G \to \widehat{\mathcal{X}}/G$ is a tame strong M-polyfold bundle and the quotient map $\pi : p^{-1}(\widehat{\mathcal{X}}) \to p^{-1}(\widehat{\mathcal{X}})/G$ is a sc-smooth strong bundle map.*

2. *s induces a proper sc-Fredholm section $\overline{s} : \widehat{\mathcal{X}}^1/G \to p^{-1}(\widehat{\mathcal{X}}^1)/G$ by $\pi^*\overline{s} = s|_{\widehat{\mathcal{X}}^1}$.*

Part (1) of Theorem 3.1 is proven in Section 3.1 and part (2) is proven in Section 3.2. The uniqueness of the M-polyfold structure is addressed in Proposition 3.30.

Remark 3.2. *A few remarks on the level shift are in order.*

- *Like polyfolds, M-polyfolds have level structures $\mathcal{X}_\infty \subset \ldots \mathcal{X}_1 \subset \mathcal{X}_0 = \mathcal{X}$, such that $\mathcal{X}_\infty$ is dense in every level $\mathcal{X}_i$. We can shift levels up by k to get $\mathcal{X}^k$, i.e. $(\mathcal{X}^k)_i := \mathcal{X}_{k+i}$. All the information needed to regularize $s^{-1}(0)$ is encoded in $\mathcal{X}_\infty$. In particular, $s^{-1}(0)$ and any of its regularizing perturbations are contained in $\mathcal{X}_\infty$.*

- *The two levels shifted in part (1) of Theorem 3.1 come from the construction of the slices to the group action, c.f. Lemma 3.27. The extra level shift in part (2) is from Lemma 3.40.*

- *The level shift in Theorem 3.1 seems to be necessary. For example, we have the sc-Banach space $\mathbb{E}$ of continuous functions on S^1, i.e. $\mathbb{E}_i := \mathrm{C}^i(S^1)$. Then we have a sc-smooth S^1-action on $\mathbb{E}$ by $\theta \cdot f := f(\theta + \cdot)$, for $f \in \mathbb{E}$ and $\theta \in S^1$. Let $\mathcal{X} := \{f \in \mathbb{E} | f(\theta + \cdot) \neq f, \forall \theta \in S^1\}$. Then $\mathcal{X}$ is an open subset of $\mathbb{E}$, hence a M-polyfold. The S^1-action restricted to $\mathcal{X}$ is free. Then we can give the quotient $\mathcal{X}^1/S^1$ a M-polyfold structure, since we need C^1-differentiability to write down the local slice conditions, see [35, Section 2.2, 4.3] for details. It is not clear whether one can give $\mathcal{X}/S^1$ a M-polyfold structure.*

- *It is not clear to us whether one can construct the quotient bundle and section by shifting only one level. Since in applications, the invariants are derived from the zero set $s^{-1}(0)$, which is contained in $\mathcal{X}_\infty$. Therefore, shifting one level and three levels do not make an essential difference.*

Remark 3.3. *The infinite dimensional condition is only used in part (2) of Theorem 3.1, see Proposition 3.42.*

3.1 Free quotients of M-polyfolds and strong M-polyfold bundles

If a compact Lie group G acts on a finite dimensional manifold M freely, then the quotient M/G is a smooth manifold, see e.g. [72]. This section proves the analogue for M-polyfolds and M-polyfold bundles. The proof also provides a local prototype for our main theorem on polyfolds.

M-polyfolds and M-polyfold bundles

This section reviews some definitions from [65] that will be crucial for our construction. Let $\mathbb{R}_+ := [0, \infty)$. We begin with the local models that generalize open subsets of $\mathbb{R}^m_+$ for manifolds with boundary and corner. A partial quadrant is defined to be $\mathbb{R}^m_+ \times \mathbb{E}$. For every $(r, e) \in \mathbb{R}^m_+ \times \mathbb{E}$, the degeneracy index $d : \mathbb{R}^m_+ \times \mathbb{E} \to \mathbb{N}$ is defined to be

$$d(r, e) := \# \{i \in \{1, \ldots, m\} | \text{the } i\text{-th coordinate of } r \text{ is zero.}\}.$$

For every $x \in \mathbb{R}^m_+ \times \mathbb{E}$, we can define the minimal linear subspace $(\mathbb{R}^m_+ \times \mathbb{E})_x \subset \mathbb{R}^m \times \mathbb{E}$ [65, Definition 2.16] as follows: If $x = (r_1, \ldots, r_m, e) \in \mathbb{R}^m_+ \times \mathbb{E}$, then

$$(\mathbb{R}^m_+ \times \mathbb{E})_x := \{(v_1, \ldots, v_m, f) | v_i = 0 \text{ if } r_i = 0\} \subset \mathbb{R}^m \times \mathbb{E}. \tag{3.1}$$

This can be understood as the tangent space of the intersection of all the faces containing x, i.e. the tangent space of the corner of degeneracy index $d(x)$.

Remark 3.4. *The original definition of partial quadrant [65, Definition 1.6] is a closed convex subset $C \subset \mathbb{E}'$, such that there exists another sc-Banach space $\mathbb{E}$ and a linear sc-isomorphism $\Psi : \mathbb{E}' \to \mathbb{R}^m \times \mathbb{E}$ satisfying $\Psi(C) = \mathbb{R}^m_+ \times \mathbb{E}$. Instead of including Ψ in the discussion, we will work with the standard model $\mathbb{R}^m_+ \times \mathbb{E}$ to simplify notation.*

Definition 3.5 ([65, Definition 2.17]). *Let U be an open subset of $\mathbb{R}^m_+ \times \mathbb{E}$. A sc^∞ map $r : U \to U$ is called a **tame sc-retraction** if the following conditions hold:*

- *$r \circ r = r$;*
- *$d(r(x)) = d(x)$ for all $x \in U$;*
- *at every x in $r(U)_\infty := r(U) \cap (\mathbb{R}^m_+ \times \mathbb{E})_\infty$, there exists a sc-subspace $\mathbb{A} \subset (\mathbb{R}^m_+ \times \mathbb{E})_x$, such that $\mathbb{R}^m \times \mathbb{E} = \mathrm{D}r_x(\mathbb{R}^m \times \mathbb{E}) \oplus \mathbb{A}$.*

*A pair $(O, \mathbb{R}^m_+ \times \mathbb{E})$ is called a **tame sc-retract** if there exists a tame sc-retraction r on an open subset $U \subset \mathbb{R}^m_+ \times \mathbb{E}$, such that $r(U) = O$.*

Remark 3.6. *In [65], a sc-retract is a tuple $(O, C, \mathbb{E}')$ with $C \subset \mathbb{E}'$ a partial quadrant. Since we fix the form of partial quadrants throughout this paper, we have simplified the notation for sc-retracts to $(O, C = \mathbb{R}^m_+ \times \mathbb{E})$.*

The notion of smoothness for maps between open subsets of $\mathbb{R}^n$ is generalized by polyfold theory in two ways: First, sc-smoothness for maps between open subset of sc-Banach spaces is defined in [65, Definition 1.9]. Second, for maps between sc-retracts, sc-smoothness is defined as follows.

Definition 3.7 ([65, Definition 2.4]). *Let $(O, \mathbb{R}^m_+ \times \mathbb{E})$ and $(O', \mathbb{R}^{m'}_+ \times \mathbb{E}')$ be two tame sc-retracts. A map $f : O \to O'$ is **sc-smooth** if $f \circ r : \mathbb{R}^m_+ \times \mathbb{E} \supset U \to \mathbb{R}^{m'}_+ \times \mathbb{E}'$ is sc-smooth, where r is a sc-retraction for $(O, \mathbb{R}^m_+ \times \mathbb{E})$ and U is an open subset of $\mathbb{R}^m_+ \times \mathbb{E}$ such that $r(U) = O$.*

This notion is well-defined by [65, Proposition 2.3], i.e. the definition does not depend on the choice of sc-retraction r and open set U.

Definition 3.8 ([65, Definition 2.8, 2.19]). *Let $\mathcal{X}$ be a topological space. A **tame M-polyfold chart** for $\mathcal{X}$ is a triple $(\mathcal{O}, \phi, (O, \mathbb{R}^m_+ \times \mathbb{E}))$, such that*

- *$(O, \mathbb{R}^m_+ \times \mathbb{E})$ is a tame sc-retract;*
- *$\phi : \mathcal{O} \to O$ is a homeomorphism from an open subset $\mathcal{O} \subset \mathcal{X}$.*

*Two tame M-polyfold charts $(\mathcal{O}, \phi, (O, \mathbb{R}^m_+ \times \mathbb{E}))$ and $(\mathcal{O}', \phi', (O', \mathbb{R}^{m'}_+ \times \mathbb{E}'))$ are compatible if $\phi' \circ \phi^{-1}$ resp. $\phi \circ \phi'^{-1}$ are sc-smooth map from $\phi(\mathcal{O} \cap \mathcal{O}')$ to $\phi'(\mathcal{O} \cap \mathcal{O}')$ resp. from $\phi'(\mathcal{O} \cap \mathcal{O}')$ to $\phi(\mathcal{O} \cap \mathcal{O}')$ in the sense of Definition 3.7. An atlas is a covering by compatible charts. A **tame M-polyfold structure** on $\mathcal{X}$ is a maximal atlas of tame M-polyfold charts for $\mathcal{X}$. A **tame M-polyfold** $\mathcal{X}$ is a paracompact Hausdorff space with a tame M-polyfold structure.*

The following remark defines the notion of the tangent spaces of M-polyfolds.

Remark 3.9. *The tangent space of a partial quadrant $T(\mathbb{R}^m_+ \times \mathbb{E})$ [65, Definition 1.8] is defined to be $(\mathbb{R}^m_+ \times \mathbb{E})^1 \times (\mathbb{R}^m \times \mathbb{E})$. The tangent space of a sc-retract $(O, \mathbb{R}^m_+ \times \mathbb{E})$ is defined to be the sc-retract $(TO, T(\mathbb{R}^m_+ \times \mathbb{E}))$, where $TO = Tr(TU)$ is the image of the tangent map $Tr : TU \to TU, (x, e) \mapsto (x, \mathrm{D}r_x(e))$ for any choice of retraction $r : U \to U$ with $r(U) = O$. The tangent space of a M-polyfold $\mathcal{X}$ is a M-polyfold $T\mathcal{X}$ [65, Proposition 2.5] with charts $(T\mathcal{O}, T\phi, (TO, T(\mathbb{R}^m_+ \times \mathbb{E})))$. Then the projection $\pi : T\mathcal{X} \to \mathcal{X}^1$ defines a M-polyfold bundle in the sense of Definition 3.11.*

Definition 3.10. *We say a tame M-polyfold $\mathcal{X}$ is infinite dimensional if for every $x \in \mathcal{X}_\infty$, the dimension of the tangent space $T_x\mathcal{X}$ is infinite.*

M-Polyfolds in all known applications [61, 62, 102, 74] are infinite dimensional and tame. Next we review the notion of (strong) M-polyfold bundles, which generalizes the notion of vector bundles over manifolds.

Definition 3.11. *Let $\mathcal{X}$ be a tame M-polyfold, $\mathcal{Y}$ a paracompact Hausdorff space and $p : \mathcal{Y} \to \mathcal{X}$ a surjection with $p^{-1}(x)$ a vector space for every $x \in \mathcal{X}$. A* ***tame bundle chart*** *for the bundle $p : \mathcal{Y} \to \mathcal{X}$ is a tuple $(\mathcal{O}, \Phi, (K, (\mathbb{R}^m_+ \times \mathbb{E}) \times \mathbb{F}))$ such that the following holds:*

- *there exists an open subset $U \subset \mathbb{R}^m_+ \times \mathbb{E}$ and a sc-smooth retraction $R : U \times \mathbb{F} \to U \times \mathbb{F}, (x, f) \mapsto (r(x), \varrho(x)f)$, where $\varrho(x)$ is a linear map from $\mathbb{F}$ to $\mathbb{F}$ and r is a tame retraction;*

- *there is homeomorphism $\phi : \mathcal{O} \to r(U)$ such that $(\mathcal{O}, \phi, (r(U), \mathbb{R}^m_+ \times \mathbb{E}))$ is compatible chart for the M-polyfold $\mathcal{X}$.*

- *$\Phi : p^{-1}(\mathcal{O}) \to K := R(U \times \mathbb{F})$ is bundle isomorphism, i.e. Φ is a homeomorphism such that $\pi \circ \Phi = \phi \circ p$ and $\Phi_x : p^{-1}(x) \to \operatorname{im} \varrho(\phi(x))$ is a linear isomorphism, where $\pi : r(U) \times \mathbb{F} \supset K \to r(U)$ is the projection.*

Two tame bundle charts $(\mathcal{O}, \Phi, (K, (\mathbb{R}^m_+ \times \mathbb{E}) \times \mathbb{F}))$ and $(\mathcal{O}', \Phi', (K', (\mathbb{R}^{m'}_+ \times \mathbb{E}') \times \mathbb{F}'))$ are compatible iff the transition maps $\Phi' \circ \Phi^{-1} : \Phi(p^{-1}(\mathcal{O} \cap \mathcal{O}')) \to \Phi'(p^{-1}(\mathcal{O} \cap \mathcal{O}'))$ and $\Phi \circ \Phi'^{-1} : \Phi'(p^{-1}(\mathcal{O} \cap \mathcal{O}')) \to \Phi(p^{-1}(\mathcal{O} \cap \mathcal{O}'))$ are sc-smooth. Then $p : \mathcal{Y} \to \mathcal{X}$ is a ***tame M-polyfold bundle*** *iff it is equipped with a strong bundle atlas.*

Since fibers of M-polyfold bundles tend to be infinite dimensional vector spaces in applications, it will be necessary to have an extra "strong" structure on the bundle, which will be used to formulate the notion of compact perturbation of a Fredholm section. In polyfold theory, it uses the following filtrations. Let $U \subset \mathbb{R}^m_+ \times \mathbb{E}$ be a open subset and $\mathbb{F}$ another sc-Banach space. We define the non-symmetric product $U \triangleleft F$ to be the set $U \times \mathbb{F}$ with extra structures of $(U \triangleleft \mathbb{F})[i]$, which are defined to be the filtrations $((U \triangleleft \mathbb{F})[i])_m := U_m \oplus F_{m+i}$. In particular, $(U \triangleleft \mathbb{F})[i]$ is an open subset of the partial quadrant $((\mathbb{R}^m_+ \times \mathbb{E}) \triangleleft \mathbb{F})[i]$.

Definition 3.12 ([65, Definition 2.26])**.** *Let $\mathcal{X}$ be a tame M-polyfold, $\mathcal{Y}$ a paracompact Hausdorff space and $p : \mathcal{Y} \to \mathcal{X}$ a surjection with $p^{-1}(x)$ a vector space for every $x \in \mathcal{X}$. A* ***tame strong bundle chart*** *for the bundle $p : \mathcal{Y} \to \mathcal{X}$ is a tuple $(\mathcal{O}, \Phi, (K, (\mathbb{R}^m_+ \times \mathbb{E}) \triangleleft \mathbb{F}))$ such that the following holds:*

- *there exists an open subset $U \subset \mathbb{R}^m_+ \times \mathbb{E}$ and a map $R : U \triangleleft \mathbb{F} \to U \triangleleft \mathbb{F}, (x, f) \mapsto (r(x), \varrho(x)f)$ satisfying $R \circ R = R$, where $\varrho(x)$ is a linear map from $\mathbb{F}$ to $\mathbb{F}$ and r is a tame retraction;*
- *$R[i] := R|_{(U \triangleleft \mathbb{F})[i]} : (U \triangleleft \mathbb{F})[i] \to (U \triangleleft \mathbb{F})[i]$ is sc-smooth for $i = 0, 1$;*
- *there is homeomorphism $\phi : \mathcal{O} \to r(U)$ such that $(\mathcal{O}, \phi, (r(U), \mathbb{R}^m_+ \times \mathbb{E}))$ is compatible chart for the M-polyfold $\mathcal{X}$.*
- *$\Phi : p^{-1}(\mathcal{O}) \to K := R((U \triangleleft \mathbb{F})[0])$ is bundle isomorphism covering ϕ.*

Two strong bundle charts $(\mathcal{V}, \Phi, (K, (\mathbb{R}^m_+ \times \mathbb{E}) \triangleleft \mathbb{F}))$ and $(\mathcal{V}', \Phi', (K', (\mathbb{R}^{s'}_+ \times \mathbb{E}') \triangleleft \mathbb{F}'))$ are compatible iff the transition maps $\Phi' \circ \Phi^{-1}[i] : \Phi(p^{-1}(\mathcal{O} \cap \mathcal{O}'))[i] \to \Phi'(p^{-1}(\mathcal{O} \cap \mathcal{O}'))[i]$ and $\Phi \circ \Phi'^{-1}[i] : \Phi'(p^{-1}(\mathcal{O} \cap \mathcal{O}'))[i] \to \Phi(p^{-1}(\mathcal{O} \cap \mathcal{O}'))[i]$ are sc-smooth for $i = 0, 1$. Then $p : \mathcal{Y} \to \mathcal{X}$ is a ***tame strong M-polyfold bundle*** *iff it is equipped with a strong bundle atlas.*

Note that a strong bundle $\mathcal{Y}$ defines two M-polyfold bundles $\mathcal{Y}[i]$ over $\mathcal{X}$ for $i = 0, 1$, and there is a sc^∞ bundle inclusion $\mathcal{Y}[1] \subset \mathcal{Y}[0] = \mathcal{Y}$ covering the identity on $\mathcal{X}$. Let $p_a : \mathcal{Y}_a \to \mathcal{X}_a$ and $p_b : \mathcal{Y}_b \to \mathcal{X}_b$ be two M-polyfold bundles, then a sc-smooth map $F : \mathcal{Y}_a \to \mathcal{Y}_b$ is a bundle map covering a map $f : \mathcal{X}_a \to \mathcal{X}_b$ iff $p_b \circ F = f \circ p_a$ and F is linear on each fiber $(\mathcal{Y}_a)_x$. If p_a, p_b are strong bundles, then F is a strong bundle map iff $F[i] : \mathcal{Y}_a[i] \to \mathcal{Y}_b[i]$ are sc-smooth bundle maps for $i = 0, 1$. A (strong) bundle map is a (strong) bundle isomorphism iff it admits a (strong) bundle map inverse.

Remark 3.13. *Sections of $\mathcal{Y}[1]$ are called sc^+ sections [65, Definition 2.27], which play the role of compact perturbations.*

Definition 3.14. *A* $\mathbf{sc}^\infty$ ***G-action*** *ρ on a tame strong M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$ is a strong bundle map $\rho : G \times \mathcal{Y} \to \mathcal{Y}$, such that $\rho(h, \rho(g, y)) = \rho(hg, y)$ for $h, g \in G$, $y \in \mathcal{Y}$.*

Given a group action ρ, since $\rho : G \times \mathcal{Y} \to \mathcal{Y}$ is a bundle map, it induces a map $\rho_{\mathcal{X}} : G \times \mathcal{X} \to \mathcal{X}$. To see $\rho_{\mathcal{X}}$ is sc-smooth near a point $(g, x) \in G \times \mathcal{X}$, we first choose sc^∞ section s defined on a neighborhood of x. Then $\rho_{\mathcal{X}}(h, y) = p(\rho(h, s(y)))$, which is sc-smooth.

Slices of free group actions

Let $p : \mathcal{Y} \to \mathcal{X}$ be a tame strong M-polyfold bundle. Assume ρ is a sc-smooth G action on p for a compact Lie group G. The core of the proof of Theorem 3.1 is finding a slice $\tilde{O} \subset \mathcal{X}$ to the group action near every point $x_0 \in \mathcal{X}_\infty$, such that $\tilde{O}$ is transverse to the orbits of the group action. When $\mathcal{X}$ is a smooth manifold, we require slices to be submanifolds. There are several notions of "smooth subset" of M-polyfolds, which generalize the notion of submanifold, e.g. [65, Definition 2.12]. We will work with the notion from [36] in Definition 3.17.

Definition 3.15 ([36, Definition 3.2, 3.4])**.** *Consider an open subset $U \subset \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{E}$ for some $n \geq 0$. A tame sc-retraction $r : U \to U$ is called $\mathbb{R}^n$**-sliced** if it satisfies $\pi_{\mathbb{R}^n} \circ r = \pi_{\mathbb{R}^n}$. A tame strong bundle retraction $R : U \triangleleft \mathbb{F} \to U \triangleleft \mathbb{F}$ is $\mathbb{R}^n$**-sliced** if it covers a $\mathbb{R}^n$-sliced sc-retraction on U.*

An important consequence of Definition 3.15 is that the restriction of r resp. R to $\mathbb{R}^m_+ \times \{v\} \times \mathbb{E}$ resp. $(\mathbb{R}^m_+ \times \{v\} \times \mathbb{E}) \triangleleft \mathbb{F}$ is a tame retraction resp. tame strong bundle retraction for every v.

Lemma 3.16 ([36, Lemma 3.3, 3.5])**.** *Given a $\mathbb{R}^n$-sliced tame retraction $r : \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{E} \supset U \to U$, let $\tilde{U} := (\mathbb{R}^m_+ \times \{0\} \times \mathbb{E}) \cap U$. Then the restriction $\tilde{r} := r|_{\tilde{U}} : \tilde{U} \to \tilde{U}$ is a tame retraction. Given a $\mathbb{R}^n$-sliced tame strong bundle retraction $R : U \triangleleft \mathbb{F} \to U \triangleleft \mathbb{F}$, the restriction $\tilde{R} := R|\tilde{U} \triangleleft \mathbb{F} : \tilde{U} \triangleleft \mathbb{F} \to \tilde{U} \triangleleft \mathbb{F}$ is a tame strong bundle retraction.*

Definition 3.17. *Let $\mathcal{X}$ be tame M-polyfold. Then a subset $\tilde{\mathcal{O}} \subset \mathcal{X}$ is called a **slice** if there exists a tame chart $(\mathcal{O}, \phi, (O, \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{E}))$ such that O is defined by a $\mathbb{R}^n$-sliced retraction $r : U \to U$ and $\tilde{\mathcal{O}} = \phi^{-1} \circ r((\mathbb{R}^m_+ \times \{0\} \times \mathbb{E}) \cap U)$*

*Let $p : \mathcal{Y} \to \mathcal{X}$ be a tame strong M-polyfold bundle and $\tilde{\mathcal{O}}$ a subset of $\mathcal{X}$. The subset $p^{-1}(\tilde{\mathcal{O}}) \subset \mathcal{Y}$ is called a **bundle slice** if there exists a tame strong bundle chart $(\mathcal{O}, \Phi, (K, (\mathbb{R}^m_+ \times \mathbb{R}^n \times E) \triangleleft \mathbb{F}))$ of such that K is defined by a $\mathbb{R}^n$-sliced bundle retraction $R : U \triangleleft \mathbb{F} \to U \triangleleft \mathbb{F}$ and $p^{-1}(\tilde{\mathcal{O}}) = \Phi^{-1} \circ R(((\mathbb{R}^m_+ \times \{0\} \times \mathbb{E}) \cap U) \triangleleft \mathbb{F})$.*

For a bundle slice $p^{-1}(\tilde{\mathcal{O}})$, $p|_{p^{-1}(\tilde{\mathcal{O}})} : p^{-1}(\tilde{\mathcal{O}}) \to \tilde{\mathcal{O}}$ is a tame strong M-polyfold bundle by Lemma 3.16. In particular, slice $\tilde{\mathcal{O}}$ is a tame M-polyfold by itself.

The following normal form for submersions on retracts from [36] will be used to construct the slices for the quotient. To state the lemma, we first recall the reduced tangent space from [65]. Let $(O, \mathbb{R}^m_+ \times \mathbb{E})$ be a sc-retract. For every $x \in O_\infty$, the reduced tangent space [65, Definition 2.15] is the subspace:

$$T^R_x O := T_x O \cap (\mathbb{R}^m_+ \times \mathbb{E})_x \subset T_x O,$$

where $(\mathbb{R}^m_+ \times \mathbb{E})_x$ is defined in (3.1). In particular, if $(0,0) \in O$, we have $T^R_{(0,0)} O \subset \{0\} \times \mathbb{E}$.

Remark 3.18. *$T^R_x O$ is invariant under sc-diffeomorphism [65, Proposition 2.8]. Therefore for a M-polyfold $\mathcal{X}$ and $x \in \mathcal{X}_\infty$, we can define $T^R_x \mathcal{X} := \mathrm{d}\phi^{-1}(T^R_y O) \subset T_x \mathcal{X}$ for any chart $(\mathcal{O}, \phi, (O, \mathbb{R}^m_+ \times \mathbb{E}))$ with $\phi(x) = y \in O$. $T^R_x \mathcal{X}$ is called the reduced tangent space [65, Definition 2.20].*

Lemma 3.19 ([36, Lemma 4.2, Remark 4.3]). *Consider a tame sc-retract $(O, \mathbb{R}^m_+ \times \mathbb{E})$ containing $x_0 := (0,0) \in O_\infty$ and a sc-smooth map $f : O \to \mathbb{R}^n$. Suppose that $f(x_0) = 0$ and the restriction of the tangent map $\mathrm{D}f_{x_0}|_{T^R_{x_0}O} : T^R_{x_0}O \to \mathbb{R}^n$ is surjective. Let $\mathbb{K}$ denote $\ker \mathrm{D}(f \circ r)_{x_0} \cap (\{0\} \times \mathbb{E})$, where r is a retraction for the sc-retract $(O, \mathbb{R}^m_+ \times \mathbb{E})$. Then we can view $\mathbb{K}$ as a subspace of $\mathbb{E}$ of codimension n. Assume $L \subset (T^R_{x_0}O)_\infty$ is a complement of $\mathbb{K}$ in $\mathbb{E}$. Then there exists neighborhoods $U \subset \mathbb{R}^m_+ \times \mathbb{E}^1$ of x_0 and $U' \subset \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}^1$ of $(0,0,0)$, such that there exists a sc-diffeomorphism $h : U \to U'$ with the following properties.*

- $h(x_0) = (0,0,0)$.
- $f \circ r \circ h^{-1} : \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}^1 \supset U' \to \mathbb{R}^n$ *is the projection to* $\mathbb{R}^n$.
- $h \circ r \circ h^{-1}$ *is a* $\mathbb{R}^n$*-sliced retraction on* U'*. In particular,* $f^{-1}(0) \cap U$ *is a slice of* O^1.
- $\mathrm{D}h_{x_0}(\{0\} \times L) = \{0\} \times \mathbb{R}^n \times \{0\}$ *and* $\mathrm{D}h_{x_0}(\{0\} \times \mathbb{K}^1) = \{0\} \times \{0\} \times \mathbb{K}^1$.

Remark 3.20. *The existence of complement L is guaranteed, see [65, Lemma 2.2].*

From the perspective of Lemma 3.19, in order to construct slices for the quotient, we need to construct submersive maps to $\mathbb{R}^{\dim G}$. Such maps will not be globally defined on $\mathcal{X}$ in general. In fact, we can only construct sc-smooth submersive maps near every smooth point $x_0 \in \mathcal{X}_\infty$.

Definition 3.21. *A M-polyfold chart $(\mathcal{O}, \phi, (O, \mathbb{R}^m_+ \times \mathbb{E}))$ is around x_0 if $\phi(x_0) = (0,0) \in \mathbb{R}^m_+ \times \mathbb{E}$.*

It is clear that the existence of M-polyfold chart around x_0 is equivalent to $x_0 \in \mathcal{X}_\infty$. Let $\rho_\mathcal{X} : G \times \mathcal{X} \to \mathcal{X}$ be a sc-smooth action on the M-polyfold $\mathcal{X}$. Given a M-polyfold chart around x_0 and a neighborhood $B \subset G$ of id, we have a sc-smooth map that parametrizes the orbit through x_0 locally,

$$\gamma : B \to \mathbb{R}^m_+ \times \mathbb{E}, \qquad g \mapsto \phi \circ \rho_\mathcal{X}(g, x_0). \tag{3.2}$$

The **infinitesimal directions** of the G-action at x_0 in this chart is the subspace

$$\mathrm{D}\gamma_{\mathrm{id}}(T_{\mathrm{id}}B) \subset (TO_{(0,0)})_\infty \subset \mathbb{R}^m \times \mathbb{E}_\infty. \tag{3.3}$$

Proposition 3.22. $\mathrm{D}\gamma_{\mathrm{id}}(T_{\mathrm{id}}B) \subset (T^R_{(0,0)}O)_\infty \subset \{0\} \times \mathbb{E}_\infty \simeq \mathbb{E}_\infty$.

Proof. For each $g \in G$, the map $\rho_\mathcal{X}(g, \cdot)$ is a sc-diffeomorphism. By [65, Proposition 2.8, 2.10], $\rho(g, \cdot)$ preserves the degeneracy index. In particular, we have $\phi \circ \rho_\mathcal{X}(g, x_0) \in \{0\} \times \mathbb{E} \subset \mathbb{R}^m_+ \times \mathbb{E}$. Therefore $\mathrm{D}\gamma_{\mathrm{id}}(T_{\mathrm{id}}B) \subset TO_{(0,0)} \cap (\{0\} \times \mathbb{E}) = T^R_{(0,0)}O \subset \{0\} \times \mathbb{E} \simeq \mathbb{E}$. □

Proposition 3.23. *If $\rho_\mathcal{X}$ is free, then infinitesimal directions at $x_0 \in \mathcal{X}_\infty$ in a chart $(\mathcal{O}, \phi, (O, \mathbb{R}^m_+ \times \mathbb{E}))$ around x_0 is of dimension $\dim G$.*

Proof. Assume otherwise, that is there exists $\xi \in T_{\mathrm{id}}G$ such that $\mathrm{D}\gamma_{\mathrm{id}}(\xi) = 0$. Then $\xi(t) := \rho_{\mathcal{X}}(\exp(t\xi), x_0) : (-\epsilon, \epsilon) \to \mathcal{X}$ is a sc-smooth map with the property that $\mathrm{D}\xi(t) = 0$ because of the group property $\rho_{\mathcal{X}}(\exp(t\xi), \rho_{\mathcal{X}}(\exp s\xi), x_0)) = \rho_{\mathcal{X}}(\exp((t+s)\xi), x_0)$. By [65, Proposition 1.7], $\phi \circ \xi : (-\epsilon, \epsilon)^1 = (-\epsilon, \epsilon) \to \mathbb{R}^m_+ \times \mathbb{E}$ is C^1. Then $\xi(t) \equiv x_0$, contradicting the free action assumption. □

Since $\mathrm{D}\gamma_{\mathrm{id}}(T_{\mathrm{id}}B) \subset \mathbb{E}_\infty$ is finite dimensional, by [65, Proposition 1.1] there exists a sc-complement $\mathbb{H}$ of $\mathrm{D}\gamma_{\mathrm{id}}(T_{\mathrm{id}}B)$ in $\mathbb{E}$. In Lemma 3.27 below, we prove that an open subset of the shifted space $\phi^{-1}(O \cap (\mathbb{R}^m_+ \times \mathbb{H}^2)) \subset \mathcal{X}^2$ is a slice. First, we use Lemma 3.19 to prove the following more general statement, which is used in Lemma 3.27 and Proposition 4.39.

Lemma 3.24. *Let $p : \mathcal{Y} \to \mathcal{X}$ be a tame M-polyfold and $(\mathcal{O}, \Phi, (P, (\mathbb{R}^m_+ \times \mathbb{E}) \triangleleft \mathbb{F}))$ a tame strong bundle chart around $x_0 \in \mathcal{X}_\infty$ covering a tame M-polyfold chart $(\mathcal{O}, \phi, (O, \mathbb{R}^m_+ \times \mathbb{E}))$. Let r be a tame retraction, R a strong bundle retraction covering r such that $O = r(U)$ and $P = R(U \triangleleft \mathbb{F})$ for an open neighborhood $U \subset \mathbb{R}^m_+ \times \mathbb{E}$ of $(0,0)$. For a neighborhood $B \subset \mathbb{R}^n$ of 0, assume $\Lambda : B \times p^{-1}(\mathcal{O}) \to \mathcal{Y}$ is a sc-smooth strong bundle map such that $\Lambda(0, \cdot)|_{p^{-1}(\mathcal{O})} = \mathrm{id}_{p^{-1}(\mathcal{O})}$. Let $\Gamma : B \times \mathcal{O} \to \mathcal{X}$ denote the map on the base covered by Λ. Suppose that $\mathrm{D}\Gamma_{(0,x_0)}(T_0 B \times \{0\}) \subset (T^R_{x_0}\mathcal{O})_\infty$ and is of dimension n. Let $\Xi := \mathrm{D}(\phi \circ \Gamma)_{(0,x_0)}(T_0 B \times \{0\}) \subset \{0\} \times \mathbb{E}_\infty \simeq \mathbb{E}_\infty$ and $\mathbb{H}$ any sc-complement of Ξ in $\mathbb{E}$. Then the following holds.*

1. *There exists open neighborhoods $\mathcal{O}' \subset \mathcal{O}^2$ of x_0,$V \subset B$ of 0 and a sc-smooth map $f : \mathcal{O}' \to V$ such that $\tilde{\mathcal{O}}' := f^{-1}(0)$ is a slice of $\mathcal{X}^2$ containing x_0 and $p^{-1}(\tilde{\mathcal{O}}')$ is a bundle slice of $p : \mathcal{Y}^2 \to \mathcal{X}^2$.*
2. *For $x \in \mathcal{O}'$, $g = f(x)$ is the unique element $g \in V$, such that $\Gamma(g, x) \in \tilde{\mathcal{O}}'$.*
3. *$\eta : \mathcal{O}' \to \tilde{\mathcal{O}}'$ defined by $x \mapsto \Gamma(f(x), x)$ is sc-smooth. $N : p^{-1}(\mathcal{O}') \to p^{-1}(\tilde{\mathcal{O}}')$ defined by $v \mapsto \Lambda(f(p(v)), v)$ is a sc-smooth strong bundle map.*
4. *Let $\mathbb{K} := \left(\mathbb{H} \cap T^R_{(0,0)}O^1\right) \oplus \left(\ker \mathrm{D}r_{(0,0)} \cap (\{0\} \times \mathbb{E}^1)\right) \subset \mathbb{E}^1$. There exist neighborhoods $U' \subset U^2$ of $(0,0)$, $U'' \subset \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}^1$ of $(0,0,0)$ and a sc-diffeomorphism $h : U' \to U''$, such that $h \circ r \circ h^{-1} : U'' \to U''$ is a $\mathbb{R}^n$-sliced retraction. Moreover $\mathcal{O}' = \phi^{-1} \circ r(U')$ and $\tilde{\mathcal{O}}' = \phi^{-1} \circ r \circ h^{-1}((\mathbb{R}^m_+ \times \{0\} \times \mathbb{K}^1) \cap U'')$.*
5. *$\mathrm{D}h_{(0,0)}(\Xi) = \{0\} \times \mathbb{R}^n \times \{0\}$, $\mathrm{D}h_{(0,0)}(\{0\} \times \mathbb{K}^1) = \{0\} \times \{0\} \times \mathbb{K}^1$.*
6. *$\tilde{\mathcal{O}}' = \mathcal{O}' \cap \phi^{-1}(O \cap (\mathbb{R}^m_+ \times \mathbb{H}))$.*

Proof. Let $\pi_\Xi, \pi_{\mathbb{H}}$ be the projections to Ξ and $\mathbb{H}$ in $\mathbb{E} = \Xi \oplus \mathbb{H}$. Let $W := \Gamma(\cdot, \phi^{-1} \circ r(\cdot))^{-1}(\mathcal{O}) \cap (U \times B)$, which is open neighborhood of $(0,0,0) \in U \times B$. We consider the well-defined sc-smooth map $q : U \times B \supset W \to \Xi$:

$$q : (x, g) \mapsto \pi_\Xi \circ \phi \circ \Gamma(g, \phi^{-1} \circ r(x)).$$

Since $\mathrm{D}q_{(0,0,0)}|_{\{0\}\times\{0\}\times T_0B} = \pi_\Xi \circ \mathrm{D}(\phi\circ\Gamma)_{(0,x_0)}|_{T_0B\times\{0\}}$, the assumption $\mathrm{D}(\phi\circ\Gamma)_{(0,x_0)}(T_0B\times\{0\}) = \Xi$ implies that

$$\mathrm{D}q_{(0,0,0)}|_{\{0\}\times\{0\}\times T_0B} \text{ is an isomorphism on to } \Xi. \tag{3.4}$$

Then we can apply Lemma 3.28 to get open neighborhoods $U^* \subset U^1$ of $(0,0)$ and $V \subset B$ of 0, and a sc-smooth function $t : U^* \to V$, such that

$$q(u,v) = 0 \text{ for } u \in U^*, v \in V \text{ iff } v = t(u); \tag{3.5}$$

$$\mathrm{D}t_{(0,0)}(u) = (\mathrm{D}q_{(0,0,0)}|_{\{0\}\times\{0\}\times T_0V})^{-1} \circ \mathrm{D}q_{(0,0,0)}(u,0), \quad \forall u \in \mathbb{R}^m\times\mathbb{E}^1. \tag{3.6}$$

Note that $\Gamma(0,\cdot) = \mathrm{id}_{\mathcal{O}}$ by assumption. Then we have

$$\mathrm{D}q_{(0,0,0)}(u,0) = \pi_\Xi \circ \mathrm{D}r_{(0,0)}(u), \quad \forall u \in \mathbb{R}^m\times\mathbb{E}^1. \tag{3.7}$$

Since $\mathrm{D}r_{(0,0)}|_{\{0\}\times\Xi} = \mathrm{id}_\Xi$, (3.6) and (3.7) imply that

$$\mathrm{D}t_{(0,0)}|_{\{0\}\times\Xi\times\{0\}} \text{ is surjective onto } T_0V. \tag{3.8}$$

Note that $q(r(x),g) = q(x,g)$, hence (3.5) implies that $t\circ r = t$ on $U^*\times V$. Hence $t|_{r(U^*)}$ is a sc-smooth function.

Since $\Xi \subset T^R_{(0,0)}O$ by assumption, (3.8) implies that $\mathrm{D}t_{(0,0)}|_{T^R_{(0,0)}O} : T^R_{(0,0)}O \to T_0V$ is surjective. We can apply Lemma 3.19 to t as follows: There exist open neighborhoods $U' \subset (U^*)^1 \subset U^2$ of $(0,0)$, $U'' \subset \mathbb{R}^m_+\times\mathbb{R}^n\times\mathbb{K}^1$ of $(0,0,0)$ and a sc-diffeomorphism $h : U' \to U''$ such that $h\circ r\circ h^{-1}$ is a $\mathbb{R}^n$-sliced retraction and $t\circ h^{-1}$ is the projection to $\mathbb{R}^n$, where

$$\mathbb{K} := \ker \mathrm{D}(t\circ r)_{(0,0)} \cap (\{0\}\times\mathbb{E}^1) = \ker \mathrm{D}t_{(0,0)} \cap (\{0\}\times\mathbb{E}^1). \tag{3.9}$$

Then (3.7) implies that $\mathbb{K} = \left(\mathbb{H}\cap T^R_{(0,0)}O^1\right)\oplus\left(\ker \mathrm{D}r_{(0,0)}\cap(\{0\}\times\mathbb{E}^1)\right)$. Let $O' := r(U')$, $O'' := h\circ r\circ h^{-1}(U'')$ and $\mathcal{O}' := \phi^{-1}(O') = \phi^{-1}\circ h^{-1}(O'') \subset \mathcal{X}^2$. We also define $H := h \lhd \mathrm{id}_{\mathbb{F}^2}$ and $R'' := H\circ R\circ H^{-1}$ and $K'' = R''(U''\lhd\mathbb{F}^2)$. Then $(\mathcal{O}', H\circ\Phi, (K'', (\mathbb{R}^m_+\times\mathbb{R}^n\times\mathbb{K}^1)\lhd\mathbb{F}^2))$ is a tame strong bundle chart for $p : \mathcal{Y}^2 \to \mathcal{X}^2$ around x_0 covering a M-polyfold chart $(\mathcal{O}', h\circ\phi, (O'', \mathbb{R}^m_+\times\mathbb{R}^n\times\mathbb{K}^1))$. Therefore by Lemma 3.16, $\tilde{\mathcal{O}}' := \phi^{-1}(t^{-1}(0)\cap O')$ is a slice of $\mathcal{X}^2$ and $p^{-1}(\tilde{\mathcal{O}}')$ is a bundle slice. So far, we have proven Property (1) and (4). Property (5) follows from Lemma 3.19.

Let $f : \mathcal{O}' \to V$ be the sc-smooth map defined by $x \mapsto t\circ\phi(x)$. Let $g \in V$ and $x \in \mathcal{O}'$. Then $\Gamma(g,x) \in f^{-1}(0) = \tilde{\mathcal{O}}'$ is equivalent to $q(0,\phi(\Gamma(g,x))) = 0$ by (3.5), that is $\pi_\Xi\circ\phi\circ\Gamma(0,\Gamma(g,x)) = \pi_\Xi\circ\phi\circ\Gamma(g,x) = 0$. Therefore $\Gamma(g,x) \in \tilde{\mathcal{O}}'$ is equivalent to $(\phi(x),g)$ is a solution to $q = 0$ in $U'\times V$. Therefore by (3.5), $g = t\circ\phi(x) = f(x)$. Hence property (2) holds.

To see $\eta(x) := \Gamma(f(x),x)$ is sc-smooth. By chain rule [65, Theorem 1.1], η is a sc-smooth map from $\mathcal{O}'$ to $\mathcal{O}'$. In chart U'', that is $h\circ\phi\circ\eta\circ\phi^{-1}\circ r\circ h^{-1} : U'' \to \mathbb{R}^m_+\times\mathbb{R}^n\times\mathbb{K}^1$ is sc-smooth. Since $\mathrm{im}(h\circ\phi\circ\eta\circ\phi^{-1}\circ r\circ h^{-1}) \subset \mathbb{R}^m_+\times\{0\}\times\mathbb{K}^1$, $h\circ\phi\circ\eta\circ\phi^{-1}\circ r\circ h^{-1}$ is also a sc-smooth map from U'' to $\mathbb{R}^m_+ \times \{0\} \times \mathbb{K}^1$. That is $\eta : \mathcal{O} \to \tilde{\mathcal{O}}'$ is sc-smooth. The sc-smoothness of $N := \rho(f(p(x)), x)$ follows from the same argument. This proves property (3).

To show property (6), for $x \in O'$, by (3.5) $t(x) = 0$ is equivalent to $0 = q(x, 0) = \pi_\Xi \circ \phi \circ (0, \phi^{-1} \circ r(x)) = \pi_\Xi(x)$, i.e. $x \in \mathbb{H}$. Since $\tilde{\mathcal{O}}' = \{\phi^{-1}(x) | t(x) = 0, x \in O\}$, hence $\tilde{\mathcal{O}}' = \mathcal{O}' \cap \phi^{-1}(O \cap (\mathbb{R}^m_+ \times \mathbb{H}))$. □

Remark 3.25. *Property (1) - (3) in Lemma 3.24 are directly used in the construction of G-slices, e.g. Lemma 3.27, Proposition 4.39 and Proposition 4.68. Property (4) and (5) is used to get G-slices which is also good in the sense of Definition 3.38, see Proposition 3.42.*

Definition 3.26. *Let ρ be a sc-smooth action on tame strong M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$. For every x_0, a **bundle G-slice of p around x_0** is a tuple $(\tilde{\mathcal{O}}, \mathcal{O}, V, f, \eta, N)$ such that the following holds.*

- *$\mathcal{O}$ is open subset of $\mathcal{X}$ and $V \subset G$ is an open neighborhood of* id.
- *$f : \mathcal{O} \to V$ is a sc-smooth map, such that $x_0 \in \tilde{\mathcal{O}} := f^{-1}(\mathrm{id})$ and $p^{-1}(\tilde{\mathcal{O}})$ is a bundle slice.*
- *For $x \in \mathcal{O}$, $g = f(x)$ is the unique element $g \in V$ such that $\rho_{\mathcal{X}}(g, x) \in \tilde{\mathcal{O}}$.*
- *$\eta : \mathcal{O} \to \tilde{\mathcal{O}}$ defined by $x \mapsto \rho_{\mathcal{X}}(f(x), x)$ is sc-smooth. $N : p^{-1}(\mathcal{O}) \to p^{-1}(\tilde{\mathcal{O}})$ defined by $v \mapsto \rho(f(p(v)), v)$ is a sc-smooth strong bundle map.*
- *$\psi : \tilde{\mathcal{O}} \to \mathcal{O} \xrightarrow{\pi} \mathcal{X}/G$ is injective.*

Lemma 3.27. *Let ρ be a sc-smooth action on the tame strong M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$ such that $\rho_{\mathcal{X}}$ is free. Then there exists a bundle G-slice of $p : \mathcal{Y}^2 \to \mathcal{X}^2$ around every $x_0 \in \mathcal{X}_\infty$.*

Proof. For $x_0 \in \mathcal{X}_\infty$, let $(\mathcal{O}, \Phi, (K, (\mathbb{R}^m_+ \times \mathbb{E}) \triangleleft \mathbb{F}))$ be a tame strong M-polyfold bundle chart around x_0 covering a tame M-polyfold chart $(\mathcal{O}, \phi, (O, \mathbb{R}^m_+ \times \mathbb{E}))$. Then we define

$$\Lambda : B \times p^{-1}(\mathcal{O}) \to \mathcal{Y}, \quad (g, v) \mapsto \rho(g, v),$$

where $B \subset G$ is a neighborhood of id. Then $\Lambda(\mathrm{id}, \cdot)|_{p^{-1}(\mathcal{O})} = \mathrm{id}_{p^{-1}(\mathcal{O})}$ and Λ covers the map

$$\Gamma : B \times \mathcal{O} \to \mathcal{X}, (g, x) \mapsto \rho_{\mathcal{X}}(g, x).$$

Since the group action is free, by Proposition 3.22 and Proposition 3.23 $\mathrm{D}\Gamma_{(\mathrm{id}, x_0)}(T_{\mathrm{id}} B \times \{0\}) = \mathrm{D}\gamma_{\mathrm{id}}(T_{\mathrm{id}} B) \subset T^R_{x_0} \mathcal{O}$ and $\dim \mathrm{D}\Gamma_{(\mathrm{id}, x_0)}(T_{\mathrm{id}} B \times \{0\}) = \dim B$. Hence we can apply Lemma 3.24 to Λ. As a consequence, we have $(\tilde{\mathcal{O}}', \mathcal{O}', V, f, \eta, N)$ satisfying all the conditions of a bundle G-slice except the injective condition. Moreover, by Definition 3.26, for every open neighborhood $\tilde{\mathcal{O}}'' \subset \tilde{\mathcal{O}}'$ of x_0, let $\mathcal{O}'' = \eta^{-1}(\tilde{\mathcal{O}}'')$, then $(\tilde{\mathcal{O}}'', \mathcal{O}'', V, f, \eta, N)$ satisfies all but the injective condition of a bundle G-slice.

We claim that there is an open neighborhood $\tilde{\mathcal{O}}'' \subset \tilde{\mathcal{O}}'$ of x_0, such that $\psi|_{\tilde{\mathcal{O}}''}$ is injective. Assume otherwise, that is there exist $x_n, y_n \in \tilde{\mathcal{O}}'$ converging to x_0 and $g_n \neq \mathrm{id} \in G$ such that $\rho_{\mathcal{X}}(g_n, x_n) = y_n$. Since G is compact, we have $\lim_{i\to\infty} g_{n_i} = g_0$ for a subsequence $\{n_i\}_{i\in\mathbb{N}}$. The continuity of $\rho_{\mathcal{X}}$ implies that $\rho_{\mathcal{X}}(g_0, x_0) = x_0$. Since the group action is free, we have $g_0 = \mathrm{id}$. In particular, there exists $n \in \mathbb{N}$ such that $g_n \in V$. Then both $x_n = \rho_{\mathcal{X}}(\mathrm{id}, x_n)$ and $y_n = \rho_{\mathcal{X}}(g_n, x_n)$ are in $\tilde{\mathcal{O}}$, which contradicts property (2) of Lemma 3.24. As a consequence, $(\eta^{-1}(\tilde{\mathcal{O}}''), \tilde{\mathcal{O}}'', V, f, \eta, N)$ satisfying all the conditions of a bundle G-slice. ☐

Although a general implicit function theorem does not exists in sc-calculus [37], the following special form holds. It is used in the proof of Lemma 3.27 and Lemma 3.40.

Lemma 3.28. *Let $q : \mathbb{R}^m_+ \times \mathbb{E} \times \mathbb{R}^n \to \mathbb{R}^n$ be a sc-smooth map. Assume $q(0,0,0) = 0$ and $\mathrm{D}q_{(0,0,0)}(\{0\} \times \{0\} \times \mathbb{R}^n) = \mathbb{R}^n$. Then there exists open neighborhoods $U \subset \mathbb{R}^m_+ \times \mathbb{E}^1$ of $(0,0)$, $V \subset \mathbb{R}^n$ of 0 and a sc-smooth map $f : U \to V$, such that*

1. *$q : \mathbb{R}^m_+ \times \mathbb{E}_{m+1} \supset U_m \to V$ is C^{m+1} for all $m \geq 0$.*
2. *$q(u, v) = 0$ for $u \in U, v \in V$ iff $v = f(u)$;*
3. *$\mathrm{D}f_{(0,0)}(u) = (\mathrm{D}q_{(0,0,0)}|_{\{0\}\times\{0\}\times\mathbb{R}^n})^{-1} \circ \mathrm{D}q_{(0,0,0)}(u, 0)$ for $u \in \mathbb{R}^m \times \mathbb{E}^1$.*

Proof. By [65, Proposition 1.7], $q : \mathbb{R}^m_+ \times \mathbb{E}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^k map. Since $q(0,0,0) = 0$ and $\mathrm{D}q_{(0,0,0)}(\{0\}\times\{0\}\times\mathbb{R}^n) = \mathbb{R}^n$, we can apply the classical implicit function theorem for partial quadrants in Banach spaces[1]. That is there exists an open neighborhoods $U \subset \mathbb{R}^m_+ \times \mathbb{E}^1$ of $(0,0)$ and $V \subset \mathbb{R}^n$ of 0, and a C^1 map $f : U \to V$, such that $q(u, v) = 0$ for $u \in U, v \in V$ iff $v = f(u)$. Moreover, $\mathrm{D}q_{(u,v)}(\{0\} \times \{0\} \times \mathbb{R}^n) = \mathbb{R}^n$. for every $(u, v) \in U \times V$ with $q(u, v) = 0$. It remains to prove that f is sc-smooth. For every $(u, v) \in U \times V$ with $q(u, v) = 0$, if $u \in \mathbb{R}^m_+ \times \mathbb{E}^k$, we can apply the classical implicit function theorem to the C^k map $q : \mathbb{R}^m_+ \times \mathbb{E}^k \times \mathbb{R}^n \to \mathbb{R}^n$ near (u, v) to get C^k map $\tilde{f}_k$ defined near u solving $q(u, v) = 0$. By the uniqueness of implicit function, we have $f = \tilde{f}_k$ in a neighborhood of u in $\mathbb{R}^m_+ \times \mathbb{E}^k \times \mathbb{R}^n$. Therefore $f : U \cap (\mathbb{R}^m_+ \times \mathbb{E}^k \times \mathbb{R}^n) \to \mathbb{R}^n$ is C^k. By [65, Proposition 1.8], $f : U \to V$ is sc-smooth. The last assertion follows from the classical implicit function theorem. ☐

Free quotients of M-polyfolds and M-polyfold bundles

We first prove that bundle G-slices give rise to a tame strong M-polyfold bundle structure on the topological quotient. In particular, part (1) of Theorem 3.1 follows directly form the following lemma.

[1] The inverse function theorem for partial quadrant was discussed in [92, Theorem 2.2.4], and the implicit function theorem for partial quadrants follows as a corollary.

Lemma 3.29. *Assume ρ acts on a tame strong M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$ such that the induced action $\rho_{\mathcal{X}}$ on $\mathcal{X}$ is free. For every $x_0 \in \mathcal{X}_\infty$, we pick a bundle G-slice $(\tilde{\mathcal{O}}_{x_0}, \mathcal{O}_{x_0}, V_{x_0}, f_{x_0}, \eta_{x_0}, N_{x_0})$ of p around x_0. Let $\widehat{\mathcal{X}} := \cup_{x_0 \in \mathcal{X}_\infty} \rho_{\mathcal{X}}(G, \mathcal{O}_{x_0})$. Then $\overline{p} : p^{-1}(\widehat{\mathcal{X}})/G \to \widehat{\mathcal{X}}/G$ is a tame strong M-polyfold bundle, such that map*

$$\Psi_{x_0} = \pi \circ \iota : p^{-1}(\tilde{\mathcal{O}}_{x_0}) \overset{\iota}{\hookrightarrow} p^{-1}(\mathcal{O}_{x_0}) \overset{\pi}{\to} p^{-1}(\widehat{\mathcal{X}})/G \tag{3.10}$$

is a strong bundle morphism. Moreover, the quotient map $\pi : p^{-1}(\widehat{\mathcal{X}}) \to p^{-1}(\widehat{\mathcal{X}})/G$ is a sc-smooth strong bundle map.

Proof. By construction, $\widehat{\mathcal{X}}$ is a G-invariant open subset of $\mathcal{X}$ containing $\mathcal{X}_\infty$. We claim that for every $x_0 \in \mathcal{X}$, the following maps

$$\psi_{x_0} : \tilde{\mathcal{O}}_{x_0} \hookrightarrow \widehat{\mathcal{X}} \overset{\pi}{\to} \widehat{\mathcal{X}}/G \tag{3.11}$$

induce a sc-smooth structure on $\widehat{\mathcal{X}}/G \subset \mathcal{X}/G$.

To prove this claim, we first show ψ_{x_0} is a homeomorphism onto the image. Since ψ_{x_0} is injective, it suffices to show that ψ_{x_0} is an open map. For every open subset $\tilde{\mathcal{U}} \subset \tilde{\mathcal{O}}_{x_0}$, we have $\psi_{x_0}(\tilde{\mathcal{U}}) = \pi(\eta_{x_0}^{-1}(\tilde{\mathcal{U}}))$. Thus it is sufficient to prove $\pi^{-1}\left(\pi(\eta_{x_0}^{-1}(\tilde{\mathcal{U}}))\right) \subset \mathcal{X}$ is open. Because $\eta_{x_0}^{-1}(\tilde{\mathcal{U}}) \subset \mathcal{X}$ is open, $\rho_{\mathcal{X}}\left(G, \eta_{x_0}^{-1}(\tilde{\mathcal{U}})\right) \subset \mathcal{X}$ is also open. Since $\pi^{-1}\left(\pi(\eta_{x_0}^{-1}(\tilde{\mathcal{U}}))\right) = \rho_{\mathcal{X}}\left(G, \eta_{x_0}^{-1}(\tilde{\mathcal{U}})\right)$, $\pi^{-1}\left(\pi(\eta_{x_0}^{-1}(\tilde{\mathcal{U}}))\right)$ is open.

Next we will prove the compatibility between ψ_{x_0} and ψ_{x_1}. Consider a point $q \in \widehat{\mathcal{X}}/G$ such that $\psi_{x_1}(x) = \psi_{x_0}(y) = q$ for $x \in \tilde{\mathcal{O}}_{x_1}, y \in \tilde{\mathcal{O}}_{x_0}$. If we view x, y as points in $\mathcal{X}$, then $\pi(x) = \pi(y) = q$, i.e. there exists $g_0 \in G$ such that $\rho_{\mathcal{X}}(g_0, y) = x$. We claim the transition map near y can be expressed as

$$\psi_{x_1}^{-1} \circ \psi_{x_0}(z) = \eta_{x_1} \circ \rho_{\mathcal{X}}(g_0, z).$$

This is because η_{x_1} is defined on a neighborhood of $x = \rho_{\mathcal{X}}(g_0, y)$ in $\mathcal{O}_{x_1}$ and $\pi \circ \eta_{x_1} \circ \rho_{\mathcal{X}}(g_0, z) = \pi \circ \rho_{\mathcal{X}}(g_0, z) = \pi(z)$. Since η_{x_1} is sc-smooth, $\psi_{x_1}^{-1} \circ \psi_{x_0}$ is sc-smooth. Since each slice $\tilde{\mathcal{O}}_x$ is a tame M-polyfold, we give $\widehat{\mathcal{X}}/G$ a tame M-polyfold structure.

By [65, Theorem 2.2], $\mathcal{X}$ is metrizable. Therefore by Lemma 3.31 below, $\mathcal{X}/G$ is again a metrizable space and so is $\widehat{\mathcal{X}}/G$. Hence $\widehat{\mathcal{X}}/G$ is a paracompact Hausdorff space with tame M-polyfold structure, i.e. a tame M-polyfold.

Finally we prove that the quotient map $\pi : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}/G$ is sc$^\infty$. Consider $x \in \widehat{\mathcal{X}}$, by our construction of $\widehat{\mathcal{X}}$, there exist $g_0 \in G$ and a slice $\tilde{\mathcal{O}}_{x_0}$, such that $\rho_{\mathcal{X}}(g_0, x) \in \tilde{\mathcal{O}}_{x_0}$. Therefore the map π can be locally expressed as

$$\pi : z \mapsto \eta_{x_0} \circ \rho_{\mathcal{X}}(g_0, z).$$

Since η_{x_0} and $\rho_{\mathcal{X}}$ are both sc-smooth, the quotient map π is sc-smooth by the chain rule.

Similarly, bundle maps $\Psi_{x_0} : p^{-1}(\tilde{\mathcal{O}}_{x_0}) \hookrightarrow p^{-1}(\widehat{\mathcal{X}}) \xrightarrow{\pi} p^{-1}(\widehat{\mathcal{X}})/G$ gives $p^{-1}(\widehat{\mathcal{X}})/G \to \widehat{\mathcal{X}}/G$ a tame strong M-polyfold bundle structure such that the quotient map $\pi : p^{-1}(\widehat{\mathcal{X}}) \to p^{-1}(\widehat{\mathcal{X}})/G$ is a sc-smooth strong bundle map. In particular,

$$\Psi_{x_1}^{-1} \circ \Psi_{x_0} \text{ is a strong bundle isomorphism locally.} \tag{3.12}$$

This fact is used in the proof of part (2) of Theorem 3.1. □

Proposition 3.30. *Assume two different sets of choices of bundle G-slices in Lemma 3.29 give rise to two quotient M-polyfolds $\widehat{\mathcal{X}}_a/G$ and $\widehat{\mathcal{X}}_b/G$. Let U be the topological quotient $(\widehat{\mathcal{X}}_a \cap \widehat{\mathcal{X}}_b)/G$, which is an open subset of $\mathcal{X}/G$ containing $\mathcal{X}_\infty/G$. Then the identity map $(\widehat{\mathcal{X}}_a/G) \cap U \to (\widehat{\mathcal{X}}_b/G) \cap U$ is a sc-diffeomorphism, where $(\widehat{\mathcal{X}}_a/G) \cap U$ is U with the M-polyfold structure from $\widehat{\mathcal{X}}_a/G$ and $(\widehat{\mathcal{X}}_b/G) \cap U$ is U with the M-polyfold structure from $\widehat{\mathcal{X}}_b/G$. Similarly, the identity map $p^{-1}(\widehat{\mathcal{X}}_a)/G \cap \overline{p}^{-1}(U) \to p^{-1}(\widehat{\mathcal{X}}_b)/G \cap \overline{p}^{-1}(U)$ is a strong bundle isomorphism.*

Proof. We will prove the claim on the base M-polyfold and the assertion on the bundle follows from a similar argument. The quotient construction in Lemma 3.29 has the following universal property: Let W be a G-invariant open subset of $\mathcal{X}$. If there is a G-invariant sc$^\infty$ map $f : \widehat{\mathcal{X}} \cap W \to \mathcal{Z}$, then there exists a unique sc$^\infty$ map $\overline{f} : (\widehat{\mathcal{X}}/G) \cap (W/G) \to \mathcal{Z}$ such that $f = \overline{f} \circ \pi$ on $\widehat{\mathcal{X}}$. The existence and uniqueness of a continuous map $\overline{f}$ follows from that $(\widehat{\mathcal{X}}/G) \cap (W/G)$ is the topological quotient of $\widehat{\mathcal{X}} \cap W$ by G. To see $\overline{f}$ is sc-smooth, we have the following commutative diagram,

$$\begin{tikzcd}
\tilde{\mathcal{O}}_{x_0} \cap W \arrow[r] \arrow[dr, "\psi_{x_0}"'] & \widehat{\mathcal{X}} \cap W \arrow[r, "f"] \arrow[d, "\pi"] & \mathcal{Z} \\
& (\widehat{\mathcal{X}}/G) \cap (W/G). \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

Recall from Lemma 3.29, $\psi_{x_0} : \tilde{\mathcal{O}}_{x_0} \cap W \to (\widehat{\mathcal{X}}/G) \cap (W/G)$ is sc-diffeomorphism onto an open set. By the diagram above, $\overline{f} \circ \psi_{x_0} = f|_{\tilde{\mathcal{O}}_{x_0} \cap W}$. Therefore $\overline{f}$ is sc-smooth.

Given the conditions of this proposition, we have the following commutative diagram by the universal property,

$$\begin{tikzcd}
& \widehat{\mathcal{X}}_a \cap \widehat{\mathcal{X}}_b \arrow[dl, "\pi_a"'] \arrow[dr, "\pi_b"] & \\
\widehat{\mathcal{X}}_a/G \cap U \arrow[rr, shift left, "\mathrm{id}=\overline{\pi_b}"] & & \widehat{\mathcal{X}}_b/G \cap U, \arrow[ll, shift left, "\mathrm{id}=\overline{\pi_a}"]
\end{tikzcd}$$

where π_a resp. π_b are the quotient maps on $\widehat{\mathcal{X}}_a$ resp. $\widehat{\mathcal{X}}_b$ and $\overline{\pi_a}, \overline{\pi_b}$ are the sc-smooth maps induced by G-invariant maps π_a resp. π_b. Since $\overline{\pi_a} = \overline{\pi_b} = \mathrm{id}_U$ as topological maps, identity map $\overline{\pi_a} : \widehat{\mathcal{X}}_a/G \cap U \to \widehat{\mathcal{X}}_b/G \cap U$ is a sc-diffeomorphism with sc-smooth inverse $\overline{\pi_b}$. □

The slice construction only gives M-polyfold structures resp. polyfold structures, which are local in nature. Globally, M-polyfolds resp. polyfolds are also metrizable spaces. The following Lemma asserts that the metrizablity passes to the quotients.

Lemma 3.31. *Let X be a metrizable space, G a compact Lie group and $\rho : G \times X \to X$ a continuous group action, then the quotient topology on X/G is also metrizable.*

Proof. Let d be a metric on X. We first show that averaging over G gives rise to a G-invariant metric d_G on X inducing the same topology. First we equip G with a left invariant Riemannian metric and define a map $d_G : X \times X \to \mathbb{R}_+$ by

$$d_G(x,y) = \int_G d(gx, gy)dg.$$

Then d_G is a G-invariant metric and d_G is continuous. We claim that d_G induces the same topology as d. The continuity implies any d_G-ball contains a d-ball. It suffices to prove that any d-ball contains a d_G-ball. Assume otherwise that there is a d-ball does not contain any d_G-ball, i.e. there exist $\{y_k\}_{k\in\mathbb{N}}$, such that $\lim d_G(x, y_k) = 0$ and $d(x, y_k) > C$. Let $f_k := d(gx, gy_k) \in \mathrm{C}^0(G)$. Then $\lim_k d_G(x, y_k) = 0$ implies that the functions f_k converge to 0 in the space of absolutely integrable functions $L^1(G)$. After passing to a subsequence, f_k converge to 0 almost everywhere on G. Then there is a $g_0 \in G$, such that $\lim_k f_k(g_0) = \lim_k d(g_0 x, g_0 y_k) = 0$. Since g_0 acts continuously on X, this implies $d(x, y_k) \to 0$, which contradicts the assumption $d(x, y_k) > C$. Hence d, d_G induce the same topology on X.

Using d_G, we can equip X/G a metric as follows. Let π denote the quotient map $X \to X/G$. Then over X/G, we define

$$d_{X/G}(\pi(x), \pi(y)) := \min_{g\in G} d_G(x, gy).$$

For fixed $x, y \in X$, $d_G(x, gy)$ is a continuous function on the compact space G. Therefore $\min_{g\in G} d_G(x, gy)$ exists. Because d_G is G-invariant, $d_{X/G}$ is well defined, i.e. it only depends on $\pi(x), \pi(y)$. We claim $d_{X/G}$ is a metric on X/G. Since $d_G \geq 0$, $d_{X/G} \geq 0$. If $d_{X/G}(\pi(x), \pi(y)) = 0$, then there exists $g_0 \in G$ such that $d_G(x, g_0 y) = 0$, hence $x = g_0 y$ and $\pi(x) = \pi(y)$. Since $d_G(x, gy) = d_G(g^{-1}, y)$, we have

$$\begin{aligned} d_{X/G}(\pi(x), \pi(y)) &= \min_{g\in G} d_G(x, gy) \\ &= \min_{g\in G} d_G(g^{-1}x, y) \\ &= \min_{g\in G} d_G(y, g^{-1}x) \\ &= \min_{g\in G} d_G(y, gx) \\ &= d_{X/G}(\pi(y), \pi(x)), \end{aligned}$$

Therefore $d_{X/G}$ is symmetric. Finally, we check the triangle inequality. For $x, y, z \in X$, pick $g, h \in G$ such that $d_{X/G}(\pi(x), \pi(y)) = d_G(x, gy)$ and $d_{X/G}(\pi(y), \pi(z)) = d_G(y, hz)$, since we have $d_G(y,hz) = d_G(gy,ghz)$, then

$$\begin{aligned} d_{X/G}(\pi(x),\pi(y)) + d_{X/G}(\pi(y),\pi(z)) &= d_G(x,gy) + d_G(y,hz) \\ &= d_G(x,gy) + d_G(gy,ghz) \\ &\geq d_G(x,ghz) \\ &\geq d_{X/G}(\pi(x),\pi(z)). \end{aligned}$$

Therefore $d_{X/G}$ is a metric.

To finish the proof, it suffices to show that $d_{X/G}$ induces the quotient topology. Assume $U \subset X/G$ is open in the quotient topology. Recall that $U \subset X/G$ is open iff $\pi^{-1}(U) \subset X$ is open. Then for every point $x \in \pi^{-1}(U)$, there exists a d_G-ball $B_r^{d_G}(x) \subset \pi^{-1}(U)$ of radius $r > 0$. We claim the $d_{X/G}$-ball $B_r^{d_{X/G}}(\pi(x))$ is contained in U. This is because for any point $\pi(y) \in B_r^{d_{X/G}}(\pi(x))$, there exists $g_0 \in G$ such that $d_G(x,g_0y) < r$. Therefore we have $g_0y \in \pi^{-1}(U)$, hence $\pi(y) \in U$. Conversely, it suffices to check that every $d_{X/G}$-ball is open in the quotient topology. For a $d_{X/G}$-ball $B_r^{d_{X/G}}(\pi(x))$, we claim $\pi^{-1}(B_r^{d_{X/G}}(\pi(x))) = \rho(G, B_r^{d_G}(x))$. This implies $B_r^{d_{X/G}}(\pi(x))$ is open in the quotient topology. To prove the claim, note that $y \in \pi^{-1}(B_r^{d_{X/G}}(\pi(x)))$ iff $d_{X/G}(\pi(x),\pi(y)) < r$, that is there exists $g \in G$ such that $d_G(x,gy) < r$, which is equivalent to $y \in \rho(G, B_r^{d_G}(x))$. □

3.2 Free quotients of Fredholm sections

Basics of sc-Fredholm sections

The most important feature of polyfold theory is that it has a Fredholm theory package [65, Chapter 3, 5], which includes an implicit function theorem and a perturbation theory. The definition of sc-Fredholm section in polyfold is more involved than the classical Fredholm theory, since the Fredholmness of linearization is not sufficient for an implicit function theorem, also see [37].

Recall from Definition 3.12, strong M-polyfold bundles are modeled on bundle retracts, which can be very complicated subset of a sc-Banach space. When analyzing zero sets of M-polyfold sections, we need to extend sections from retracts to the underlying sc-Banach spaces so that we can apply the implicit function theorem [65, Theorem 3.7] developed for sc-Banach spaces. Moreover, such extensions should not change the zero sets. Therefore the following concept was introduced in [65, Definition 3.4].

Definition 3.32. *Let $R : U \lhd \mathbb{F} \to U \lhd \mathbb{F}, (x,e) \mapsto (r(x), \varrho(x)e)$ be a tame strong bundle retraction and $s : r(U) \to R(U \lhd F)$ a sc-smooth section. A filled section s^f at $(0,0) \in U \subset \mathbb{R}^m_+ \times \mathbb{E}$ is a section $s^f : U \to U \lhd \mathbb{F}$ with the following properties.*

1. *$s(x) = s^f(x)$ for $x \in r(U)$.*
2. *If $s^f(y) = \varrho(r(y))s^f(y)$ for a point $y \in U$, then $y \in \operatorname{im} r$.*

3. *Let* $S(y) := s^f(y) - \varrho(r(y))s^f(y)$. *Then* $\mathrm{D}S_{(0,0)}|_{\ker \mathrm{D}r_{(0,0)}} : \ker \mathrm{D}r_{(0,0)} \to \mathbb{F}$ *is an isomorphism onto* $\ker \varrho(0,0)$.

We first show that fillings induce fillings for any restriction of the section to a slice in the sense of Definition 3.17.

Proposition 3.33. *Let* $R = (r, \varrho)$ *be a* $\mathbb{R}^n$*-sliced bundle retraction on* $U \lhd \mathbb{F}$ *for an open neighborhood* $U \subset \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}$ *of* $(0,0,0)$ *covering a* $\mathbb{R}^n$*-sliced retraction* r. *Let* $\tilde{U} := (\mathbb{R}^m_+ \times \{0\} \times \mathbb{K}) \cap U$, $\tilde{R} := R|_{\tilde{U} \lhd F}$ *and* $\tilde{r} := r|_{\tilde{U}}$. *Assume* $s : r(U) \to R(U \lhd \mathbb{F})$ *be a section with filling* $s^f : U \to \mathbb{F}$. *Then the restriction to the slice* $\tilde{s} := s|_{r(\tilde{U})} : \tilde{r}(\tilde{U}) \to \tilde{R}(\tilde{U} \lhd \mathbb{F})$ *has a filling* $\tilde{s}^f := s^f|_{\tilde{U}}$.

Proof. It suffices to check the three properties of Definition 3.32. Property (1) holds because $s^f|_{\tilde{U}}$ is the restriction of s^f. For property (2), if $\tilde{s}^f(y) = \varrho(\tilde{r}(y))\tilde{s}^f(y)$ for a point $y \in \tilde{U}$, then $s^f(y) = \varrho(r(y))s^f(y)$. By the property (2) of the filling s^f, we have $y \in \operatorname{im} r$. Since $\operatorname{im} \tilde{r} = \tilde{U} \cap \operatorname{im} r$, we have $y \in \operatorname{im} \tilde{r}$.

For property (3), let $\tilde{S}(y) := \tilde{s}^f(y) - \varrho(\tilde{r}(y))\tilde{s}^f(y)$ for $y \in \tilde{U}$ and $S(y) := s^f(y) - \varrho(r(y))s^f(y)$ for $y \in U$. Then $\tilde{S} = S|_{\tilde{U}}$, hence we have $\mathrm{D}\tilde{S}_{(0,0)} = \mathrm{D}S_{(0,0,0)}|_{\mathbb{R}^m \times \{0\} \times \mathbb{K}}$. Since r is $\mathbb{R}^n$-sliced, i.e. $\pi_{\mathbb{R}^n} \circ r = \pi_{\mathbb{R}^n}$, we have $\ker \mathrm{D}r_{(0,0,0)} = \ker \mathrm{D}\tilde{r}_{(0,0)} \subset \mathbb{R}^m \times \{0\} \times \mathbb{K}$. Therefore, we have

$$\mathrm{D}\tilde{S}_{(0,0)}|_{\ker \mathrm{D}(r_0)_{(0,0)}} = \mathrm{D}S_{(0,0,0)}|_{\ker \mathrm{D}r_{(0,0,0)}}.$$

Therefore $\mathrm{D}\tilde{S}_{(0,0)}|_{\ker \mathrm{D}\tilde{r}_{(0,0)}}$ is an isomorphism onto $\ker \varrho(0,0,0)$ due to property (3) of the filling s^f. □

The key ingredient of the Fredholm property in polyfold theory is the following contracting property, so that implicit function theorem [65, Theorem 3.7] holds. Note that for a C^1 map between Banach spaces, the following contracting property holds after a C^1 change of coordinate as long as the map has a Fredholm linearization.

Definition 3.34 ([65, Definition 3.6])**.** *Let* $C := \mathbb{R}^m_+ \times \mathbb{R}^k$ *and let* $\mathbb{W}$ *be a sc-Banach space. A* ***basic germ*** *is a sc-smooth germ at* $(0,0)$ *represented by a sc-smooth section*

$$U \to U \lhd \mathbb{R}^N \times \mathbb{W}, \quad x \mapsto (x, f(x)),$$

on an open neighborhood $U \subset C \times \mathbb{W}$ *of* $(0,0)$, *such that* $f(0,0) = 0$ *and*

$$B(c,w) := \pi_{\mathbb{W}} f(c,w) - w, \quad \forall (c,w) \in U$$

has the contracting property: For every $m \in \mathbb{N}, \epsilon > 0$, *there exists a neighborhood* $U_{\epsilon,m} \subset U_m$ *of* $(0,0)$, *so that*

$$||B(c,w) - B(c,w')||_m \le \epsilon ||w - w'||_m, \qquad \forall (c,w),(c,w') \in U_{\epsilon,m}. \tag{3.13}$$

Then the definition of sc-Fredholm sections of strong M-polyfold bundles is the combination of Definition 3.32 and Definition 3.34.

Definition 3.35 ([65, Definition 3.8]). *Let $p : \mathcal{Y} \to \mathcal{X}$ be a M-polyfold bundle and $s : \mathcal{X} \to \mathcal{Y}$ a sc-smooth section. A bundle chart $(\mathcal{O}, \Phi, (P, (\mathbb{R}^m_+ \times \mathbb{E}) \lhd \mathbb{F}))$ around $x_0 \in \mathcal{X}_\infty$ along with a strong bundle isomorphism $A : U \lhd \mathbb{F} \to U' \lhd (\mathbb{R}^N \times \mathbb{W}), (x, v) \mapsto (\alpha(x), \Lambda(x)v)$ is a* ***Fredholm chart*** *of s around x_0 if the following holds.*

1. *$U \subset \mathbb{R}^m_+ \times \mathbb{E}, U' \subset C \times \mathbb{W} := \mathbb{R}^m_+ \times \mathbb{R}^k \times \mathbb{W}$ are open neighborhoods of $(0, 0)$, such that $R(U \lhd \mathbb{F}) = P$.*
2. *$\Phi_* s$ has a filling $s^f : U \to \mathbb{F}$.*
3. *$\alpha(0, 0) = (0, 0)$ and there exists a sc^+-section $\gamma : U' \to U' \lhd (R^N \times \mathbb{W})$ such that $A_* s^f - \gamma$ is a basic germ at $(0, 0)$.*

The section s is ***sc-Fredholm section*** *iff the following two conditions hold.*

- ***Regularization property****: if $s(x) \in \mathcal{Y}[1]_m$, then $x \in \mathcal{X}_{m+1}$.*
- ***Basic germ property****: for every $x_0 \in \mathcal{X}_\infty$, there exists a Fredholm chart around x_0 (Definition 3.35).*

A sc-Fredholm section is proper, if $s^{-1}(0)$ is compact in the $\mathcal{X}_0$ topology.

Remark 3.36. *Note that the regularization property implies that $s^{-1}(0) \subset \mathcal{X}_\infty$. By [65, Theorem 5.3], when $s^{-1}(0)$ is compact in the $\mathcal{X}_0$ topology, then $s^{-1}(0)$ is compact in the $\mathcal{X}_\infty$ topology*[2]*. As a consequence $s^{-1}(0)$ is compact in $\mathcal{X}_i$ topology for every i. Therefore we will not specify the topology for the compactness going forward.*

Sc-Fredholm property of quotients

The goal of this subsection is to prove part 2 of Theorem 3.1, namely that a proper G-equivariant sc-Fredholm section descends to a proper sc-Fredholm section on the free quotient of an M-polyfold bundle. The proof can be found at the end of this subsection. Most parts of this proof work with the quotient of the section in local coordinates, i.e. with restrictions of the sc-Fredholm section to G-slices (Definition 3.26) established in Lemma 3.27. Thus we will more generally consider restrictions of sc-Fredholm sections to slices of M-polyfolds, as defined in Definition 3.17. The following proposition follows directly from the definition of regularization property in Definition 3.35.

Proposition 3.37. *Let $p : \mathcal{Y} \to \mathcal{X}$ be a tame strong M-polyfold bundle with $s : \mathcal{X} \to \mathcal{Y}$ a sc-Fredholm section. Assume $p^{-1}(\tilde{\mathcal{O}})$ is a bundle slice (Definition 3.26), then $\tilde{s} := s|_{\tilde{\mathcal{O}}} : \tilde{\mathcal{O}} \to p^{-1}(\tilde{\mathcal{O}})$ has the regularization property in Definition 3.35.*

[2]The $\mathcal{X}_\infty$ topology is the inverse limit topology of $\mathcal{X}_i$ topology on $\mathcal{X}_\infty$.

Next we need to address the question of whether the restriction $\tilde{s}$ has the basic germ property in Definition 3.35. Since the main condition of the basic germ property is the contracting property (3.13) after a bundle isomorphism A in Definition 3.35, the basic germ property of $\tilde{s}$ requires some compatibility between the bundle isomorphism A and the slice $\tilde{\mathcal{O}}$. To be more precise, we introduce the following notion of good slice to ensure the restriction $s|_{\tilde{\mathcal{O}}}$ inherits the basic germ property from s, see Lemma 3.40

Definition 3.38. *A slice $\tilde{\mathcal{O}}$ of $\mathcal{X}$ around $x_0 \in \mathcal{X}_\infty$ is **good** with respect to a sc-Fredholm section $s : \mathcal{X} \to \mathcal{Y}$, if we have the following.*

1. *There is a Fredholm chart $(\mathcal{O}, \Phi, (P, (\mathbb{R}^m_+ \times \mathbb{E}) \lhd \mathbb{F}), A)$ around x_0 for s covering a M-polyfold chart $(\mathcal{O}, \phi, (O, \mathbb{R}^m_+ \times \mathbb{E}))$. R is a strong bundle retraction covering a tame traction r such that $P = R(U \lhd \mathbb{F})$ and $O = r(U)$ for an open neighborhood $U \subset \mathbb{R}^m_+ \times \mathbb{E}$ of $(0,0)$.*

2. *The strong bundle isomorphism $A : U \lhd \mathbb{F} \to U' \lhd (\mathbb{R}^N \times \mathbb{W})$ is in the form of $(x, v) \mapsto (\alpha(x), \Lambda(x)v)$, where α is a sc-diffeomorphism from U to a neighborhood $U' \subset C \times \mathbb{W} := \mathbb{R}^m_+ \times \mathbb{R}^k \times \mathbb{W}$ of $(0,0)$ and $\Lambda(x)$ is a linear isomorphism from $\mathbb{F}$ to $\mathbb{R}^N \times \mathbb{W}^N$.*

3. *There is a sc-diffeomorphism $h : U \to U'' \subset \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}$ such that $h \circ r \circ h^{-1} : U'' \to U''$ is $\mathbb{R}^n$-sliced, $h(0,0) = (0,0,0)$ and $\tilde{\mathcal{O}} = \phi^{-1} \circ r \circ h^{-1}((\mathbb{R}^m_+ \times \{0\} \times \mathbb{K}) \cap U'')$.*

4. *$\pi_{\mathbb{R}^n} \circ \mathrm{D}(h \circ \alpha^{-1})_{(0,0)}|_{\{0\} \times \mathbb{W}} : \mathbb{W} \to \mathbb{R}^n$ is surjective.*

Remark 3.39. *Suppose the slice is constructed as in Lemma 3.19 from a submersive map f. Then the following two equivalent conditions imply goodness. Let $E := T^R_{x_0}\mathcal{X}, F := \ker \mathrm{D} f_{x_0} \cap T^R_{x_0}\mathcal{X}$.*

1. *There exists a n dimensional subspace $L \subset T^R_{x_0}\mathcal{X}_\infty$, such that it is a complement of F in E and $\mathrm{D}(\alpha \circ \phi)_{x_0}(L) \subset \{0\} \times \mathbb{W}$. This is the s-compatibly transversality condition in [36, Definition 5.8]*

2. *$\mathrm{D}(\alpha \circ \phi)_{x_0}(F) \cap \{0\} \times \mathbb{W}_\infty$ is a n-codimensional subspace in $\mathrm{D}(\alpha \circ \phi)_{x_0}(E) \cap \{0\} \times \mathbb{W}_\infty$.*

The second condition is generically satisfied if we can perturb F, which is exactly the proof in Proposition 3.42 of the existence of good slices.

Proof. By (3.19) of Lemma 3.19, (1) implies Definition 3.38. Next we will prove (1)$\Leftrightarrow$(2). If $L \subset T^R_{x_0}\mathcal{X}_\infty$ is a complement of F in E, such that $\mathrm{D}(\alpha \circ \phi)_{x_0}(L) \subset \{0\} \times \mathbb{W}$, then $\mathrm{D}(\alpha \circ \phi)_{x_0}(L)$ is the complement of $\mathrm{D}(\alpha \circ \phi)_{x_0}(F) \cap \{0\} \times \mathbb{W}_\infty$ in $\mathrm{D}(\alpha \circ \phi)_{x_0}(E) \cap \{0\} \times \mathbb{W}_\infty$. Hence (1)$\Rightarrow$(2). If $\mathrm{D}(\alpha \circ \phi)_{x_0}(F) \cap \{0\} \times \mathbb{W}_\infty$ is n-codimensional in $\mathrm{D}(\alpha \circ \phi)_{x_0}(E) \cap \{0\} \times \mathbb{W}\infty$. Let V be the n-dimensional complement, then there exists $L \subset E_\infty$ such that $\mathrm{D}(\alpha \circ \phi)_{x_0}(L) = V$. Then L is the complement of F in E satisfying condition (1). $\square$

Lemma 3.40. *Let $x_0 \in \mathcal{X}_\infty$ and $\tilde{\mathcal{O}}$ a good slice of $\mathcal{X}$ around x_0 with respect to a sc-Fredholm section $s : \mathcal{X} \to \mathcal{Y}$. Then there is a Fredholm chart around x_0 for $\tilde{s} := s|_{\tilde{\mathcal{O}}^1} : \tilde{\mathcal{O}}^1 \to p^{-1}(\tilde{\mathcal{O}}^1)$.*

Proof. By the assumption of $\tilde{\mathcal{O}}$ being a good slice, we have the following structures and properties:

1. a strong bundle chart $(\mathcal{O}, \Phi, (P, (\mathbb{R}^m_+ \times \mathbb{E}) \triangleleft \mathbb{F}))$ around x_0 covering a M-polyfold chart $(\mathcal{O}, \phi, (O, \mathbb{R}^m_+ \times \mathbb{E}))$, where $P = R(U \triangleleft \mathbb{F})$ and $O = r(U)$ for bundle retraction R covering tame retraction r and open neighborhood $U \subset \mathbb{R}^m_+ \times \mathbb{E}$ of $(0,0)$;

2. $\Phi_* s : O \to P$ has a filling $s^f : U \to U \triangleleft \mathbb{F}$ as in Definition 3.32;

3. a strong bundle isomorphism $A : U \triangleleft \mathbb{F} \to U' \triangleleft (\mathbb{R}^N \times \mathbb{W})$ for an open neighborhood $U' \subset C \times \mathbb{W} := \mathbb{R}^m_+ \times \mathbb{R}^k \times \mathbb{W}$ of $(0,0)$, where $A = (\alpha, \Lambda)$ consisting of a sc-diffeomorphism $\alpha : U \to U'$ and a sc$^\infty$ family of linear isomorphism $\Lambda(x) : \mathbb{F} \to \mathbb{R}^N \times \mathbb{W}$ parametrized by $x \in U$;

4. a sc$^+$-section $\gamma : U' \to U' \triangleleft (\mathbb{R}^N \times \mathbb{W})$ such that $A_* s^f - \gamma$ is a basic germ at $(0,0)$;

5. a sc-diffeomorphism $h : U \to U'' \subset \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}$ such that $h \circ r \circ h^{-1} : U'' \to U''$ is $\mathbb{R}^n$-sliced, $h(0,0) = (0,0,0)$ and $\tilde{\mathcal{O}} = \phi^{-1} \circ r \circ h^{-1}((\mathbb{R}^m_+ \times \{0\} \times \mathbb{K}) \cap U'')$;

6. $\pi_{\mathbb{R}_n} \circ \mathrm{D}(h \circ \alpha^{-1})_{(0,0)}|_{\{0\}\times\mathbb{W}} : \mathbb{W} \to \mathbb{R}^n$ is surjective.

Sc-smooth map $H := h \triangleleft \mathrm{id}_{\mathbb{F}} : U \triangleleft \mathbb{F} \to U'' \triangleleft \mathbb{F}$ is a strong bundle isomorphism. Let $R'' := H \circ R \circ H^{-1}$, which is a $\mathbb{R}^n$-sliced bundle retraction on $U'' \triangleleft \mathbb{F}$. Then $(\mathcal{O}, H \circ \Phi, (R''(U'' \triangleleft \mathbb{F}), (\mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}) \triangleleft \mathbb{F}))$ is another strong bundle chart around x_0. Then $H_* s^f$ is a filling of $(H \circ \Phi)_* s$ by [36, proof of part (III) Lemma 4.2]. Let $\tilde{U}'' := (\mathbb{R}^m_+ \times \{0\} \times \mathbb{K}) \cap U''$ and $\tilde{s}^f := H_* s^f|_{\tilde{U}''}$. By Proposition 3.33, $\tilde{s}^f$ is a filling of $\tilde{s} : \tilde{\mathcal{O}} \to p^{-1}(\tilde{\mathcal{O}})$ in the chart induced by $(\mathcal{O}, H \circ \Phi, (R''(U'' \triangleleft \mathbb{F}), (\mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}) \triangleleft \mathbb{F}))$.

It remains to verify that $\tilde{s}^f$ is basic germ after a bundle isomorphism. We claim that there exist a sc-Banach subspace $\mathbb{T} \subset \mathbb{W}$ with a finite dimensional complement $T^\perp$, a sc-diffeomorphism $q : U_\vartriangle \to U_\blacktriangle$ for open neighborhoods $U_\vartriangle \subset C \times \mathbb{T}^1, U_\blacktriangle \subset (\tilde{U}'')^1$ of $(0,0)$ and a strong bundle isomorphism $Q : U_\vartriangle \triangleleft (\mathbb{R}^N \times \mathbb{W}^1) \to U_\blacktriangle \triangleleft \mathbb{F}^1$ covering q such that $Q^* \tilde{s}^f$ is a basic germ. For better visualization of all the structures, we have the following diagram.

$$
\begin{tikzcd}
C \times \mathbb{T}^1 \supset U_\vartriangle \arrow[r, "Q^* \tilde{s}^f"] \arrow[d, "q"] & U_\vartriangle \lhd (\mathbb{R}^N \times \mathbb{W}^1) = U_\vartriangle \lhd (\mathbb{R}^N \times T^\perp \times \mathbb{T}^1) \arrow[d, "Q"] & & & & & \\
\mathbb{R}^m_+ \times \{0\} \times \mathbb{K}^1 \supset U_\blacktriangle \arrow[r, "\tilde{s}^f"] & U_\blacktriangle \lhd \mathbb{F}^1 & & & & & \\
\cap & \cap & & & & & \\
\mathbb{R}^m_+ \times \{0\} \times \mathbb{K} \supset \tilde{U}'' \arrow[r, "\tilde{s}^f"] \arrow[d, "\iota"] & \tilde{U}'' \lhd \mathbb{F} \arrow[d, "\iota \lhd \mathrm{id}_\mathbb{F}"] & \supset & \tilde{r}''(\tilde{U}'') \arrow[r, "(H\circ\Phi)_*\tilde{s}"] \arrow[d] & \tilde{R}''(\tilde{U}'' \lhd \mathbb{F}) \arrow[d] & \tilde{\mathcal{O}} \arrow[l, dotted, "H\circ\Phi"'] \arrow[r, "\tilde{s}"] \arrow[d] & p^{-1}(\tilde{\mathcal{O}}) \arrow[d] \\
\mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K} \supset U'' \arrow[r, "H_* s^f"] & U'' \lhd \mathbb{F} & \supset & r''(U'') \arrow[r, "(H\circ\Phi)_* s"] & R''(U'' \lhd \mathbb{F}) & \mathcal{O} \arrow[l, dotted, "H\circ\Phi"'] \arrow[r, "s"] & p^{-1}(\mathcal{O}) \\
\mathbb{R}^m_+ \times \mathbb{E} \supset U \arrow[r, "s^f"] \arrow[u, "h"] \arrow[d, "\alpha"] & U \lhd \mathbb{F} \arrow[u, "H"] \arrow[d, "A"] & \supset & r(U) \arrow[r, "\Phi_* s"] \arrow[u, "h"] & R(U \lhd \mathbb{F}) \arrow[u, "H"] & \mathcal{O} \arrow[l, dotted, "\Phi"'] \arrow[r, "s"] \arrow[u] & p^{-1}(\mathcal{O}) \arrow[u] \\
C \times \mathbb{W} \supset U' \arrow[r, "A_* s^f"] & U' \lhd (\mathbb{R}^N \times \mathbb{W}) & & & & &
\end{tikzcd}
$$

Filled sections on partial quadrants Sections on retracts Sections on M-polyfolds

Here $\iota : \tilde{U}'' \to U''$ is the inclusion and $\tilde{r}'' := r''|_{\tilde{U}''}$ and $\tilde{R}'' := R''|_{\tilde{U}''}$ are the retractions on the slice.

Let $\beta := \alpha \circ h^{-1}$. Since $\pi_{\mathbb{R}^n} \circ \mathrm{D}\beta^{-1}_{(0,0)}|_{\{0\}\times\mathbb{W}} : \mathbb{W} \to \mathbb{R}^n$ is surjective,

$$\mathbb{T} := \ker \left(\pi_{\mathbb{R}^n} \circ \mathrm{D}\beta^{-1}_{(0,0)}|_{\{0\}\times\mathbb{W}} \right)$$

is a subspace in $\mathbb{W}$ of codimension n. Because $\pi_{\mathbb{R}^n} \circ \mathrm{D}\beta^{-1}_{(0,0)}(\{0\} \times \mathbb{W}_\infty)$ is a dense subspace of $\pi_{\mathbb{R}^n} \circ \mathrm{D}\beta^{-1}_{(0,0)}(\{0\} \times \mathbb{W}) = \mathbb{R}^n$, we have $\pi_{\mathbb{R}^n} \circ \mathrm{D}\beta^{-1}_{(0,0)}(\mathbb{W}_\infty) = \mathbb{R}^n$. Therefore we can pick n preimages of the basis of $\mathbb{R}^n$ in $\mathbb{W}_\infty$. They form a subspace $T^\perp \subset \mathbb{W}_\infty$. Then we have a splitting $\mathbb{W} = \mathbb{T} \oplus T^\perp$ and $\pi_{\mathbb{R}^n} \circ \mathrm{D}\beta^{-1}_{(0,0)}|_{\{0\}\times T^\perp} : T^\perp \to \mathbb{R}^n$ is an isomorphism. For $(u, e) \in \mathbb{R}^m_+ \times \mathbb{K}^1$ close to $(0, 0)$, we define a sc-smooth map q' by

$$q' : (u, e) \mapsto (\pi_C \circ \beta \circ \iota(u, e), \pi_\mathbb{T} \circ \pi_\mathbb{W} \circ \beta \circ \iota(u, e)). \tag{3.14}$$

That is q' is the composition of the ι, β and the projection from $C \times \mathbb{W}^1$ to $C \times \mathbb{T}^1$. We claim q' has an inverse given by

$$q : (c, t) \mapsto \beta^{-1}(c, t + \theta(c, t)),$$

where $(c, t) \in C \times \mathbb{T}^1$ close $(0, 0)$ and $\theta(c, t) \in T^\perp$ solves the following equation,

$$\pi_{\mathbb{R}^n} \circ \beta^{-1}(c, t + \theta) = 0. \tag{3.15}$$

Since $\omega(c,t,\theta) := \pi_{\mathbb{R}^n} \circ \beta^{-1}(c,t+\theta)$ is a sc-smooth function from an open neighborhood of $(0,0,0)$ in $C\times\mathbb{T}\times T^{\perp}$ to $\mathbb{R}^n$ and $\mathrm{D}\omega_{(0,0,0)}(0,0,u) = \pi_{\mathbb{R}^n}\circ \mathrm{D}_{(0,0)}\beta^{-1}(u)$, i.e. $\mathrm{D}\omega_{(0,0,0)}$ is invertible on $T^{\perp}$. By Lemma 3.28, there is a sc-smooth map $\theta(c,t)$ from an open neighborhood $U_{\square} \subset C\times\mathbb{T}^1$ of $(0,0)$ to $T^{\perp}$, such that $\theta(c,t)$ solves the equation $\omega(c,t,\theta)=0$ for $(c,t)\in U_{\square}$ and $\theta \in D$, where $D\subset T^{\perp}$ is an open neighborhood of 0. Moreover, $\theta(0,0)=0$. This shows q is sc-smooth near $(0,0)$. We now prove q is the inverse to q'. First for $(c,t)\in U_{\square}$

$$\begin{aligned} q'\circ q(c,t) &= \pi_{C\times\mathbb{T}}\circ\beta\circ\iota\circ\beta^{-1}(c,t+\theta(c,t)) \\ &= \pi_{C\times\mathbb{T}}\circ\beta\circ\beta^{-1}(c,t+\theta(c,t)) \\ &= (c,t). \end{aligned}$$

On the other hand, if $(u,e)\in(\mathbb{R}^m_+\times\mathbb{K}^1)\cap q'^{-1}(U_{\square})\cap\pi_{T^{\perp}}\circ\beta\circ\iota^{-1}(D)$ and let $(c,t):=q'(u,e)=\pi_{C\times\mathbb{T}}\circ\beta\circ\iota(u,e)$. Then $(c,t+\pi_{T^{\perp}}\circ\beta\circ\iota(u,e))=\beta(u,e)$, which means $\theta:=\pi_{T^{\perp}}\circ\beta\circ\iota(u,e)$ solves equation (3.15) in $U_{\square}\times D$. Then we have on $(\mathbb{R}^m_+\times\mathbb{K}^1)\cap q^{-1}(U_{\square})\cap\pi_{T^{\perp}}\circ\beta\circ\iota^{-1}(D)$,

$$\begin{aligned} q\circ q'(u,e) &= \beta^{-1}\left(q'(r,k)+(0,\pi_{T^{\perp}}\circ\beta\circ\iota(u,e))\right) \\ &\overset{(3.14)}{=} \beta^{-1}\circ\beta\circ\iota(u,e) \\ &= (u,e). \end{aligned}$$

Therefore q is the inverse to q' locally. Moreover, q has the following property

$$\beta\circ\iota\circ q(c,t)=\beta\circ q(c,t)=(c,t+\theta(c,t)). \tag{3.16}$$

We choose open neighborhoods $U_{\blacktriangle}\subset\tilde{U}'', U_{\triangle}\subset U_{\square}$, such that $q':U_{\blacktriangle}\to U_{\triangle}, q:U_{\triangle}\to U_{\blacktriangle}$ are inverse to each other.

The strong bundle isomorphism $Q:U_{\triangle}\triangleleft(\mathbb{R}^N\times\mathbb{W}^1)\to U_{\blacktriangle}\triangleleft\mathbb{F}^1$ is defined by $(x,v)\mapsto(q(x),\Lambda(h^{-1}\circ\iota\circ q(x))^{-1}v)$. It has an inverse $(x,v)\mapsto(q'(x),\Lambda(h^{-1}\circ\iota(x))v)$. It remains to show that $Q^*\tilde{s}^f$ is a basic germ at $(0,0)$.

First, we have

$$\begin{aligned} Q^*\tilde{s}^f(c,t) &= \Lambda(h^{-1}\circ\iota\circ q(c,t))\tilde{s}^f(q(c,t)) \\ &= \Lambda(h^{-1}\circ\iota\circ p(c,t))s^f(h^{-1}\circ\iota\circ q(c,t)) \\ &= A_*s^f(\alpha\circ h^{-1}\circ\iota\circ q(c,t)). \end{aligned}$$

Let $\theta:=Q^*\iota_*H_*A^*\lambda:U_{\triangle}\to U_{\triangle}\triangleleft(\mathbb{R}^N\times\mathbb{W}^1)$, which is a sc$^+$-section on $U_{\triangle}$. Let $B(c,w)$ denote $\pi_{\mathbb{W}}\circ A_*s^f(c,w)-\pi_{\mathbb{W}}\gamma(c,w)-w$ and $\tilde{B}(c,t)$ denote $\pi_{\mathbb{T}}\circ Q^*\tilde{s}^f(c,t)-\pi_{\mathbb{T}}\circ\theta(t,w)-t=\pi_{\mathbb{T}}\circ A_*s^f\circ\beta\circ\iota\circ q(c,t)-\pi_{\mathbb{T}}\circ\gamma\circ\beta\circ\iota\circ q(c,t)-t$. To show $Q^*\tilde{s}^f$ is a basic germ, it suffices to prove $\tilde{B}(c,t)$ has the contracting property (3.13). By (3.16) we have

$$\begin{aligned} \tilde{B}(c,t) &= \pi_{\mathbb{T}}\circ A_*s^f\circ\beta\circ\iota\circ q(c,t)-\pi_{\mathbb{T}}\circ\gamma\circ\beta\circ\iota\circ q(c,t)-t \\ &= \pi_{\mathbb{T}}\circ A_*s^f(c,t+\theta(c,t))-\pi_{\mathbb{T}}\circ\gamma(c,t+\theta(c,t))-t \\ &= \pi_{\mathbb{T}}\left(B(c,t+\theta(c,t))\right). \end{aligned}$$

Since $B(c, w)$ has contracting property (3.13) and $\theta(0,0) = 0$, $\forall \epsilon > 0, m \geq 1 \in \mathbb{N}$ there exist an open neighborhood $U_{\epsilon,m} \subset C \times \mathbb{T}_m$ of $(0,0)$, over which we have:

$$||\tilde{B}(c,t_1) - \tilde{B}(c,t_2)||_{\mathbb{T}_m} \leq \epsilon||\theta(c,t_1) + t_1 - \theta(c,t_2) - t_2||_{\mathbb{W}_m}.$$

Since $\theta(c,t)$ is C^1 on $C \times \mathbb{T}_m$ for $m \geq 1$ by Lemma 3.28, $||\theta(c,t_1) - \theta(c,t_2)||_{\mathbb{W}_m}$ is bounded by $C_m||t_1 - t_2||_{\mathbb{T}_m}$ on $U(m) \subset C \times \mathbb{T}_m$. Since $||t_1 - t_2||_{\mathbb{W}_m} = ||t_1 - t_2||_{\mathbb{T}_m}$ for $t_1, t_2 \in \mathbb{T}_m$, we have

$$||\tilde{B}(c,t_1) - \tilde{B}(c,t_2)||_{\mathbb{T}_m} \leq \epsilon(C_m + 1)||t_1 - t_2||_{\mathbb{T}_m}, \quad \forall (c,t_1),(c,t_2) \in U_{\epsilon,m} \cap U(m).$$

This proves that $Q^*\tilde{s}^f$ is a basic germ at $(0,0)$. □

Remark 3.41. *In the proof of Lemma 3.40, a good slice is used to get a sc-diffeomorphism q such that (3.16) holds. The importance of (3.16) is that $\beta \circ \iota \circ q$ does not change the coordinate in C, so that we can use the contracting property of $A_* s^f$.*

Next we prove the existence of good slices. We first prove a more general result that will also be used in later in the construction of polyfold quotient. When constructing slices using Lemma 3.24, we can choose different complements $\mathbb{H}$ to construct different changes of coordinates h. When the base M-polyfold is infinite dimensional, then it is always possible to find a complement $\mathbb{H}$ such that the slice constructed in Lemma 3.24 is good. To be more specific, we have the following proposition.

Proposition 3.42. *Let $((\mathcal{O}, \Phi, (P, (\mathbb{R}^m_+ \times \mathbb{E}) \triangleleft \mathbb{F})), A)$ be a Fredholm chart (Definition 3.35) of strong M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$ around $x_0 \in \mathcal{X}_\infty$ with respect to a sc-Fredholm section $s : \mathcal{X} \to \mathcal{Y}$. Suppose the covered chart on the base is $(\mathcal{O}, \phi, (O, \mathbb{R}^m_+ \times \mathbb{E}))$. Assume the bundle isomorphism A is the form of*

$$U \triangleleft \mathbb{F} \to U' \triangleleft (\mathbb{R}^N \times \mathbb{W}), \quad (x,v) \mapsto (\alpha(x), \Lambda(x)v),$$

where $U \subset \mathbb{R}^m_+ \times \mathbb{E}$ and $U' \subset C \times \mathbb{W} := \mathbb{R}^m_+ \times \mathbb{R}^k \times \mathbb{W}$ are neighborhoods of $(0,0)$. Under the assumptions of Lemma 3.24, that is

1. *for a neighborhood $B \subset \mathbb{R}^n$ of 0 we have a strong bundle map $\Lambda : B \times p^{-1}(\mathcal{O}) \to \mathcal{Y}$ such that $\Lambda(0, \cdot) = \mathrm{id}_{p^{-1}(\mathcal{O})}$;*
2. *let $\Gamma : B \times \mathcal{O} \to \mathcal{X}$ denote the map on the base covered by Λ, then $\Xi := \mathrm{D}(\phi \circ \Gamma)_{(0,x_0)}(T_0 B \times \{0\}) \subset (T^R_{x_0}(O))_\infty \subset \{0\} \times \mathbb{E}_\infty$ and $\dim \Xi = n$.*

If $\mathcal{X}$ is infinite dimensional, then there exists a sc-complement $\mathbb{H}$ of Ξ in $\mathbb{E}$, such that the slice $\tilde{\mathcal{O}}' \subset \mathcal{X}^2$ yield by Lemma 3.24 is good with respect to s.

Remark 3.43. *Recall that in Lemma 3.17, G-slices (Definition 3.26) are constructed by applying Lemma 3.24 to $\Lambda : B \times p^{-1}(\mathcal{O}) \to \mathcal{Y}, (g,v) \mapsto \rho(g,v)$, where $B \subset G$ is a neighborhood* id *and $\rho : G \times \mathcal{Y} \to \mathcal{Y}$ is the action. Therefore by Proposition 3.42 and Lemma 3.27, if the base M-polyfold $\mathcal{X}$ is infinite dimensional, then there exists G-slice $(\tilde{\mathcal{O}}, \mathcal{O}, V, f, \eta, N)$ around $x_0 \in \mathcal{X}_\infty$ such that $\tilde{\mathcal{O}} \subset \mathcal{X}^2$ is good.*

Proof. If we pick a sc-complement $\mathbb{H}$ of Ξ in $\mathbb{E}$, then Lemma 3.24 yields a slice $\tilde{\mathcal{O}}' \subset \mathcal{X}^2$ around x_0. And by property (4) of Lemma 3.24, there is a change of coordinate $h : U' \to U''$ for neighborhoods $U' \subset U^2$ of $(0,0)$ and $U'' \subset \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}^1$ of $(0,0,0)$, such that the slice $\tilde{\mathcal{O}}' = \phi^{-1} \circ r \circ h^{-1}((\mathbb{R}^m_+ \times \{0\} \times \mathbb{K}^1) \cap U'')$. Recall from Definition 3.38, to show $\tilde{\mathcal{O}}'$ is a good slice, it suffices to prove $\pi_{\mathbb{R}^n} \circ \mathrm{D}(h \circ \alpha^{-1})_{(0,0)}|_{\{0\}\times\mathbb{W}^2}$ is surjective onto $\mathbb{R}^n$. Since $\mathrm{D}\alpha^{-1}|_{(0,0)}(\{0\} \times \mathbb{W}^2) \subset \mathbb{R}^m \times \mathbb{E}^2$ is finite codimensional in $\mathbb{E}^2$ and $T^R_{(0,0)}O^2 \subset \{0\} \times \mathbb{E}^2$ is infinite dimensional by assumption, then

$$\mathbb{N} := T^R_{(0,0)}O^2 \cap \mathrm{D}\alpha^{-1}|_{(0,0)}(\{0\} \times \mathbb{W}^2) \subset T^R_{(0,0)}O^2 \subset \{0\} \times \mathbb{E}^2$$

is infinite dimensional and it is sufficient to prove

$$\pi_{\mathbb{R}^n} \circ \mathrm{D}h_{(0,0)}(\mathbb{N}) = \mathbb{R}^n \tag{3.17}$$

By property (4) and (5) of Lemma 3.24, we have

$$\pi_{\mathbb{R}^n} \circ \mathrm{D}h_{(0,0)}(\{0\} \times \Xi) = \mathbb{R}^n, \quad \pi_{\mathbb{R}^n} \circ \mathrm{D}h_{(0,0)}(T^R_{(0,0)}O^2 \cap (\{0\} \times \mathbb{H})) = 0. \tag{3.18}$$

Note that $\Xi \oplus T^R_{(0,0)}O^2 \cap (\{0\} \times \mathbb{H}) = T^R_{(0,0)}O^2$, then by (3.18), (3.17) is equivalent to $\mathbb{N} + T^R_{(0,0)}O^2 \cap (\{0\} \times \mathbb{H}) = T^R_{(0,0)}O^2$. Let π_Ξ denote the projection of $\mathbb{E} = \Xi \oplus \mathbb{H}$ to Ξ. Then it suffices to prove $\pi_\Xi(\mathbb{N}) = \Xi$.

We claim that there exists a projection $\pi_\epsilon : \mathbb{E} \to \Xi$ such that $\pi_\epsilon(\mathbb{N}) = \Xi$. Then by the discussion above $\mathbb{H} := \ker \pi_\epsilon$ gives rise to a good slice. To prove the claim, we first pick any projection $\pi_0 : \mathbb{E} \to \Xi$ and then we will perturb π_0 to π_ϵ so that $\pi_\epsilon(\mathbb{N}) = \Xi$. Since $\mathbb{N}$ is infinite dimensional, we can pick linearly independent vectors $\theta_1, \ldots, \theta_n \in \mathbb{N}$ such that $\langle \theta_1, \ldots, \theta_n \rangle \cap \Xi = \{0\}$. Then by the Hahn-Banach theorem, there are continuous linear functionals (hence sc-smooth) $l_i : \mathbb{E} \to \mathbb{R}$ such that $l_i(\theta_j) = \delta_{ij}$ and $\Xi \subset \ker l_i$. Let $\xi_1, \ldots, \xi_n$ be a basis of Ξ, then we define for $\epsilon := (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n$,

$$\pi_\epsilon : \mathbb{E} \to \Xi, \quad e \mapsto \pi_0(e) + \sum_{i=1}^n \epsilon_i l_i(e) \xi_i.$$

Therefore $\pi_\epsilon(\xi_i) = \xi_i$ and $\pi_\epsilon \circ \pi_\epsilon = \pi_\epsilon$ for every ϵ, that is π_ϵ is a projection from $\mathbb{E}$ to Ξ. If we consider $\pi_\epsilon|_{\langle \theta_1,\ldots,\theta_n \rangle}$, then

$$\det(\pi_\epsilon|_{\langle \theta_1,\ldots,\theta_n \rangle}) = \prod_{i=1}^n \epsilon_i + \text{ lower order terms},$$

when we use $\{\theta_1, \ldots, \theta_n\}$ and $\{\xi_1, \ldots, \xi_n\}$ as basis. Therefore there exists ϵ such that $\pi_\epsilon|_{\langle \theta_1,\ldots,\theta_n \rangle}$ is an isomorphism. Hence the claim is proven for such π_ϵ. ∎

Proof of part (2) of Theorem 3.1. Let $\{(\tilde{\mathcal{O}}_x, \mathcal{O}_x, V_x, f_x, \eta_x, N_x)\}_{x \in \mathcal{X}_\infty}$ be the bundle G-slices used in Lemma 3.29 for the construction of $\widehat{\mathcal{X}}$. Therefore by Lemma 3.29, $\Psi_x : p^{-1}(\tilde{\mathcal{O}}_x) \xhookrightarrow{\iota}$

$p^{-1}(\mathcal{O}_x) \xrightarrow{\pi} p^{-1}(\widehat{\mathcal{X}})/G$ in (3.10) are strong bundle isomorphisms, where $\pi : p^{-1}(\widehat{\mathcal{X}}) \to p^{-1}(\widehat{\mathcal{X}})/G$ is the quotient map and $\iota : p^{-1}(\tilde{\mathcal{O}}_x) \to p^{-1}(\mathcal{O}_x)$ is the inclusion.

Since the section s is G-equivariant, s induces a continuous section $\overline{s} : \widehat{\mathcal{X}}^1/G \to p^{-1}(\widehat{\mathcal{X}}^1)/G$ by $s|_{\widehat{\mathcal{X}}^1} = \pi^*\overline{s}$. To see $\overline{s}$ is sc-smooth and has the regularization property, since $\Psi_x^*\overline{s} = \iota^*\pi^*\overline{s} = \iota^*s = \tilde{s} := s|_{\tilde{\mathcal{O}}_x^1}$, it suffices to show $\tilde{s}$ is sc-smooth and has the regularization property. Then by Proposition 3.37, $\overline{s}$ is sc-smooth and has the regularization property.

Next, we claim that $\overline{s} : \widehat{\mathcal{X}}^1/G \to p^{-1}(\widehat{\mathcal{X}}^1)/G$ has the basic germ property. It suffices to show that there is a Fredholm chart in $\mathcal{O}_x^1$ around x for $\tilde{s}$. By Remark 3.43, for every $x \in \mathcal{X}_\infty$ there exists a bundle G-slice $(\tilde{\mathcal{O}}'_x, \mathcal{O}'_x, V'_x, f'_x, \eta'_x, N'_x)$ around x such that $\tilde{\mathcal{O}}'_x \subset \mathcal{X}^2$ is a good slice. In particular, by Lemma 3.15 there is a Fredholm chart around x in $(\tilde{\mathcal{O}}'_x)^1$ for $\tilde{s}' := s|_{(\tilde{\mathcal{O}}'_x)^1} : (\tilde{\mathcal{O}}'_x)^1 \to p^{-1}((\tilde{\mathcal{O}}'_x)^1)$. The good slice $\tilde{\mathcal{O}}'_x$ is not necessarily $\tilde{\mathcal{O}}_x$ in the construction of $\widehat{\mathcal{X}}$. However, by (3.12) $\Psi_x^{-1} \circ \Psi'_x$ is a strong bundle isomorphism, where Ψ'_x are the map in (3.10) for slices $\tilde{\mathcal{O}}'_x$. Since s is G-equivariant, we have

$$\tilde{s}' = s|_{(\tilde{\mathcal{O}}'_x)^1} = (\Psi'_x)^*\overline{s} = (\Psi_x^{-1} \circ \Psi'_x)^*\Psi_x^*\overline{s} = (\Psi_x^{-1} \circ \Psi'_x)^*\tilde{s}$$

in a neighborhood of x where the strong bundle isomorphism $\Psi_x^{-1} \circ \Psi'_x$ is defined. Therefore there is a Fredholm chart in $\tilde{\mathcal{O}}_x^1$ around x for $\tilde{s}$.

Since $s^{-1}(0)$ is compact in $\mathcal{X}_0$ topology, by Remark 3.36, $s^{-1}(0)$ is compact in $\mathcal{X}^3$ topology. Because $\overline{s}^{-1}(0) = s^{-1}(0)/G$ and G is compact, $\overline{s}^{-1}(0)$ is compact in $\mathcal{X}^3/G$ topology, i.e. the $\widehat{\mathcal{X}}^1/G$ topology. That is $\overline{s} : \widehat{\mathcal{X}}^1/G \to p^{-1}(\widehat{\mathcal{X}}^1)/G$ is a proper sc-Fredholm section. □

Chapter 4

Finite Isotropy quotient of Polyfolds

In this section, we prove the main result Theorem 1.1 except the claims on orientation and good position. Polyfolds are modeled on groupoids as in [84], hence should be thought of as the orbifold version of M-polyfolds. Since a polyfold can have finite isotropy, the quotient theorem for polyfolds also allows the group action to have finite isotropy. If one works on general polyfolds, some pathological polyfolds, e.g Example 4.19 and 4.20, obstruct the construction in this paper. In particular, the Hausdorffness of quotients might fail, see Remark 4.51 and Example 4.52. Therefore we introduce regular polyfolds and bundles in Definition 4.13 and Definition 4.57, the quotient construction works for regular polyfolds and regular polyfold bundles. Polyfolds in current applications are all regular and regularity can be checked easily by Proposition 4.16, Proposition 4.17 and Corollary 4.18. We also point out that orbifolds are always regular. Hence such constrain only exists in polyfold theory.

We review the basics of polyfolds and introduce regular polyfolds in Section 3.1. Section 3.2 introduces the definition of group actions on polyfolds and Section 3.3 discusses properties of group actions. We construct the polyfold quotients in Section 3.4. Section 3.5 is about quotients of polyfold bundles and sc-Fredholm sections.

4.1 Regular polyfolds

We first recall definitions and properties of polyfolds that will be used in this paper from [65]. For that purpose, we first recall the definition of an ep-groupoid.

Ep-groupoids and regular ep-groupoids

Definition 4.1 ([65, Def 7.3]). *An* ***ep-groupoid*** $(\mathcal{X}, \boldsymbol{X})$ *is a groupoid with M-polyfold structures on the object set* $\mathcal{X}$ *and the morphism set* $\boldsymbol{X}$*. Moreover, all the structure maps: source* s*, target* t*, composition* m*, unit* u *and inverse* i *are sc-smooth and the following two properties hold,*

- ***étale****:* s, t *are surjective local sc-diffeomorphisms;*

- ***proper***: *for every* $x \in \mathcal{X}$, *there exists an open neighborhood* $\mathcal{V}$ *of* x, *such that* $t : s^{-1}(\overline{\mathcal{V}}) \to \mathcal{X}$ *is proper, where* $\overline{\mathcal{V}}$ *is the closure of* $\mathcal{V}$ *in* $\mathcal{X}$.[1]

An ep-groupoid is ***tame*** *if the object space* $\mathcal{X}$ *is a tame M-polyfold.*

The proper condition implies that for every $x \in \mathcal{X}$, the isotorpy group stab_x is finite [65, Proposition 7.4]. The étale condition implies that every $\phi \in \boldsymbol{X}$ there exists neighborhoods $\mathcal{U}_{s(\phi)} \subset \mathcal{X}, \mathcal{U}_{t(\phi)} \subset \mathcal{X}$ and $\boldsymbol{U}_\phi \subset \boldsymbol{X}$ of $s(\phi), t(\phi)$ and ϕ, such that both $s : \boldsymbol{U}_\phi \to \mathcal{U}_{s(\phi)}$ and $t : \boldsymbol{U}_\phi \to \mathcal{U}_{t(\phi)}$ are diffeomorphisms. The following definition was introduced in [65, Section 7.1].

Definition 4.2. *For every* $\phi \in \boldsymbol{X}$, *we define*

$$L_\phi := t \circ s^{-1}, \quad \mathcal{U}_{s(\phi)} \to \mathcal{U}_{t(\phi)}, \tag{4.1}$$

which is a sc-diffeomorphism.

Definition 4.3. *Let* G *be a finite group. Assume there is a sc-smooth group action* $\rho : G \times \mathcal{O} \to \mathcal{O}$ *on a M-polyfold* $\mathcal{O}$. *The* ***translation groupoid*** $G \ltimes \mathcal{O}$ *is defined to be:*

$$\mathrm{obj}(G \ltimes \mathcal{O}) := \mathcal{O}, \qquad \mathrm{mor}(G \ltimes \mathcal{O}) := G \times \mathcal{O}.$$

The structure maps are defined by:

$$s : (g, x) \mapsto x, \qquad t : (g, x) \mapsto \rho(g, x).$$

The composition for (h, y) *and* (g, x) *satisfying* $y = \rho(g, x)$ *is defined to be* (hg, x). *The inverse map is given by* $i(g, x) = (g^{-1}, \rho(g, x))$ *and the unit map is given by* $u(x) = (\mathrm{id}, x)$.

It is clear that the source and target maps of $G \ltimes \mathcal{O}$ are étale and the $G \ltimes \mathcal{O}$ is proper by Proposition 4.9 below. Therefore $G \ltimes \mathcal{O}$ is an ep-groupoid. The following theorem, as another consequence of the étale and proper properties in Definition 4.1, essentially says that an ep-groupoid is locally isomorphic to a translation groupoid.

Theorem 4.4 ([65, Thm 7.1])**.** *Given an ep-groupoid* $(\mathcal{X}, \boldsymbol{X})$, *an object* $x \in \mathcal{X}$ *and an open neighborhood* $\mathcal{V} \subset \mathcal{X}$ *of* x. *Then there exists an open neighborhood* $\mathcal{U} \subset \mathcal{V}$ *of* x *and a group homomorphism defined by*

$$\mathrm{stab}_x \to \mathrm{Diff}_{\mathrm{sc}}(\mathcal{U}), \quad \phi \mapsto L_\phi.$$

where $\mathrm{Diff}_{\mathrm{sc}}(\mathcal{U})$ *is the group of sc-diffeomorphism from* $\mathcal{U}$ *to itself. Moreover, there is a sc-smooth map*

$$\Sigma : \mathrm{stab}_x \times \mathcal{U} \to \boldsymbol{X},$$

such that they have the following properties:

[1] Hence for every open set $\mathcal{W}$ with $\overline{\mathcal{W}} \subset \overline{\mathcal{V}}$, $t : s^{-1}(\overline{\mathcal{W}}) \to \mathcal{X}$ is proper.

- *$\Sigma(\phi, x) = \phi$;*
- *$s(\Sigma(\phi, y)) = y$ and $t(\Sigma(\phi, y)) = L_\phi(y)$ for all $y \in \mathcal{U}$ and $\phi \in \mathrm{stab}_x$;*
- *If $\psi : y \to z$ is a morphism connecting two objects $y, z \in \mathcal{U}$, then there is a unique $\phi \in \mathrm{stab}_x$ such that $\Sigma(\phi, y) = \psi$.*

In other words, $\mathrm{stab}_x \ltimes \mathcal{U} \xrightarrow{(i_\mathcal{U}, \Sigma)} (\mathcal{X}, \boldsymbol{X})$ is a fully faithful functor, where $i_\mathcal{U} : \mathcal{U} \to \mathcal{X}$ is the inclusion.

The following corollary is a direct consequence of Theorem 4.4 and Definition 4.2.

Corollary 4.5. *Let $(\mathcal{X}, \boldsymbol{X})$ be an ep-groupoid and $\mathcal{U} \subset \mathcal{X}$ a neighborhood of $x \in \mathcal{X}$ such that Theorem 4.4 holds. Let $\psi \in \boldsymbol{X}$ such that $s(\psi), t(\psi) \in \mathcal{U}$. Then L_ψ is defined on $\mathcal{U}$ and there exists $\phi \in \mathrm{stab}_x$ such that $L_\psi = L_\phi$ on $\mathcal{U}$.*

Remark 4.6. *If $(\mathcal{X}, \boldsymbol{X})$ is only an étale groupoid with finite isotropy. Then for every $x \in \mathcal{X}$ and open neighborhood $\mathcal{V}$ containing x, there exists an open neighborhood $\mathcal{U}$ containing x such that*

$$\mathrm{stab}_x \to \mathrm{Diff}_{\mathrm{sc}}(\mathcal{U}), \quad g \mapsto L_g$$

is a group homomorphism, see the proof of [65, Thm 7.1].

The fully faithful functor $\mathrm{stab}_x \ltimes \mathcal{U} \xrightarrow{(i_\mathcal{U}, \Sigma)} (\mathcal{X}, \boldsymbol{X})$ can be thought of as the local charts on an ep-groupoid. To be more precise, the following definition was introduced in [65]. For an ep-groupoid $(\mathcal{X}, \boldsymbol{X})$, there is an equivalence relation $\sim_{\boldsymbol{X}}$ on $\mathcal{X}$ such that $x \sim_{\boldsymbol{X}} y$ iff there exists $\phi \in \boldsymbol{X}$ with $s(\phi) = x$ and $t(\phi) = y$. Let $|\mathcal{X}| := \mathcal{X}/\sim_{\boldsymbol{X}}$. Then $|\mathcal{X}|$ is a topological space equipped with the quotient topology. We use $|\cdot|$ to denote the quotient map $\mathcal{X} \to |\mathcal{X}|$.

Definition 4.7 ([65, Definition 7.9])**.** *Let $(\mathcal{X}, \boldsymbol{X})$ be an ep-groupoid and x be an object in $\mathcal{X}$. A **local uniformizer around x** is a fully faithful functor*

$$\Psi_x : \mathrm{stab}_x \ltimes \mathcal{U}_x \to (\mathcal{X}, \boldsymbol{X}),$$

where $\mathcal{U}_x \subset \mathcal{X}$ is an open neighborhood of x, such that the following properties hold:

1. *on the object level $\Psi_x^0 : \mathcal{U}_x \to \mathcal{X}$ is an inclusion of open subset, on the morphism level $\Psi_x^1 : \mathrm{stab}_x \times \mathcal{U}_x \to \boldsymbol{X}$ is sc-smooth;*
2. *on the orbit space $|\Psi_x| : \mathcal{U}_x / \mathrm{stab}_x \to |\mathcal{X}|$ is homeomorphism onto an open subset of $|\mathcal{X}|$, i.e. $|\mathcal{U}_x|$ is open in $|\mathcal{X}|$.*

The existence of local uniformizers is guaranteed by Theorem 4.4 and the assertion on the homeomorphism of the orbit space was proven in [65, Proposition 7.6]. From the perspective of Theorem 4.4, a local uniformizer is essentially determined by the open neighborhood $\mathcal{U}_x \subset \mathcal{X}$ of x. Hence we will call $\mathcal{U}_x$ a local uniformizer when there is no confusion.

In constructions of ep-groupoids, we need to verify the properness in Definition 4.1. For that purpose, we introduce the following characterization of properness in Proposition 4.9

Definition 4.8. *An* ***orbit set*** *in an open subset $\mathcal{U} \subset \mathcal{X}$ of a point $x \in \mathcal{X}$ is defined to be the set $S_{x,\mathcal{U}} := \{\phi \in \boldsymbol{X} | s(\phi) \in \mathcal{U}, t(\phi) = x\}$.*

Note that we do not require $x \in \mathcal{U}$ and the orbit set $S_{x,\mathcal{U}}$ only depends on $|x| \in |\mathcal{X}|$ and $\mathcal{U}$. Then Theorem 4.4 implies that if $\mathcal{U}_x$ is a local uniformizer around x and $y \in \mathcal{U}_x$, the cardinality of the orbit set $S_{y,\mathcal{U}_x}$ is $|\operatorname{stab}_x|$. Also note that $t : s^{-1}(\overline{\mathcal{V}}) \to \mathcal{X}$ being proper implies that $t(s^{-1}(\overline{\mathcal{V}})) \subset \mathcal{X}$ is sequentially closed. Since M-polyfold $\mathcal{X}$ is metrizable, $t(s^{-1}(\overline{\mathcal{V}}))$ is closed. In fact, those two properties imply properness of an étale groupoid, i.e. we have the following proposition.

Proposition 4.9. *Let $(\mathcal{X}, \boldsymbol{X})$ be an étale groupoid with finite isotropy. Assume for every $x \in \mathcal{X}$, there exists an open neighborhood $\mathcal{U}_x \subset \mathcal{X}$ of x, such that the following two conditions hold.*

1. *For every $y \in \mathcal{U}_x$, $|S_{y,\mathcal{U}_x}| = |\operatorname{stab}_x|$.*
2. *There exists an open neighborhood $\mathcal{V}_x \subset \overline{\mathcal{V}_x} \subset \mathcal{U}_x$ of x, such that $t(s^{-1}(\overline{\mathcal{V}_x}))$ is a closed subset of $\mathcal{X}$.*

Then $(\mathcal{X}, \boldsymbol{X})$ is an ep-groupoid.

Proof. We claim that $t : s^{-1}(\overline{\mathcal{V}_x}) \to \mathcal{X}$ is proper. Let $K \subset \mathcal{X}$ be compact set. It suffices to prove $t^{-1}(K) \cap s^{-1}(\overline{\mathcal{V}_x}) \subset \boldsymbol{X}$ is compact. We will prove compactness by a finite cover of compact sets.

Let $n := |\operatorname{stab}_x|$. By assumption (1), for $y \in t(s^{-1}(\mathcal{U}_x))$, we have the orbit set $S_{y,\mathcal{U}_x} = \{\phi_1, \ldots, \phi_n\}$, such that $t(\phi_i) = y$ and $s(\phi_i) \in \mathcal{U}_x$. Because the source and target maps s, t are local sc-diffeomorphisms, $t(s^{-1}(\mathcal{U}_x))$ is open and there exist open neighborhood $\mathcal{O} \subset t(s^{-1}(\mathcal{U}_x))$ of y and open neighborhoods $\boldsymbol{O}(1), \ldots, \boldsymbol{O}(n)$ of $\phi_1, \ldots, \phi_n$ in $\boldsymbol{X}$, such that $\boldsymbol{O}(i) \cap \boldsymbol{O}(j) = \emptyset$ for all $i \neq j$ and both $t|_{\boldsymbol{O}(i)} : \boldsymbol{O}(i) \to \mathcal{O}$, $s|_{\boldsymbol{O}(i)} : \boldsymbol{O}(i) \to s(\boldsymbol{O}(i)) \subset \mathcal{U}_x$ are sc-diffeomorphisms for all i. For every $z \in \mathcal{O}$, by assumption (1) there are also exactly n morphisms $\psi_1, \ldots, \psi_n \in \boldsymbol{X}$, such that $s(\psi_i) \in \mathcal{U}_x$ and $t(\psi_i) = z$. Since $\left(t|_{\boldsymbol{O}(i)}\right)^{-1}(z)$ already provides n such elements, we must have $\psi_i \in \boldsymbol{O}(i)$ (after a permutation). Therefore for every set $\mathcal{W} \subset \mathcal{O}$, the set of all the morphisms from $\mathcal{U}_x$ to $\mathcal{W}$ is completely described by:

$$t^{-1}(\mathcal{W}) \cap s^{-1}(\mathcal{U}_x) = \coprod_{i=1}^{n} \boldsymbol{O}(i) \cap t^{-1}(\mathcal{W}) = \coprod_{i=1}^{n} t|_{\boldsymbol{O}(i)}^{-1}(\mathcal{W}).$$

As a consequence, for every open neighborhood $\mathcal{O}' \subset \mathcal{X}$ of y with $\overline{\mathcal{O}'} \subset \mathcal{O}$, we have

$$t^{-1}(\overline{\mathcal{O}'} \cap K) \cap s^{-1}(\mathcal{U}_x) = \coprod_{i=1}^{n} (t|_{\boldsymbol{O}(i)})^{-1}(\overline{\mathcal{O}'} \cap K)$$

is compact, since $\overline{\mathcal{O}'} \cap K \subset \mathcal{O}$ is compat. Since $s^{-1}(\overline{\mathcal{V}_x})$ is closed and $s^{-1}(\overline{\mathcal{V}_x}) \subset s^{-1}(\mathcal{U}_x)$, we have $t^{-1}(\overline{\mathcal{O}'} \cap K) \cap s^{-1}(\overline{\mathcal{V}_x})$ is compact. To sum up, we prove that for every $y \in t(s^{-1}(\mathcal{U}_x))$

there exists an open neighborhood $\mathcal{O}' \subset t(s^{-1}(\mathcal{U}_x))$ of y such that $(t|_{s^{-1}(\overline{\mathcal{V}_x})})^{-1}(\overline{\mathcal{O}'} \cap K) = t^{-1}(\overline{\mathcal{O}'} \cap K) \cap s^{-1}(\overline{\mathcal{V}_x})$ is compact.

By assumption (2) that $t(s^{-1}(\overline{\mathcal{V}_x}))$ is closed, $K \cap t(s^{-1}(\overline{\mathcal{V}_x}))$ is compact. Therefore we can cover the compact set $K \cap t(s^{-1}(\overline{\mathcal{V}_x}))$ by finitely many such $\mathcal{O}'$. Then $(t|_{s^{-1}(\overline{\mathcal{V}_x})})^{-1}(K) = t^{-1}(K) \cap s^{-1}(\overline{\mathcal{V}_x})$ is a union of finitely many compact sets, hence is also compact. □

The next proposition asserts that one only needs to verify the first condition in Proposition 4.9 for very few points.

Proposition 4.10. *Let $(\mathcal{X}, \boldsymbol{X})$ be an étale groupoid with finite isotropy. Suppose $\mathcal{U} \subset \mathcal{X}$ is an open neighborhood of x and $|\operatorname{stab}_{y,\mathcal{U}}| = |\operatorname{stab}_x|$ for all $y \in \mathcal{U}$. Then for every $y \in \mathcal{U}$, there exists an open neighborhood $\mathcal{V} \subset \mathcal{U}$ of y, such that $|S_{z,\mathcal{V}}| = \operatorname{stab}_y$ for $z \in \mathcal{V}$.*

Proof. Let $n := |\operatorname{stab}_x|$. Since $|S_{y,\mathcal{U}}| = |\operatorname{stab}_x| = n$ for $y \in \mathcal{U}$, there exists $\phi_1, \ldots, \phi_n \in \boldsymbol{X}$ such that $s(\phi_i) \in \mathcal{U}$ and $t(\phi_i) = y$. We assume $\phi_i \in \operatorname{stab}_y$ iff $i \le k$. By the étable property, there exist open neighborhood $\mathcal{O} \subset \mathcal{U}$ of y and disjoint open neighborhoods $\boldsymbol{O}(i) \subset \boldsymbol{X}$ of ϕ_i, such that $s : \boldsymbol{O}(i) \to s(\boldsymbol{O}(i)) \subset \mathcal{U}$ and $t : \boldsymbol{O}(i) \to \mathcal{O}$ are diffeomorphisms. Moreover, we can assume $\mathcal{O} \cap s(\boldsymbol{O}(i)) = \emptyset$ and $s(\boldsymbol{O}(j)) \cap s(\boldsymbol{O}(i)) = \emptyset$ for $i > k$ and $j \le k$. By Remark 4.6, there exists open neighborhood $\mathcal{V} \subset \mathcal{O} \cap_{i=1}^k s(\boldsymbol{O}(i))$ of y admitting a stab_y action defined by $\phi_i \mapsto L_{\phi_i}$. Then $|S_{z,\mathcal{V}}| \ge |\operatorname{stab}_y| = k$ by construction of $\mathcal{V}$. By the same argument used in Proposition 4.9, we have $S_{z,\mathcal{U}} = \{t|^{-1}_{\boldsymbol{O}(i)}(z)\}$ for $z \in \mathcal{O}$. Note that $s(t^{-1}_{\boldsymbol{O}(j)}(z)) \in s(\boldsymbol{O}(j))$ and $s(\boldsymbol{O}(j)) \cap \mathcal{V} = \emptyset$ for $j > k$. Since $S_{z,\mathcal{V}} \subset S_{z,\mathcal{U}}$. we have $|S_{z,\mathcal{V}}| \le k$ for every $z \in \mathcal{V}$. Hence the proposition holds. □

Next we introduce the concept of regular ep-groupoid, the motivation of such definition is to rule out pathological ep-groupoids in Example 4.19 and Example 4.20.

Definition 4.11. *For $x \in \mathcal{X}$, a local uniformizer $\operatorname{stab}_x \ltimes \mathcal{U}$ around x is **regular** if the following two conditions are met:*

1. *if $L_\phi|_{\mathcal{V}} = \operatorname{id}_{\mathcal{V}}$ for some open subset $\mathcal{V} \subset \mathcal{U}$ and $\phi \in \operatorname{stab}_x$, then $L_\phi = \operatorname{id}$ on $\mathcal{U}$;*
2. *for every connected[2] uniformizer $\mathcal{W} \subset \mathcal{U}$ around x, if $\Phi : \mathcal{W} \to \operatorname{stab}_x$ is a map such that $L_{\Phi(\cdot)}(\cdot) : \mathcal{W} \to \mathcal{X}, z \mapsto L_{\Phi(z)}(z)$ is sc-smooth, then there exists $\phi \in \operatorname{stab}_x$, such that $L_{\Phi(z)}(z) = L_\phi(z)$ for all $z \in \mathcal{W}$.*

By Definition 4.11, any smaller uniformizer around x inside a regular uniformizer around x is also regular.

Remark 4.12. *The second condition in Definition 4.11 can be weakened from all $\mathcal{W} \subset \mathcal{U}$ to a (countable) local topology basis $\{\mathcal{W}_i \subset \mathcal{U}\}_{i \in \mathbb{N}}$. All the arguments used in this paper go through. For the simplicity of notation, we only work with Definition 4.11 in this paper.*

[2] Since M-polyfolds are locally path connected, path connectedness is equivalent to connectedness.

Definition 4.13. *An ep-groupoid $(\mathcal{X}, \boldsymbol{X})$ is **regular**, if for any point $x \in \mathcal{X}$, there exists a regular local uniformizer around x.*[3]

Remark 4.14. *It is not clear whether the regularity of $(\mathcal{X}, \boldsymbol{X})$ implies the regularity of $(\mathcal{X}^i, \boldsymbol{X}^i)$ for other $i > 0$. However, if the condition in Corollary 4.18 holds, then $(\mathcal{X}^i, \boldsymbol{X}^i)$ are regular for all $i \geq 0$.*

In most of the applications, we can use one of the following propositions to confirm regularity. We first recall the definition of effective ep-groupoid. Let $\mathrm{Diff}_{\mathrm{sc}}(x)$ be the group of germs of diffeomorphisms fixing x. Then an action $\mathrm{stab}_x \to \mathrm{Diff}_{\mathrm{sc}}(\mathcal{U})$ in Theorem 4.4 descends to a group homomorphism $\mathrm{stab}_x \to \mathrm{Diff}_{\mathrm{sc}}(x)$. The **effective part of $\mathbf{stab}_{\boldsymbol{x}}$** is defined to be

$$\mathrm{stab}_x^{\mathrm{eff}} := \mathrm{stab}_x / (\ker(\mathrm{stab}_x \to \mathrm{Diff}_{\mathrm{sc}}(x))). \tag{4.2}$$

Definition 4.15 ([65, Definition 7.11])**.** *An ep-groupoid is effective if the homomorphism $\mathrm{stab}_x \to \mathrm{Diff}_{sc}(x)$ is injective for any $x \in \mathcal{X}$, i.e. $\mathrm{stab}_x = \mathrm{stab}_x^{\mathrm{eff}}$.*

The reason of passing to germs is that there might exist neighborhoods $\mathcal{V} \subset \mathcal{U} \subset \mathcal{X}$ of x, such that $\mathrm{stab}_x \to \mathrm{Diff}_{\mathrm{sc}}(\mathcal{U})$ is injective but $\mathrm{stab}_x \to \mathrm{Diff}_{\mathrm{sc}}(\mathcal{V})$ is not injective, e.g. Example 4.19.

Proposition 4.16. *Let $(\mathcal{X}, \boldsymbol{X})$ be an effective groupoid. Assume for every $x \in \mathcal{X}$, there exists a local uniformizer $\mathcal{U}$ around x, such that for any connected uniformizer $\mathcal{V} \subset \mathcal{U}$ around x, we have $\mathcal{V} \backslash \cup_{\phi \neq \mathrm{id} \in \mathrm{stab}_x} \mathrm{Fix}(\phi)$ is connected, where $\mathrm{Fix}(\phi)$ is the fixed set of L_ϕ. Then $(\mathcal{X}, \boldsymbol{X})$ is regular.*

Proposition 4.17. *If $(\mathcal{X}, \boldsymbol{X})$ has the property that for any $x \in \mathcal{X}_\infty$ and $\phi \in \mathrm{stab}_x$ if $\mathrm{D}L_\phi : T_x\mathcal{X} \to T_x\mathcal{X}$ is the identity map, then L_ϕ is the identity map near x. Then $(\mathcal{X}, \boldsymbol{X})$ is regular.*

As a simple corollary of Proposition 4.17, we have the following.

Corollary 4.18. *If the linearized action $\mathrm{stab}_x \to \hom(T_x\mathcal{X}, T_x\mathcal{X})$ defined by $\phi \mapsto (\mathrm{D}L_\phi)_x$ is injective for every point $x \in \mathcal{X}_\infty$, then $(\mathcal{X}, \boldsymbol{X})$ is a regular ep-groupoid.*

By Proposition 4.17, if the ep-groupoid is modeled on finite dimensional manifolds, i.e. an orbifold, then it is automatically regular. However, the dimension jump phenomenon in M-polyfolds may destroy the regular property, see Example 4.19 and 4.20. The reason of introducing regularity is related to the Hausdorffness of the quotient polyfolds, see Remark 4.51 and Example 4.52. The second property is used to get Proposition 4.33, so that we can control the isotropy of the quotient ep-groupoid, see (4.8). For the other consequences of the second property, see Proposition A.6 and Theorem A.7

[3]The regularity of an ep-groupoid should be differed from the regularity of the topology on the orbits space.

Proposition 4.16 is proven in the appendix. Proposition 4.17 was proven in [65, Lemma 10.2]. In fact, [65, Lemma 10.2] only stated the second property of a regular uniformizer. However the conditions in Proposition 4.17 imply that the set $S := \{x \in \mathcal{U}_\infty | \mathrm{D}L_\phi = \mathrm{id}\}$ is both open and closed in $\mathcal{U}_\infty$ for a uniformizer $\mathcal{U}$. Hence when $\mathcal{U}_\infty$ is connected, $S \neq \emptyset$ implies $S = \mathcal{U}_\infty$. Therefore if $L_\phi = \mathrm{id}$ on some $\mathcal{V} \subset \mathcal{U}$, $L_\phi = \mathrm{id}$ on $\mathcal{U}_\infty$. Since $\mathcal{U}_\infty$ is dense in $\mathcal{U}$, $L_\phi = \mathrm{id}$ on $\mathcal{U}$. This proves the first property of a regular uniformizer.

In applications [61, 62, 102, 74], Corollary 4.18 holds. From another perspective, all the related ep-groupoids are effective and the fixed set $\mathrm{Fix}(\phi)$ of a nontrivial element ϕ has "infinite codimension". Hence it is easy to verify the connectedness of the complement. Therefore the polyfolds appear in current applications are regular.

Unlike the orbifold case, irregular ep-groupoids exist. The following two examples shows that there exist ep-groupoids do not satisfy any one of the two properties of regularity.

Example 4.19. *Consider the following retraction from [64, Example 1.22]. Let the sc-Hilbert space $\mathbb{E}_m := H^m_{\delta_m}(\mathbb{R})$, where $0 = \delta_0 < \delta_1 < \ldots < \delta_m < \ldots$ are the exponential decay weights. Choose a smooth compactly supported positive function β, such that $\int_{\mathbb{R}} \beta^2 = 1$. Define $r_t : \mathbb{E} \to \mathbb{E}$ to be:*

$$r_t(f) := \begin{cases} 0, & t \leq 0; \\ \int_{\mathbb{R}} f(x)\beta(x + e^{\frac{1}{t}})dx \cdot \beta(x + e^{\frac{1}{t}}), & t > 0. \end{cases}$$

It was checked in [64] that $\pi : \mathbb{R} \times \mathbb{E} \to \mathbb{R} \times \mathbb{E}, (t, f) \mapsto (t, r_t(f))$ defines a sc^∞ retraction. Then the retract $\mathrm{im}\,\pi$ defines an M-polyfold $\mathcal{M}$. Pictorially, $\mathcal{M}$ is a closed ray attached to an open half plane at $(0, 0)$. There is a $\mathbb{Z}_2$ action on the M-polyfold $\mathcal{M}$ induced by multiplying -1 on the $\mathbb{E}$ component. The translation groupoid $\mathbb{Z}_2 \ltimes \mathcal{M}$ is an ep-groupoid, but it does not have the regular property at $(0, 0)$, since $\phi \neq \mathrm{id} \in \mathrm{stab}_{(0,0)}$ fixes the half line part, but does not fix the half plane part.

Example 4.20. *Let $\mathbb{E}$ and r_t denote the same objects as in Example 4.19. Then we have a sc-retraction $\pi : \mathbb{R} \times \mathbb{E} \to \mathbb{R} \times \mathbb{E}, (t, f) \mapsto (t, r_{|t|}(f))$ by the same argument in [64]. Topologically, the retract $\mathrm{im}\,\pi$ is two open half spaces $(-\infty, 0) \times \mathbb{R}$ and $(0, \infty) \times \mathbb{R}$ connected at the origin. $\mathbb{Z}_2$ can act on $\mathbb{R} \times \mathbb{E}$ by sending (t, x) to $(t, -x)$, such action induces a sc-smooth $\mathbb{Z}_2$ action on $\mathrm{im}\,\pi$. We claim the second property of regularity fails for the action ep-groupoid $\mathbb{Z}_2 \ltimes \mathrm{im}\,\pi$. Let $\Psi : \mathrm{im}\,\pi \to \mathbb{Z}_2$ be the map such that $\Psi = \mathrm{id} \in \mathbb{Z}_2$ when $t \leq 0$ and Ψ is the nontrivial element of $\mathbb{Z}_2$ when $t > 0$, then $\eta : \mathrm{im}\,\pi \to \mathrm{im}\,\pi, z \mapsto L_{\Psi(z)}(z)$ is sc-smooth. This is because $\eta \circ \pi$ can be expressed as*

$$\eta \circ \pi(t, f) = \begin{cases} (t, r_{-t}(f)), & t \leq 0; \\ (t, -r_t(f)), & t > 0. \end{cases}$$

This map is sc-smooth by the same argument in [64]. Therefore the second property of regularity does not hold for the action groupoid $\mathbb{Z}_2 \ltimes \mathrm{im}\,\pi$. It is easy to check directly that neither Proposition 4.16 nor Proposition 4.17 hold for $\mathbb{Z}_2 \ltimes \mathrm{im}\,\pi$.

Generalized maps between ep-groupoids

In this subsection, we review the construction of the category of ep-groupoids. A functor $F : (\mathcal{X}, \boldsymbol{X}) \to (\mathcal{Y}, \boldsymbol{Y})$ between ep-groupoids is sc-smooth if both the map on objects $F^0 : \mathcal{X} \to \mathcal{Y}$ and the map on morphisms $F^1 : \boldsymbol{X} \to \boldsymbol{Y}$ are sc-smooth.

Definition 4.21 ([65, Definition 10.1])**.** *A sc-smooth functor $F : (\mathcal{X}, \boldsymbol{X}) \to (\mathcal{Y}, \boldsymbol{Y})$ is an* ***equivalence*** *provided it has the following properties:*

- *F is fully faithful;*
- *the induced map $|F| : |\mathcal{X}| \to |\mathcal{Y}|$ between the orbit spaces is a homeomorphism;*
- *F^0 is a local sc-diffeomorphism.*

When there is no ambiguity, we will also abbreviate F^0 and F^1 to F.

Definition 4.22 ([65, Definition 10.2])**.** *Two sc-smooth functors F, G from $(\mathcal{X}, \boldsymbol{X})$ to $(\mathcal{Y}, \boldsymbol{Y})$ are called* ***naturally equivalent*** *if there exists a sc-smooth map $\tau : \mathcal{X} \to \boldsymbol{Y}$ such that $\tau(x) \in \boldsymbol{Y}$ is a morphism from $F(x)$ to $G(x)$ and the following diagram commutes for any morphism $\phi \in \boldsymbol{X}$ from x to y,*

$$\begin{tikzcd} F(x) \arrow[r, "F(\phi)"] \arrow[d, "\tau(x)"] & F(y) \arrow[d, "\tau(y)"] \\ G(x) \arrow[r, "G(\phi)"] & G(y). \end{tikzcd}$$

The sc-smooth map $\tau : \mathcal{X} \to \boldsymbol{Y}$ is called the natural transformation from F to G and symbolically referred to as $\tau : F \Rightarrow G$.

It was shown in [65, Section 10.1] that "naturally equivariant" is an equivalence relation between sc-smooth functors. If two functors F and G are naturally equivalent, then $|F| = |G|$. Hence two naturally equivalent functors should be thought of as the "same map" on ep-groupoids. However, if we only use sc-smooth functors (up to natural equivalence) as morphisms between the ep-groupoids, we will not have enough morphisms. In particular, the equivalences in Definition 4.21 do not necessarily have inverses. Therefore we need to invert equivalence formally, which is the process of localization. Then the following definition was introduced in [65].

Definition 4.23 ([65, Definition 10.8])**.** *A* ***diagram*** *from ep-groupoid $(\mathcal{X}, \boldsymbol{X})$ to ep-groupoid $(\mathcal{Y}, \boldsymbol{Y})$ is*

$$(\mathcal{X}, \boldsymbol{X}) \xleftarrow{F} (\mathcal{Z}, \boldsymbol{Z}) \xrightarrow{\Phi} (\mathcal{Y}, \boldsymbol{Y}),$$

where F is an equivalence and Φ is a sc-smooth functor. A digram $(\mathcal{X}, \boldsymbol{X}) \xleftarrow{F'} (\mathcal{Z}', \boldsymbol{Z}') \xrightarrow{\Phi'} (\mathcal{Y}, \boldsymbol{Y})$ is a ***refinement*** *of $(\mathcal{X}, \boldsymbol{X}) \xleftarrow{F} (\mathcal{Z}, \boldsymbol{Z}) \xrightarrow{\Phi} (\mathcal{Y}, \boldsymbol{Y})$, if there is an equivalence H :*

$(\mathcal{Z}', \boldsymbol{Z}') \to (\mathcal{Z}, \boldsymbol{Z})$ *such that* $H \circ F \Rightarrow F'$ *and* $H \circ \Phi \Rightarrow \Phi'$. *Equivalently, we have the following 2-commutative diagram of groupoids:*

$$\begin{tikzcd}
(\mathcal{X},\boldsymbol{X}) & (\mathcal{Z},\boldsymbol{Z}) \arrow[l, "F"'] \arrow[r, "\phi"] & (\mathcal{Y},\boldsymbol{Y}) \\
& (\mathcal{Z}',\boldsymbol{Z}') . \arrow[ul, "F'"] \arrow[u, "H"'] \arrow[ur, "\Phi'"'] &
\end{tikzcd}$$

Two diagrams $(\mathcal{X}, \boldsymbol{X}) \overset{F}{\leftarrow} (\mathcal{Z}, \boldsymbol{Z}) \overset{\Phi}{\rightarrow} (\mathcal{Y}, \boldsymbol{Y})$ *and* $(\mathcal{X}, \boldsymbol{X}) \overset{F'}{\leftarrow} (\mathcal{Z}', \boldsymbol{Z}') \overset{\Phi'}{\rightarrow} (\mathcal{Y}, \boldsymbol{Y})$ *are equivalent if they admit a common refinement. A* ***generalized map*** *from* $(\mathcal{X}, \boldsymbol{X})$ *to* $(\mathcal{Y}, \boldsymbol{Y})$ *is an equivalence class* $[d]$ *of a diagram* d *from* $(\mathcal{X}, \boldsymbol{X})$ *to* $(\mathcal{Y}, \boldsymbol{Y})$.

Common refinement defines an equivalence relation by [65, Lemma 10.4]. Using generalized maps as the morphisms between ep-groupoids, we form a category $\mathscr{E}\mathscr{P}(\boldsymbol{E}^{-1})$ [65, Theorem 10.3], where $\boldsymbol{E}$ stands for the class of equivalences between ep-groupoids. We call isomorphisms in $\mathscr{E}\mathscr{P}(\boldsymbol{E}^{-1})$ generalized isomorphisms. The following important property holds for generalized isomorphisms.

Theorem 4.24 ([65, Theorem 10.4])**.** *A generalized map* $[a] = [(\mathcal{X}, \boldsymbol{X}) \overset{F}{\leftarrow} (\mathcal{Z}, \boldsymbol{Z}) \overset{G}{\rightarrow} (\mathcal{Y}, \boldsymbol{Y})]$ *is a generalized isomorphism iff* G *is an equivalence. Moreover, the inverse is* $[a]^{-1} = [(\mathcal{Y}, \boldsymbol{Y}) \overset{G}{\leftarrow} (\mathcal{Z}, \boldsymbol{Z}) \overset{F}{\rightarrow} (\mathcal{X}, \boldsymbol{X})]$.

As a special case, a sc-smooth functor $\Phi : (\mathcal{X}, \boldsymbol{X}) \to (\mathcal{Y}, \boldsymbol{Y})$ is an isomorphism in $\mathscr{E}\mathscr{P}(\boldsymbol{E}^{-1})$ iff Φ is an equivalence.

Basics of polyfolds

Definition 4.25 ([65, Definition 16.1])**.** *A* ***polyfold structure*** *for a topological space* Z *is a pair* $((\mathcal{X}, \boldsymbol{X}), \alpha)$*, where* $(\mathcal{X}, \boldsymbol{X})$ *is an ep-groupoid and* α *is a homeomorphism from the orbits space* $|\mathcal{X}|$ *to* Z.

Definition 4.26 ([65, Definition 16.2,16.3])**.** *Two polyfold structures* $((\mathcal{X}, \boldsymbol{X}), \alpha)$, $((\mathcal{Y}, \boldsymbol{Y}), \beta)$ *are equivalent, if there exist a third ep-groupoid* $(\mathcal{Z}, \boldsymbol{Z})$ *and equivalences* $(\mathcal{X}, \boldsymbol{X}) \overset{F}{\leftarrow} (\mathcal{Z}, \boldsymbol{Z}) \overset{G}{\rightarrow} (\mathcal{Y}, \boldsymbol{Y})$*, such that* $\alpha \circ |F| = \beta \circ |G|$. *A* ***polyfold*** (Z, c) *is a paracompact space* Z *with an equivalence class of polyfold structures* c. *We will suppress* c *when there is no confusion. A polyfold is* ***regular*** *resp.* ***tame*** *resp.* ***infinite dimensional*** *if it has a polyfold structure* $((\mathcal{X}, \boldsymbol{X}), \alpha)$ *such that* $(\mathcal{X}, \boldsymbol{X})$ *is regular resp. tame resp. infinite dimensional* [4].

The polyfold definition in [65, Definition 16.3] does not require paracompactness. Since partition of unit is very important in application, we only work with paracompact poly-

[4]By [65, Lemma 16.2], all polyfold structures are regular resp. tame resp. infinite dimensional.

fold[5]. By [65, Proposition 16.1], a topological space admitting a polyfold structure is locally metrizable and regular Hausdorff. Then being paracompact implies that the polyfold is also metrizable. Then by [65, Proposition 7.12], Z^k and any open subsets of Z^k are also paracompact polyfolds.

To get a category of polyfolds $\mathscr{P}$, we need to define the morphisms between polyfolds. Given two polyfolds (Z,c) and (W,d), we can choose two polyfold structures $((\mathcal{X},\boldsymbol{X}),\alpha)$ and $((\mathcal{Y},\boldsymbol{Y}),\beta)$ respectively. Then we define a **morphism between those two polyfold structures** is a pair $(\mathfrak{f},f)$, where $\mathfrak{f}$ is a generalized map from $(\mathcal{X},\boldsymbol{X})$ to $(\mathcal{Y},\boldsymbol{Y})$ and $f: Z\to W$, such that $f\circ\alpha=\beta\circ|\mathfrak{f}|$. Suppose we have another pair of polyfold structures $((\mathcal{X}',\boldsymbol{X}'),\alpha')$ and $((\mathcal{Y}',\boldsymbol{Y}'),\beta')$ respectively and a morphism $(\mathfrak{f}',f')$ between them. Then $(\mathfrak{f},f)$ is equivalent to $(\mathfrak{f}',f')$ if we have the following commutative diagram:

$$\begin{tikzcd}
((\mathcal{X},\boldsymbol{X}),\alpha) \arrow[r, "(\mathfrak{f},f)"] \arrow[d, "(\mathfrak{h},\mathrm{id}_Z)"] & ((\mathcal{Y},\boldsymbol{Y}),\beta) \arrow[d, "(\mathfrak{g},\mathrm{id}_W)"] \\
((\mathcal{X}',\boldsymbol{X}'),\alpha') \arrow[r, "(\mathfrak{f}',f')"] & ((\mathcal{Y}',\boldsymbol{Y}'),\beta'),
\end{tikzcd}$$

where $\mathfrak{h}$ and $\mathfrak{g}$ are generalized isomorphisms and the commutativity is in the sense of $\mathfrak{g}\circ\mathfrak{f}=\mathfrak{f}'\circ\mathfrak{h}$ as generalized maps and $f=f'$. A morphism between two polyfolds is an equivalence class of morphisms between polyfold structures. Therefore we have a category $\mathscr{P}$ of polyfolds, see [65, Definition 16.8].

To simplify notations, **we will abbreviate $\boldsymbol{\alpha}$ and write $\boldsymbol{Z=|\mathcal{X}|}$ from now on.** Then a morphism between polyfolds $Z=|\mathcal{X}|$ and $W=|\mathcal{Y}|$ is represented an equivalence class of a generalized map $\mathfrak{f}:(\mathcal{X},\boldsymbol{X})\to(\mathcal{Y},\boldsymbol{Y})$.

4.2 Group actions on polyfolds

Now we give a tentative definition of a sc^∞ group action on a polyfold.

Definition 4.27. *Let Z be a polyfold, a* **sc^∞ *G-action*** *on Z is pair $(\rho,\mathfrak{P})$ with the following structures.*

1. *ρ is a continuous action $G\times Z\to Z$.*

2. *$\mathfrak{P}:(\star,G)\to\mathscr{P}$ is a functor sending $\star$ to Z, such that $\rho(g,z)=|\mathfrak{P}(g)|(z)$, here we use $|\mathfrak{P}(g)|$ to denote the underlying topological map of the polyfold morphism $\mathfrak{P}(g)$.*

3. *Let $g\in G$ and $p,q\in Z$ such that $\rho(g,p)=q$. Then there exist a neighborhood U of g, two equivalent polyfold structures $(\mathcal{X},\boldsymbol{X})$ and $(\mathcal{Y},\boldsymbol{Y})$, two points $x\in\mathcal{X},y\in\mathcal{Y}$*

[5] The partition of unity on polyfold requires more than paracompactness, see [65, Theoreom 7.4]. If the M-polyfold structures on the object space is built on sc-Hilbert space, then we will have sc-smooth partition unity, see [65, Theorem 5.12, Corollary 5.2, Theorem 7.4]. Also note that the construction in Lemma 3.24 is closed in the sc-Hilbert space category.

with $|x| = p$, $|y| = q$ and a local uniformizer $\mathcal{U} \subset \mathcal{X}$ around x. So that the U-family of polyfold maps $U \to \mathrm{Mor}_{\mathscr{P}}(|\mathcal{U}|, Z)$ defined by $h \mapsto \mathfrak{P}(h)|_{|\mathcal{U}|}$ is represented by the following sc-smooth functor

$$\rho_{g,x} : U \times (\mathrm{stab}_x \ltimes \mathcal{U}) \to (\mathcal{Y}, \boldsymbol{Y}),$$

satisfying $\rho_{g,x}(g, x) = y$.

Remark 4.28. *The continuous action ρ is completely determined by $\mathfrak{P}$, where the continuity follows from (3) in Definition 4.27.*

Remark 4.29. *Assume G acts on Z by $(\rho, \mathfrak{P})$. Since for each $h \in G$ morphism $\mathfrak{P}(h)$ is an isomorphism of polyfold, i.e. $\mathfrak{P}(h)$ is represented by generalized isomorphisms. Then we have $\rho_{g,x}(h, \cdot)$ is fully faithful by Theorem 4.24. By Proposition A.4, $\rho_{g,x}(h, \cdot) : \mathcal{U} \to \mathcal{Y}$ is a sc-diffeomorphism onto the open image set for every $h \in U$.*

The following proposition provides a natural characterization of group action.

Proposition 4.30. *Let $\mathfrak{p} : G \times Z \to Z$ be a polyfold morphism such that*

1. *$\mathfrak{p}(\mathrm{id}, \cdot) = \mathrm{id}_Z$ as polyfold morphism;*
2. *$\mathfrak{p}(gh, \cdot) = \mathfrak{p}(g, \cdot) \circ \mathfrak{p}(h, \cdot)$ as polyfold morphism.*

Then $\mathfrak{p}$ induces a G-action $(\rho, \mathfrak{P})$, where $\rho = |\mathfrak{p}|$ and $\mathfrak{P}(g) = \mathfrak{p}(g, \cdot)$.

Proof. It suffices to prove the existence of local representations of $\mathfrak{P}$. Note that $\mathfrak{p}$ can be represented the following digram $G \times (\mathcal{X}, \boldsymbol{X}) \xleftarrow{F} (\mathcal{W}, \boldsymbol{W}) \xrightarrow{\Phi} (\mathcal{Y}, \boldsymbol{Y})$ with F an equivalence. For every $(g, x) \in G \times \mathcal{X}$, there are local uniformizers of $G \times (\mathcal{X}, \boldsymbol{X})$ around (g, x) in the form of $U \times (\mathrm{stab}_x \ltimes \mathcal{U})$, where $\mathcal{U}$ is a local uniformizer for $(\mathcal{X}, \boldsymbol{X})$ around x and $U \subset G$ is a neighborhood of g. Since F is an equivalence, for small enough U and $\mathcal{U}$, we can assume a local uniformizer of $\mathcal{W}$ also in the form of $U \times (\mathrm{stab}_x \ltimes \mathcal{U})$. Then $\mathfrak{P}$ is locally represented by $\Phi|_{U \times (\mathrm{stab}_x \ltimes \mathcal{U})}$. □

4.3 Properties of group actions

In this subsection, we study the properties of group actions on polyfolds. Recall from Definition 4.27, we have local representatives of group actions $\rho_{g,x} : U \times (\mathrm{stab}_x \times \mathcal{U}) \to (\mathcal{Y}, \boldsymbol{Y})$. If we want to compose such local actions, we need to modify $\rho_{g,x}$ so that it is defined (partially) on one polyfold structure $(\mathcal{X}, \boldsymbol{X})$. For this purpose, we fix a polyfold structure $(\mathcal{X}, \boldsymbol{X})$. Let $x, y \in \mathcal{X}$ such that $\rho(g, |x|) = |y|$ in $Z = |\mathcal{X}|$. Then by Definition 4.27, the action can be locally represented by

$$\rho_{g,x_a} : U_a \times (\mathrm{stab}_{x_a} \ltimes \mathcal{U}_a) \to (\mathcal{X}_b, \boldsymbol{X}_b) \tag{4.3}$$

where $(\mathcal{X}_a, \boldsymbol{X}_a), (\mathcal{X}_b, \boldsymbol{X}_b)$ are polyfold structures of Z and $\mathrm{stab}_{x_a} \ltimes \mathcal{U}_a$ is a local uniformizer of $(\mathcal{X}_a, \boldsymbol{X}_a)$ around x_a with $|x_a| = |x| \in Z$. Since $(\mathcal{X}, \boldsymbol{X}), (\mathcal{X}_a, \boldsymbol{X}_a), (\mathcal{X}_b, \boldsymbol{X}_b)$ are equivalent polyfold structures, then we have the following sequence of equivalences of ep-groupoids, note that we suppress the morphism spaces since there are no ambiguities,

$$\begin{tikzcd} & & \mathcal{X}_a \arrow[r, "\rho_{g,x_a}(g,\cdot)"] & \mathcal{X}_b & \mathcal{W}_b \arrow[l, "F_b"'] \arrow[r, "G_b"] & \mathcal{X}, \\ \mathcal{X} & \mathcal{W}_a \arrow[l, "F_a"'] \arrow[r, "G_a"] & \mathcal{U}_a \arrow[u, "\subset"'] \end{tikzcd} \tag{4.4}$$

where F_a, G_a, F_b, G_b are all equivalences and $|G_b| \circ |F_b|^{-1} \circ |\rho_{g,x_a}(g,\cdot)| \circ |G_a| \circ |F_a|^{-1} = \rho(g,\cdot)$, i.e. over the orbit space $|\mathcal{U}_a|$ the composition of the sequence covers the action $\rho(g,\cdot)$. Moreover, there exist $w_a \in \mathcal{W}_a, w_b \in \mathcal{W}_b, \phi, \psi \in \boldsymbol{X}, \delta \in \boldsymbol{X}_a, \eta \in \boldsymbol{X}_b$, such that $F_a(w_a) = \phi(x)$, $\delta(G_a(w_a)) = x_a$, $\eta(\rho_{g,x_a}(g, x_a)) = F_b(w_b)$ and $G_b(w_b) = \psi(y)$, i.e.

$$\mathcal{X} \xleftarrow{F_a} \mathcal{W}_a \xrightarrow{G_a} \mathcal{X}_a \xrightarrow{\rho_{g,x_a}(g,\cdot)} \mathcal{X}_b \xleftarrow{F_b} \mathcal{W}_b \xrightarrow{G_b} \mathcal{X}$$

$$\begin{tikzcd} x \arrow[d, "\phi"'] & & x_a \arrow[r] & \rho_{g,x_a}(g,x_a) \arrow[d, "\eta"'] & & y \arrow[d, "\psi"'] \\ F_a(w_a) & w_a \arrow[l] \arrow[r] & G_a(w_a) \arrow[u, "\delta"] & F_b(w_b) & w_b \arrow[l] \arrow[r] & G_b(w_b) \end{tikzcd}$$

Here ρ_{g,x_a} is only partially defined on $\mathcal{X}_a$. The first row here just indicates where the elements are from and the directions of equivalences. We point out that arrows in the lower half of the diagram have two different meanings, the vertical arrows are morphisms in one ep-groupoid and the horizontal arrows are the maps on objects of equivalences. Let F_{a,w_a} and F_{b,w_b} denote the local sc-diffeomorphisms F_a and F_b near w_a in $\mathcal{W}_a$ and w_b in $\mathcal{W}_b$ respectively. Then by Remark 4.29, we have the following proposition.

Proposition 4.31. *There exist neighborhoods $\mathcal{V} \subset \mathcal{X}$ of x and $V \subset U \subset G$ of g, we have for every $h \in V$,*

$$\Gamma(h, z) := L_\psi^{-1} \circ G_b \circ {F_{b,w_b}}^{-1} \circ L_\eta \circ \rho_{g,x_a}(h,\cdot) \circ L_\delta \circ G_a \circ {F_{a,w_a}}^{-1} \circ L_\phi(z) \tag{4.5}$$

defines a local sc-diffeomorphism from $\mathcal{V}$.

Since on the orbit space, $|\Gamma(h,\cdot)| = \rho(h,\cdot)$. We call Γ a **local lift of the action at (g, x, y)**. Γ is not unique, it depends on choices of $w_a, w_b, \phi, \psi, \delta, \eta$ as well as the polyfold structures $\mathcal{W}_a, \mathcal{W}_b, \mathcal{X}_a, \mathcal{X}_b$.

Definition 4.32. *Let Z be a polyfold with a G-action $(\rho, \mathfrak{P})$ and $(\mathcal{X}, \boldsymbol{X})$ a polyfold structure of $Z = |\mathcal{X}|$. Let $x, y \in \mathcal{X}, g \in G$ such that $\rho(g, |x|) = |y|$. We define $L(x, y, g)$ to be the set of germs of all local lifts at (g, x, y).*

Although there are infinitely many different choices involved in the construction of $L(x, y, g)$, the following proposition asserts that $L(x, y, g)$ is a finite set for regular polyfolds.

Proposition 4.33. *Let Z be a regular polyfold with a G-action and $(\mathcal{X}, \boldsymbol{X})$ a polyfold structure of $Z = |\mathcal{X}|$. Then for $x, y \in \mathcal{X}$ and $g \in G$, the isotropy stab_x acts on $L(x, y, g)$ by precomposing the local sc-diffeomorphism L_ϕ for $\phi \in \mathrm{stab}_x$ and this action is transitive. The isotropy stab_y also acts on $L(x, y, g)$ by post-composing L_ϕ for $\phi \in \mathrm{stab}_y$, this action is also transitive.*

Roughly speaking, this proposition is a consequence of that all the functors in (4.5) are equivalences. Regularity plays an important role in the proof of Proposition 4.33. Indeed, the irregular polyfold in Example 4.20 provides an example such that Proposition 4.33 does not hold, see Example 4.52.

Corollary 4.34. $|L(x, y, g)| = |\,\mathrm{stab}_x^{\mathrm{eff}}\,| = |\,\mathrm{stab}_y^{\mathrm{eff}}\,|$*, where* $\mathrm{stab}_x^{\mathrm{eff}}, \mathrm{stab}_y^{\mathrm{eff}}$ *are the effective parts defined in* (4.2).

$L(x, y, g)$ also has the following group properties.

Proposition 4.35. *Let Z be a regular polyfold with a G-action and $(\mathcal{X}, \boldsymbol{X})$ a polyfold structure of $Z = |\mathcal{X}|$. Then for $x, y \in \mathcal{X}$ and $g \in G$, $L(x, y, g)$ has the following structures.*

1. *For every $\phi \in \mathrm{mor}(y, z)$, $\Gamma \mapsto L_\phi \circ \Gamma$ defines a bijection from $L(x, y, g)$ to $L(x, z, g)$. And $\Gamma \mapsto \Gamma \circ L_\phi$ defines a bijection from $L(z, x, g)$ to $L(y, x, g)$.*

2. *There is a well-defined multiplication $\circ : L(y, z, g) \times L(x, y, h) \to L(x, z, gh)$ with the property that if $\Gamma_1 \in L(y, z, g)$ and $\Gamma_2 \in L(x, y, h)$, then*
$$\Gamma_1 \circ \Gamma_2(\epsilon gh, z) = \Gamma_1(\epsilon g, \Gamma_2(h, z)) = \Gamma_1(g, \Gamma_2(g^{-1}\epsilon gh, z)) \tag{4.6}$$
for ϵ in a neighborhood of $\mathrm{id} \in G$ and z in a neighborhood of x.

3. *There is a unique identity element $\Gamma_{\mathrm{id}} \in L(x, x, \mathrm{id})$, such that $\Gamma_{\mathrm{id}}(\mathrm{id}, z) = z$ for z in an open neighborhood of x. This identity is both left and right identity in the multiplication structure.*

4. *There is an (both right and left) inverse map $L(x, y, g) \to L(y, x, g^{-1})$ with respect to the multiplication and identity structures above.*

5. *For $x \in \mathcal{X}, g \in G$, there exists an open neighborhood $\mathcal{U} \times \mathcal{O} \times V$ of (x, y, g) in $\mathcal{X} \times \mathcal{C} \times G$ such that all the local lifts in $L(x, y, g)$ is defined on $V \times \mathcal{U}$ with image in $\mathcal{O}$. Moreover, for any $(x', g') \in \mathcal{U} \times V$ and $y' \in \mathcal{O}$ such that $\rho(g', |x'|) = |y'|$, every element $\Gamma' \in L(x', y', g')$ is represented by the restriction of a unique element $\Gamma \in L(x, y, g)$ to a neighborhood of (g', x') as germ.*

We prove Proposition 4.33 and Proposition 4.35 in the appendix.

4.4 Quotients of polyfolds

In this subsection, we construct quotients of polyfolds. We first construct local uniformizers of the quotient ep-groupoids in Proposition 4.39. Then we assemble such local uniformizers in Theorem 4.42 to form quotient ep-groupoids, which give polyfold structures on the quotients.

Definition 4.36. *Let Z be a polyfold with a sc-smooth G-action $(\rho, \mathfrak{P})$ and $(\mathcal{X}, \boldsymbol{X})$ a polyfold structure. For each $z \in Z$, the* ***isotropy*** *G_z is defined to be the isotorpy of ρ at z. For each $x \in \mathcal{X}$, the* ***isotropy*** *G_x is defined to be $G_{|x|}$. An action has* ***finite isotropy*** *iff for every $z \in Z$, $|G_z| < \infty$.*

For $x_0 \in \mathcal{X}_\infty$, let $\Gamma_{x_0} \in L(x_0, x_0, \mathrm{id})$ be the identity element as asserted in Proposition 4.35. Then we can define the **infinitesimal directions** of the group action at x to be

$$\mathfrak{g}_{x_0} := \mathrm{D}(\Gamma_{x_0})_{(\mathrm{id}, x_0)}(T_{\mathrm{id}}G \times \{0\}).$$

Then we have analogue of Proposition 3.22 and Proposition 3.23 as follows.

Proposition 4.37. *If the action has finite isotropy, then $\mathfrak{g}_{x_0} \subset (T^R_{x_0}\mathcal{X})_\infty$ and $\dim \mathfrak{g}_{x_0} = \dim G$.*

Proof. By (4.4), $\Gamma_{x_0}(\epsilon, \Gamma_{x_0}(\delta, x)) = \Gamma_{x_0}(\epsilon\delta, x)$ for ϵ, δ close to id and x close to x_0. Then the proof of Proposition 3.23 applies here and $\dim \mathfrak{g}_{x_0} = \dim G$. Since $\Gamma_{x_0}(\epsilon, \cdot)$ is a local sc-diffeomorphism for ϵ close to id by Proposition 4.31, the arguments in Proposition 3.22 prove that $\mathfrak{g}_{x_0} \subset (T^R_{x_0}\mathcal{X})_\infty$. □

We point out here a property of $\mathfrak{g}_{x_0}$ that will be used in Remark 4.53. Let $g_0 \in G_{x_0}$ and a local lift $\Gamma \in L(x_0, x_0, g_0)$, then

$$\mathrm{D}\Gamma_{(g_0, x_0)}(\{0\} \times \mathfrak{g}_{x_0}) = \mathfrak{g}_{x_0}. \tag{4.7}$$

By (4.6), for $\epsilon \in G$ close to id, we have

$$\Gamma(g_0, \Gamma_{x_0}(g_0^{-1}\epsilon g_0, x_0)) = \Gamma(\epsilon g_0, x_0) = \Gamma_{x_0}(\epsilon, \Gamma(g_0, x_0)) = \Gamma_{x_0}(\epsilon, x_0).$$

Then (4.7) is proven by taking derivatives in ϵ.

To construct quotient polyfolds, we need to introduce the analogue of G-slices in the ep-groupoid setup. Of course slices for ep-groupoids should be translation ep-groupoids. We first discuss the isotropy of the quotients. For every $x \in \mathcal{X}$, let $\mathrm{stab}^Q_x := \cup_{g \in G_x} L(x, x, g)$. Then stab^Q_x is a group by Proposition 4.35. Moreover, we have an exact sequence of groups:

$$1 \to \mathrm{stab}^{\mathrm{eff}}_x \to \mathrm{stab}^Q_x \to G_x \to 1, \tag{4.8}$$

where the first map is the inclusion $\mathrm{stab}^{\mathrm{eff}}_x \simeq L(x, x, \mathrm{id}) \hookrightarrow \mathrm{stab}^Q_x$ and the second map is the projection. As a consequence, stab^Q_x is a finite group with $|\,\mathrm{stab}^Q_x\,| = |\,\mathrm{stab}^{\mathrm{eff}}_x\,| \cdot |G_x|$.

Definition 4.38. *Let Z be a regular polyfold with a sc-smooth G-action with finite isotropy and $(\mathcal{X}, \boldsymbol{X})$ a polyfold structure. For every $x_0 \in \mathcal{X}_\infty$, a G-slice for the polyfold structure $(\mathcal{X}, \boldsymbol{X})$ around x_0 is a tuple $(\mathcal{U}, \tilde{\mathcal{U}}, V, f, \eta)$ such that the following holds.*

1. *$\mathcal{U} \subset \mathcal{X}$ is an open neighborhood of x_0 and $V \subset G$ is an open neighborhood of id.*
2. *Let Γ_0 denote the identity element in $L(x_0, x_0, \mathrm{id})$, Γ_0 is defined on $V \times \mathcal{U}$.*
3. *$\tilde{\mathcal{U}} := f^{-1}(0)$ is slice (Definition 3.17) of $\mathcal{X}$ containing x_0.*
4. *For $x \in \mathcal{U}$, $g = f(x)$ is the unique element $g \in V$ such that $\Gamma_0(g, x) \in \tilde{\mathcal{U}}$.*
5. *$\eta : \mathcal{U} \to \tilde{\mathcal{U}}$ defined by $x \mapsto \Gamma_0(f(x), x)$ is sc-smooth.*
6. *There is a sc-smooth action of $\mathrm{stab}^Q_{x_0}$ on $\tilde{\mathcal{U}}$ defined by*
$$\Gamma \odot y := \eta(\Gamma(g_0, y)), \quad \forall \Gamma \in L(x, x, g_0), y \in \tilde{\mathcal{U}}. \tag{4.9}$$
7. *For every $y \in \tilde{\mathcal{U}}$, the set $S_{y,\tilde{\mathcal{U}}} := \{(z, g, \Gamma) | z \in \tilde{\mathcal{U}}, g \in G, \Gamma \in L(y, z, g)\}$ has exactly $|\,\mathrm{stab}^Q_{x_0}\,|$ elements.*

Proposition 4.39. *Let Z be a regular polyfold with a sc-smooth G-action with finite isotropy and $(\mathcal{X}, \boldsymbol{X})$ a polyfold structure. For every $x_0 \in \mathcal{X}_\infty$, there exists a G-slice for the polyfold structure $(\mathcal{X}^2, \boldsymbol{X}^2)$ of Z^2 around x_0.*

Proof. By Proposition 4.37, we can apply Lemma 3.24 to the identity element $\Gamma_0 \in L(x_0, x_0, \mathrm{id})$. Therefore we have a tuple $(\mathcal{W}, \tilde{\mathcal{W}}, V, f, \eta)$ such that the following holds.

(i) $\mathcal{W} \subset \mathcal{X}^2$ is an open neighborhood x_0 and $V \subset G$ is an open neighborhood id.

(ii) Γ_0 is defined on $V \times \mathcal{W}$.

(iii) $\tilde{\mathcal{W}} := f^{-1}(0)$ is slice of $\mathcal{X}$ containing x_0.

(iv) For $x \in \mathcal{W}$, $g = f(x)$ is the unique element $g \in V$ such that $\Gamma_0(g, x) \in \tilde{\mathcal{W}}$.

(v) $\eta : \mathcal{W} \to \tilde{\mathcal{W}}$ defined by $x \mapsto \Gamma_0(f(x), x)$ is sc-smooth.

(vi) Property (5) of Proposition 4.35 holds for every $\Gamma \in \mathrm{stab}^Q_{x_0}$ on $\mathcal{W} \times V$.

Moreover, for every open neighborhood $\tilde{\mathcal{W}}' \subset \tilde{\mathcal{W}}$ of x_0, let $\mathcal{W}' = \eta^{-1}(\tilde{\mathcal{W}}')$. Then the tuple $(\mathcal{W}', \tilde{\mathcal{W}}', V, f|_{\mathcal{W}'}, \eta|_{\mathcal{W}'})$ also has property (i)-(vi).

Next we will construct an neighborhood $\tilde{\mathcal{V}} \subset \tilde{\mathcal{W}}$ of x_0 so that (4.9) defines a sc-smooth group action on $\tilde{\mathcal{V}}$. First, we pick a small enough neighborhood $\tilde{\mathcal{W}}' \subset \tilde{\mathcal{W}}$ of x_0, such that (4.9) is well-defined for every $\Gamma \in \mathrm{stab}^Q_{x_0}$, i.e. $\Gamma(g_0, \tilde{\mathcal{W}}') \subset \mathcal{W}$ for every $\Gamma \in L(x_0, x_0, g_0)$. Since
$$\Gamma \odot y = \eta(\Gamma(g_0, y)) = \Gamma_0(f(\Gamma(g_0, y)), \Gamma(g_0, y)), \tag{4.10}$$

and the inverse to Γ_0 is itself, we can require the neighborhood $\tilde{\mathcal{W}}'$ is small enough such that we can invert (4.10) to get

$$\Gamma(g_0, y) = \Gamma_0(f(\Gamma(g_0, y))^{-1}, \Gamma \odot y), \quad \forall \Gamma \in \operatorname{stab}^Q_{x_0}, y \in \tilde{\mathcal{W}}'. \tag{4.11}$$

Moreover, we can shrink $\tilde{\mathcal{W}}'$ further such that the following equations are defined and hold for every $\Gamma_1 \in L(x_0, x_0, g_1)$ and $\Gamma_2 \in L(x_0, x_0, g_2)$ and $y \in \tilde{\mathcal{W}}'$,

$$\begin{aligned}
(\Gamma_1 \circ \Gamma_2) \odot y &= \eta(\Gamma_1 \circ \Gamma_2(g_1 g_2, y)) \\
&\overset{(4.6)}{=} \eta(\Gamma_1(g_1, \Gamma_2(g_2, y)) \\
&\overset{(4.11)}{=} \eta(\Gamma_1(g_1, \Gamma_0(f(\Gamma_2(g_2, y))^{-1}, \Gamma_2 \odot y))) \\
&\overset{(4.6)}{=} \eta(\Gamma_1(g_1 f(\Gamma_2(g_2, y))^{-1}, \Gamma_2 \odot y) \\
&\overset{(4.6)}{=} \eta(\Gamma_0(g_1 f(\Gamma_2(g_2, y))^{-1} g_1^{-1}, \Gamma_1(g_1, \Gamma_2 \odot y))) \\
&\overset{(iv)}{=} \eta(\Gamma_1(g_1, \Gamma_2 \odot y))) \\
&= \Gamma_1 \odot (\Gamma_2 \odot y)
\end{aligned} \tag{4.12}$$

Since

$$\Gamma_0 \odot y = \eta(\Gamma_0(\mathrm{id}, y)) = \eta(y) = y, \quad \forall y \in \tilde{\mathcal{W}}', \tag{4.13}$$

(4.12) implies that $\Gamma^{-1} \odot (\Gamma \odot y) = y$ for every $y \in \tilde{\mathcal{W}}'$. Hence $\Gamma \odot$ is injective on $\tilde{\mathcal{W}}'$. Pick an open neighborhood $\tilde{\mathcal{O}} \subset \tilde{\mathcal{W}}'$, such that $\Gamma \odot (\tilde{\mathcal{O}}) \subset \tilde{\mathcal{W}}'$ for all Γ. Then the injectivity of $\Gamma^{-1} \odot$ on $\tilde{\mathcal{W}}'$ and $\Gamma^{-1} \odot (\Gamma \odot y) = y$ on $\tilde{\mathcal{O}}$ imply that $\Gamma \odot (\tilde{\mathcal{O}}) = (\Gamma^{-1} \odot)|^{-1}_{\tilde{\mathcal{W}}'}(\tilde{\mathcal{O}})$. As a consequence, $\Gamma \odot (\tilde{\mathcal{O}})$ is open in $\tilde{\mathcal{W}}'$ for every Γ. We define

$$\tilde{\mathcal{V}} := \cap_{\Gamma \in \operatorname{stab}^Q_{x_0}} \Gamma \odot (\tilde{\mathcal{O}}).$$

Then $\tilde{\mathcal{V}} \subset \tilde{\mathcal{W}}'$ is an open neighborhood of x_0. Therefore (4.12) implies that (4.9) defines a sc-smooth group action of $\operatorname{stab}^Q_{x_0}$ on $\tilde{\mathcal{V}}$. Moreover, for every open neighborhood $\tilde{\mathcal{V}}' \subset \tilde{\mathcal{V}}$ of x_0, the $\operatorname{stab}^Q_{x_0}$ action can be restricted to $\tilde{\mathcal{V}}'' := \cap_{\Gamma \in \operatorname{stab}^Q_{x_0}} \Gamma \odot (\tilde{\mathcal{V}}') \subset \tilde{\mathcal{V}}'$.

Finally, we claim that we can find a neighborhood $\tilde{\mathcal{U}} \subset \tilde{\mathcal{V}}$ of x_0 invariant under the $\operatorname{stab}^Q_{x_0}$ action, such that $|S_{y,\tilde{\mathcal{U}}}| = |\operatorname{stab}^Q_{x_0}|$ for every $y \in \tilde{\mathcal{U}}$. First we pick a smaller neighborhood $\tilde{\mathcal{V}}' \subset \tilde{\mathcal{V}}$ such that for all $y \in \tilde{\mathcal{V}}'$ and $\Gamma \in L(x_0, x_0, g_0)$,

$$\Gamma \odot y = \eta(\Gamma(g_0, y)) = \Gamma_0(f(\Gamma(g_0, y)), \Gamma(g_0, y)) \overset{(4.6)}{=} \Gamma(f(\Gamma(g_0, y))g_0, y). \tag{4.14}$$

For every open neighborhood $\tilde{\mathcal{U}}' \subset \tilde{\mathcal{V}}'$ of x_0, let $\tilde{\mathcal{U}} := \cap_{\Gamma \in \operatorname{stab}^Q_{x_0}} \Gamma \odot (\tilde{\mathcal{U}}') \subset \tilde{\mathcal{U}}'$ and Γ_y denotes the restriction of Γ to $L(y, \Gamma \odot y, f(\Gamma(g_0, y))g_0)$. Then $(\Gamma \odot y, f(\Gamma(g_0, y))g_0, \Gamma_y) \in S_{y,\tilde{\mathcal{U}}}$ by (4.14). By the uniqueness of the restriction in the property (5) of Proposition 4.35, different $\Gamma \in \operatorname{stab}^Q_{x_0}$ gives different elements in $S_{y,\tilde{\mathcal{U}}}$, thus $|S_{y,\tilde{\mathcal{U}}}| \geq |\operatorname{stab}^Q_{x_0}|$.

Assume otherwise that the last assertion does not hold for any of such $\tilde{\mathcal{U}}$. Then there exists a shrinking sequence of open neighborhoods $\tilde{\mathcal{U}}(k) \subset \tilde{\mathcal{V}}'$ invariant under the $\operatorname{stab}^Q_{x_0}$ action , such that the last assertion does not hold on $\tilde{\mathcal{U}}(k)$. In other words, we have $y_k \in \tilde{\mathcal{U}}(k)$ such that $\lim_k y_k \to x_0$ and for each k there are at least $|\operatorname{stab}^Q_x|+1$ tuples $\{(z^i_k, g^i_k, \Gamma^i_k)\}_{1\le i\le |\operatorname{stab}^Q_x|+1}$ with $\Gamma^i_k(g^i_k, y_k) = z^i_k \in \tilde{\mathcal{U}}(k)$ and $\lim_k z^i_k \to x_0$. After passing to a subsequence, we can assume $\lim_k g^i_k \in G_{x_0}$ for all i. Therefore there is a set $I \subset \{1, \ldots, |\operatorname{stab}^Q_{x_0}| + 1\}$ with at least $|\operatorname{stab}^Q_{x_0}|/|G_{x_0}| + 1 = |\operatorname{stab}^{\text{eff}}_{x_0}| + 1$ elements, such that for all $i \in I$, $\lim_k g^i_k = g_0 \in G_x$. By property (vi), Γ^i_k is a restriction of some $\Gamma \in L(x, x, g_0)$. After passing to a subsequence, we can assume Γ^i_k is the restriction of a fixed $\Gamma^i \in L(x, x, g_0)$ for a fixed i. Since $|L(x_0, x_0, g_0)| = |\operatorname{stab}^{\text{eff}}_{x_0}|$, there is a subset $J \subset I$ containing at least two elements such that for $j \in J$, Γ^j_k is the restriction of a common $\Gamma \in L(x, x, g_0)$. Then we have $z^{j_1}_k = \Gamma(g^{j_1}_k, y_k)$, $z^{j_2}_k = \Gamma(g^{j_2}_k, y_k)$ for $j_1, j_2 \in J$. Therefore for $k \gg 0$, we have

$$z^{j_1}_k = \Gamma(g^{j_1}_k, y_k) = \Gamma_0(g^{j_1}_k g_0^{-1}, \Gamma(g_0, y_k)).$$

Since $z^{j_1}_k \in \tilde{\mathcal{U}}(k) \subset \tilde{\mathcal{W}}$, by property (iv) we have $g^{j_1}_k g_0^{-1} = f(\Gamma(g_0, y_k))$. Therefore $z^{j_1}_k = \Gamma \odot y_k$. Similarly, we have $g^{j_2}_k g_0^{-1} = f(\Gamma(g_0, y_k))$ and $z^{j_2}_k = \Gamma \odot y_k$. This contradicts that $(z^{j_1}_k, g^{j_1}_k, \Gamma^{j_1}_k)$ and $(z^{j_2}_k, g^{j_2}_k, \Gamma^{j_2}_k)$ are different. $\square$

We will see later that $S_{z,\tilde{\mathcal{U}}}$ is the orbit set (Definition 4.8) for the quotient ep-groupoids. The last assertion in Proposition 4.39 will be used to prove the properness of the quotient ep-groupoids. We also need the following two results to prove the paracompactness of the morphism space for the quotient ep-groupoids.

Lemma 4.40 ([65, Lemma 2.7])**.** *A regular Hausdorff space Y is paracompact iff for any open covers of Y, there exists a locally finite refinement consisting of closed sets.*

Proposition 4.41 ([65, Proposition 2.17])**.** *Let Y be a regular Hausdorff space, let $(Y_i)_{i\in I}$ be a locally finite family of closed subsets of Y such that $Y = \cup_{i\in I} Y_i$. If every Y_i is paracompact, then Y is paracompact.*

Now we have all the prerequisites for the proof of the following theorem.

Theorem 4.42. *Let Z be a regular tame polyfold and $(\mathcal{X}, \boldsymbol{X})$ a polyfold structure. Assume compact Lie group G acts on Z sc-smoothly and the action only has finite isotropy. For every $x_0 \in \mathcal{X}_\infty$, let $(\mathcal{U}_{x_0}, \tilde{\mathcal{U}}_{x_0}, V_{x_0}, f_{x_0}, \eta_{x_0})$ be a G-slice for $(\mathcal{X}^2, \boldsymbol{X}^2)$ around x_0. We define sets*

$$\begin{aligned} \mathcal{Q} &= \coprod_{x_0 \in \mathcal{X}_\infty} \tilde{\mathcal{U}}_{x_0}, \\ \boldsymbol{Q} &= \{(x, y, g, \Gamma) | x, y \in \mathcal{Q}, g \in G, \Gamma \in L(x, y, g)\}, \end{aligned}$$

and maps

$$\begin{array}{rrcll} s_Q: & \boldsymbol{Q} & \to & \mathcal{Q}: & (x,y,g,\Gamma) \mapsto x, \\ t_Q: & \boldsymbol{Q} & \to & \mathcal{Q}: & (x,y,g,\Gamma) \mapsto y, \\ m_Q: & \boldsymbol{Q}_s \times_t \boldsymbol{Q} & \to & \boldsymbol{Q}: & ((y,z,g,\Gamma),(x,y,h,\Pi)) \mapsto (x,z,gh,\Gamma\circ\Pi), \\ u_Q: & \mathcal{Q} & \to & \boldsymbol{Q}: & x \mapsto (x,x,\mathrm{id},\Gamma_x), \\ i_Q: & \boldsymbol{Q} & \to & \boldsymbol{Q}:, & (x,y,g,\Gamma) \mapsto (y,x,g^{-1},\Gamma^{-1}), \end{array}$$

where $\Gamma_x \in L(x,x,\mathrm{id})$ is the identity element. Then $\mathcal{Q},\boldsymbol{Q}$ can be equipped with M-polyfold structures so that $(\mathcal{Q},\boldsymbol{Q})$ is an ep-groupoid.

Let $\widehat{Z} := \cup_{x_0\in\mathcal{X}_\infty}\rho(G,|\mathcal{U}_{x_0}|) \subset Z^2$. Then $\widehat{Z}$ is a G-invariant open set of Z^2 containing Z_∞ and $(\mathcal{Q},\boldsymbol{Q})$ defines a polyfold structure on $\widehat{Z}/G$ such that the topological quotient map $\pi_G:\widehat{Z}\to\widehat{Z}/G$ is covered by a sc-smooth polyfold map $\mathfrak{q}:\widehat{Z}\to\widehat{Z}/G$.

Proof. From the definition of $\mathcal{Q},\boldsymbol{Q}$ and the structure maps, $(\mathcal{Q},\boldsymbol{Q})$ is a groupoid. $\mathcal{Q}$ is a tame M-polyfold since it is a disjoint union of tame M-polyfolds. The remaining part of the proof is divided into several claims.

Claim 4.43. *$\boldsymbol{Q}$ has a M-polyfold structure.*

Proof. We first give $\boldsymbol{Q}$ a topology. Let $\mathcal{U}\subset\mathcal{Q},\mathcal{W}\subset\mathcal{Q},V\subset G$ be open subsets and Γ a local lift defined on $V\times\mathcal{U}$. We claim all such $(\mathcal{U}\times\mathcal{W}\times V\times\{\Gamma\})\cap\boldsymbol{Q}$ form a topological basis, here Γ is thought of as the germ restricted to the point in $V\times\mathcal{U}$. For any $(x,y,g,\Gamma)\in(\mathcal{U}'\times\mathcal{W}'\times V'\times\{\Gamma'\})\cap(\mathcal{U}''\times\mathcal{W}''\times V''\times\{\Gamma''\})\cap\boldsymbol{Q}$, we have the restrictions of Γ' and Γ'' to (g,x) is the same as Γ as a germ. Hence we can find smaller neighborhood $\mathcal{U}\subset\mathcal{U}'\cap\mathcal{U}'',\mathcal{W}\subset\mathcal{W}'\cap\mathcal{W}''$ and $V\subset V'\cap V''$, such that $(x,y,g,\Gamma)\subset(\mathcal{U}\times\mathcal{W}\times V\times\{\Gamma\})\cap\boldsymbol{Q}\subset(\mathcal{U}'\times\mathcal{W}'\times V'\times\{\Gamma'\})\cap(\mathcal{U}''\times\mathcal{W}''\times V''\times\{\Gamma''\})\cap\boldsymbol{Q}$. This proves the claim and we can use this topological basis to put a topology on $\boldsymbol{Q}$.

Next we equip $\boldsymbol{Q}$ with a M-polyfold structure. Let $(x,y,g,\Gamma)\in\boldsymbol{Q}$. If $y\in\tilde{\mathcal{U}}_w$ for $w\in\mathcal{X}_\infty$ then map

$$s_Q^{-1}: z\mapsto (z,\eta_w(\Gamma(g,z)),f_w(\Gamma(g,z))g,\Gamma_{f_w(\Gamma(g,z))g,z})\in\boldsymbol{Q} \tag{4.15}$$

is well-defined on a neighborhood $\mathcal{V}\subset\mathcal{Q}$ of x to $\boldsymbol{Q}$, where $\Gamma_{f_w(\Gamma(g,z))g,z}$ is the restriction of Γ to a neighborhood of $(f_w(\Gamma(g,z)),z)$. We chose $\mathcal{V}$ small enough such that the following conditions hold for every $z\in\mathcal{V}$.

1. $\Gamma(g,z)$ is defined and $\Gamma(g,z)\in\mathcal{U}_w$.
2. Let $g_z := f_w(\Gamma(g,z))g$, then g_z is in a neighborhood $V'\subset G$ of g, such that property (5) of Proposition 4.35 holds on $\mathcal{V}\times V'$ for $L(x,y,g)$.
3. For $g'\in V'$ and $z\in\mathcal{V}$, $\Gamma_{g',z}(g',z)=\Gamma_w(g'g^{-1},\Gamma(g,z))$, where Γ_w is the identity element in $L(w,w,\mathrm{id})$ defined on $\mathcal{U}_w$.

We claim s_Q^{-1} is the local inverse to s_Q near (x,y,g,Γ) as the notation indicates. Since $s_Q \circ s_Q^{-1} = \mathrm{id}_{\mathcal{V}}$, it is sufficient to prove s_Q^{-1} is an open map. For every open subset $\mathcal{O} \subset \mathcal{V}$, let $(x',y',g',\Gamma') \in (\mathcal{O} \times \tilde{\mathcal{U}}_w \times V' \times \{\Gamma\}) \cap \boldsymbol{Q}$. Since $\Gamma'(g',x') = y'$, by (3) $y' = \Gamma_w(g'g^{-1}, \Gamma(g,x'))$. By property (4) of Definition 4.38 and (1) here, we have $g'g^{-1} = f_w(\Gamma(g,x'))$ and $y' = \eta_w(\Gamma(g,x'))$. That is $(x',y',g',\Gamma') \in s_Q^{-1}(\mathcal{O})$. Hence $s_Q^{-1}(\mathcal{O}) = (\mathcal{O} \times \tilde{\mathcal{U}}_w \times V' \times \{\Gamma\}) \cap \boldsymbol{Q}$ which is open in $\boldsymbol{Q}$. Therefore s_Q^{-1} gives $\boldsymbol{Q}$ a tame M-polyfold structures locally. The transition maps are the identity map on $\mathcal{Q}$, thus $\boldsymbol{Q}$ has a tame M-polyfold structure and the source map $s_Q : \boldsymbol{Q} \to \mathcal{Q}$ is a local sc-diffeomorphism. The target map $t_Q : \boldsymbol{Q} \to \mathcal{Q}$ is sc$^\infty$, since the composition of target map t_Q with (4.15) is

$$z \to \eta_w(\Gamma(g,z)),$$

which is a local sc-diffeomorphism, as one can write down a local sc$^\infty$ inverse just like (4.15). Similarly the unit map u_Q, the inverse map i_Q and the multiplication m_Q are sc$^\infty$. This proves the étale property. $\square$

Claim 4.44. *$\boldsymbol{Q}$ is Hausdorff and paracompact and $(\boldsymbol{Q}, \mathcal{Q})$ is an étale ep-groupoid.*

Proof. By our definition of the topology on $\boldsymbol{Q}$, we have a continuous projection:

$$\pi : \boldsymbol{Q} \to \mathcal{Q} \times \mathcal{Q} \times G.$$

Because $\mathcal{Q} \times \mathcal{Q} \times G$ is Hausdorff, any two points in $\boldsymbol{Q}$ with different projections are separated by open sets. If two points have same projection, i.e. (x,y,g,Γ_1) and (x,y,g,Γ_2), such that $\Gamma_1 \neq \Gamma_2$. Then by Proposition 4.33, $\Gamma_2 = \Gamma_1 \circ L_\phi$ for some $\phi \in \mathrm{stab}_x$. Since $\Gamma_1 \neq \Gamma_2$ and the ep-groupoid $(\mathcal{X}, \boldsymbol{X})$ is regular, thus L_ϕ does not fix any open set near x. Then for z in a neighborhood of x and h in a neighborhood of g, we have $\Gamma_{1,h,z} \neq \Gamma_{2,h,z}$, where $\Gamma_{1,h,z}, \Gamma_{2,h,z}$ are the restrictions of Γ_1, Γ_2 to (h,z). Thus (x,y,g,Γ_1) and (x,y,g,Γ_2) can be separated by $\mathcal{U} \times \mathcal{W} \times V \times \{\Gamma_1\}$ and $\mathcal{U} \times \mathcal{W} \times V \times \{\Gamma_2\}$ for some open neighborhoods $\mathcal{U}, \mathcal{W}, V$ of x, y, g.

We will use Proposition 4.41 to prove that $\boldsymbol{Q}$ is paracompact. We first prove that the map $\pi : \boldsymbol{Q} \to \mathcal{Q} \times \mathcal{Q} \times G$ is closed map. Let $\boldsymbol{C}$ be a closed subset of $\boldsymbol{Q}$. Note that if we have $(x_k, y_k, g_k) \in \mathcal{Q} \times \mathcal{Q} \times G$ such that $(x_k, y_k, g_k) \to (x,y,g)$ and $(x_k, y_k, g_k, \Gamma_k) \in \boldsymbol{C}$, then by Proposition 4.35, there is a $\Gamma \in L(x,y,g)$ such that after passing to a subsequence Γ_k is the restriction of a Γ. This shows that $(x,y,g,\Gamma) \in \boldsymbol{Q}$ is a limit point of $\boldsymbol{C}$, hence $(x,y,g,\Gamma) \in \boldsymbol{C}$ and $(x,y,g) \in \pi(\boldsymbol{C})$. Therefore π is a closed map. Next we claim that $\boldsymbol{Q}$ is a regular space. That is for $p = (x,y,g,\Gamma) \in \boldsymbol{Q}$ and a closed set $\boldsymbol{C} \subset \boldsymbol{Q}$ not containing p, we can separate them by non-intersecting open sets. If $\pi(p) \notin \pi(\boldsymbol{C})$, since $\mathcal{Q} \times \mathcal{Q} \times G$ is regular, we can separate p and $\boldsymbol{C}$. If $\pi(p) \in \pi(\boldsymbol{C})$, by Proposition 4.33 $\boldsymbol{C} \cap \pi^{-1}(\pi(p))$ is a finite set. Since $\boldsymbol{Q}$ is Hausdorff, we can find an open neighborhood $\boldsymbol{U} \subset \boldsymbol{Q}$ of p and $\boldsymbol{V} \subset \boldsymbol{Q}$ of $\boldsymbol{C} \cap \pi^{-1}(\pi(p))$ separating p and $\boldsymbol{C} \cap \pi^{-1}(\pi(p))$. Note that p and $\boldsymbol{C} \backslash \boldsymbol{V}$ can be separated since $\pi(p) \notin \pi(\boldsymbol{C} \backslash \boldsymbol{V})$. Therefore p and $\boldsymbol{C}$ can be separated hence $\boldsymbol{Q}$ is regular.

Next we apply Proposition 4.41 as follows. Observe that for every $(x,y,g) \in \mathcal{Q} \times \mathcal{Q} \times G$, the preimage $\pi^{-1}(x,y,g) \subset \boldsymbol{Q}$ is a finite set. Then we can find small neighborhood $\mathcal{U} \times \mathcal{W} \times$

$V \subset \mathcal{Q} \times \mathcal{Q} \times G$ of (x, y, g), such that $\pi^{-1}(\mathcal{U} \times \mathcal{W} \times V)$ has finitely many components and over each component s_Q is a sc-diffeomorphism onto its image. Therefore $\pi^{-1}(\mathcal{U} \times \mathcal{W} \times V)$ is a paracompact space hence metrizable. Since $\mathcal{Q} \times \mathcal{Q} \times G$ is a metrizable space and is covered by such open sets $\mathcal{U} \times \mathcal{W} \times V$. By Lemma 4.40, we can find a locally finite refinement $\{C_i\}_{i\in I}$ consisting closed sets. Then $\{\pi^{-1}(C_i)\}_{i\in I}$ is a locally finite covering of $\boldsymbol{Q}$ by closed sets. Since $\pi^{-1}(C_i)$ is a closed subset of some metrizable space $\pi^{-1}(\mathcal{U} \times \mathcal{W} \times V)$, $\pi^{-1}(C_i)$ is also metrizable and hence paracompact. We have $\boldsymbol{Q}$ is also paracompact by Proposition 4.41. Therefore $\boldsymbol{Q}$ is a M-polyfold and $(\mathcal{Q}, \boldsymbol{Q})$ is an étable groupoid. □

Claim 4.45. *$(\mathcal{Q}, \boldsymbol{Q})$ is an ep-groupoid.*

Proof. From our construction stab_x^Q in (4.8) is the isotropy group of $(\mathcal{Q}, \boldsymbol{Q})$ for $x \in \mathcal{Q}$. By Definition 4.38, the orbit set $S_{y,\tilde{\mathcal{U}}_x}$ has $|\,\mathrm{stab}_x^Q\,|$ elements for $y \in \tilde{\mathcal{U}}_x$. Then by Proposition 4.10, the first condition of Proposition 4.9 holds. Thus to prove the properness, we need to prove that for any $z \in \mathcal{Q}$ there exists a arbitrary small open neighborhood $\mathcal{V} \subset \mathcal{Q}$ of z with the property that $t_Q(s_Q^{-1}(\overline{\mathcal{V}}))$ is a closed set of $\mathcal{Q}$. For every open neighborhood $\mathcal{O} \subset \tilde{\mathcal{U}}_x$ of z, we can pick $\mathcal{V} \subset \mathcal{O}$, such that $\mathcal{V}$ is contained in an open set $\mathcal{W} \subset \mathcal{U}_x \subset \mathcal{X}^2$ which has the property that $t : s^{-1}(\overline{\mathcal{W}}) \to \mathcal{X}^2$ is proper. Moreover, for small enough $\mathcal{V}$, we can assume there is neighborhood $V \subset G$ of id, such that $\mathcal{W} = \Gamma_x(V, \mathcal{V})$. We claim that

$$\overline{\mathcal{W}} = \Gamma_x(\overline{V}, \overline{\mathcal{V}}). \tag{4.16}$$

This is because a sequence $x_i \in \mathcal{W}$ converge x_∞, iff $\eta_x(x_i) \to \eta_x(x_\infty)$ in $\overline{\mathcal{V}}$ and $f_x(x_i) \to f_x(x_\infty)$, that is $x_\infty \in \Gamma_x(\overline{V}, \overline{\mathcal{V}})$. By (4.16), $\overline{\mathcal{W}} \subset \eta_x^{-1}(\overline{\mathcal{V}})$. Since $\mathcal{Q}$ is metrizable, to show $t_Q(s_Q^{-1}(\overline{\mathcal{V}}))$ is a closed set, it is enough to show it is sequentially closed. We pick a sequence $y_k \in t_Q(s_Q^{-1}(\overline{\mathcal{V}}))$ converging to $y_\infty \in \tilde{\mathcal{U}}_y$. It suffices to show that $y_\infty \in t_Q(s_Q^{-1}(\overline{\mathcal{V}}))$. Since $y_k \in t_Q(s_Q^{-1}(\overline{\mathcal{V}}))$, there is $x_k \in \overline{\mathcal{V}} \subset \tilde{\mathcal{U}}_x$ and $g_k \in G$ such that $\rho(g_k, |x_k|) = |y_k|$. Since G is compact, we can assume $\lim_k g_k \to g_0 \in G$ by choosing a subsequence. We choose any $\tilde{y}_\infty \in \mathcal{X}^2$ such that $\rho(g_0^{-1}, |y_\infty|) = |\tilde{y}_\infty|$ and pick any $\Gamma \in L(y_\infty, \tilde{y}_\infty, g_0^{-1})$. Let $\tilde{y}_k := \Gamma(g_0^{-1}, y_k)$, then $\lim_k \tilde{y}_k = \tilde{y}_\infty$ in $\mathcal{X}^2$. We also define $\tilde{x}_k := \Gamma_x(g_0^{-1}g_k, x_k)$. Then for k big enough, $\tilde{x}_k \in \overline{\mathcal{W}}$ by (4.16). Note that in the orbits space $|\mathcal{X}^2|$, we have

$$\rho(g_0^{-1}g_k, |x_k|) = |\tilde{x}_k|, \quad \rho(g_0^{-1}, |y_k|) = |\tilde{y}_k|.$$

Therefore $|\tilde{x}_k| = |\tilde{y}_k|$ in $|\mathcal{X}^2|$. By the properness of $t : s^{-1}(\overline{\mathcal{W}}) \to \mathcal{X}^2$, there exists $\tilde{x}_\infty \in \overline{\mathcal{W}}$ and a morphism $\phi : \tilde{x}_\infty \to \tilde{y}_\infty$. Since $\eta_x(\tilde{x}_\infty) \in \overline{\mathcal{V}}$, we have a morphism in $\mathbf{Q}$ from $\eta_x(\tilde{x}_\infty)$ to y_∞ defined by

$$(\eta_x(\tilde{x}_\infty), y_\infty, g_0 g_x(\tilde{x}_\infty), \Gamma^{-1} \circ L_\phi \circ \Gamma_x).$$

Here Γ_x is understood as the restriction of the identity element in $L(x, x, \mathrm{id})$ to $L(\eta(\tilde{x}_\infty), \tilde{x}_\infty, f_x(\tilde{x}_\infty))$. Therefore y_∞ is in $t_Q(s_Q^{-1}(\overline{\mathcal{V}}))$. Thus we prove that $t_Q(s_Q^{-1}(\overline{\mathcal{V}}))$ is a closed set of $\mathcal{Q}$ and the properness follows. Hence $(\mathcal{Q}, \boldsymbol{Q})$ is an ep-groupoid. □

Claim 4.46. *$\widehat{Z}/G$ is a tame polyfold.*

Proof. By the definition of $\widehat{Z}$, $\widehat{Z} \subset Z^2$ is a G-invariant open neighborhood Z_∞. By Lemma 3.31, $\widehat{Z}/G$ is metrizable space. Let β denote the composition of maps

$$\beta : \mathcal{Q} \xrightarrow{i} \mathcal{X}^2 \xrightarrow{\pi_{\mathcal{X}}} |\mathcal{X}^2| = Z^2 \xrightarrow{\pi_G} Z^2/G,$$

where $i : \mathcal{Q} \to \mathcal{X}^2$ is defined by inclusion of the slices $\tilde{\mathcal{U}}_x \hookrightarrow \mathcal{X}^2$. We claim β induces a homeomorphism $|\beta| : |\mathcal{Q}| \to \widehat{Z}/G$. First, for $x, y \in \mathcal{Q}$, the existence of $(x, y, g, \Gamma) \in \boldsymbol{Q}$ is equivalent to that $|x|$ and $|y|$ are in the same G-orbit in Z. As a consequence $|\beta|$ is well-defined and is a bijection. Since β is continuous, $|\beta|$ is also continuous. Therefore to show that $|\beta|$ is homeomorphism, it suffices to show that $|\beta|$ is an open map. Let U be an open set of $|\mathcal{Q}|$. Let $\pi_{\mathcal{Q}} := \mathcal{Q} \to |\mathcal{Q}|$, $\eta : \coprod_{x_0 \in \mathcal{X}_\infty} \mathcal{U}_{x_0} \to \mathcal{Q}$ defined by $\eta|_{\mathcal{U}_{x_0}} = \eta_{x_0}$ and $\iota : \coprod_{x_0 \in \mathcal{X}_\infty} \mathcal{U}_{x_0} \to \mathcal{X}^2$ defined by the inclusions $\mathcal{U}_{x_0} \hookrightarrow \mathcal{X}^2$. Then

$$|\beta|(U) = \pi_G \circ \pi_{\mathcal{Q}} \circ i(\pi_{\mathcal{Q}}^{-1}(U)) = \pi_G(\rho(G, \pi_{\mathcal{X}} \circ \iota(\eta^{-1}(\pi_{\mathcal{Q}}^{-1}(U))))).$$

Since ι and $\pi_{\mathcal{X}}$ are open maps by [65, Proposition 7.6]. Then $\pi_{\mathcal{X}} \circ \iota(\eta^{-1}(\pi_{\mathcal{Q}}^{-1}(U)))$ is open, therefore $\rho(G, \pi_{\mathcal{X}} \circ \iota(\eta^{-1}(\pi_{\mathcal{Q}}^{-1}(U))))$ is a G-invariant open set of Z^2. Hence $|\beta|(U)$ is open. □

Claim 4.47. *There is sc-smooth polyfold map $\mathfrak{q}$ covering the topological quotient $\pi_G : \widehat{Z} \to \widehat{Z}/G$.*

Proof. Let $(\widehat{\mathcal{X}}, \widehat{\boldsymbol{X}})$ be the full subcategory $(\mathcal{X}^2, \boldsymbol{X}^2)$ with orbit space $\widehat{Z}$. Then $(\widehat{\mathcal{X}}, \widehat{\boldsymbol{X}})$ gives $\widehat{Z}$ a polyfold structure. But we will construct another equivalent polyfold structure on $\widehat{Z}$ to write down the quotient map. Let $z \in \widehat{\mathcal{X}}$, then there is an open neighborhood $\mathcal{V}_z \subset \widehat{\mathcal{X}}$ with the following structures.

1. There exists $u \in \mathcal{X}_\infty$, $v \in \tilde{\mathcal{U}}_u$ and $g \in G$ such that $\rho(g, |z|) = |v|$.
2. There is $\Gamma \in L(z, v, g)$ with $\Gamma(g, z) = v$ and

$$\Gamma(f_u(\Gamma(g, x))g, x) = \Gamma_u(f_u(\Gamma(g, x), \Gamma(g, x)) = \eta_u(\Gamma(g, x)) \in \tilde{\mathcal{U}}_u$$

   is defined for $x \in \mathcal{V}_z$.

For each $z \in \mathcal{X}_\infty$, we pick a $\mathcal{V}_z$ along with the corresponding u, v, g and Γ. Then we define an ep-groupoid $(\mathcal{T}, \boldsymbol{T})$ as follows:

$$\mathcal{T} := \coprod \mathcal{V}_z, \quad \boldsymbol{T} := \{(x, \phi, y) | x \in \mathcal{V}_z, y \in \mathcal{V}_w, \phi \in \boldsymbol{X}, \phi(x) = y\}$$

The structures maps are:

$$\begin{array}{rrcl}
s: & (x,\phi,y) & \mapsto & x \\
t: & (x,\phi,y) & \mapsto & y \\
m: & ((y,\psi,z),(x,\phi,y)) & \mapsto & (x,\psi\circ\phi,z) \\
u: & (x,\phi,y) & \mapsto & (y,\phi^{-1},x) \\
i: & x & \mapsto & (x,\mathrm{id},x).
\end{array}$$

The natural functor $\mathcal{T} \to \widehat{\mathcal{X}}$ by sending $x \in \mathcal{V}_z$ to $x \in \widehat{\mathcal{X}}$ and (x, ϕ, y) to ϕ is an equivalence. Hence $(\mathcal{T}, \boldsymbol{T})$ is also a polyfold structure for $\widehat{Z}$.

The record of u, v, g and Γ for each $\mathcal{V}_z$ is then used to construct a functor from $(\mathcal{T}, \boldsymbol{T})$ to $(\mathcal{Q}, \boldsymbol{Q})$. We claim there is a functor $Q : \mathcal{T} \to \mathcal{Q}$ covering the topological quotient map $\pi_G : \widehat{Z} \to \widehat{Z}/G$ as follows. For $x \in \mathcal{V}_z$, we define $\rho_x := f_u(\Gamma(g, x))g$ and Λ_x is the restriction of Γ in $L(x, \Gamma(\rho_x, x), \rho_x)$. Let $(x, \phi, y) \in \boldsymbol{T}$ be a morphism. Then we define

$$Q : \begin{array}{c} x \mapsto \Gamma_u(\rho_x, x) \in \tilde{\mathcal{U}}_w; \\ (x, \phi, y) \mapsto (Q(x), Q(y), \rho_y \rho_x^{-1}, \Lambda_y \circ L_\phi \circ \Lambda_x^{-1}). \end{array} \tag{4.17}$$

It is a functor by the definition of our structure maps and is sc$^\infty$ as every component in (4.17) are sc$^\infty$. □

□

Remark 4.48. *If for every $x_0 \in \mathcal{X}_\infty$, we pick two different G-slices around x_0 to construct two quotient ep-groupoids $(\mathcal{Q}_a, \boldsymbol{Q}_a), (\mathcal{Q}_b, \boldsymbol{Q}_b)$ and give $\widehat{Z}_a/G, \widehat{Z}_b/G$ polyfold structures. Then we can take union of two sets of slices to form an ep-groupoid $(\mathcal{Q}_{ab}, \boldsymbol{Q}_{ab})$, which gives $(\widehat{Z}_a \cup \widehat{Z}_b)/G$ a polyfold structure. The natural inclusions $(\mathcal{Q}_a, \boldsymbol{Q}_a) \to (\mathcal{Q}_{ab}, \boldsymbol{Q}_{ab})$ and $(\mathcal{Q}_b, \boldsymbol{Q}_b) \to (\mathcal{Q}_{ab}, \boldsymbol{Q}_{ab})$ induce the open inclusions of polyfolds $\widehat{Z}_a/G \to (\widehat{Z}_a \cup \widehat{Z}_b)/G$ and $\widehat{Z}_b/G \to (\widehat{Z}_a \cup \widehat{Z}_b)/G$. As a consequence, $\widehat{Z}_a/G$ and $\widehat{Z}_b/G$ restrict to the same polyfold on $(\widehat{Z}_a \cap \widehat{Z}_b)/G$.*

Proposition 4.49. *Let Z be a regular tame polyfold and $(\mathcal{X}_a, \boldsymbol{X}_a), (\mathcal{X}_b, \boldsymbol{X}_b)$ two polyfold structures. Assume compact Lie group G acts on Z sc-smoothly and the action only has finite isotropy. Let $\widehat{Z}_a, \widehat{Z}_b \subset Z^2$ be two open G-invariant open sets containing Z_∞ such that $\mathbb{Z}_a/G, \mathbb{Z}_b/G$ are the quotient polyfolds constructed from $(\mathcal{X}_a, \boldsymbol{X}_a)$ resp. $(\mathcal{X}_b, \boldsymbol{X}_b)$ from Theorem 4.69. Then there exists a G-invariant open set $\widehat{Z} \subset \mathbb{Z}_a \cap \mathbb{Z}_b$ containing Z_∞, such that $\mathbb{Z}_a/G, \mathbb{Z}_b/G$ restricted to $\widehat{Z}/G$ are the same polyfold.*

Proof. Since $(\mathcal{X}_a, \boldsymbol{X}_a), (\mathcal{X}_b, \boldsymbol{X}_b)$ are equivalent polyfold structures, we have equivalences $(\mathcal{X}_a, \boldsymbol{X}_a) \xleftarrow{F} (\mathcal{W}, \boldsymbol{W}) \xrightarrow{G} (\mathcal{X}_b, \boldsymbol{X}_b)$. We apply Theorem 4.42 to $(\mathcal{W}, \boldsymbol{W})$, i.e. we pick G-slices $(\mathcal{U}_{x_0}, \tilde{\mathcal{U}}_{x_0}, V_{x_0}, f_{x_0}, \eta_{x_0})$ for $(\mathcal{W}^2, \boldsymbol{W}^2)$ around $x_0 \in \mathcal{W}_\infty$ to construct a quotient ep-groupoid $(\mathcal{Q}^{\mathcal{W}}, \boldsymbol{Q}^{\mathcal{W}})$ and gives $\widehat{Z}_{\mathcal{W}}/G$ a polyfold structure. We can also require that $F|_{\mathcal{U}_{x_0}}, G|_{\mathcal{U}_{x_0}}$ are sc-diffeomorphisms. Then $(F(\mathcal{U}_{x_0}), F(\tilde{\mathcal{U}}_{x_0}), V_{x_0}, F_* f_{x_0}, F_* \eta_{x_0})$ and $(G(\mathcal{U}_{x_0}), G(\tilde{\mathcal{U}}_{x_0}), V_{x_0}, G_* f_{x_0}, G_* \eta_{x_0})$ are G-slices for $(\mathcal{X}_a^2, \boldsymbol{X}_a^2)$ and $(\mathcal{X}_b^2, \boldsymbol{X}_b^2)$. By Remark 4.48, $(F(\mathcal{U}_{x_0}), F(\tilde{\mathcal{U}}_{x_0}), V_{x_0}, F_* f_{x_0}, F_* \eta_{x_0})$ and $\widehat{Z}_a/G$ induce the equivalent polyfold structures on $(\widehat{Z}_{\mathcal{W}} \cap \widehat{Z}_a)/G$, $(G(\mathcal{U}_{x_0}), G(\tilde{\mathcal{U}}_{x_0}), V_{x_0}, G_* f_{x_0}, G_* \eta_{x_0})$ and $\widehat{Z}_b/G$ induce the equivalent polyfold structures on $(\widehat{Z}_{\mathcal{W}} \cap \widehat{Z}_b)/G$. Therefore $\widehat{Z}_a/G, \widehat{Z}_b/G$ restricted to $(\widehat{Z}_{\mathcal{W}} \cap \widehat{Z}_a \cap \widehat{Z}_b)/G$ are the same polyfold. □

Remark 4.50. *The challenge of constructing a quotient ep-groupoid is the construction of the morphism space. Our approach in Theorem 4.42 is geometric in the sense that two morphisms are different iff they are represented by different geometric data (x, y, g, Γ). The regular property of polyfold is to control local lifts $\Gamma \in L(x, y, g)$. It is not clear to us how to* *construct a quotient ep-groupoid* $(\mathcal{Q}, \boldsymbol{Q})$ *with the isotorpy* stab^Q_x *satisfying an exact sequence* $1 \to \mathrm{stab}_x \to \mathrm{stab}^Q_x \to G_x \to 1$.

Remark 4.51. *When G is the trivial group, the construction in Theorem 4.42 gives an effective polyfold structure on $\widehat{Z} \subset Z^2$. One can still apply the construction in Theorem 4.42 to Example 4.19, then the new morphism space contains two open half planes and one half-line with the attaching point is a double point, thus not Hausdorff. This illustrates the role of the first property of regularity*[6].

The following example shows the importance of the second property in the definition of regularity (Definition 4.11).

Example 4.52. *If we apply the construction in Theorem 4.42 to Example 4.20 with the trivial group action. Since the sc-diffeomorphism $\eta : \mathrm{im}\,\pi \to \mathrm{im}\,\pi$ in Example 4.20 can be completed to an equivalence between $\mathbb{Z}_2 \ltimes \mathrm{im}\,\pi$ and itself. Hence η induces a local lift of the trivial action near $(0,0)$ and $\eta \circ \eta = \mathrm{id}$. Moreover $\eta \neq L_\phi$, where ϕ is the nontrivial element of $\mathbb{Z}_2 = \mathrm{stab}_{(0,0)}$. The isotropy group $\mathrm{stab}^Q_{(0,0)}$ for the quotient construction is $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by η and L_ϕ.Therefore (4.8) does not holds anymore.*

In this example, one can still carry out the remaining construction. If we use $\mathrm{im}\,\pi$ as the only slice for the construction, then the resulted morphism space $\boldsymbol{Q}$ is two copies of $\mathrm{im}\,\pi$ with two extra points p, q corresponding to η and $\eta \circ L_\phi$, such that p connects the negative side of one copy with the positive side of the other copy and q connects the remaining two sides. As a result, $\boldsymbol{Q}$ is no longer Hausdorff due to p and q. We see from this example that the second property of regularity is crucial for the control of the isotropy of the quotient. It is not clear to us where there exists an example with $|\,\mathrm{stab}^Q_x\,| = \infty$.

Remark 4.53. *If one wants to take another quotient on top of a quotient, we need to study whether the quotient is regular. The situation depends on the external group action. For example, if we think of the action groupoids in Example 4.19 and Example 4.20 as quotients of $\mathbb{Z}_2$ actions, then the quotient polyfolds are not regular. In light of Corollary 4.18, we give a sufficient condition for the quotient to be regular. Following the discussion in (4.7), $\mathrm{D}\Gamma(g_0, \cdot)_{x_0}$ preserve the infinitesimal direction $\mathfrak{g}_{x_0}$ for $g_0 \in G_{x_0}$ and $\Gamma \in L(x_0, x_0, g_0)$. Then for every $x_0 \in \mathcal{X}_\infty$, we can define the map $\mathrm{stab}^Q_{x_0} \to \mathrm{hom}(T_{x_0}\mathcal{X}/\mathfrak{g}_{x_0}, T_{x_0}\mathcal{X}/\mathfrak{g}_{x_0})$ by $\Gamma \in L(x_0, x_0, g_0) \mapsto [\mathrm{D}\Gamma(g_0, \cdot)_{x_0}]$. If such map is injective for every $x_0 \in \mathcal{X}_\infty$, then the quotient polyfold is regular.*

4.5 Quotients of polyfold bundles and sections

In this section, we prove Theorem 1.1 up to the claims on good position and orientation The proof is analogous to the proof of Theorem 4.42. The construction of quotient polyfold

[6] However, to get an effective ep-groupoid out of a general ep-groupoid, one only needs the first condition in the regular property (Definition 4.11), see the proof of [65, Proposition 7.9].

bundles is analogous to the construction of quotient polyfolds in Theorem 4.42, once we set up the counterpart of Proposition 4.39 in the bundle case.

Regular strong polyfold bundles and sc-Fredholm sections

We first recall the basics of polyfold bundles and sc-Fredholm sections that will be used here. Like the polyfold case, we review the bundles over ep-groupoids first.

Definition 4.54 ([65, Definition 8.4]). *A sc-smooth functor $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ between two tame ep-groupoids is a* ***tame strong ep-groupoid bundle****, if the following properties hold.*

1. *On the object space $P^0 : \mathcal{E} \to \mathcal{X}$ is a tame strong M-polyfold bundle.*

2. *The morphism space $\boldsymbol{E}$ is the fiber product $\boldsymbol{X}_s \times_{P^0} \mathcal{E}$. And $P^1 : \boldsymbol{E} \to \boldsymbol{X}$ is the projection to the $\boldsymbol{X}$ component.*

3. *The structure maps on $\boldsymbol{E}$ are given by*

$$\begin{aligned} s(g,e) &= e; \\ t(g,e) &= \mu(g,e); \\ m((g,e),(h,a)) &= (g \circ h, a); \\ u(e) &= (1,e); \\ i(g,e) &= (i(g), \mu(g,e)). \end{aligned}$$

4. *The target map μ has following properties:*
   - $t_{\mathcal{X}} \circ P^1 = P^0 \circ \mu$, *i.e. μ is a bundle map $\boldsymbol{E} \to \mathcal{E}$ covering the target map $t_{\mathcal{X}} : \boldsymbol{X} \to \mathcal{X}$;*
   - *μ is surjective local sc-diffeomorphism and linear from fiber $\boldsymbol{E}_\phi$ to $\mathcal{E}_{t_{\mathcal{X}}(\phi)}$, for $\phi \in \boldsymbol{X}$;*
   - *$\mu(1_x, e) = e$, for $e \in \mathcal{E}_x$;*
   - *$\mu(g \circ h, e) = \mu(g, \mu(h, e))$.*

Note that for $\phi \in \boldsymbol{X}$, there exist open neighborhoods $\mathcal{U}_{s_{\mathcal{X}}(\phi)}, \mathcal{U}_{t_{\mathcal{X}}(\phi)} \subset \mathcal{X}$ and $\boldsymbol{U}_\phi \subset \boldsymbol{X}$ of $s_{\mathcal{X}}(\phi), t_{\mathcal{X}}(\phi)$ and ϕ, such that $t_{\mathcal{X}} : \boldsymbol{U}_\phi \to \mathcal{U}_{t_{\mathcal{X}}(\phi)}$ and $s_{\mathcal{X}} : \boldsymbol{U}_\phi \to \mathcal{U}_{s_{\mathcal{X}}(\phi)}$ are sc-diffeomorphisms. Then $t : (P^1)^{-1}(\boldsymbol{U}_\phi) \to (P^0)^{-1}(\mathcal{U}_{t_{\mathcal{X}}(\phi)})$ and $s : (P^1)^{-1}(\boldsymbol{U}_\phi) \to (P^0)^{-1}(\mathcal{U}_{s_{\mathcal{X}}(\phi)})$ are both strong bundle isomorphisms. In particular, we can define the following strong bundle isomorphism covering L_ϕ (Definition 4.2),

$$R_\phi := t \circ s^{-1} : (P^0)^{-1}(\mathcal{U}_{s(\phi)}) \to (P^0)^{-1}(\mathcal{U}_{t(\phi)}).$$

R_ϕ is a special case of local action $L_{(\phi,0)}$ on the ep-groupoid $(\mathcal{E}, \boldsymbol{E})$ for $(\phi, 0) \in \boldsymbol{E}$.

To introduce the definition of regular strong bundle, we first define a local uniformizer for a strong polyfold bundle.

Proposition 4.55. *Let $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ be a tame strong ep-groupoid bundle and $x \in \mathcal{X}$. Assume $\phi_x : \operatorname{stab}_x \ltimes \mathcal{U} \to (\mathcal{X}, \boldsymbol{X})$ is a local uniformizer of $(\mathcal{X}, \boldsymbol{X})$ around x. Then there exists a diagram of sc-smooth functors*

$$\begin{tikzcd}
\operatorname{stab}_x \ltimes (P^0)^{-1}(\mathcal{U}) \arrow[r, "\Psi_x"] \arrow[d, "P"] & (\mathcal{E}, \boldsymbol{E}) \arrow[d, "P"] \\
\operatorname{stab}_x \ltimes \mathcal{U} \arrow[r, "\psi_x"] & (\mathcal{X}, \boldsymbol{X}),
\end{tikzcd}$$

such that the following holds.

1. stab_x *acts on* $(P^0)^{-1}(\mathcal{U})$ *by bundle isomorphisms.*
2. Ψ_x *is fully faithful and on the object level* $\Psi_x^0 : (P^0)^{-1}(\mathcal{U}) \to \mathcal{E}$ *is the inclusion.*
3. *On the orbit space* $|\Psi_x| : (P^0)^{-1}(\mathcal{U}) / \operatorname{stab}_x \to |(P^0)^{-1}(\mathcal{U})| \subset |\mathcal{E}|$ *is a homeomorphism.*

Proof. stab_x acts on $(P^0)^{-1}(\mathcal{U})$ by $\phi \in \operatorname{stab}_x \mapsto R_\phi$. We write $\psi_x = (\mathrm{id}_{\mathcal{U}}, \Sigma) : \operatorname{stab}_x \ltimes \mathcal{U} \to (\mathcal{X}, \boldsymbol{X})$. Then we can define $\Psi_x : \operatorname{stab}_x \ltimes (P^0)^{-1}(\mathcal{U}) \to (\mathcal{E}, \boldsymbol{E})$ by

$$\Psi_x^0 : (P^0)^{-1}(\mathcal{U}) \hookrightarrow \mathcal{E},$$

$$\Psi_x^1 : \operatorname{stab}_x \times (P^0)^{-1}(\mathcal{U}) \to \boldsymbol{E}, \quad (\phi, e) \mapsto (\Sigma(\phi, P^0(e)), e) \in \boldsymbol{E} = \boldsymbol{X}_s \times_{P^0} \mathcal{E}.$$

Then Ψ is fully faithful and on object level $|\Psi_x| : (P^0)^{-1}(\mathcal{U}) / \operatorname{stab}_x \to |(P^0)^{-1}(\mathcal{U})| \subset |\mathcal{E}|$ is a homeomorphism. □

From the proof above, we see that $(P^0)^{-1}(\mathcal{U})$ is a local uniformizer of $(\mathcal{E}, \boldsymbol{E})$ around $(x, 0) \in \mathcal{E}$ and is also a strong bundle over $\mathcal{U}$. We will call $\Psi_x : \operatorname{stab}_x \ltimes (P^0)^{-1}(\mathcal{U}) \to (\mathcal{E}, \boldsymbol{E})$ a **local uniformizer for tame strong ep-groupoid bundle $\boldsymbol{P}$ around $\boldsymbol{x}$**. When there is no ambiguity, we will call $(P^0)^{-1}(\mathcal{U})$ a local uniformizer of P around x. Let $\operatorname{Diff}^\pi_{\mathrm{sc}}(x)$ denotes the group of germs of bundle isomorphism fixing x. Then the action in Proposition 4.55 defines a homomorphism $\operatorname{stab}_x \to \operatorname{Diff}^\pi_{\mathrm{sc}}(x)$. We define the **effective part**

$$\operatorname{stab}_x^{\pi,\mathrm{eff}} := \operatorname{stab}_x / \ker(\operatorname{stab}_x \to \operatorname{Diff}^\pi_{\mathrm{sc}}(x)).$$

Note that $\operatorname{stab}_x^{\pi,\mathrm{eff}}$ might be different from $\operatorname{stab}_x^{\mathrm{eff}}$ defined in (4.2) for the base.

Definition 4.56. *Let $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ be a strong ep-groupoid bundle. For $x \in \mathcal{X}$, a local uniformizer $\operatorname{stab}_x \ltimes P^{-1}(\mathcal{U})$ around x is* ***regular*** *if the following two conditions are met:*

1. *if $R_\phi|_{P^{-1}(\mathcal{V})} = \mathrm{id}_{P^{-1}(\mathcal{V})}$ for some open subset $\mathcal{V} \subset \mathcal{U}$, then $R_\phi = \mathrm{id}$ on $P^{-1}(\mathcal{U})$;*
2. *for every path connected uniformizer $\mathcal{W} \subset \mathcal{U}$ of $\mathcal{X}$ around x, if $\Phi : P^{-1}(\mathcal{W}) \to \operatorname{stab}_x$ is a a map such that $R_{\Phi(w)}(w) : P^{-1}(\mathcal{W}) \to P^{-1}(\mathcal{W})$ is sc-smooth, then there exists $\phi \in \operatorname{stab}_x$, such that $R_{\Phi(w)}(w) = R_\phi(w)$ for all $w \in P^{-1}(\mathcal{W})$.*

Definition 4.57. *A strong ep-groupoid bundle $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ is **regular**, if for any point $x \in \mathcal{X}$, there exists a regular local uniformizer of P around x.*

Note that the a strong ep-groupoid bundle being regular is different from the underlying ep-groupoid being regular. It seems that being regular as a bundle is weaker than being regular as an ep-groupoid, since a regular bundle uniformizer around x is a regular uniformizer around $(x, 0)$. The regularity of a bundle maybe only imply the regularity around the zero section.

The counterparts of Proposition 4.16, Proposition 4.17 and Corollary 4.18 hold for strong ep-groupoid bundles for the same reason. That is we have the following.

Proposition 4.58. *Let $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ be a strong ep-groupoid bundle such that $\mathrm{stab}_x^{\pi,\mathrm{eff}} = \mathrm{stab}_x$ for all $x \in \mathcal{X}$. Assume for every $x \in \mathcal{X}$, there exists a local uniformizer $\mathcal{U}$ around x, such that for any connected uniformizer $\mathcal{V} \subset \mathcal{U}$ around x, we have $P^{-1}(\mathcal{V}) \backslash \cup_{\phi \neq \mathrm{id} \in \mathrm{stab}_x} \mathrm{Fix}(\phi)$ is connected, where $\mathrm{Fix}(\phi)$ is the fixed set of R_ϕ. Then P is regular.*

Proposition 4.59. *If the strong ep-groupoid bundle $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ has the property that for any $x \in \mathcal{X}_\infty$ and $\phi \in \mathrm{stab}_x$ if $\mathrm{D}R_\phi : T_{(x,0)}\mathcal{E} \to T_{(x,0)}\mathcal{E}$ is the identity map, then $R_\phi = \mathrm{id}$ locally. Then P is regular.*

We point out that the regularity of a bundle is not related to the regularity of the base. For example we can think of the ep-groupoid $\mathbb{Z}_2 \ltimes \mathcal{M}$ in Example 4.19 as an ep-groupoid bundle over the trivial action groupoid $\mathbb{Z}_2 \ltimes \mathbb{R}$. Then $\mathbb{Z}_2 \ltimes \mathcal{M}$ is not regular bundle, but the base is regular. On the other hand, $\mathbb{Z}_2 \ltimes (\mathcal{M} \times \mathbb{R})$ is an ep-groupoid bundle over $\mathbb{Z}_2 \ltimes \mathcal{M}$, where $\mathbb{Z}_2$ acts on the $\mathbb{R}$ coordinate by multiplying -1. Then $\mathbb{Z}_2 \ltimes (\mathcal{M} \times \mathbb{R})$ is a regular bundle over the irregular ep-groupoid $\mathbb{Z}_2 \times \mathcal{M}$.

A strong bundle functor $F : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{F}, \boldsymbol{F})$ between two strong ep-groupoid bundles $(\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ and $(\mathcal{F}, \boldsymbol{F}) \to (\mathcal{Y}, \boldsymbol{Y})$ is a sc-smooth functor, such that $F^0 : \mathcal{E} \to \mathcal{F}$ and $F^1 : \boldsymbol{E} \to \boldsymbol{F}$ are both strong bundle maps. As a consequence, F induces a sc-smooth functor $f : (\mathcal{X}, \boldsymbol{X}) \to (\mathcal{Y}, \boldsymbol{Y})$ on the bases.

Definition 4.60 ([65, Definition 10.10])**.** *A **strong bundle equivalence***

$$F : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{F}, \boldsymbol{F})$$

is a strong bundle functor, moreover the following two conditions hold.

1. *$F[i] : (\mathcal{E}[i], \boldsymbol{E}[i]) \to (\mathcal{F}[i], \boldsymbol{F}[i])$ are equivalences between ep-groupoids for $i = 0, 1$, covering an equivalence $f : (\mathcal{X}, \boldsymbol{X}) \to (\mathcal{Y}, \boldsymbol{Y})$.*

2. *F^0, F^1 are local strong bundle isomorphisms.*

Just like the ep-groupoid situation, one can define generalized strong bundle maps between strong ep-groupoid bundles [65, Section 10.5], i.e. an equivalence of diagram of strong bundle functors $\mathcal{E} \xleftarrow{F} \mathcal{H} \xrightarrow{\Phi} \mathcal{F}$, where F is a strong bundle equivalence. Then one can define the category of strong ep-groupoid bundles $\mathscr{SEP}(\boldsymbol{F}^{-1})$ [65, Definition 10.4], where $\boldsymbol{F}$ is the class of strong bundle equivalences. The objects of $\mathscr{SEP}(\boldsymbol{F}^{-1})$ are strong ep-groupoid bundles and the morphisms of $\mathscr{SEP}(\boldsymbol{F}^{-1})$ are generalized strong bundle maps. Sections of strong ep-groupoids can be pulled back through generalized strong bundle maps and can be pushed forward if the generalized bundle map is an isomorphism [65, Theorem 10.9].

Definition 4.61 ([65, Definition 8.7])**.** *A **sc-Fredholm section** of a strong bundle $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ is a sc-smooth functor $S : (\mathcal{X}, \boldsymbol{X}) \to (\mathcal{E}, \boldsymbol{E})$, such that $P \circ S = \mathrm{id}_{(\mathcal{X}, \boldsymbol{X})}$ and $S^0 : \mathcal{X} \to \mathcal{E}$ is a sc-Fredholm section of the strong M-polyfold bundle $P^0 : \mathcal{E} \to \mathcal{X}$. The section is called proper if $|(S^{-1}(0)| \subset |\mathcal{X}|$ is compact in the $|\mathcal{X}_0|$ topology*[7].

Definition 4.62 ([65, Definition 16.16])**.** *Let $p : W \to Z$ be a surjection between paracompact Hausdorff spaces, a strong polyfold bundle structure $((\mathcal{E}, \boldsymbol{E}) \xrightarrow{P} (\mathcal{X}, \boldsymbol{X}), \Gamma, \gamma)$ is the following structure:*

1. *$P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ is a strong ep-groupoid bundle;*
2. *$\Gamma : |\mathcal{E}| \to W$ and $\gamma : |\mathcal{X}| \to Z$ are homeomorphisms, such that $\pi \circ \Gamma = \gamma \circ |P|$.*

The equivalence between strong polyfold bundle structures are defined similarly as the equivalence between polyfold structures, see [65, Definition 16.17]. A strong polyfold bundle $p : W \to Z$ a surjection between paracompact Hausdorff spaces with an equivalence class of strong polyfold bundle structures, see [65, Definition 16.18]. Then one can define the category of strong polyfold bundles, see [65, Section 16.3] for more details. Let $\mathscr{SPB}$ denotes the category of strong polyfold bundles [65, Definition 16.18, 16.22]. The sections of strong polyfold bundle [65, Definition 16.27] are represented by sections of a strong polyfold bundle structure up to the equivalences induced by the pullbacks of strong polyfold bundle structure equivalences. A section is sc-Fredholm [65, Definition 16.40] if one of the representative sections (hence all) is sc-Fredholm. By [65, Proposition 10.7], a section of a strong polyfold bundle has a unique representative for a strong polyfold bundle structure. To simplify notations, we will **abbreviate a strong polyfold bundle structure $((\mathcal{E}, \boldsymbol{E}) \xrightarrow{P} (\mathcal{X}, \boldsymbol{X}), \Gamma, \gamma)$ to $(\mathcal{E}, \boldsymbol{E}) \xrightarrow{P} (\mathcal{X}, \boldsymbol{X})$** in the remaining part of this paper. That is $Z = |\mathcal{X}|$ and $W = |\mathcal{E}|$. And a strong polyfold bundle map is represented by an equivalence class of generalized strong bundle maps.

Group action on strong polyfold bundles

Definition 4.63. *A **sc$^\infty$ G-action** on a strong polyfold bundle $p : W \to Z$ is a pair $(\rho, \mathfrak{P})$ such that following holds.*

[7] As a consequence, $|S^{-1}(0)|$ is compact in all $|\mathcal{X}_i|$ topology and $|\mathcal{X}_\infty|$ topology, by the same proof as [65, Theorem 5.3]

1. *$\rho : G \times W \to W$ is a continuous group action.*

2. *$\mathfrak{P} : (\star, G) \to \mathscr{SPB}$ is a functor sending $\star$ to W, such that $\rho(g, w) = |\mathfrak{P}(g)|(w)$.*

3. *Let $g \in G$ and $p, q \in Z$ such that $\rho(g, (p, 0)) = (q, 0)$. Then there exist a neighborhood U of g, two equivalent polyfold bundle structures $(\mathcal{E}, \boldsymbol{E}) \xrightarrow{P} (\mathcal{X}, \boldsymbol{X}), (\mathcal{F}, \boldsymbol{F}) \xrightarrow{Q} (\mathcal{Y}, \boldsymbol{Y})$, two points $x \in \mathcal{X}, y \in \mathcal{Y}$ with $|x| = p, |y| = q$ and a local uniformizer $\mathcal{U}$ around x. So that the U-family of polyfold bundle maps $U \to \mathrm{Mor}_{\mathscr{SPB}}(|P^{-1}(\mathcal{U})|, W)$ defined by $h \mapsto \mathfrak{P}(h)$ is represented by the following sc-smooth bundle map*
$$\rho_{g,x} : U \times (\mathrm{stab}_x \ltimes P^{-1}(\mathcal{U})) \to (\mathcal{F}, \boldsymbol{F}),$$
satisfying $\rho_{g,x}(g, (x, 0)) = (y, 0)$.

*Assume G acts on $p : W \to Z$ by $(\rho, \mathfrak{P})$. A section $s : Z \to W$ is **G-equivariant** iff it is invariant under the pullback of strong bundle isomorphism $\mathfrak{P}(g)$ for all $g \in G$.*

It is clear that a group action $(\rho, \mathfrak{P})$ on $p : W \to Z$ induces a group action $(\rho_Z, \mathfrak{P}_Z)$ on the base Z in the sense of Definition 4.27.

Remark 4.64. *By an analogue of Proposition A.4, $\rho_{g,x}(h, \cdot)$ is a local strong bundle isomorphism from $P^{-1}(\mathcal{U})$.*

We can define local lifts for an action on a strong polyfold bundle like before. Let $(\rho, \mathfrak{P})$ acts on $p : W \to Z$ and $(\mathcal{E}, \boldsymbol{E}) \xrightarrow{P} (\mathcal{X}, \boldsymbol{X})$ a strong bundle structure on p. Let $x, y \in \mathcal{X}$ and assume $\rho_Z(g, |x|) = |y|$ in $Z = |\mathcal{X}|$. Then by assumption, the action can be locally represented by
$$\rho_{g,x_a} : U \times (\mathrm{stab}_{x_a} \ltimes P_a^{-1}(\mathcal{U})) \to (\mathcal{E}_b, \boldsymbol{E}_b), \tag{4.18}$$
where $(\mathcal{E}_a, \boldsymbol{E}_a) \xrightarrow{P_a} (\mathcal{X}_a, \boldsymbol{X}_a)$, $(\mathcal{E}_b, \boldsymbol{E}_b) \xrightarrow{P_b} (\mathcal{X}_b, \boldsymbol{X}_b)$ are two strong polyfold bundle stronges and $\mathrm{stab}_{x_a} \ltimes P_a^{-1}(\mathcal{U}_a)$ is a local uniformizer of P_a with $|x_a| = |x| \in Z$. Then we have the following sequence of equivalences of ep-groupoids bundles, note that we suppress the morphism spaces since there are no ambiguities,
$$\begin{tikzcd} & & \mathcal{E}_a \arrow[r, "\rho_{g,x_a}(g,\cdot)"] & \mathcal{E}_b & \mathcal{H}_b \arrow[l, "F_b"'] \arrow[r, "G_b"] & \mathcal{E} \\ \mathcal{E} & \mathcal{H}_a \arrow[l, "F_a"'] \arrow[r, "G_a"] & P_a^{-1}(\mathcal{U}_a) \arrow[u, "\subset"'] \end{tikzcd} \tag{4.19}$$
where F_a, G_a, F_b, G_b are all strong bundle equivalences and $|G_b| \circ |F_b|^{-1} \circ |\rho_{g,x_a}(g, \cdot)| \circ |G_a| \circ |F_a|^{-1} = \rho(g, \cdot)$, i.e. over the orbit space $|P_a^{-1}(\mathcal{U})|$ the composition of the sequence covers the action $\rho(g, \cdot)$. Let f_a, g_a, f_b, g_b be the induced equivalences on the bases of F_a, G_a, F_b, G_b.

There exist $w_a \in \mathcal{W}_a, w_b \in \mathcal{W}_b, \phi, \psi \in \boldsymbol{X}, \delta \in \boldsymbol{X}_a, \eta \in \boldsymbol{X}_b$, such that $f_a(w_a) = \phi(x)$, $\delta(g_a(w_a)) = x_a$, $\eta(\rho_{g,x_a}(g, x_a)) = f_b(w_b)$ and $g_b(w_b) = \psi(y)$, i.e.

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "P"'] & \mathcal{H}_a \arrow[l, "F_a"'] \arrow[r, "G_a"] \arrow[d, "Q_a"'] & \mathcal{E}_a \arrow[r, "\rho_{g,x_a}(g,\cdot)"] \arrow[d, "P_a"'] & \mathcal{E}_b \arrow[d, "P_b"] & \mathcal{H}_b \arrow[l, "F_b"'] \arrow[r, "G_b"] \arrow[d, "Q_b"] & \mathcal{E} \arrow[d, "P"] \\
\mathcal{X} & \mathcal{W}_a \arrow[l, "f_a"'] \arrow[r, "g_a"] & \mathcal{X}_a \arrow[r, "\rho_{Z,g,x_a}(g,\cdot)"] & \mathcal{X}_b & \mathcal{W}_b \arrow[l, "f_b"'] \arrow[r, "g_b"] & \mathcal{X}
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[d, "\phi"'] & & x_a \arrow[r] & \rho_{Z,g,x_a}(g, x_a) \arrow[d, "\eta"'] & & y \arrow[d, "\psi"'] \\
f_a(w_a) & w_a \arrow[l] \arrow[r] & g_a(w_a) \arrow[u, "\delta"] & f_b(w_b) & w_b \arrow[l] \arrow[r] & g_b(w_b)
\end{tikzcd}$$

Here ρ_{g,x_a} is only partially defined on $\mathcal{E}_a$. Let F_{a,w_a} and F_{b,w_b} denote the local bundle diffeomorphisms F_a and F_b near w_a in $\mathcal{W}_a$ and w_b in $\mathcal{W}_b$ respectively. Then by Remark 4.64, there exists a open neighborhood $\mathcal{V} \subset \mathcal{X}$ of x, such that for h close to g,

$$\Lambda(h, w) := R_\psi^{-1} \circ G_b \circ {F_{b,w_b}}^{-1} \circ R_\eta \circ \rho_{g,x_a}(h, \cdot) \circ R_\delta \circ G_a \circ {F_{a,w_a}}^{-1} \circ R_\phi(w) \tag{4.20}$$

defines a local bundle isomorphism from $P^{-1}(\mathcal{V})$. We call Λ a **local lift of the bundle action at (g, x, y)**. Moreover, every local lift Λ induces a local lift of the action on the base:

$$\Gamma(h, z) := L_\psi^{-1} \circ g_b \circ {f_{b,w_b}}^{-1} \circ L_\eta \circ \rho_{Z,g,x_a}(h, \cdot) \circ L_\delta \circ g_a \circ {f_{a,w_a}}^{-1} \circ L_\phi(z). \tag{4.21}$$

We will use (Λ, Γ) to denote a local lift of the bundle actions, although Γ is determined by Λ. (Λ, Γ) is not unique, but the only ambiguity comes from isotropy. That is we have the following analogue of Proposition 4.33 with an identical proof. Let $L^\pi(x, y, g)$ denotes set of germs local lifts of the bundle action at (g, x, y). The germ here is in the sense of the base topology, i.e. $\Lambda_1 = \Lambda_2$ as germs iff $\Lambda_1|_{V\times P^{-1}(\mathcal{U})} = \Lambda_2|_{V\times P^{-1}(\mathcal{U})}$ where $V \subset G$ and $\mathcal{U} \subset \mathcal{X}$ are neighborhoods of g and x respectively.

Proposition 4.65. *Let $p : W \to Z$ be a regular strong polyfold bundle with a G-action and $(\mathcal{E}, \boldsymbol{E}) \xrightarrow{P} (\mathcal{X}, \boldsymbol{X})$ a strong polyfold bundle structure. Then for $x, y \in \mathcal{X}$ and $g \in G$, the isotropy* stab_x *acts on $L^\pi(x, y, g)$ by precomposing with the local bundle diffeomorphism R_ϕ for $\phi \in \mathrm{stab}_x$, and this action is transitive.* stab_y *also acts on $L^\pi(x, y, g)$ by post-composing R_ϕ for $\phi \in \mathrm{stab}_y$, this action is also transitive.*

Proof. The claim follows from s similar diagram chasing in the proof of Proposition 4.33 by replacing Lemma A.2 with Lemma A.3. □

We point out here that there exist (Λ_1, Γ_1), (Λ_2, Γ_2) such that $\Lambda_1 \neq \Lambda_2$ but $\Gamma_1 = \Gamma_2$. For example, let $\pi : \mathbb{R} \to \mathrm{pt}$ be a trivial bundle over a point. $\mathbb{Z}_2$ acts on π by multiplying -1.

Then the translation groupoid $Z_2 \ltimes \pi$ is an ep-groupoid bundle over $\mathbb{Z}_2 \ltimes \mathrm{pt}$. Then there are two local lifts $(\Lambda_1, \Gamma_1), (\Lambda_2, \Gamma_2)$ of the action of the trivial group with $\Gamma_1 = \Gamma_2$. The key observation here is that the effective isotropy $\mathrm{stab}_x^{\pi,\mathrm{eff}}$ of the bundle might be larger than the effective isotropy $\mathrm{stab}_x^{\mathrm{eff}}$ of the base.

The local lifts in $L^\pi(x,y,g)$ also have the group properties like Proposition 4.35.

Proposition 4.66. *Let $p : W \to Z$ be a regular strong polyfold bundle with a G action and $(\mathcal{E}, \boldsymbol{E}) \overset{P}{\to} (\mathcal{X}, \boldsymbol{X})$ a strong polyfold bundle structure. Then for $x, y \in \mathcal{X}$ and $g \in G$, $L^\pi(x,y,g)$ has the following structures.*

1. *For every $\phi \in \mathrm{mor}(y,z)$, $\Lambda \mapsto R_\phi \circ \Lambda$ defines a bijection from $L^\pi(x,y,g)$ to $L^\pi(x,z,g)$. And $\Lambda \mapsto \Lambda \circ R_\phi$ defines a bijection from $L^\pi(z,x,g)$ to $L^\pi(y,x,g)$.*

2. *There is a well-defined multiplication $\circ : L^\pi(y,z,g) \times L^\pi(x,y,h) \to L^\pi(x,z,gh)$. Moreover, if $\Lambda_1 \in L^\pi(y,z,g)$ and $\Lambda_2 \in L^\pi(x,y,h)$, then*
$$\Lambda_1 \circ \Lambda_2(\epsilon gh, w) = \Lambda_1(\epsilon g, \Lambda_2(h,w)) = \Lambda_1(g, \Lambda_2(g^{-1}\epsilon gh, w))$$
for ϵ in a neighborhood of $\mathrm{id} \in G$, and $P(w)$ in a neighborhood of x. Therefore the induced local lifts on the base also have the property
$$\Gamma_1 \circ \Gamma_2(\epsilon gh, z) = \Gamma_1(\epsilon g, \Gamma_2(h,z)) = \Gamma_1(g, \Gamma_2(g^{-1}\epsilon gh, z)), \tag{4.22}$$
for ϵ in a neighborhood of $\mathrm{id} \in G$ and z in a neighborhood of x.

3. *There is a unique identity element $\Lambda_{\mathrm{id}} \in L^\pi(x,x,\mathrm{id})$, such that $\Lambda_{\mathrm{id}}(\mathrm{id}, w) = w$. This identity is both left and right identity in the multiplication structure.*

4. *There is an (both right and left) inverse map $L^\pi(x,y,g) \to L^\pi(y,x,g^{-1})$ with respect to the multiplication and identity structures above.*

5. *There exists an open neighborhood $\mathcal{U} \times V$ of (x,g) in $\mathcal{X} \times G$, such that all the local lifts in $L^\pi(x,y,g)$ is defined on $V \times P^{-1}(\mathcal{U})$. Moreover, for any $(x',g') \in \mathcal{U} \times V$ and let $y' := \Gamma_0(g',x')$ for $(\Lambda_0, \Gamma_0) \in L^\pi(x,y,g)$, then every element $\Lambda' \in L^\pi(x',y',g')$ is represented by the restriction of a unique element $\Lambda \in L^\pi(x,y,g)$.*

Proof. The claim follows from an analogues proof of Proposition 4.35 by replacing Proposition 4.33 with Proposition 4.65. □

Note that for every $x \in \mathcal{X}$, $\cup_{g \in G_x} L^\pi(x,x,g)$ forms a group by Proposition 4.66. Let stab_x^{EQ} denote the group $\cup_{g \in G_x} L^\pi(x,x,g)$. It is clear that we have an exact sequence of groups:
$$1 \to \mathrm{stab}_x^{\pi,\mathrm{eff}} \to \mathrm{stab}_x^{EQ} \to G_x \to 1, \tag{4.23}$$
where the first map is the inclusion $\mathrm{stab}_x^{\pi,\mathrm{eff}} \simeq L^\pi(x,x,\mathrm{id}) \hookrightarrow \mathrm{stab}_x^{EQ}$ and the second map is the projection. In particular, stab_x^{EQ} is a finite group with $|\,\mathrm{stab}_x^{EQ}\,| = |\,\mathrm{stab}_x^{\pi,\mathrm{eff}}\,| \cdot |G_x|$. For

$x \in \mathcal{X}_\infty$, let (Λ_0, Γ_0) be the identity element in $L^\pi(x_0, x_0, \mathrm{id})$. Then we define the **infinitesimal directions** to be

$$\mathfrak{g}_{x_0} = \mathrm{D}(\Gamma_0)_{(\mathrm{id},x_0)}(T_gG \times \{0\}) \tag{4.24}$$

By the same argument in Proposition 4.37, we have $\mathfrak{g}_{x_0} \subset (T^R_{x_0}\mathcal{X})_\infty$ and $\dim \mathfrak{g}_{x_0} = \dim G$. As a consequence, we have the existence of G-slice defined below, which is a analogue of Proposition 4.39.

Definition 4.67. *Let $p : W \to Z$ be a regular strong polyfold bundle with a sc-smooth G-action with finite isotropy and $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ a polyfold bundle structure. Let (Λ_0, Γ_0) be the idenity element in $R(x_0, x_0, \mathrm{id})$. A **G-slice for P around $x_0 \in \mathcal{X}_\infty$** is a tuple $(\mathcal{U}, \tilde{\mathcal{U}}, V, f, \eta, N)$ such that the following holds.*

1. *$\mathcal{U} \subset \mathcal{X}$ is an open neighborhood x_0 and $V \subset G$ is an open neighborhood* id.
2. *Λ_0 is defined on $V \times P^{-1}(\mathcal{U})$.*
3. *$\tilde{\mathcal{U}} := f^{-1}(0)$ is slice of $\mathcal{X}$ containing x_0 and $P^{-1}(\tilde{\mathcal{U}})$ is a bundle slice.*
4. *For $x \in \mathcal{U}$, $g = f(x)$ is the unique element $g \in V$ such that $\Gamma_0(g, x) \in \tilde{\mathcal{U}}$.*
5. *$\eta : \mathcal{U} \to \tilde{\mathcal{U}}$ defined by $x \mapsto \Gamma_0(f(x), x)$ is sc-smooth and $N : P^{-1}(\mathcal{U}) \to P^{-1}(\tilde{\mathcal{U}})$ defined by $v \mapsto \Lambda_0(f(P(v)), v)$ is a sc-smooth strong bundle map.*
6. *There is sc-smooth action of $\mathrm{stab}^{EQ}_{x_0}$ acts on $P^{-1}(\tilde{\mathcal{U}})$ defined by*
$$\Lambda \odot v := N(\Lambda(g_0, v)), \quad \forall \Lambda \in L^\pi(x_0, x_0, g_0), v \in P^{-1}(\tilde{\mathcal{U}}). \tag{4.25}$$
7. *For every $v \in P^{-1}(\tilde{\mathcal{U}})$, the set*
$$S_{v,P^{-1}(\tilde{\mathcal{U}})} := \{(u, g, \Lambda) | u \in P^{-1}(\tilde{\mathcal{U}}), g \in G, \Lambda \in L^\pi(P(v), P(u), g), \Lambda(g, v) = u\}$$
has exactly $|\,\mathrm{stab}^{EQ}_{x_0}\,|$ elements. As a consequnce, for every $y \in \tilde{\mathcal{U}}$, the set
$$S_{y,\tilde{\mathcal{U}}} := \{(z, g, \Lambda) | z \in \tilde{\mathcal{U}}, g \in G, \Lambda \in L^\pi(y, z, g)\}$$
also has exactly $|\,\mathrm{stab}^{EQ}_{x_0}\,|$ elements.

The following proposition follows from the same proof of Proposition 4.39 and the assertion on good slices follows from Proposition 3.42.

Proposition 4.68. *Let $p : W \to Z$ be a regular strong polyfold bundle with a sc-smooth G-action such the induced action on Z has finite isotropy. Let $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ be a polyfold bundle structure. Then for every $x_0 \in \mathcal{X}_\infty$, there exists a G-slice $(\mathcal{U}, \tilde{\mathcal{U}}, V, f, \eta, N)$ for $P : \mathcal{E}^2 \to \mathcal{X}^2$ around x_0.*

If $\mathcal{X}$ is infinite dimensional and there exists a sc-Fredholm section $s : \mathcal{X} \to \mathcal{E}$, then $\tilde{\mathcal{U}}$ can be chosen to be good with respect to s (Definition 3.38).

Quotient of strong polyfold bundles and sections

Theorem 4.69. *Let $p : W \to Z$ be a regular tame strong polyfold bundle such that Z is infinite dimensional. A compact Lie group G acts on p sc-smoothly. If the G-action on Z only has finite isotropy, then there is a G-invariant open dense set $\widehat{Z} \subset Z^2$ containing Z_∞ such that $p^{-1}(\widehat{Z})/G \to \widehat{Z}/G$ can be equipped with a strong tame polyfold bundle structure. Moreover, the topological quotient map $\pi_G : p^{-1}(\widehat{Z}) \to p^{-1}(\widehat{Z})/G$ is covered by sc-smooth strong polyfold bundle map $\mathfrak{q} : p^{-1}(\widehat{Z}) \to p^{-1}(\widehat{Z})/G$.*

If $s : Z \to W$ is a G-equivariant proper sc-Fredholm section. Then s induces a proper sc-Fredholm section $\overline{s} : \widehat{Z}^1/G \to p^{-1}(\widehat{Z}^1)/G$ by $\mathfrak{q}^*\overline{s} = s|_{\widehat{Z}^1}$.

Proof. The proof is analogous to the proof of Theorem 4.42 by replacing Proposition 4.39 with Proposition 4.68. Let $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ be strong polyfold bundle structure of $p : W \to Z$. If we pick a G-slice $(\mathcal{U}_x, \tilde{\mathcal{U}}_x, V_x, f_x, \eta_x, N_x)$ for $P : \mathcal{E}^2 \to \mathcal{X}^2$ around every $x \in \mathcal{X}_\infty$, then we can construct two ep-groupoids by

$$\mathcal{E}\mathcal{Q} := \coprod_{x \in \mathcal{X}_\infty} P^{-1}(\tilde{\mathcal{U}}_x), \quad \mathcal{Q} := \coprod_{x \in \mathcal{X}_\infty} \tilde{\mathcal{U}}_x,$$

and

$$\boldsymbol{EQ} := \{(u, v, g, (\Lambda, \Gamma)) | (u, v, g) \in \mathcal{E}\mathcal{Q} \times \mathcal{E}\mathcal{Q} \times G, (\Lambda, \Gamma) \in L^\pi(P(u), P(v), g), \Lambda(g, u) = v\},$$

$$\boldsymbol{Q} := \{(x, y, g, (\Lambda, \Gamma)) | (x, y, g) \in \mathcal{Q} \times \mathcal{Q} \times G, (\Lambda, \Gamma) \in L^\pi(x, y, g)\}.$$

The structure maps are defined similarly as in Theorem 4.42. By the same argument of Theorem 4.42, $(\mathcal{E}\mathcal{Q}, \boldsymbol{EQ})$ and $(\mathcal{Q}, \boldsymbol{Q})$ are ep-groupoids. The obvious projection $P_Q : (\mathcal{E}\mathcal{Q}, \boldsymbol{EQ}) \to (\mathcal{Q}, \boldsymbol{Q})$ defines a strong ep-groupoid bundle and gives $p^{-1}(\widehat{Z})/G \to \widehat{Z}/G$ a strong polyfold bundle structure, where $\widehat{Z} = \cup_{x \in \mathcal{X}_\infty} \rho_Z(G, |\mathcal{U}_x|) \subset Z^2$.

Let $S : (\mathcal{X}, \boldsymbol{X}) \to (\mathcal{E}, \boldsymbol{E})$ be the representative of the section s, such representative is unique by [65, Proposition 10.7]. Let $(\Lambda, \Gamma) \in L^\pi(x, y, g)$ and assume Λ is defined on $V \times P^{-1}(\mathcal{U})$ for neighborhoods $V \subset G$ of g and $\mathcal{U} \subset \mathcal{X}$ of x. Since s is G-equivariant and by the uniqueness of representative S on $\mathcal{X}$ (also on $\mathcal{U}$), we have:

$$S^0(\Gamma(h, z)) = \Lambda(h, S^0(z)), \quad \forall (h, z) \in V \times \mathcal{U} \tag{4.26}$$

Let $S^0_Q : \mathcal{Q} \to \mathcal{E}\mathcal{Q}$ denote the restriction $S|_{\tilde{\mathcal{U}}_x} : \tilde{\mathcal{U}}_x \to P^{-1}(\tilde{\mathcal{U}}_x)$. We define $S^1_Q : \boldsymbol{Q} \to \boldsymbol{EQ}$ to be

$$(x, y, g, (\Lambda, \Gamma)) \mapsto (S^0_Q(x), S^0_Q(y), g, (\Lambda, \Gamma)). \tag{4.27}$$

(4.27) is well-defined by (4.26). We claim $S_Q := (S^0_Q, S^1_Q)$ is a functor from $(\mathcal{Q}, \boldsymbol{Q})$ to $(\mathcal{E}\mathcal{Q}, \boldsymbol{EQ})$. It is clear that $s_{\mathcal{E}\mathcal{Q}} \circ S^1_Q = S^0_Q \circ s_{\mathcal{Q}}$ and $t_{\mathcal{E}\mathcal{Q}} \circ S^1_Q = S^0_Q \circ t_{\mathcal{Q}}$, where $s_{\mathcal{E}\mathcal{Q}}, s_{\mathcal{Q}}$ are the source maps for $(\mathcal{E}\mathcal{Q}, \boldsymbol{EQ})$ and $(\mathcal{Q}, \boldsymbol{Q})$, $t_{\mathcal{E}\mathcal{Q}}, t_{\mathcal{Q}}$ are the target maps for $(\mathcal{E}\mathcal{Q}, \boldsymbol{EQ})$ and

$(\mathcal{Q}, \boldsymbol{Q})$. It remains to prove that the compatibility with composition. Let $m_{\mathcal{EQ}}$ and $m_{\mathcal{Q}}$ denote the composition in $(\mathcal{EQ}, \boldsymbol{EQ})$ and $(\mathcal{Q}, \boldsymbol{Q})$. Then we have

$$\begin{aligned}
& S_Q^1(m_Q(y,z,g,(\Lambda_1,\Gamma_1)),(x,y,h,(\Lambda_2,\Gamma_2))) \\
= \ & S_Q^1((x,z,gh,(\Lambda_1\circ\Lambda_2,\Gamma_1\circ\Gamma_2))) \\
= \ & (S_Q^0(x),S_Q^0(z),gh,(\Lambda_1\circ\Lambda_2,\Gamma_1\circ\Gamma_2)) \\
= \ & m_{\mathcal{EQ}}((S_Q^0(y),S_Q^0(z),g,(\Lambda_1,\Gamma_1)),(S_Q^0(x),S_Q^0(y),h,(\Lambda_2,\Gamma_2))) \\
= \ & m_{\mathcal{EQ}}(S_Q^1(y,z,g,(\Lambda_1,\Gamma_1)),S_Q^1(x,y,h,(\Lambda_2,\Gamma_2)).
\end{aligned}$$

This finishes the proof of the claim. By Proposition 4.68, for every $x \in \mathcal{X}_\infty$, there exists G-slice $\tilde{\mathcal{U}}_x'$ such that $S_Q^0|_{(\tilde{\mathcal{U}}_x')^1}$ has a Fredholm chart around x by Lemma 3.40. Although $\tilde{\mathcal{U}}_x$ may be different from $\tilde{\mathcal{U}}_x'$, $S_Q^0|_{\tilde{\mathcal{U}}_x}$ is equivalent to $S_Q^0|_{\tilde{\mathcal{U}}_x'}$ by a strong bundle isomorphism like the proof of Theorem 3.1. Hence $S_Q^0|_{\tilde{\mathcal{U}}_x^1}$ has a Fredholm chart around x. By Proposition 3.37, S_Q has the regularizing property. Therefore $S_Q : \mathcal{Q}^1 \to \mathcal{EQ}^1$ is a sc-Fredholm functor, hence the induced section $\overline{s} : \widehat{Z}^1/G \to \pi^{-1}(\widehat{Z}^1)/G$ on the quotient polyfold is sc-Fredholm. $\overline{s}^{-1}(0)$ is compact because $\overline{s}^{-1}(0) = s^{-1}(0)/G$ and both $s^{-1}(0)$ and G are compact. □

Remark 4.70. *By the same argument in Proposition 4.49, if one uses two sets of different choices of G-slice and polyfold bundle structure to construct two quotient polyfold bundle $p^{-1}(\widehat{Z}_a)/G$ and $p^{-1}(\widehat{Z}_b)/G$, then there exists a G-invariant open set $\widehat{Z} \subset \widehat{Z}_a \cap \widehat{Z}_b$ containing Z_∞ such that the restrictions of $p^{-1}(\widehat{Z}_a)/G, p^{-1}(\widehat{Z}_b)/G$ give equivalent polyfold structures on $p^{-1}(\widehat{Z})/G$.*

Remark 4.71. *Theorem 4.69 provides a quotient of the base polyfold Z even if Z is not regular. Assume the base is also regular, the constructions in Theorem 4.69 and Theorem 4.42 may yield different quotients. The reason is that $\mathrm{stab}_x^{\pi,\mathrm{eff}}$ and $\mathrm{stab}_x^{\mathrm{eff}}$ may be different.*

Chapter 5

Orientation and Good Position

5.1 Orientation

Orientations of Fredholm sections were discussed in [65, Chapter 6, Section 9.3 and 18.5]. In the case of M-polyfold, let $s : \mathcal{X} \to \mathcal{Y}$ be a sc-Fredholm section of a tame strong M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$. Then there exists a $\mathbb{Z}_2$ bundle $\det s$ over $\mathcal{X}_\infty$, which is essentially the $\mathbb{Z}_2$ reduction of the "determinant bundle" of the linearization $\mathrm{D}s$, see [65, Chapter 6] for details. The sc-Fredholm section is orientable if $\det s$ admits a continuous section and an orientation of s is a continuous section of $\det s$.

Theorem 5.1 ([65, Theorem 6.3]). *If the sc-Fredholm section s is oriented, then $(s+p)^{-1}(0)$ is an oriented manifold for every transverse sc^+-perturbation p.*

In the case of ep-groupoids and polyfolds, let $S : (\mathcal{X}, \boldsymbol{X}) \to (\mathcal{E}, \boldsymbol{E})$ be a sc-Fredholm functor of a tame strong ep-groupoid bundle $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$. Then we have an orientation bundle $\det S$ on $\mathcal{X}_\infty$. By [65, Theorem 12.11], for every $\phi \in \boldsymbol{X}_\infty$, there is an induced map $\hat{\phi} : \det S_{s(\phi)} \to \det S_{t(\phi)}$. Therefore $\det S$ can be made into a $\mathbb{Z}_2$ bundle over $(\mathcal{X}, \boldsymbol{X})$ such that $\hat{\phi}$ are the morphisms.

Definition 5.2 ([65, Definition 12.12]). *An orientation of S is a continuous section of $\det S$, such that it is invariant under $\hat{\phi}$ for all $\phi \in \boldsymbol{X}_\infty$.*

Proposition 5.3 ([65, Proposition 12.4]). *A sc-Fredholm section S is orientable iff the following two condition holds.*

1. $|\det S| \to |\mathcal{X}_\infty|$ *is a two to one cover.*
2. $|\det S| \to |\mathcal{X}_\infty|$ *admits a continuous section.*

The orientations can be transferred through generalized strong bundle isomorphisms, see [65, Theorem 12.12]. Then an orientation for polyfold sc-Fredholm section is an equivalence class of orientations on strong polyfold bundle structures, where two orientations are equivalent iff they are connected by a generalized strong bundle isomorphism.

Proposition 5.4. *Under the assumptions of Theorem 4.69, assume the sc-Fredholm section $s : Z \to W$ is oriented. Let $\det s := |\det S|$ is the trivial $\mathbb{Z}_2$ bundle over Z_∞ with a continuous section σ as asserted in Proposition 5.3. Then G acts on $\det s$ continuously and $\det s/G = \det \overline{s}$ on the quotient Z_∞/G.*

Proposition requires some setup for the orientation bundles on polyfolds, we will prove Proposition 5.4 in our future work.

Definition 5.5. *Under the assumptions of Proposition 5.4, the G-action preserves the s-orientation if the G-action on the trivial $\mathbb{Z}_2$ bundle $\det s$ preserve the section σ.*

By Proposition 5.4, when G is connected, every G-action preserves the s-action. As another straightforward corollary of Proposition 5.4, we have the following.

Proposition 5.6. *If the G-action preserves the s-orientation, then $\det \overline{s}$ is a two to one covering of Z_∞/G and admits a continuous section, i.e. $\overline{s}$ is oriented.*

5.2 Good position

Tame polyfolds have well-behaved boundary and corner structures, which play important roles in Floer-type theories like Hamiltonian-Floer homology and SFT. The discussions on the boundary and corners of tame M-polyfold resp. polyfold can be found in [65, Section 2.3].

Definition 5.7 ([65, Definition 5.9])**.** *Given a sc-Fredholm section $s : \mathcal{X} \to \mathcal{Y}$ of a tame strong M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$, we say s is in **general position** if the linearization $\mathrm{D}s_x : T_x\mathcal{X} \to \mathcal{Y}_x$ for $x \in s^{-1}(0)$ is surjective and $\ker \mathrm{D}s_x$ has a sc-complement in the reduced tangent space $T^R_x\mathcal{X}$. Equivalently, s is in general position if for all $x \in s^{-1}(0)$ the restricted linearization $\mathrm{D}s_x|_{T^R_x\mathcal{X}} : T^R_x\mathcal{X} \to \mathcal{Y}_x$ is surjective. In other words, s restricted to all the boundary and corner is transverse to 0.*

A sc-Fredholm section $s : Z \to W$ of a tame strong polyfold bundle $p : W \to Z$ is in general position, s has a representative $S : (\mathcal{X}, \boldsymbol{X}) \to (\mathcal{E}, \boldsymbol{E})$ for a strong polyfold structure $(\mathcal{X}, \boldsymbol{X}) \xrightarrow{P} (\mathcal{E}, \boldsymbol{E})$, such that S^0 is in general position.

Remark 5.8. *Under the assumptions of Theorem 4.69, if s is in general position, then $\overline{s}$ is also in general position. Let $(\tilde{\mathcal{U}}, \mathcal{U}, V, f, \eta, N)$ be a G-slice around $x \in \mathcal{X}_\infty$, since the infinitesimal directions Ξ of the gorp action is contained in $T^R_x\mathcal{U}$. Moreover, $T^R\mathcal{U} = \Xi \oplus T^R_x\tilde{\mathcal{U}}$. Since s is G-equivariant, we have $\mathrm{D}s_x(\Xi) = 0$. Therefore $\mathrm{D}s_x|_{T^R_x\mathcal{U}} \to \mathcal{Y}_x$ being surjective implies that $\mathrm{D}s_x|_{T^R_x\tilde{\mathcal{U}}} \to \mathcal{Y}_x$ is surjective, that is $\overline{s}$ is general position.*

[65, Theorem 5.6, Theorem 15.6] assert that there exist sc$^+$- perturbations resp. sc$^+$-multisection perturbations so that the perturbed sections are in general position in the M-polyfold resp. polyfold case. Moreover, the zero set of a sc-Fredholm section resp. multi-section in general position is a manifold resp. branched ep$^+$-subgroupoid with boundary and corner [65, Definition 9.1], and the degeneracy index on the zero set equals to the degeneracy index of the base M-polyfold resp. polyfold. However, if one imposes extra conditions on the perturbations, general position may not be possible. Therefore the concept of good position was introduced in [65, Definition 3.16]. When a sc-Fredholm (multi)section is in good position on the boundary, the zero set still has boundary and corner structure by [65, Theorem 3.13]. In this subsection, we prove the good position is preserved under the quotient construction.

Definition 5.9 ([65, Definition 3.9]). *A closed linear subspace N of $\mathbb{R}^m \times \mathbb{E}$ is in* ***good position*** *to the partial quadrant $C := \mathbb{R}^m_+ \times \mathbb{E}$, if $N \cap C$ has a nonempty interior in N and there exist a sc-complement $N^\perp$ of N in $\mathbb{R}^m \times \mathbb{E}$ and positive constant c, so that for any point $(n_0, n_1) \in N \oplus N^\perp$ satisfying*

$$|n_1|_0 \leq c|n_0|_0,$$

the statements $n_0 + n_1 \in C$ and $n_0 \in C$ are equivalent.

The boundary of a M-polyfold resp. polyfold $\partial\mathcal{X}$ resp. ∂Z is the set of points with degeneracy index greater than one.

Definition 5.10 ([59, Definition 5.15],[65, Definition 3.16]). *Let $s : \mathcal{X} \to \mathcal{Y}$ be a sc-Fredholm section of a tame strong M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$. Then s is called* ***in good position on the boundary*** *if for every $x \in s^{-1}(0) \cap \partial\mathcal{X}$, the linearization $\mathrm{D}s_x : T_x\mathcal{X} \to \mathcal{Y}_x$ is surjective and the kernel of $\mathrm{D}s^f_x$ is in good position for a filler s^f of s on a partial quadrant $\mathbb{R}^n_+ \times \mathbb{E}$. A sc-Fredholm section of a strong polyfold bundle is in good position on the boundary if there is a representative in good position on the boundary.*

Proposition 5.11. *Let $s : \mathcal{X} \to \mathcal{Y}$ be a sc-Fredholm section of a tame strong M-polyfold bundle $p : \mathcal{Y} \to \mathcal{X}$. If $s : \mathcal{X} \to \mathcal{Y}$ is in good on the boundary then$s : \mathcal{X}^k \to \mathcal{Y}^k$ is in good position on the boundary for every $k \geq 1$.*

Proposition 5.12. *Under the same conditions in Theorem 4.69, assume s is in good position on the boundary. Then the induced section $\overline{s}$ on the quotient is also in good position on the boundary.*

Proof of Proposition 5.12. By the constriction in Theorem 4.69, it suffices to verify the good position on every G-slice. Let $P : (\mathcal{E}, \boldsymbol{E}) \to (\mathcal{X}, \boldsymbol{X})$ be a strong polyfold bundle structure for $p : W \to Z$. For every $y \in \overline{s}^{-1}(0) \in Z_\infty/G$, there exists $x \in \mathcal{X}_\infty$ such that $|x| \in s^{-1}(0)$ and $\pi(|x|) = y$, where $\pi : Z \to Z/G$ is the projection. Let $(\tilde{\mathcal{U}}, \mathcal{U}, V, f, \eta, \mathbb{N})$ be the G-slice of

P around x that used in the construction of the quotient in Theorem 4.69.

$$\begin{tikzcd}
(\mathbb{R}^m_+ \times \{0\} \times \mathbb{K}^3) \supset U' \arrow[r, "\overline{s}^f"] \arrow[d, "\iota"] & U' \lhd \mathbb{F}^3 \arrow[d, "\mathrm{id}"] \\
\mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}^3 \supset U \arrow[r, "H_* s^f"] \arrow[d, "h^{-1}"] & U \lhd \mathbb{F}^3 \arrow[d, "H^{-1}"] \\
\mathbb{R}^m_+ \times \mathbb{E}^3 \supset O \arrow[r, "s^f"] & O \lhd \mathbb{F}^3,
\end{tikzcd}$$

where s^f is filling of s on open neighborhood $O \subset \mathbb{R}^m_+ \times \mathbb{E}^3$ of $(0,0)$ such that $\mathcal{U}_x$ is homeomorphic to a retract $r(O)$ by some chart map, $h : O \to U \subset \mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}^3$ is the change of coordinate to a $\mathbb{R}^n$-sliced retract asserted by Lemma 3.27 and $\overline{s}^f$ is the filler of $\overline{s}$ on the slice $\tilde{\mathcal{U}}_x$.

By assumption, we can assume s^f is good position at x, that is $\mathrm{D}s^f_{(0,0)}$ is surjective and $N := \ker \mathrm{D}s^f_{(0,0)} \subset \mathbb{R}^m \times \mathbb{E}_\infty$ has a complement $N^\perp \subset \mathbb{R}^m \times \mathbb{E}$ such that conditions in Definition 5.9 hold for constant c, note that good position condition for s^f initially does not have level shift. Since all norms on finite dimensional space N are equivalent, there exists $B > 0$ such that $|n_1|_3 < B|n_0|_3$ implies $|n_1|_0 < c|n_0|_0$ for $n_0 \in N, n_1 \in N^\perp_3$. Therefore if $|n_1|_3 < B|n_0|_3$, $n_0 + n_1 \in C := \mathbb{R}^+_m \times \mathbb{E}^3$ is equivalent to $n_0 \in C$.

Let Ξ be the infinitesimal directions of the group action at x and $\mathbb{H}$ is the complement of Ξ in $\mathbb{E}$ used in Lemma 3.27. Note that $\Xi \subset N$ because s is equivariant. Since $\mathrm{D}h_{(0,0)}(\Xi) = \{0\} \times \mathbb{R}^n \times \{0\}$ by Lemma 3.19, we have $\mathrm{D}\overline{s}^f_{(0,0)}$ is also surjective. So we only need to verify the condition on the complement.

Let $\overline{N} := \ker \mathrm{D}\overline{s}^f_{(0,0)} = \mathrm{D}h_{(0,0)}(N) \cap (\mathbb{R}^m \times \{0\} \times \mathbb{K}^3)$. We define $\tau : (N^\perp)^3 \to \mathbb{R}^m \times \{0\} \times \mathbb{K}^3$ by $v \mapsto \pi_{\mathbb{R}^m \times \{0\} \times \mathbb{K}^3} \circ \mathrm{D}h_{(0,0)} v$. By Lemme 3.19, $\tau(v) = 0$ iff $v \in \Xi \cap (N^\perp)^3$. Note that $\Xi \subset N$, therefore τ is injective.

We claim $\operatorname{im} \tau \oplus \overline{N} = \mathbb{R}^m \times \{0\} \times \mathbb{K}^3$. If $e \in \operatorname{im} \tau \cap \overline{N}$, in particular $\mathrm{D}h^{-1}_{(0,0,0)} e \in N$. Assume $e = \pi_{\mathbb{R}^m \times \{0\} \times \mathbb{K}^3} \circ \mathrm{D}h_{(0,0)}(v)$ for $v \in (N^\perp)^3$. Then $\pi_{\mathbb{R}^m \times \{0\} \times \mathbb{K}^3} \circ \mathrm{D}h_{(0,0)}(v) \in \mathrm{D}h_{(0,0)}(N)$, that is $\mathrm{D}h_{(0,0)}(v - u) \in \{0\} \times \mathbb{R}^n \times \{0\}$ for some $u \in N$. Hence $v - u \in \Xi$. Since $\Xi \subset N$, $v \in N$. Therefore $v = 0$, which implies $e = 0$. This proves $\operatorname{im} \tau \cap \overline{N} = \{0\}$. Next, for any vector $v \in \mathbb{R}^m \times \{0\} \times \mathbb{K}^3$, we have $v = \mathrm{D}h_{(0,0)} n_0 + \mathrm{D}h_{(0,0)} n_1$ for $n_0 \in N, n_1 \in (N^\perp)^3$. Then $v = \mathrm{D}h_{(0,0)} n_0 + \pi_{\{0\} \times \mathbb{R}^n \times \{0\}} \mathrm{D}h_{(0,0)} n_1 + \tau(n_1)$. There exists $\xi \in \Xi$, such that $\pi_{\{0\} \times \mathbb{R}^n \times \{0\}} \mathrm{D}h_{(0,0)} n_1 = \mathrm{D}h_{(0,0)} \xi$. Then $\mathrm{D}h_{(0,0)} n_0 + \pi_{\{0\} \times \mathbb{R}^n \times \{0\}} \mathrm{D}h_{(0,0)} n_1 = \mathrm{D}h_{(0,0)}(n_0 + \xi)$. Since $\mathrm{D}h_{(0,0)} n_0 + \pi_{\{0\} \times \mathbb{R}^n \times \{0\}} \mathrm{D}h_{(0,0)} n_1 \in \mathbb{R}^m \times \{0\} \times \mathbb{K}^3$. Note that $n_0 + \xi \in N$, hence we have $\mathrm{D}h_{(0,0)} n_0 + \pi_{\{0\} \times \mathbb{R}^n \times \{0\}} \mathrm{D}h_{(0,0)} n_1 \in \overline{N}$. This shows v has a splitting in $\operatorname{im} \tau$ and $\overline{N}$. Hence the claim holds.

We claim $\operatorname{im} \tau$ is a complement of $\overline{N}$ so that the conditions in Definition 5.9 hold. First, there exists $D > 0$, such that $|n_1|_0 \le D|n_0|_0$ for $n_0 \in \overline{N}, n_1 \in \operatorname{im} \tau$ implies that $|v_1|_3 \le B|v_0|_3$ for $v_0 \in N, v_1 \in (N^\perp)^3$ with $\mathrm{D}h_{(0,0)}(v_0) = n_0, \tau(v_1) = n_1$. Therefore for $|n_1|_0 \le D|n_0|_0$, $v_0 + v_1 \in C$ iff $v_0 \in C$. Since $\mathbb{R}^m_+ \times \mathbb{R}^n \times \mathbb{K}^3 = \mathrm{D}h_{(0,0)} C$, $n_0 + n_1 \in \mathbb{R}^m_+ \times \{0\} \times \mathbb{K}^3$ iff

$\mathrm{D}h^{-1}_{(0,0,0)}n_0 + \mathrm{D}h^{-1}_{(0,0,0)}(n_1) \in C$, that is $v_0 + v_1 + \xi \in C$ for $\xi := -\mathrm{D}h^{-1}_{(0,0,0)} \circ \pi_{\{0\}\times\mathbb{R}^n\times\{0\}} \circ \mathrm{D}h_{(0,0)}(v_1) \in \Xi$. Since $\xi \in \Xi \subset \{0\} \times \mathbb{E}^3$, hence $v_0 + v_1 + \xi \in C$ iff $v_0 + v_1 \in C$. To sum up, for $|n_1|_0 \leq D|n_0|_0$, $n_0 + n_1 \in \mathbb{R}^m_+ \times \{0\} \times \mathbb{K}^3$ iff $v_0 \in C$, which is equivalent to that $n_0 \in \mathbb{R}^m_+ \times \{0\} \times \mathbb{K}^3$. □

Combining Theorem 4.69, Proposition 5.6 and Proposition 5.12, we get a proof of the main Theorem 1.1.

Chapter 6

Equivariant Transversality

As corollaries of Theorem 1.1 and the perturbation theory [65, Theorem 5.6, Theorem 15.4] on polyfolds, we have the following basic equivariant transversality results.

Corollary 6.1. *Let $p:\mathcal{Y}\to\mathcal{X}$ be a tame strong M-polyfold bundle equipped with a sc-smooth G-action and a G-equivariant proper sc-Fredholm section s, such that the induced action on $\mathcal{X}$ is free. Assume $\mathcal{X}$ is infinite dimensional and supports sc-smooth bump functions. Then there exists an G-invariant open neighborhood $\widehat{\mathcal{X}}\subset\mathcal{X}^3$ containing $\mathcal{X}_\infty$ and an equivariant sc^+-perturbation γ on $\widehat{\mathcal{X}}$, such that $s+\gamma$ is proper and in general position. Moreover, if s is already in general position on a G-invariant closed set $C\subset\widehat{\mathcal{X}}$, then γ can be chosen to satisfy $\gamma|_C=0$.*

Proof. By Theorem 3.1, $\overline{s}$ is a sc-Fredholm section of $p^{-1}(\widehat{\mathcal{X}})/G\to\widehat{\mathcal{X}}/G$ for a G-invariant open set $\widehat{\mathcal{X}}\subset\mathcal{X}^3$ containing $\mathcal{X}_\infty$. By [65, Theorem 5.6], there exists a sc^+-perturbation $\overline{\gamma}$ on $\widehat{\mathcal{X}}/G$ such that $\overline{s}+\overline{\gamma}$ is proper and in general position. Let $\gamma:=\pi^*\overline{\gamma}$, then $s+\gamma$ is G-equivariant, proper and transverse. By Remark 5.8, $s+\gamma$ is in general position. For the last assertion, we can chose $\overline{\gamma}|_{C/G}=0$ by the proof of [65, Theorem 5.6]. Then $\gamma|_C=0$. □

Remark 6.2. *Since the quotient construction preserves good position by Proposition 14.3, by [65, Theorem 5.7] we can require $s+\gamma$ is in good position on the boundary given s is in good position on the boundary.*

Remark 6.3. *Since $\widehat{\mathcal{X}}$ contains of neighborhood of $s^{-1}(0)$ in $\mathcal{X}^3$, if we choose the $\operatorname{supp}\gamma$ small enough, then γ can be extended to $\mathcal{X}^3$ by 0. However, if there are infinite M-polyfolds with free group actions, keeping the property on the supports in a coherent way might be challenging. Since in applications, we only need regular moduli spaces, which are contained in $\mathcal{X}_\infty$, having γ defined on a neighborhood of $\mathcal{X}_\infty$ is sufficient.*

By [65, Theorem 15.4], we have the following polyfold version with an identical proof.

Corollary 6.4. *Let $p:W\to Z$ be a regular tame strong polyfold bundle equipped with a sc-smooth G-action and a G-equivariant proper sc-Fredholm section s, such that the induced* *action on Z has finite isotropy. Assume Z is infinite-dimensional and supports sc-smooth bump functions. Then there exists a G-invariant neighborhood $\widehat{Z}$ of $\mathbb{Z}^3$ containing Z_∞ and an equivariant sc^+-multisection perturbation κ defined on $\widehat{Z}$, such that $s+\kappa$ is proper and in general position. Moreover, if s is already in general position on a G-invariant closed set $C \subset \widehat{Z}$, then κ can be chosen to satisfy $\kappa|_C = 0$.*

Remark 6.5. *[65, Theorem 5.6, Theorem 15.4] assert the abundance of transverse perturbations and the perturbation can be chosen to be supported in arbitrarily small neighborhoods of $s^{-1}(0)$. One can get similar properties for equivariant perturbations under the conditions in Corollary 6.1 and Corollary 6.4.*

For more general group actions, equivariant transversality is often obstructed. The obstructions usually arise from points with larger isotropy. Therefore the first step to analyze equivariant transversality would be understanding those with biggest isotropy group, namely the fixed locus. Moreover, the study of fixed locus is important for the localization theorem 10.4 and 10.7. We will first discuss the finite dimensional case to motivate the discussion in the polyfold case.

6.1 Manifold case

In this subsection, we discuss equivariant transversality near the fixed locus in the case of manifolds. We first show that there is a standard local model for the equivariant transversality problem near the fixed locus. We call a vector bundle $p : E \to M$ a ***G*-vector bundle** iff G acts on p such that the G-action on M is trivial. Then the fibers of G-vector bundle is a G-representation. Let $\{V^\lambda\}_{\lambda\in\Lambda}$ denote the set of all the nontrivial irreducible G-representations.

Proposition 6.6. *Let $p : E \to M$ be a G-vector bundle and ρ denotes the G-action. Then there is a decomposition of G-vector bundles $E = E^G \oplus_{\lambda\in\Lambda} E^\lambda$ such that E^G is fixed by the G-action and the fibers of E^λ are sums of the irreducible representation V^λ.*

Proof. Since G is compact, every finite dimensional representation can be decomposed into direct sum of irreducible representations. Since each fiber E_x can be decomposed into irreducible representations, it suffices to show that the V^λ component of E_x forms a smooth subbundle. Let $\chi^\lambda : G \to \mathbb{R}$ denote the character of the irreducible representation V^λ. We equip G with a Haar measure μ such that $\mu(G) = 1$, then

$$P^\lambda : E \to E, \quad v \mapsto \frac{\dim V^\lambda}{\dim_{\mathbb{R}} \mathrm{End}(V^\lambda)} \int_G \chi^\lambda(g)\rho(g, v)d\mu$$

defines a smooth projection on E, e.g see [19]. On each fiber $P^\lambda_x : E_x \to E_x$ is the projection to the V^λ component. Therefore $E^\lambda := \ker(\mathrm{id} - P^\lambda)$ is a smooth subbundle of E. $\square$

Proposition 6.7. *Let $p : E \to M$ be a vector bundle over a closed manifold with a G-action ρ for a compact group G. Suppose $M^G \subset M$ is the fixed point set of the induced G-action ρ_M on M. Then we have the following.*

1. *M^G is a submanifold of M.*
2. *Let N denote the normal bundle of $M^G \subset M$, then the linearization of ρ_M induces a G-action on N such that N is G-vector bundle.*
3. *There exists a G-invariant open neighborhood $U \subset M$ of M^G, such that there is a G-equivariant bundle isomorphism*
$$\Phi : E|_U \to \pi^*(E|_{M^G}),$$
where π is the projection from N to M^G.

Proof. We equip M with a G-invariant metric d. If $x \in M^G$, then there is a linear representation:
$$\mathrm{D}(\rho_M)_x : G \to O(T_xM),$$
where $O(T_xM)$ is the orthogonal group. By the uniqueness of geodesic, we have
$$\rho_M(g, \exp(\xi)) = \exp(\mathrm{D}(\rho_M)_x(g) \cdot \xi), \quad \forall \xi \in T_xM.$$
Therefore M^G is a submanifold and the tangent T_xM^G is the fixed subspace $(T_xM)^G$. Let $\pi : N \to M^G$ denote the normal bundle of $M^G \subset M$. Linearization $\mathrm{D}(\rho_M)x$ induces a G-action on each fiber N_x. Then by the equivariant tubular neighborhood theorem [34, Theorem 3.18], there is a G-invariant neighborhood of M^G equivariantly diffeomorphic to the normal bundle N. With a little abuse of notation, we will call this tubular neighborhood N.

Fixing a connection, we have a bundle isomorphism $\Psi : E|_N \to \pi^*(E|_{M^G})$ by parallel transportation, such that Ψ is equivariant on $E|_{M^G}$. We can write Ψ as $(x, v) \mapsto (x, \psi(x, v))$, where $\psi(x, v) \in E|_{\pi(x)}$. We equip G with a Haar measure μ such that $\mu(G) = 1$, then we define
$$\Phi : E|_N \to \pi^*(E|_{M^G}), (x, v) \to \left(x, \int_G g\psi\left(\rho(g^{-1}, (x, v))\right) \mathrm{d}\mu\right).$$
Then Φ is an equivariant bundle map and $\Phi|_{E|_{M^G}} = \Psi|_{E|_{M^G}}$. Hence Φ is an equivariant bundle isomorphism for a smaller G-invariant tubular neighborhood of M^G in N. □

As a corollary of Proposition 6.7, the equivariant transversality problem on the fixed locus M^G is equivalent to the equivariant transversality problem on $\pi^*(E|_{M^G}) \to N$. Since $N, E|_{M^G}$ are G-vector bundles over M^G, by Proposition 6.6 we have decompositions of G-vector bundles:
$$N = \oplus_{\lambda \in \Lambda} N^\lambda, \qquad E|_{M^G} := E^G \oplus_{\lambda \in \Lambda} E^\lambda.$$

Let $s : N \to \pi^*(E|_{M^G})$ be a G-equivariant section. If $s(x) = 0$ for $x \in M^G$, then the linearized operator $\mathrm{D}s_x$ can be decomposed into

$$\mathrm{D}s_x = \mathrm{D}^G s_x \oplus_{\lambda\in\Lambda} \mathrm{D}^\lambda s_x,$$

where $\mathrm{D}^G s_x : T_x M^G \to E^G_x$ is the linearization of the fixed part $s|_{M^G} : M^G \to E^G$ and $\mathrm{D}^\lambda s_x : N^\lambda_x \to E^\lambda_x$ is G-equivariant. Thus G-equivariant transversality on M^G means that $s|_{M^G} : M^G \to E^G$ is transverse to 0 and all $\mathrm{D}^\lambda s_x$ are surjective for all $x \in s|^{-1}_{M^G}(0)$. If $\operatorname{rank} E^\lambda > \operatorname{rank} N^\lambda$ for some λ, then we can not expect equivariant transversality, unless $s|^{-1}_{M^G}(0) = \emptyset$.

Even if $\operatorname{rank} E^\lambda \leq \operatorname{rank} N^\lambda$ for all $\lambda \in \Lambda$ appearing in the decomposition, there are still some other obstructions. The spaces of equivariant linear maps $\operatorname{Hom}_G(N^\lambda_x, E^\lambda_x)$ form a vector bundle over manifold M^G. We define

$$S^\lambda := \{(x,h) \in \operatorname{Hom}_G(N^\lambda, E^\lambda) | x \in M^G, h \in \operatorname{Hom}_G(N^\lambda_x, E^\lambda_x) \text{ is not surjective.}\}. \tag{6.1}$$

Then $S^\lambda \to M^G$ is a fiber bundle. Therefore the equivariant transversality near fixed locus can be characterized by

1. $s|_{M^G} : M^G \to E^G$ is transverse to 0;
2. $\mathrm{D}^\lambda_x \notin S^\lambda$ for all $x \in s|^{-1}_{M^G}(0)$ and $\lambda \in \Lambda$.

For the irreducible representation V^λ, by Schur's Lemma the endomorphism ring $\operatorname{End}_G(V^\lambda)$ is a finite-dimensional division ring over $\mathbb{R}$. Hence by Frobenius Theorem, there are only three possibilities for $\operatorname{End}_G(V^\lambda)$, namely $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. The not-onto maps in $\operatorname{Hom}_G((V^\lambda)^n, (V^\lambda)^m) = \operatorname{Hom}_{\operatorname{End}_G(V^\lambda)}(\operatorname{End}_G(V^\lambda)^n, \operatorname{End}_G(V^\lambda)^m)$, i.e. $\operatorname{End}_G(V^\lambda)$-coefficient matrices with rank smaller than m. Such matrices form a determinantal variety S and S has real codimension $(n - m + 1)\dim_{\mathbb{R}} \operatorname{End}_G(V^\lambda)$ in $\operatorname{Hom}_{\operatorname{End}_G(V^\lambda)}(\operatorname{End}_G(V^\lambda)^n, \operatorname{End}_G(V^\lambda)^m)$ [54, Proposition 12.2]. The singularities come those matrices with rank than $m-1$, whose real codimension in S is $(m-n+3)\dim_{\mathbb{R}} \operatorname{End}_G(V^\lambda)$. Therefore when $m \geq n$, this singularities are of real codimension bigger than $3\dim_{\mathbb{R}} \operatorname{End}_G(V^\lambda) \geq 3$. Since the fibers of S^λ are those determinantal varieties, then there is a well-defined Euler class $e(S^\lambda) \in H^{\operatorname{codim} S^\lambda}(M^G)$ represented by a pseudo-cycle $t^{-1}(S^\lambda)$ for a generic section t of $\operatorname{Hom}_G(N^\lambda, E^\lambda) \to M^G$.[1] Then if $e(E^G \to M^G) \cup e(S^\lambda) \neq 0$ for some λ, it is impossible find equivariant transverse sections on M^G, here $e(E^G \to M^G)$ is the Euler class of $E^G \to M^G$.

However in the following simple case, we can use a dimension argument to get equivariant transversality near the fixed locus.

Proposition 6.8. *Following the notation in Proposition 6.7, let $s : N \to \pi^*(E|_{M^G})$ be a G-equivariant section. Let S_λ be the fiber bundle defined in (6.1). If*

$$\dim M^G - \operatorname{rank} E^G < \operatorname{codim} S^\lambda = (\frac{\operatorname{ind} \mathrm{D}^\lambda s}{\dim V^\lambda} + 1)\dim_{\mathbb{R}} \operatorname{End}_G(V^\lambda)$$

[1] See [80] for details on pseudo-cycles.

for all λ such that $\dim E^\lambda > 0$, then there exists G-equivariant section γ such that $s+\gamma$ is transverse to 0 on M^G.

Proof. Let $\gamma : M^G \to E^G$ be a perturbation such that $s|_{M^G} + \gamma : M^G \to E^G$ is transverse to 0. Then $s+\pi^*\gamma$ is a G-equivariant section such that $(s+\pi^*\gamma)|_{M^G} : M^G \to E^G$ is transverse to 0. Hence without loss of generality, we can assume $s|_{M^G} : M^G \to E^G$ is transverse to 0. In particular $s^{-1}(0)\cap M^G$ is a smooth manifold with dimension $\dim M^G - \operatorname{rank} E^G$.

For each λ such that $\dim E^\lambda > 0$, we can find a section

$$t^\lambda : s^{-1}(0)\cap M^G \to \operatorname{Hom}_G(N^\lambda, E^\lambda)|_{s^{-1}(0)\cap M^G},$$

such that $\mathrm{D}^\lambda s_x + t^\lambda(x) \notin S^\lambda$ for every $x \in s^{-1}(0)\cap M^G$. This is because $\dim(s^{-1}(0)\cap M^G) < \operatorname{codim} S^\lambda$. We can extend t^λ to a section of $\operatorname{Hom}_G(N^\lambda, E^\lambda) \to M^G$. For all other λ such that $\dim E^\lambda = 0$, we define $t^\lambda = 0$. Since $\oplus_{\lambda\in\Lambda} t^\lambda$ can be understood as an equivariant section $N \to \pi^*(E|_{M^G})$ vanishing on M^G, then $s + \oplus_{\lambda\in\Lambda} t^\lambda$ is a G-equivariant transverse section near M^G. □

6.2 Polyfold case

Let $p : W \to Z$ be a strong polyfold bundle with G-action and $s : Z \to W$ an equivariant proper sc-Fredholm section. We need an analogue of Proposition 6.7 in the polyfold case, thus we introduce the following assumptions. Like before, we define ***G*-tame (strong) M-polyfold bundle** to be a tame (strong) M-polyfold bundle $\pi : \mathcal{N} \to \mathcal{X}$ with a sc-smooth action ρ, such that the induced action $\rho_{\mathcal{X}}$ is trivial. Let Λ be the index set for all the nontrivial irreducible representations V_λ of G. **To guarantee the existence of bump functions and partition of unity, we assume from now on that all the M-polyfolds are modeled on sc-Hilbert spaces**[2]**.** Such assumption also plays a role in Proposition A.8.

Definition 6.9. *We say the G-action on $p : W \to Z$ satisfies the **tubular neighborhood assumption on the fixed locus without isotropy** if the following conditions hold.*

1. *The fixed part Z^G and W^G are tame subpolyfolds [65, Definition 16.10] of Z and W, such that W^G is a strong polyfold bundle over Z^G. Since s is equivariant, we have $s^G := s|_{Z^G} : Z^G \to W^G$. We also require s^G is a proper sc-Fredholm section.*

2. *Z^G has no isotropy, i.e. Z^G is a tame M-polyfold. Hence W^G is a tame strong M-polyfold bundle over Z^G.*

3. *There exists a G-invariant neighborhood N of Z^G in Z, such that there is a projection $\pi : N \to Z^G$ making N a G-tame M-polyfold bundle. We have a decomposition of*

[2] In fact, it is sufficient to assume the first level spaces are Hilbert spaces. Such assumption is also used to get sc-smooth bump functions, see [65, Corollary 5.2]

G-tame M-polyfold bundles $N = \widehat{\oplus}_{\lambda\in\Lambda} N^\lambda$, such that each fiber of N^λ is $V^\lambda \otimes \mathbb{H}$, where $\mathbb{H}$ is a sc-Hilbert space. Here $\widehat{\oplus}$ stands for some completion of the direct sum, see Remark 6.12

4. *The G-tame strong M-polyfold bundle $W|_{Z^G}$ has a similar decomposition, i.e. we have G-tame strong M-polyfold bundles W^λ over Z^G for $\lambda \in \Lambda$, such that $W[i]|_{Z^G} = W^G[i]\widehat{\oplus}_{\lambda\in\Lambda} W^\lambda[i]$ for $i = 0, 1$. Moreover, there is a G-equivariant strong bundle isomorphism $W|_N \to \pi^*(W|_{Z^G})$.*

5. *If W^λ is not the rank zero bundle, then $\dim(W^\lambda)_x = \infty$ for every $x \in Z^G$.*

Remark 6.10. *A less restrictive assumption is dropping (2) in Definition 6.9, so that the fixed locus Z^G is a polyfold and all $W^G, N^\lambda, W^\lambda$ are polyfold bundles. Condition (2) guarantees the existences of a global stabilization in Proposition 6.15, which is important for us to actually get the equivariant transversality when possible.*

In the following, we will abbreviate tubular neighborhood assumption on the fixed locus without isotropy (Definition 6.9) to **tubular neighborhood assumption**.

Remark 6.11. *The tubular neighborhood assumption does not always hold for general polyfolds. In fact, the tubular neighborhood Theorem does not hold for M-polyfolds. For example, there is no neighborhood of the origin with linear structures in the object M-polyfold in Example 4.19. Such phenomenon is due to that M-polyfolds are modeled on retracts. Although one may expect a tubular neighborhood of retract fiberation, we need linear structures in the proof of Theorem 6.17.*

Remark 6.12. *Let $\mathbb{E}^\lambda$ be a family of sc-Hilbert spaces for $\lambda \in \Lambda$ and $\{k^{i,\lambda}\}$ a family of positive integers for $i \in \mathbb{N}, \lambda \in \Lambda$. One can define Hilbert space F_i as the completion of $\oplus_{\lambda\in\Lambda}\mathbb{E}_i^\lambda$ using the norm $|\sum e^\lambda|_{F_i} := \sum k^{i,\lambda}|e^\lambda|_i$ for $e^\lambda \in \mathbb{E}_i^\lambda$. If $\ldots \subset F_i \subset \ldots \subset F_0$ is a sc-Hilbert space, we use $\widehat{\oplus}_{\lambda\in\Lambda}\mathbb{E}^\lambda$ to denote $\ldots \subset F_i \subset \ldots \subset F_0$. Although $\widehat{\oplus}$ depends on the weights $\{k^{i,\lambda}\}$, we will not emphasis the weights since we only need the fact that $\oplus_{\lambda\in\Lambda}\mathbb{E}^\lambda$ is a subspace of $\widehat{\oplus}_{\lambda\in\Lambda}E^\lambda$ for the arguments in this paper. We point out here that in condition (4) of Definition 6.9 the weights for $W[i]|_{Z^G} = W^G[i]\widehat{\oplus}_{\lambda\in\Lambda}W_\lambda[i]$ could be different for $i = 0$ and 1.*

Remark 6.13. *The condition (11.3) in Definition 6.9 is used in the proof of Proposition 6.15. Such condition is satisfied in all known applications.*

Definition 6.9 without (2) is usually satisfied in applications. The no-isotorpy assumption holds for the S^1-action on the Hamilton Floer homology polyfold, see Section 7 for details.

Proposition 6.14. *Suppose the tubular neighborhood assumption (Definition 6.9) holds, then there exists G-equivariant sc^+ perturbation γ, such that $(s+\gamma)|_{Z^G} : Z^G \to W^G$ is transverse to 0 and $s+\gamma$ is proper. In particular, $(s+\gamma)^{-1}(0) \cap Z^G$ is a compact manifold. Moreover, one can choose γ such that $\operatorname{supp}\gamma$ is contained in arbitrarily small neighborhood of $s^{-1}(0) \cap Z^G$ in Z.*

Proof. By [65, Theorem 5.6], we can find a sc$^+$-perturbation τ such that $s^G+\tau : Z^G \to W^G$ is a proper transverse section. For small neighborhood $U \subset Z$ of $s^{-1}(0)\cap Z^G$, we can assume $\operatorname{supp}\tau$ is contained in $U\cap Z^G$. Using the tubular neighborhood assumptions, $\pi^*\tau$ is a G-equivariant sc$^+$-perturbation on N. By Corollary A.15, there exists a G-invariant bump function $f: N \to [0,1]$ such that $f|_{(s^G+\tau)^{-1}(0)} = 1$ and $\operatorname{supp} f \subset U$. Then $f\pi^*\tau$ is G-equivariant sc$^+$-perturbation on Z and $(s+f\pi^*\tau)|_{Z^G} := s^G+\tau : Z^G \to W^G$ is transverse to 0 and $\gamma := f\pi^*\tau$ is supported in U. Since γ can be chosen to be arbitrarily small in some auxiliary norm ([65, Definition 5.1]) by a similar argument, then $s+\gamma$ is proper by [65, Theorem 5.1] □

Let $x \in Z^G_\infty$ such that $s(x)=0$. Since Z^G and N are M-polyfolds by assumption, we can take linearization of s on Z^G instead of on a polyfold structure. Since s is equivariant, we have

$$\mathrm{D}s_x = \mathrm{D}^G s_x \oplus_{\lambda\in\Lambda} \mathrm{D}^\lambda s_x,$$

where $\mathrm{D}^G s_x$ is the linearization of $s_{Z^G} : Z^G \to W^G$ at x and $\mathrm{D}^\lambda s$ are G-equivariant linear Fredholm operators from $N^\lambda_x \to W^\lambda_x$. To get equivariant transversality near Z^G, we need at least $\operatorname{ind}\mathrm{D}^\lambda s_x \geq 0$ for all λ and $x \in s^{-1}(0)\cap Z^G$. To translate the discussion in the finite dimensional case to the polyfold world, we need to understand $\operatorname{Hom}_G(N^\lambda, W^\lambda)$. In the following, we show that we can find a finite dimensional substitute of $\operatorname{Hom}_G(N^\lambda, W^\lambda)$.

Proposition 6.15. *Under the tubular neighborhood assumption (Definition 6.9), assume $s^{-1}(0)\cap Z^G$ is a compact manifold. Then there is a G-invariant finite-dimensional trivial subbundle $\overline{W}^\lambda \subset W^\lambda_\infty$ over $s^{-1}(0)\cap Z^G$ with constant rank, such that $\overline{W}^\lambda$ covers* $\operatorname{Coker}\mathrm{D}^\lambda s$ *over $s^{-1}(0)\cap Z^G$.*

For every $x \in Z^G_\infty$, there exists sc$^+$ perturbation $\tau : Z^G \to W^G$ such that $s(x)+\tau(x)=0$. Then $f\pi^*\tau$ is a G-equivariant perturbation defined on Z for an equivariant bump function f supported in a neighborhood x. Then we can define $\mathrm{D}^\lambda s_x$ to be $\mathrm{D}^\lambda(s+f\pi^*\tau)_x$, i.e. the λ-component of $\mathrm{D}(s+f\pi^*\tau)_x$. Since $\mathrm{D}^\lambda(f_1\pi^*\tau_1 - f_2\pi^*\tau_2)_x = 0$ for different choices of f and τ, $\mathrm{D}^\lambda s_x$ is well-defined for every $x \in Z^G_\infty$.

Proposition 6.16. *Under the tubular neighborhood assumption (Definition 6.9), we have the following.*

1. *Suppose that $s^G : Z^G \to W^G$ is transverse to 0. Let $\overline{W}^\lambda$ be a trivial bundle asserted in Proposition 6.15, then $\overline{N}^\lambda := (\mathrm{D}^\lambda s)^{-1}(\overline{W}^\lambda)$ is smooth subbundle in N^λ_∞ over $s^{-1}(0)\cap Z^G$. Moreover,* $\operatorname{rank}\overline{N}^\lambda_x - \operatorname{rank}\overline{W}^\lambda_x = \operatorname{ind}\mathrm{D}^\lambda s_x$
2. $\operatorname{ind}\mathrm{D}^\lambda s$ *is locally constant on Z^G_∞.*
3. *For every $k\in\mathbb{N}$, $\{x \in s^{-1}(0)\cap Z^G \,|\, \dim\operatorname{Coker}\mathrm{D}^\lambda s_x \leq k\}$ is an open subset of Z^G_∞.*

We prove Proposition 6.15 and Proposition 6.16 in the appendix.

After applying Proposition 6.16, we choose a G-invariant decomposition $N^\lambda = \overline{N}^\lambda \oplus \underline{N}^\lambda$ by Proposition A.8. Take $\tau^\lambda \in \mathrm{Hom}_G(\overline{N}^\lambda, \overline{W}^\lambda)$, then $\mathrm{D}^\lambda s + \tau^\lambda \circ \pi_{\overline{N}^\lambda} : N^\lambda \to W^\lambda$ is surjective iff $\mathrm{D}^\lambda s|_{\overline{N}^\lambda} + \tau^\gamma : \overline{N}^\lambda \to W^\lambda$ is surjective onto $\overline{W}^\lambda$. Such τ^λ provides the sc^+-perturbation $\tau^\lambda \circ \pi_{\overline{N}^\lambda}$ in the λ-direction.

Theorem 6.17. *Suppose the tubular neighborhood assumption (Definition 6.9) holds. If for all $\lambda \in \Lambda$ and $x \in s^{-1}(0) \cap Z^G$ such that $\mathrm{D}^\lambda s_x$ is not surjective, we have*

$$\mathrm{ind}\, s_x^G < (\frac{\mathrm{ind}\,\mathrm{D}^\lambda s_x}{\dim V^\lambda} + 1) \dim_{\mathbb{R}} \mathrm{End}_G(V^\lambda), \tag{6.2}$$

then there exists a G-invariant sc^+-perturbation γ supported in N, such that $s + \gamma$ is proper and transverse to 0 on Z^G.

Proof. By Proposition 6.14 and Proposition 6.16, we can assume $s^G : Z^G \cap W^G$ is transverse to 0 and (6.2) holds for every $x \in M := s^{-1}(0) \cap Z^G$ such that $\mathrm{D}^\lambda s_x$ is not surjective. By Proposition 6.15, there exists G-equivariant trivial subbundle $\overline{W}^\lambda \subset W^\lambda$ covering the cokernel of $\mathrm{D}^\lambda s$, and $\overline{N}^\lambda := D^\lambda s^{-1}(\overline{N}^\lambda)$ is a finite rank bundle over M. Since the fiber bundle (6.1) in $\mathrm{Hom}_G(\overline{N}^\lambda, \overline{W}^\lambda)$ has codimension $(\frac{\mathrm{ind}\,\mathrm{D}^\lambda s_x}{\dim V^\lambda} + 1) \dim_{\mathbb{R}} \mathrm{End}_G(V^\lambda)$. Therefore we can find a section τ^λ in $\mathrm{Hom}_G(\overline{N}^\lambda, \overline{W}^\lambda) \to M$, such that $\mathrm{D}^\lambda s_x|_{\overline{N}^\lambda} + \tau^\lambda(x)$ is surjective for every point in $x \in M$. Using the projection $N^\lambda \to \overline{N}^\lambda$ guaranteed by Proposition A.8, then $\tau^\lambda \pi_{\overline{N}^\lambda}$ is a sc^∞ bundle map $N^\lambda \to \overline{W}^\lambda \subset W_\lambda[1]$ over M. Since M is the zero set of the transverse sc-Fredholm section $s^G : Z^G \to W^G$, then by [65, Theorem 3.13] $M \subset Z^{S^1}$ is sub-M-polyfold in the sense of [65, Definition 2.12]. That is for every $x \in M$, there exists a neighborhood $V \subset Z^G$ of x and a sc-smooth retraction $r : V \to V$ such that $r(V) = V \cap M$. Then $r^*(\tau^\lambda \circ \pi_{\overline{N}^\lambda})|_{V \cap M} : N^\lambda|_V \to W^\lambda[1]|_V$ defines a sc-smooth bundle map by Proposition 6.19. By a partition of unity argument, we can find a sc-smooth bundle map $\theta^\lambda : N^\lambda|_U \to W^\lambda[1]|_U$ for a neighborhood $U \subset Z^G$ of M. By Corollary A.15 and Corollary A.16, we can assume $\theta^\lambda : N^\lambda|_U \to W^\lambda[1]|_U$ is a G-equivariant bundle map and $\theta^\lambda|_{N^\lambda|_M} = \tau^\lambda \circ \pi_{\overline{N}^\lambda}$. Using the tubular neighborhood assumption, θ^λ can be treated as a sc^+-section of $W \to Z$ on $N|_U \subset Z$. We apply the same procedure to all λ, such that $\mathrm{D}^\lambda s_x$ is not surjective for some $x \in M$. By Proposition 6.16 and the compactness of M, there are only finitely many λ such that $D^\lambda s$ is not surjective for some point $x \in M$. Then $s + f \sum_{\lambda \in \Lambda} \theta^\lambda$ is an equivariant section and transverse on Z^G, where f is a G-invariant bump function on N such that $f|_{M \times \{0\}} = 1$ and $\mathrm{supp}\, f \subset N|_U$. We can chose f such that $f \sum_{\lambda \in \Lambda} \theta^\lambda$ is supported in arbitrarily small neighborhood of $M \times \{0\}$ in Z and arbitrarily small in some auxiliary norm, then $s + f \sum_{\lambda \in \Lambda} \theta^\lambda$ is proper on any closed subset of N. □

In the case of S^1-action, every nontrivial irreducible representation is two-dimensional and classified by a non-negative integer weight. The endomorphism ring is isomorphic to $\mathbb{C}$.

Corollary 6.18. *Assume $G = S^1$ and the tubular neighborhood assumption(Definition 6.9) holds. Suppose for all weights $\lambda \in \mathbb{N}^+$ and $x \in s^{-1}(0) \cap Z^{S^1}$ such that $\mathrm{D}^\lambda s_x$ is not surjective, we have $\operatorname{ind} \mathrm{D}^\lambda s_x + 2 > \operatorname{ind} \mathrm{D}^{S^1} s_x$. Then there exists a S^1-invariant neighborhood $\widehat{Z} \subset Z^3$ containing Z_∞ and an equivariant sc^+-multisection perturbation κon $\widehat{Z}$, such that $s + \kappa$ is proper and in general position.*

Proof. By Theorem 6.17, there is a S^1-equivariant sc$^+$-perturbation γ supported in N such that $s + \gamma$ is transverse to 0 on Z^{S^1}. By [37, Corollary 3.3], there is a S^1-invariant neighborhood $U \subset Z^3$ of Z^{S^1}, such that $s + \tau$ is proper and transverse to 0 on $\overline{U} \subset Z^3$. Let $C := \overline{U} \backslash Z^{S^1}$, then C is a closed subset in $Z^3 \backslash Z^{S^1}$. Since the S^1-action on $Z \backslash Z^{S^1}$ has finite isotropy, by Corollary 6.4 there exists an equivariant transverse sc^+-multisection perturbation τ on $V \subset Z^3 \backslash Z^{S^1}$ containing $Z_\infty \backslash Z_\infty^{S^1}$ such that $\tau|_{C \cap V} = \gamma|_{C \cap V}$ and $s + \tau$ is proper on $Z^3 \backslash U$. Then

$$\kappa(z) = \begin{cases} \gamma(z), & z \in U; \\ \tau(z), & z \in V \backslash U; \end{cases}$$

is an equivariant transverse perturbation defined on the G-invariant open set $\widehat{Z} := V \cup U \subset Z^3$ containing Z_∞. □

The following Proposition is a direct consequence of the chain rule [65, Theorem 1.1].

Proposition 6.19. *Let $(K, \mathbb{R}_+^m \times \mathbb{E}) \times \mathbb{F})$ and $(P, (\mathbb{R}_+^m \times \mathbb{E}) \times \mathbb{H})$ be two tame M-polyfold bundle retracts over one tame M-polyfold retract $(O, \mathbb{R}_+^m \times \mathbb{E})$. Let R_1, R_2 be the bundle retractions defining K and P and assume both R_1 and R_2 cover the tame tractions r on the base. Suppose we have sc-smooth retraction $\pi : O \to O$ and a sc-smooth bundle map $f : K|_\pi(O) \to P|_{\pi(O)}$, then*

$$\pi^* f : K \to P, \quad (x, v) \mapsto R_2(x, \pi_{\mathbb{H}} \circ f(R_1(\pi(x), v)))$$

is sc-smooth bundle map extending f.

Chapter 7

Application: Hamiltonian Floer Homology for Autonomous Hamiltonians

Floer's original proof [40] of the weak Arnold's conjecture was constructing Hamiltonian Floer homology using a C^2 small Morse function as the Hamiltonian. If one can achieve transversality using a time independent almost complex structure for both Floer homology and Morse homology, then there is an identification between Floer chain complex and Morse chain complex. This is because that there is an S^1 action on the Floer trajectories by rotation, then any 0-dimensional moduli space of Floer trajectories must be independent of the S^1 coordinate. Hence they can be identified with the 0-dimensional moduli spaces of gradient trajectories. When sphere bubbles can be excluded, the equivariant transversality was proven in [40, 95] using different strategies. Since polyfold theory provides a framework to deal with the bubbling phenomena, a proof of weak Arnold's conjecture for general symplectic manifolds is essentially setting up the equivariant transversality on the Hamiltonian Floer homology polyfolds. In this section, we give a sketch of such proof.

7.1 Hamiltonian Floer Homology Polyfolds

Let H be a C^2 small Morse function on the closed symplectic manifold (M, ω), such that all the time-1 period orbits of the Hamiltonian vector field X_H are critical points of H. Let $C(H)$ denote the set of critical points of H. We can assume H is also self-indexing in the sense that $H(x) > H(y)$ iff $\operatorname{ind} x > \operatorname{ind} y$ for all $x, y \in C(H)$, where $\operatorname{ind} x, \operatorname{ind} y$ are the Morse-index. We also pick an almost complex structure J such that $\omega(\cdot, J\cdot)$ satisfies Sard-Smale condition for Morse function H. Let $\hbar := \min\{\int_{S^2} u^*\omega | u \text{ is a nontrivial } J \text{ holomorphic sphere}\}$, which is a positive number. We rescale H so that $\max H - \min H < \hbar$.

Assumption 7.1. *We make the following assumptions on the Hamiltonian Floer homology polyfolds.*

1. *For every pair of critical points x, y and a homology class $A \in H_2(M)$ in the image of the Hurewicz map $\pi_2(M) \to H_2(M)$ such that $H(x) - H(y) + \omega(A) > 0$, there is regular tame strong polyfold bundle $p_{x,y,A} : W_{x,y,A} \to Z_{x,y,A}$ with a sc-Fredholm section $s_{x,y,A} : Z_{x,y,A} \to W_{x,y,A}$. Moreover, $\operatorname{ind} s_{x,y,A} = \operatorname{ind} x - \operatorname{ind} y + 2c_1(A) - 1$. The maximal degeneracy index on each polyfold $Z_{x,y,A}$ is finite. $Z_{x,y,0}$ is a M-polyfold for every $x, y \in C(H)$.*

2. *Let $\partial_1 Z$ denote the boundary points with degeneracy index 1 and $\partial_0 Z$ denote the interior points. Then*
$$W_{x,y,A}|_{\partial_1 Z_{x,y,A}} = \bigsqcup_{\substack{z\in C(H)\\ A_0+A_1=A}} W_{x,z,A_0}|_{\partial_0 Z_{x,z,A_0}} \times W_{x,z,A_0}|_{\partial_0 Z_{z,y,A_1}}.$$
Moreover, $s_{x,y,A}|_{\partial_0 Z_{x,z,A_0} \times \partial_0 Z_{z,y,A_1}} = s_{x,z,A_0} \times s_{z,y,A_1}$.

3. *The reparametrization in the S^1 direction gives a S^1-action on $p_{x,y,A}$ such that $s_{x,y,A}$ is S^1-equivariant. The S^1-action on $Z_{x,y,A}$ has finite isotropy unless $A = 0$. The S^1-action on $p_{x,y,0} : W_{x,y,0} \to Z_{x,y,0}$ satisfies the tubular neighborhood assumption (Definition 6.9) when $\operatorname{ind} x - \operatorname{ind} y = 1, 2$. The fixed part $s_{x,y,0}|_{Z^{S^1}_{x,y,0}} : Z^{S^1}_{x,y,0} \to W^{S^1}_{x,y,0}$ is the sc-Fredholm section on the Morse homology M-polyfold for $(H, \omega(\cdot, J\cdot))$. Since we assume the Sard-Smale condition for $(H, \omega(\cdot, J\cdot))$, $s_{x,y,0}|_{Z^{S^1}_{x,y,0}} : Z^{S^1}_{x,y,0} \to W^{S^1}_{x,y,0}$ is transverse to 0.*

4. *For every x, y such that $\operatorname{ind} x - \operatorname{ind} y = 1, 2$ and $i \in \mathbb{N}^+$, $\operatorname{ind} \mathrm{D}^i s_{x,y,0} = 0$ on $s^{-1}_{x,y,0}(0) \cap Z^{S^1}_{x,y,0}$.*

Assumption (1) and (2) are expected to be derived from SFT polyfolds in [62]. Since $\max H - \min H < \hbar$, there is no sphere bundle in $Z_{x,y,0}$ hence $Z_{x,y,0}$ is a M-polyfold. We verify assumption (3) and (4) in Subsection 7.2 for the naive Hamiltonian Floer homology polyfolds, which was described briefly in [102]. Therefore to verify Assumption 7.1, one can either verify assumption (3) and (4) for the SFT polyfolds using a similar argument sketched in Subsection 7.2 or complete the construction of Hamilton Floer homology polyfolds in [102], i.e. include sphere bubbles.

To define the Hamiltonian Floer homology, we need perturbations satisfying the following condition.

Definition 7.2. *A family of perturbations $\{p_{x,y,A}\}$ is transverse and coherent iff*

1. *For all x, y, A, $p_{x,y,A}$ is defined on neighborhood of $(Z_{x,y,A})_\infty$.*

2. *$s_{x,y,A} + p_{x,y,A}$ is transverse to 0 for all x, y, A such that $\operatorname{ind} s_{x,y,A} \le 1$;*

3. *For every $x, y, z \in C(H)$ and $A = A_0 + A_1$, we have*
$$p_{x,z,A} = p_{x,y,A_0} \times p_{y,z,A_1}, \tag{7.1}$$
on a neighborhood of $(Z_{x,y,A_0})_\infty \times (Z_{y,z,A_1})_\infty$ in $Z_{x,y,A_0} \times Z_{y,z,A_1}$.

If we have a transverse and coherent perturbation family $\{p_{x,y,A}\}$, then one can define an operator ∂ on $C_*^{HF} := \oplus_{x\in C(H)}\Lambda\langle x\rangle$, where $\Lambda := \{\sum_{A\in H^2(A)} f_A q^A | \forall K \in \mathbb{R}, \#\{f_A \neq 0 | \omega(A) < K\} < \infty\}$ is the Novikov coefficient.

$$\partial x := \sum_{\substack{y\in C(H),A \\ \mathrm{ind}\, s_{x,y,A}=0}} (\int_{(s_{x,y,A}+p_{x,y,A})^{-1}(0)} 1) q^A y. \tag{7.2}$$

It can be shown that $\partial\circ\partial = 0$. The corresponding homology is expected to be independent of the abstract perturbations and defining data like H and J. The homology is called the Hamiltonian Floer homology $HF_*(M)$. To get coherent perturbations satisfying (7.1), one usually needs to apply some inductive argument. For this purpose, we need to extend a sc^+ perturbation from the boundary. We assume the following for now.

Assumption 7.3. *A sc^+-multisection on the boundary ∂Z can be extended to Z.*

The induction is usually based on the maximal degeneracy index of the polyfold, i.e. we first find perturbations on those polyfolds without boundary and we choose the perturbations to be transverse if the index is not bigger than 1 by [65, Theorem 15.4]. Then by coherent condition (7.1), we have perturbations on the boundaries of polyfolds with maximal degeneracy index 1. Then one can extend the perturbation from the boundary into the interior. We can also require transversality if the index is smaller than 1. To see such requirement is feasible, note that on $Z_{x,y,A_0}\times Z_{y,z,A_1}\subset \partial Z_{x,z,A_0+A_1}$ we have $\mathrm{ind}_{x,y,A_0}+\mathrm{ind}_{y,z,A_1} = \mathrm{ind}_{x,z,A_0+A_1} - 1$. Hence if $\mathrm{ind}_{x,z,A_0+A_1}\leq 1$, then either both ind_{x,y,A_0} and ind_{y,z,A_1} are zero or one of them is negative. In either case, by induction hypothesis, $p_{x,y,A_0}\times p_{x,y,A_1}$ is a transverse perturbation on $Z_{x,y,A_0}\times Z_{y,z,A_1}$. Therefore by Assumption 7.3 and th proof of [65, Theorem 15.5], one can find transverse extension in the interior. The induction argument goes on with respect to the maximal degeneracy index.

Theorem 7.4. *Suppose Assumption 7.1 and Assumption 7.3 hold, then there exists a transverse and coherent perturbation family $\{p_{x,y,A}\}$ such that $p_{x,y,A}$ is S^1-equivariant for all x, y, A.*

As a corollary of Theorem 7.4, one have the following.

Corollary 7.5. *Using the perturbations in Theorem 7.4, let ∂_M denote the Morse differential, i.e.*

$$\partial_M x := \sum_{\substack{y\in C(H) \\ \mathrm{ind}\, x - \mathrm{ind}\, y=1}} \#(s_{x,y,0}|_{Z^{S^1}_{x,y,0}})^{-1}(0) y.$$

Then the differential in (7.2) satisfies $\partial x = \partial_M x$. In particular, we have $HF^*(M) \cong H^*(M)\otimes\Lambda$ and the weak Arnold conjecture holds.

Proof. For every x, y, A such that $\mathrm{ind}_{x,y,A} = 0$, if $A \neq 0$ then equivariant transversality implies that $(s_{x,y,A} + p_{x,y,A})^{-1}(0) = \emptyset$; if $A = 0$ then equivariant transversality implies that $(s_{x,y,0} + p_{x,y,0})^{-1}(0) = (s_{x,y,0}|_{Z^{S^1}_{x,y,0}})^{-1}(0)$. Hence $\partial x = \partial_M x$ and the remaining claim follows. $\square$

Proof of Theorem 7.4. We apply the induction argument to find such perturbations. The self-index condition implies that $Z_{x,y,0} = \emptyset$ for every x, y such that $\mathrm{ind}\, x \leq \mathrm{ind}\, y$. We find sc$^+$-perturbations on polyfold $Z_{x,y,A}$ without boundary, such that the following properties hold.

1. By Corollary 6.4, if $A \neq 0$ then we can assume $s_{x,y,A} + p_{x,y,A}$ is S^1-equivariant and is transverse to 0.

2. By Corollary 6.18, If $A = 0$, we can assume $s_{x,y,0} + p_{x,y,0}$ is S^1-equivariant and is transverse to 0 if $\mathrm{ind}\, s_{x,y,0} = 0, 1$.

3. For other cases, we simply require $p_{x,y,0}$ is S^1-equivariant, for example $p_{x,y,0} = 0$.

Assume we have found perturbations satisfying the properties in Theorem 7.4 for all polyfolds with maximal degeneracy index at most k. We will prove that we can perturbations satisfying the properties in Theorem 7.4 on for all polyfolds $Z_{x,y,A}$ with maximal degeneracy index $k+1$.

1. If $\mathrm{ind}\, s_{x,y,A} > 1$, if $A \neq 0$ then the S^1-action on $Z_{x,y,A}$ has no fixed point. We can use quotient to find a S^1-equivariant extension. If $A = 0$, then $Z_{x,y,0}$ is a M-polyfold, then we can find one extension and by Lemma A.13 we can use an averaging argument to find a S^1-equivariant extension.

2. If $\mathrm{ind}_{x,y,A} \leq 1$ and $A \neq 0$, since the S^1-action on the boundary $\partial Z_{x,y,A}$ has no fixed locus and by induction hypothesis the perturbation on the boundary is transverse, by Corollary 6.4 and the extension Theorem [65, Theorem 15.5], we can assume $s_{x,y,A} + p_{x,y,A}$ is S^1-equivariant and transverse to 0.

3. For the remaining case, i.e. $\mathrm{ind}\, s_{x,y,0} = 0, 1$, by Corollary 6.18 and [65, Theorem 15.5], we can assume $s_{x,y,0} + p_{x,y,0}$ is S^1-equivariant and transverse to 0.

Hence Theorem 7.4 holds by induction. $\square$

The extension process of sc$^+$-perturbations was discussed in [65, Chapter 14]. [65, Theorem 14.1] showed the existence of extensions for structurable sc$^+$-multisections and the extensions can be chosen to be structurable, where structurable multisections were defined in [65, Definition 13.17]. Then Assumption 7.3 in Theorem 7.4 can be dropped, if the following statement holds.

Conjecture 7.6. *Under the conditions of Theorem 1.1, let $\mathfrak{q} : p^{-1}(\widehat{Z}) \to p^{-1}(\widehat{Z})/G$ be the quotient map.*

1. *If κ is structurable* sc^+*-multisection on $\widehat{Z}/G$, then $\mathfrak{q}^*\kappa$ is structurable.*

2. *If κ is G-equivariant structurable* sc^+*-multisection on $\widehat{Z}$, then κ induces a structurable* sc^+*-multisection on $\widehat{Z}/G$ by $\kappa = \mathfrak{q}^*\overline{\kappa}$.*

7.2 Assumption 7.1 in the naive construction

In stead of using the polyfolds for SFT, one can construct the Hamiltonian Floer homology polyfolds in a more direct way. Part of this construction can be found in [102].

Let γ be a Morse flow line connecting x, y such that $\operatorname{ind} x - \operatorname{ind} y = 1$, then a neighborhood of γ in $Z^{S^1}_{x,y,0}$ is modeled on:

$$\mathbb{H}_\gamma = \{u \in H^{3+i}_{\delta_i}(\mathbb{R}, \gamma^*TM) | u(0) \in H\}$$

with chart map:

$$u \in \mathbb{H}_\gamma \to \exp_\gamma(u)$$

where H is a small hyperplane in $T_{\gamma(0)}M$ transversing to γ at $\gamma(0)$, and δ_i is a small weight for the exponential decay at two ends. Similarly, there is a polyfold $Z_{x,y,0}$ of Floer flow lines connecting x, y. There is an S^1 action by reparametrization in the S^1 direction on the cylinder part. The sc-smoothness of this action is checked in [35, 58]. In particular, the fixed set is exactly the flow lines constant in the S^1 direction without any bubbling. To verify Assumption 7.1, we need to understand the local picture of the polyfold near the fixed locus.

Like the Morse theory case, a neighborhood of γ in $Z_{x,y,0}$ is modeled on:

$$u \in \overline{\mathbb{H}}^F_\gamma = \{u \in H^{3+i}_{\delta_i}(\mathbb{R} \times S^1, \gamma^*TM) | u(0,0) \in H\}$$

with chart map

$$u \in \overline{\mathbb{H}}^F_\gamma \to \exp_\gamma u$$

This is not very good for our purpose, instead, we consider the following sc-Hilbert space:

$$\mathbb{H}^F_\gamma = \{u \in H^{3+i}_{\delta_i}(\mathbb{R} \times S^1, \gamma^*TM) | \int_{S^1} u(0,s)ds \in H\}$$

with chart map:

$$u \in \mathbb{H}^F_\gamma \to \exp_\gamma u$$

These two local charts are compatible. One easy way to see it is to apply the proof of theorem 3.1 to the M-polyfolds of parametrized flow lines with translation $\mathbb{R}$ action, since both models can be viewed as a slice to the $\mathbb{R}$-action

The S^1-action on $Z_{x,y,0}$ gives rise to the decomposition:

$$\mathbb{H}^F_\gamma = \mathbb{H}_\gamma \widehat{\oplus}_{i \in \mathbb{N}^+} (\mathbb{H} \otimes \langle \sin it, \cos it \rangle),$$

where $\mathbb{H}$ is $\mathbb{H}_\gamma$ dropping the hyperplane constraint. $\widehat{\oplus}$ needs some explanation: write $\mathbb{E}^{(0)} = \mathbb{H}_\gamma$, $\mathbb{E}^{(i)} = \mathbb{H} \otimes \langle \sin it, \cos it \rangle$, the sc-structure on $\mathbb{H}$ induces *sc*-structure on $\mathbb{E}^{(i)}$. $\widehat{\oplus}$ means the m-th level is a completion of the direct sum with respect to norm induced by the bilinear form

$$\langle (e_0, e_1, \ldots), (f_0, f_1, \ldots) \rangle_m = \sum \langle e_i, f_i \rangle_{\mathbb{E}^{(i)}_m} + \sum i^{2m} \langle e_i, f_i \rangle_{\mathbb{E}^{(i)}_0}.$$

Similar construction exists in the fiber direction, the only difference is index shifted by one and the hyperplane condition is dropped completely. The fixed locus is exactly the M-polyfold in Morse theory, hence tubular neighborhood assumptions near the fixed locus without isotropy holds.

The linearized operator of the Floer equation at γ is given by

$$\mathrm{D}_\gamma \xi = \nabla_s \xi + J \nabla_t \xi + \nabla_\xi \operatorname{grad} H$$

where ∇ is the Levi-Civita connection for metric $\omega(\cdot, J\cdot)$. After a trivialization of $\gamma^* TM$, one can think of ξ as a map from $\mathbb{R} \times S^1$ to $\mathbb{C}^n$ and

$$\mathrm{D}_\gamma \xi = \partial_s \xi + i \partial_t \xi + A(s) \xi$$

where $A(s)$ depends on C^2 norm of H. Since we can write

$$\xi(s,t) = \xi_0(s) + \sum_{n=1}^{\infty} (\xi_n(s) \sin nt + \eta_n(s) \cos nt),$$

we have

$$\begin{aligned} \mathrm{D}_\gamma \xi &= \mathrm{D}^0_\gamma \xi_0 + \sum_{n=1}^{\infty} (\xi_n'(s) \sin nt + \eta_n'(s) \cos nt + i \xi_n(s) n \cos nt - i \eta_n(s) n \sin nt) \\ &\quad + A(s)(\xi_n \sin nt + \eta_n \cos nt), \end{aligned}$$

where D^0_γ is the linearized operator of gradient flow equation. Therefore

$$D^n_\gamma(\xi_n, \eta_n) = (\xi_n' - in\eta_n + A(s)\xi_n, \eta_n' + in\xi_n + A(s)\eta_n)$$

When H is C^2 small, $A(s)$ is small in C^0 norm, then D^n_γ has index 0 [91]. This verifies condition (4) of Assumption 7.1 for x, y such that $\operatorname{ind} x - \operatorname{ind} y = 1$. Similar argument can be used to verify the case of $\operatorname{ind} x - \operatorname{ind} y = 2$.

Part II

Equivariant Fundamental Class and Localization

Chapter 8

Equivariant de Rham Theory on Polyfolds

8.1 De Rham theory on polyfolds

Differential forms on M-polyfolds

We will review the de Rham theory and integration theory on polyfold from [63] and [65]. Recall that the tangent space $T\mathbb{E}$ of an sc-Banach space $\mathbb{E}$ is $\mathbb{E}^1 \oplus \mathbb{E}$. In general, we have the tangent bundle $T\mathcal{X} \to \mathcal{X}^1$. Since $\mathcal{X}^{i+1} \subset \mathcal{X}^i$ is an sc^∞ map, we can pull back the tangent bundle to get a M-polyfold bundle $T\mathcal{X}^{i-1}|_{\mathcal{X}_{i+1}} \to \mathcal{X}^{i+1}$ with a sc^∞ embedding $T\mathcal{X}^{i-1}|_{\mathcal{X}^{i+1}} \subset T\mathcal{X}^{i-1}$. In general, we can define $T\mathcal{X}^j|_{\mathcal{X}^i}$ for all $j < i$. When $j = i-1$, it is the usual tangent bundle $T\mathcal{X}^{i-1}$. Moreover, there are sc^∞ embedding $T\mathcal{X}^j|_{\mathcal{X}^i} \subset T\mathcal{X}^k|_{\mathcal{X}^l}$, whenever $j > k, i > l$.

Definition 8.1 ([65, definition 4.9]). *A sc-differential k form of class $(i,j), i > j$ form on M-polyfold $\mathcal{X}$ is a sc-smooth map $\omega : \oplus_k T\mathcal{X}^j|_{\mathcal{X}^i} \to \mathbb{R}$, which is linear in each argument separately and skew-symmetric. Let $\Omega^k(\mathcal{X}^{i,j})$ denote the space of all such forms.*

When $i > l, j > m$, there is a natural inclusion $\Omega^*(\mathcal{X}^{i,j}) \to \Omega^*(\mathcal{X}^{l,m})$ induced by the sc-smooth inclusion $\oplus_k T\mathcal{X}^j|_{\mathcal{X}^i} \subset \oplus_k T\mathcal{X}^m|_{\mathcal{X}^l}$. Thus We define:

$$\Omega^*_\infty(\mathcal{X}) = \varinjlim \Omega^*(\mathcal{X}^{i,j})$$

Locally the exterior differential is defined just like the normal case:

$$\begin{aligned} \mathrm{d}\omega(A_0,\ldots,A_k) &= \sum(-1)^i \mathrm{D}(\omega(A_0,\ldots,\hat{A}_i,\ldots,A_k))A_i \\ &+ \sum_{i<j}(-1)^{i+j}\omega([A_i,A_j],A_0,\ldots,\hat{A}_i\ldots\hat{A}_j,\ldots,A_k) \end{aligned}$$

The Lie bracket will need a little more differentiability to define, hence the exterior differential d will send $\Omega^*(\mathcal{X}^{i,j})$ to another space with higher index, details are can be found in [63], we also include the discussion in the appendix. The upshot is over $\Omega^*_\infty(\mathcal{X})$, d is well defined and $\mathrm{d}^2 = 0$, and the following functorial property holds:

Proposition 8.2. *If $f : \mathcal{X} \to \mathcal{Y}$ is a sc-smooth map between two M-polyfolds, then there is a functorial pull back on differential forms $f^* : \Omega^*_\infty(\mathcal{Y}) \to \Omega^*_\infty(\mathcal{X})$, such $f^*\mathrm{d} = \mathrm{d}f^*$.*

Remark 8.3. *[63] uses only $\Omega^*(\mathcal{X}^{i+1,i})$, since $\{(i+1,i)\}$ are co final in the directed system, thus two definitions give same space.*

Differential forms as sections of a sheaf

For our purpose, we need to modify the definition of differential forms. We want to show the equivariant counterpart can be defined through Borel construction as well as the G^* algebra structure. We would like to interpolate the cohomology as a sheaf cohomology. Although, such complication may not be necessary for applications, c.f. remark 9.3.

Since the quotient Theorem 1.1 only asserts polyfold structures on a dense set on level 3 containing the ∞ level, we will modify the definition to make a sheaf of differential forms on $\mathcal{X}_\infty$. Let τ_i be the $\mathcal{X}_i$ topology. Let $k \geq i, k > l$, we have a sheaf $\Omega^*_{\infty,\tau_i,k,l}(\cdot)$ on $\mathcal{X}_\infty$ with the τ_i topology defined by:

$$\Omega^*_{\infty,\tau_i,k,l}(\mathcal{O}) = \varinjlim_{\mathcal{R}} \Omega^*(\mathcal{R}^{k,l}),$$

where $\mathcal{R}$ is an open set in $\mathcal{X}_i$ such that $\mathcal{R} \cap \mathcal{X}_\infty = \mathcal{O}$. To see it defines a sheaf, one only needs to show that one can glue up local sections once they give the same intersection sections. Assume $\omega_\alpha \in \Omega^*_{\infty,\tau_i,k,l}(\mathcal{O}_\alpha)$ for $\alpha \in A$, such that $\omega_\alpha = \omega_\beta$ on $\mathcal{O}_{\alpha\beta} := \mathcal{O}_\alpha \cap \mathcal{O}_\beta$ for every α, β. Then we can find open sets $\mathcal{R}_\alpha, \mathcal{R}_\beta$ and $\mathcal{R}_{\alpha\beta}$ in $\mathcal{X}_i$, such that $\mathcal{O}_\alpha = \mathcal{R}_\alpha \cap \mathcal{X}_\infty, \mathcal{O}_\beta = \mathcal{R}_\beta \cap \mathcal{X}_\infty, , \mathcal{O}_{\alpha\beta} = \mathcal{R}_{\alpha\beta} \cap \mathcal{X}_\infty$ and $\omega_\alpha, \omega_\beta$ are defined on $\mathcal{R}_\alpha, \mathcal{R}_\beta$. Moreover, $\omega_\alpha = \omega_\beta$ on $\mathcal{R}_{\alpha\beta}$. Although we can not assume $\mathcal{R}_{\alpha\beta} = \mathcal{R}_\alpha \cap \mathcal{R}_\beta$, but we can assume $\mathcal{R}_{\alpha\beta} \subset \mathcal{R}_\alpha \cap \mathcal{R}_\beta$. Since $\mathcal{R}_\alpha \cap \mathcal{R}_\beta$ and $\mathcal{R}_{\alpha\beta}$ contain $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ as dense subsets, $\omega_\alpha = \omega_\beta$ over $\mathcal{R}_{\alpha\beta}$ implies $\omega_\alpha = \omega_\beta$ over $\mathcal{R}_\alpha \cap \mathcal{R}_\beta$. Therefore ω_α defines a unique form over $\cup_\alpha \mathcal{R}_\alpha$. Hence $\Omega^*_{\infty,\tau_i,k,l}$ is a sheaf.

Then we have the direct limit of the sheaves[1] $\varinjlim_{k,l} \Omega^*_{\infty,\tau_i,k,l}$, denote it by Ω^*_{∞,τ_i}. Then we can define the space of differential forms by

$$\Omega^*(\mathcal{X}, \tau_i) := \Omega^*_{\infty,\tau_i}(\mathcal{X}_\infty)$$

It is not hard to see the exterior differential is defined on $\Omega^*(\mathcal{X}, \tau_i)$, making $(\Omega^*(\mathcal{X}, \tau_i), \mathrm{d})$ into a cochain complex. The cohomology is denoted by $H^*(\mathcal{X}, \tau_i)$.

Assume the M-polyfold is built on sc-Hilbert spaces. By the partition of unity [60, Theorem 5.8], then $\Omega^*_{\infty,\tau_i}(\cdot)$ is a fine sheaf. And by the Poincaré lemma proved in [60] for $x \in \mathcal{X}_\infty$. Therefore

$$\underline{\mathbb{R}} \to \Omega^0_{\infty,\tau_i} \to \Omega^1_{\infty,\tau_i} \to \dots$$

[1] i.e. the sheafification of the direct limit presheaf.

is a resolution by acyclic sheaves. By [49, Théorème 5.10.1], the sheaf cohomology is equivalent to Čech cohomology for paracompact spaces, thus $H^*(\mathcal{X},\tau_i) = \check{H}^*(\mathcal{X}_\infty,\tau_i,\mathbb{R})$. Since M-polyfold is locally contractible, thus $H^*(\mathcal{X},\tau_i)$ is isomorphic to the singular cohomology $H^*_{\text{sing}}(\mathcal{X}_\infty,\tau_i)$.

Since there are level shifts in the quotient construction, we prefer a theory without the level topology τ_i. Therefore we define:

$$\Omega^*(\mathcal{X},\tau_\infty) := \varinjlim \Omega^*(\mathcal{X},\tau_i).$$

Moreover, we have the following sequence of cochain complexes, which is also functorial:

$$\Omega^*_\infty(\mathcal{X}) \longrightarrow \Omega^*(\mathcal{X},\tau_0) \longrightarrow \Omega^*(\mathcal{X},\tau_1) \longrightarrow \ldots \longrightarrow \Omega^*(\mathcal{X},\tau_\infty) \tag{8.1}$$

Differential forms on polyfolds

Since polyfolds can be viewed analogs of orbifolds, the differential forms should be those invariant under the local symmetries. Let $(\mathcal{X},\boldsymbol{X})$ be an ep-groupoid. Recall that every $\phi\in\boldsymbol{X}$ induces a sc-diffeomorphism L_ϕ from a neighborhood of the source $s(\phi)$ to a neighborhood of the target $t(\phi)$.

Definition 8.4 ([65, Definition 8.2])**.** *A differential form of class (i,j) on $(\mathcal{X},\boldsymbol{X})$ is a differential form ω on $\mathcal{X}$ of class (i,j), such that for every $\phi\in\boldsymbol{X}$, $L^*_\phi\omega_{t(\phi)} = \omega_{s(\phi)}$. The space of differential forms of class (i,j) is denoted by $\Omega^*((\mathcal{X},\boldsymbol{X})^{i,j})$*

Definition 8.5. *Let Z be a polyfold and $(\mathcal{X},\boldsymbol{X})$ a polyfold structure with $Z=|\mathcal{X}|$. Let τ_i denotes the Z_i topology. For $k\geq i$ and $k>l$, we define a sheaf $\Omega^*_{\infty,\tau_i,k,l}$ on Z_∞ with τ_i topology to be*

$$\Omega^*_{\infty,\tau_i,k,l}(U) := \varinjlim_{(\mathcal{R},\boldsymbol{R})} \Omega^*((\mathcal{R},\boldsymbol{R})^{i,j}),$$

where $\mathcal{R}$ is an open subset of $\mathcal{X}^i$ and $(\mathcal{R},\boldsymbol{R})$ is a full subcategory of $(\mathcal{X}^i,\boldsymbol{X}^i)$ such that $|\mathcal{R}|\cap Z_\infty = U$. Then we define

$$\Omega^*_{\infty,\tau_i} := \varinjlim_{k,l} \Omega^*_{\infty,\tau_i,k,l}$$

The exterior differential d is defined on Ω^*_{∞,τ_i} satisfying $\mathrm{d}\circ\mathrm{d}=0$. Definition 8.5 makes sense because of the following theorem, i.e. independent of the polyfold structure.

Theorem 8.6. *[65, Theorem 11.2] Let $\mathfrak{h}:(\mathcal{X},\boldsymbol{X})\to(\mathcal{Y},\boldsymbol{Y})$ be a generalized isomorphism between ep-groupoids. Then $\mathfrak{h}$ induces pushforward $\mathfrak{h}_*:\Omega^*((\mathcal{X},\boldsymbol{X}),\tau_i)\to\Omega^*((\mathcal{Y},\boldsymbol{Y}),\tau_i)$ and pullback $\mathfrak{h}^*:\Omega^*((\mathcal{Y},\boldsymbol{Y}),\tau_i)\to\Omega^*((\mathcal{X},\boldsymbol{X}),\tau_i)$ such that $\mathfrak{h}^*=\mathfrak{h}_*^{-1}$.*

We define $\Omega^*_\infty(Z) = \varinjlim \Omega^*(Z^{i,j})$. Then we define $\Omega^*(Z,\tau_i) = \Omega^*_{\infty,\tau_i}(Z_\infty)$ and $\Omega^*(Z,\tau_\infty) = \varinjlim \Omega^*(Z,\tau_i)$. The cohomology are denoted by $H^*(Z,\tau_i)$. We also have the following sequence:

$$\Omega^*_\infty(Z) \longrightarrow \Omega^*(Z,\tau_0) \longrightarrow \Omega^*(Z,\tau_1) \longrightarrow \ldots \longrightarrow \Omega^*(Z,\tau_\infty). \tag{8.2}$$

Since polyfolds are paracompact, $H^*(Z,\tau_i) = \check{H}^*(Z_\infty,\tau_i,\mathbb{R})$. The relation with singular cohomology depends on whether (Z_∞,τ_i) is locally contractible, which relies on the understanding of the finite group action on an M-polyfold.

Integration theory

In the case of M-polyfold [59], let $\alpha \in \Omega^*(\mathcal{X},\tau_i)$, one can integrate α over a smooth manifold $M \subset \mathcal{X}_\infty$, which usually comes from zero set of a transverse Fredholm section. The integration commutes with (8.1), moreover Stokes' theorem holds for this integration theory.

In the case of polyfold, there is also a well-defined integration theory [63] of differential forms on the orbit space of the branched ep-groupoid $\Theta : (\mathcal{X},\boldsymbol{X}) \to \mathbb{Q}^+$. The integration theory is compatible with the sequence (8.2), and is also invariant under the equivalence of polyfold structures and the Stoke's theorem holds.

Definition 8.7 (definition 1.19 [63])**.** *A branched ep-subgroupoid of the ep-groupoid $(\mathcal{X},\boldsymbol{X})$ is a functor to $\mathbb{Q}^+$ having the following properties:*

1. *The support of Θ defined by $\operatorname{supp}\Theta = \{x \in \mathcal{X} | \Theta(x) > 0\}$ is contained in $\mathcal{X}_\infty$*
2. *Every point $x \in \operatorname{supp}\Theta$ is contained a uniformizer around x, such that $\operatorname{supp}\Theta \cap \mathcal{U} = \cup_{i\in I} M_i$ where I is a finite index set and M_i are finite dimensional submanifolds of $\mathcal{X}$ in good position with respect to the boundary of $\partial\mathcal{X}$.*
3. *There exist positive rational number $\sigma_i, i \in I$ such that if $y \in \operatorname{supp}\Theta \cap \mathcal{U}$, $\Theta(y) = \sum_{\{i|y\in M_i\}} \sigma_i$*
4. *$M_i \to \mathcal{U}$ is proper.*

The integration is defined as follows, pick a finite open cover $\{|\mathcal{U}_\alpha|\}_{\alpha\in A}$ of $|\operatorname{supp}\Theta|$ and a partition $\{\phi_\alpha\}$ of unity subordinate to the open cover, where $\mathcal{U}_\alpha$ are local uniformizers around $\alpha \in \mathcal{X}$ with decompositions of $\operatorname{supp}\Theta$. The integration of ω is defined to be

$$\int_\Theta \omega := \sum_{\alpha\in A} \frac{1}{|\operatorname{stab}^{\text{eff}}_\alpha|} \sum_{i\in I_\alpha} \sigma_i \int_{M_i} \phi_\alpha \omega.$$

8.2 Borel construction

Given a compact Lie group G, one can find an embedding $G \subset U(n)$. By $H(m,n)$, we mean the compact orientable smooth manifold of n orthogonal vectors in $\mathbb{C}^m$. Then $U(n)$ acts on $H(m,n)$ with quotient $\operatorname{Gr}(m,n)$, they serve as a finite dimensional approximation of the classifying principle bundle $EU(n) \to BU(n)$. Then $EG \to BG$ can be approximated by $H(m,n) \to H(m,n)/G$. If we G acts on finite-dimensional smooth manifold M smoothly, then one can define equivariant cohomology by $H^*_G(M) := H^*(M \times_G EG) = \varprojlim H^*(M \times_G H(m,n))$. We first clarify the meaning of approximation of classifying spaces.

Definition 8.8. *A chain of smooth maps $E_0 \to E_1 \to \dots$ between finite dimensional closed manifold is called* ***approximation to the classifying space $EG \to BG$****, if the following conditions hold.*

1. *E_n are principle G-bundles and $E_n \to E_{n+1}$ are G-equivariant embeddings.*

2. *For all k, there exists N_k, such that for $n > N_k$ E_n is $k-$connected.*

3. *$\Omega(E) = \varprojlim \Omega^*(E_n)$ satisfies condition (C), see Definition 8.12.*

$H(m,n)$ is indeed an approximation of BG [53]. Given any approximation $E_0 \to E_1 \dots$ of $EG \to BG$, we have $H^*_G(M) := H^*(M \times_G EG) = \varprojlim H^*(M \times_G E_n)$.

In the polyfold case, one also would like to approximate the homotopy quotient of the polyfold, i.e. one needs to make $Z \times_G E_n$ a polyfold. Since the group action G on $Z \times E_n$ is free, we can apply Theorem 1.1. Therefore there exists a G-invariant open set R_n of $Z^3 \times \mathbb{E}_n$ containing $Z_\infty \times E_n$, such that R_n/G is a polyfold. Therefore it makes sense to talk about $\Omega^*(Z_\infty \times_G E_n, \tau_i), 3 \le i \le \infty$. Moreover, the inclusion $i : Z \times E_n \subset Z \times E_{n+1}$ induces a map $i^* : \Omega^*(Z_\infty \times_G E_{n+1}, \tau_i) \to \Omega^*(Z_\infty \times_G E_n, \tau_i)$. Then we can define

$$\Omega^*(Z_\infty \times_G E, \tau_i) := \varprojlim \Omega^*(Z_\infty \times_G E_n, \tau_i), 3 \le i \le \infty,$$

and define $H^*_G(Z, \tau_i) := H^*(\Omega^*(Z_\infty \times_G E, \tau_i))$. Since i^* is surjective, the following $\varprojlim^1$ exact sequence (c.f.[104]) holds:

$$0 \to \varprojlim{}^1 H^{p-1}(Z_\infty \times_G E_n, \tau_i) \to H^p(Z \times_G E, \tau_i) \to \varprojlim H^p(Z_\infty \times_G E_n, \tau_i) \to 0. \tag{8.3}$$

The sequence (8.2) then induces the following

$$H^*_G(Z, \tau_3) \to \dots \to H^*_G(Z, \tau_\infty). \tag{8.4}$$

8.3 Equivariant Cohomology from G^* Module

To see the definition of equivariant cohomology is actually independent of the approximation, we recall the algebraic treatment from [53] to give an alternative definition which does not involve the approximation. This section is essentially from [53]. For a Lie algebra $\mathfrak{g}$, we can associate it with a Lie superalgebra $\tilde{\mathfrak{g}} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Choose a basis $\xi_1, \dots, \xi_n$ of $\mathfrak{g}$, with structure constants c^k_{ij}. Then $\mathfrak{g}_0$ is generated by $L_{\xi_1}, \dots, L_{\xi_n}$ (Lie derivative), $\mathfrak{g}_{-1}$ is generated by $\iota_{\xi_1}, \dots, \iota_{\xi_n}$(interior multiplication), and $\mathfrak{g}_1$ is generated by d (exterior differential). Such that the super Lie bracket is defined as follows:

$$\begin{array}{rclrclrcl} [\iota_i, \iota_j] &=& 0, & [L_i, \iota_j] &=& c^k_{ij}\iota_k, & [L_i, L_j] &=& c^k_{ij}L_k, \\ [d, \iota_i] &=& 0, & [d, L_i] &=& 0, & [d, d] &=& 0. \end{array}$$

Definition 8.9 (G^* Algebra/Module). *A **G^* algebra/module** is a commutative superalgebra/module A, together with a representation ρ of G on A and an action of $\tilde{\mathfrak{g}}$ as superderivation on A. Moreover, the structures are consistent in the sense that*

$$\begin{aligned} \tfrac{d}{dt}\rho(\exp t\xi)|_{t=0} &= L_\xi; \\ \rho(a)L_\xi\rho(a^{-1}) &= L_{Ad_a\xi}; \\ \rho(a)\iota_\xi\iota(a^{-1}) &= \iota_{Ad_a\xi}; \\ \rho(a)d\rho(a^{-1}) &= d. \end{aligned}$$

A G^* morphism is an algebra/module morphism commuting with all the actions.

Remark 8.10. *The derivative in the first equation only needs to exist in a very week sense: e.g. A has a topology to make sense of it, or element in A is G finite, i.e. the space spanned by the orbit is finite dimensional. The upshot is this derivative should be linear and zero over constant path, see [53] for more detail.*

If G acts on manifold M, then $\Omega^*(M)$ is equipped with a G^* algebra structure. Let $\xi \in \mathfrak{g}$ and$\xi_M \subset \Gamma(TM)$ is defined to be $\frac{d}{dt}|_{t=0}\exp(-t\xi)$. Then $L_\xi\alpha := L_{\xi_M}\alpha$, $\iota_\xi\alpha := \iota_{\xi_M}\alpha$, d is the exterior differential and $\rho(g)\alpha := (g^{-1})^*\alpha$ give a G^* algebra structure on $\Omega^*(M)$.

Proposition 8.11. *If G acts sc-smoothly on polyfold Z, then $\Omega^*_\infty(Z)$, $\Omega^*(Z,\tau_i)$, Ω^*_{Z,τ_∞} are G^* algebras. $\Omega^*(Z,\tau_i) \to \Omega^*(Z,\tau_j)$ is a G^* algebra morphism for $i \leq j$. And any equivariant polyfold morphism $F : Z \to Y$ induces G^* algebra morphism $F^* : \Omega^*(Y,\tau_i) \to \Omega^*(Z,\tau_i)$.*

Definition 8.12. *A G^* algebra is said to be of type (C), if there exist $\theta^i \in A_1$ (connection elements), such that $\iota_j\theta^i = \delta^i_j$ and the space spanned by θ^i is $G-$invariant.*

For a G^* module A, we can define equivariant cohomology $H_G(A)$ by picking an acyclic G^* module E satisfying condition (C), and taking cohomology of $(A \otimes E)_{\mathrm{bas}}$, where subscript bas stands for basic complex, meaning those annihilated by ι and G invariant. If one can make sense of averaging over G to get an invariant element in A. then the following is proven in [53]. The averaging process is allowed in the case of polyfold, c.f. Appendix B.

Theorem 8.13. *$H_G(A)$ does not depend on the choice of E*

One of the most important examples of acyclic G^* algebra satisfying condition (C) is the Weil algebra, which is also the “smallest” one.

Definition 8.14 (Weil Algebra). *$W := \bigwedge \mathfrak{g}^* \otimes S(\mathfrak{g}^*)$, where in the wedge product the generator θ^i has grading 1 and the generator (linear polynomial) z^k in the symmetric product has grading 2. The action of G is given by dual adjoint action ad^*. Since $\tilde{\mathfrak{g}}$ should acts as* *derivation, it's enough to specify the action on the generators:*

$$\begin{aligned} L_i\theta^j &= -c^j_{ik}\theta^k; \\ L_i z^j &= -c^j_{il} z^k; \\ \iota_i\theta^j &= \delta^j_i; \\ \iota_i z^j &= L_i\theta^j; \\ d\theta^i &= z^i; \\ dz^i &= 0; \end{aligned}$$

Being the smallest G^* algebra can be made precise by showing every G^* algebra satisfying condition (C) maps into a W^* module

Definition 8.15 (W^* Module)**.** *A is G^* module and there is a G^* module homomorphism $W \otimes A \to A$, then A is called a W^* module.*

If A is a W^* module, then $H_G(A) = H(A_{\text{bas}})$.

Definition 8.16. *Let W be the Weil algebra for G and A a G^* algebra/module. Then **Weil model** is the chain complex $((A \otimes W)_{bas}, d)$.*

Another model which will compute the same cohomology is as follows.

Definition 8.17. *Let $S(\mathfrak{g}^*)$ be the ring of polynomial over $\mathfrak{g}^*$, then **Cartan model** $(S(\mathfrak{g}^*)\otimes A)^G$ is the G invariant A valued polynomial over $\mathfrak{g}^*$. The differential $d_G = d_A - z^i \otimes \iota_i$.*

These two models are related by Mathai-Quillen isomorphism. There is a spectral sequence associated to Cartan model, such that $E_0^{p,q} = (S^p(\mathfrak{g}^*)\otimes A^{q-p})^G$. The spectral sequence converges to $H_G(A)$ and the first page is $E_1^{p,q} = (S^p(\mathfrak{g}^*) \otimes H^{q-p}(A))^G$. If G is connected then $E_1^{p,q} = S^p(\mathfrak{g}^*)^G \otimes H^{q-p}(A)$. As corollaries of spectral sequence, we have the following:

Proposition 8.18. *If G is a finite group, then $H_G(A) = H(A)^G$*

Proposition 8.19. *If G is connected, and T is a maximal torus, W is its weyl group, then $H_G(A) = H_T(A)^W$*

The main result on this section is that the equivariant cohomology defined in the previous section is isomorphic to the equivariant cohomology defined using the G^* algebra structure:

Theorem 8.20. *For every finite-dimensional approximation of EG, we have*

$$H^*(\Omega^*(Z \times_G E, \tau_i)) = H^*_G(\Omega^*(Z, \tau_i)), 3 \leq i \leq \infty$$

As a corollary, Proposition 8.18 and 8.19 holds for equivariant cohomology.

Proof. Let $= \Omega^*(Z_\infty \times E, \tau_i) = \varprojlim \Omega^*(Z_\infty \times E_n, \tau_i)$, and $\Omega^*(E) = \varprojlim \Omega^*(E_n)$. Then $\Omega^*(E)$ satisfies condition (C). Hence by the definition in this section, $H_G^*(\Omega^*(Z, \tau_i)) = H^*((\Omega^*(Z, \tau_i) \otimes \Omega^*(E))_{\mathrm{bas}})$, on the other hand, $\Omega^*(Z \times_G E, \tau_i) = \Omega^*(Z \times E, \tau_i)_{\mathrm{bas}}$. We only need to show

$$H^*((\Omega^*(Z \times E, \tau_i))_{\mathrm{bas}}) = H^*((\Omega^*(Z, \tau_i) \otimes \Omega^*(E))_{\mathrm{bas}}) \tag{8.5}$$

Because $\Omega^*(E)$ is a W^* module(c.f. [53]), thus $\Omega^*(Z \times E, \tau_i)$ also a W^* module, hence the cohomology of the basic complex is just equivariant cohomology of the W^* module. To show (8.5) holds, it is equivalent to show

$$\Omega^*(Z, \tau_i) \otimes \Omega^*(E) \to \Omega^*(Z \times E, \tau_i)$$

induces an isomorphism on $G-$equivariant cohomology. By the Cartan spectral sequence, it suffices to show that the morphism above induces an isomorphism on ordinary cohomology. Since $\Omega^*(E)$ is acyclic, it suffices to show $\Omega^*(Z, \tau_i) \to \Omega^*(Z \times E, \tau_i)$ induces isomorphism on cohomology.

Since $\Omega^*(Z \times E_{n+1}, \tau_i) \to \Omega^*(Z \times E_n, \tau_i)$ is surjective, we have $\varprojlim^1$ exact sequence:

$$0 \to \varprojlim{}^1 H^{*-1}(Z \times E_n, \tau_i) \to H^*(Z \times E, \tau_i) \to \varprojlim H^*(Z \times E_n, \tau_i) \to 0$$

By the Kunneth formula (Proposition B.8), $H^*(Z \times E_n, \tau_i) = H^*(Z, \tau_i) \otimes H^*(E_n)$. Since $\varprojlim^1 H^*(E_n) = 0$, thus $H^*(Z \times E, \tau_i) \simeq \varprojlim H^*(Z \times E_n, \tau_i) = H^*(Z, \tau_i)$. In particular, $\Omega^*(Z, \tau_i) \to \Omega^*(Z \times E, \tau_i)$ induces an isomorphism on cohomology. $\square$

Chapter 9

Equivariant Fundamental Class

If B is an oriented closed manifold with a G-action, then pushforward induced by $B \to \mathrm{pt}$ defines a $H^*(BG)$ module morphism

$$\int_B : H^*_G(B) \to H^{*-\dim B}_G(pt) = H^{*-\dim B}(BG).$$

The morphism $\int_B$ can be understood as the equivariant fundamental class. One the other hand, let $E \to B$ be a finite dimensional oriented vector bundle over a closed oriented manifold. Let e be the Euler class and s any transverse section, then

$$\int_B \alpha \wedge e = \int_{s^{-1}(0)} i^*\alpha.$$

As a generalization of these two facts, when G acts on the vector bundle $E \to B$, then there is map:

$$H^*_G(B) \to H^*_G(pt) : \alpha \mapsto \int_B \alpha \wedge e_G,$$

where e_G is the equivariant Euler class. When there is an equivariant transverse section s, then it is the same as $\int_{s^{-1}(0)} i^*\alpha$. Such map will be referred to as the equivariant fundamental class. Our goal of this chapter is to construct a such map in polyfold setting, where both Euler class(as a cohomology class) and equivariant transversality are beyond reach.

Theorem 9.1. *Let G be a compact Lie group and $\pi : W \to Z$ a regular strong polyfold bundle with a sc-smooth G-action. Assume the base Z has no boundary and is infinite dimensional. Suppose $s : Z \to W$ is a G-equivariant oriented proper Fredholm section, such that G preserves the orientation. Then there is $H^*(BG)$ module homomorphism,*

$$s_* : H^*_G(Z, \tau_i) \to H^{*-k}(BG), \forall i \geq 3$$

where $k = \operatorname{ind} s \in \mathbb{Z}$. Moreover, s_* is compatible with (8.4).

Remark 9.2 (Orientation convention)**.** *Let* $\det(s)$ *be the determinant bundle of* $s : Z \to W$*, and* $\det s^n$ *be the determinant bundle* $s^n : Z \times E_n \to W \times E_n$*, then there is a natural map* $\det(s) \otimes \det(E_n) \to \det s^n$ *(c.f.[106]). Hence orientations of* s *and* E_n *determine an orientation of* $\det s^n$*. The* G *action on* $Z \times E_n$ *is given by* $(z, e) \cdot g = (g^{-1} \cdot z, e \cdot g)$*, and the infinitesimal directions form a finite rank subbundle* V*. Let* s_n *be the induced section* $Z \times_G E_n \to W \times_G E_n$*. By* $\det(s_n) \otimes \det V \to \det(s^n)$*, orientations of* G, E_n *and* $\det(s)$ *induce an orientation of* $\det(s_n)$*. We will twist the induced orientation by* $(-1)^{\dim B_n \cdot \dim G}$*, heuristically, we quotient out the* G *from the top of* E_n*. We orient* B_n *by* $TG \oplus TB_n = TE_n$*. When* G *acts on* M *trivially, the induced orientation on* $M \times_G E_n$ *is the product orientation of* $M \times B_n$*. This orientation convention is to make sure that* $s_* = \int_{s^{-1}(0)}$ *when transversality is granted.*

Proof of Theorem 9.1 in M-polyfold case. Let $\mathcal{X}$ be an M-polyfold with G-action, pick an oriented approximation $E_n \to B_n$ to $EG \to BG$. By Theorem 1.1, there is a quotient M-polyfold $\mathcal{R}_n \subset \mathcal{X}^3 \times_G E_n$ with strong M-polyfold bundle $\mathcal{Y}^3 \times_G E_n|_{\mathcal{R}_n}$ and an oriented proper Fredholm section s_n induced by s. Then there exists an sc^+ perturbation p such that $s_n + p$ is a transverse section on $\mathcal{Y} \times_G E_n|_{\mathcal{R}_n} \to \mathcal{R}_n$. Besides, there is natural projection $\pi_n : \mathcal{R}_n \to E_n/G = B_n$, then we can define s_{n*} as the composition of π_{n*} and i^*,

$$s_{n*} H^*(\mathcal{X} \times_G E_n, \tau_i) \xrightarrow{i^*} H^*((s_n + p)^{-1}(0)) \xrightarrow{\pi_{n*}} H^{*-k}(B_n)$$

where pushforward π_{n*} is defined to by $\int_{B_n} \pi_{n*}\theta \wedge \eta = \int_{(s_n+p)^{-1}(0)} \theta \wedge \pi_n^*\eta$ for $\eta \in H^*(B_n)$. If there are two different small perturbations p_1, p_2, then $(s_n + p_1)^{-1}(0)$ and $(s_n + p_2)^{-1}(0)$ are cobordant [59]. Therefore s_{n*} is independent of p.

To show the limit of s_{n*} exists, we need to show the following commutative diagram:

$$\begin{tikzcd} H^*(\mathcal{X} \times_G E_{n+1}, \tau_i) \arrow[r, "s_{n+1*}"] \arrow[d, "i^*"] & H^{*-k}(B_{n+1}) \arrow[d, "i^*"] \\ H^*(\mathcal{X} \times_G E_n, \tau_i) \arrow[r, "s_{n*}"] & H^{*-k}(B_n) \end{tikzcd} \tag{9.1}$$

Let N denote the tubular neighborhood of $B_n \subset B_{n+1}$, we orient the fibers F by $TB_n \oplus TF = N = TB_{n+1}|_N$. Pick $\tau_k \in \Omega^*(B_{n+1})$, supported in a radius $\frac{1}{k}$ tubular neighborhood of B_n in B_{n+1}, moreover $\tau_k|_F$ is positive and $\int_F \tau_k = 1$. Then $\tau_k \to \delta_{B_k} = \int_{B_n}$ in the sense of current, more precisely:

$$\lim_{k\to\infty} \int_{B_{n+1}} \alpha \wedge \tau_k \to \int_{B_n} i^*\alpha, \quad \forall \alpha \in \Omega^*(B_{n+1})$$

Pull N back to E_{n+1}, we get a G-invariant tubular neighborhood $\tilde{N}$ of $E_n \subset E_{n+1}$. The projection $\pi : \mathcal{Y}^3 \times_G \tilde{N} \to \mathcal{Y}^3 \times_G E_n$ is sc-smooth and is a bundle map. If $s_n + p$ is transverse, then π^*p defines an sc^+ perturbation, such that $s_{n+1} + \pi^*p$ is transverse over $\mathcal{X}^3 \times_G \tilde{N}$, with the zero set $U := \pi^{-1}((s_n + p)^{-1}(0))$. We can extend the perturbation π^*p to a global transverse sc$^+$ perturbation $\tilde{p}$, thus $(s_{n+1}+\tilde{p})^{-1}(0)$ contains U as a tubular neighborhood of $(s_n+p)^{-1}(0)$. Therefore $\pi_{n+1}^*\tau_k$ converges to $\delta_{(s_n+p)^{-1}(0)}$, i.e. for every $\alpha \in \Omega^*((s_{n+1}+\tilde{p})^{-1}(0))$:

$$\lim_{k\to\infty}\int_{(s_{n+1}+\tilde{p})^{-1}(0)} \alpha \wedge \pi_{n+1}^*\tau_k \to \int_{(s_n+p)^{-1}(0)} i^*\alpha.$$

If the projection $U \to (s_n+p)^{-1}(0)$, $N \to B_n$ are both denoted by π, then $\pi\circ\pi_{n+1} = \pi_n \circ \pi$.

To show the diagram (9.1) is commutative, i.e. $s_{n*}i^*\theta = i^*s_{n+1*}\theta$, we pair it with $\eta \in H^*(B_n)$:

$$\begin{aligned}
\int_{B_n} s_{n*}i^*\theta\wedge\eta &= \int_{(s_n+p)^{-1}(0)} i^*\theta\wedge\pi_n^*\eta \\
&= \lim_{k\to\infty}\int_{(s_{n+1}+\tilde{p})^{-1}(0)} \theta\wedge\pi^*\pi_n^*\eta\wedge\pi_{n+1}^*\tau_k \\
&= \lim_{k\to\infty}\int_{(s_{n+1}+\tilde{p})^{-1}(0)} \theta\wedge\pi_{n+1}^*\pi^*\eta\wedge\pi_{n+1}^*\tau_k \\
&= \lim_{k\to\infty}\int_{B_{n+1}} s_{n+1*}\theta\wedge\pi^*\eta\wedge\tau_k.
\end{aligned}$$

On the other hand:

$$\int_{B_n} i^*s_{n+1*}\theta\wedge\eta = \lim_{k\to\infty}\int_{B_{n+1}} s_{n+1*}\theta\wedge\pi^*\eta\wedge\tau_k.$$

Therefore the diagram (9.1) is commutative, and there is a map s_* as the composition of the following two maps:

$$H^*_G(\mathcal{X},\tau_i) = H^*(\varprojlim\Omega^*(\mathcal{X}\times_G E_n,\tau_i)) \to \varprojlim H^*(\mathcal{X}\times_G E_n,\tau_i) \xrightarrow{\varprojlim s_{n*}} \varprojlim H^{*-k}(B_n) = H^{*-k}(BG)$$

Moreover, for each n, π_n^* gives rise to a $H^*(B_n)$ module structure on $H^*(\mathcal{X}\times_G E_n,\tau_i)$, and this is compatible with the commutative diagram above by the same argument. Therefore s_* is $H^*(BG)$ module homomorphism $H^*_G(\mathcal{X},\tau_i)\to H^{*-k}(BG)$.

We still have to show s_* is independent of the choice of approximation, given E_n, E'_n are two approximations, hence so is $E_n\times E'_n$. Then by a similar argument we the commutative diagram:

$$\begin{tikzcd}
H^*(\Omega^*(\mathcal{X}\times_G E,\tau_i)) \arrow[r] \arrow[d] & H^*(\mathcal{X}\times_G E_n,\tau_i) \arrow[r, "s_{n*}"] \arrow[d, "\pi^*"'] & H^{*-k}(B_n) \arrow[d, "\pi^*"'] \\
H^*(\Omega^*(\mathcal{X}\times_G (E\times E'),\tau_i)) \arrow[r] & H^*(\mathcal{X}\times_G (E_n\times E'_n),\tau_i) \arrow[r, "s_{n*}"] & H^{*-k}(E_n\times E'_n/G) \\
H^*(\Omega^*(\mathcal{X}\times_G E',\tau_i)) \arrow[r] \arrow[u] & H^*(\mathcal{X}\times_G E'_n,\tau_i) \arrow[r, "s_{n*}"] \arrow[u, "\pi^*"] & H^{*-k}(B'_n) \arrow[u, "\pi^*"]
\end{tikzcd}$$

which is also compatible with the directed system. The arrows in first and third columns are isomorphism by Theorem 8.13 and 8.20, after we take inverse limit. Thus s_* is well-defined. $\square$

The proof for polyfold case is almost same, expect we have to work with branched sub-orbifolds:

Proof of Theorem 9.1 in polyfold case. Θ_1 is defined by $s_n + \Gamma$ on $Z^1 \times_G E_n$ for suitable multivalued sc^+ perturbation Γ. Then $\pi^*\Gamma$ defines a transverse multivalued sc^+ perturbation of s_{n+1} over $Z^1 \times_G \tilde{N}$. We can extend $\pi^*\Gamma$ to a global perturbation $\tilde{\Gamma}$, let Θ_2 denote the branched suborbifold $(s_{n+1}+\tilde{\Gamma})^{-1}(0)$. Then $\pi^{-1}|\mathrm{Supp}\Theta_1| \subset |\mathrm{Supp}\Theta_2|$ is an open set. For each point x in $|\mathrm{Supp}\Theta_1|$, there is a neighborhood $\mathcal{V}$ in a polyfold structure of $Z^1 \times_G E_n$, such that $\mathrm{Supp}\Theta_1$ is represented by $\cup(M_i, \sigma_i)$. Then $\pi^{-1}(\mathcal{V})$ is an open set of a polyfold structure of $Z^1 \times_G \tilde{N}$, and over $\pi^{-1}(\mathcal{V})$, $\pi^{-1}|\mathrm{Supp}\Theta_1|$ is represented by $\cup(U_i, \sigma_i)$, with $U_i = \pi^{-1}(M_i)$.

Let τ_k be the classes as before, such that $\tau_k \to \delta_{B_n}$, then $\pi^*_{n+1}\tau_k$ has same property over $U_i \to M_i$. Let $\{\mathcal{V}_\alpha\}$ cover $\mathrm{Supp}\Theta_1$ with partition of unity $\{\phi_\alpha\}$, over each $\mathcal{V}_\alpha$, $\mathrm{Supp}\Theta_1$ has local representation $(M_i^\alpha, \sigma_i^\alpha)$, and over $\pi^{-1}(\mathcal{V}_\alpha)$, $\pi^{-1}|\mathrm{Supp}\Theta_1|$ has local representation $(U_i^\alpha, \sigma_i^\alpha)$. Let $\eta \in H^*(B_n)$, we have:

$$\begin{aligned}
\int_{B_n} i^* s_{n+1*}\theta \wedge \eta &= \lim_{k\to\infty} \int_{B_{n+1}} s_{n+1*}\theta \wedge \pi^*\eta \wedge \tau_k \\
&= \lim_{k\to\infty} \int_{(s_{n+1}+\tilde{\Gamma})^{-1}(0)} \theta \wedge \pi^*_{n+1}\pi^*\eta \wedge \pi^*_{n+1}\tau_k \\
&= \lim_{k\to\infty} \int_{(s_{n+1}+\tilde{\Gamma})^{-1}(0)} \theta \wedge \pi^*\pi^*_n\eta \wedge \pi^*_{n+1}\tau_k \\
&= \lim_{k\to\infty} \sum_\alpha \frac{1}{|G_\alpha|} \int_{U_i^\alpha} \sigma_i \pi^*\phi_\alpha \theta \wedge \pi^*\pi^*_n\eta \wedge \pi^*_{n+1}\tau_k \\
&= \sum_\alpha \frac{1}{|G_\alpha|} \int_{M_i^\alpha} \sigma_i \phi_\alpha i^*\theta \wedge \pi^*_n\eta \\
&= \int_{(s_n+\Gamma)^{-1}(0)} i^*\theta \wedge \pi^*_n\eta \\
&= \int_{B_n} s_{n*} i^*\theta \wedge \eta
\end{aligned}$$

This proves the digram (9.1) for polyfold, hence s_* is well defined for polyfold. $\square$

Remark 9.3. *In applications[61], there is usually an sc^∞ G-equivariant map $ev : Z \to M$. Instead of evaluating all class in $H^*_G(Z, \tau_i)$, we only evaluate those pulled back from $H^*_G(M)$. Since $H^*_G(M)$ is defined independent of the approximation of the classifying spaces, thus one can verify directly that $s_* \circ ev^* : H^*_G(M) \to H^{*-\mathrm{ind}\, s}_G(pt)$ is independent of approximation. Thus there is no need to going to the discussion in Section 8.1 in applications.*

Proposition 9.4. *If s is already transverse, then $s_* : H_G^*(Z, \tau_i) \to H_G^{*-k}(pt)$ is equivalent to integration over G-orbifold $s^{-1}(0)$*

Proof. In this case, we can define s_* without perturbation. □

Proposition 9.5. *If $W \to Z$ is actually an M-polyfold bundle, G action is free,and G is not discrete, then $s_* = 0$*

Proof. Free action implies the quotient exists as an M-polyfold, hence there exists equivariant transverse perturbation p, such that $(s+p)^{-1}(0)$ is a closed manifold with a free G action, then $\int_{(s+p)^{-1}(0)}$ being zero on $H_G^*((s+p)^{-1}(0))$ implies $s_* = 0$. □

The following proposition follows from direct check:

Proposition 9.6. *$W_1 \to Z_1$ and $W_2 \to Z_2$ are two M-polyfold strong bundles with sc-smooth G_1, G_2 action respectively, and s_1, s_2 are two orientable equivariant proper Fredholm sections. Denote $s_1 \times s_2$ as the section from $Z_1 \times Z_2$ to $W_1 \times W_2$, then*

$$(s_1 \times s_2)_*(a \wedge b) = (-1)^{|a| \operatorname{ind} s_2 + \operatorname{ind} s_1 \operatorname{ind} s_2} s_{1*}a \wedge s_{2*}b \in H^*_{G_1 \times G_2}(pt)$$

where $a \in H^*_{G_1}(Z_1, \tau_i)$ and $b \in H^*_{G_2}(Z_2, \tau_i)$.

Proposition 9.7. *If H is a closed subgroup of G, then the following diagram is commutative:*

$$\begin{tikzcd} H_G^*(Z, \tau_i) \arrow[r, "s_*"] \arrow[d] & H^{*-k}(BG) \arrow[d] \\ H_H^*(Z, \tau_i) \arrow[r, "s_*"] & H^{*-k}(BH) \end{tikzcd}$$

Proof. Assume $\{E_n\}$ is an approximation to EG, then it is also an approximation to EH, and $E_nH \to E_nG$ is a G/H bundle. A transverse section s_n^Gover $Z^1 \times_G E_n$ will induce a transverse section s_n^H over $Z^1 \times_H E_n$. Let p denote both projections $E_n/H \to E_n/G$ and $Z^1 \times_H E_n \to Z^1 \times_G E_n$. Therefore we have the following commutative diagram of G/H fiberations.

$$\begin{tikzcd} {s_n^H}^{-1}(0) \arrow[r, "\pi"] \arrow[d, "p"] & E_n/H \arrow[d, "p"] \\ {s_n^G}^{-1}(0) \arrow[r, "\pi"] & E_n/G \end{tikzcd}$$

We claim that the following is commutative

$$\begin{tikzcd} H^*({s_n^H}^{-1}(0)) \arrow[r, "\pi_*"] & H^{*-k}(E_n/H) \\ H^*({s_n^G}^{-1}(0)) \arrow[r, "\pi_*"] \arrow[u, "p^*"] & H^{*-k}(E_n/G) \arrow[u, "p^*"'] \end{tikzcd}$$

Let $\alpha \in H^*(s_n^{G^{-1}}(0))$, then for all $\eta \in H^*(E_n/H)$ we have

$$\begin{aligned}
\int_{E_n/H} \pi_* \circ p^* \alpha \wedge \eta &= \int_{s_n^{H^{-1}}(0)} p^* \alpha \wedge \pi^* \eta \\
&= \int_{s_n^{G^{-1}}(0)} \alpha \wedge \pi^* p_* \eta \\
&= \int_{E_n/G} \pi_* \alpha \wedge p_* \eta \\
&= \int_{E_n/H} p^* \pi_* \alpha \wedge \eta
\end{aligned}$$

where p_* is the integration along the fiber. This diagram is also compatible with the direct system. Passing to the inverse limits proves the proposition. $\square$

Chapter 10

Localization Theorem

The localization theorem [3] for torus action on a closed manifold M asserts the integration of an equivariant closed form on M, can be computed by restricting it to the fixed locus M^T, i.e. we have

$$\int_M \alpha = \int_{M^T} \alpha \wedge e_T^{-1}, \alpha \in \mathrm{H}_G^*(M),$$

where e_T is the equivariant Euler class of the normal bundle of M^T and it is invertible. When the torus T acts on a finite-dimensional bundle $V \to M$,

$$s_*(\alpha) = \int_M \alpha \wedge e_T(V) = \int_{M^T} \alpha \wedge e_T(V) \wedge e_T^{-1},$$

where $e_T(V)$ is the equivariant Euler class of V. Since a torus representation can be classified by its weight $\mathbb{Z}^n/\mathbb{Z}_2$, therefore the normal bundle N of M^T in M can be decomposed as:

$$N = \bigoplus_{i \in (\mathbb{Z}^n - \{0,\ldots,0\})/\mathbb{Z}_2} N_i.$$

Let V^T be the fixed part of $V|_{M^T}$, similarly we have decomposition.

$$V_{M^T} = V^T \bigoplus_{i \in (\mathbb{Z}^n - \{0,\ldots,0\})/\mathbb{Z}_2} V_i.$$

Then $e_T = \wedge_i e_T(N_i)$, and $e_T(V)|_{M^T} = e(V^T) \wedge_i e_T(V_i)$, thus

$$s_*(\alpha) = \int_{M^T} \alpha \wedge e_T(V) \wedge e_T^{-1} = \int_{M^T} \frac{\alpha \wedge e(V^T) \wedge_i e_T(V_i)}{\wedge_i e_T(N_i)} = s^*_{M^T}\left(\alpha \wedge_i \frac{e_T(V_i)}{e_T(N_i)}\right). \tag{10.1}$$

We can also get (10.1) by analyzing the equivariant transversality near M^T. Let $s : M \to V$ be an equivariant section, for simplicity, we assume s is equivariantly transverse to 0, then s_* is just $\int_{s^{-1}(0)}$. We need to understand the normal bundle of $s^{-1}(0) \cap M^T$ in $s^{-1}(0)$. s being equivariant implies $\mathrm{D}s = \sum \mathrm{D}_i s$, where $\mathrm{D}_i s \in \mathrm{Hom}_T(N_i, V_i)$. If s is transverse, the normal bundle of $s^{-1}(0) \cap M^T$ in $s^{-1}(0)$ is $\oplus \ker \mathrm{D}_i s|_{s^{-1}(0)\cap M^T}$. In general, the normal bundle can be understood as the virtual bundle $\oplus(N_i - V_i)|_{s^{-1}(0)\cap M^T}$, and the equivariant Euler class should be

$$\wedge_i \frac{e^T(N_i)}{e^T(V_i)}.$$

In the polyfold case, N_i and V_i are infinite dimensional with the equivariant Fredholm operator $\mathrm{D}_i s$. Therefore we make the tubular neighborhood assumption without isotropy as in Definition 6.9 to make sense of $N_i - V_i$ by substituting N_i, V_i by finite dimensional bundles. In particular, by Proposition 6.15, we can assume $s|_{Z^G} : Z^G \to W^G$ is transverse and there exists a finite dimensional G-invariant trivial bundle $F_i \subset W_i$ over a neighborhood of $s^{-1}(0) \cap Z^G$ in $\mathcal{X}$, covers the cokernel of $\mathrm{D}_i s$ over $s^{-1}(0) \cap Z^G$.

Remark 10.1 (Orientation). *Since there is a natural map* $\det(s|_{Z^G}) \otimes_i \det(D_i s) = \det(s)$. *Once we fix the representative of the weight in* $\mathbb{Z}^n$, *the complex structure of* E_i *is also fixed, therefore* $\det \mathrm{D}_i s$ *comes with a natural orientation, and they determine an orientation of* $\det(s|_{Z^G})$.

Recall from Chapter 4, a G-action on a polyfold induces sets of local lifts of actions $L(x, y, g)$ satisfying Proposition 4.33 and Proposition 4.35.

10.1 S^1 localization

To prove the localization theorem, we still need another ingredient, i.e. localizing s_* to a neighborhood of $s^{-1}(0) \cap Z^T \subset Z$. Let $\mathrm{L}H^*_T(M)$ denote the localization of $H^*_T(M)$ by nontrivial element of $H^*(BT)$. In classical case, we have $\mathrm{L}H^*_T(M) = H^*(M^T)\otimes\mathbb{R}(u_1, \ldots, u_n)$. If U is T invariant neighborhood of M^T, then $\mathrm{L}H^*_T(M-U) = 0$, i.e. there exists homogeneous polynomial p in $\mathbb{R}[u_1, \ldots, u_n] = H^*(BT)$, such that the pull back of p is zero in $H^*_T(M-U)$, then the pull back of p in $H^*_T(M)$ can be represented by form supported in U. So we have $\int_M \omega \wedge p = \int_U \omega \wedge p, \omega \in H^*_T(M)$. Therefore we only need to understand the tubular neighborhood of M^T. In polyfold case, we will find such p directly. We divide the proof into two cases, in S^1 case, such conclusion holds. However in general torus T^n case, we will need a geometric assumption, which can be understood as the analog of the neighborhood theorem of an orbit.

We fix $S^{2n+1} \to \mathbb{CP}^n$ as the approximation to $ES^1 \to BS^1$, the right S^1 action is given by $v \cdot \theta = e^{-2\pi i\theta} \cdot v \in S^{2n+1} \subset \mathbb{C}^{n+1}$, and S^{2n+1} is oriented such that $[S^1][\mathbb{CP}^n] = [S^{2n+1}]$ induces the complex orientation on $\mathbb{CP}^n$. This convention makes the first Chern class u of the S^1 bundle is represented by the hyperplane divisor. The pull back to $Z \times_{S^1} S^{2n+1}$ is denoted by u_n.

Proposition 10.2. *Assume* $K \subset Z_\infty$ *is a compact set in* τ_3 *topology with no fixed points, then there is* S^1 *invariant open set* U *of* K *in* Z^3. *Such that* $u_n^k = 0$ *in* $H^*(U \times_{S^1} S^{2n+1}, \tau_3)$ *for all* n.

Proof. Let $x \in K - Z^G$, then there exists a slice S_x satisfies conditions in Proposition 4.39. Then using the S^1 action and morphisms to move S_x around, we get a full subcategory $\mathcal{U}_x$ of $\mathcal{X}^3$, whose orbits correspond to a S^1 equivariant neighborhood U_x of x in Z^3. We can find a neighborhood O of id in G, such that $O \times S_x \to \mathcal{X}^1 : (g, v) \to \Gamma_{id}(g, v)$ is a local sc-diffeomorphism.

We can define α on $O \times S_x$ by $\alpha(TS_x) = 0$ and α is the Maurer-Cartan form on O. We can push-forward it to an open set of $\mathcal{X}^1$, the new form is still denoted by α. Then we can define β on $\mathcal{U}_x$ by

$$\beta_p = \frac{1}{mk} \sum_{\substack{g \in S^1, g[p]=[y], y \in S_x \\ \Gamma \in L(p,y,g)}} \Gamma^* \alpha$$

where m is the size of the isotropy group at x of the S^1 action and k is the size of effective part of the isotropy from the polyfold structure. Since α is closed, and β locally is just average of km terms of pull back of α by Proposition 4.39, hence β is sc^∞ and closed and S^1 invariant. And $\beta(\xi) = 1$, for the infinitesimal direction ξ of S^1 action.

Over S^{2n+1}, u is exact with a S^1 invariant primitive ρ such that $\rho(\xi) = 1$. Then over $U_x \times S^{2n+1}$, $\rho + \beta$ is S^1 invariant and $\rho + \beta$ vanishes on the infinitesimal direction of the S^1 action on $U_x \times S^{2n+1}$, hence $\rho + \beta$ defines an element in $\Omega^*_\infty(U^1_x \times_{S^1} S^{2n+1}, \tau_3)$, besides $d(\rho + \beta) = d\rho = u$, this show $u_n \in H^2(U^1_x \times_{S^1} S^{2n+1}, \tau_3)$ is zero for any n.

We have find a finite S^1 invariant cover $\{U_i\}_{i \in I}$of K, such that $u_n = 0$ in $H^*(U_i \times_{S^1} S^{2n+1}, \tau_3)$ for all n. Therefore $u_n^{|I|} = 0$ in $H^*(\cup U_i \times_{S^1} S^{2n+1}, \tau_3)$ for all n. $\square$

Because s_* is a $H^*(BG)$ module map, we have:

Corollary 10.3. *Given S^1 acts Z, if Z does not have fixed point, then $s_* = 0$*

Theorem 10.4. *Suppose tubular neighborhood assumption (Definition 6.9) holds, we can assume $s|_{Z^{S^1}} Z^{S^1} \to W^{S^1}$ is already transverse, and M^{S^1} denotes the zero set in the fixed locus $s^{-1}(0) \cap Z^{S^1}$. Then there exists an element $e \in \mathrm{L}H^*(M^{S^1}) = H^*(M^{S^1}) \otimes \mathbb{R}(u)$, such that e is invertible and the following localization theorem holds:*

$$s_*(\theta) = \int_{M^{S^1}} i^*\theta \wedge e^{-1}$$

The strategy of the proof is first finding transverse perturbation p of s_n, such that there is a manifold $N \subset \mathbb{CP}^n$ and $M^{S^1} \times N \subset (s_n + p)^{-1}(0)$. Then we can analyze the Thom class.

Proof of theorem 10.4. Over M^{S^1}, $\mathrm{D}s$ can be decomposed to $\sum_{d \in \mathbb{N}} \mathrm{D}_d s$. For any $d > 0$ that $\mathrm{D}_d s$ is not surjective over M^{S^1}, there exists a trivial S^1 bundle $F_d \subset W_d$ over an invariant open neighborhood O of M^{S^1} in $\mathcal{X}$ covers $\operatorname{Coker} \mathrm{D}_d s$ by Proposition 6.15. Assume first we only have one such d, let π_d denote the projection $F_d \to O$. By Proposition A.8, we have a S^1 invariant complement M-polyfold bundle $F_d^\perp$, such that $W_d = F_d \oplus F_d^\perp$. The projection to F_d is denoted by π_F. Since $\pi_F s$ is an sc^+ section, then $s - \pi_F s$ is still Fredholm[59]. We also have Fredholm sections

$$(s - \pi_F \circ s) \circ \pi_d : F_d \to W$$

and

$$(s - \pi_F \circ s) \circ \pi_d + \mathrm{id}_F : F_d \to W$$

It is easy to see $(s - \pi_F \circ s) \circ \pi_d + \mathrm{id}_F : F_d \to W$ is equivariant transverse at $\pi_d^{-1}(M^{S^1})$, therefore $\tilde{M} = ((s - \pi_F \circ s) \circ \pi_d + \mathrm{id}_F)^{-1}(0)$ is cut out transversely and contains M^{S^1} as submanifold. Moreover, there is diffeomorphism from $(s - \pi_F \circ s)^{-1}(0)$ to $\tilde{M}$ defined by $x \mapsto (x, \pi_F \circ s(x))$.

Let $E_d = (\mathrm{D}_d s)^{-1} F_d$, then the normal bundle of $M^{S^1} \subset \tilde{M}$ is equivariant isomorphic to E_d. Therefore one can find a tubular neighborhood of M^{S^1} in $\tilde{M}$ equivariant diffeomorphic to E_d. By shrinking O, we can assume $\tilde{M}$ is the equivariant tubular neighborhood.

Over $O^1 \times_{S^1} S^{2n+1}$, $\sigma := s - \pi_F \circ s$ induces a operator σ_n. It is clear that $\sigma_n - s_n$ is an sc^+ section. Moreover we have

$$\sigma_n{}^{-1}(0) = \sigma^{-1}(0) \times_{S^1} S^{2n+1} \cong \tilde{M} \times_{S^1} S^{2n+1} \cong E_d \times_{S^1} S^{2n+1} = E_d \boxtimes \mathcal{O}(d)$$

where $E_d \boxtimes \mathcal{O}(d)$ is the exterior tensor product of $E_d \to M^{S^1}$ and $\mathcal{O}(d) \to \mathbb{CP}^n$. But σ_n is not transverse, since the linearized operator is zero in the F_d direction, but surjective onto the complement. The projection from $\sigma_n{}^{-1}(0)$ to $M^{S^1} \times \mathbb{CP}^n$ is denoted by π. Thus

$$F_d|_{\sigma^{-1}(0) \times_{S^1} S^{2n+1}} = \pi^*(F_d \boxtimes \mathcal{O}(d))$$

If $\dim_{\mathbb{C}} F_d = l$, then there is a holomorphic transverse section $t : \mathbb{CP}^n \to \mathcal{O}(d)^l$ such that $[t^{-1}(0)] = (d[H])^l$. The section t can be extended to $\sigma^{-1}(0) \times_{S^1} S^{2n+1} \cong \tilde{M} \times_{S^1} S^{2n+1} \cong E_d \boxtimes \mathcal{O}(d)$ by pullback. Since $\sigma^{-1}(0) \cong \tilde{M}$ and $\tilde{M}$ is the zero set of a transverse section $(s - \pi_F \circ s) \circ \pi_d + \mathrm{id}_F$, we can use the local picture of transverse solution described in [59] to extend $t : \sigma^{-1}(0) \times_{S^1} S^{2n+1} \to F_d|_{\sigma^{-1}(0) \times_{S^1} S^{2n+1}}$ to a section $\tilde{t} : O^1 \times_{S^1} S^{2n+1} \to F_d \times_{S^1} S^{2n+1}$, such that $\sigma_n + \tilde{t}$ is transverse to 0 and $(\sigma_n + \tilde{t})^{-1}(0) \cong E_d \boxtimes \mathcal{O}(d)|_{M^{S^1} \times t^{-1}(0)}$. Let U denote $(\sigma_n + \tilde{t})^{-1}(0)$ which is an equivariant tubular neighborhood of $M^{S^1} \times t^{-1}(0)$. We can keep extending $\tilde{t}$ outside of $O^1 \times_{S^1} S^{2n+1}$ to the whole quotient.

The Thom class of $\pi : U \to M^{S^1} \times t^{-1}(0)$ is denoted by η_n, then $i^*\eta_n \in H^*(M^{S^1} \times t^{-1}(0))$ is the Euler class of the normal bundle, where $I : M^{S^1} \times t^{-1}(0) \to U$ is the inclusion. Since the normal bundle can be identified with $E_d \boxtimes \mathcal{O}(d)$, $i^*\eta_n$ is the Euler class of $E_d \boxtimes \mathcal{O}(d)$ over $M^{S^1} \times t^{-1}(0)$. Note that the Euler class of $E_d \boxtimes \mathcal{O}(d) \to M^{S^1} \times \mathbb{CP}^n$ converges to the equivariant Euler class of $E_d \to M^{S^1}$, in particular it is fixed for as an element in $H^*(M^{S^1}) \otimes \mathbb{R}[u]$ for n big enough. By Lefschetz hyperplane theorem, $H^*(M^{S^1} \times \mathbb{CP}^n) \to H^*(M^{S^1} \times t^{-1}(0))$ is isomorphism for $* < n + 1 - l$. Hence for n big enough, $i^*\eta_n$ does not change and equals to the equivariant Euler class $e \in H^*(M^{S^1}) \otimes \mathbb{R}[u]$ of $E_d \to M^{S^1}$.

The Euler class $i^*\eta_n$ has leading term $(du)^{\dim_{\mathbb{C}} E_d}$. Moreover there is an element e^{-1} in $H^*(M^{S^1}) \otimes \mathbb{R}(u)$, such that $e \wedge e^{-1} = 1$ and $u^a \wedge e^{-1}$ is in $H^*(M^{S^1}) \otimes \mathbb{R}[u]$, for some positive integer a which only depends on $\dim E_d$ and $\dim M^{S^1}$. For n big, one can assume $u^a \wedge e^{-1}$ is in $H^*(\mathcal{M}^{S^1} \times t^{-1}(0))$, such that $u^a \wedge e^{-1} \wedge i^*\eta_n = u^a$.

Since projection π is a homotopy equivalence, we have the following diagram:

$$\begin{tikzcd}
H^*(U) \arrow[d, shift left, "i^*"] & H^*(\mathbb{CP}^n) \arrow[l, "\pi_n^*"'] \arrow[d, leftrightarrow, "\mathrm{id}"] \\
H^*(M^{S^1} \times t^{-1}(0)) \arrow[u, shift left, "\pi^*"] & H^*(\mathbb{CP}^n) \arrow[l, "\pi_n^*"']
\end{tikzcd}$$

Since $i^*(\pi^*(e^{-1} \wedge \pi_n^* u^a) \wedge \eta_n) = \pi_n^* u^a$, and i^* is an isomorphism, hence $\pi^*(e^{-1} \wedge \pi_n^* u^a) \wedge \eta_n = \pi_n^* u^a$ in $H^*(U)$.

For a fixed integer s, let $\theta \in H^s(\mathbb{Z} \times_{S^1} S^{2n+1})$. Consider $\int_{(\sigma_n+\tilde{t})^{-1}(0)} \theta \wedge \pi^*(e^{-1} \wedge \pi_n^* u^a) \wedge \eta_n \wedge \pi_n^* u^{i-a}$ where $s + 2i = k + 2n$. By Proposition 10.2, there is $m \in \mathbb{N}$ such that $\pi_n^* u^m$ is cohomologous to an element ω supported outside $O^1 \times_{S^1} S^{2n+1}$ for any n. Then for $i - a - m \geq 0$

$$\begin{aligned}
\int_{(\sigma_n+\tilde{t})^{-1}(0)} \theta \wedge \pi^*(e^{-1} \wedge \pi_n^* u^a) \wedge \eta_n \wedge \pi_n^* u^{i-a} &= \int_{(\sigma_n+\tilde{t})^{-1}(0)} \theta \wedge \pi^*(e^{-1} \wedge \pi_n^* u^a) \wedge \eta_n \wedge \pi_n^* u^m \wedge \pi_n^* u^{i-a-m} \\
&= \int_U \theta \wedge \pi^*(e^{-1} \wedge \pi_n^* u^a) \wedge \eta_n \wedge \omega \wedge \pi_n^* u^{i-a-m} \\
&= \int_U \theta \wedge \pi_n^* u^a \wedge \omega \wedge \pi_n^* u^{i-a-m} \\
&= \int_{s_n^{-1}(0)} \theta \wedge \pi_n^* u^i
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\int_{\sigma_n+\tilde{t}^{-1}(0)} \theta \wedge \pi^*(e^{-1} \wedge \pi_n^* u^a) \wedge \eta_n \wedge \pi_n^* u^{i-a} &= \int_{\mathcal{M}^{S^1} \times (dH)^l} i^*(\theta \wedge p_n^*(e^{-1} \wedge \pi_n^* u^a) \wedge \pi_n^* u^{i-a}) \\
&= \int_{M^{S^1} \times t^{-1}(0)} i^*\theta \wedge (e^{-1} \wedge u^a) \wedge u^{i-a} \\
&= \int_{M^{S^1} \times \mathbb{CP}^n} i^*\theta \wedge (e^{-1} \wedge u^a) \wedge u^{i-a} \wedge (du)^l
\end{aligned}$$

Hence for fixed s and large enough n,

$$s_{n*}(\theta \wedge u^a) = \int_{\mathcal{M}^{S^1}} i^*\theta \wedge (e^{-1} \wedge u^a) \wedge (du)^l$$

Let $n \to \infty$, we have

$$s_*(\theta \wedge u^a) = \int_{M^{S^1}} i^*\theta \wedge (e^{-1} \wedge u^a) \wedge (du)^l.$$

Since s_* is a $\mathbb{R}[u]$ module map, we can localize u hence finish the proof for case when only one $\mathrm{D}_d s$ is not surjective.

If there are several d_i, such that $\mathrm{D}_{d_i} s$ are not surjective. Since we can only have finitely many such d_i, we can replace the holomorphic section $t : \mathbb{CP}^n \to \mathcal{O}(d)^l$ by $t : \mathbb{CP}^n \to \oplus \mathcal{O}(d_i)^{l_i}$. Then the proof follows from the same argument. $\square$

Remark 10.5. *The rule of weight being positive is artificial, we can certainly work with negative weight d, since $\mathcal{O}(d)$ and $\mathcal{O}(-d)$ are isomorphic as real bundle, hence the proof still works. Note that such modification changes the complex structures on E_d, F_d and the induced orientation for $\det s_{ZG}$, but the localization theorem still holds without any extra signs.*

10.2 T^n localization

Fix a point $x \in Z_\infty$, let T_x denote the isotropy group from the T^n-action. Then the tangent space $TT_{\mathrm{id}} \subset T_{\mathrm{id}} T^n = \mathbb{R}^n$ is generated rational vectors. The orthogonal complement of $T_{\mathrm{id}} T_x$ is also generated by rational vectors, and the complement generates a subgroup $T_\perp$, with $T_\perp \to T^n/T_x$ is a covering. To prove the analogue of Proposition 10.2 for the T^n-action, we need the following assumption:

Definition 10.6. *A T-action satisfies the slice condition if for $x \in \mathcal{X}_\infty$, there is a slice S_x of the $T_\perp$ action around x, such that S_x contains the infinitesimal directions of T_x action on it.*

In finite dimensional cases, by the slice theorem (cf. [4]), for any orbit $T^n \cdot x$ there is tubular neighborhood which is equivariant diffeomorphic to $V \times_{T_x} T^n$, where V is the normal bundle of $T^n \cdot x$ and it is equipped with a T_x action. Therefore V is the slice we need. This assumption holds for polyfolds in applications, as long as we have a similar geometric picture near a orbit.

When **A5** holds, then one can use the $T_\perp$ action and morphisms to move the slice around to get a full subcategory $\mathcal{U}$ in $\mathcal{X}$, such that $|\mathcal{U}|$ is a $T_\perp$ invariant neighborhood of the orbit $T^n x$ in Z. Moreover, the $T_\perp$ action on $\mathcal{U}$ has only finite isotropy, since it is so on the slice. Moreover, Proposition 4.39 holds on S_x for the $T_\perp$ action. Let u_i represents the hyperplane divisor in $H^2(\mathbb{CP}^m)$ of the ith $\mathbb{CP}^m$ in the approximations of the classifying space $B_m = (\mathbb{CP}^m)^n$.

Proposition 10.7. *Assume the slice condition holds. Let $K \subset Z_\infty$ be a compact set in τ_1 topology with no fixed points, then there is T^n invariant neighborhood U of K, there exist a homogeneous polynomial p in $\mathbb{R}[u_1, \dots, u_n]$, such that $\pi_m^* p = 0$ in $H^*(U \times_T (S^{2m+1})^n)$, for all m.*

Proof. Let $x \in K$, then there exits a $T_\perp$ slice S_x containing all the infinitesimal directions of T_x, such that Proposition 4.39 holds. Pick a neighborhood O of id in $T_\perp$, then we have local sc^∞ diffeomorphism:

$$O \times S_x \to \mathcal{U} : (g, v) \to \Gamma_{\mathrm{id}}(g, v) \tag{10.2}$$

Pick a $S^1 \subset T_\perp$ with a tangent vector of the S^1 represented by $(a_1, \ldots, a_n) \in \mathbb{R}^n$. Then we can define a differential form α on $O \times S_x$, such that $\alpha(TS_x) = 0$ and α restricted infinitesimal directions on T^n is $\langle (a_1, \ldots, a_n), \cdot \rangle_{\mathbb{R}^n}$. The second requirement can be achieved because S_x contains T_x infinitesimal directions. The pushforward of α by (10.2) to an open subset of $\mathcal{U}$ is still denoted by α. Then we can define a differential form β on $\mathcal{U}$:

$$\beta_p = \frac{1}{mk} \sum_{\substack{t \in T_\perp, t\cdot[p]=[y], y \in S_x \\ \Gamma \in L(p,y,t)}} \Gamma^* \alpha_p$$

where m is the size of the isotropy group of $T_\perp$ action on x, and k is the size of effective isotropy form the polyfold structure. Then β is closed, T' invariant and on the infinitesimal direction of the T^n-action is $\langle (a_1, \ldots, a_n), \cdot \rangle$. Since $T_\perp \cdot S_x$ contains the whole orbit $T^n \cdot x$, hence we can find T^n invariant neighborhood $V_x \subset T_\perp \cdot S_x$ of $T^n \cdot x$. If we average β over T^n, we can get a form γ on V_x, which is closed, T^n invariant and having the prescribed form on the infinitesimal directions of T^n action.

u_i has S^1 invariant primitive ρ_i in $\Omega^1(S^{2m+1})$, such that $\rho_i(\xi) = 1$, where ξ is the infinitesimal direction of the S^1 action. Then over $V_x \times_{T^n} (S^{2m+1})^n$, $\sum a_i \rho_i + \beta \in \Omega^1_\infty(V_x \times (S^{2m+1})^n, \tau_i)$ defines a form on $V_x \times_{T^n} (S^{2m+1})^n$, besides $d(\sum a_i \rho_i + \beta) = \sum a_i u_i$, hence $\sum a_i u_i$ is exact on $\Omega^2_\infty(V_x \times_{T^n} (S^{2m+1})^n, \tau_i)$. If $\sum a_i u_i$ is exact on $V_x \times_T (S^{2m+1})^n$ and $\sum b_i u_i$ is exact on $V_y \times_T (S^{2m+1})^n$, then $\sum a_i u_i \wedge \sum b_i u_i$ is exact on $(V_x \cup V_y) \times_T (S^{2m+1})^n$, thus we can find p and U such that the conclusion holds. □

With Proposition 10.7, the following theorem can be proven with an identical proof as in the S^1 case.

Theorem 10.8. *Under assumption* $\boldsymbol{A1-5}$, *and* $s|_{Z^T} : Z^T \to W^T$ *is already transverse, and* $M^T = s|_{Z^T}^{-1}(0)$. *Then there exists an invertible element* $e \in H^*(M^T) \otimes \mathbb{R}(u_1, \ldots, u_n)$, *such that the following localization theorem holds:*

$$s_*(\theta) = \int_{M^T} i^*\theta \wedge e^{-1}$$

The proof also indicates that the equivariant Euler class is of the following form:

Corollary 10.9.

$$e = \bigwedge_{d \in (\mathbb{Z}^n - 0)/\mathbb{Z}_2} \frac{e_{T^n}(E_d)}{e_{T^n}(F_d)}$$

Corollary 10.10. *If* M^T *is cut out by equivariant section* s *transversely, then* e *is the equivariant Euler class of* M^T *in* $s^{-1}(0)$.

Corollary 10.11. *If M^T is a set of finite points, then*

$$e = \sum_{p \in M^T} \prod_{\lambda \in (\mathbb{Z}^n - 0)/Z_2} \pm \left(\sum \lambda_i u_i\right)^{\frac{k^p_\lambda}{2}} 1_p$$

where k^p_λ is the index for D_λ at point p in M^T and $\lambda = \{\lambda_1, \dots, \lambda_n\}$ is a nonzero weight, the signs are determined by explicit orientations.

Example 10.12. *Consider the Hirzebruch's surface $J_{-2} = P(\mathbb{C} \oplus \mathcal{O}(-2))$, where $\mathcal{O}(-2)$ is the -2 line bundle over $\mathbb{CP}^1$. There is a S^1 action on J_{-2} sending $\{x, [a : b]\} \to \{x, [e^{-i\theta}a, b]\}$. Moreover, there exists symplectic structure compatible with the standard complex structure.*

Consider the holomorphic sphere C given by $\{x, [1 : 0]\}$, then it's normal bundle is $\mathcal{O}(-2)$, hence $c_1(C) = 0$, so the Fredholm index for it will be $4 + 2c_1(C) - 6 = -2$, besides by adjunction formula, any J curve in this homology class will be an embedding, hence by perturbing J, one can get rid of this curve.

Since the standard almost complex structure is invariant under the S^1 action, we actually have an equivariant theory. If we integrate 1 over the equivariant fundamental class, the background polyfold for this moduli problem is an M-polyfold with one chart only and **A1 − 4** *are satisfied. M^{S^1} is one point representing C, hence we only need to figure out the weight and index in the normal direction, which is 1*[1] *and -2. Therefore by Corollary 10.11, $s^*(1) = u \neq 0$, which means this curve is equivariantly rigid. This also indicates that the equivariant transversality is actually obstructed here.*

Example 10.13. *Consider a C^2 small Morse function H, and J is time independent compatible almost complex structure for symplectic manifold (M, ω), such that Riemannian metric $\omega(\cdot, J\cdot)$ satisfies Morse-Smale condition for H. Let x, y be two critical points and $Z(x, y, A)$ be the polyfold of cylinders from x to y with homology class $A \in H_2(M)$. Then as long as $A \neq 0$, the S^1 action on $Z(x, y, A)$ has no fixed points. That is we can take quotient $Z(x, y, A)/S^1$, and equivariant transversality is unobstructed. Therefore, we can choose equivariant perturbations such that the contribution from $Z(x, y, A)$ to the Floer differential ∂_{HF} is zero. Therefore we only need to count the contributions from $Z(x, y, 0)$ where $\operatorname{ind} x - \operatorname{ind} y = 1$. A priori $Z(x, y, 0)$ is a polyfold has boundary and corner, however the boundary is either $Z(x, z, 0) \times Z(z, y, 0)$ or $Z(x, z, A) \times Z(x, y, -A)$ for $A \neq 0$. In the latter case, since one of the polyfolds has Fredholm index ≤ 0, we can assume there exists equivariant perturbation such that the zero set in $Z(x, z, A) \times Z(x, y, -A)$ is empty. For the formal one, we can assume $\operatorname{ind} x - \operatorname{ind} z \leq 0$, then $Z(x, z, 0)$ has no zeros in the fixed locus. Then we can find equivariant perturbation such that the zero set on $Z(x, z, 0)$ is empty. Therefore after equivariant perturbations, we can assume that Floer differential is defined only using $Z(x, y, 0)$ with $\operatorname{ind} x - \operatorname{ind} y = 1$, and $Z(x, y, 0)$ has no zeros on its boundary.*

[1] Since the orientation of the moduli space comes from the complex structure of $\ker \bar{\partial}$, hence our weight will be 1 instead of -1.

$Z(x,y,0)$ has no zeros on its boundary, then we can compute its non-equivariant/equivariant fundamental class. It is clear that the Floer differential $\langle \partial_{HF} x, y \rangle = s_{Z(x,y,0)_*}(1)$, where $s_{Z(x,y,0)_*}(\cdot)$ is the non-equivariant fundamental class associated with the sc-Fredholm section $s_{Z(x,y,0)} : Z(x,y,0) \to W(x,y,0)$. By Proposition 9.7, $s_{Z(x,y,0)_*}(1) \equiv s_{Z(x,y,0),S^1{}_*}(1) \mod u$. To compute the equivariant count $s_{Z(x,y,0),S^1{}_*}(1)$, we can apply the localization formula. Note that the fixed locus of $Z(x,y,0)$ is the M-polyfold $\mathcal{X}(x,y)$ of curves from x to y in the Morse theory of $(H, \omega(\cdot, J\cdot))$. Then the zero set in the fixed locus M^{S^1} is a set of points, such that $\#M^{S^1} = \langle \partial_M x, y \rangle$, where ∂M is the Morse differential. It is verified in Chapter 7 that the tubular neighborhood assumptions are satisfied for $Z(x,y,0)$ and $\operatorname{ind} D_i s = 0$ for $i \in \mathbb{N}$. Therefore by Corollary 10.11, $s_{Z(x,y,0),S^1{}_*}(1) = \#M^{S^1} = \langle \partial_M x, y \rangle$. Hence $\partial_{HF} = \partial_M$, this proves that $HF(M) \cong H^*(M, \Lambda)$ without going into the discussion of the equivariant transversality on $Z(x,y,0)$.

Part III

Morse-Bott Cohomology and Equivariant Cohomology

Chapter 11

Motivation From Homological Perturbation Theory

11.1 Notations in differential topology

Unless stated otherwise, all manifolds considered in this part are manifolds possibly with boundaries and corners [83], i.e. for every point in the manifold, there is an open neighborhood diffeomorphic to an open subset of $\mathbb{R}_+^n$, where $\mathbb{R}_+ := [0, \infty)$.

Definition 11.1. *Let M be a manifold and $x \in M$ be a point, by choosing a chart $\phi : \mathbb{R}_+^n \supset U \to M$ near $x \in M$, the* ***degeneracy index*** *$d(x)$ of the point x is defined to be $\#\{v_i | v_i = 0\}$, where $\phi(v_1, \ldots, v_n) = x \in M$.*

The degeneracy index d does not depend on the local chart ϕ. For $i \geq 0$, we define the depth-i boundary $\partial^i M$ to be

$$\partial^i M := \{x \in M | d(x) = i\}.$$

Then $\partial^0 M$ is the set of interior points of M. Note that all $\partial^i M$ are manifolds without boundary, and in most cases they are noncompact.

In this part, unless stated otherwise, a submanifold N of a manifold M is defined as follows:

Definition 11.2. *$N \subset M$ is a* ***submanifold*** *of M, iff N is a manifold, such that the inclusion $N \to M$ is smooth and for all $i \geq 0$, $\partial^i N = N \cap \partial^i M$.*

Note that $\partial^i M$ are not submanifolds of M for $i \geq 1$ in our sense, because $\partial^i M$ itself is a manifold without boundary and corner, $\partial^i M \cap \partial^i M \neq \partial^i(\partial^i M) = \emptyset$.

Definition 11.3. *We define transversality as follows:*

- *Let C be a* ***manifold without boundary****, B a submanifold of C and M a manifold. A smooth map $f : M \to C$ is* ***transverse to B****, iff $f|_{\partial^k M} \pitchfork B$ for all k, i.e. $\mathrm{D}f_x(T\partial^k M) + T_{f(x)}B = T_{f(x)}C$ for all k and $x \in \partial^k M$ such that $f(x) \in B$.*

- *Let M be a manifold, N_1, N_2 be two submanifolds, then we say* ***N_1 is transverse to N_2*** *iff for all $k \geq 0$ and any $x \in \partial^k N_1 \cap \partial^k N_2$, we have $T_x\partial^k N_1 + T_x\partial^k N_2 = T_x\partial^k M$, i.e. $\partial^k N_1$ is transverse to $\partial^k N_2$ in $\partial^k M$ in the classical sense.*

If $f : M \to C$ is transverse B, then $f^{-1}(B)$ is a submanifold in M in the sense of Definition 11.2. If N_1 is transverse to N_2 in M, then $N_1 \cap N_2$ is a submanifold of M.

Since measure-zero sets on differentiable manifolds are well defined, and our construction is based on integration, thus errors over a measure-zero set can be tolerated. In particular, we have the following useful notation:

Definition 11.4 (Diffeomorphism up to zero-measure)**.** *Let M, N be two manifolds. $f : M \to N$ is a* ***diffeomorphism up to zero-measure*** *iff there exist measure zero closed sets $M_1 \subset M, N_1 \subset N$, such that $f|_{M\backslash M_1} : M\backslash M_1 \to N\backslash N_1$ is a diffeomorphism.*

Orientations

This paragraph fixes our orientation conventions. Given an **oriented** vector bundle E over a manifold M, the determinant bundle $\det E$ is a trivial line bundle. $\det E$ can be reduced further to a trivial $\mathbb{Z}_2$ bundle $\operatorname{sign} E$. Moreover, we can assign a $\mathbb{Z}_2$ grading $|E| = \dim E - \dim M$ to $\operatorname{sign} E$. The fibers of $\operatorname{sign} E$ can be understood as equivalence classes of ordered basis $(e_1, \ldots, e_n)$ of each fiber E_x, two ordered bases are equivalent if the transformation matrix has positive determinant. The orientation of E is a section of $\operatorname{sign} E$, and we use $[E]$ to denote the orientation of the oriented vector bundle E.

Given two vector bundles E, F over M, we fix a bundle isomorphism:

$$m_{E,F} : \begin{array}{rcl} \operatorname{sign}(E) \otimes_{\mathbb{Z}_2} \operatorname{sign}(F) & \to & \operatorname{sign}(E \oplus F) \\ (e_1, \ldots, e_n) \otimes (f_1, \ldots, f_m) & \mapsto & (e_1, \ldots, e_n, f_1, \ldots, f_m). \end{array}$$

Therefore orientations $[E]$ and $[F]$ determine an orientation of $E \oplus F$ through $m_{E,F}$. We define $[E][F]$ to be the induced orientation $m_{E,F}([E], [F])$ of $E \oplus F$. Since $(e_1, \ldots, e_n, f_1, \ldots, f_m) = (-1)^{nm}(f_1, \ldots, f_m, e_1, \ldots, e_n)$, we have:

$$[E][F] = (-1)^{|F||E|}[F][E].$$

Definition 11.5. *For simplicity, we introduce the following notations:*

- *A manifold M is oriented iff the tangent bundle TM is oriented, and we use $[M]$ to denote $[TM]$ for the oriented manifold M.*

- *$\partial[M]$ denotes the induced orientation on the depth-1 boundary $\partial^1 M$ for an oriented manifold M [13].*

- *Unless stated otherwise, the product $M \times N$ is oriented by the product orientation of M and N. Let $[M \times N]$ denote the product orientation, then $\partial[M \times N] = (\partial[M]) \times [N] + (-1)^{\dim M}[M] \times (\partial[N])$*
- *If $f : M \to N$ is a diffeomorphism, then $f([M])$ is the orientation on N induced by $\mathrm{D}f : TM \to TN$ and $[M]$.*
- *Let $E \to N$ be an oriented vector bundle, $f : M \to N$ is a smooth map, then the bundle map $f^*E \to E$ induces a bundle map $\mathrm{sign}(f^*E) \to \mathrm{sign}(E)$. Through this map, the orientation $[E]$ induces an orientation on f^*E over M, the induced orientation is denoted by $f^*[E]$.*

As an example of such notations, we explain our orientation convention for the normal bundle N of the diagonal $\Delta \subset C \times C = C^1 \times C^2$ for an oriented closed manifold C. Δ is oriented by $\pi_1([\Delta]) = [C^1]$.[1] Then there exists a unique orientation of N, such that when restricted to Δ:

$$[T\Delta][N] = [TC^1][TC^2]|_{\Delta}.$$

For simplicity, we suppress the restrictions and the superscripts[2], and the equation becomes

$$[\Delta][N] = [C][C] \text{ or equivalently } [N][\Delta] = (-1)^{(\dim C_i)^2}[C][C]. \tag{11.1}$$

11.2 Flow categories.

First, we recall the definition of flow category from [26], which is the central concept in this part.

Definition 11.6. *A* ***flow category*** *is a small category $\mathcal{C}$ with the following properties:*

(F1) The objects space $Obj_{\mathcal{C}} = C$ is a manifold without boundary. $C = \sqcup_{i\in\mathbb{Z}} C_i$ is a disjoint union of closed oriented manifolds C_i.

(F2) The morphism space $Mor_{\mathcal{C}} = \mathcal{M}$ is a manifold. The source and target maps $s, t : \mathcal{M} \to C$ are smooth.

(F3) Let $\mathcal{M}_{i,j}$ denote $(s \times t)^{-1}(C_i \times C_j)$. Then $\mathcal{M}_{i,i} = C_i$, corresponding to the identity morphisms, and s, t restricted to $\mathcal{M}_{i,i}$ are identities. $\mathcal{M}_{i,j} = \emptyset$ for $j < i$, and $\mathcal{M}_{i,j}$ is a compact manifold for $j > i$.

(F4) Let $s_{i,j}, t_{i,j}$ denote $s|_{\mathcal{M}_{i,j}}, t|_{\mathcal{M}_{i,j}}$. For all strictly increasing sequence $i_0 < i_1 < \ldots < i_k$, $t_{i_0,i_1} \times s_{i_1,i_2} \times t_{i_1,i_2} \times \ldots \times s_{i_{k-1},i_k} : \mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,i_2} \times \ldots \times \mathcal{M}_{i_{k-1},i_k} \to C_{i_1} \times C_{i_1} \times C_{i_2} \times C_{i_2} \times \ldots \times C_{i_{k-1}} \times C_{i_{k-1}}$ is transverse to the submanifold $\Delta_{i_1} \times \ldots \times \Delta_{i_{k-1}}$ in the sense

[1] It is equivalent to $\pi_2([\Delta]) = [C^2]$

[2] We will never switch the order of the two copies of C, throughout this paper.

of definition 11.2. Therefore the fiber product $\mathcal{M}_{i_0,i_1} \times_{i_1} \mathcal{M}_{i_1,i_2} \times_{i_2} \ldots \times_{i_{k-1}} \mathcal{M}_{i_{k-1},i_k} = (t_{i_0,i_1} \times s_{i_1,i_2} \times t_{i_1,i_2} \times \ldots \times s_{i_{k-1},i_k})^{-1}(\Delta_{i_1} \times \Delta_{i_2} \times \ldots \times \Delta_{i_{k-1}}) \subset \mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,i_2} \times \ldots \times \mathcal{M}_{i_{k-1},i_k}$ *is a submanifold.*

(F5) The composition $m : \mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k} \to \mathcal{M}_{i,k}$ *is a smooth injective map into the boundary of* $\mathcal{M}_{i,k}$.

(F6) $m : \sqcup_{i<j<k} \partial^0(\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}) \to \partial^1 \mathcal{M}_{i,k}$ *is a diffeomorphism.*

In particular, by (F6), $\mathcal{M}_{i,i+1}$ are closed manifolds. Also (F6) guarantees the following form of Stokes' theorem with signs determined by orientation conventions which will be specified later:

$$\int_{\mathcal{M}_{i,k}} d\alpha = \sum_{i<j<k} \pm \int_{\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}} m^*\alpha, \forall \alpha \in \Omega^*(\mathcal{M}_{i,k}).$$

Let $\alpha \in \Omega^*(C_i), \beta \in \Omega^*(C_k)$ and $i < j < k$, because $s_{i,k} \circ m = s_{i,j}$ and $t_{i,k} \circ m = t_{j,k}$, then

$$\int_{m(\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k})} s_{i,k}^*\alpha \wedge t_{i,k}^*\beta = \int_{\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}} m^* s_{i,k}^*\alpha \wedge m^* t_{i,k}^*\beta = \int_{\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}} s_{i,j}^*\alpha \wedge t_{j,k}^*\beta. \quad (11.2)$$

Since we will only consider pullback of forms by source and target maps, it is convenient to think that $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$ is contained in $\partial \mathcal{M}_{i,k}$, and suppress the composition map m.

Example 11.7 (Flow category from a Morse-Bott function on a closed manifold)**.** *Fixing a Morse-Bott function f on a closed manifold M, then there are finitely many critical values $v_1 < \ldots < v_n$. Let C_i denote the critical manifold corresponding to the critical value v_i. Let $\mathcal{M}_{i,j}$ be the compactified moduli spaces of unparametrized gradient flow lines from C_i to C_j. The source map s and target map t are the evaluation maps at the two ends of the flow lines in $\mathcal{M}_{i,j}$. The composition map m is the concatenation of flow lines. It's a folklore theorem that $\mathcal{M}_{i,j}$ are smooth manifolds with boundary and corners if one chooses a generic metric, c.f. [6, 44] and chapter 16. Then $\{C_i, \mathcal{M}_{i,j}\}$ form a flow category.*

Similar construction also exists in the Floer theory, as long as there is a background "Morse-Bott" function in the theory and all the transversality conditions are met. For example, [26] gives an explicit construction of the flow category for the Hamiltonian Floer cohomology theory on $\mathbb{CP}^n$, where the background Morse-Bott function is the symplectic action functional with Hamiltonian $H = 0$. There are also flow categories without background Morse-Bott function, for example, the flow category for Khovanov homology [75].

Remark 11.8. *In the context of Floer theory, the moduli spaces may not be manifolds in general, but some spaces with local symmetries, e.g. branched suborbifolds in [60]. Every argument in this paper holds for branched suborbifolds, since there is a Stokes' theorem for branched suborbifolds [63].*

Remark 11.9. *When the flow category comes from a Morse-Bott function f, but f is not single valued[3], we need to lift f to $\tilde{f}$ over the cyclic cover [26]. We will see the cochain*

[3] For example, Hamiltonian Floer cohomology on (M, ω) with $\omega|_{\pi_2(M)} \neq 0$

complex we give in Definition 12.6 is exactly the Novikov cochain complex.

Remark 11.10. *In the definition of flow category, we require C_i to be compact, this is only for simplicity. In fact, we only need each component of C_i is closed, but C_i can have infinite components. The compactness of $\mathcal{M}_{i,j}$ can be weakened to either one of the following two conditions:*

- *For each component C_i^α of C_i, $(s_{i,j} \times t_{i,j})^{-1}(C_i^\alpha, C_j)$ is compact.*
- *For each component C_j^α of C_j, $(s_{i,j} \times t_{i,j})^{-1}(C_i \times C_j^\alpha)$ is compact.*

All results in this part hold for these settings.

Remark 11.11. *The construction of the minimal Morse-Bott cochain complex can be generalized to the case that C_i^α are noncompact manifolds, and we need to require the properness of the source or target maps. Such generalization will appear in our future work.*

Remark 11.12. *If there is a background Morse-Bott function f, sometimes it is impossible to partition critical manifolds by $\mathbb{Z}$ and in the order of increasing critical values, i.e. critical values may accumulate. For example, Hamiltonian Floer cohomology with Novikov coefficient will have this problem, if the symplectic form is irrational. However, by Gromov compactness for Hamiltonian Floer cohomology, there is action gap $\hbar$, such that there are no non-constant flow lines for energy smaller than $\hbar$. Then we can still divide all critical manifolds into groups indexed by $\mathbb{Z}$, such that there are no non-constant flow lines inside each group, and flow category can still be defined.*

Remark 11.13. *The assumption of C_i being oriented can be dropped with the price of working with local systems. We discuss this in chapter 14.*

All the formulas in this paper work component-wisely for components in $C_i, \mathcal{M}_{i,j}$. Therefore for simplicity, we assume $\dim \mathcal{M}_{i,j}$ and $\dim C_i$ are well defined, denote them by $m_{i,j}$ and c_i. We formally define $m_{i,i} := c_i - 1$. Since $t_{i,j} \times s_{j,k} : \mathcal{M}_{i,j} \times \mathcal{M}_{j,k} \to C_j \times C_j$ is transverse to Δ_j and $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$ is part of the boundary of $\mathcal{M}_{i,k}$, we have:

$$m_{i,j} + m_{j,k} - c_j + 1 = m_{i,k}, \forall i \leq j \leq k. \tag{11.3}$$

Definition 11.14. *Assume $\dim C_i$ and $\dim \mathcal{M}_{i,j}$ are well-defined. A flow category is **graded** if for each $i \in \mathbb{Z}$, there is an integer d_i, such that $d_i = d_j + c_j - m_{i,j} - 1$ for all $i < j$*

Remark 11.15. *When $\dim C_i$ and $\dim \mathcal{M}_{i,j}$ are not well-defined, we can still define graded flow category, which is a flow category with an assignment of the index d_α to each component C_α of C, such that they satisfy similar relations.*

In the case of finite dimensional Morse-Bott theory, d_i can be the dimension of the negative eigenspace of $Hess(f)$ on C_i. For Hamiltonian-Floer cohomology, d_i is related to the generalized Conley-Zehnder index [90].

Next, we define the orientations on a flow category. Since $t_{i,j}\times s_{j,k}:\mathcal{M}_{i,j}\times\mathcal{M}_{j,k}\to C_j\times C_j$ is transverse to the diagonal Δ_j, the pullback $(t_{i,j}\times s_{j,k})^*N_j$ of the normal bundle N_j of Δ_j by $t_{i,j}\times s_{j,k}$ is the normal bundle of $\mathcal{M}_{i,j}\times_j\mathcal{M}_{j,k}=(t_{i,j}\times s_{j,k})^{-1}(\Delta_j)$ in $\mathcal{M}_{i,j}\times\mathcal{M}_{j,k}$. Since N_j is oriented using $[N_j][\Delta_j]=(-1)^{(\dim C_j)^2}[C_j][C_j]$, we can pull back this orientation to orient the normal bundle of $\mathcal{M}_{i,j}\times_j\mathcal{M}_{j,k}$. Then we define the coherent orientations on a flow category as follows:

Definition 11.16 (Orientation). *A coherent orientation on a flow category is an assignment of orientations for each $\mathcal{M}_{i,j}$ and $\mathcal{M}_{i,j}\times_j\mathcal{M}_{j,k}$, such that the following two relations hold:*

$$\begin{aligned}(t_{i,j}\times s_{j,k})^*[N_j][\mathcal{M}_{i,j}\times_j\mathcal{M}_{j,k}] =&(-1)^{c_jm_{i,j}}[\mathcal{M}_{i,j}][\mathcal{M}_{j,k}]\\ \partial[\mathcal{M}_{i,k}] =&\sum_j(-1)^{m_{i,j}}m\left([\mathcal{M}_{i,j}\times_j\mathcal{M}_{j,k}]\right).\end{aligned}$$

Or equivalently we assign orientations on $\mathcal{M}_{i,j}$ such that

$$(t_{i,j}\times s_{j,k})^*[N_j]m^{-1}\left(\partial[\mathcal{M}_{i,k}]|_{m(\mathcal{M}_{i,j}\times_j\mathcal{M}_{j,k})}\right)=(-1)^{(c_j+1)m_{i,j}}[\mathcal{M}_{i,j}][\mathcal{M}_{j,k}].$$

We discuss how coherent orientations arise in Floer theories in Chapter 14.

11.3 Review of existing constructions

Throughout this section, we fix a flow category $\mathcal{C}=\{C_i,\mathcal{M}_{i,j}\}$, such that there are finitely many C_i. Before giving our construction of the minimal Morse-Bott cochain complex in section 12.2. We review the existing literature on cohomology theories for flow categories. For simplicity, we neglect the signs and orientations completely in this section.

Austin-Braam's Morse-Bott cochain complex (BC^{AB},d^{AB})

In [6], Austin and Braam defined the Morse-Bott cochain complex of a flow category to be

$$(BC^{AB}:=\oplus_i\Omega^*(C_i),d^{AB}),$$

where $\Omega^*(C_i)$ is the space of differential forms on C_i. The differential d^{AB} is defined as $\sum_{k\geq 0}d_k$, where d_k is defined by:

$$\begin{aligned}d_0:=\mathrm{d} &: \Omega^*(C_i)\to\Omega^*(C_i) \text{ is the usual exterior differential on differential forms;}\\ d_k &: \Omega^*(C_i)\to\mathcal{D}^*(C_{i+k}), d_k(\alpha)=t_{i,i+k_*}s^*_{i,i+k}(\alpha),\text{ for } k\geq 1..\end{aligned}\tag{11.4}$$

$\mathcal{D}^*(C)$ is the space of currents on C. The operator d_k taking value in currents instead of differential forms causes difficulties to prove that (BC^{AB},d^{AB}) is a cochain complex. Thus to make the cochain complex (BC^{AB},d^{AB}) well-defined, the target maps $t_{i,j}$ are assumed to be fiberations in Austin-Braam's model. Under such assumptions, t_{i,j_*} is the integration along the fiber. However it was noticed in [70] that the fiberation condition is obstructed for some Morse-Bott functions, i.e. there exists a Morse Bott function, such that the fiberation property fails for all metrics.

Fukaya's Morse-Bott chain complex

Fukaya [44] gave a construction similar to Austin-Braam's model, but using singular chains instead of differential forms. The chain complex is defined to be

$$(BC^F := \oplus_i C_*(C_i), \partial^F),$$

where $C_*(C_i)$ is the space of singular chains on C_i. $\partial^F = \sum_{k\geq 0} \partial_k$, where ∂_k is defined by:

$$\begin{aligned} \partial_0 = \partial \quad &: \quad C_*(C_i) \to C_*(C_i) \text{ is the usual boundary operator on singular chains;} \\ \partial_k \quad &: \quad C_*(C_{i+k}) \to C_*(C_i), \partial_k(P) = s_{i,i+k_*} t^*_{i,i+k}(P), \text{ for } k \geq 1. \end{aligned}$$

The pushforward is well-defined. The pullback is defined as follows: Let $P : \Delta \to C_{i+k}$ be a singular chain. Assume transversality, $\Delta \times_{C_{i+k}} \mathcal{M}_{i,i+k}$ is a manifold with boundary and corner, hence the projections to the second factor[4]

$$\pi_{\mathcal{M}_{i,i+k}} : \Delta \times_{C_{i+k}} \mathcal{M}_{i,i+k} \to \mathcal{M}_{i,i+k}$$

can be understood as a singular chain in $\mathcal{M}_{i,i+k}$, and the pullback $t^*_{i,i+k}(P)$ is defined to be this chain.

To guarantee this pullback is well-defined for all singular chains in C_{i+k}, one also needs to assume the target map $t_{i,i+k}$ is a fibration. To drop this constraint, Fukaya constructed a subset $C_{geo}(C_i)$ of the singular chain complex, such that the fiber products in pullback are defined over $C_{geo}(C_i)$, and the operators ∂_k are closed on $C_{geo}(C_i)$. Then

$$(\oplus_i C_{geo}(C_i), \sum_{k\geq 0} \partial_k)$$

defines a chain complex.

Cascades

The cascades construction was first introduced in [14]. For each C_i we choose a Morse-Smale pair (f_i, g_i). Then the cascades cochain complex is defined to be

$$(BC^C := \oplus_i MC(f_i, g_i), d^C),$$

where $MC(f_i, g_i)$ is the Morse cochain complex for (C_i, f_i, g_i). The differential d^C is defined to be $\sum_{k\geq 0} d^C_k$, where d^C_k is defined by:

$$\begin{aligned} d^C_0 = d_M \quad &: \quad MC(f_i, g_i) \to MC(f_i, g_i) \text{ is the usual Morse differential for } (f_i, g_i); \\ d^C_k \quad &: \quad MC(f_i, g_i) \to MC(f_{i+k}, g_{i+k}) \text{ for } k \geq 1 \\ & \qquad \text{defined by counting rigid cascades from } C_i \text{ to } C_{i+k}. \end{aligned}$$

[4] To be more precise, we need to choose a triangulation of $\Delta \times_{C_{i+k}} \mathcal{M}_{i,i+k}$

A 0-cascade is a unparameterized gradient flow line for (f_i, g_i). For $k \geq 1$, A k-cascade from $a \in crit(f_i)$ to $b \in crit(f_j)$ for $i < j$ is a tuple for $r_0 := i < r_1 < \ldots < r_k < r_{k+1} := j$:

$$(\gamma_i, t_i^+, m_{i,r_1}, \gamma_{r_1}, t_{r_1}^-, t_{r_1}^+, \ldots, m_{r_{k-1},r_k}, \gamma_{r_k}, t_{r_k}^-, t_{r_k}^+, m_{r_k,j}, \gamma_j, t_j^-),$$

where γ_* is a gradient flow line in C_*, and $m_{*,*}$ is a point in $\mathcal{M}_{*,*}$, t_*^-, t_*^+ are real numbers, such that $-\infty < t_*^- < t_*^+ < \infty$, $\gamma_i(-\infty) = a, \gamma_j(+\infty) = b$ and $\gamma_{r_s}(t_{r_s}^+) = s(m_{r_s,r_{s+1}})$, $\gamma_{r_s}(t_{r_s}^-) = t(m_{r_{s-1},r_s})$.

When appropriate transversality assumptions are met, the moduli space of all cascades from a to b is a manifold. Moreover, there is a natural compactification of the moduli space by including the "broken" cascades. Then the differential d^C for cascades cochain complex comes from counting the zero dimensional moduli space of cascades.

11.4 Homological perturbation theory and reductions of the cochain complexes

If we want to generalize Austin-Braam's construction beyond fiberations, we need to confront the challenge of d_k taking value in the space of current. The operator d_k is defined using pushforward of differential forms, since pushforward is defined as the dual operator of pullback. Therefore the problem is rooted in the fact that the dual space of differential forms $\Omega^*(C^i)$ is the space of currents $\mathcal{D}^*(C_i)$ instead of itself. While this is never a problem for finite dimensional vector spaces. On the other hand, homological perturbation lemma provides a method for constructing small cochain complexes from larger ones. This suggests one might be able to "apply" the homological perturbation lemma to the "almost existing" Austin-Braam's cochain complex $(\bigoplus_i \Omega^*(C_i), \sum_{k\geq 0} d_k)$, and constructs a much smaller cochain complex, which is well-defined for any flow category.

A homological perturbation theorem

Roughly speaking, homological perturbation lemma is a procedure that takes in a cochain complex and some perturbation data (in most cases, some projections and some homotopies) and spits out another cochain complex, which is quasi-isomorphic to the input cochain complex. In our case here, consider a cochain complex in the form of $A = \oplus_{i=1}^n A_i$, where A_i are $\mathbb{Z}_2$ linear spaces. The differential d is defined as $\sum_{k\geq 0} d_k$ with $d_k : A_i \to A_{i+k}$ for $k \geq 0$. Then $d^2 = 0$ implies that (A_i, d_0) is also a cochain complex for all i. Moreover, if we are given projections $p_k : A_k \to A_k$, such that there exist homotopies $H_k : A_k \to A_k$ between the identity and p_k, i.e.

$$\mathrm{id} - p_k = d_0 \circ H_k + H_k \circ d_0. \tag{11.5}$$

We will call the projection-homotopy family $\{(p_i, H_i)\}$ a perturbation data. With a perturbation data, we have the following homological perturbation lemma.

Lemma 11.17. *There is a differential on $\bigoplus_i p_i(A_i)$, such that the natural inclusion $\bigoplus_i p_i(A_i) \hookrightarrow A$ is a quasi-isomorphism.*

The lemma holds for general coefficient rings, once appropriate signs are assigned. Since we do not have a rigorously defined cochain complex $(\oplus_i \Omega^*(C_i), \sum_{k\geq 0} d_k)$ to start with, we will not apply this lemma and we omit the proof. What is relevant to our purpose is the formula for the differential on $\bigoplus p_i(A_i)$, which can be viewed as an analog of the perturbation theorem for A_∞ structures proved in [68]. For a strictly increasing sequence of integers $T = \{i_0 = 0, i_1, \ldots, i_r = k\}$, we define the an operator $D_{k,T} : p_i(A_i) \to p_{i+k}(A_{i+k})$ for all integer i:

$$D_{k,T} = p_{i+k} \circ d_{i_r - i_{r-1}} \circ H_{i+i_{r-1}} \circ \ldots \circ H_{i+i_2} \circ d_{i_2-i_1} \circ H_{i+i_1} \circ d_{i_1 - i_0} \circ \iota_i, \tag{11.6}$$

where $\iota_i : p_i(A_i) \to A_i$ denotes the inclusion. The new differential D on $\bigoplus_i p_i(A_i)$ is defined as

$$D = \sum_{k=0}^{\infty} D_k,$$

where $D_k = \sum_T D_{k,T}$ is a summation over all strictly increasing sequence T from 0 to k. (11.6) will motivate the formula for the differential (Definition 12.6) we put on our minimal Morse-Bott cochain complex.

Cascades from homological perturbation

Unlike the Austin-Braam's cochain complex, the cascades construction does not require the fiberation condition in general. It turns out the reason of this is that cascades construction can be understood as "apply" Lemma 11.17 to the Austin-Braam's Morse-Bott cochain complex. In this subsection, we will explain this claim heuristically. To be more specific, we need to choose the perturbation data, i.e. a family of projections and homotopies (p_i, H_i) on $\Omega^*(C_i)$. We require that the image $\operatorname{im} p_i$ is a finite dimensional subspace of $\Omega^*(C_i)$. Given such perturbation data, we need to check that operators $D_{k,T}$ from (11.6) is well-defined, and that $(\oplus_i \operatorname{im} p_i, D)$ is a cochain complex. $D_{K,T}$ tends to be well-defined due to the finite dimensional condition, and $(\oplus_i \operatorname{im} p_i, D)$ tends to be a cochain complex because of the homological perturbation lemma.

One choice of perturbation data (p_i, H_i) is given by the construction in [55]. Before giving the construction, we set up some notations first:

- $\mathcal{D}^*(C)$ denotes the space of currents on C, for basics of currents, we refer readers to [51]. Here, we only fix the sign for the inclusion $\iota : \Omega^*(C) \to \mathcal{D}^*(C)$, for $\alpha, \beta \in \Omega^*(C)$,

$$\iota(\alpha)(\beta) = \int_C \alpha \wedge \beta.$$

- Let any current $k \in \mathcal{D}^*(C \times C)$, then the induced integral operator $I_k : \Omega^*(C_i) \to \mathcal{D}^*(C_i)$ is defined as:

$$I_k(\alpha)(\beta) = (-1)^{\dim C} k(\pi_1^*\alpha \wedge \pi_2^*\beta) \text{ for } \alpha \in \Omega^*(C_i), \forall \beta \in \Omega^*(C_i), \tag{11.7}$$

  where π_1/π_2 are projections of $C_i \times C_i$ to the first/second factor.

- For a closed manifold C, any oriented submanifold (possibly with boundary and corner) B defines a current $[B] \in \Omega^*(C)$ by $[B](\alpha) = \int_B \alpha|_B$ for $\alpha \in \Omega^*(C)$.

To construct the perturbation data, we consider a Morse-Smale pair (f_i, g_i) on the critical manifold C_i. $Crit(f_i)$ is the set of critical points of f_i. Let $\phi_t^i : C_i \to C_i$ denote the time-t flow of the gradient vector field $\nabla_{g_i} f_i$ on C_i. Then the pullback operator ${\phi_{-t}^i}^* : \Omega^*(C_i) \to \Omega^*(C_i)$ can also be understood as the integral operator $I_{[\operatorname{graph} \phi_t^i]}$ of $\operatorname{graph} \phi_t^i := \{(x, \phi_t^i(x))\} \subset C_i \times C_i$. The submanifold $\cup_{0<t'<t} \operatorname{graph} \phi_{t'}^i \subset C_i \times C_i$ defines an integral operator $H_t^i := I_{[\cup_{0<t'<t} \operatorname{graph} \phi_{t'}^i]}$, and they satisfy

$$\mathrm{id} - {\phi_{-t}^i}^* = \mathrm{d} \circ H_t^i + H_t^i \circ \mathrm{d}. \tag{11.8}$$

It was proven in [55] that when $t \to \infty$, (11.8) converges to a projection-homotopy relations like (11.5) . To be more specific, let U_x, S_x denote the unstable and stable submanifolds of critical point $x \in Crit(f_i)$, i.e.

$$\begin{aligned} U_x &= \{y \in C_i | \lim_{t \to -\infty} \phi_t^i(y) = x\}; \\ S_x &= \{y \in C_i | \lim_{t \to \infty} \phi_t^i(y) = x\}. \end{aligned}$$

Then $S_x \times U_x$ is a submanifold of $C_i \times C_i$ for critical point $x \in Crit(f_i)$. It was shown in [55], in the sense of current, we have[5] :

$$\lim_{t \to \infty} \left[\operatorname{graph} \phi_t^i\right] = \sum_{x \in Crit(f_i)} [S_x \times U_x]; \tag{11.9}$$

$$\lim_{t \to \infty} \left[\bigcup_{0<t'<t} \operatorname{graph} \phi_{t'}^i\right] = \left[\bigcup_{0<t'<\infty} \operatorname{graph} \phi_{t'}^i\right]. \tag{11.10}$$

In particular, (11.9) (11.10) define two integral operators ${\phi_{-\infty}^i}^*, H_\infty^i : \Omega^*(C_i) \to \mathcal{D}^*(C_i)$, such that

$$\iota - {\phi_{-\infty}^i}^* = \mathrm{d} \circ H_\infty^i + H_\infty^i \circ \mathrm{d}, \tag{11.11}$$

where ι is the natural embedding $\Omega^*(C_i) \hookrightarrow \mathcal{D}^*(C_i)$. Notice that

$${\phi_{-\infty}^i}^*(\alpha) := \sum_{x \in Crit(f_i)} \left(\int_{C_i} \alpha \wedge [S_x]\right) \cdot [U_x] = \sum_{x \in Cr(f_i)} \left(\int_{S_x} \alpha|_{S_x}\right) \cdot [U_x]$$

[5] For orientations, c.f. [55].

can be viewed as the projection from a differential from to the Morse complex. (11.11) shows that H^i_∞ defines a homotopy between ι and the projection ${\phi^i_{-\infty}}^*$. Strictly speaking, (11.11) is not a genuine projection-homotopy relation (11.5), since ${\phi^i_{-\infty}}^*$ lands in space of currents instead of differential forms. From now on, we will neglect such issues, and show formally that the cascades construction can be understood as applying the construction in (11.6) using the perturbation data $({\phi^i_{-\infty}}^*, H^i_\infty)$.

By (11.6), for $x \in Crit(f_i)$, the first term D_0 in $D = \sum_{k\geq 0} D_k$ is defined by

$$D_0([U_x]) = {\phi^i_{-\infty}}^*(d_0([U_x])) = {\phi^i_{-\infty}}^*(\mathrm{d}([U_x])) = \sum_{y\in Crit(f_i)} \left(\int_{C_i} \mathrm{d}([U_x]) \wedge [S_y]\right) \cdot [U_y].$$

It was proven in [55] that $\int_{C_i} \mathrm{d}([U_x]) \wedge [S_y]$ is well-defined and equals to the signed counts of rigid gradient flow lines from x to y. Therefore D_0 recovers the Morse differential on C_i. There are several issues of applying formula (11.6) for higher terms, one of them is pullback of currents. We will neglect such technical issues. Let $x \in Crit(f_i)$. Then

$$\begin{aligned} D_1([U_x]) = {\phi^i_{-\infty}}^* d_1[U_x] &= \sum_{y\in Crit(f_{i+1})} \left(\int_{C_{i+1}} d_1[U_x] \wedge [S_y]\right) \cdot [U_y] \\ &\overset{(11.4)}{=} \sum_{y\in Crit(f_{i+1})} \left(\int_{\mathcal{M}_{i,i+1}} s^*_{i,i+1}[U_x] \wedge t^*_{i,i+1}[S_y]\right) \cdot [U_y]. \end{aligned}$$

When $s^{-1}_{i,i+1}(U_x) \pitchfork t^{-1}_{i,i+1}(S_y)$, $\int_{\mathcal{M}_{i,i+1}} s^*_{i,i+1}[U_x] \wedge t^*_{i,i+1}[S_y]$ can be interpreted as $\#\{s^{-1}_{i,i+1}(U_x) \cap t^{-1}_{i,i+1}(S_y)\}$, thus D_1 counts the 1-cascade as defined in [14, 43]. By the same argument $D_{2,\{0,2\}}$ counts rigid 1-cascade from C_i to C_{i+2}.

Next, we consider operator $D_{2,\{0,1,2\}}$, $D_{2,\{0,1,2\}}([U_x])$ equals to:

$$\begin{aligned} & {\phi^i_{-\infty}}^* \circ d_1 \circ H^{i+1}_\infty \circ d_1([U_x]) \\ = & \sum_{y\in Crit(f_{i+2})} \left(\int_{C_{i+2}} (d_1 \circ H^{i+1}_\infty \circ d_1[U_x]) \wedge [S_y]\right) \cdot [U_y] \\ \overset{(11.4)}{=} & \sum_{y\in Crit(f_{i+2})} \left(\int_{\mathcal{M}_{i+1,i+2}} s^*_{i+1,i+2}(H_{i+1} \circ d_1[U_x]) \wedge t^*_{i+1,i+2}[S_y]\right) \cdot [U_y] \\ \overset{(11.10)}{=} & \sum_{y\in Crit(f_{i+2})} \left(\int_{\mathcal{M}_{i,i+1}\times\mathcal{M}_{i+1,i+2}} s^*_{i,i+1}[U_x] \wedge (t_{i,i+1} \times s_{i+1,i+2})^* \left[\bigcup_{0<t'<\infty} \mathrm{graph}\, \phi^{i+1}_{t'}\right]\right. \\ & \left. \wedge t^*_{i+1,i+2}[S_y]\right) \cdot [U_y]. \end{aligned}$$

When transversality holds, it equals to

$$\sum_{y\in Cr(f_{i+2})} \# \left\{ \left(s^{-1}_{i,i+1}(U_x) \times t^{-1}_{i+1,i+2}(S_y)\right) \pitchfork \left((t_{i,i+1} \times s_{i+1,i+2})^{-1} (\bigcup_{0<t'<\infty} \mathrm{graph}\, \phi^{i+1}_{t'}) \right) \right\} \cdot [U_y].$$

It can be interpreted as 2-cascades from C_i to C_{i+2}. In general, assume the transversality for the cascades moduli spaces, we shall recover the whole cascades construction from (11.6), thus the cascades construction fits into the homological perturbation philosophy.

Remark 11.18. *The cascades cochain complex is defined for any flow category, i.e. there is no need for the fiberation condition. But we can not choose the auxiliary Morse-Smale pair* (f_i, g_i) *freely. The choices depend on the structures of the flow category. Roughly speaking, this is because* (11.11) *is not a genuine projection-homotopy relation, it can be made into a genuine projection-homotopy relation, if we extend* $\Omega^*(C_i)$ *by adding some currents. But the extension depends on the structures of the flow category, just like the geometric chain complex* $C_{geo}(C_i)$ *depends on the structures of the flow category.*

Chapter 12

The Minimal Morse-Bott Cochain Complexes

In this chapter, we construct the minimal Morse-Bott cochain complex for a general oriented flow category, which can be applied to both finite dimensional Morse-Bott theory and Floer theories. It can be viewed as another application of the homological perturbation Lemma 11.17 based on a different perturbation data.

This chapter is organized as follows. Section 12.1 constructs the perturbation data for the minimal Morse-Bott cochain complex, which motivates the formula for the differential on the minimal cochain complex. Section 12.2 constructs the Morse-Bott cochain complexes for every oriented flow category. Section 12.3 defines flow-morphisms which can be viewed as the generalizations of the continuation maps and shows that flow-morphisms induce morphisms between Morse-Bott cochain complexes. Section 12.4 discusses the compositions of flow-morphisms. Section 12.5 defines flow-homotopies and proves that flow-homotopies induce homotopies between morphisms. Section 12.6 establishes that our theory is canonical on the cochain complex level, and is independent of all the choices. The proofs in this chapter involve a lot of index and sign computations, we provide the detailed proof of the coboundary map d_{BC} in section 12.2, but we defer the proofs of other results in sections 12.3, 12.4 and 12.5 to section 12.7.

12.1 Perturbation data for the minimal Morse-Bott cochain complexes

In this section, we construct the perturbation data $\{(p_i, H_i)\}$ for our minimal Morse-Bott cochain complex for an oriented flow category $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$, and then we write down the operators $D_{k,T}$ in the spirit of formula (11.6). Such operators are important ingredients for the differential on the minimal Morse-Bott cochain complex.

Projection p_i

We define the pairing on $\Omega^*(C_i)$:

$$\langle \alpha, \beta \rangle_i := (-1)^{\dim C_i \cdot |\beta|} \int_{C_i} \alpha \wedge \beta \text{ for } \alpha, \beta \in \Omega^*(C_i). \tag{12.1}$$

We pick representatives $\{\theta_{i,a}\}_{1 \le a \le \dim H^*(C_i)}$ of a basis of $H^*(C_i)$ in $\Omega^*(C_i)$, such choices give us a quasi-isomorphic embedding:

$$H^*(C_i) \to \Omega^*(C_i).$$

Let $h(i)$ denote the image of the embedding above. $\{\theta^*_{i,a}\}_{1 \le a \le \dim H^*(C_i)} \subset h(i)$ is the dual basis to the basis $\{\theta_{i,a}\}$ in the sense that

$$\langle \theta^*_{i,a}, \theta_{i,b} \rangle_i = \delta_{ab}.$$

Then we can define projection $p_i : \Omega^*(C_i) \to h(i) \subset \Omega^*(C_i)$:

$$p_i(\alpha) := \sum_{a=1}^{\dim H^*(C_i)} \langle \alpha, \theta_{i,a} \rangle_i \cdot \theta^*_{i,a}.$$

We can identify $H^*(C_i)$ with $h(i)$, therefore p_i can be thought as a projection from $\Omega^*(C_i)$ to $H^*(C_i)$.

Homotopy H_i

The Poincaré dual of the diagonal Δ_i can be represented by Thom classes. If we identify a tubular neighborhood of the diagonal $\Delta_i \subset C_i \times C_i$ with the unit disk bundle[1] of the normal bundle N of Δ_i, then we can consider the following sequence of Thom classes of the diagonal Δ_i supported in the tubular neighborhood:

$$\delta^n_i := \mathrm{d}(\rho_n \psi),$$

where ψ_i is the angular form of the sphere bundle $S(N)$, c.f. [13], and $\rho_n : \mathbb{R}^+ \to \mathbb{R}$ are smooth functions, such that ρ_n is increasing, supported in $[0, \frac{1}{n}]$ and is -1 near 0.

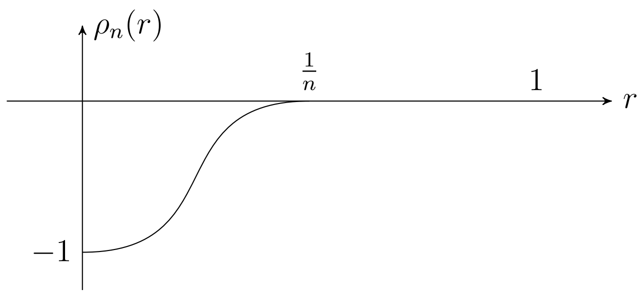


Figure 12.1: Graph of ρ_n

[1]with respect to some auxiliary metric

For details of this construction, we refer readers to [13]. We also include a brief discussion of this construction and properties of it in Appendix C.1.

Lemma 12.1 (Appendix C.1)**.** *δ_i^n converges to the Dirac current δ_i of the diagonal Δ_i in the sense of current, i.e. $\forall \alpha \in \Omega^*(C_i \times C_i)$*

$$\lim_{n\to\infty}\int_{C_i\times C_i}\alpha\wedge\delta_i^n=\int_{C_i\times C_i}\alpha\wedge\delta_i:=\int_{\Delta_i}\alpha|_{\Delta_i}=[\Delta_i](\alpha).$$

By (11.7), $\int_{C_i\times C_i}\alpha\wedge\delta_i^n=(-1)^{\dim C_i}\int\delta_i^n\wedge\alpha=I_{\delta_i^n}$, then Lemma 12.1 asserts that

$$\lim_{n\to\infty}I_{\delta_i^n}=I_{\delta_i}=\mathrm{id}\,.$$

On the other hand, if we fix a basis $\{\theta_{i,a}\}$ and a dual basis $\{\theta^*_{i,a}\}$, under the orientation convention (11.1), $\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta^*_{i,a}$ is cohomologous to δ_i^n for all n, where π_1/π_2 are the projections to the first/second factor of $C_i\times C_i$.

Proposition 12.2. *$\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta^*_{i,a}$ is cohomologous to δ_i^n for all n.*

Proof. Since δ_i^n are cohomologous to each other for different n, therefore Lemma 12.1 implies that if $\alpha\in\Omega^*(C_i\times C_i)$ is closed, then for all n,

$$\int_{C_i\times C_i}\alpha\wedge\delta_i^n=\int_{\Delta_i}\alpha|_{\Delta_i}.$$

In view of the Poincaré duality, it suffices to show that for all closed $\alpha\in\Omega^*(C_i\times C_i)$,

$$\int_{C_i\times C_i}\alpha\wedge\left(\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta^*_{i,a}\right)=\int_{\Delta_i}\alpha|_{\Delta_i}. \tag{12.2}$$

Since the cohomology of $C_i\times C_i$ is spanned by $\{\pi_1^*\theta^*_{i,a}\wedge\pi_2^*\theta_{i,b}\}$, thus it is enough to verify (12.2) for $\alpha=\pi_1^*\theta^*_{i,c}\wedge\pi_2^*\theta_{i,d}$. Since $\left\langle\theta_{i,a},\theta^*_{i,b}\right\rangle_i=\delta_{ab}$, if $c\neq d$,

$$\int_{C_i\times C_i}\pi_1^*\theta^*_{i,c}\wedge\pi_2^*\theta_{i,d}\wedge\left(\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta^*_{i,a}\right)=0=\int_{C_i}\theta^*_{i,c}\wedge\theta_{i,d}.$$

When $c=d$,

$$\begin{aligned}\int_{C_i\times C_i}\pi_1^*\theta^*_{i,c}\wedge\pi_2^*\theta_{i,c}\wedge\left(\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta^*_{i,a}\right) &= \int_{C_i\times C_i}\pi_1^*\theta^*_{i,c}\wedge\pi_2^*\theta_{i,c}\wedge\pi_1^*\theta_{i,c}\wedge\pi_2^*\theta^*_{i,c}\\ &= (-1)^{|\theta_{i,c}|^2+|\theta_{i,c}|\cdot|\theta^*_{i,c}|}\int_{C_i\times C_i}\pi_1^*\theta^*_{i,c}\wedge\pi_1^*\theta_{i,c}\wedge\pi_2^*\theta^*_{i,c}\wedge\pi_2^*\theta_{i,c}\\ &= (-1)^{|\theta_{i,c}|^2+|\theta_{i,c}|\cdot|\theta^*_{i,c}|+\dim C_i|\theta_{i,c}|}\left(\int_{C_i}\theta^*_{i,c}\wedge\theta_{i,c}\right)\left\langle\theta^*_{i,c},\theta_{i,c}\right\rangle\\ &= \int_{C_i}\theta^*_{i,c}\wedge\theta_{i,c}.\end{aligned}$$

Thus (12.2) is proven. $\square$

By Proposition 12.2, there exist primitives $f_i^n \in \Omega^*(C_i \times C_i)$ such that

$$\begin{aligned} \mathrm{d}f_i^n &= \delta_i^n - \sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta_{i,a}^*; \\ f_i^n - f_i^m &= (\rho_n - \rho_m)\psi_i. \end{aligned} \tag{12.3}$$

Since the integral operator I_{δ_i} of the Dirac current δ_i is the identity map from $\Omega^*(C_i)$ to itself. The integral operator $I_{\sum_a \pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*}$ is the projection p_i we constructed in the previous paragraph. Therefore by (12.3), the integral operator $I_{f_i^n}$ of the primitive f_i^n satisfies the relation:

$$I_{\delta_i^n} - I_{\sum_a \pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*} = \mathrm{d} \circ I_{f_i^n} + I_{f_i^n} \circ \mathrm{d}. \tag{12.4}$$

It is proven in the appendix C.1 that f_i^n converges to a current $f_i \in \mathcal{D}^*(C_i \times C_i)$, and the corresponding integral operator I_{f_i} satisfies the following relation:

$$\mathrm{id} - p_i = \mathrm{d} \circ I_{f_i} + I_{f_i} \circ \mathrm{d},$$

which is the limit of (12.4). Therefore the integral operator $I_{f_i} := \lim I_{f_i^n}$ plays the role of homotopy in our perturbation data. We define:

Definition 12.3. *A* ***defining-data*** Θ *for an oriented flow category* $\mathcal{C}$ *consists of:*

- *quasi-isomorphic embeddings* $H^*(C_i) \to \Omega^*(C_i)$. *The image is denoted by* $h(\mathcal{C}, i)$ *and we fix basis* $\{\theta_{i,a}\}$ *of* $h(\mathcal{C}, i)$. *We also fix a dual basis* $\{\theta_{i,a}^*\}$ *in the sense that* $\langle \theta_{i,a}^*, \theta_{i,b}\rangle_i := (-1)^{\dim C_i \cdot |\theta_{i,b}|} \int_{C_i} \theta_{i,a}^* \wedge \theta_{i,b} = \delta_{ab}$;
- *a sequence of Thom classes* $\delta_i^n = \mathrm{d}(\rho_n\psi_i)$ *of the diagonal* Δ_i *of* $C_i \times C_i$ *for all* i;
- *primitives* f_i^n, *such that* $\mathrm{d}f_i^n = \delta_i^n - \sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta_{i,a}^*$ *and* $f_i^n - f_i^m = (\rho_n - \rho_m)\psi_i$ *for all* i.

Fixing a defining data Θ gives a perturbation data to apply the homological perturbation Lemma 11.17.

The perturbed operator $D_{k,T,\Theta}$

Throughout this paragraph, we fix a defining data Θ. From the induced perturbation data, we are able to write down the operator $D_{k,T,\Theta}$ in the spirit of (11.6), which is the main component of the differential for our minimal Morse-Bott cochain complex. **From now on, we will be very specific about the signs**. Before writing down the formula for $D_{k,T,\Theta}$, we set up the following notations:

- $[\alpha]$ is the cohomology class of a closed form $\alpha \in h(\mathcal{C}, i)$, and $|\alpha|$ is the degree of the form.

- $\mathcal{M}^{s,k}_{i_1,\ldots,i_r} := \mathcal{M}_{s,s+i_1} \times \ldots \times \mathcal{M}_{s+i_r,s+k}$ for $0 < i_1 < i_2 < \ldots < i_r < k$ with product orientation.

- For $b \in \Omega^*(C_s), c \in \Omega^*(C_{s+k}),\ h_{s+i_j} \in \Omega^*(C_{s+i_j} \times C_{s+i_j}), 1 \leq j \leq r$, we define the pairing $\mathcal{M}^{s,k}_{i_1,\ldots,i_r}[b, h_{s+i_1}, \ldots, h_{s+i_r}, c]$ to be

$$\int_{\mathcal{M}^{s,k}_{i_1,\ldots,i_r}} s^*_{s,s+i_1} b \wedge (t_{s,s+i_1} \times s_{s+i_1,s+i_2})^* h_{s+i_1} \wedge \ldots \wedge (t_{s+i_{r-1},s+i_r} \times s_{s+i_r,s+k})^* h_{s+i_r} \wedge t^*_{s+i_r,s+k} c. \tag{12.5}$$

- For $\alpha \in h(\mathcal{C}, s)$,

$$\begin{aligned} \dagger(\mathcal{C}, \alpha, k) &:= (|\alpha| + m_{s,s+k})(c_{s+k} + 1); \\ \ddagger(\mathcal{C}, \alpha, k) &:= (|\alpha| + m_{s,s+k} + 1)(c_{s+k} + 1). \end{aligned}$$

By (11.6), for an increasing sequence $T = \{0, i_1, \ldots, i_r, k\}$, we write operator $D_{k,T,\Theta}$ as:

$$\langle D_{k,T,\Theta}[\alpha], [\gamma] \rangle_{s+k} = (-1)^\star \lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, f^n_{s+i_r}, \gamma], \tag{12.6}$$

where $\star = |\alpha|(c_s + 1) + \sum_{j=1}^r \ddagger(\mathcal{C}, \alpha, i_j)$.

Remark 12.4. *One way to understand the signs above is to think of $D_{k,T,\Theta}$ as a composition of certain operators. Let $\alpha \in \Omega^*(C_i)$ and $f \in \Omega^*(C_j \times C_j)$, then $\mathcal{M}_{i,j}$ defines an operator:*

$$\mathcal{M}_{i,j}(\alpha, f) = (-1)^{|\alpha|(c_i+1)} \int_{\mathcal{M}_{i,j}} s^*_{i,j}\alpha \wedge (t_{i,j} \times \mathrm{id}_j)^* f \in \Omega^*(C_j).$$

where $t_{i,j} \times \mathrm{id}_j : \mathcal{M}_{i,j} \times C_j \to C_j \times C_j$. If $|f| = c_j - 1$, then $|\mathcal{M}_{i,j}(\alpha, f)| = |\alpha| + c_j - 1 - m_{i,j}$. For $g \in \Omega^*(C_k \times C_k)$, we have

$$\begin{aligned} \mathcal{M}_{j,k}(\mathcal{M}_{i,j}(\alpha, f), g) &= (-1)^{(|\alpha|+c_j-1-m_{i,j})(c_j+1)} \int_{\mathcal{M}_{j,k}} s^*_{j,k}\mathcal{M}_{i,j}(\alpha, f) \wedge (t_{j,k} \times \mathrm{id}_k)^* g \\ &= (-1)^{|\alpha|(c_i+1)+(|\alpha|+m_{i,j}+1)(c_j+1)} \int_{\mathcal{M}_{i,j}\times\mathcal{M}_{j,k}} s^*_{i,j}\alpha \wedge (t_{i,j} \times s_{j,k})^* f \wedge (t_{j,k} \times \mathrm{id}_k)^* g. \end{aligned}$$

Therefore $(-1)^\star \mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, f^n_{s+i_r}, \gamma]$ is the integral of the wedge product of compositions of such operators and $t^*_{s+i_r,s+k}\gamma$.

The following lemma asserts that (12.6) is well-defined and will be used in the proof of the main theorem. We prove it in appendix C.1.

Lemma 12.5. $\lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, f^n_{s+i_r}, \gamma] \in \mathbb{R}$ *exists.*

12.2 Minimal Morse-Bott cochain complexes

The main theorem of this section is that we can get a well-defined cochain complex out of an oriented flow category. This cochain complex is called the minimal Morse-Bott cochain complex.

Definition 12.6. *Given a defining data Θ, **the minimal Morse-Bott complex** of a flow category $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ is defined to be*

$$BC = \varinjlim_{q \to -\infty} \prod_{j=q}^{\infty} H^*(C_j),$$

i.e. the direct sum near the negative end and direct product near the positive end. To be more precise, every element in BC is a function $A : \mathbb{Z} \to \prod_{j\in\mathbb{Z}} H^*(C_j)$, such that $A(i) \in H^*(C_i)$, and there exists $N_A \in \mathbb{Z}$, such that $A(i) = 0\ \forall i < N_A$. The differential $d_{BC,\Theta} : BC \to BC$ is defined as $\prod_{k\geq 1} d_{k,\Theta}$, where $d_{k,\Theta} : H^*(C_s) \to H^*(C_{s+k})$ is defined as

$$d_{k,\Theta} = \sum_{T} D_{k,T,\Theta},$$

for all sequences $T = \{0, i_1, \ldots, i_r, k\}$ with $0 < i_1 < \ldots < i_r < k$. In other words,

$$\langle d_{k,\Theta}[\alpha], [\gamma]\rangle_{s+k} := (-1)^{|\alpha|(c_s+1)} \left(\mathcal{M}^{s,k}[\alpha,\gamma] + \lim_{n\to\infty} \sum_{0<i_1<\ldots<i_r<k} (-1)^{\star} \mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, f^n_{s+i_r}, \gamma]\right). \tag{12.7}$$

for $\alpha \in h(\mathcal{C}, i)$, $\gamma \in h(\mathcal{C}, i+k)$ and $\star = \sum_{j=1}^{r} \ddagger(\mathcal{C}, \alpha, i_j)$. If we formally define $d_{i,\Theta} = 0$ for $i \leq 0$, then for $A \in BC$

$$(d_{BC,\Theta}A)(i) = \sum_{j} d_{i-j,\Theta} A(j).$$

Note that it is a finite sum. If moreover the flow category is graded by d_i, then BC is also graded, and the grading of an element $\alpha \in H^*(C_i)$ is $|\alpha| + d_i$.

The main theorem of this chapter is

Theorem 12.7. *Given an oriented flow category $\mathcal{C}$. If we fix a defining-data Θ, then $(BC, d_{BC,\Theta})$ is a cochain complex. The cohomology $H^*(BC, d_{BC,\Theta})$ is independent of the defining-data Θ. If in addition, the flow category is graded, then $d_{BC,\Theta}$ is a differential of degree 1.*

Remark 12.8. *To illustrate that Theorem (12.7) is nontrivial, we prove in chapter 16 that when the flow category comes from a Morse-Bott function f on a closed manifold M, the cohomology of the minimal Morse-Bott cochain complex is the regular cohomology $H^*(M, \mathbb{R})$.*

Remark 12.9. *If all the critical manifolds C_i are points, then there is a unique defining data Θ_0. Assume for simplicity, each C_i consists of just one point, the minimal Morse-Bott cochain complex BC is generated by the critical points, and equals to the usual Morse cochain complex:*

$$BC = \varinjlim_{q\to-\infty} \prod_{j=q}^{\infty} H^*(C_j) = \varinjlim_{q\to-\infty} \prod_{j=q}^{\infty} \mathbb{R}.$$

Since $|f_i^n| = -1$, $d_{k,\Theta_0} : H^*(C_s) \to H^*(C_{s+k})$ only has the leading term:

$$\langle d_{k,\Theta_0}[1],[1]\rangle_{s+k} = \mathcal{M}^{s,k}[1,1] = \int_{\mathcal{M}_{s,s+k}} 1.$$

Therefore the differential $d_{BC,\Theta_0} = \sum_{k\geq 1} d_{k,\Theta_0}$ is just the signed counting of all zero-dimensional moduli spaces $\mathcal{M}_{s,s+k}$, which is usual Morse cochain differential.

Remark 12.10. *When $C_i = \sqcup_\alpha C_i^\alpha$ is not compact, i.e. C_i has infinitely many components. We need to make one of the compactness assumptions in remark 11.10. If we assume $(s_{i,j}\times t_j)^{-1}(C_i^\alpha \times C_j)$ are compact, then $H^*(C_i)$ is defined to be the direct sum $\oplus_\alpha H^*(C_i^\alpha)$. If we assume $(s_{i,j}\times t_j)^{-1}(C_i\times C_j^\alpha)$ are compact, then $H^*(C_i)$ is defined to be the direct product $\prod_\alpha H^*(C_i^\alpha)$. Such rules make the definition of $d_{BC,\Theta}$ well-defined on the minimal Morse-Bott cochain complex.*

Remark 12.11. *Theorem 12.7 is the most simplified version, and the a more general version is Theorem 14.10, which allows more flexible data and works for flow categories that are not oriented.*

We first show that $(BC, d_{BC,\Theta})$ is a cochain complex, the invariance is deferred to the next section. For simplicity, we will suppress the subscript Θ in the proof.

Proposition 12.12.

$$d^2_{BC,\Theta} = 0$$

Before proving Proposition 12.12, we first introduce some simplified notations.

- For $0 < i_1 < i_2 \ldots < i_r < k$, we define

$$\mathcal{M}^{s,k}_{i_1,\ldots,\overline{i_p},\ldots,i_r} := \mathcal{M}_{s,s+i_1}\times\ldots\times(\mathcal{M}_{s+i_{p-1},s+i_p}\times_{s+i_p}\mathcal{M}_{s+i_p,s+i_{p+1}})\times\ldots\times\mathcal{M}_{s+i_r,s+k} \quad (12.8)$$

  with the product orientation.

- For $b\in\Omega^*(C_s), c\in\Omega^*(C_{s+k}), h_{s+i_j}\in\Omega^*(C_{s+i_j}\times C_{s+i_j})$, we define $\mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\mathrm{d}(b,h_{s+i_1},\ldots,h_{s+i_r},c)]$ to be:

$$\int_{\mathcal{M}^{s,k}_{i_1,\ldots,i_r}} \mathrm{d}\left(s^*_{s,s+i_1}b\wedge(t_{s,s+i_1}\times s_{s+i_1,s+i_2})^*h_{s+i_1}\wedge\ldots\wedge(t_{s+i_{r-1},s+i_r}\times s_{s+i_r,s+k})^*h_{s+i_r}\wedge t^*_{s+i_r,s+k}c\right). \quad (12.9)$$

- We define the pairing $\mathcal{M}^{s,k}_{i_1,\dots,\overline{i_p},\dots,i_r}[b, h_{s+i_1}, \dots, h_{s+i_{p-1}}, h_{s+i_{p+1}}, \dots, h_{s+i_r}, c]$ over $\mathcal{M}^{s,k}_{i_1,\dots,\overline{i_p},\dots,i_r}$ to be:

$$\int_{\mathcal{M}^{s,k}_{i_1,\dots,\overline{i_p},\dots,i_r}} s^*_{s,s+i_1} b \wedge (t_{s,s+i_1} \times s_{s+i_1,s+i_2})^* h_{s+i_1} \wedge \dots \wedge (t_{s+i_{p-2},s+i_{p-1}} \times s_{s+i_{p-1},s+i_{p+1}})^* h_{s+i_{p-1}}$$
$$\wedge (t_{s+i_{p-1},s+i_{p+1}} \times s_{s+i_{p+1},s+i_{p+2}})^* h_{s+i_{p+1}} \wedge \dots \wedge (t_{s+i_{r-1},s+i_r} \times s_{s+i_r,s+k})^* h_{s+i_r} \wedge t^*_{s+i_r,s+k} c. \tag{12.10}$$

- When we compose two operators, a term involving trace will appear. Therefore we introduce $\mathrm{Tr}^{s+i_p} \mathcal{M}^{s,k}_{i_1,\dots,i_r}[b, h_{s+i_1}, \dots, h_{s+i_{p-1}}, \theta\theta^*_{s+i_p}, h_{s+i_{p+1}}, \dots, h_{s+i_r}, c]$ to denote the trace term below:

$$\begin{aligned}\int_{\mathcal{M}^{s,k}_{i_1,\dots,i_r}} & s^*_{s,s+i_1} b \wedge (t_{s,s+i_1} \times s_{s+i_1,s+i_2})^* h_{s+i_1} \wedge \dots \\ & \wedge (t_{s+i_{p-1},s+i_p} \times s_{s+i_p,s+i_{p+1}})^* (\sum_a \pi_1^* \theta_{s+i_p,a} \wedge \pi_2^* \theta^*_{s+i_p,a}) \wedge \dots \\ & \wedge (t_{s+i_r} \times s_{s+i_r})^* f_{s+i_r} \wedge t^*_{s+k} \gamma,\end{aligned} \tag{12.11}$$

  where π_1, π_2 are the projections of $C \times C$ to the first and second factor.

The following Lemma is used in the proof of Proposition 12.12 and is proven in Appendix C.1.

Lemma 12.13. *For an oriented flow category $\mathcal{C}$,*

$$\lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^n_{s+i_1}, \dots, \delta^n_{s+i_p}, \dots, f^n_{s+i_r}, \gamma] = (-1)^* \lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\dots,i_{p-1},\overline{i_p},i_{p+1},\dots,i_r}[\alpha, f^n_{s+i_1}, \dots, f^n_{s+i_r}, \gamma],$$

where $* = (a + m_{s,s+i_p}) c_{s+i_p}$.

Proof of proposition 12.12. It suffices to show that for all $\alpha \in h(\mathcal{C}, s)$ and $\gamma \in h(\mathcal{C}, s+k)$

$$\left\langle \sum_{i=1}^{k-1} d_{k-i} \circ d_i[\alpha], [\gamma] \right\rangle_{s+k} = 0. \tag{12.12}$$

We define the leading term D_k of d_k to be:

$$\langle D_k[\alpha], [\gamma] \rangle_{s+k} := (-1)^{|\alpha|(c_s+1)} \mathcal{M}^{s,k}[\alpha, \gamma].$$

The remaining term is denoted by $E_k := d_k - D_k$. Thus (12.12) is equivalent to:

$$\left\langle \sum_i (D_{k-i} \circ D_i + E_{k-i} \circ E_i + E_{k-i} \circ D_i + D_{k-i} \circ E_i)[\alpha], [\gamma] \right\rangle_{s+k} = 0. \tag{12.13}$$

We will compute $\boldsymbol{S := \langle \sum_i D_{k-i} \circ D_i[\alpha], [\gamma]\rangle_{s+k}}$ first by induction and then compare it with the remaining terms. Recall that the representatives of basis and dual basis $\{\theta_{i,a}\}, \{\theta^*_{i,a}\}$ have the property $\langle \theta^*_{i,a}, \theta_{i,b}\rangle_i = \delta_{ab}$, thus

$$D_k[\alpha] = \sum_a \langle D_k[\alpha], [\theta_{s+k,a}]\rangle_{s+k} [\theta^*_{s+k,a}]. \tag{12.14}$$

Therefore

$$\begin{aligned} S &= \left\langle \sum_i D_{k-i} \circ D_i[\alpha], [\gamma] \right\rangle_{s+k} \\ &= \left\langle \sum_i D_{k-i} \left(\sum_a \langle D_i[\alpha], [\theta_{s+i,a}]\rangle_{s+i} [\theta^*_{s+i,a}] \right), [\gamma] \right\rangle_{s+k} \\ &= \left\langle \sum_i D_{k-i} \left(\sum_a (-1)^{|\alpha|(c_s+1)} \left(\mathcal{M}^{s,i}[\alpha, \theta_{s+i,a}]\right) [\theta^*_{s+i,a}] \right), [\gamma] \right\rangle_{s+k}. \end{aligned} \tag{12.15}$$

Since we have the degree relation $|\mathcal{M}^{s,i}[\alpha, \theta_{s+i,a}][\theta^*_{s+i,a}]| = |D_i[\alpha]| = |d_i[\alpha]|$.

$$\begin{aligned} (12.15) &= \sum_i \sum_a (-1)^{|\alpha|(c_s+1)} \mathcal{M}^{s,i}[\alpha, \theta_{s+i,a}] \left\langle D_{k-i}[\theta^*_{s+i,a}], [\gamma]\right\rangle_{s+k} \\ &= \sum_i \sum_a (-1)^{|\alpha|(c_s+1)+|d_i[\alpha]|(c_{s+i}+1)} \mathcal{M}^{s,i}[\alpha, \theta_{s+i,a}] \mathcal{M}^{s+i,k-i}[\theta^*_{s+i,a}, \gamma] \\ &= \sum_i (-1)^{|\alpha|(c_s+1)+|d_i[\alpha]|(c_{s+i}+1)} \operatorname{Tr}^i \mathcal{M}^{s,k}_i[\alpha, \theta\theta^*_{s+i}, \gamma]. \end{aligned} \tag{12.16}$$

Using $\delta^n_i - \sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta^*_{i,a} = \mathrm{d}f^n_i$ for any $n \in \mathbb{N}$:

$$\begin{aligned} (12.16) &= \sum_i (-1)^{|\alpha|(c_s+1)+|d_i[\alpha]|(c_{s+i}+1)} \mathcal{M}^{s,k}_i[\alpha, \delta^n_{s+i} - \mathrm{d}f^n_{s+i}, \gamma] \\ &= \lim_{n\to\infty} \sum_i (-1)^{|\alpha|(c_s+1)+|d_i[\alpha]|(c_{s+i}+1)} \mathcal{M}^{s,k}_i[\alpha, \delta^n_{s+i} - \mathrm{d}f^n_{s+i}, \gamma] \\ &= \lim_{n\to\infty} \sum_i (-1)^{|\alpha|(c_s+1)+|d_i[\alpha]|(c_{s+i}+1)} \mathcal{M}^{s,k}_i[\alpha, \delta^n_{s+i}, \gamma] && (12.17) \\ &\quad + \lim_{n\to\infty} \sum_i (-1)^{|\alpha|(c_s+1)+|d_i[\alpha]|(c_{s+i}+1)+1} \mathcal{M}^{s,k}_i[\alpha, \mathrm{d}f^n_{s+i}, \gamma]. && (12.18) \end{aligned}$$

By Lemma 12.13:

$$(12.17) = \sum_i (-1)^{|\alpha|(c_s+1)+|d_i[\alpha]|(c_{s+i}+1)+(a+m_{s,s+i})c_{s+i}} \mathcal{M}^{s,k}_{\bar{i}}[\alpha, \gamma]. \tag{12.19}$$

We will show first (12.19) vanishes. Because $|d_i[\alpha]| \equiv |\alpha| + m_{s,s+i} + c_{s+i} \mod 2$, we have

$$|d_i[\alpha]|(c_{s+i}+1) \equiv (|\alpha| + m_{s,s+i})(c_{s+i}+1) = \dagger(\mathcal{C}, \alpha, i) \mod 2. \tag{12.20}$$

Therefore the sign in (12.19) is

$$(-1)^{|\alpha|(c_s+1)+|d_i\alpha|(c_{s+i}+1)+(a+m_{s,s+i})c_{s+i}} = (-1)^{|\alpha|(c_s)+m_{s,s+i}}. \tag{12.21}$$

Because $\partial[\mathcal{M}_{ik}] = \sum(-1)^{m_{ij}}[\mathcal{M}_{ij}] \times_j [\mathcal{M}_{jk}]$, by Stokes' theorem

$$\begin{aligned}
(12.19) &= \sum_i (-1)^{|\alpha|c_s+m_{s,s+i}} \int_{\mathcal{M}_{s,s+i}\times_{s+i}\mathcal{M}_{s+i,s+k}} s^*_{s,s+i}\alpha \wedge t^*_{s+i,s+k}\gamma \\
&= (-1)^{|\alpha|c_s} \int_{\partial\mathcal{M}_{s,s+k}} s^*_{s,s+k}\alpha \wedge t^*_{s,s+k}\gamma \\
&= (-1)^{|\alpha|c_s} \int_{\mathcal{M}_{s,s+k}} \mathrm{d}\left(s^*_{s,s+k}\alpha \wedge t^*_{s,s+k}\gamma\right) \\
&= 0.
\end{aligned}$$

So far, we prove that

$$S = (12.18) = \lim_{n\to\infty} \sum_i (-1)^{|\alpha|(c_s+1)+\dagger(\mathcal{C},\alpha,i)+1} \mathcal{M}_i^{s,k}[\alpha, \mathrm{d}f^n_{s+i}, \gamma]. \tag{12.22}$$

It is an integration of an exact form, thus we can apply Stokes' theorem. Since $\mathcal{M}_i^{s,k} = \mathcal{M}_{s,s+i} \times \mathcal{M}_{s+i,s+k}$ is a product of two manifolds, there are two types of boundaries coming from the boundary of each component. Because $\partial[\mathcal{M}_i^{s,k}] = (\partial[\mathcal{M}_{s,s+i}]) \times [\mathcal{M}_{s+i,s+k}] + (-1)^{m_{s,s+i}}[\mathcal{M}_{s,s+i}] \times (\partial[\mathcal{M}_{s+i,s+k}])$, we have (for simplicity, we suppress the wedge and pullback notations):

$$\begin{aligned}
& \lim_{n\to\infty} \mathcal{M}_i^{s,k}[\alpha, \mathrm{d}f^n_{s+i}, \gamma] \\
= & \lim_{n\to\infty} (-1)^{|\alpha|} \mathcal{M}_i^{s,k}[\mathrm{d}(\alpha, f^n_{s+i}, \gamma)] \\
= & \lim_{n\to\infty} (-1)^{|\alpha|} \int_{\partial(\mathcal{M}_{s,s+i}\times\mathcal{M}_{s+i,s+k})} \alpha f^n_{s+i}\gamma \\
= & \lim_{n\to\infty} \sum_{0<j<i} (-1)^{|\alpha|+m_{s,s+j}} \int_{\mathcal{M}_{s,s+j}\times_{s+j}\mathcal{M}_{s+j,s+i}\times\mathcal{M}_{s+i,s+k}} \alpha f^n_{s+i}\gamma \\
& + \lim_{n\to\infty} \sum_{i<l<k} (-1)^{|\alpha|+m_{s,s+i}+m_{s+i,s+l}} \int_{\mathcal{M}_{s,s+i}\times\mathcal{M}_{s+i,s+l}\times_{s+l}\mathcal{M}_{s+l,s+k}} \alpha f^n_{s+i}\gamma \\
= & \lim_{n\to\infty} \left(\sum_{0<j<i} (-1)^{|\alpha|+m_{s,s+j}} \mathcal{M}^{s,k}_{\bar{j},i}[\alpha, f^n_{s+i}, \gamma] \right. \\
& \left. + \sum_{i<l<k} (-1)^{|\alpha|+m_{s,s+i}+m_{s+i,s+l}} \mathcal{M}^{s,k}_{i,\bar{l}}[\alpha, f^n_{s+i}, \gamma] \right).
\end{aligned} \tag{12.23}$$

Applying Lemma 12.13, we have

$$\begin{aligned}(12.23) \;=\; & \lim_{n\to\infty}\sum_{0<j<i}(-1)^{|\alpha|+m_{s,s+j}+(|\alpha|+m_{s,s+j})c_{s+j}}\mathcal{M}^{s,k}_{j,i}[\alpha,\delta^n_{s+j},f^n_{s+i},\gamma]\\ & +\lim_{n\to\infty}\sum_{i<l<k}(-1)^{|\alpha|+m_{s,s+i}+m_{s+i,s+l}+(|\alpha|+m_{s,s+l})c_{s+l}}\mathcal{M}^{s,k}_{i,l}[\alpha,f^n_{s+i},\delta^n_{s+l},\gamma].\end{aligned}$$

By $\delta^n_i-\sum_a \pi_1^*\theta_{i,a}\wedge\pi_2^*\theta^*_{i,a}=\mathrm{d}f^n_i$, (12.23) can be written as:

$$\begin{aligned}(12.23) \;=\; & \lim_{n\to\infty}\sum_{j,0<j<i}(-1)^{\dagger(\mathcal{C},\alpha,j)}\left(\mathrm{Tr}^{s+j}\,\mathcal{M}^{s,k}_{j,i}[\alpha,\theta\theta^*_{s+j},f^n_{s+i},\gamma]+\mathcal{M}^{s,k}_{j,i}[\alpha,\mathrm{d}f^n_{s+j},f^n_{s+i},\gamma]\right)\\ & +\lim_{n\to\infty}\sum_{l,i<l<k}(-1)^{\dagger(\mathcal{C},\alpha,l)+c_{s+i}+1}\left(\mathrm{Tr}^{s+l}\,\mathcal{M}^{s,k}_{i,l}[\alpha,f^n_{s+i},\theta\theta^*_{s+l},\gamma]+\mathcal{M}^{s,k}_{i,l}[\alpha,\mathrm{d}f^n_{s+i},f^n_{s+l},\gamma]\right).\end{aligned}$$

Therefore (12.22) can be expressed as the sum of the following two terms:

$$\lim_{n\to\infty}\sum_i\sum_{0<j<i}(-1)^{|\alpha|(c_s+1)+\dagger(\mathcal{C},\alpha,j)+\dagger(\mathcal{C},\alpha,i)+1}\left(\mathrm{Tr}^{s+j}\,\mathcal{M}^{s,k}_{j,i}[\alpha,\theta\theta^*_{s+j},f^n_{s+i},\gamma]+\mathcal{M}^{s,k}_{j,i}[\alpha,\mathrm{d}f^n_{s+j},f^n_{s+i},\gamma]\right),$$
$$\lim_{n\to\infty}\sum_i\sum_{i<l<k}(-1)^{|\alpha|(c_s+1)+\ddagger(\mathcal{C},\alpha,i)+\dagger(\mathcal{C},\alpha,l)+1}\left(\mathrm{Tr}^{s+l}\,\mathcal{M}^{s,k}_{i,l}[\alpha,f^n_{s+i},\theta\theta^*_{s+l},\gamma]+\mathcal{M}^{s,k}_{j,i}[\alpha,f^n_{s+i},\mathrm{d}f^n_{s+l},\gamma]\right).$$

Thus (12.22) can be rewritten as:

$$\begin{aligned}S \;=\; & \lim_{n\to\infty}\sum_i\sum_{0<j<i}(-1)^{|\alpha|(c_s+1)+\dagger(\mathcal{C},\alpha,j)+\dagger(\mathcal{C},\alpha,i)+1}\,\mathrm{Tr}^{s+j}\,\mathcal{M}^{s,k}_{j,i}[\alpha,\theta\theta^*_{s+j},f^n_{s+i},\gamma]\\ & +\lim_{n\to\infty}\sum_i\sum_{i<l<k}(-1)^{|\alpha|(c_s+1)+\ddagger(\mathcal{C},\alpha,i)+\dagger(\mathcal{C},\alpha,l)+1}\,\mathrm{Tr}^{s+l}\,\mathcal{M}^{s,k}_{i,l}[\alpha,f^n_{s+i},\theta\theta^*_{s+l},\gamma]\\ & +\lim_{n\to\infty}\sum_i\sum_{0<j<i}(-1)^{|\alpha|(c_s+1)+\dagger(\mathcal{C},\alpha,j)+\dagger(\mathcal{C},\alpha,i)+1}\mathcal{M}^{s,k}_{j,i}[\alpha,\mathrm{d}f^n_{s+j},f^n_{s+i},\gamma] \qquad (12.24)\\ & +\lim_{n\to\infty}\sum_i\sum_{i<l<k}(-1)^{|\alpha|(c_s+1)+\ddagger(\mathcal{C},\alpha,i)+\dagger(\mathcal{C},\alpha,l)+1}\mathcal{M}^{s,k}_{j,i}[\alpha,f^n_{s+i},\mathrm{d}f^n_{s+l},\gamma]. \qquad (12.25)\end{aligned}$$

The last two parts (12.24) and (12.25) sum up to the following integrals of exact forms:

$$\lim_{n\to\infty}\sum_{i,j,i<j}(-1)^{|\alpha|c_s+\dagger(\mathcal{C},\alpha,i)+\dagger(\mathcal{C},\alpha,j)+1}\mathcal{M}^{s,k}_{i,j}[\mathrm{d}(\alpha,f^n_{s+i},f^n_{s+j},\gamma)]. \qquad (12.26)$$

So far, we prove that

$$\begin{aligned}S \;=\; & \lim_{n\to\infty}\sum_i\sum_{0<j<i}(-1)^{|\alpha|(c_s+1)+\dagger(\mathcal{C},\alpha,j)+\dagger(\mathcal{C},\alpha,i)+1}\,\mathrm{Tr}^{s+j}\,\mathcal{M}^{s,k}_{j,i}[\alpha,\theta\theta^*_{s+j},f^n_{s+i},\gamma]\\ & +\lim_{n\to\infty}\sum_i\sum_{i<l<k}(-1)^{|\alpha|(c_s+1)+\ddagger(\mathcal{C},\alpha,i)+\dagger(\mathcal{C},\alpha,l)+1}\,\mathrm{Tr}^{s+l}\,\mathcal{M}^{s,k}_{i,l}[\alpha,f^n_{s+i},\theta\theta^*_{s+l},\gamma]\\ & +\lim_{n\to\infty}\sum_{i,j,i<j}(-1)^{|\alpha|c_s+\dagger(\mathcal{C},\alpha,i)+\dagger(\mathcal{C},\alpha,j)+1}\mathcal{M}^{s,k}_{i,j}[\mathrm{d}(\alpha,f^n_{s+i},f^n_{s+j},\gamma)]. \qquad (12.27)\end{aligned}$$

We can apply Stokes' theorem again to (12.27), and use the trick above, i.e. whenever we see a fiber product, which is part of boundary of some other $\mathcal{M}_{*,*}$, we can replace it by Cartesian product and insert a Dirac current, then replace the Dirac current by $\sum \pi_1^*\theta \wedge \pi_2^*\theta^* + \mathrm{d}f$. We claim that the terms involving $\mathrm{d}f$ sum up to an exact form, and the process can keep going. In particular, we will prove the following statement by induction, for $r \geq 2$:

$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{\substack{1\le p\le q\le r, q>1\\ 0<i_1<\ldots<i_q<k}} (-1)^{\star_2} \operatorname{Tr}^{s+i_p} \mathcal{M}^{s,k}_{i_1,\ldots,i_q}[\alpha, f^n_{s+i_1}, \ldots, f^n_{s+i_{p-1}}, \theta\theta^*_{s+i_p}, f^n_{s+i_{p+1}}, \ldots, f^n_{s+i_q}, \gamma] \\ &\quad + \lim_{n\to\infty} \sum_{0<i_1<\ldots<i_r<k} (-1)^{\star_1} \mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\mathrm{d}(\alpha, f^n_{s+i_1}, \ldots, f^n_{s+i_r}, \gamma)], \end{aligned} \tag{12.28}$$

where

$$\star_1 := 1 + |\alpha|c_s + \sum_{j=1}^{r} \dagger(\mathcal{C}, \alpha, i_j),$$

$$\star_2 := 1 + |\alpha|(c_s+1) + \sum_{j=1}^{p-1} \ddagger(\mathcal{C}, \alpha, i_j) + \sum_{j=p}^{q} \dagger(\mathcal{C}, \alpha, i_j).$$

Proof of the claim. The $r = 2$ case is exactly (12.27). Assume the claim holds for r, we will show the claim also holds for $r+1$. We need to apply Stokes' theorem to the exact term (12.28) in the induction hypothesis. The boundary $\partial(\mathcal{M}_{s,s+i_1} \times \ldots \times \mathcal{M}_{s+i_r,s+k})$ comes from fiber product at $s+w$ for all t, w such that $0 < i_1 < \ldots < i_t < w < i_{t+1} < \ldots i_r < k$. Consider the boundary coming from the fiber product at $s+w$, after applying Stokes' theorem to (12.28), the contribution from integration over the $\mathcal{M}^{s,k}_{i_1,\ldots,i_t,\overline{w},\ldots,i_r} \subset \mathcal{M}^{s,k}_{i_1,\ldots,i_r}$ is

$$(-1)^{\star_3} \lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\ldots,i_t,\overline{w},\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, f^n_{s+i_r}, \gamma], \tag{12.29}$$

where $\star_3 := 1 + |\alpha|c_s + \sum_{j=1}^r \dagger(\mathcal{C}, \alpha, i_j) + m_{s,s+i_1} + \ldots + m_{s+i_t,s+w}$. By replacing the fiber product in $\mathcal{M}^{s,k}_{i_1,\ldots,i_t,\overline{w},\ldots,i_r}$ with the Cartesian product $\mathcal{M}^{s,k}_{i_1,\ldots,i_t,w,\ldots,i_r}$,

$$(12.29) = (-1)^{\star_3+(|\alpha|+m_{s,s+w})c_{s+w}} \lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\ldots,i_t,w,\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, \delta^n_{s+w}, \ldots, f^n_{s+i_r}, \gamma]. \tag{12.30}$$

We replace the Thom classes δ^n_* by $\sum_a \pi_1^*\theta_{*,a} \wedge \pi_2^*\theta^*_{*,a} + \mathrm{d}f^n_*$,

$$\begin{aligned} (12.30) &= (-1)^{\star_3+(|\alpha|+m_{s,s+w})c_{s+w}} \lim_{n\to\infty} \operatorname{Tr}^{s+w} \mathcal{M}^{s,k}_{i_1,\ldots,i_t,w,\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, \theta\theta^*_{s+w}, \ldots, f^n_{s+i_r}, \gamma] \\ &\quad + (-1)^{\star_3+(|\alpha|+m_{s,s+w})c_{s+w}} \lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\ldots,i_t,w,\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, \mathrm{d}f^n_{s+w}, \ldots, f^n_{s+i_r}, \gamma]. \end{aligned} \tag{12.31}$$

Let $\star_4$ denote $\star_3 + (|\alpha| + m_{s,s+w})c_{s+w}$. By (11.3) we have:

$$\star_4 = 1 + |\alpha|(c_s+1) + \sum_{j=1}^{t} \ddagger(\mathcal{C}, \alpha, i_j) + \dagger(\mathcal{C}, \alpha, w) + \sum_{j=t+1}^{r} \dagger(\mathcal{C}, \alpha, i_j).$$

By induction hypothesis:

$$
\begin{aligned}
S \;=\; & \lim_{n\to\infty} \sum_{\substack{1\le p\le q\le r, q>1\\ 0<i_1<\ldots<i_q<k}} (-1)^{\star_2}\,\mathrm{Tr}^{s+i_p}\,\mathcal{M}^{s,k}_{i_1,\ldots,i_q}[\alpha, f^n_{s+i_1},\ldots,f^n_{s+i_{p-1}},\theta\theta^*_{s+i_p}, f^n_{s+i_{p+1}},\ldots,f^n_{s+i_q},\gamma]\\
& + \lim_{n\to\infty} \sum_{0<i_1<\ldots<i_r<k} (-1)^{\star_1}\mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\mathrm{d}(\alpha, f^n_{s+i_1},\ldots,f^n_{s+i_r},\gamma)]. \qquad (12.32)
\end{aligned}
$$

By (12.31), (12.32) is the sum of the following two terms:

$$
\lim_{n\to\infty} \sum_{\substack{0<i_1<\ldots<i_t<w\\ <i_{t+1}<i_r<k}} (-1)^{\star_4}\mathcal{M}^{s,k}_{i_1,\ldots,i_t,w,i_{t+1},\ldots,i_r}[\alpha, f^n_{s+i_1},\ldots,\mathrm{d}f^n_{s+w},\ldots,f^n_{s+i_r},\gamma] \qquad (12.33)
$$

$$
\sum_{\substack{0<i_1<\ldots<i_t<w\\ <i_{t+1}<i_r<k}} (-1)^{\star_4}\,\mathrm{Tr}^{s+w}\,\mathcal{M}^{s,k}_{i_1,\ldots,i_t,w,i_{t+1},\ldots,i_r}[\alpha, f^n_{s+i_1},\ldots,\theta\theta^*_{s+w},\ldots,f^n_{s+i_r},\gamma]. \qquad (12.34)
$$

Because $\star_5 := \star_4 + |\alpha| + \sum_{j=1}^t (c_{s+i_j}+1) \equiv 1 + |\alpha| c_s + \sum_{j=1}^r \dagger(\mathcal{C},\alpha,i_j) + \dagger(\mathcal{C},\alpha,w) \mod 2$ and $|f^n_{s+i_j}| \equiv c_{s+i_j}+1 \mod 2$,

$$
(12.33) = \sum_{0<i_1<\ldots<i_t<w<i_{t+1}<i_r<k} (-1)^{\star_5}\mathcal{M}^{s,k}_{i_1,\ldots,i_t,w,i_{t+1},\ldots,i_r}[\mathrm{d}(\alpha, f^n_{s+i_1},\ldots,f^n_{s+w},\ldots,f^n_{s+i_r},\gamma)]. \qquad (12.35)
$$

Therefore

$$
\begin{aligned}
S \;=\; & \lim_{n\to\infty} \sum_{\substack{1\le p\le q\le r, q>1\\ 0<i_1<\ldots<i_q<k}} (-1)^{\star_2}\,\mathrm{Tr}^{s+i_p}\,\mathcal{M}^{s,k}_{i_1,\ldots,i_q}[\alpha, f^n_{s+i_1},\ldots,f^n_{s+i_{p-1}},\theta\theta^*_{s+i_p}, f^n_{s+i_{p+1}},\ldots,f^n_{s+i_q},\gamma]\\
& + \sum_{0<i_1<\ldots<i_t<w<i_{t+1}<i_r<k} (-1)^{\star_4}\,\mathrm{Tr}^{s+w}\,\mathcal{M}^{s,k}_{i_1,\ldots,i_t,w,i_{t+1},\ldots,i_r}[\alpha, f^n_{s+i_1},\ldots,\theta\theta^*_{s+w},\ldots,f^n_{s+i_r},\gamma]\\
& + \lim_{n\to\infty} \sum_{0<i_1<\ldots<i_t<w<i_{t+1}<i_r<k} (-1)^{\star_5}\mathcal{M}^{s,k}_{i_1,\ldots,i_t,w,i_{t+1},\ldots,i_r}[\mathrm{d}(\alpha, f^n_{s+i_1},\ldots,f^n_{s+w},\ldots,f^n_{s+i_r},\gamma)].
\end{aligned}
$$

This is the $r+1$ case, so we prove the claim. $\square$

Therefore in the case of $r = k-1$, $S = \langle \sum_i D_{k-i}\circ D_i[\alpha], [\gamma]\rangle_{s+k}$ is the sum of the following two terms:

$$
\lim_{n\to\infty} (-1)^{\star_1}\mathcal{M}^{s,k}_{1,\ldots,k-1}[\mathrm{d}(\alpha, f^n_{s+1},\ldots,f^n_{s+k-1},\gamma)], \qquad (12.36)
$$

$$
\lim_{n\to\infty} \sum_{\substack{1\le p\le q\le k-1, q>1\\ 0<i_1<\ldots<i_q<k}} (-1)^{\star_2}\,\mathrm{Tr}^{s+i_p}\,\mathcal{M}^{s,k}_{i_1,\ldots,i_q}[\alpha, f^n_{s+i_1},\ldots,f^n_{s+i_{p-1}},\theta\theta^*_{s+i_p}, f^n_{s+i_{p+1}},\ldots,f^n_{s+i_q},\gamma], \qquad (12.37)
$$

where

$$\star_1 := 1 + |\alpha| c_s + \sum_{j=1}^{k-1} \dagger(\mathcal{C}, \alpha, j),$$

$$\star_2 := 1 + |\alpha|(c_s + 1) + \sum_{j=1}^{p-1} \ddagger(\mathcal{C}, \alpha, i_j) + \sum_{j=p}^{q} \dagger(\mathcal{C}, \alpha, i_j).$$

Since $\mathcal{M}^{s,k}_{1,\ldots,k-1}$ is a closed manifold, (12.36) is 0 by Stokes' theorem. For the remaining term, we claim that

$$(12.37) = -\left\langle \sum_i (E_{k-i} \circ E_i + E_{k-i} \circ D_i + D_{k-i} \circ E_i)[\alpha], [\gamma] \right\rangle_{s+k}. \tag{12.38}$$

Once this claim is proven, we have

$$S = (12.37) = -\left\langle \sum_i (E_{k-i} \circ E_i + E_{k-i} \circ D_i + D_{k-i} \circ E_i)[\alpha], [\gamma] \right\rangle_{s+k},$$

that is

$$\langle d_{k-i} \circ d_i[\alpha], [\gamma] \rangle_{s+k} = 0.$$

Therefore the proposition is proven. □

Proof of the claim (12.37). Let $S_1 := \langle \sum_i E_{k-i} \circ E_i[\alpha], [\gamma] \rangle_{s+k}$, $S_2 := \langle \sum_i E_{k-i} \circ D_i[\alpha], [\gamma] \rangle_{s+k}$, $S_3 := \langle \sum_i D_{k-i} \circ E_i[\alpha], [\gamma] \rangle_{s+k}$. By Definition 12.6,

$$\langle E_i[\alpha], [\gamma] \rangle_{s+i} = \lim_{n\to\infty} \sum_{0<i_1\ldots<i_r<i} (-1)^{|\alpha|(c_s+1)+\sum_{j=1}^{r} \ddagger(\mathcal{C},\alpha,i_j)} \mathcal{M}^{s,i}_{i_1,\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, f^n_{s+i_r}, \gamma]. \tag{12.39}$$

Therefore we can write S_1 as a trace term:

$$S_1 = \lim_{n\to\infty} \sum_i \sum_{\substack{0<i_1<\ldots<i_r<i \\ 0<j_1<\ldots<j_t<k-i}} (-1)^{\star_7} \operatorname{Tr}^{s+i} \mathcal{M}^{s,k}_{i_1,\ldots,i_r,i,i+j_1,\ldots,i+j_t}[\alpha, f^n_{s+i_1}, \ldots, \theta\theta^*_{s+i}, \ldots, f^n_{i+j_t}, \gamma], \tag{12.40}$$

where

$$\star_7 \;\; := \;\; |\alpha|(c_s+1) + \sum_{l=1}^{r} \ddagger(\mathcal{C}, \alpha, i_l) + |E_i(\alpha)|(c_{s+i}+1) + \sum_{l=1}^{t} \ddagger(\mathcal{C}, E_i(\alpha), j_l).$$

Because $|E_i(\alpha)|(c_{s+i}+1) \equiv (|\alpha| + m_{s,s+i})(c_{s+i}+1) \mod 2$ and $\ddagger(\mathcal{C}, E_i(\alpha), t) \equiv \dagger(\mathcal{C}, \alpha, i+t) \mod 2$,

$$\star_7 \;\; = \;\; |\alpha|(c_s+1) + \sum_{l=1}^{r} \ddagger(\mathcal{C}, \alpha, i_l) + \dagger(\mathcal{C}, \alpha, i) + \sum_{l=1}^{t} \dagger(\mathcal{C}, \alpha, i + j_l)$$

Moreover, the other two terms are:

$$S_2 = \lim_{n\to\infty} \sum_i \sum_{0<j_1<\ldots<j_t<k-i} (-1)^{\star_8} \operatorname{Tr}^{s+i} \mathcal{M}^{s,k}_{i,i+j_1,\ldots,i+j_t}[\alpha, \theta\theta^*_{s+i}, \ldots, f^n_{i+j_t}, \gamma], \tag{12.41}$$

$$S_3 = \lim_{n\to\infty} \sum_i \sum_{0<i_1<\ldots<i_r<i} (-1)^{\star_9} \operatorname{Tr}^{s+i} \mathcal{M}^{s,k}_{i_1,\ldots,i_r,i}[\alpha, f^n_{i_1}, \ldots, f^n_{i_r}, \theta\theta^*_{s+i}, \gamma], \tag{12.42}$$

where $\star_8 := |\alpha|(c_s+1)+\dagger(\mathcal{C},\alpha,i)+\sum_{l=1}^t \dagger(\mathcal{C},\alpha,i+j_l)$ and $\star_9 := |\alpha|(c_s+1)+\sum_{l=1}^r \ddagger(\mathcal{C},\alpha,i_l)+\dagger(\mathcal{C},\alpha,i)$. The sum of (12.40),(12.41) and (12.42) is (12.37) with a negative sign, thus we prove the claim. □

Remark 12.14. *The argument used in the proof of proposition 12.12 will be used repeatedly in the constructions of morphisms and homotopies. Roughly speaking, the general process is as follows: We divide the operators into two parts: the leading term, i.e. the one contains no f_i, and the error term. Then we compose only the leading terms first. Similar to (12.14), a composition can be written as a trace term, we replace the trace term by a Thom class δ_i^n plus $\mathrm{d}f_i^n$ in the composition of the leading terms. The part with Thom class will converge to an integration over the fiber products, i.e. an integration over the boundary of a bigger moduli space, thus Stokes' theorem can be applied. While the part with $\mathrm{d}f$ is an integration of exact forms, thus by Stokes' theorem, it can be reduced to an integration over the boundary, i.e. a fiber product. Then replace the fiber product by Cartesian product through inserting the Dirac current. Since we can rewrite the Dirac current as the limit of a $\mathrm{d}f^n$ plus a trace term, then usually the terms with $\mathrm{d}f$ will form an exact form and the replacing argument keeps repeating. Eventually, we will get to a manifold without boundary, thus the integration of the final exact form is 0, and we are left with some trace terms. Then we can compare those trace terms directly to the composition of error term with leading term and error term with error term to prove the algebraic equation.*

From the proof of proposition 12.12, we see that we can suppress the index n and $\lim\limits_{n\to\infty}$ by Lemma 12.5 12.13. We write f_i as the limit of f_i^n in the space of currents, then

$$\delta_i = \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta^*_{i,a} + \mathrm{d}f_i, \tag{12.43}$$

where δ_i is the Dirac current. We can use (12.43) to do formal computations.

12.3 Flow morphisms induce cochain morphisms

In this section, we introduce the flow-morphisms between flow categories, which are the generalizations of the continuation maps in the Morse case [5]. We assign every flow category an identity flow-morphism from the flow category to itself. Using the identity flow-morphism, we show that $H^*(BC, d_{BC,\Theta})$ is independent of the defining-data Θ, thus finish the proof of Theorem 12.7.

Definition 12.15. *An **oriented flow-morphism** H from an oriented flow categories $\mathcal{C} := \{C_i, \mathcal{M}^C_{i,j}\}$ to another oriented flow category $\mathcal{D} := \{D_i, \mathcal{M}^D_{i,j}\}$ is a family of compact oriented manifolds $H_{i,j}$, such that*

1. *there are two smooth evaluation maps $s, t : H_{i,j} \to C_i, D_j$;*
2. *$\exists N \in \mathbb{Z}$, such that when $i - j > N$, $H_{i,j} = \emptyset$;*
3. *fiber products $\mathcal{M}^C_{i_0,i_1} \times_{i_1} \ldots \times_{i_k} H_{i_k,j_0} \times_{j_0} \ldots \times_{j_{m-1}} \mathcal{M}^D_{j_{m-1},j_m}$ are cut out transversely, for all $i_0 < i_1 < \ldots < i_k, j_0 < \ldots < j_{m-1} < j_m$;*
4. *there are smooth maps $m_L : \mathcal{M}^C_{i,j} \times_j H_{j,k} \to H_{i,k}$ and $m_R : H_{i,j} \times_j \mathcal{M}^D_{j,k} \to H_{i,k}$, such that*
$$\begin{aligned} s \circ m_L(a,b) &= s^{\mathcal{C}}(a), \\ t \circ m_L(a,b) &= t(b), \\ s \circ m_R(a,b) &= s(a), \\ t \circ m_R(a,b) &= t^{\mathcal{D}}(b), \end{aligned}$$
where map $s^{\mathcal{C}}$ is the source map for flow category $\mathcal{C}$ and map $t^{\mathcal{D}}$ is the target map for flow category $\mathcal{D}$;
5. *the map $m_L \cup m_R : \left(\cup_j \partial^0(\mathcal{M}^C_{i,j} \times_j H_{j,k})\right) \cup \left(\cup_j \partial^0(H_{i,j} \times_j \mathcal{M}^{\mathcal{D}}_{j,k})\right) \to \partial^1 H_{i,k}$ is a diffeomorphism up to measure-zero sets;*
6. *the orientation $[H_{i,j}]$ has the following properties:*
$$\partial[H_{i,j}] = \sum_{p>0} (-1)^{m^C_{i,i+p}} m_L\left([\mathcal{M}^C_{i,i+p} \times_{i+p} H_{i+p,j}]\right) + \sum_{p>0} (-1)^{h_{i,j}} m_R\left([H_{i,j-p} \times_{j-p} M^D_{j-p,j}]\right),$$
$$\begin{aligned} (t^{\mathcal{C}} \times s)^*[N_j][\mathcal{M}^C_{i,j} \times_j H_{j,k}] &= (-1)^{c_j m^C_{i,j}} [\mathcal{M}^C_{i,j}][H_{j,k}], \\ (t \times s^{\mathcal{D}})^*[N_j][H_{i,j} \times_j \mathcal{M}^D_{j,k}] &= (-1)^{d_j h_{i,j}} [H_{i,j}][\mathcal{M}^D_{j,k}], \end{aligned}$$
where $c_i := \dim C_i, m^C_{i,j} := \dim \mathcal{M}^C_{i,j}, d_j := \dim D_j$ and $h_{i,j} := \dim H_{i,j}$.

By property 4 of flow morphism, we have formula similar to (11.2). Thus it is convenient to use m_L, m_R to identify $\mathcal{M}^{\mathcal{C}}_{i,j} \times_j H_{j,k}, H_{i,j} \times_j \mathcal{M}^{\mathcal{D}}_{j,k}$ with the corresponding parts of $\partial H_{i,k}$. We will suppress m_L, m_R, and treat $\mathcal{M}^{\mathcal{C}}_{i,j} \times_j H_{j,k}, H_{i,j} \times_j \mathcal{M}^{\mathcal{D}}_{j,k}$ as they are contained in $\partial H_{i,k}$.

Remark 12.16. *In the context of Floer theory, the existence of N in condition 2 usually comes from some energy estimates: There exists some notion of energy $E(u)$ for curve u in the moduli space $H_{i,j}$, such that $E(u) \geq 0$. If the energy $E(u)$ satisfies inequality $E(u) \leq g(D_j) - f(C_i) + C$, where f and g are the background Morse-Bott functional for $\mathcal{C}$ and $\mathcal{D}$ and C is a universal constant depending on the data we used to define $H_{i,j}$. When $j \ll i$, $E(u) < 0$,[2] i.e. there is no curve in $H_{i,j}$.*

[2] assuming the critical values do not accumulate

Remark 12.17. *When C_i has infinite components, the compactness of $H_{i,j}$ can be weaken to similar conditions as in remark 11.10.*

Notations

- $\mathcal{C} := \{C_i, \mathcal{M}^{\mathcal{C}}_{i,j}\}$, $\mathcal{D} := \{D_i, \mathcal{M}^{\mathcal{D}}_{i,j}\}$ are oriented flow categories.
- $c_i := \dim C_i, d_i := \dim D_i, m^C_{i,j} := \dim \mathcal{M}^C_{i,j}$ and $m^D_{i,j} := \dim \mathcal{M}^D_{i,j}$.
- $h_{i,j} := \dim H_{i,j}$, then $h_{i,j} + m^D_{j,k} - d_j + 1 = h_{i,k}$ and $m^C_{i,j} + h_{j,k} - c_i + 1 = h_{i,k}$.
- For $k \in \mathbb{Z}$ and $0 < i_1 < \ldots < i_p$ and $j_1 < \ldots < j_q < k$,

  $$H^{s,k}_{i_1,\ldots,i_p|j_1\ldots,j_q} := \mathcal{M}^C_{s,s+i_1} \times \ldots \times \mathcal{M}^C_{s+i_{p-1},s+i_p} \times H_{s+i_p,s+j_1} \times \mathcal{M}^D_{s+j_1,s+j_2} \times \ldots \times \mathcal{M}^D_{s+j_q,s+k}$$

  with the product orientation.
- g^n_i are the counterparts of primitives f^n_i for flow category $\mathcal{D}$.
- $H^{*,*}_{\ldots|\ldots}[\alpha, f, \ldots, g, \ldots, \gamma]$ is defined similarly to $\mathcal{M}^{*,*}_{\ldots}[\alpha, f, \ldots, \gamma]$ in (12.5).
- For $\alpha \in h(\mathcal{C}, s)$, $\ddagger(H, \alpha, k) = (|\alpha| + h_{s,s+k} + 1)(d_{s+k} + 1)$.

The counterparts of Lemma 12.5,12.13 hold for H by the same argument. We will suppress the index n and $\lim\limits_{n\to\infty}$ and use formal relations like (12.43). The main result of this section is that a flow-morphism induces a cochain map between the Morse-Bott cochain complexes:

Theorem 12.18. *Let H be an oriented flow-morphism between two oriented flow categories $\mathcal{C}$ and $\mathcal{D}$. If we fix defining-data Θ_1, Θ_2 for $\mathcal{C}, \mathcal{D}$. Then we have map $\phi^H_{\Theta_1,\Theta_2} : BC^{\mathcal{C}} \to BC^{\mathcal{D}}$ given by (12.44), such that*

$$\phi^H_{\Theta_1,\Theta_2} \circ d^C_{BC,\Theta_1} - d^D_{BC,\Theta_2} \circ \phi^H_{\Theta_1,\Theta_2} = 0.$$

In particular, $\phi^H_{\Theta_1,\Theta_2}$ induces a map $H^*(BC^{\mathcal{C}}, d^{\mathcal{C}}_{BC,\Theta_1}) \to H^*(BC^{\mathcal{D}}, d^{\mathcal{D}}_{BC,\Theta_2})$

For simplicity, we suppress the subscripts Θ_1, Θ_2. The map ϕ^H is defined as $\prod_k \phi^H_k$, i.e. for $A \in BC^{\mathcal{C}}$:

$$\phi^H(A)(j) = \sum_{k\in\mathbb{Z}} \phi^H_k(A(j-k)), \tag{12.44}$$

where $\phi^H_k : H^*(C_s) \to H^*(D_{s+k})$ is defined as

$$\begin{aligned}\left\langle \phi^H_k[\alpha],[\gamma]\right\rangle_{s+k} &:= (-1)^{|\alpha|c_s+h_{s,s+k}} H^{s,k}_{|}[\alpha,\gamma] \\ &\quad + \sum_{\substack{0<i_1<\ldots<i_p \\ j_1<\ldots<j_q<k}} (-1)^* H^{s,k}_{i_1,\ldots,i_p|j_1,\ldots,j_q}[\alpha, f_{s+i_1},\ldots,f_{s+i_p},g_{s+j_1},\ldots,g_{s+j_q},\gamma] \\ &:= (-1)^{|\alpha|c_s+h_{s,s+k}} H^{s,k}_{|}[\alpha,\gamma] \\ &\quad + \lim_{n\to\infty} \sum_{\substack{0<i_1<\ldots<i_p \\ j_1<\ldots<j_q<k}} (-1)^* H^{s,k}_{i_1,\ldots,i_p|j_1,\ldots,j_q}[\alpha, f^n_{s+i_1},\ldots,f^n_{s+i_p},g^n_{s+j_1},\ldots,g^n_{s+j_q},\gamma],\end{aligned} \tag{12.45}$$

where

$$* := |\alpha|c_s + h_{s,s+k} + \sum_{w=1}^{p} \ddagger(\mathcal{C},\alpha,i_w) + \sum_{w=1}^{q} \ddagger(H,\alpha,j_w) + d_{s+j_1} + 1 + m^{\mathcal{D}}_{s+j_1,s+k}.$$

When $q=0$, we formally define $j_1 := k$ and define $m^D_{k,k} := d_k - 1$. The existence of N in the condition 2 of the Definition 12.15 implies that $\phi^H_k = 0$ for $k < -N$, thus (12.44) (12.45) are finite sums.

Proof of Theorem 12.18. The proof is analogous to the proof of Proposition 12.12, basically careful index computing. We defer the proof to Section 12.7. □

Identity morphism

In this subsection, we show that for every oriented flow category $\mathcal{C}$, there is an oriented flow-morphism I from $\mathcal{C}$ to $\mathcal{C}$, called the identity flow-morphism. Roughly speaking, when the flow category has a background Morse-Bott function, the identity flow morphism comes from the compactified moduli space of parametrized gradient flow lines, i.e. flow lines without quotient by the translation action of $\mathbb{R}$. Using the identity flow-morphism, we show the Morse-Bott cohomology is independent of the defining data Θ.

Definition/Lemma 12.19 (Identity Flow-Morphism)**.** *For an oriented flow category $\mathcal{C}$, there is a canonical oriented flow morphism I from $\mathcal{C}$ to itself. Given by $I_{i,j} := \mathcal{M}_{i,j}\times[0,j-i]$ with the product orientation, for $i\le j$, and $I_{i,j}=\emptyset$ for $i>j$. The source and target maps $s,t : I_{i,j}\to C_i, C_j$ are defined as*

$$s,t := s^{\mathcal{C}}, t^{\mathcal{C}}.$$

And m_L, m_R are defined as follows:

$$\begin{aligned} m_L &: \mathcal{M}_{i,k}\times_k I_{k,j} \to I_{i,k}, (a,b,t)\mapsto (m(a,b), t+k-i) \\ m_R &: I_{i,k}\times_k \mathcal{M}_{k,j} \to I_{i,k}, (a,t,b)\mapsto (m(a,b), t)\end{aligned}$$

where m is the composition in $\mathcal{C}$.

Before proving this lemma, we will first use it to finish the proof of Theorem 12.7.

Proof of Theorem 12.7. We have shown in proposition 12.12 that (BC, d_{BC,Θ_1}) and (BC, d_{BC,Θ_2}) are cochain complexes. By (12.45), $\phi^I_{\Theta_1,\Theta_2}$ can be written as $\mathrm{id}+N$, where N is strictly upper triangular, i.e. N sends $H^*(C_s)$ to $\prod_{t=s+1}^{\infty} H^*(C_t)$. Since $\sum_{n=0}^{\infty}(-N)^n$ is well defined on the cochain complex BC, therefore $\sum_{n=0}^{\infty}(-N)^n$ is the inverse to $\mathrm{id}+N$. Thus $\phi^I_{\Theta_1,\Theta_2}$ induces an isomorphism on cohomology for any Θ_1, Θ_2. □

Proof of Lemma 12.19. Condition 2 of Definition 12.15 follows from that $I_{i,j} = \emptyset$, for $i > j$. Condition 3 holds for I due to the transversality property of flow category $\mathcal{C}$. Since $m_L(\mathcal{M}_{i,k} \times_j I_{k,j}) = \mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k} \times [k-i, j-i]$ and $m_R(I_{i,k} \times_j \mathcal{M}_{k.j}) = \mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k} \times [0, k-i]$, therefore condition 4, 5 of flow-morphism are satisfied by I. Therefore only the orientation condition 6 remains to check.

Unless stated otherwise, product of manifolds are always equipped with the product orientation. For $i < j$:

$$\begin{aligned}
\partial[I_{i,j}] &= \partial[\mathcal{M}_{i,j} \times [0, j-i]] \\
&= (-1)^{m_{i,j}+1}[\mathcal{M}_{i,j} \times \{0\}] + (-1)^{m_{i,j}}[\mathcal{M}_{i,j} \times \{j-i\}] \\
&\quad + \sum_{i<k<j} (-1)^{m_{i,k}}[\mathcal{M}_{i,k} \times_k \mathcal{M}_{k,j} \times [0, j-i]] \\
&= (-1)^{m_{i,j}+1}[\mathcal{M}_{i,j} \times \{0\}] + (-1)^{m_{i,j}}[\mathcal{M}_{i,j} \times \{j-i\}] && (12.46)\\
&\quad + \sum_{i<k<j} (-1)^{m_{i,k}}[\mathcal{M}_{i,k} \times_k \mathcal{M}_{k,j} \times [0, k-i]] && (12.47)\\
&\quad + \sum_{i<k<j} (-1)^{m_{i,k}}[\mathcal{M}_{i,k} \times_k \mathcal{M}_{k,j} \times [k-i, j-i]]. && (12.48)
\end{aligned}$$

Since the flow category $\mathcal{C}$ is oriented, for $i < k < j$,

$$(t^{\mathcal{C}} \times s^{\mathcal{C}})^*[N_k][\mathcal{M}_{i,k} \times_k \mathcal{M}_{k,j}] = (-1)^{c_k m_{i,k}}[\mathcal{M}_{i,k}][\mathcal{M}_{k,j}]. \tag{12.49}$$

Let π be the projection $I_{i,j} \to \mathcal{M}_{i,j}$ for $i < j$, then

$$\begin{aligned}
(t \times s^{\mathcal{C}})^* N_k &= \pi^*(t^{\mathcal{C}} \times s^{\mathcal{C}})^* N_k|_{\mathcal{M}_{i,k} \times_k \mathcal{M}_{k,j} \times [0,k-i]}, \\
(t^{\mathcal{C}} \times s)^* N_k &= \pi^*(t^{\mathcal{C}} \times s^{\mathcal{C}})^* N_k|_{\mathcal{M}_{i,k} \times_k \mathcal{M}_{k,j} \times [k-i,j-i]}.
\end{aligned}$$

Therefore (12.49) implies

$$\begin{aligned}
(t \times s^{\mathcal{C}})^*[N_k][\mathcal{M}_{i,k} \times_k \mathcal{M}_{k,j} \times [0, k-i]] &= (-1)^{c_{i,k} m_{i,k} + m_{k,j}}[\mathcal{M}_{i,k} \times [0, k-i]][\mathcal{M}_{k,j}] \\
&= (-1)^{c_{i,k} m_{i,k} + m_{k,j}}[I_{i,k}][\mathcal{M}_{j,k}], && (12.50)\\
(t^{\mathcal{C}} \times s)^*[N_k][\mathcal{M}_{i,k} \times_k \mathcal{M}_{k,j} \times [k-i, j-i]] &= (-1)^{c_k m_{i,j}}[\mathcal{M}_{i,k}][\mathcal{M}_{k,j} \times [k-i, j-i]] \\
&= (-1)^{c_k m_{i,j}}[\mathcal{M}_{i,k}][I_{k,j}]. && (12.51)\\
& && (12.52)
\end{aligned}$$

If we orient $I_{i,k}\times_k \mathcal{M}_{k,j}$ by $(-1)^{m_{k,j}+c_k}[\mathcal{M}_{i,k}\times_k \mathcal{M}_{k,j}]\times[[0,k-i]]$ and $[\mathcal{M}_{i,k}\times_k I_{k,j}]$ by $[\mathcal{M}_{i,k}\times_k \mathcal{M}_{k,j}]\times[[k-i,j-i]]$, then (12.50) implies that

$$\begin{aligned}(12.47)=(-1)^{m_{i,k}}[\mathcal{M}_{i,k}\times_k \mathcal{M}_{k,j}\times[0,k-i]] &= (-1)^{m_{i,j}+1}[I_{i,k}\times_k \mathcal{M}_{k,j}], && (12.53)\\ (t\times s^{\mathcal{C}})^*[N_k][\mathcal{M}_{i,k}\times_j I_{k,j}] &= (-1)^{c_k(m_{i,k}+1)}[I_{i,k}][\mathcal{M}_{k,j}]. && (12.54)\end{aligned}$$

And (12.51) implies that

$$\begin{aligned}(12.48)=(-1)^{m_{i,k}}[\mathcal{M}_{i,k}\times_k \mathcal{M}_{k,j}\times[k-i,j-i]] &= (-1)^{m_{i,k}}[\mathcal{M}_{i,k}\times_k I_{k,j}], && (12.55)\\ (t^{\mathcal{C}}\times s)^*[N_k][\mathcal{M}_{i,k}\times_k I_{k,j}] &= (-1)^{c_k m_{i,k}}[\mathcal{M}_{i,k}][I_{k,j}]. && (12.56)\end{aligned}$$

We still have to consider the first two copies of $\mathcal{M}_{i,j}$ in (12.46). Since $m_L: C_i\times_i\mathcal{M}_{i,j}\to\mathcal{M}_{i,j}$ and $m_R:\mathcal{M}_{i,j}\times_j C_j\to\mathcal{M}_{i,j}$ are diffeomorphisms. Therefore we can orient $I_{i,i}\times_i\mathcal{M}_{i,j}=C_i\times_i\mathcal{M}_{i,j}$ and $\mathcal{M}_{i,j}\times_j C_j=\mathcal{M}_{i,j}\times_j I_{j,j}$ by $m_L^{-1}([\mathcal{M}_{i,j}])$ and $m_R^{-1}([\mathcal{M}_{i,j}])$. Then by Lemma 12.20 below we have

$$\begin{aligned}(t\times s^{\mathcal{C}})^*[N_i][C_i\times\mathcal{M}_{i,j}] &= (-1)^{c_i^2}[C_i][\mathcal{M}_{i,j}],\\ (t^{\mathcal{C}}\times s)^*[N_j][\mathcal{M}_{i,j}\times_j C_j] &= (-1)^{c_j m_{i,j}}[\mathcal{M}_{i,j}][C_j].\end{aligned}$$

Therefore, we have

$$\begin{aligned}(-1)^{m_{i,j}+1}[\mathcal{M}_{i,j}\times\{0\}] &= (-1)^{m_{i,j}+1}m_R([I_{i,i}\times_i\mathcal{M}_{i,j}]),\\ [(t\times s^{\mathcal{C}})^*N_j][I_{i,i}\times_i\mathcal{M}_{i,j}] &= (-1)^{c_i^2}[I_{i,i}][\mathcal{M}_{i,j}],\\ (-1)^{m_{i,j}}[\mathcal{M}_{i,j}\times\{j-i\}] &= (-1)^{m_{i,j}}m_L([\mathcal{M}_{i,j}\times_j I_{j,j}]),\\ [(t^{\mathcal{C}}\times s)^*N_i][\mathcal{M}_{i,j}\times_j I_{j,j}] &= (-1)^{c_j m_{i,j}}[\mathcal{M}_{i,j}][I_{j,j}].\end{aligned}\qquad(12.57)$$

(12.53), (12.54), (12.55), (12.56) and (12.57) prove the orientation property 6 of Definition 12.15. □

To state the Lemma 12.20, we need to set up some notations. Let E,F be two oriented finite dimensional vector spaces, and $l:E\to F$ be a linear map. Δ_F denotes the diagonal subspace of $F\times F$. Let ordered basis $(f_1,\ldots,f_n)$ represents orientation $[F]$ of F, and ordered basis $(e_1,\ldots,e_m)$ represents orientation of E. Then $((f_1,f_1),\ldots,(f_n,f_n))$ determines an orientation $[\Delta_F]$ of Δ_F. Just like (11.1), we orient the quotient bundle (i.e. the normal bundle) $(F\times F)/\Delta_F$, such that $[\Delta_F][(F\times F)/\Delta_F]=[F][F]$. The fiber product $E\times_l F$ is graph of l in $E\times F$, then $((e_1,l(e_1)),\ldots,(e_m,l(e_m))$ determines an orientation $[E\times_l F]$ on $E\times_l F=\operatorname{graph} l$. The projection $\pi:E\times_l F\to E$ is an isomorphism, and the orientation we put on $E\times_l F$ has the property that $\pi([E\times_l F])=[E]$. Since $(l,\mathrm{id}):(E\times F)/E\times_l F\to(F\times F)/\Delta_F$ is an isomorphism, thus we can orient $(E\times F)/E\times_l F$ by $(l,\mathrm{id})([(E\times F)/E\times_l F])=[(F\times F)/\Delta_F]$. What we describe here is the tanget picture of $\mathcal{M}_{i,j}\times_j C_j$: let $(m,c)\in\mathcal{M}_{i,j}\times_j C_j$, then the correspondences are $E=T_m\mathcal{M}_{i,j}$, $F=T_cC_j$, $l=Ds|_m$, and the orientations match up.

Lemma 12.20. *Following the notation above, we have*

$$[(E\times F)/E\times_l F][E\times_l F]=(-1)^{\dim E\dim F}[E][F].$$

Proof. Since ordered basis $((0_F,f_1),\ldots,(0_F,f_n))$ represents a bases for $(F\times F)/\Delta_F$ as well as the orientation $[(F\times F)/\Delta_F]$. Since $((0_E,f_1),\ldots,(0_E,f_n))$ represents a basis for $(E\times F)/E\times_l F$, and is mapped to $((0_F,f_1),\ldots,(0_F,f_n))$ through the map (l,id), thus $((0_E,f_1),\ldots,(0_E,f_n))$ represents the orientation on $(E\times F)/E\times_l F$. Since

$$\{(e_1,l(e_1)),\ldots,(e_m,l(e_m)),(0_E,f_1),\ldots,(0_E,f_n)\}$$

represents the orientation $[E][F]$, so we have

$$[E\times_l F][(E\times F)/E\times_l F]=[E][F] \text{ or } [(E\times F)/E\times_l F][E\times_l F]=(-1)^{\dim E\dim F}[E][F].$$

□

Similarly, if we consider $F\times_l E$, and it is oriented by $((l(e_1),e_1),\ldots,(l(e_m),e_m))$, and we orient $(F\times E)/F\times_l E$ by $(\mathrm{id},l)([(F\times E)/F\times_l E])=[(F\times F)/\Delta_F]$, then

$$[(F\times E)/F\times_l E][F\times_l E]=(-1)^{(\dim F)^2}[F][E].$$

12.4 Composition of flow-morphisms

In this section, we discuss how to compose two flow-morphisms, and how do the induced cochain morphisms behave under the compositions.

Definition 12.21 (Composable flow-morphisms). *Two flow-morphisms $H:\mathcal{C}\to\mathcal{D}$, $F:\mathcal{D}\to\mathcal{E}$ are called composable iff the fiber products $H_{i,j}\times_j F_{j,k}$ are cut out transversely for all i,j,k.*

Heuristically, one can define the composition $F\circ H$ of two composable morphisms F and H to be $F\circ H_{i,k}=\cup_j H_{i,j}\times_j F_{j,k}$, where the orientation is determined by

$$(t^H\times s^F)^*[N_j][H_{i,j}\times_j F_{j,k}]=(-1)^{d_j h_{i,j}}[H_{i,j}][F_{j,k}] \tag{12.58}$$

However, this is no longer a flow-morphism, since the boundary can also come from the fiber products in the middle in addition to fiber products at two ends.

Definition 12.22. *An oriented pre-flow-morphism F between two flow category $\mathcal{C}$ and $\mathcal{D}$ is a family of compact oriented manifolds $F_{i,j}$, with smooth maps $s:F_{i,j}\to C_i, t:F_{i,j}\to D_j$. Moreover, there exists N, such for $i-j>N$, $F_{i,j}=\emptyset$, and the fiber products $\mathcal{M}^C_{i_0,i_1}\times_{i_1}\ldots\times_{i_k}F_{i_k,j_0}\times_{j_0}\ldots\times_{j_{l-1}}\mathcal{M}^D_{j_{l-1},j_l}$ are cut out transversely for all $i_0<\ldots<i_k, j_0<\ldots<\ldots j_l$*

Given a pre-flow-morphism F, one can still define ϕ^F by (12.45). Let F and H be two composable flow-morphisms, then $F\circ H$ defines a pre-flow-morphism. We need to understand the relation between $\phi^{F\circ H}$ and $\phi^F\circ\phi^H$, it turns out they are differed by a homotopy.

Notations

- $\mathcal{E} := \{E_i, \mathcal{M}^E_{i,j}\}$ is an oriented flow category, $e_i := \dim E_i$, $m^E_{i,j} := \dim \mathcal{M}^E_{i,j}$ and $f_{i,j} := \dim F_{i,j}$.
- h_i are the counterparts of primitives f_i for $\mathcal{E}$.
- For $k \in \mathbb{Z}$, $0 < i_1 < \ldots < i_p$, $j_1 < \ldots < j_q$ and $k_1 < \ldots < k_r < k$, we define $F \times H^{s,k}_{i_1,\ldots,i_p|j_1\ldots,j_q|k_1,\ldots,k_r}$ to be:

$$\mathcal{M}^C_{s,s+i_1} \times \ldots H_{s+i_p,s+j_1} \times \mathcal{M}^D_{s+j_1,s+j_2} \times \ldots F_{s+j_q,s+k_1} \times \ldots \mathcal{M}^E s + k_r, s+k.$$

  Note that we must have $q \geq 1$ to define it.

- $F \times H^{s,k}_{i_1,\ldots,i_p|j_1\ldots,j_q|k_1,\ldots,k_r}[\alpha, f_{s+i_1}, \ldots, f_{s+i_p}, g_{s+j_1}, \ldots, g_{s+j_q}, h_{s+k_1}, \ldots, h_{s+k_r}, \gamma]$ is defined similarly to (12.5).

Theorem 12.23. *Let F, H be composable oriented flow-morphisms from $\mathcal{C}$ to $\mathcal{D}$ and from $\mathcal{D}$ to $\mathcal{E}$, fix defining-data Θ_1, Θ_2 and Θ_3 for $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, then there exists an operator $P_{\Theta_1,\Theta_2,\Theta_3} : BC^{\mathcal{C}} \to BC^{\mathcal{E}}$ defined by (12.59), such that*

$$\phi^{F\circ H}_{\Theta_1,\Theta_3} - \phi^F_{\Theta_2,\Theta_3} \circ \phi^H_{\Theta_1,\Theta_2} + P_{\Theta_1,\Theta_2,\Theta_3} \circ d_{BC,\Theta_1} + d_{BC,\Theta_3} \circ P_{\Theta_1,\Theta_2,\Theta_3} = 0$$

As a corollary, $\phi^{F\circ H}_{\Theta_1,\Theta_3}$ is a cochain map between $(BC^{\mathcal{C}}, d^C_{BC,\Theta_1})$ and $(BC^{\mathcal{E}}, d^E_{BC,\Theta_3})$ and homotopic to $\phi^F_{\Theta_2,\Theta_3} \circ \phi^H_{\Theta_1,\Theta_2}$. For simplicity, we will suppress the subscripts Θ. For $k \in \mathbb{Z}$, $\alpha \in h(\mathcal{C}, s), \gamma \in h(\mathcal{E}, s+k)$, the operator P is defined by

$$\begin{aligned}\langle P[\alpha], [\gamma]\rangle_{s+k} := \sum (-1)^{\bigstar} F \times H^{s,k}_{i_1,\ldots,i_p|j_1,\ldots,j_q|k_1,\ldots,k_r}[\alpha, & f_{s+i_1}, \ldots, f_{s+i_p}, g_{s+j_1}, \ldots \\ & \ldots, g_{s+j_q}, h_{s+k_1}, \ldots, h_{s+k_r}, \gamma]\end{aligned} \tag{12.59}$$

where

$$\begin{aligned}\bigstar := |\alpha|(c_s+1) + \dim F \circ H_{s,s+k} + 1 + & \sum_{w=1}^{p} \ddagger(\mathcal{C}, \alpha, i_w) + h_{s,s+j_1} + \sum_{w=1}^{q} \ddagger(H, \alpha, j_w) \\ & + \sum_{w=1}^{r} \dagger(F \circ H, \alpha, k_w) + e_{s+k_1} + m^E_{s,s+k_1,s+k} + 1\end{aligned}$$

When $r = 0$, the convention is $k_1 := k$ and $m^E_{s+k,s+k} := e_{s+k} - 1$.

Proof of Theorem 12.23. The proof is deferred to the section 12.7. □

12.5 Flow homotopies induce cochain homotopies

In this section, we introduce the flow-homotopies between pre-flow-morphisms, which can be understood as generalizations of the homotopies between continuation maps in classical Morse theory [5].

Definition 12.24. *An **oriented flow-homotopy** K between two pre-flow-morphisms F and H from $\mathcal{C}$ to $\mathcal{D}$ is a family of oriented compact manifold $K_{i,j}$ with source and target maps $s : K_{i,j} \to C_i$ and $t : K_{i,j} \to D_j$, such that the following hold.*

1. *There are smooth injective maps $\iota_F, \iota_H : F_{i,j}, H_{i,j} \to K_{i,j}$, such that $s \circ \iota_{F/H} = s^{F/H}$, $t \circ \iota_{F/H} = t^{F/H}$, where $s^{F/H}, t^{F/H}$ are the source and target maps for F, H.*
2. *$\exists N$, such that when $i - j > N$, $K_{i,j} = \emptyset$.*
3. *Fiber products $\mathcal{M}^C_{i_0,i_1} \times_{i_1} \ldots \times_{i_k} K_{i_k,j_0} \times_{j_0} \ldots \times_{j_{l-1}} \mathcal{M}^D_{j_{l-1},j_l}$ are cut out transversely for all $i_0 < \ldots < i_k, j_0 < \ldots < j_l$.*
4. *There are smooth maps $m_L : \mathcal{M}^C_{i,j} \times_j K_{j,k} \to K_{i,k}$ and $m_R : K_{i,j} \times_j \mathcal{M}^D_{j,k} \to K_{i,k}$. Such that*
$$\begin{aligned} s \circ m_L(a,b) &= s^{\mathcal{C}}(a), \\ t \circ m_L(a,b) &= t(b), \\ s \circ m_R(a,b) &= s(a), \\ t \circ m_R(a,b) &= t^{\mathcal{D}}(b), \end{aligned}$$
where $s^{\mathcal{C}}$ is the source map for $\mathcal{C}$ and $t^{\mathcal{D}}$ is the source map for $\mathcal{D}$.
5. *$\iota_F \cup \iota_H \cup m_L \cup m_R : \partial^0 F_{i,k} \cup \partial^0 H_{i,k} \cup \left(\cup_j \partial^0(\mathcal{M}^{\mathcal{C}}_{i,j} \times_j K_{j,k})\right) \cup \left(\cup_j \partial^0(K_{i,j} \times_j \mathcal{M}^{\mathcal{D}}_{j,k})\right) \to \partial^1 K_{i,k}$ is a diffeomorphism up to measure-zero sets.*
6. *The orientation $[K_{i,j}]$ has the following properties:*
$$\begin{aligned} \partial[K_{i,j}] = \iota_F([F_{i,j}]) - \iota_H([H_{i,j}]) &+ \sum_{p>0} (-1)^{c_{i+p}+1} m_L([\mathcal{M}^C_{i,i+p} \times_{i+p} K_{i+p,j}]) \\ &+ \sum_{p>0} (-1)^{k_{i,j}} m_R([K_{i,j-p} \times_{j-p} M^D_{j-p,j}]), \end{aligned}$$
$$(t^{\mathcal{C}} \times s)^*[N_j][\mathcal{M}^C_{i,j} \times_j K_{j,k}] = (-1)^{c_j m^C_{i,j}} [\mathcal{M}^C_{i,j}][K_{j,k}],$$
$$(t \times s^{\mathcal{D}})^*[N_j][H_{i,j} \times_j \mathcal{M}^D_{j,k}] = (-1)^{d_j k_{i,j}} [K_{i,j}][\mathcal{M}^D_{j,k}],$$
where $k_{i,j} := \dim K_{i,j}$.

Notations

- $k_{i,j} := \dim K_{i,j}$, then $k_{i,j} + m^D_{j,k} - d_j + 1 = k_{i,k}$ and $m^C_{i,j} + k_{j,k} - c_i + 1 = k_{i,k}$.
- For $k \in \mathbb{Z}$ and $0 < i_1 < \ldots < i_p$ and $j_1 < \ldots < j_q < k$,

  $$K^{s,k}_{i_1,\ldots,i_p|j_1\ldots,j_q} := \mathcal{M}^C_{s,s+i_1} \times \ldots \times \mathcal{M}^C_{s+i_{p-1},s+i_p} \times K_{s+i_p,s+j_1} \times \mathcal{M}^D_{s+j_1,s+j_2} \times \ldots \times \mathcal{M}^D_{s+j_q,s+k}.$$
- $K^{*,*}_{\ldots}[\alpha, f, \ldots, g \ldots, \gamma]$ is defined similarly as (12.5).
- For $\alpha \in h(\mathcal{C}, s)$, $\ddagger(K, \alpha, k) := (|\alpha| + k_{s,s+k} + 1)(d_{s+k} + 1)$.

Theorem 12.25. *Given an oriented flow-homotopy K between two pre-flow-categories $F, H : \mathcal{C} \to \mathcal{D}$. Fixing all the defining-data Θ_1, Θ_2 for $\mathcal{C}, \mathcal{D}$, there exists operator $\Lambda^K_{\Theta_1,\Theta_2} : BC^{\mathcal{C}} \to BC^{\mathcal{D}}$ defined by (12.60), such that*

$$d^D_{BC,\Theta_2} \circ \Lambda^K_{\Theta_1,\Theta_2} + \Lambda^K_{\Theta_1,\Theta_2} \circ d^C_{BC,\Theta_1} + \phi^F_{\Theta_1,\Theta_2} - \phi^H_{\Theta_1,\Theta_2} = 0.$$

We will suppress the subscripts. Let $\alpha \in h(\mathcal{C}, s)$ and $\gamma \in h(\mathcal{D}, s+k)$, then $\left\langle \Lambda^K[\alpha], [\gamma] \right\rangle_{s+k}$ is defined to be:

$$(-1)^{|\alpha|(c_s+1)+k_{s,s+k}} \left(K^{s,s+k}_{|}[\alpha,\gamma] + \sum_{\substack{0<i_1<\ldots<i_p \\ j_1<\ldots<j_q<k}} (-1)^{\clubsuit} I^{s,k}_{i_1,\ldots,i_p|j_1,\ldots,j_q}[\alpha, f_{s+i_1}, \ldots, g_{s+j_q}, \gamma] \right), \tag{12.60}$$

where $\clubsuit := \sum_{w=1}^p \ddagger(\mathcal{C}, \alpha, i_w) + c_{s+i_p} + m^C_{s,s+i_p} + 1 + \sum_{w=1}^q \ddagger(K, \alpha, j_w) + d_{s+j_1} + m^D_{s+j_1,s+k} + 1$. Formally i_0 is defined to be 0, i_{q+1} is defined to be k and $m^{C/D}_{i,i}$ is defined to be $c(d)_i - 1$.

Proof of Theorem 12.25. The proof is deferred in the last subsection. $\square$

12.6 Morse-Bott cochain complex is canonical

Unlike the Morse case, where the defining-data is unique, the minimal Morse-Bott cochain complex $(BC, d_{BC,\Theta})$ depends on the defining-data Θ, i.e. choices of representatives, choices of Thom classes and choices of f^n_i. Morphism $\phi^H_{\Theta,\Theta'}$ from (12.45) also depends on Θ, Θ'. Theorem 12.7 asserts that the cohomology is independent of the defining-data. In fact, the invariance can be stated on the cochain level. In this section, we prove the assignment of the homotopy type of minimal Morse-Bott cochain complex $(BC, d_{BC,\Theta})$ is natural with respect to Θ. Moreover, we will show in this subsection, the cochain morphism $\phi^H_{\Theta,\Theta'}$ from (12.45) is also canonical in a suitable sense.

For an oriented flow category $\mathcal{C}$, there is an associated category $\mathcal{D}ata^{\mathcal{C}}$. An object of $\mathcal{D}ata^{\mathcal{C}}$ is a defining-data Θ for $\mathcal{C}$, and there is exactly one morphism from one object to another object. Then for every object Θ in $\mathcal{D}ata^{\mathcal{C}}$, we can associate it with a cochain complex $(BC, d_{BC,\Theta})$. The following theorem says this assignment can be completed to a functor from $\mathcal{D}ata^{\mathcal{C}} \to \mathcal{K}(\mathcal{C}h)$, where $\mathcal{K}(\mathcal{C}h)$ is the homotopy category of cochain complexes.

Theorem 12.26.

$$\begin{aligned} \Theta &\mapsto (BC, d_{BC,\Theta}) \\ (\Theta_1 \to \Theta_2) &\mapsto \left(\phi^I_{\Theta_1,\Theta_2} : (BC, d_{BC,\Theta_1}) \to (BC, d_{BC,\Theta_2})\right) \end{aligned}$$

defines a functor $\mathcal{BC}^{\mathcal{C}} : \mathcal{D}ata^{\mathcal{C}} \to \mathcal{K}(\mathcal{C}h)$, where I is the identity flow morphism.

Proof. Step 1, $\phi^I_{\Theta,\Theta}$ is homotopic to identity. For $i < j$, $I_{i,j} = \mathcal{M}_{i,j} \times [0, j-i]$ and $(I \circ I)_{i,j} = \cup_{k,i<k<j} I_{i,k} \times_k I_{k,j}$ have a interval direction. Since the pullback of differential forms by source and target maps can not cover that interval direction. Therefore when $p \neq q$

$$I^{s,k}_{\ldots,p|q,\ldots}[\ldots, f_{s+p}, f_{s+q}, \ldots] = I \circ I^{s,k}_{\ldots,p|q,\ldots}[\ldots, f_{s+p}, f_{s+q}, \ldots] = 0$$

It is not hard to check that $\phi^{I\circ I}_{\Theta,\Theta}$ can be written as $\mathrm{id} + M$ with M strictly upper triangular. To be more specific, for $k \in \mathbb{N}^+$, $\alpha \in h(\mathcal{C}, s)$ and $\gamma \in h(\mathcal{C}, s+k)$

$$\begin{aligned} \langle M[\alpha], [\gamma] \rangle_{s+k} \quad = \quad & \sum_{\substack{1 \le p \le q \le k \\ 0 < i_1 < \ldots < i_q < k}} (-1)^{\spadesuit_1} I \circ I^{s,k}_{i_1,\ldots,i_p|i_p,\ldots,i_q}[\alpha, f_{s+i_1}, \ldots, f_{s+i_p}, f_{s+i_p}, \ldots, f_{s+i_q}, \gamma] \\ + & \sum_{\substack{1 \le p \\ 0 < i_1 < \ldots < i_p = k}} (-1)^{\spadesuit_2} I \circ I^{s,k}_{i_1,\ldots,i_p|}[\alpha, f_{s+i_1}, \ldots, f_{s+i_p}, f_{s+i_p}, \gamma] \\ + & \sum_{\substack{1 \le p \\ 0 = i_1 < \ldots < i_p < k}} (-1)^{\spadesuit_3} I \circ I^{s,k}_{|i_1,\ldots,i_p}[\alpha, f_{s+i_1}, f_{s+i_1}, \ldots, f_{s+i_p}, \gamma], \end{aligned}$$

where

$$\begin{aligned} \spadesuit_1 &= \sum_{w=1}^{p} \ddagger(\mathcal{C}, \alpha, i_w) + \sum_{w=p}^{q} \dagger(\mathcal{C}, \alpha, i_w) + |\alpha| c_s + m_{s,s+k} + c_{s+i_p} + m_{s+i_p,s+k}, \\ \spadesuit_2 &= \sum_{w=1}^{p} \ddagger(\mathcal{C}, \alpha, i_w) + |\alpha| c_s + m_{s,s+k} + 1, \\ \spadesuit_3 &= \sum_{w=1}^{p} \dagger(\mathcal{C}, \alpha, i_w) + (|\alpha| + 1) c_s. \end{aligned}$$

Similarly, we have decompositions $\phi^I_{\Theta,\Theta} = \mathrm{id} + N$ with and N strictly upper triangular. Because

$$\begin{aligned} & I \circ I^{s,k}_{i_1,\ldots,i_p|i_p,\ldots,i_q}[\alpha, f_{s+i_1}, \ldots, f_{s+i_p}, f_{s+i_p}, \ldots, f_{s+i_q}, \gamma] \\ = & I^{s,k}_{i_1,\ldots,i_p|i_p,\ldots,i_q}[\alpha, f_{s+i_1}, \ldots, f_{s+i_p}, f_{s+i_p}, \ldots, f_{s+i_q}, \gamma]. \end{aligned}$$

Thus $N = M$. Then by Theorem 12.23,

$$(Id + M) - (Id + M)^2 = P \circ d_{BC,\Theta} + d_{BC,\Theta} \circ P.$$

Since $Id + M$ is a cochain isomorphism, we have:

$$Id - (Id + M) = (Id + M)^{-1} \circ P \circ d_{BC,\Theta} + d_{BC,\Theta} \circ (Id + M)^{-1} \circ P.$$

Thus $Id + M = \phi^I_{\Theta,\Theta}$ is homotopic to identity.

Step 2, functoriality. Given three defining-data $\Theta_1, \Theta_2, \Theta_3$, by the same argument as above we have

$$\phi^I_{\Theta_1,\Theta_3} = \phi^{I\circ I}_{\Theta_1,\Theta_3}$$

By Theorem 12.23,

$$\phi^{I\circ I}_{\Theta_1,\Theta_3} - \phi^I_{\Theta_2,\Theta_3} \circ \phi^I_{\Theta_1,\Theta_2} = P \circ d_{BC,\Theta_1} + d_{BC,\Theta_3} \circ P$$

Thus $\phi^I_{\Theta_1,\Theta_3}$ is homotopic to $\phi^I_{\Theta_2,\Theta_3} \circ \phi^I_{\Theta_1,\Theta_2}$. □

To explain the story for flow-morphisms, we introduce the category $(\mathcal{D}ata^{\mathcal{C}} \to \mathcal{D}ata^{\mathcal{D}})$ for two oriented flow categories $\mathcal{C}$ and $\mathcal{D}$. $(\mathcal{D}ata^{\mathcal{C}} \to \mathcal{D}ata^{\mathcal{D}})$ is a union of $\mathcal{D}ata^{\mathcal{C}}$ and $\mathcal{D}ata^{\mathcal{D}}$ plus a unique morphism from each defining-data of $\mathcal{C}$ to each defining-data of $\mathcal{D}$. The next theorem states that the assignment of $\phi^H_{\Theta,\Theta'}$ is natural:

Theorem 12.27. *For an oriented flow-morphism H, there is a functor*

$$\mathcal{H} : (\mathcal{D}ata^{\mathcal{C}} \to \mathcal{D}ata^{\mathcal{D}}) \to \mathcal{K}(\mathcal{C}h)$$

which extends $\mathcal{BC}^{\mathcal{C}}$ and $\mathcal{BC}^{\mathcal{D}}$: let Θ^C, Θ^D be defining data for $\mathcal{C}, \mathcal{D}$ respectively, the morphism $\Theta^C \to \Theta^D$ is sent to $\phi^H_{\Theta^C,\Theta^D}$.

Proof. We only need to prove the functoriality. We use Θ^C, Θ^D to stand for defining data for $\mathcal{C}, \mathcal{D}$ respectively.

Case 1, functoriality for compositions $\Theta^C_1 \to \Theta^C_2$ and $\Theta^C_2 \to \Theta^D$. By Theorem 12.23:

$$\Phi^{H\circ I}_{\Theta^C_1,\Theta^D} \text{ is homotopic to } \phi^H_{\Theta^C_2,\Theta_D} \circ \Phi^I_{\Theta^C_1,\Theta^C_2},$$

and

$$\Phi^{H\circ I}_{\Theta^C_1,\Theta^D} \text{ is homotopic to } \phi^H_{\Theta^C_1,\Theta_D} \circ \Phi^I_{\Theta^C_1,\Theta^C_1}.$$

Since $\Phi^I_{\Theta^C_1,\Theta^C_1}$ is homotopic to identity by Theorem 12.26, thus $\phi^H_{\Theta^C_2,\Theta_D} \circ \Phi^I_{\Theta^C_1,\Theta^C_2}$ is homotopic to $\phi^H_{\Theta^C_1,\Theta_D}$.

Case 2, functoriality for compositions $\Theta^C \to \Theta^D_1$ and $\Theta^D_1 \to \Theta^D_2$. Follows from the same argument. □

12.7 Proofs

Before proving Theorem 12.18 and 12.23, we recall some notations and introduce some new notations. Let H be an oriented flow-morphism from flow category $\mathcal{C}$ to flow category $\mathcal{D}$, and F be an oriented flow-morphism from $\mathcal{D}$ to $\mathcal{E}$.

Notations for proofs of Theorem 12.18 and 12.23

- $\mathcal{C} := \{C_i, \mathcal{M}^C_{i,j}\}$, $\mathcal{D} := \{D_i, \mathcal{M}^D_{i,j}\}$ and $\mathcal{E} := \{E_i, \mathcal{M}^E_{i,j}\}$ are oriented flow categories.
- $c_i := \dim C_i, d_i := \dim D_i, e_i := \dim E_i, m^C_{i,j} := \dim \mathcal{M}^C_{i,j}, m^D_{i,j} := \dim \mathcal{M}^D_{i,j}$ and $m^E_{i,j} := \dim \mathcal{M}^E_{i,j}$.
- $h_{i,j} := \dim H_{i,j}, f_{i,j} := \dim F_{i,j}$, then by Definition 12.3, they satisfy the following relations:

$$\begin{aligned} h_{i,j} + m^D_{j,k} - d_j + 1 &= h_{i,k}, \\ m^C_{i,j} + h_{j,k} - c_i + 1 &= h_{i,k}, \\ f_{i,j} + m^E_{j,k} - e_j + 1 &= f_{i,k}, \\ m^D_{i,j} + f_{j,k} - d_i + 1 &= f_{i,k}. \end{aligned}$$

- For $k \in \mathbb{Z}$, $0 < i_1 < \ldots < i_p$ and $j_1 < \ldots < j_q < k$,

  $$H^{s,k}_{i_1,\ldots,i_p|j_1\ldots,j_q} := \mathcal{M}^C_{s,s+i_1} \times \ldots \times \mathcal{M}^C_{s+i_{p-1},s+i_p} \times H_{s+i_p,s+j_1} \times \mathcal{M}^D_{s+j_1,s+j_2} \times \ldots \times \mathcal{M}^D_{s+j_q,s+k}.$$

- $H^{s,k}_{i_1,\ldots,\overline{i_t},\ldots,i_p|j_1\ldots,j_q}$ and $H^{s,k}_{i_1,\ldots,i_p|j_1\ldots,\overline{j_t},\ldots,j_q}$ are the fiber product versions at $s+i_t$ and $s+j_t$ similar to (12.8).
- For $k \in \mathbb{Z}$, $0 < i_1 < \ldots < i_p$, $j_1 < \ldots < j_q$ and $k_1 < \ldots < k_r < k$, we define $F \times H^{s,k}_{i_1,\ldots,i_p|j_1\ldots,j_q|k_1,\ldots,k_r}$ to be:

  $$\mathcal{M}^C_{s,s+i_1} \times \ldots H_{s+i_p,s+j_1} \times \mathcal{M}^D_{s+j_1,s+j_2} \times \ldots F_{s+j_q,s+k_1} \times \ldots \mathcal{M}^E_{s+k_r,s+k}.$$

  Note that we must have $q \geq 1$ to define it.
- Similarly, we define $F \circ H^{s,k}_{i_1,\ldots,i_p|j|k_1,\ldots,k_q}$ to be:

  $$\mathcal{M}^C_{s,s+i_1} \times \ldots \times \mathcal{M}^C_{s+i_{p-1},s+i_p} \times \left(H_{s+i_p,s+j} \times_{s+j} F_{s+j,s+k_1}\right) \times \mathcal{M}^E_{s+k_1,s+k_2} \times \ldots \mathcal{M}^E_{s+k_q,s+k}.$$

- The counterparts of the primitives f_i for flow categories $\mathcal{D}$ and $\mathcal{E}$ are denoted by g_i and h_i.

- $H^{*,*}_{\dots}[\alpha, f, \dots, g \dots, \gamma]$, $H^{*,*}_{\dots}[\mathrm{d}(\alpha, f, \dots, g \dots, \gamma)]$, $H^{*,*}_{\dots,\bar{;},\dots}[\alpha, f, \dots, g \dots, \gamma]$ and $\mathrm{Tr}^*_{C/D} H^{*,*}_{\dots}[\alpha, f, \dots, \theta\theta^{C/D^*} g \dots, \gamma]$ are defined similarly to (12.5), (12.9), (12.10) and (12.11).
- $F \times H^{s,k}_{i_1,\dots,i_p|j_1\dots,j_q|k_1,\dots,k_r}[\alpha, f_{s+i_1}, \dots, f_{s+i_p}, g_{s+j_1}, \dots, g_{s+j_q}, h_{s+k_1}, \dots, h_{s+k_r}, \gamma]$, $F \circ H^{s,k}_{i_1,\dots,i_p|j|k_1,\dots,k_q}[\alpha, f_{s+i_1}, \dots, f_{s+i_p}, h_{s+k_1}, \dots, h_{s+k_r}, \gamma]$ are defined similarly to (12.5).
- For $\alpha \in h(\mathcal{C}, s)$, $\dagger(H, \alpha, k) := (|\alpha| + h_{s,s+k})(d_{s+k} + 1)$ and $\ddagger(H, \alpha, k) := (|\alpha| + h_{s,s+k} + 1)(d_{s+k} + 1)$, similarly for F and $F \circ H$.

Lemma 12.5,12.13 also hold for flow-morphisms by the same argument. As we have seen in the proof of Proposition 12.12, we can take limit in any order, thus we will suppress the index n and $\lim\limits_{n\to\infty}$ and use the formal relation (12.43).

Proof of Theorem 12.18. The proof is analogous to the proof of Proposition 12.12. Following the in the proof of Proposition 12.12, there are decompositions

$$d^C_k = D^C_k + E^C_k, d^D_k = D^D_k + E^D_k.$$

We define the leading term Φ^H_k:

$$\left\langle \Phi^H_k[\alpha], [\gamma] \right\rangle_{s+k} := (-1)^{|\alpha| c_s + h_{s,s+k}} H^{s,k}_{|}[\alpha, \gamma]$$

then the error term $\epsilon^H_k := \phi^H_k - \Phi^H_k$. To prove Theorem 12.18, it suffices to prove that for all $k \in \mathbb{Z}$, $\alpha \in h(\mathcal{C}, s)$ and $\gamma \in h(\mathcal{D}, s+k)$,

$$\left\langle \sum_i \left(\phi^H_{k-i} \circ d^C_i - d^D_{k-i} \circ \phi^H_i\right)[\alpha], [\gamma] \right\rangle_{s+k} = 0.$$

For this purpose, we will first compute the leading term $\left\langle \sum_i \left(\Phi^H_{k-i} \circ D^C_i - D^D_{k-i}\Phi^H_i\right)[\alpha], [\gamma] \right\rangle_{s+k}$,

$$\begin{aligned}
& \left\langle \sum_i \left(\Phi^H_{k-i} \circ D^C_i - D^D_{k-i} \circ \Phi^H_i\right)[\alpha], [\gamma] \right\rangle_{s+k} \\
= \; & \sum_{j>0} (-1)^{*_1} \mathrm{Tr}^{s+j} H^{s,k}_{j|}[\alpha, \theta\theta^C_{s+j}{}^*, \gamma] \\
& + \sum_{j<k} (-1)^{*_2} \mathrm{Tr}^{s+j} H^{s,k}_{|j}[\alpha, \theta\theta^D_{s+j}{}^*, \gamma] \\
= \; & \sum_{j>0} (-1)^{1+|\alpha|+*_1} H^{s,k}_{j|}[\mathrm{d}(\alpha, f_{s+j}, \gamma)] && (12.61) \\
& + \sum_{j<k} (-1)^{1+|\alpha|+*_2} H^{s,k}_{|j}[\mathrm{d}(\alpha, g_{s+j}, \gamma)] && (12.62) \\
& + (-1)^{1+|\alpha|(c_s+1)+h_{s,s+k}} \int_{\partial H_{s,s+k}} s^*\alpha \wedge t^*\gamma, && (12.63)
\end{aligned}$$

where

$$\begin{aligned} *_1 &= 1+|\alpha|c_s+h_{s,s+k}+\dagger(\mathcal{C},\alpha,j), \\ *_2 &= 1+|\alpha|c_s+h_{s,s+k}+\dagger(H,\alpha,j)+d_{s+j}+1+m^D_{s+j,s+k}. \end{aligned}$$

(12.63) is 0 by Stokes' theorem. We claim after applying Stokes' theorem to (12.61) and (12.62), then the replacing trick in remark 12.14 $r-1$ more times, we have

$$\left\langle \sum_i \left(\Phi^H_{k-i}\circ D^C_i + D^D_{k-i}\circ\Phi^H_i\right)[\alpha],[\gamma]\right\rangle_{s+k} =$$

$$\sum_{\substack{0\le p\le r\\ 0<i_1<\ldots<i_p\\ j_1<\ldots<j_{r-p}<k}} (-1)^{*_3} H^{s,k}_{i_1,\ldots,i_p|j_1,\ldots,j_{r-p}}[\mathrm{d}(\alpha,f_{s+i_1},\ldots,f_{s+i_p},g_{s+j_1},\ldots,g_{s+j_{r-p}},\gamma)] \tag{12.64}$$

$$+\sum_{\substack{0\le p\le q\le r\\ 1\le t\le p\\ 0<i_1\ldots<i_p\\ j_1<\ldots<j_{q-p}<k}} (-1)^{*_4}\,\mathrm{Tr}^{s+i_t}\, H^{s,k}_{i_1,\ldots,i_p|j_1,\ldots,j_{q-p}}[\alpha,f_{s+i_1},\ldots,\theta^C\theta^C_{s+i_t}{}^*,\ldots,g_{s+i_{q-p}},\gamma] \tag{12.65}$$

$$+\sum_{\substack{0\le p\le q\le r\\ 1\le t\le q-p\\ 0<i_1\ldots<i_p\\ j_1<\ldots<j_{q-p}<k}} (-1)^{*_5}\,\mathrm{Tr}^{s+j_t}\, H^{s,k}_{i_1,\ldots,i_p|j_1,\ldots,j_{q-p}}[\alpha,f_{s+i_1},\ldots,\theta^D\theta^D_{s+j_t}{}^*,\ldots,g_{s+i_{q-p}},\gamma] \tag{12.66}$$

where

$$\begin{aligned} *_3 &:= |\alpha|(c_s+1)+h_{s,s+k}+\sum_{w=1}^{p}\dagger(\mathcal{C},\alpha,i_w)+\sum_{w=1}^{r-p}\dagger(H,\alpha,j_w)+d_{s+j_1}+1+m^D_{s+j_1,s+k}, \\ *_4 &:= |\alpha|c_s+h_{s,s+k}+\sum_{w=1}^{t-1}\ddagger(\mathcal{C},\alpha,i_w)+\sum_{w=t}^{p}\dagger(\mathcal{C},\alpha,i_w)+\sum_{w=1}^{q-p}\dagger(H,\alpha,j_w)+(d_{s+j_1}+m^D_{s+j_1,s+k}+1), \\ *_5 &:= |\alpha|c_s+h_{s,s+k}+\sum_{w=1}^{p}\ddagger(\mathcal{C},\alpha,i_w)+\sum_{w=1}^{t-1}\ddagger(H,\alpha,j_w)+\sum_{w=t}^{q-p}\dagger(H,\alpha,j_w)+(d_{s+j_1}+1+m^D_{s+j_1,s+k}). \end{aligned}$$

The claim is proven by induction just like the proof of Proposition 12.12, thus we skip the induction. When $r>k+N$, the exact term of (12.64) is zero, since $H^{s,k}_{i_1,\ldots i_p|j_1,\ldots,j_{r-p}}=\emptyset$. The remaining trace terms (12.65) and (12.66) are exactly:

$$\begin{aligned} &-\left\langle \sum_i \left(\Phi^H_{k-i}\circ E^C_i - E^D_{k-i}\circ\Phi^H_i\right)[\alpha],[\gamma]\right\rangle_{s+k} - \left\langle \sum_i \left(\epsilon^H_{k-i}\circ D^C_i - D^D_{k-i}\circ\epsilon^H_i\right)[\alpha],[\gamma]\right\rangle_{s+k} \\ &-\left\langle \sum_i \left(\epsilon^H_{k-i}\circ E^C_i - E^D_{k-i}\circ\epsilon^H_i\right)[\alpha],[\gamma]\right\rangle_{s+k} \\ =&\left\langle \sum_i \left(\Phi^H_{k-i}\circ D^C_i - D^D_{k-i}\circ\Phi^H_i\right)[\alpha],[\gamma]\right\rangle_{s+k} - \left\langle \sum_i \left(\phi^H_{k-i}\circ d^C_i - d^D_{k-i}\circ\phi^H_i\right)[\alpha],[\gamma]\right\rangle_{s+k}. \end{aligned}$$

That is we have shown $\left\langle \sum_i \left(\phi^H_{k-i} \circ d^C_i - d^D_{k-i} \circ \phi^H_i\right)[\alpha], [\gamma]\right\rangle_{s+k} = 0$, hence the Theorem 12.18 holds. $\square$

Proof of Theorem 12.23. We write $\left\langle \phi^F \circ \phi^H[\alpha], [\gamma]\right\rangle = T_1 + T_2$, where

$$\begin{aligned}
T_1 &= \sum_{l\geq 0} T_1^l, \\
T_1^l &= \sum_{p+r=l} \sum_j \pm \operatorname{Tr}_D^{s+j} F \times H^{s,k}_{i_1,\dots,i_p|j|k_1,\dots,k_r}[\alpha f_{s+i_1}, \dots f_{s+i_p}, \theta\theta^D_{s+j}{}^*, h_{s+k_1}, \dots, h_{s+k_r}, \gamma].
\end{aligned}$$

We claim that the sum $\sum_{l=0}^t T_1^l$ equals to the sum of the following terms:

$$\sum_{A_1} (-1)^{\bigstar_1} F \circ H^{s,k}_{i_1,\dots,i_p|k_1,\dots k_{r-p}}[\alpha, f_{s+i_1}, \dots, f_{s+i_p}, h_{s+k_1}, \dots, h_{s+k_{r-p}}, \gamma] \tag{12.67}$$

$$+ \sum_{A_2} (-1)^{\bigstar_2} F \circ H^{s,k}_{i_1,\dots,i_p|j_1,\dots,j_q|k_1,\dots,k_{t+1-p-q}}[\mathrm{d}(\alpha, f_{s+i_1}, \dots, f_{s+i_p}, g_{s+j_1}, \dots$$
$$\dots, g_{s+j_q}, h_{s+k_1}, \dots, h_{s+k_{t+1-p-q}} \gamma))] \tag{12.68}$$

$$+ \sum_{A_3} (-1)^{\bigstar_3} \operatorname{Tr}_C^{s+i_u} F \circ H^{s,k}_{i_1,\dots,i_p|j_1,\dots,j_q|k_1,\dots,k_{r-p-q}}[\alpha, f_{s+i_1}, \dots, \theta\theta^C_{s+i_u}{}^*, \dots$$
$$\dots, f_{s+i_p}, g_{s+j_1}, \dots g_{s+j_q}, h_{s+k_1}, \dots h_{s+k_{r-p-q}}, \gamma] \tag{12.69}$$

$$+ \sum_{A_4} (-1)^{\bigstar_4} \operatorname{Tr}_D^{s+j_u} F \circ H^{s,k}_{i_1,\dots,i_p|j_1,\dots,j_q|k_1,\dots,k_{r-p-q}}[\alpha, f_{s+i_1}, \dots, f_{s+i_p}, g_{s+j_1}, \dots, \theta\theta^D_{s+j_u}{}^*, \dots$$
$$\dots, g_{s+j_q}, h_{s+k_1}, h_{s+k_{r-p-q}}, \gamma] \tag{12.70}$$

$$+ \sum_{A_5} (-1)^{\bigstar_5} \operatorname{Tr}_E^{s+k_u} F \circ H^{s,k}_{i_1,\dots,i_p|j_1,\dots,j_q|k_1,\dots,k_{r-p-q}}[\alpha, f_{s+i_1}, \dots, f_{s+i_p}, g_{s+j_1}, \dots, g_{s+j_q}, h_{s+k_1}, \dots$$
$$\dots, \theta\theta^E_{s+k_u}{}^*, \dots, h_{s+k_{r-p-q}}, \gamma], \tag{12.71}$$

where the subscripts are

$$A_1 := \{(p, r, i_1, \ldots, i_p, k_1, \ldots, k_{r-p}) \,|\, 0 \le p \le r \le t, 0 < i_1 < \ldots < i_p, k_1 < \ldots k_{r-p} < k\},$$

$$A_2 := \left\{ (p, q, i_1, \ldots, i_p, j_1, \ldots, j_q, k_1, \ldots, k_{t+1-p-q}) \,\middle|\, \begin{array}{c} p \ge 0, q \ge 1, p+q \le t+1 \\ 0 < i_1 < \ldots < i_p, j_1 < \ldots < j_q \\ k_1 < \ldots k_{t+1-p-q} < k \end{array} \right\},$$

$$A_3 := \left\{ (r, p, q, u, i_1, \ldots, i_p, j_1, \ldots, j_q, k_1, \ldots, k_{r-p-q}) \,\middle|\, \begin{array}{c} r \le t+1, 0 \le p, 1 \le q \\ p+q \le r, 1 \le u \le p \\ 0 < i_1 < \ldots < i_p, j_1 < \ldots < j_q \\ k_1 < \ldots < k_{r-p-q} < k \end{array} \right\},$$

$$A_4 := \left\{ (r, p, q, u, i_1, \ldots, i_p, j_1, \ldots, j_q, k_1, \ldots, k_{r-p-q}) \,\middle|\, \begin{array}{c} r \le t+1, 0 \le p, 2 \le q \\ p+q \le r, 1 \le u \le q \\ 0 < i_1 < \ldots < i_p, j_1 < \ldots < j_q \\ k_1 < \ldots < k_{r-p-q} < k \end{array} \right\},$$

$$A_5 := \left\{ (r, p, q, u, i_1, \ldots, i_p, j_1, \ldots, j_q, k_1, \ldots, k_{r-p-q}) \,\middle|\, \begin{array}{c} r \le t+1, 0 \le p, 1 \le q \\ p+q \le r, 1 \le u \le r-p-q \\ 0 < i_1 < \ldots < i_p, j_1 < \ldots < j_q \\ k_1 < \ldots < k_{r-p-q} < k \end{array} \right\}.$$

And the signs are:

$$
\begin{aligned}
\bigstar_1 &:= |\alpha|c_s + \dim F\circ H_{s,s+k} + \sum_{w=1}^{p} \ddagger(\mathcal{C},\alpha,i_w) + \sum_{w=1}^{r-p} \ddagger(F\circ H,\alpha,k_w + e_{s+k_1} + 1 + m^E_{s+k_1,s+k},\\
\bigstar_2 &:= |\alpha|c_s + \dim F\circ H_{s,s+k} + 1 + \sum_{w=1}^{p} \dagger(\mathcal{C},\alpha,i_w) + h_{s,s+j_1} + \sum_{w=1}^{q} \dagger(H,\alpha,j_w)\\
&\quad + \sum_{w=1}^{t+1-p-q} \ddagger(F\circ H,\alpha,k_w) + e_{s+k_1} + 1 + m^E_{s+k_1,s+k},\\
\bigstar_3 &:= |\alpha|(c_s+1) + \dim F\circ H_{s,s+k} + 1 + \sum_{w=1}^{u-1} \ddagger(\mathcal{C},\alpha,i_w) + \sum_{w=u}^{p} \dagger(\mathcal{C},\alpha,i_w) + h_{s,s+j_1}\\
&\quad + \sum_{w=1}^{q} \dagger(H,\alpha,j_w) + \sum_{w=1}^{r-p-q} \ddagger(F\circ H,\alpha,k_w) + e_{s+k_1} + 1 + m^E_{s+k_1,s+k},\\
\bigstar_4 &:= |\alpha|(c_s+1) + \dim F\circ H_{s,s+k} + 1 + \sum_{w=1}^{p} \ddagger(\mathcal{C},\alpha,i_w) + h_{s,s+j_1} + \sum_{w=1}^{u-1} \ddagger(H,\alpha,j_w)\\
&\quad + \sum_{w=u}^{q} \dagger(H,\alpha,j_w) + \sum_{w=1}^{r-p-q} \ddagger(F\circ H,\alpha,k_w) + e_{s+k_1} + 1 + m^E_{s+k_1,s+k},\\
\bigstar_5 &:= |\alpha|(c_s+1) + \dim F\circ H_{s,s+k} + 1 + \sum_{w=1}^{p} \ddagger(\mathcal{C},\alpha,i_w) + h_{s,s+j_1}\\
&\quad + \sum_{w=1}^{q} \ddagger(H,\alpha,j_w) + \sum_{w=1}^{u-1} \dagger(F\circ H,\alpha,k_w) + \sum_{w=u}^{r-p-q} \ddagger(F\circ H,\alpha,k_w) + e_{s+k_1} + 1 + m^E_{s+k_1,s+k}.
\end{aligned}
$$

Assume the claim holds, let $t\to\infty$, then (12.68) becomes 0 as it is an integration over an empty set for t big enough. (12.67) is $\langle \phi^{F\circ H}[\alpha],[\gamma]\rangle_{s+k}$, (12.69) is $\langle P\circ d_{BC}[\alpha],[\gamma]\rangle_{s+k}$, (12.70) is $-T_2$, and (12.71) is $\langle d_{BC}\circ P[\alpha],[\gamma]\rangle_{s+k}$. Thus Theorem 12.23 is proven.

The claim (12.67)–(12.71) is proven by induction again. First of all T_1^0 is

$$\sum_j (-1)^{|\alpha|(c_s+1)+\dim F\circ H_{s,s+k}+\dagger(H,\alpha,j)+h_{s,s+j}} \operatorname{Tr}_D^{s+j} F\times H^{s,k}_{|j|}[\alpha,\theta{\theta^D_{s+j}}^*,\gamma].$$

Replacing the trace term by $\delta - \mathrm{d}g$, we have T_1^0 is actually

$$
\begin{aligned}
T_1^0 &= \sum_j (-1)^{|\alpha|c_s+\dim F\circ H_{s,s+k}} F\circ H^{s,k}_{|j|}[\alpha,\gamma]\\
&\qquad + \sum_j (-1)^{|\alpha|c_s+\dim F\circ H_{s,s+k}+1+\dagger(H,\alpha,j)+h_{s,s+j}} F\times H^{s,k}_{|j|}[\mathrm{d}(\alpha,g_{s+j},\gamma)].
\end{aligned}
\tag{12.72}
$$

This is exactly the $t=0$ case of the claim. In general, the claim can be proven by induction. By induction hypothesis, we can apply Stokes' theorem to the exact term (12.68), thus the integration is reduced to integration along the boundary, i.e. some fiber products. Then we can replace the fiber products by Cartesian products with Dirac currents, then we can replace the Dirac current by trace $\theta\theta^*$ plus $\mathrm{d}f/\mathrm{d}g/\mathrm{d}h$. For the extra term T_1^{t+1}, we replace the trace term by $\delta + \mathrm{d}g$ just like what we did to T_1^0, then all $\mathrm{d}f/\mathrm{d}g/\mathrm{d}h$ terms form the exact form in the $t+1$ case of the claim.

□

Notations for proof of Theorem 12.25

Let H, F be two oriented flow-morphisms from $\mathcal{C}$ to $\mathcal{D}$, and K be an oriented flow-homotopy from H to F.

- $k_{i,j} := \dim K_{i,j}$, then by Definition 12.24,
$$\begin{aligned} k_{i,j} + m^D_{j,k} - d_j + 1 &= k_{i,k}, \\ m^C_{i,j} + k_{j,k} - c_i + 1 &= k_{i,k}. \end{aligned}$$
- For $k \in \mathbb{Z}$, $0 < i_1 < \ldots < i_p$ and $j_1 < \ldots < j_q < k$,
$$K^{s,k}_{i_1,\ldots,i_p|j_1\ldots,j_q} := \mathcal{M}^C_{s,s+i_1} \times \ldots \times \mathcal{M}^C_{s+i_{p-1},s+i_p} \times K_{s+i_p,s+j_1} \times \mathcal{M}^D_{s+j_1,s+j_2} \times \ldots \times \mathcal{M}^D_{s+j_q,s+k}.$$
- $K^{s,k}_{i_1,\ldots,\overline{i_t},\ldots,i_p|j_1\ldots,j_q}$ and $K^{s,k}_{i_1,\ldots,i_p|j_1\ldots,\overline{j_t},\ldots,j_q}$ are the fiber product versions at $s+i_t$ and $s+j_t$ defined similarly to (12.8).
- $K^{*,*}_{\ldots}[\alpha, f, \ldots, g\ldots, \gamma]$ and the exact/trace/fiber product versions are defined similarly as in (12.5), (12.9), (12.10) and (12.11).
- For $\alpha \in h(\mathcal{C}, s)$,
$$\begin{aligned} \dagger(K,\alpha,k) &:= (|\alpha| + k_{s,s+k})(d_{s+k}+1), \\ \ddagger(K,\alpha,k) &:= (|\alpha| + k_{s,s+k}+1)(d_{s+k}+1). \end{aligned}$$

Proof of Theorem 12.25. Similar to the proof of Proposition 12.12 and Theorem 12.18, we can compute the leading term in $\left\langle \left(d^D_{BC} \circ \Lambda^K + \Lambda^K \circ d^C_{BC}\right)[\alpha], [\gamma] \right\rangle_{s+k}$ first. Thus we write $\Lambda^K = \Gamma^K + \lambda^K$, where
$$\left\langle \Gamma^K[\alpha], [\lambda] \right\rangle_{s+k} := (-1)^{|\alpha|(c_s+1)+k_{s,s+k}} K^{s,s+k}_{|}[\alpha, \gamma].$$

We claim for $r \geq 1$, $k \in \mathbb{Z}$, $\alpha \in h(\mathcal{C}, s)$ and $\gamma \in h(\mathcal{D}, s+k)$, $\left\langle \left(D^D \circ \Gamma^K + \Gamma^K \circ D^C\right)[\alpha], [\gamma]\right\rangle_{s+k}$ equals to

$$\sum_{0 \leq p \leq r} (-1)^{\clubsuit_1} K^{s,k}|_{i_1,\ldots,i_p|j_1,\ldots,j_{r-p}}[\mathrm{d}(\alpha, f_{s+i_1}, \ldots, g_{s+j_{r-p}}, \gamma)] \tag{12.73}$$

$$+ \sum_{\substack{0 \leq p \leq q \leq r \\ q>1, 1 \leq u \leq p}} (-1)^{\clubsuit_2} \operatorname{Tr}_C^{s+i_u} K^{s,k}_{i_1,\ldots,i_p|j_1,\ldots,j_{q-p}}[\alpha, f_{s+i_1}, \ldots, \theta\theta^C_{s+i_u}{}^*, \ldots, g_{s+i_{q-p}}, \gamma] \tag{12.74}$$

$$+ \sum_{\substack{0 \leq p \leq q \leq r \\ q>1, 1 \leq u \leq q-p}} (-1)^{\clubsuit_3} \operatorname{Tr}_D^{s+j_u} K^{s,k}_{i_1,\ldots,i_p|j_1,\ldots,j_{q-p}}[\alpha, f_{s+i_1}, \ldots, \theta\theta^D_{s+j_u}{}^*, \ldots, g_{s+i_{q-p}}, \gamma] \tag{12.75}$$

$$+ \sum_{\substack{0 \leq p \leq q < r \\ q>1}} (-1)^{\clubsuit_4} \left(F^{s,k}|_{i_1,\ldots,i_p|j_1,\ldots,j_{q-p}} - H^{s,k}|_{i_1,\ldots,i_p|j_1,\ldots,j_{q-p}}\right)[\alpha, f_{s+i_1}, \ldots, g_{s+j_{q-p}}, \gamma], \tag{12.76}$$

where

$$\begin{aligned}
\clubsuit_1 &:= 1 + |\alpha|c_s + k_{s,s+k} + \sum_{w=1}^{p} \dagger(\mathcal{C}, \alpha, i_w) + c_{s+i_p} + 1 + m^C_{s,s+i_p} + 1 + \sum_{w=1}^{r-p} \dagger(K, \alpha, j_w) \\
&\quad + d_{s+j_1} + m^D_{s+j_1,s+k} + 1 + \ddagger(K, \alpha, k), \\
\clubsuit_2 &:= 1 + |\alpha|(c_s+1) + k_{s,s+k} + \sum_{w=1}^{u-1} \ddagger(\mathcal{C}, \alpha, i_w) + \sum_{w=u}^{p} \dagger(\mathcal{C}, \alpha, i_w) + c_{s+i_p} + 1 + m_{s,s+i_p} \\
&\quad + \sum_{w=1}^{q-p} \dagger(K, \alpha, j_w) + d_{s+j_1} + m^D_{s+j_1,s+k} + 1 + \ddagger(K, \alpha, k), \\
\clubsuit_3 &:= 1 + |\alpha|(c_s+1) + k_{s,s+k} + \sum_{w=1}^{p} \ddagger(\mathcal{C}, \alpha, i_w) + c_{s+i_p} + 1 + m^C_{s,s+i_p} + \sum_{w=1}^{u-1} \ddagger(K, \alpha, j_w) \\
&\quad + \sum_{w=u}^{q-p} \dagger(K, \alpha, j_w) + d_{s+j_1} + 1 + m^D_{s+j_1,s+k}, \\
\clubsuit_4 &:= |\alpha|c_s + k_{s,s+k} + \sum_{w=1}^{p} \ddagger(\mathcal{C}, \alpha, i_w) + \sum_{w=1}^{q-p} \dagger(K, \alpha, j_w) + d_{s+j_1} + 1 + m^D_{s+j_1,s+k}.
\end{aligned}$$

The claim is proven by induction like before. Let $r \to \infty$, (12.73) vanishes. (12.74) plus (12.75) equals to

$$-\left\langle \left(D^D \circ \lambda^K + \lambda^K \circ D^C + E^D \circ \Gamma^K + \Gamma^K \circ E^C + E^D \circ \lambda^K + \lambda^K \circ E^C\right)[\alpha], [\gamma]\right\rangle_{s+k},$$

and (12.76) equals to $-\left\langle \left(\phi^F - \phi^H\right)[\alpha], [\gamma]\right\rangle_{s+k}$, this proves the theorem. $\square$

Chapter 13

Action Spectral Sequence

Austin-Braam's construction [6] on Morse-Bott cohomology comes with a spectral sequence, which is induced by the action filtration. Similar consideration for Floer theory can be found in many places, e.g. [97]. The spectral sequence is a useful computational tool. Moreover, it was shown in [6], under the fiberation condition, the spectral sequence from Austin-Braam's construction (from page one) is isomorphic to the Bott spectral sequence. This implies, in particular, the spectral sequence from page one only depends on the Morse-Bott function. In the case of flow categories, there is a natural spectral sequence corresponding to the action spectral sequence above. For basics of spectral sequences, we refer readers to [78, 104].

Given a flow category $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$, we have the following action filtration on the minimal Morse-Bott cochain complex BC:

$$F_pBC = \prod_{i \geq p} H^*(C_i).$$

The associated spectral sequence can be described explicitly as follows. We define Z^p_{k+1} to be space of $\alpha_0 \in H^*(C_p)$, such that there exist $\alpha_1, \alpha_2 \ldots \alpha_{k-1} \in H^*(C_*)$, such that

$$\begin{array}{rcl} d_1\alpha_0 & = & 0, \\ d_2\alpha_0 + d_1\alpha_1 & = & 0, \\ d_3\alpha_0 + d_2\alpha_1 + d_1\alpha_2 & = & 0, \\ \ldots & & \\ d_k\alpha_0 + d_{k-1}\alpha_1 + \ldots d_1\alpha_{k-1} & = & 0. \end{array} \tag{13.1}$$

We define B^p_{k+1} to be space of $\alpha \in H^*(C_p)$, such that there exist $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in H^*(C_*)$ and

$$\begin{array}{rcl} \alpha & = & d_k\alpha_0 + d_{k-1}\alpha_1 + \ldots d_1\alpha_{k-1}, \\ 0 & = & d_{k-1}\alpha_0 + d_{k-2}\alpha_1 + \ldots d_1\alpha_{k-2}, \\ & & \ldots \\ 0 & = & d_1\alpha_0. \end{array} \tag{13.2}$$

On Z_{k+1}/B_{k+1}, there is a map $\partial_{k+1}\alpha_0 = d_{k+1}\alpha_0 + d_k\alpha_1 + \ldots d_2\alpha_{k-1}$. Since the differential on the minimal Morse-Bott cochain complex has the special form of $\prod d_i$, unwrapping Larey's theorem on spectral sequence associated to a filtered module, we have the following.

Proposition 13.1 ([78]).

$$B_1^p \subset B_2^p \subset \ldots B_k^p \subset B_\infty^p := \cup_k B_k^p \subset Z_\infty^p := \cap_k Z_k^p \ldots \subset Z_k^p \subset \ldots \subset Z_2^p \subset Z_1^p$$

In addition ∂_k is a well-defined map from Z_k^p/B_k^p to Z_k^{p+k+1}/B_k^{p+k+1}, such that $\partial_k^2 = 0$, and $Z_{k+1}^p/B_{k+1}^p \simeq H^p(Z_k/B_k, \partial_k)$. Hence we have a spectral sequence $(E_k^p := Z_k^p/B_k^p, \partial_k)$, with

$$E_\infty^p = Z_\infty^p/B_\infty^p = F_pH(BC, d_{BC})/F_{p+1}H(BC, d_{BC}).$$

The spectral sequence (E_k^p, ∂_k) is the spectral sequence induced from the filtration F_pBC.

The second page of the spectral sequence is computed by taking the cohomology with respect to d_1 in (12.7). Since d_1 is computed using $\mathcal{M}_{*,*+1}$, which are manifolds without boundary, thus d_1 is the pullback and pushforward of cohomology, it might be computable in good cases.

Proposition 13.2. *Every page of the spectral sequence is independent of the defining-data*

Proof. Since the identity flow-Morphism I induces a cochain map $\phi^I_{\Theta_1,\Theta_2} : (BC, d_{BC,\Theta_1}) \to (BC, d_{BC,\Theta_2})$. The cochain map ϕ^I preserves the filtrations, thus it induces a morphism between spectral sequences. Since the induced map on the zeroth page is identity, thus it induces isomorphism on every page. □

Remark 13.3. *The action spectral sequence (from page one) is an invariant of the Morse-Bott functional/flow category, i.e. it is not only independent of the defining data, but also independent of the constructions, i.e. Austin-Braam's construction (when defined), Fukaya's construction, cascades construction and our minimal Morse-Bott construction.*

We have to worry about the convergence of the spectral sequence, following [78], we have the following exact sequence:

$$0 \to \varprojlim_p F_pH(BC, d_{BC}) \to H(BC, d_{BC}) \to E_\infty \to \varprojlim_p{}^1 F_pH(BC, d_{BC}) \to 0$$

In some good case like $F_pBC = 0$ for p big, the convergence of the spectral sequence is granted, and E_∞ is isomorphic to the Morse-Bott cohomology. For example, the symplectic cohomology considered in [16, 97] satisfies such condition.

Chapter 14

Orientations and Local Systems

The aim of this chapter is to discuss how the orientation conventions in definition 11.16,12.15 and 12.24 arise in applications. We need to use extra structures from Floer/Morse theory. In particular, our discussion is based on linear gluing theorem for the determinant line bundles over broken trajectories. For the Hamiltonian-Floer-(co)homology, the linear gluing theorem was discussed in [42]. Most of or all Floer theories should share this property. In this chapter, we discuss the counterpart of the linear gluing theorem in Morse-Bott setting without proof, and we explain how do the orientations in Definition 11.16,12.15 and 12.24 from the gluing properties.

14.1 Orientations for flow categories

Review of orientations in the Morse case

Take Hamiltonian Floer cohomology for (M,ω) as an example, suppose we are in the Morse case, i.e. all the contractible periodic orbits are nondegenerate. To avoid the Novikov coefficient, we assume $\omega|_{\pi_2(M)}=0$. The orientation problem is independent of the transversality, that is the discussion below can be lifted to the underlying Banach manifolds/polyfolds. But for simplicity, we assume that we are in a transverse situation, that is we have a flow category $\{x_i,\mathcal{M}_{i,j}\}$, where x_i are the nondegenerate contractible periodic orbits, and $\mathcal{M}_{i,j}$ are the moduli spaces of Floer trajectories from x_i to x_j.

Following [42], we can assign each periodic orbit x_i an orientation line o_i with a $\mathbb{Z}_2$ grading. Such line is constructed from the determinant line of a perturbed $\overline{\partial}$ operator over $\mathbb{C}$ with one positive end at the infinity, and the grading is the index of the operator. For every point in $\mathcal{M}_{i,j}$, there is an orientation line with a $\mathbb{Z}_2$ grading coming from the determinant line bundle of the Floer equation. All these lines from a line bundle $o_{i,j}$ over $\mathcal{M}_{i,j}$. We refer readers to [106] for how to topologize the determinant line bundles. By the gluing theorem for linear operators [42], we have an isomorphism over $\mathcal{M}_{x,y}$:

$$\rho_{i,\{i,j\}}: s^*o_i\otimes o_{i,j}\to t^*o_j, \tag{14.1}$$

and over $\mathcal{M}_{i,j} \times \mathcal{M}_{j,k} \subset \mathcal{M}_{i,k}$ there is an isomorphism:

$$\rho_{\{i,j\},\{j,k\}} : \pi_1^* o_{i,j} \otimes \pi_2^* o_{j,k} \to o_{i,k},$$

where π_1, π_2 are the two projections. $\rho_{i,\{i,j\}}$ and $\rho_{\{i,j\},\{i,j\}}$ are compatible in the sense that there is commutative diagram over $\mathcal{M}_{i,j} \times \mathcal{M}_{j,k}$ up to multiplying a positive number:

$$\begin{tikzcd}
s^* o_i \otimes \pi_1^* o_{i,j} \otimes \pi_2^* o_{j,k} \arrow[r, "\mathrm{id} \otimes \rho_{\{i,j\},\{j,k\}}"] \arrow[d, "\rho_{i,\{i,j\}} \otimes \mathrm{id}"] & s^* o_i \otimes o_{i,k} \arrow[dddd, "\rho_{i,\{i,k\}}"] \\
\pi_1^* t^* o_j \otimes \pi_2^* o_{j,k} \arrow[d, "="] & \\
\pi_2^* s^* o_j \otimes \pi_2^* o_{j,k} \arrow[d, "\pi_2^* \rho_{j,\{j,k\}}"] & \\
\pi_2^* t^* o_k \arrow[d, "="] & \\
t^* o_k \arrow[r] & t^* o_k .
\end{tikzcd}$$

Let $\underline{\mathbb{R}}$ be the trivial line bundle coming from the translation $\mathbb{R}$ action with grading 1. By the fact that $\det(T\mathcal{M}_{i,j}) \otimes \underline{\mathbb{R}} = o_{i,j}$, the above commutative diagram becomes (we suppress the pull backs):

$$\begin{tikzcd}
o_i \otimes \det(T\mathcal{M}_{i,j}) \otimes \underline{\mathbb{R}}_r \otimes \det(T\mathcal{M}_{j,k}) \otimes \underline{\mathbb{R}}_s \arrow[r] \arrow[d] & o_i \otimes \det(T\mathcal{M}_{i,k}) \otimes \underline{\mathbb{R}}_t \arrow[dd] \\
o_j \otimes \det(T\mathcal{M}_{j,k}) \otimes \underline{\mathbb{R}}_s \arrow[d] & \\
o_k \arrow[r] & o_k .
\end{tikzcd} \tag{14.2}$$

If we fix the orientations of o_i for all x_i, there are induced orientations on $o_{i,j}$ through the map (14.1), thus there are induced orientations $[\mathcal{M}_{i,j}]$ on $\mathcal{M}_{i,j}$. By standard gluing argument [1, Lemma 1.5.7.]:

$$\begin{aligned} &\langle r, s\rangle \text{ is pointing in the } t \text{ direction}, \\ &\langle -r, s\rangle \text{ is pointing out along } \mathcal{M}_{i,j} \times \mathcal{M}_{j,k} \subset \partial \mathcal{M}_{i,k}. \end{aligned} \tag{14.3}$$

We fix such notations for later use. Thus (14.2) implies

$$[\mathcal{M}_{i,j}][\mathcal{M}_{j,k}] = (-1)^{m_{i,j}+1} \partial [\mathcal{M}_{i,k}]|_{\mathcal{M}_{i,j} \times \mathcal{M}_{j,k}}$$

This orientation relation can be used to prove $d^2 = 0$ for the Hamiltonian Floer cohomology in the Morse case, and orientations $-[\mathcal{M}_{i,j}]$ fit into the orientation convention in Definition 11.16.

Orientations in the Morse-Bott case

We should expect similar structures and properties in Morse-Bott theories. We phrase the structures as a definition and explain how to get an oriented flow category from there.

Definition 14.1 (Orientation Structure). *An **orientation structure** on a flow category $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ is the following structures.*

1. *There are topological line bundles o_i over C_i with $\mathbb{Z}_2$ gradings and topological line bundles $o_{i,j}$ over $\mathcal{M}_{i,j}$ with $\mathbb{Z}_2$ gradings for every C_i and $\mathcal{M}_{i,j}$.*

2. *There are isomorphisms:*
$$\det(\mathcal{M}_{i,j}) \otimes \underline{\mathbb{R}} = s^* \det C_i \otimes o_{i,j} \otimes t^* \det C_j. \tag{14.4}$$

3. *There are bundle isomorphisms over $\mathcal{M}_{i,j}$:*
$$\rho_{i,\{i,j\}} : s^* o_i \otimes s^* \det C_i \otimes o_{i,j} \to t^* o_j, \tag{14.5}$$
and bundle isomorphisms over over $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k} \subset \mathcal{M}_{i,j} \times \mathcal{M}_{j,k}$:
$$\rho_{\{i,j\},\{j,k\}} : \pi_1^* o_{i,j} \otimes (t \times s)^* \det T\Delta_j \otimes \pi_2^* o_{j,k} \to o_{i,k}|_{\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}}. \tag{14.6}$$

4. *The bundle isomorphisms are compatible in the sense that the following diagram over $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$ is commutative up to multiplying a positive number:*
$$\begin{tikzcd} s^* o_i \otimes s^* \det C_i \otimes \pi_1^* o_{i,j} \otimes (t \times s)^* \det \Delta_j \otimes \pi_2^* o_{j,k} \arrow[r, "\mathrm{id} \otimes \rho_{\{i,j\},\{j,k\}}"] \arrow[d, "\rho_{i,\{i,j\}} \otimes \mathrm{id}"] & s^* o_i \otimes s^* \det C_i \otimes o_{i,k} \arrow[dd, "\rho_{i,\{i,k\}}"] \\ \pi_2^* s^* o_j \otimes \pi_2^* s^* \det C_j \otimes \pi_2^* o_{j,k} \arrow[d, "\pi_2^* \rho_{j,\{j,k\}}"] & \\ t^* o_k \arrow[r] & t^* o_k. \end{tikzcd} \tag{14.7}$$
The diagram makes sense, because over the fiber product $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$, $\pi_1^* t^* o_j = \pi_2^* s^* o_j$, and $(t \times s)^* \det \Delta_j = \pi_2^* s^* \det C_j$.

5. *(14.3) holds.*

In applications, the topological line bundle o_i usually comes from the determinant line bundle of a perturbed Floer equation with exponential decay at the end over a domain with one positive end. For the details on exponential decay, we refer readers to [14, 43]. The topological line bundle $o_{i,j}$ usually comes from the determinant bundle of the Floer equation with exponential decays at two ends over a cylinder. The bundle isomorphisms and their compatible diagram are from a version of the linear gluing theorem for Fredholm operators [42]. At last, condition 5 follows from the same argument used in [1].

Proposition 14.2. *Assume the flow category $\mathcal{C}$ has an orientation structure, and all the line bundles o_i are oriented. Then $\mathcal{C}$ can be oriented.*

Proof. By (14.5), if o_i are oriented, and C_i are oriented, then there are an induced orientations $[o_{i,j}]$ on $o_{i,j}$. By (14.7), over the fiber product $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$ we have:

$$s^*[\det C_i]\pi_1^*[o_{i,j}](t \times s)^*[\det \Delta_j]\pi_2^*[o_{j,k}] = s^*[\det C_i][o_{i,k}]$$

By (14.4), there is an induced orientation $[\mathcal{M}_{i,j}]$ on each $\mathcal{M}_{i,j}$. Using the fact $[\Delta_j][N_j] = [C_j][C_j]$ and condition 5, we have

$$(t \times s)^*[N_j]\partial[\mathcal{M}_{i,k}]|_{\mathcal{M}_{i,j}\times_j \mathcal{M}_{j,k}} = (-1)^{c_j m_{i,j} + m_{i,j}+1}[\mathcal{M}_{i,j}][\mathcal{M}_{j,k}]$$

Then orientations $-[\mathcal{M}_{i,j}]$ satisfy Definition 11.16. The extra minus sign makes signs in (12.7) factorize nicely. □

Orientations for flow-morphisms

In this subsection, we explain how does the orientation convention in Definition 12.15 arise. Like the flow category case, we define the structures to get orientations, which we expect to hold in most of the applications.

Definition 14.3. *Assume $H = \{H_{i,j}\}$ is a flow-morphism from flow category $\mathcal{C}$ to $\mathcal{D}$, $\mathcal{C}$ and $\mathcal{D}$ have orientation structures. An orientation structure on H is the following structures.*

1. *There are $\mathbb{Z}_2$ graded topological line bundles $o^H_{i,j}$ over $H_{i,j}$.*
2. *We have bundle isomorphisms:*
$$\det TH_{i,j} = s^* \det C_i \otimes o^H_{i,j} \otimes t^* \det D_j. \tag{14.8}$$
3. *Over $H_{i,j}$, we have isomorphism $\rho^H_{i,\{i,j\}}$:*
$$\rho^H_{i,\{i,j\}} : s^* o^C_i \otimes s^* \det C_i \otimes o^H_{i,j} \to t^* o^D_j. \tag{14.9}$$
4. *At the fiber product $H_{i,j} \times_j \mathcal{M}^D_{j,k}$ we have bundle isomorphism:*
$$\rho^{H,\mathcal{D}}_{\{i,j\},\{j,k\}} : \pi_1^* o^H_{i,j} \otimes (t \times s)^* \det \Delta^D_j \otimes \pi_2^* o^D_{j,k} \to o^H_{i,k}. \tag{14.10}$$

5. *(14.9), (14.10) are compatible in the sense that we have a commutative diagram over $H_{i,j} \times_j \mathcal{M}^D_{j,k}$:*

$$\begin{tikzcd}
s^* o_i^C \otimes s^* \det C_i \otimes \pi_1^* o^H_{i,j} \otimes (t \times s)^* \det \Delta_j \otimes \pi_2^* o^D_{j,k} \arrow[r, "\mathrm{id} \otimes \rho^{H,\mathcal{D}}_{\{i,j\},\{j,k\}}"] \arrow[d, "\rho^H_{i,\{i,j\}} \otimes \mathrm{id}"] & s^* o_i^C \otimes s^* \det C_i \otimes o^H_{i,k} \arrow[dd, "\rho^H_{i,\{i,k\}}"] \\
\pi_2^* s^* o_j^D \otimes \pi_2^* s^* \det D_j \otimes \pi_2^* o^D_{j,k} \arrow[d, "\pi_2^* \rho^{\mathcal{D}}_{j,\{j,k\}}"] & \\
t^* o_k \arrow[r] & t^* o_k.
\end{tikzcd} \tag{14.11}$$

6. *The $\mathbb{R}$ component for $o^D_{j,k}$ in (14.4) points out on the boundary.*

7. *There are similar structures on the other type of fiber product $\mathcal{M}^C_{i,j} \times_j H_{j,k}$.*

In the Morse case, take Hamiltonian Floer cohomology [5] as an example, the bundle $o^H_{i,j}$ is the determinant line bundle of the "time-dependent" Floer equation [5, p. 384]. In the Morse-Bott case, then $o^H_{i,j}$ is the determinant line bundle of the "time-dependent" Floer equation with exponential decays at both ends. By the same argument used in Proposition 14.2, we have the following.

Proposition 14.4. *Let $\mathcal{C}$, $\mathcal{D}$ be two flow categories with orientation structures, and H be a flow-morphism from $\mathcal{C}$ to $\mathcal{D}$ with an orientation structure. Assume o_i^C, o_i^D are oriented, and $\mathcal{C}$ and $\mathcal{D}$ are oriented by Proposition 14.2, then (14.8) (14.9) determine orientations on $H_{i,j}$, such that H is an oriented flow-morphism.*

Remark 14.5. *An orientation structure on a pre-flow-morphism is defined by the first three conditions in Definition 14.4.*

Orientations for flow-homotopies

In applications, the flow-homotopy from H to F usually comes from throwing in an extra $[0,1]_z$ parameter [5, p. 414], such that when $z = 0$, the equation defines flow-morphism H, when $z = 1$, the equation defines flow-morphism F. Therefore we define a flow homotopy with orientation structures to be:

Definition 14.6. *Let H, F be two flow-morphisms with orientation structures from $\mathcal{C}$ to $\mathcal{D}$, a flow-homotopy K between H and F is said to have an orientation structure if the following properties hold.*

1. *There is a $\mathbb{Z}_2$ graded line bundle $o^K_{i,j}$ over $K_{i,j}$, and*

$$\det K_{i,j} \simeq \underline{\mathbb{R}_z} \otimes s^* \det C_i \otimes o^K_{i,j} \otimes t^* \det D_j. \tag{14.12}$$

The z coordinate is pointing in along $H_{i,j}$ and pointing out along $F_{i,j}$.

2. $o^K_{i,k}|_{H_{i,j}} = o^H_{i,j}$, $o^K_{i,j}|_{F_{i,j}} = o^F_{i,j}$.

3. *Over* $K_{i,j}$, *we have:*

$$\rho^K_{i,\{i,j\}} : s^* o^C_i \otimes s^* \det C_i \otimes o^K_{i,j} \to t^* o^D_j. \tag{14.13}$$

   And $\rho^K_{i,\{i,j\}} = \rho^H_{i,\{i,j\}}$ *over* $H_{i,j} \subset K_{i,j}$, *and* $\rho^K_{i,\{i,j\}} = \rho^F_{i,\{i,j\}}$ *over* $F_{i,j} \subset K_{i,j}$

4. *At the fiber product/breaking* $K_{i,j} \times_j \mathcal{M}^D_{j,k}$ *we have*

$$\rho^{K,\mathcal{D}}_{\{i,j\},\{j,k\}} : \pi_1^* o^K_{i,j} \otimes (t \times s)^* \det \Delta^D_j \otimes \pi_2^* o^D_{j,k} \to o^K_{i,k}. \tag{14.14}$$

5. (14.13), (14.14) *are natural in the sense that there is a commutative diagram over* $K_{i,j} \times_j \mathcal{M}^D_{j,k}$*:*

$$\begin{tikzcd}
s^* o^C_i \otimes s^* \det C_i \otimes \pi_1^* o^K_{i,j} \otimes (t \times s)^* \det \Delta_j \otimes \pi_2^* o^D_{j,k} \arrow[r] \arrow[d] & s^* o^C_i \otimes s^* \det C_i \otimes o^K_{i,k} \arrow[dd] \\
\pi_2^* s^* o^D_j \otimes \pi_2^* s^* \det D_j \otimes \pi_2^* o^D_{j,k} \arrow[d] & \\
t^* o_k \arrow[r] & t^* o_k.
\end{tikzcd} \tag{14.15}$$

   Moreover, the $\mathbb{R}$ *component of* $o^D_{j,k}$ *coordinate points out on the boundary component* $K_{i,j} \times_j \mathcal{M}^D_{j,k} \subset \partial K_{i,k}$.

6. *There are similar structures on the other type of fiber product* $\mathcal{M}^C_{i,j} \times_j K_{j,k}$.

Fixing orientations of o^C_i and o^D_i if possible, (14.12), (14.13) imply the induced orientations of $K_{i,j}$, $H_{i,j}$ and $F_{i,j}$ satisfies

$$\partial[K_{i,j}|_{H_{i,j}}] = [H_{i,j}],$$
$$\partial[K_{i,j}|_{F_{i,j}}] = -[F_{i,j}].$$

In general, we have the analog of Proposition 14.2 14.4:

Theorem 14.7. *Let* K *be a flow-homotopy between two pre-flow-morphisms* F, H*. Both* F, H *are from* $\mathcal{C}$ *to* $\mathcal{D}$ *and everything is equipped with orientation structures. Assume* o^C_i, o^D_i *are oriented, and* $\mathcal{C}, \mathcal{D}, F, H$ *are oriented by Proposition 14.2 14.4, then* $K_{i,j}$ *are oriented by* (14.12) (14.13)*, such that* K *is an oriented flow-homotopy between* F *and* H*.*

14.2 Local systems

From the discussion above, to get an oriented flow category/flow-morphism/flow-homotopy, one needs orientations on o_i, however this is not always the case in the Morse-Bott case, since there are nonorientable line bundles over closed manifolds. So we need to upgrade the minimal Morse-Bott cochain complex to a version with local system. In fact, Definition 14.1,14.3 and 14.6 are exactly the structures we need to define a cochain complex/cochain map/cochain homotopy.

In the case of finite dimensional Morse-Bott theory, let C be a critical manifold with stable bundle S, then in view of the Thom isomorphism, the contribution from a critical manifold C to the cohomology should be the cohomology with local system $H^*(C, \det S)$. In the abstract setting, if a flow category has an orientation structure, then the line bundle o_i plays the role of $\det S$.

We first recall the de Rham theory with local systems. The de Rham complex $\Omega^*(C_i, o_i)$ with local system o_i is defined as sections of $\wedge T^*C_i \otimes_{\mathbb{Z}_2} o_i$, and the usual exterior differential lifts to $\Omega^*(C_i, o_i)$ to a differential d_{o_i}. The associated cohomology is denoted by $H^*(C_i, o_i)$. Wedge product defines $\Omega^*(C_i, o_i) \times \Omega^*(C_i, o_i') \to \Omega^*(C_i, o_i \otimes o_i')$, which induces a map on cohomology. In particular, since $o_i^* \otimes o_i$ is canonically the trivial bundle, we have paring $H^*(C, o^*) \times H^*(C, o) \to \mathbb{R}$ by integrating over oriented C_i. It is a nondegenerate pairing just like usual case.

We define $o_i^* \boxtimes o_i$ over $C_i \times C_i$ to be $\pi_1^* o_i^* \otimes \pi_2^* o_i$, then $o_i^* \boxtimes o_i$ restricted to the diagonal Δ_i is trivial. Therefore for any $\omega \in \Omega^*(C_i \times C_i, o_i^* \boxtimes o_i)$, $\int_{\Delta_i} \omega$ is well defined, in particular, it defines a linear map $H^*(C_i \times C_i, o_i^* \boxtimes o_i) \to \mathbb{R}$, therefore it can be represented by a class in $H^*(C_i \times C_i, o_i \boxtimes o_i^*) = H^*(C_i, o_i) \otimes H^*(C_i, o_i^*)$, i.e. the twisted Poincaré dual of the diagonal. There are two ways to represent this class: The usual Thom classes δ_i^n of the diagonal are closed forms in $\Omega^*(C_i \times C_i, o_i \boxtimes o_i^*)$, since it is supported in a tubular neighborhood, and $o_i \boxtimes o_i^*$ is trivial over that tubular neighborhood. They represent the twisted Poincaré dual of the diagonal. One the other hand, we can find quasi-isomorphic embeddings

$$H^*(C_i, o_i) \to \Omega^*(C_i, o_i), H^*(C_i, o_i^*) \to \Omega^*(C_i, o_i^*)$$

by choosing representatives $\{\theta_{i,a}\}$ of $H^*(C_i, o_i)$, and the dual basis in $\Omega^*(C_i, o_i^*)$ are $\{\theta_{i,a}^*\}$ in the sense that $\langle \theta_{i,a}^*, \theta_{i,b} \rangle = (-1)^{c_i|\theta_{i,b}|} \int \theta_{i,a}^* \wedge \theta_{i,b} = \delta_{ab}$. Then it can be checked that $\sum_a \pi_1^* \theta_{i,a} \wedge \pi_2^* \theta_{i,a}^*$ also represents the twisted Poincaré dual of the diagonal. Therefore one can find $f_i^n \in \Omega^*(C_i \times C_i, o_i \boxtimes o_i^*)$ such that $df_i^n = \delta_i^n - \sum_a \pi_1^* \theta_{i,a} \wedge \pi_2^* \theta_{i,a}^*$. We define the Morse-Bott cochain of a flow category with an orientation structure to be

$$BC := \varinjlim_{q \to -\infty} \prod_{j=q}^{\infty} H^*(C_j, o_j^*) = \varinjlim_{q \to -\infty} \prod_{j=q}^{\infty} H^*(C_j, o_j) \text{ (since } o_i = o_i^*). \tag{14.16}$$

We need to argue that (12.7) still make sense. Let $\alpha \in \Omega^*(C_s, o_s^*), \gamma \in \Omega^*(C_{s+k}, o_{s+k})$, then $s_{s,s+k}^* \alpha \wedge t_{s,s+k}^* \gamma \in \Omega^*(\mathcal{M}_{s,s+k}, s_{s,s+k}^* o_i^* \otimes t_{s,s+k}^* o_j)$. By (14.5), $\rho_{s,\{s,s+k\}}$ induces

$$L_1 : s_{s,s+k}^* o_s^* \otimes t_{s,s+k}^* o_{s+k} \to s^* \det C_s \otimes o_{s,s+k}.$$

Since C_{s+k} is oriented, then by (14.4), we have bundle isomorphism

$$L_2 : s^*_{s,s+k} \det C_s \otimes o_{s,s+k} \to \det \mathcal{M}_{s,s+k}.$$

Then $-L_2 \circ L_1$ defines bundle isomorphism from $s^*_{s,s+k} o^*_s \otimes t^*_{s,s+k} o_{s+k}$ to $\det \mathcal{M}_{s,s+k}$. The negative sign is to match up that the negative sign in the proof of Proposition 14.2. Using $-L_1 \circ L_1$, we can interpret $s^*_{s,s+k}\alpha \wedge t^*_{s,s+k}\gamma$ as an from in $\Omega(\mathcal{M}_{s,s+k}, \det \mathcal{M}_{s,s+k})$. Since integration theory is well-defined for $\Omega^*(\mathcal{M}_{s,s+k}, \det \mathcal{M}_{s,s+k})$ for every compact manifold $\mathcal{M}_{s,s+k}$ oriented or not, thus $\int_{\mathcal{M}_{s,s+k}} s^*_{s,s+k}\alpha \wedge t^*_{s,s+k}\gamma$ is well-defined.

Next, we consider $\mathcal{M}^{s,k}_i[\alpha, f_{s+i}, \gamma]$. Since $s^*_{s,s+i}\alpha \wedge (t_{s,s+i} \times s_{s+i,s+k})^* f_{s+i} \wedge t^*_{s+i,s+k}\gamma$ is a form in $\Omega^*(\mathcal{M}_{s,s+i} \times \mathcal{M}_{s+i,s+k}, s^*_{s,s+i} o^*_s \otimes (t_{s,s+i} \times s_{s+i,s+k})^* (o_{s+i} \boxtimes o^*_{s+i}) \otimes t^*_{s+i,s+k} o_{s+k})$. Since $s^*_{s,s+i} o^*_s \otimes (t_{s,s+i} \times s_{s+i,s+k})^* \,(o_{s+i} \boxtimes o^*_{s+i}) \otimes t^*_{s+i,s+k} o_{s+k}$ equals to $\big(s^*_{s,s+i} o^*_s \otimes t^*_{s,s+i} o_{s,s+i}\big) \boxtimes \big(s^*_{s+i,s+k} o^*_{s+i} \otimes t^*_{s+i,s+k} o_{s+k}\big)$. Then $-L_2 \circ L_1$ for $s^*_{s,s+i} o^*_s \otimes t^*_{s,s+i} o_{s,s+i}$ and $s^*_{s+i,s+k} o^*_{s+i} \otimes t^*_{s+i,s+k} o_{s+k}$ induces bundle isomorphism:

$$s^* o^*_s \otimes (t \times s)^* (o_{s+i} \boxtimes o^*_{s+i}) \otimes t^* o_{s+k} \to \det \mathcal{M}_{s,s+i} \otimes \det \mathcal{M}_{s+i,s+k} \to \det(\mathcal{M}_{s,s+i} \times \mathcal{M}_{s+i,s+k}).$$

Thus $\mathcal{M}^{s,k}_i[\alpha, f_{s+i}, \gamma]$ is defined. Similarly, $\mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, f^n_{s+i_r}, \gamma]$ makes sense in the local system setting. Thus the differential $d_{BC} = \prod d_k$ (12.7) is well-defined and $d^2_{BC} = 0$ by an identical proof. The situation for flow-morphisms and flow-homotopies are the same.

C_i nonorientable case

The discussion above can be generalized to the case when C_i is not orientable. In that case, we can not orient $\mathcal{M}_{i,j}$ using (14.4). Therefore we need to apply the construction with local systems. Assume that $\mathcal{C}$ is a flow category with an orientation structure, where C_i can be nonorientable. Since integration is well-defined for $\Omega^*(C_i, \det C_i)$, thus we have the pairing $H^*(C_i, o^*_i) \times H^*(C_i, o_i \otimes \det C_i) \to \mathbb{R}$, and it is the nondegenerate by the twisted Poincaré duality in [13].

For any form $\omega \in \pi^*_1 o^*_i \boxtimes \pi^*_2(o_i \otimes \det C_i)$, since $\pi^*_2 \det C_i$ is canonically isomorphic $\det \Delta_i$ over the diagonal Δ_i, thus $\int_{\Delta_i} \omega$ is well defined. Thus $\int_{\Delta_i}$ defines a map from $H^*(C_i \times C_i, \pi^*_1 o^*_i \boxtimes \pi^*_2(o_i \otimes \det C_i))$ to $\mathbb{R}$, thus it can be represented the twisted Poincaré dual in $H^*(C_i \times C_i, \pi^*_1(o_i \otimes \det C_i) \boxtimes \pi^*_2 o^*_i)$.

If we choose the representatives $\{\theta_{i,a}\}$ for a basis of $H^*(C, o_i \otimes \det C_i)$, and the representatives of dual basis in $H^*(C_i, o^*_i)$ are denoted by $\{\theta^*_{i,a}\}$, then the twisted Poincaré dual can be represented by $\sum_a \pi^*_1\theta_{i,a} \wedge \pi^*_2\theta^*_{i,a}$. On the other hand, we would like to interpret the twisted Poincaré dual by Thom classes. There is natural isomorphism $\pi^*_1 \det C_i \otimes \pi^*_2 \det C_i \simeq \det \Delta_i \otimes \det N_i$ over the Δ_i, induced by the same convention as in (11.1). Using the natural identification $\pi^*_2 \det C_i \to \det \Delta_i$, then there is an induced isomorphism $\pi^*_1 \det C_i \to N_i$. Using this isomorphism, if a form in $\Omega^*(C_i \times C_i, \pi^*_1(o_i \otimes \det C_i) \boxtimes \pi^*_2 o^*_i)$ is supported in the tubular neighborhood of Δ_i, then it can be viewed as a form $\Omega^*(N_i, \det N_i)$. Using the twisted Thom isomorphism in [99], we get another representative of the twisted Poincaré class by the Thom classes.

Then d_{BC} (12.7) is a well-defined coboundary map on BC (14.16) by the same reason as before, similarly for flow-morphisms and flow-homotopies.

14.3 Generalizations of the construction

It should be clear now, that the only essential ingredient in our construction is the equation:

$$\delta_i^n = \mathrm{d}f_i^n + \sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta_{i,a}^*.$$

In fact, it is not necessary to construct our cochain complex on the cohomology of the critical manifolds, we will extract the properties we used in the proofs to get a generalized construction. Such generalization provides some freedom in applications, for example one can use the generalized construction to prove Gysin exact sequence for sphere bundles over flow categories. Moreover, this generalization will "almost contain" cascades construction as an example.

Reductions

The following discussion works for local systems and non-orientable manifolds, but for simplicity of notation, we only state the definitions in the oriented cases.

Definition 14.8. *For an oriented compact manifold C, a reduction of $\Omega^*(C)$ is a pair (A, A^*), such that A, A^* are finite dimensional subspaces of $\Omega^*(C)$ with $\dim A = \dim A^*$, and there exists a basis $\{\theta_a\}$ of A and a basis $\{\theta_a^*\}$ of A^*, such that $\langle\theta_a^*, \theta_b\rangle = \delta_{ab}$, and $\sum_a \pi_1^*\theta_a \wedge \pi_2^*\theta_a^*$ is cohomologous to the Thom classes δ^n.*

Example 14.9. *In our construction of minimal Morse-Bott cochain complex, we use the reduction $A = A^*$ equals to the image of chosen quasi-embedding $H^*(C) \to \Omega^*(C)$.*

Let $\mathcal{C}$ be an oriented flow category, and for every C_i, we fix a reduction (A_i, A_i^*), then we can define a cochain complex

$$BC^A := \varinjlim_{j\to-\infty} \prod_{i=j}^{\infty} A_i^*. \tag{14.17}$$

The differential is defined as $d^A = \prod_{i=0} d_i^A$, where

$$d_0^A\alpha = (-1)^{(|\alpha|+1)(c_s+1)} \sum_a \langle \mathrm{d}\alpha, \theta_{s,a}\rangle_s \theta_{s,a}^*, \tag{14.18}$$

with d is the normal exterior differential, and $\alpha \in A_s^*$. For $k \ge 1$ and $\gamma \in A_{s+k}$

$$\left\langle d_k^A\alpha, \gamma\right\rangle_{s+k} := (-1)^{|\alpha|(c_s+1)} \left(\mathcal{M}^{s,k}[\alpha,\gamma] + \lim_{n\to\infty} \sum (-1)^{\star} \mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, f^n_{s+i_r}, \gamma]\right), \tag{14.19}$$

where $\star = \sum_{j=1}^{r} \ddagger(\mathcal{C}, \alpha, i_j)$.

Following the discussion in the previous sections of Chapter 14, the constructions in Chapter 12 generalize to the following form by identical proofs.

Theorem 14.10. *The following statements hold.*

1. *Let $\mathcal{C}$ be a flow category*[1] *with an orientation structure, and (A, A^*) be a reduction. Then (14.17), (14.18) and (14.19) define a cochain complex (BC^A, d^A) and the homotopy type of (BC^A, d^A) is independent of the reduction.*

2. *Let $\mathcal{D}$ be another flow category with an orientation structure, (B, B^*) a reduction for $\mathcal{D}$ and H a flow-morphism from $\mathcal{C}$ to $\mathcal{D}$ with an orientation structure. Then (12.45) defines a cochain morphism $\phi^H_{A,B} : (BC^A, d^A) \to (BC^B, d^B)$ and the homotopy type of ϕ^H is independent of the reductions.*

3. *Let $\mathcal{E}$ be another flow category with an orientation structure, (C, C^*) a reduction for $\mathcal{E}$ and F a flow-morphism from $\mathcal{D}$ to $\mathcal{E}$ with an orientation structure. Assume H and F are composable, then $\phi^{F\circ H}_{A,C}$ and $\phi^F_{B,C} \circ \phi^H_{A,B}$ are homotopic.*

4. *Let G be another flow-morphism from $\mathcal{C}$ to $\mathcal{D}$ with an orientation structure. Assume there exists a flow-homotopy K from G to H with an orientation structure, then $\phi^H_{A,B}$ is homotopic $\phi^G_{A,B}$.*

One important feature of our construction is that the choices we make on the critical manifolds C_i, i.e. defining data or reductions, are independent of the structures of the flow categories, flow-morphisms and flow-homotopies.

Example 14.11 (Cascades)**.** *Let's neglect the difference between differential forms and currents for now. For a Morse-Smale pair (f, g) on critical manifold C, let $A = \{[S_x]\}_{x\in Cr(f)}$, $A^* = \{[U_x]\}_{x\in Cr(f)}$. Then we have*

$$[\Delta_C] - \sum [S_x][U_x] = d \lim_{t\to\infty} [\bigcup_{0<t'<t} graph\phi^i_{t'}]$$

Thus $\{A, A^*\}$ is a reduction. Note that the "homotopy operator" f^n_i in our construction might be different from the "homotopy operator" $[\bigcup_{t'<t} graph\phi^i_{t'}]$ in cascades construction, but the homotopy operator in our construction is quite irrelevant as long as the convergence results in section 12 hold.

One should be able to modify our construction to make the argument above rigorous. We need an extension of the space of differential forms by adding in some currents, so that $[S_x], [U_x]$ are in the extension. The currents that can be added into the extension depends on the flow category. Roughly speaking, this extension is the dual of the restraint from singular chain complex $C_*(C_i)$ to the geometric chain complex $C_{geo}(C_i)$ in Fukaya's model.

[1] C_i can be nonorientable.

Example 14.12 (Gysin exact sequence)**.** *Assume we have two oriented flow categories $\mathcal{E}$ and $\mathcal{C}$, and there is a functor $\pi : \mathcal{E} \to \mathcal{C}$, such that π maps E_i to C_i and $\mathcal{M}^E_{i,j}$ to $\mathcal{M}^C_{i,j}$. Moreover, $\pi : E_i \to C_i$ and $\pi : \mathcal{M}^E_{i,j} \to \mathcal{M}^C_{i,j}$ are both n-sphere bundles. We will call $\mathcal{E}$ is a n-sphere bundle over $\mathcal{C}$. We claim there is a Gysin exact sequence associated to the sphere bundle:*

$$\ldots \to H^*(\mathcal{C}) \xrightarrow{\pi^*} H^*(\mathcal{E}) \xrightarrow{\pi_*} H^{*-n}(\mathcal{C}) \to H^{*+1}(\mathcal{C}) \to \ldots \tag{14.20}$$

To prove the claim, we construct two flow-morphisms Π_* and Π^*. Π^* is a flow-morphism from $\mathcal{C}$ to $\mathcal{E}$, on the space level Π^* is same as the identity flow-morphism for $\mathcal{E}$, the only difference is the source map on Π^* is the projection to C_i. Similarly, Π_* is flow-morphism from $\mathcal{E}$ to $\mathcal{C}$, and on the space level it is same as the identity flow-morphism for $\mathcal{E}$, but the target map for Π_* is the projection to C_i.

If the flow categories $\mathcal{E}$ and $\mathcal{C}$ are actual spaces, then Π^* and Π_* induce π^*, π_* on cohomology. In general, to prove the exact sequence, the minimal construction will not work. Instead, we will consider the following reduction: For each sphere bundle $\pi : E_i \to C_i$, we fix an angular form $\psi_i \in \Omega^*(E)$, such that $\mathrm{d}\psi_i = \pi^* e_i$, where e_i is the Euler class of the sphere bundle. We can pick representatives $\{\theta_{i,a}\}$ of $H^*(C_i)$ in $\Omega^*(C_i)$, such that $e_i \in \langle \theta_{i,1}, \ldots \theta_{i,\dim H^*(C_i)} \rangle$. We use $\{\theta_{i,1}, \ldots, \theta_{i,\dim H^*(C_i)}\}$ as part of the defining data Θ for $\mathcal{C}$. Then $A = A^* = \langle \pi^*\theta_{i,1}, \ldots, \pi^*\theta_{i,\dim H^*(C_i)}, \pi^*\theta_{i,1} \wedge \psi_i, \ldots, \pi^*\theta_{i,\dim H^*(C_i)} \wedge \psi_i \rangle$ is a reduction for $\Omega^*(E_i)$, using them as a defining data Ξ for $\mathcal{E}$. Then the flow-morphisms Π^*, Π_* induces cochain maps:

$$0 \to BC^{\mathcal{C}}_{\Theta} \xrightarrow{\phi^{\Pi^*}} BC^{\mathcal{E}}_{\Xi} \xrightarrow{\phi^{\Pi_*}} BC^{\mathcal{C}}_{\Theta} \to 0. \tag{14.21}$$

It can be proven that (14.21) is a short exact sequence, and the induced long exact sequence is the Gysin exact sequence (14.20).

Chapter 15

Equivariant Theory

The aim of this chapter is to construct an equivariant theory for a flow category with a smooth group action. Our method of construction is based on the approximation of the homotopy quotient. In the context of Floer theory, a construction in this spirit can be found in [18]. All the results in this section, namely Theorem 15.1 and 15.9, can be generalized to flow categories with local systems, but for simplicity, we only consider the oriented case.

15.1 Parametrized cohomology

Similar to the construction of parametrized symplectic homology in [18], we need to introduce the parametrized cohomology of an oriented flow category, i.e. we need to take the product of a flow category $\mathcal{C}$ with a closed oriented manifold B. Since taking product with B automatically falls into the Morse-Bott case, using the theory developed in the previous chapters, we have a direct and geometric construction. We will see that all we have to do are some orientation checks.

Let $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ be an oriented flow category and B an oriented compact manifold through out this section. We will construct the product flow category $\mathcal{C} \times B$ first and the parametrized cohomology is defined to be the cohomology of $\mathcal{C} \times B$. Each map $f : B_1 \to B_2$ will induce an oriented flow-morphism H^f from $\mathcal{C} \times B_2$ to $\mathcal{C} \times B_1$ and a homotopy induces a flow-homotopy. The main result of this section is that, after taking the minimal Morse-Bott cochain complex, we have a contravariant functor by this product construction:

Theorem 15.1. *Let $\mathcal{C}$ be an oriented flow category, then we have a contravariant functor $\mathcal{C}\times$*

$$\mathcal{C}\times : \mathcal{K}(\mathcal{M}an) \to \mathcal{K}(\mathcal{C}h),$$

where $\mathcal{K}(\mathcal{M}an)$ is the category with objects are closed oriented manifolds and morphisms are homotopy class of smooth maps.

Product flow categories

The first step towards the construction of functor $\mathcal{C}\times$ is to construct the functor on the objects, i.e. the product flow categories:

Definition/Proposition 15.2. *If we orient $C_i \times B, \mathcal{M}_{i,j} \times B$ by $[C_i \times B] = [C_i][B]$, $[\mathcal{M}_{i,j} \times B] = (-1)^{\dim B}[\mathcal{M}_{i,j}][B]$. Then $\mathcal{C} \times B = \{C_i \times B, \mathcal{M}_{i,j} \times B\}$ is an oriented flow category.*

Let $E_1 \to M_1, E_2 \to M_2$ are two vector bundles, $E_1 \boxplus E_2$ is defined to $\pi_1^* E_1 \oplus \pi_2^* E_2$ over $M_1 \times M_2$, where $\pi_1 : M_1 \times M_2 \to M_1, \pi_2 : M_1 \times M_2 \to M_2$ are the projections.

Proof of Proposition 15.2. It is clear that we only need to verify $\mathcal{C}\times B$ satisfies the orientation property in Definition 11.16. Since

$$\partial[\mathcal{M}_{i,k} \times B] = \sum_j (-1)^{\dim B + m_{i,j}}[\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}][B].$$

Let N_B be the normal bundle of Δ_B in $B \times B$, and we orient it by $[\Delta_B][N_B] = [B][B]$. Then the normal bundle of $\Delta_{C_j \times B}$ is $N_j \boxplus N_B$, if we orient $N_j \boxplus N_B$ by the product orientation, then $[\Delta_{C_j\times B}][N_j \boxplus N_B] = [C_j \times B][C_j \times B]$, i.e. $[N_j \boxplus N_B]$ satisfying our orientation convention (11.1) for $C_j \times B$.

Then

$$\begin{aligned}
& [N_i \boxplus N_B]\partial[\mathcal{M}_{i,k} \times B|_{\mathcal{M}_{i,j}\times_j \mathcal{M}_{j,k}\times B}] \\
= \; & (-1)^{\dim B + m_{i,j}}[N_i \boxplus N_B][\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}][B] \\
= \; & (-1)^{\dim B + m_{i,j} + \dim B(m_{i,k}-1) + \dim B^2}[N_i][\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}][\Delta_B][N_B] \\
= \; & (-1)^{\dim B + m_{i,j} + \dim B(m_{i,k}-1) + c_j m_{i,j} + \dim B^2}[\mathcal{M}_{i,j}][\mathcal{M}_{j,k}][B][B] \\
= \; & (-1)^{\dim B + m_{i,j} \dim B(m_{i,k}-1) + c_j m_{i,j} + \dim B m_{j,k} + \dim B^2}[\mathcal{M}_{i,j} \times B][\mathcal{M}_{j,k} \times B].
\end{aligned}$$

Because $\dim B + m_{i,j} + \dim B(m_{i,k}-1) + c_j m_{i,j} + \dim B \cdot m_{j,k} + \dim B^2 = \dim B + m_{i,j} + (m_{i,j} + \dim B)(c_j + \dim B)$, by Definition 11.16, $\mathcal{C} \times B$ is an oriented flow category. □

It is very natural to expect the following Künneth formula for the product construction. The proof involves some index computation. Since we will not use it, we will give the sketch of the proof in the appendix.

Lemma 15.3. *$H^*(\mathcal{C} \times B) \simeq H^*(\mathcal{C}) \otimes H^*(B)$.*

Flow-morphisms between product flow categories

The second step is to construct the map between morphisms, i.e. for every smooth map $f : B_1 \to B_2$, we want to associate it with a cochain map $BC^{\mathcal{C}\times B_2} \to BC^{\mathcal{C}\times B_1}$. To that end, we will first construct a flow-morphism H^f from $\mathcal{C} \times B_2 \to \mathcal{C} \times B_1$. Then the associated cochain map is the cochain map ϕ^{H^f} defined in (12.44). The flow-morphism H^f is defined as follows:

- $H^f_{i,j} = \mathcal{M}_{i,j} \times [0, j-i] \times B_1$ with product orientation when $i \leq j$, and $H^f_{i,j} = \emptyset$ when $i > j$.
- The source and target maps s, t are defined by:

$$\begin{array}{rrcl} s: & H^f_{i,j} & \to & C_i \times B_2 \\ & (m,t,b) & \mapsto & (s^{\mathcal{C}}(m), f(b)), \\ t: & H^f_{i,j} & \to & C_j \times B_1 \\ & (m,t,b) & \mapsto & (t^{\mathcal{C}}(m), b), \end{array}$$

  for $m \in \mathcal{M}_{i,j}, t \in [0, j-i], b \in B_1$.

- For $m \in \mathcal{M}_{i,k}, n \in \mathcal{M}_{k,j}, t \in [0, k-j], b_1 \in B_1, b_2 \in B_2$, such that $(m,n) \in \mathcal{M}_{i,k} \times_k \mathcal{M}_{k,j}, f(b_1) = b_2$,

$$m_L: \begin{array}{rcl} (\mathcal{M}_{i,k} \times B_2) \times_k H^f_{k,j} & \to & H^f_{i,k} \\ (m, b_2, n, t, b_1) & \mapsto & (m, n, t+k-i, b_1). \end{array}$$

- For $m \in \mathcal{M}_{i,k}, n \in \mathcal{M}_{k,j}, t \in [0, k-i], b_1 \in B_1$, such that $(m,n) \in \mathcal{M}_{i,k} \times_k \mathcal{M}_{k,j}$,

$$m_R: \begin{array}{rcl} H^f_{i,k} \times_k (\mathcal{M}_{k,j} \times B_1) & \to & H^f_{i,j} \\ (m, t, b_1, n, b_1) & \mapsto & (m, n, t, b_1). \end{array}$$

Proposition 15.4. *Let $f : B_1 \to B_2$ be a smooth map between two closed oriented manifolds, then $H^f_{i,j}$ defines a oriented flow-morphism from $\mathcal{C} \times B_2 \to \mathcal{C} \times B_1$.*

Proof. All we need to do is the orientation check, it is analogous to the proof of Lemma 12.19. $\square$

Proposition 15.5. *Under the isomorphism in Lemma 15.3, $\phi^{H^f} : H^*(\mathcal{C} \times B_2, d_{BC}) \to H^*(\mathcal{C} \times B_1, d_{BC})$ equals to* $\mathrm{id} \otimes f^*$.

For a given oriented flow category $\mathcal{C}$, we now have enough prerequisites to define the functor $\mathcal{C}\times : \mathcal{K}(\mathcal{M}an) \to \mathcal{K}(\mathcal{C}h)$:

$$\begin{array}{lrcl} \text{On objects:} & B & \mapsto & BC^{\mathcal{C}\times B} \\ \text{On morphisms:} & (B_1 \xrightarrow{f} B_2) & \mapsto & (BC^{\mathcal{C}\times B_2} \xrightarrow{\phi^{H^f}} BC^{\mathcal{C}\times B_1}) \end{array}$$

To finish the proof of Theorem 15.1, we still need to show that homotopic smooth maps induces homotopic cochain maps, and $\mathcal{C}\times$ is functorial.

Flow-homotopies between product flow categories

Let $f, g : B_1 \to B_2$ be two smooth maps, and $D : [0,1] \times B_1 \to B_2$ be a homotopy between them, such that $D|_{\{0\}\times B_1} = f$ and $D_{\{1\}\times B_1} = g$. We claim there is a flow-homotopy K^D between the H^f and H^g. Then K^D is defined as:

- For $i \leq j$, $K^D_{i,j} = [0,1]_s \times \mathcal{M}_{i,j} \times [0, j-i]_t \times B_1$ with the product orientation. For $i < j$, $K^D_{i,j} = \emptyset$.
- The source map s is defined as:
$$s : \begin{array}{rcl} [0,1]_s \times \mathcal{M}_{i,j} \times [0,j-i]_t \times B_1 & \to & C_i \times B_2 \\ (s, m, t, b) & \mapsto & (s^{\mathcal{C}}(m), D_s(b)). \end{array}$$
- The target map t is defined as:
$$t : \begin{array}{rcl} [0,1]_s \times \mathcal{M}_{i,j} \times [0,j-i]_t \times B_1 & \to & C_i \times B_1 \\ (s, m, t, b) & \mapsto & (t^{\mathcal{C}}(m), b). \end{array}$$
- $\iota_{H^f} : H^f \xrightarrow{=} \{0\} \times \mathcal{M}_{i,j} \times [0, j-i]_t \times B_1 \subset K^D_{i,j}$.
- $\iota_{H^g} : H^g \xrightarrow{=} \{1\} \times \mathcal{M}_{i,j} \times [0, j-i]_t \times B_1 \subset K^D_{i,j}$.
- m_R is defined by
$$\begin{array}{rcl} ([0,1]_s \times \mathcal{M}_{i,j} \times [0,j-i]_t \times B_1) \times_j (\mathcal{M}_{j,k} \times B_1) & \to & [0,1]_s \times \mathcal{M}_{i,k} \times [0,k-i]_t \times B_1 \subset K^D_{i,k} \\ (s, m, t, b, n, b) & \mapsto & (s, (m,n), t, b). \end{array}$$
- m_L is defined by
$$\begin{array}{rcl} (\mathcal{M}_{i,j} \times B_2) \times_j ([0,1]_s \times \mathcal{M}_{j,k} \times [0,k-j] \times B_1) & \to & [0,1]_s \times \mathcal{M}_{i,k} \times [0,k-i] \times B_1 \subset K^D_{i,k} \\ (m, b_2, s, n, t, b_1) & \mapsto & (s, (m,n), t+j-i, b_1). \end{array}$$

Proposition 15.6. *K^D is an oriented flow-homotopy from H^f to H^g.*

Proof. We only need to check the orientations, and it is analogous to the proof of Lemma 12.19. □

To complete the proof of Theorem 15.1, we still have to prove the functoriality. Let $g : B_1 \to B_2, f : B_2 \to B_3$. It is not hard to see H^f and H^g can be composed. We claim there is a homotopy K^c from $H^f \circ H^g$ to $H^{f\circ g} \circ I$ is defined by the following.

- $K^c_{i,j} = [0,2]_s \times \mathcal{M}_{i,j} \times [0, j-i]_t \times B_1$ with product orientation, for $i \leq j$.

- The source map s is defined as:

$$s: \begin{array}{rcl} [0,2]_s \times \mathcal{M}_{i,j} \times [0, j-i]_t \times B_1 & \to & C_i \times B_3 \\ (s, m, t, b) & \mapsto & (s^{\mathcal{C}}(m), f \circ g(b)). \end{array}$$

- The target map t is defined as:

$$t: \begin{array}{rcl} [0,2]_s \times \mathcal{M}_{i,j} \times [0, j-i]_t \times B_1 & \to & C_i \times B_1 \\ (s, m, t, b) & \mapsto & (t^{\mathcal{C}}(m), b). \end{array}$$

- $\iota_{H^{f\circ g} \circ I} = \begin{cases} \pi_{H^{f\circ g}_{i,j}}: & (C_i \times B_3) \times_i H^{f\circ g}_{i,j} \to \{0\} \times H^{f\circ g}_{i,j} \subset K^c_{i,j}; \\ (\frac{t}{j-i}, m_L^{H^{f\circ g}}) & (\mathcal{M}^C_{i,j} \times [0, j-i]_t \times B_3) \times_k H^{f\circ g}_{j,k} \to [0,1] \times H^{f\circ g}_{i,k} \subset K^c_{i,k}. \end{cases}$
  where $m_L^{H^{f\circ g}}$ is the left-multiplication m_L for flow-morphism H^g.

- When $k \geq j$, $\iota_{H^g \circ H^f}|_{H^f_{i,j} \times_j H^g_{j,k}}$ is defined as:

$$\begin{array}{rcl} (\mathcal{M}_{i,j} \times [0, j-i] \times B_2) \times_j (\mathcal{M}_{j,k} \times [0, k-j] \times B_1) & \to & [1,2] \times \mathcal{M}_{i,k} \times [0, k-i] \times B_1 \\ (m, t_1, b_2, n, t_2, b_1) & \to & (\frac{t_2}{k-j} + 1, (m,n), t_1 + k - j, b_1), \end{array}$$

  when $k = j$, t_2 must be zero, and $\frac{t_2}{k-j}$ is defined to be 1.

- m_R is defined by

$$\begin{array}{rcl} ([0,2] \times \mathcal{M}_{i,j} \times [0, j-i] \times B_1) \times_j (\mathcal{M}_{j,k} \times B_1) & \to & [0,1] \times \mathcal{M}_{i,k} \times [0, k-i] \times B_1 \subset K^c_{i,k} \\ (s, m, t, b, n, b) & \to & (\frac{s}{2}, (m,n), t, b). \end{array}$$

- m_L is defined by

$$\begin{array}{rcl} (\mathcal{M}_{i,j} \times B_3) \times_j ([0,2] \times \mathcal{M}_{j,k} \times [0, k-j] \times B_1) & \to & [1,2] \times \mathcal{M}_{i,k} \times [0, k-i] \times B_1 \subset K^c_{i,k} \\ (m, b_3, s, n, t, b_1) & \to & (\frac{s}{2} + 1, (m,n), t + j - i, b_1). \end{array}$$

Proposition 15.7. *K^c is an oriented flow-homotopy from $H^f \circ H^g$ to $H^{f\circ g} \circ I$.*

Proof. The proof is analogous to proof of Lemma 12.19. □

Proof of Theorem 15.1. Proposition 15.2,15.4,15.6,15.7 imply Theorem 15.1. □

Remark 15.8. *There is a generalization of the construction above: Let $B_1 \xleftarrow{f} B \xrightarrow{g} B_2$ be maps between closed oriented manifolds, then there is morphism from $\mathcal{C} \times B_2$ to $\mathcal{C} \times B_1$ with $H_{i,j} = \mathcal{M}_{i,j} \times [0, j-i] \times B$, with the source and target map are induced by g, f. The homotopy type of the induced cochain map is determined by the oriented bordism groups $\Omega^*_{SO}(B_1, B_2)$, which is defined as follows: an element in $\Omega^n_{SO}(B_1, B_2)$ is represented a closed oriented n-manifold M and two maps f, g from M to B_1, B_2, (M, f, g) and (N, f', g') are equivalent iff there is an oriented bordism D from M to N and two maps F, G from D to B_1, B_2 extending f, g, f', g'.*

15.2 Equivariant cohomology

The functor $\mathcal{C}\times$ is not very interesting, because it is quite independent of the flow category $\mathcal{C}$. However, if $\mathcal{C}$ has a compact Lie group G acting on it, then the Borel construction, which is just a product module the G action, merges some of the information of $\mathcal{C}$ into the "homotopy quotient". Thus nontrivial phenomena may arise from such construction. The first step towards the Borel construction is to upgrade Theorem 15.1 to the following form:

Theorem 15.9. *Let compact Lie group G acts on $\mathcal{C}$ in an orientation-preserving way (Definition 15.10), then there is a contravariant functor $\mathcal{C}\times_G$:*

$$\mathcal{C}\times_G : \mathcal{K}(\mathcal{P}rin_G) \to \mathcal{K}(\mathcal{C}h),$$

where $\mathcal{K}(\mathcal{P}rin_G)$ is the category whose objects are closed oriented principal G bundle, and morphisms are G-equivariant homotopy class of G-equivariant maps.

Since the classifying space $EG \to BG$ can be approximated by a sequence of closed oriented manifolds $E_n \to B_n$, such that $\ldots \subset E_n \subset E_{n+1} \subset \ldots$. $EG \to BG$ can be understood as the "G-equivariant homotopy colimit" of the diagram $\ldots \subset E_n \subset E_{n+1} \subset \ldots$. Therefore the classical Borel construction of equivariant cohomology [53] suggests that the equivariant cochain complex of a flow category should be the composition of homotopy limit and the functor $C\times_G$ to the diagram $\ldots \subset E_n \subset E_{n+1} \subset \ldots$. We will construct this theory in this section, in particular, we will show this construction is independent of the approximation $\{E_n \to B_n\}$.

The functor $\mathcal{C}\times_G$

Definition 15.10. *A **G action on an oriented flow category** $\mathcal{C}$ is G actions on C_i and $\mathcal{M}_{i,j}$, such that source, target and multiplication maps are G-equivariant. We say the G-action preserves the orientation, if the G-actions on C_i and $\mathcal{M}_{i,j}$ preserve the orientations.*

Let $E \to B$ be an oriented G-bundle, such that the G action preserves the orientation. Assume G acts on $\mathcal{C}$ in a orientation preserving manner, then G acts from right on $C_i \times E/\mathcal{M}_{i,j} \times E$ by $(c, e)g = (g^{-1}c, eg)$. Let $C_i \times_G E/\mathcal{M}_{i,j} \times_G E$ denote the quotient of the G action. If we orient B, $C_i \times_G E$ and $\mathcal{M}_{i,j} \times_G E$ by $[G][B] = [E]$, $[C_i \times_G E][G] = (-1)^{\dim G \cdot \dim B}[C_i][E]$ and $[\mathcal{M}_{i,j} \times_G E][G] = (-1)^{\dim B + \dim G \cdot \dim B}[\mathcal{M}_{i,j}][E]$, then Proposition 15.2 can be generalized to the following statement by an analogous proof.

Proposition 15.11. *$\mathcal{C} \times_G E = \{C_i \times_G E, \mathcal{M}_{i,j} \times_G E\}$ is an oriented flow category.*

Moreover, Proposition 15.4, 15.6 and 15.7 can be generalized to the equivariant settings.

Proposition 15.12. *Assume G acts on the oriented flow category $\mathcal{C}$ and preserves the orientation. Let $E_1 \to B_2, E_2 \to B_2$ be two oriented G-principle bundles and the G-actions on E_1, E_2 preserve orientations.*

1. *Let f be a smooth G-equivariant map $E_1 \to E_2$, then there is an oriented flow-morphism H^f_G from $\mathcal{C} \times_G E_2$ to $\mathcal{C} \times_G E_1$, with $[H^f_{G,i,i}] = [C_i \times_G E_1]$, $[H^f_{G,i,j}] = [\mathcal{M}_{i,j} \times_G E_1][[0, j-i]]$ for $i < j$.*

2. *Let g be another G-equivariant map $E_1 \to E_2$, and $D : [0,1] \times E_1 \to E_2$ be an equivariant homotopy between f and g, then there is an oriented flow-homotopy K^D_G between H^f_G and H^g_G.*

3. *Let $h : E_2 \to E_3$ be another equivariant map between two oriented G-principle bundles, then there is an oriented flow-homotopy K^c_G from $H^h_G \circ H^f_G$ to $H^{h\circ f}_G \circ I$.*

Theorem 15.9 follows form Proposition 15.12.

Homotopy limit

Since our construction uses an approximation, we need to take limit in the end. Consider a directed system of cochain-complexes:

$$\ldots \to A_3 \to A_2 \to A_1 \to A_0$$

Then the limit $\varinjlim A_i$ is also a cochain complex. However, this limit is not very nice from homotopy-theoretic point of view, i.e. if we change the maps in the directed system by homotopic maps, then the homotopy type of $\varinjlim A_i$ may change. In our setting, the cochain map is constructed only up to homotopy (Section 12.6), thus we need to apply a better limit called the homotopy limit, whose homotopy type is invariant under the replacement of homotopic maps. We recall some of the basic definitions and properties of homotopy limit form [86].

Let $\mathbb{N}^{op}$ be the inverse directed set $\{\ldots \to 2 \to 1 \to 0\}$, $\{A_n, \mu_{nm} : A_n \to A_m\}$ be an inverse system of chain complex over this directed set, i.e:

$$\ldots \xrightarrow{\mu_4} A_3 \xrightarrow{\mu_3} A_2 \xrightarrow{\mu_2} A_1 \xrightarrow{\mu_1} A_0$$

Then there is a map $v : \prod A_i \to \prod A_i$, such that over the basis $a_n \in A_n, v(a_n) = \mu_n(a_n)$. Then $\operatorname{holim} A_n$ is defined to the homotopy kernel of $1 - v$, i.e. $\Sigma^{-1}C(1 - v)$, where $C(\cdot)$ denotes the mapping cone and Σ is shifting by 1. Then we have a triangle in $\mathcal{K}(\mathcal{C}h)$:

$$\begin{tikzcd} \prod A_n \arrow[rr, "1-v"] & & \prod A_n \arrow[dl, "+1"] \\ & \operatorname{holim} A_n \arrow[ul] & \end{tikzcd} \tag{15.1}$$

This construction is the infinite telescope construction. It is clear the homotopy limits of any final subsets of $\mathbb{N}^{op}$ are homotopic to each other.

There is a commutative diagram in $\mathcal{K}(\mathcal{C}h)$:

$$\begin{tikzcd} \operatorname{holim} A_n \arrow[r] & \prod A_n \\ \varprojlim A_n \arrow[u] \arrow[ur] & \end{tikzcd} \tag{15.2}$$

When $\varprojlim^1 A_n = 0$, i.e. Mittag-Leffler condition holds for A_n, then $\varprojlim A_n \to \operatorname{holim} A_n$ is a quasi-isomorphism. This is the reason why we sometimes can use limit instead of homotopy limit in applications [18].

The long exact sequence from the triangle (15.1) implies we have the short exact sequence:

$$0 \to \varprojlim^1 H^{*-1}(A_n) \to H^*(\operatorname{holim} A_n) \to \varprojlim H^*(A_n) \to 0.$$

Equivariant cochain complexes

Now, we are ready to define the equivariant cochain complex of a flow category with group action. Pick an approximation $E_0 \subset \ldots \subset E_i \subset \ldots$ of the classifying space in the sense of Definition 8.8, such that E_i is oriented and G preserves the orientation. Then applying the functor $\mathcal{C}\times_G$ to this sequence, we get an inverse system in $\mathcal{K}(\mathcal{C}h)$:

$$\ldots \to BC^{\mathcal{C}\times_G E_2} \to BC^{\mathcal{C}\times_G E_1} \to BC^{\mathcal{C}\times_G E_0}.$$

Definition 15.13. *Equivariant cochain complex BC_G is defined to* $\operatorname{holim} BC^{\mathcal{C}\times_G E_n}$.

The result in Section 12.6 implies the homotopy type of BC_G is independent of the auxiliary defining data. To get a canonical theory, we still need to check BC_G does not depend on the choice of the approximation $E_n \to B_n$.

Independence of approximations

Assume there is another approximation $E_n' \to B_n'$ of the classifying space, we want to form a new sequence of approximation containing both $E_n' \to B_n'$ and $E_n \to B_n$ as final subsets. As preparation, we state the following two propositions, the proposition below is a simple application of obstruction theory.

Proposition 15.14. *Let $Y \to X$ be a smooth fiber bundle, with fiber F is k-connected, and X is a k dimensional manifold. Then there is a cross section for $Y \to X$, and any two cross sections are homotopic.*

By this proposition, [53, Proposition 1.1.1.] can be modified into the following:

Proposition 15.15. *Let $E \to B$ be a G-principle bundle, with E is k-connected. Then for any closed manifold M with* $\dim M \leq k$*, the principle bundles over M are classified by $[M, B]$, i.e. the set of homotopy classes of maps from M to B.*

Therefore by Definition 8.8 and Proposition 15.15, there exists $n_1 \in \mathbb{N}$, such that there is an equivariant map $E_1 \to E'_{n_1}$. Moreover, there exists $m_1 \in \mathbb{N}$, such that there is an equivariant map $E'_{n_1} \to E_{m_1}$, and the composition $E_1 \to E'_{n_1} \to E_{m_1}$ is equivariant homotopic to $E_1 \subset E_{m_1}$. We can keep applying this argument to get a directed system in the equivariant homotopy category of spaces:

$$E_1 \to E'_{n_1} \to E_{m_1} \to E'_{n_2} \to E_{m_2} \to \dots$$

which is also compatible with the two approximations $\{E_{m_i}\}$ and $\{E'_{n_i}\}$ up to equivariant homotopy.

Then Theorem 15.9 implies there is a well-defined inverse directed system in the homotopy category of cochain complex:

$$\dots \to BC_{\mathcal{C}\times_G E_{m_2}} \to BC_{\mathcal{C}\times_G E'_{n_2}} \to BC_{\mathcal{C}\times_G E_{m_1}} \to BC_{\mathcal{C}\times_G E'_{n_1}} \to BC_{\mathcal{C}\times_G E_1} \tag{15.3}$$

Let H denote the homotopy limit of (15.3). Since both $BC_{\mathcal{C}\times_G E'_{n_i}}$ and $BC_{\mathcal{C}\times_G E_{m_i}}$ are final in the inverse directed systems above, thus

$$\operatorname{holim} BC_{\mathcal{C}\times_G E'_n} = \operatorname{holim} BC_{\mathcal{C}\times_G E'_{n_i}} = H = \operatorname{holim} BC_{\mathcal{C}\times_G E_{m_i}} = \operatorname{holim} BC_{\mathcal{C}\times_G E_m}$$

Therefore the homotopy type of BC_G is independent of the approximation, i.e. we have the following theorem.

Theorem 15.16. *Let $\mathcal{C}$ be an oriented flow category, and the compact Lie group G acts on $\mathcal{C}$ and preserves the orientation. Then the homotopy type of the equivariant cochain complex BC_G in Definition 15.13 is well-defined, and is independent of all the choices, in particular, the choice of the approximation $\{E_n \to B_n\}$.*

Spectral sequences

From (15.1), the homotopy limit is the shifted mapping cone of $1-v$. Thus the action spectral sequence in Proposition 13.1 on $BC^{\mathcal{C}\times_G E_n}$ induces spectral sequence on the homotopy limit. In particular, we need to apply the following result from [104]:

Proposition 15.17. *Let $f : B \to C$ be a map of filtered cochain complexes. For a fixed integer $r \geq 0$, there is a filtration on the mapping cone $C(f)$ defined by:*

$$F_pC(f) = F_{p+r}B_{n+1} \oplus F_pC_n.$$

Then rth page $E_r(C(f))$ of the induced spectral sequence is the mapping cone of $f^r : E_r(B) \to E_r(C)$.

By Proposition 15.17, let $r = 1$, there is a spectral sequence for BC_G induced from the action filtration on $\Pi BC^{\mathcal{C}\times_G E_n}$. Since $E_1^p(\Pi BC_{\mathcal{C}\times_G E_n}) = \Pi H^*(C_p \times_G E_n)$ with the differential coming from the d_1 term (12.7) for each $\mathcal{C} \times_G E_n$. By Proposition 15.17, $E_1(BC_G)$ is the (shifted) mapping cone of the cochain morphism

$$1 - v : \prod_n \varinjlim_{q \to -\infty} \prod_{p=q}^{\infty} H^*(C_p \times_G E_n) \to \prod_n \varinjlim_{q \to -\infty} \prod_{p=q}^{\infty} H^*(C_p \times_G E_n)$$

Since $\varprojlim^1 H^*(C_p \times_G E_n) = 0$, i.e. Mittag-Leffler condition for inverse system

$$\ldots H^*(C_p \times_G E_n) \to H^*(C_p \times_G E_{n-1}) \ldots$$

Thus the natural map (15.2)

$$\varinjlim_{q \to -\infty} \prod_{p=q}^{\infty} H^*_G(C_p) = \varprojlim_n \varinjlim_{q \to -\infty} \prod_{p=q}^{\infty} H^*(C_p \times_G E_n) \to E_1(BC_G)$$

is quasi-isomorphism. The induced differential d_1^G on $\varinjlim_{q \to -\infty} \prod_{p=q}^{\infty} H^*_G(C_p)$ is the limit of d_1 for $\mathcal{C} \times_G E_n$. Since d_1 comes from the moduli spaces without boundary, i.e. the pullback and pushforward on cohomology. Therefore d_1^G is $t_* \circ s^* : H^*_G(C_p) \to H^*_G(C_{p+1})$ up to sign, i.e. the pullback and pushforward on equivariant cohomology. The polyfold theoretic version of d_1^G is the analog of the equivariant fundamental class in Theorem 9.1.

Corollary 15.18. *There is a spectral sequence for BC_G, such that*

$$E_2^p(BC_G) \simeq H^*(\varinjlim_{q \to -\infty} \prod_{p=q}^{\infty} H^*_G(C_p), d_1^G).$$

Chapter 16

Basic Example: Finite Dimensional Morse-Bott Cohomology

The aim of this chapter is to construct a flow category for finite dimensional Morse-Bott theory. The existence of such flow category is a folklore theorem, stated in various places, e.g, [6, 44]. The Morse version of the flow category was introduced in [26], and [103] provided a detailed construction for the flow category of a Morse function for metrics which are standard near critical points. In this chapter, we prove that there is a flow category for any Morse-Bott function if we use a generic metric. The local analysis in our case is just a family version of the analysis in [103].

In the Morse case, [5] provides an argument to reduce the continuation maps and homotopies to some gradient flow counting. Similarly, we can construct the flow-morphisms and flow-homotopies by looking at some flow-categories. Therefore, just like the Morse case, we can prove the cohomology of the flow-category is independent of the Morse-Bott function. The main theorem of this chapter is

Theorem 16.1. *Let f be a Morse-Bott function on a closed manifold M, then there exists metric g, such that the compactified moduli spaces of (unparametrized) gradient flow lines form a flow category with an orientation structure. The cohomology of the flow category is independent of the Morse-Bott function and equals the regular cohomology $H^*(M,\mathbb{R})$.*

Let f be a Morse-Bott function on M throughout this chapter, and the critical manifolds are $C_1,\ldots,C_n$, such that $f(C_i)<f(C_j)$ whenever $i<j$. We can fix a real number $\delta>0$, such that δ is strictly smaller than the absolute values of the nonzero eigenvalues of $Hess(f)$ over all critical manifolds C_i.

16.1 Fredholm problem for finite dimensional Morse-Bott theory

Like the Morse case, the moduli spaces of parametrized gradient flow line from C_i to C_j is a zero set of a Fredholm operator over some Banach space $\mathcal{B}_{i,j}$. The construction of $\mathcal{B}_{i,j}$ was included in the appendix of [43] as part of the Banach manifolds of the cascades construction, we review the construction briefly.

First we fix an auxiliary metric g_0 on M. Let γ be a smooth curve defined over $\mathbb{R}$, such that

$$\lim_{t\to-\infty}\gamma(t) = x \in C_i \text{ and } \lim_{t\to+\infty}\gamma(t) = y \in C_j, \tag{16.1}$$

$$|\frac{d}{dt}\gamma|_{g_0} < Ce^{-\delta|t|} \text{ for } |t| \ll 0 \text{ and some constant } C. \tag{16.2}$$

Let $P(C_i, C_j)$ be the space of continuous path defined over $\mathbb{R}$, connecting C_i and C_j. The Banach manifold $\mathcal{B}_{i,j}$ will be a subspace of $P(C_i, C_j)$, we will first describe the neighborhood of γ in $\mathcal{B}_{i,j}$. For this purpose, we fix the following things.

1. Fix a smooth function $\chi : \mathbb{R} \to \mathbb{R}$, such that $\chi(t) = |t|$ for $|t| \ll 0$. Then we can define the weighted Sobolev space $H^k_\delta(\mathbb{R}, \gamma^*TM)$ with norm $|f|_{H^k_\delta} = |e^{\delta\chi(t)}f|_{H^k}$, for $k \geq 1$.

2. $\rho_\pm(t)$ are smooth functions which are 1 near $\pm\infty$ and 0 near $\mp\infty$, such that (16.3) makes sense.

3. Fix local charts of M near x, y, such that C_i near x is a radius r ball in $x_1, \ldots, x_{c_i}$ coordinates, and C_j near y is a radius r ball in $y_1, \ldots, y_{c_j}$ coordinates.

Then there exists K, such that when $f \in H^k_\delta(\mathbb{R}, \gamma^*TM)$ with $|f|_{H^k_\delta} < K$, then $|f|$ is pointwise smaller than the injective radius of the metric g_0. Let exp denote the exponential map associated to the metric g_0. Then there is a map:

$$B_K(H^k_\delta(\mathbb{R}, \gamma^*TM)) \times B_r(\mathbb{R}^{c_i}) \times B_r(\mathbb{R}^{c_j}) \to P(C_i, C_j)$$

$$(f, x_1, \ldots, x_{c_1}, y_1, \ldots, y_{c_j}) \mapsto \exp_\gamma f + \sum_1^{c_i} \rho_- x_i + \sum_1^{c_j} \rho_+ y_i. \tag{16.3}$$

$\mathcal{B}_{i,j}$ consists of images of all such maps in $P(C_i, C_j)$ for all curves γ satisfying (16.1) and (16.2). Let $\mathcal{E}_{i,j} \to \mathcal{B}_{i,j}$ be the vector bundle, with the fiber over $\gamma \in \mathcal{B}_{i,j}$ is $H^{k-1}_\delta(\mathbb{R}, \gamma^*TM)$, it was proven in [43] that:

Proposition 16.2 ([43]). *$\mathcal{B}_{i,j}$ is a Banach manifold. $\mathcal{E}_{i,j} \to \mathcal{B}_{i,j}$ is a Banach bundle.*

Since $\mathcal{B}_{i,j} \to C_i \times C_j$ are submersions for all $i < j$, the fiber products $\mathcal{B}_{i,j} \times_j \ldots \times_k \mathcal{B}_{k,l}$ are Banach manifolds. Moreover, $\mathcal{E}_{i_0,i_1} \times_{i_1} \ldots \times_{i_{k-1}} \mathcal{E}_{i_{k-1},i_k} \to \mathcal{B}_{i_0,i_1} \times_{i_1} \ldots \times_{i_{k-1}} \mathcal{B}_{i_{k-1},i_k}$ are Banach bundles for all $i_0 < i_1 < \ldots < i_k$. Given a metric g, then there is a section $s_{i,j} : \mathcal{B}_{i,j} \to \mathcal{E}_{i,j}$ defined by $s(\gamma) = \gamma' - \nabla_g f(\gamma)$.

Proposition 16.3 ([43])**.** *$s_{i,j}$ is a Fredholm operator with index* $\mathrm{ind}_j - \mathrm{ind}_i + c_i + c_j$*, where* ind_i *is the dimension of negative eigenspace of $Hess(f)$ on C_i.*

Proposition 16.4. *For a generic metric g, $s_{i,j}$ is transverse to* 0*, and the fiber products $s^{-1}_{i_0,i_1}(0) \times_{i_1} \ldots \times_{i_{k-1}} s^{-1}_{i_{k-1},i_k}(0)$ are cut out transversely for all $i_0 < \ldots < i_k$.*

Proof. The proof follows from a standard Sard-Smale argument by considering the universal moduli space of all metrics, and the results for the fiber products follows from applying the Sard-Smale argument to $s_{i_0,i_1} \times_{i_1} \ldots \times_{i_{k-1}} s_{i_{k-1},i_k} : \mathcal{B}_{i_0,i_1} \times_{i_1} \ldots \times_{i_{k-1}} \mathcal{B}_{i_{k-1},i_k} \to \mathcal{E}_{i_0,i_1} \times_{i_1} \ldots \times_{i_{k-1}} \mathcal{E}_{i_{k-1},i_k}$. □

We call such pair (f, g) a Morse-Bott-Smale pair. Let $M_{i,j}$ denote $s^{-1}_{i,j}(0)/\mathbb{R}$, then $\mathcal{M}_{i,j} = \cup_{i<i_1<\ldots<i_k<j} M_{i,i_1} \times_{i_1} \ldots \times_{i_k} M_{i_k,j}$ can be made into a compact topological space. For the topology one puts on this space, we refer readers to [103, 5] for details.

16.2 Flow categories of Morse-Bott functions

The main theorem of this section is that we can put smooth structures on $\mathcal{M}_{i,j}$, such that:

Theorem 16.5. *$\{C_i, \mathcal{M}_{i,j}\}$ is a flow category.*

To prove this theorem, we need to equip $\mathcal{M}_{i,j}$ a smooth structure with boundary and corner. One strategy is using a gluing map [96], which can be generalized to Floer theory. However, we will adopt a simpler method from [26, 103, 5], which only exists in finite dimensional Morse-Bott theory.

Lemma 16.6 ([87])**.** *Let C_i be a critical manifold of the Morse-Bott function f, then there is a tubular neighborhood of C_i in M diffeomorphic to the normal bundle N of C_i. Moreover, N can be decomposed into stable and unstable bundles N^s, N^u, and there are metrics g^s, g^u on N^s, N^u, such that $f(v)|_N = f(C_i) - |v^s|^2_{g^s} + |v^u|^2_{g^u}$, where $v \in N$, and v^s, v^u are the stable and unstable components of v.*

Let g_{C_i} be a metric on C_i, If a metric g near C_i has the form $\pi^* g_{C_i} + g^s + g^u$, where π is the projection $N \to C_i$, we say the metric g is standard near C_i. In fact, we can require the Morse-Bott-Smale pair to have standard metric near all critical manifolds. For a standard metric, the gradient vector in N is contained in the fibers of the tubular neighborhood. Therefore the local picture of gradient flow line is just family of the Morse flow lines in each fiber. When restricted to a fiber F with coordinate $x_1, \ldots, x_s, y_1, \ldots, y_u$, the pair (f, g) is standard and is in the following form:

$$\begin{aligned} f|_F &= -x_1^2 - \ldots - x_s^2 + y_1^2 + \ldots + y_u^2 + C \\ g|_F &= \mathrm{d}x_1 \otimes \mathrm{d}x_1 + \ldots + \mathrm{d}x_s \otimes \mathrm{d}x_2 + \mathrm{d}y_1 \otimes \mathrm{d}y_1 + \ldots + \mathrm{d}y_u \otimes \mathrm{d}y_u \end{aligned}$$

Inside the fiber F, we define

$$\begin{aligned} S^r_s &:= \{(x_1,\ldots,x_s)|x_1^2+\ldots+x_s^2=r^2\}, \\ S^r_u &:= \{(y_1,\ldots,y_u)|y_1^2+\ldots+y_u^2=r^2\}, \\ D^r_s &:= \{(x_1,\ldots,x_s)|x_1^2+\ldots+x_s^2<r^2\}, \\ D^r_u &:= \{(y_1,\ldots,y_u)|y_1^2+\ldots+y_u^2<r^2\}. \end{aligned}$$

Let $\mathcal{M}$ be the moduli space of gradient flow lines and broken gradient flow lines of $(f|_F, g|_F)$ from $S^r_s\times D^r_u$ to $D^r_s\times S^r_u$, let ev_-, ev_+ be the two evaluation maps of $\mathcal{M}$, the following lemma is essentially contained in [103]:

Lemma 16.7. $\mathrm{im}(ev_-\times ev_+)(\mathcal{M})\subset(S^r_s\times D^r_u)\times(D^r_s\times S^r_u)$ *is a submanifold with boundary.*

Proof. Since the gradient flow lines are $(e^{-2t}x, e^{2t}y)$, thus the image of unbroken flow lines are $(x,y,\frac{|y|}{r}x,\frac{r}{|y|}y)$, it is a submanifold in $(S^r_s\times D^r_u)\times(D^r_s\times S^r_u)$. The image of broken flow lines are $(x,0,0,y)$, it is also a submanifold in $(S^r_s\times D^r_u)\times(D^r_s\times S^r_u)$. And the boundary chart is given by $(t,x,0,0,y)\to(x,ty,tx,y)$, thus the lemma is proven. □

Remark 16.8. *Lemma 4.4 of [103] composes the* $ev_-\times ev_+$ *with projection* $(x,y',x',y)\to(\frac{|x|'+|y'|}{2r},x,y)$ *to get a homeomorphism from* $\mathcal{M}$ *to* $[0,1)\times S^r_s\times S^r_u$*. This homomorphism was used in [103] to construct the smooth structures with boundary and corner on* $\mathcal{M}$*. Since the projection restricted to* $\mathrm{im}(ev_-\times ev_+)(\mathcal{M})$ *is a diffeomorphism, we can also use the smooth structure on* $\mathrm{im}(ev_-\times ev_+)(\mathcal{M})$ *to make* $\mathcal{M}$ *manifold with boundary and corners.*

Since $S^r_s\times D^r_u$ and $D^r_s\times S^r_u$ are transverse to the gradient flow, then Lemma 16.7 also holds if we replace $S^r_s\times D^r_u$ and $D^r_s\times S^r_u$ by open sets in $f|_F^{-1}(C-\epsilon)$ and $f|_F^{-1}(C+\epsilon)$.

Now we return to the Morse-Bott case with standard metric near C_i. Let ϕ^t be the flow for ∇f, then the stable manifold S_i of C_i is defined to be:

$$S_i=\{x\in M|\lim_{t\to\infty}\phi^t(x)\in C_i\}.$$

And the unstable manifold U_i is defined to be

$$U_i=\{x\in M|\lim_{t\to-\infty}\phi^t(x)\in C_i\}.$$

Both S_i and U_i are equipped with smooth evaluation maps to C_i. Then we have the family version of Lemma 16.7 as follows.

Lemma 16.9. *Given a standard metric near* C_i*, let* N_r *be the radius* r *tube of* C_i*. Suppose* ϵ *is a positive real number, and* $v_i^{\pm\epsilon}$ *denotes* $f(C_i)\pm\epsilon$*. Let* $\mathcal{M}_{i,\epsilon,r}$ *denote the moduli space of flow lines and broken flow lines from* $f^{-1}(v_i^{-\epsilon})\cap N_r$ *to* $f^{-1}(v_i^{+\epsilon})\cap N_r$*. Then there exist* $\epsilon, r>0$*, such that image of* $ev_-\times ev_+|_{\mathcal{M}_{i,\epsilon,r}}$ *is a submanifold with boundary in* $(f^{-1}(v_i^{-\epsilon})\cap N_r)\times(f^{-1}(v_i^{+\epsilon}))\cap N_r)$*, and the boundary is* $(f^{-1}(v_i^{-\epsilon})\cap S_i)\times_{C_i}(v_i^{+\epsilon})\cap U_i)$*.*

Proposition 16.10. *$M_{i,j} \times_j M_{j,k} \cup M_{i,k}$ can be given a structure of manifold with boundary.*

Proof. Since

$$M_{i,j} \simeq U_i \cap S_j \cap f^{-1}(v_j^{-\epsilon})$$
$$M_{j,k} \simeq U_j \cap S_k \cap f^{-1}(v_j^{+\epsilon})$$

The Morse-Bott-Smale condition is equivalent to the intersections are transverse. On the other hand, let $M_{i,k} \cap \mathcal{M}_{j,\epsilon,r}$ be the set of flow lines in $M_{i,k}$ which contains a flow line in $\mathcal{M}_{j,\epsilon,r}$, then it is an open set of $M_{i,k}$, and we have embedding

$$ev_- \times ev_+ : M_{i,k} \cap \mathcal{M}_{j,\epsilon,r} \to (f^{-1}(v_j^{-\epsilon}) \cap N_r) \times (f^{-1}(v_j^{+\epsilon}) \cap N_r).$$

The image is

$$\mathrm{im}(ev_- \times ev_+)(M_{i,k} \cap \mathcal{M}_{j,\epsilon,r}) = \mathrm{im}(ev_- \times ev_+)(\partial^0 \mathcal{M}_{j,\epsilon,r}) \cap ((U_i \cap f^{-1}(v_j^{-\epsilon})) \times (S_k \cap f^{-1}(v_j^{+\epsilon}))).$$

The Morse-Bott-Smale condition implies that the intersection is transverse. Moreover $\partial\, \mathrm{im}(ev_- \times ev_+)(\mathcal{M}_{j,\epsilon,r}) = (f^{-1}(v_j^{-\epsilon}) \cap S_i) \times_{C_i} (f^{-1}(v_j^{+\epsilon}) \cap U_i)$ is also transverse to $(U_i \cap f^{-1}(v_j^{-\epsilon})) \times (S_k \cap f^{-1}(v_j^{+\epsilon}))$, since fiber product $M_{i,j} \times_j M_{j,k}$ is transverse. Thus $\mathrm{im}(ev_- \times ev_+)(M_{i,k} \cap \mathcal{M}_{j,\epsilon,r})$ can be completed with boundary by the boundary structure of $\mathrm{im}(ev_- \times ev_+)(\mathcal{M}_{j,\epsilon,r})$, that is we can add in $(U_i \cap S_j \cap f^{-1}(v_j^{-\epsilon})) \times_{C_j} (S_k \cap U_j \cap f^{-1}(v_j^{+\epsilon})) \simeq M_{i,j} \times_j M_{j,k}$ as the boundary of $M_{i,k} \cap \mathcal{M}_{j,\epsilon,r}$. The topology check is analogous to [103]. □

Therefore we have gluing map $\rho_j : [0,1) \times M_{i,j} \times_j M_{j,l} \to \mathcal{M}_{i,l}$, such that $\rho_j(0)$ is identity on $M_{i,j} \times_j M_{j,l}$, and $\rho_j : (0,1) \times M_{i,j} \times_j M_{j,l} \to M_{i,l} \subset \mathcal{M}_{i,l}$. Let ρ_k denote the gluing map $[0,1) \times M_{j,k} \times_k M_{k,l} \to \mathcal{M}_{j,k}$, thus we have map:

$$[0,1) \times [0,1) \times M_{i,j} \times_j M_{j,k} \times_k M_{k,l} \to \mathcal{M}_{i,l} : (s,t) \to \rho_j(t) \circ \rho_k(s),$$

which defines the corner structures of $\mathcal{M}_{i,l}$. For corner structures with higher degeneracy index, we can use the similar constructions. This proves Theorem 16.5.

Let o_i is the determinant line bundle of the stable line bundle over C_i, then $\{C_i, \mathcal{M}_{i,j}\}$ defines a flow category $\mathcal{C}_{f,g}$ with an orientation structure. Thus we have an associated Morse-Bott cohomology.

16.3 Morphisms and homotopies

To derive the flow-morphisms between different Morse-Bott functions and flow-homotopies between them, we will use the argument from [5] to reduce the construction for flow-morphisms and flow-homotopies back to the flow categories.

Flow-morphisms

Let (f_1, g_1) and (f_2, g_2) be two locally standard Morse-Bott-Smale pairs, let $\mathcal{C}^1 = \{C^1_i, \mathcal{M}^1_{i,j}\}$ and $\mathcal{C}^2 = \{C^2_i, \mathcal{M}^2_{i,j}\}$ denote the associated flow categories. We can find a smooth function $F : \mathbb{R} \times M \to \mathbb{R}$, such that:

$$F(t,x) = \begin{cases} f_1(x) & t < \frac{1}{3}, \\ \\ f_2(x) & t > \frac{2}{3}. \end{cases}$$

We consider a Morse function h on $\mathbb{R}$, such that it only has two critical points, one local minima at 0, and one local maxima at 1, and

$$\forall x \in M, t \in (0,1), \frac{\partial F}{\partial t} + \frac{dh}{dt} > 0.$$

Then $F + h$ defines a Morse-Bott function on $\mathbb{R} \times M$, with critical manifolds $C^1_i \times \{0\}$ and $C^2_i \times \{1\}$, we can find a locally standard metric G such that

$$G(t,x) = \begin{cases} g_1 + \mathrm{d}t \otimes \mathrm{d}t & t < \frac{1}{3}, \\ \\ g_2 + \mathrm{d}t \otimes \mathrm{d}t & t > \frac{2}{3}. \end{cases}$$

also with the property (F, G) is a locally standard Morse-Bott-Smale pair. Then by Theorem 16.5, we can associate $(F + h, G)$ a flow category with an orientation structure. Let $F_{i,j}$ denote the compactified moduli space of flow lines from $C^1_i \times \{0\}$ to $C^2_j \times \{1\}$, then $F_{i,j}$ form a flow-morphism from $\mathcal{C}^1$ to $\mathcal{C}^2$. When $F(t,x) = f(x)$, and we can choose $G = g + dt^2$, then $F_{i,i} = C_i$ and $F_{i,j} \simeq \mathcal{M}_{i,j} \times [0,1] \simeq I_{i,j}$, for $i < j$.

Flow-homotopies

Assume we have continuations F, G, H from f_1 to f_2, f_2 to f_3 and f_1 to f_3 respectively, then we can find $K : \mathbb{R}_s \times \mathbb{R}_t \times M \to \mathbb{R}$, such that:

$$K(s,t,x) = \begin{cases} H(t,x) & s < \frac{1}{3}, \\ \\ F(s,x) & t < \frac{1}{3}, \\ \\ G(t,x) & s > \frac{2}{3}, \\ \\ f_3(x) & t > \frac{2}{3}. \end{cases}$$

We can find h with one local minima at 0 and local maxima at 1, such that

$$\forall (s,t,x) \in (0,1) \times \mathbb{R} \times M, \frac{\partial K}{\partial s} + h'(s) > 0,$$

$$\forall (s,t,x) \in \mathbb{R} \times (0,1) \times M, \frac{\partial K}{\partial t} + h'(t) > 0.$$

Then $K + h(s) + h(t)$ defines a Morse-Bott function, with critical manifolds $C_i^1 \times \{(0,0)\}$, $C_i^2 \times \{(1,0)\}$, $C_i^3 \times \{(0,1)\}$ and $C_i^3 \times \{(1,1)\}$, and we can find a locally standard Morse-Bott-Smale metric extending the locally standard metrics used in F, G, H and f_3. Then the flow lines from $C_i^1 \times \{(0,0)\}$ to $C_j^3 \times \{(1,1)\}$ give rise to a flow-homotopy between $G \circ F$ and $I \circ H$.

Proof of Theorem 16.1. By Theorem 16.5, we have a flow category $\mathcal{C}_{f,g}$ with an orientation structure for any locally standard Morse-Bott-Smale pair (f,g). Using the flow-morphisms and flow-homotopies above, we can see that cohomology of $\mathcal{C}_{f,g}$ does not depend on (f,g). Thus we can choose $f \equiv C$, and g be any metric, then (f,g) is a locally standard Morse-Bott-Smale pair. The corresponding flow category has object space and morphism space are both M, thus the cohomology of the flow category equals to the cohomology $H^*(M,\mathbb{R})$. $\square$

Since Morse-Smale pair is a special case of Morse-Bott-Smale pair, and our definition of cochain complex recovers the Morse cochain complex when the function is Morse. As a corollary, the $\mathbb{R}$ coefficient Morse cohomology equals to the de Rham cohomology of M.

Chapter 17

Transversality by Polyfold Theory

With the theory on flow categories developed in the previous chapters. The remaining problem is to get a flow category in applications, i.e. we need to solve the transversality problems. For this purpose, we will adopt the polyfold theory developed by Hofer-Wysocki-Zehnder [58, 60, 59]. This chapter outlines some ideas on combining our construction with polyfold theory, details will appear in future work.

Polyflow categories

The main result of chapter 12 is that for any oriented (with an orientation structure) flow category, we can associate a well-defined cochain complex up to homotopy. If we want to write down a representative cochain complex of the homotopy class, we need to fix a defining data Θ. In applications, take Hamiltonian Floer cohomology as an example, the flow category is the zero sets of some Fredholm sections over some polyfolds [102]. A natural consideration is that we replace every manifold $\mathcal{M}_{i,j}$ in the flow category by strong polyfold bundle $W_{i,j} \to Z_{i,j}$ with a Fredholm section $s_{i,j}$, and all $W_{i,j} \to Z_{i,j}, s_{i,j}$ are organized just like a flow category, and when all $s_{i,j}$ are transverse to 0, then $s_{i,j}^{-1}(0)$ defines a flow category. We wish to assign a well-defined cochain complex to such system of polyfolds up to homotopy, and when we want to write down an explicit representative cochain complex for the homotopy class, we need to fix a family of perturbations and a defining data.

Definition 17.1 (Polyflow category)**.** *A polyflow category is a small category $\mathcal{Z}$ with following properties.*

(P1) The object space $Obj_{\mathcal{Z}} = C$ is a manifold without boundary. $C = \sqcup_{i\in\mathbb{Z}} C_i$ is the disjoint union of manifolds C_i, such that each connected component of C_i is a closed manifold.

(P2) The morphism space $Mor_{\mathcal{Z}} = Z$ is a polyfold. The source and target maps $s, t : Z \to C$ are sc-smooth. Let $Z_{i,j}$ denote $(s \times t)^{-1}(C_i \times C_j)$.

(P3) $Z_{i,i} \simeq C_i$ (i.e. the identity morphisms), $Z_{i,j} = \emptyset$ for $j < i$, and $Z_{i,j}$ is a polyfold for $j > i$.

(P4) *The fiber product* $Z_{i_0,i_1} \times_{i_1} Z_{i_1,i_2} \times_{i_2} \ldots \times_{i_{k-1}} Z_{i_{k-1},i_k}$ *is cut transversely, for all increasing sequence of* $i_0 < i_1 < \ldots < i_k$.

(P5) *The composition* $m : Z_{i,j} \times_j Z_{j,k} \to Z_{i,k}$ *is an sc-smooth injective map into the boundary of* $Z_{i,k}$.

(P6) $m : \coprod_{i<j<k} \partial^0 (Z_{i,j} \times_j Z_{j,k}) \to \partial^1 Z_{i,k}$ *is an sc-diffeomorphism.*

(P7) *There are strong polyfold bundles* $W_{i,j} \to Z_{i,j}$ *and Fredholm sections* $s_{i,j}$*, such that both the bundle and section are compatible with* m*, i.e.* $m^* W_{i,k}|_{Z_{i,j} \times_j Z_{j,k}} = W_{i,j} \times W_{j,k}$ *and* $s_{i,k}|_{m(Z_{i,j} \times_j Z_{j,k})} = m(s_{i,j}, s_{j,k})$.

(P8) *For each connected component* C_i^α *of* C_i*, when restricted to the subset of* $(s \times t)^{-1}(C_i^\alpha, C_j)$*,* $s_{i,j}$ *is proper.*

(P8) *(Alternative for P8)For each connected component* C_j^α *of* C_j*, when restricted to the subset of* $(s \times t)^{-1}(C_i, C_j^\alpha)$*,* $s_{i,j}$ *is proper (or simply require* $s_{i,j}$ *is proper, when* C_i, C_j *are compact).*

(P4) can be replaced by a convenient condition: $(s \times t)|_{Z_{i,j}}$ are submersions. The index $\operatorname{ind} s_{i,j}$ plays the role of $m_{i,j}$, and the determinant bundle $\det s_{i,j}$ plays the role of $\det \mathcal{M}_{i,j}$. We can define graded and oriented (or with orientation structures) polyflow category in a similar way. When all $s_{i,j}$ are transverse to 0, the zero sets form a flow category. So the problem becomes: can we find a family of $sc+$ perturbation $p_{i,j}$, such that $s_{i,j} + p_{i,j}$ is transverse and consistent with the composition m. The consistency with m is called coherence of the perturbation and it depends on the combinatorics of the problem. In the case of polyflow category, the combinatorics are relatively simple and we can have a perturbation scheme:

Claim 17.2. *There exists coherent perturbations* $p_{i,j}$*, such that* $s_{i,j} + p_{i,j}$ *are transverse to* 0 *and in general position.*

Remark 17.3. *The claim is not correct when there are inner symmetries of the polyflow category and we want to preserve the symmetries in the perturbation. To be more precise, we may have* $Z_{i,j}$ *is same as another* $Z_{k,l}$*, and if we want* $p_{i,j} = p_{k,l}$ *then we may not be able to find transverse perturbations in general position. In application, e.g, Hamiltonian Floer cohomology, we will see such phenomenon if we are forced to use Novikov coefficient. In the Morse case, such phenomenon also causes problems (aka. self-gluing) in the homotopy argument. The homotopy argument can be viewed as a Morse-Bott problem with critical manifolds copies of* $\mathbb{R}$*. In fact, such problem of inner symmetry cause problems as long as the polyflow category has critical manifolds other than points. However, under certain assumptions*[1] *of the polyflow category, we can actually perturb the source and target maps*

[1] Basically, we require a collar neighborhood near the boundary and corner of polyfolds, such assumptions are satisfied in all known examples.

consistently to destroy all the inner symmetries. We will discuss this in detail in our future work.

Although the polyfold perturbation theory only provides branched suborbifolds as the transverse zero set. Since the convergence results, i.e. Lemma 12.5,12.13, are local. The only thing we need about $\mathcal{M}_{i,j}$ is the Stokes' theorem, which was proven in [63]. Thus all the proofs in Chapter 12 go through. We can define polyflow-morphisms and polyflow-homotopies similarly by replacing the manifolds by polyfolds with Fredholm sections. Once the perturbation scheme is given for those structures, we can generate flow-morphisms and flow-homotopies.

The enrichment to polyflow category causes more choices, i.e. the choice of perturbation. We would like to have the cohomology independent of sc+perturbation. Such invariance can be proven using a homotopy argument in the flavor of our Morse-Bott constructions.

Claim 17.4. *$\mathcal{Z}$ is a polyflow category without inner symmetry (or with some extra assumptions on the polyfolds, if there are inner symmetries), then we can associate it with a minimal Morse-Bott cochain complex $(BC^{\mathcal{Z}}, d_{BC})$, such that the homotopy type of the cochain complex is independent of defining data and sc+ perturbations.*

Equivariant theory

In chapter 15, we discuss the equivariant theory when the flow category is equipped with group action. However requiring G acts on the flow category is equivalent to requiring G-equivariant transversality, which is known to be obstructed sometimes. So we need to lift the Borel construction to the polyflow category instead of applying it directly to the flow category.

Definition 17.5 (Polyflow category with G action)**.** *G acts on polyflow category $\mathcal{C}$ iff G acts on C_i and $W_{i,j} \to Z_{i,j}$, all $s_{i,j}$ are G-invariant, and the structure maps s, t, m are G-equivariant.*

If we fix an approximation E_n of EG, then we can form a sequence of polyflow categories $\mathcal{Z} \times_G E_n$ by the quotient construction in Theorem 4.42. Under little extra assumptions on $\mathcal{Z}$, we will have a sequence of polyflow-morphisms connecting $\mathcal{Z} \times_G E_n$. Then we have a directed system in the "category" of polyflow categories, and we can get an inverse system of cochain complexes by applying claim 17.4, then the equivariant cochain complex will be the homotopy limit of this inverse system. The details will appear in our future work.

Part IV

Vanishing of Symplectic Homology

Chapter 18

Vanishing of Symplectic Homology and Obstruction to Flexible Fillability

18.1 Preliminaries on fillings and symplectic homology

Symplectic fillings

We assume throughout this note that (Y, ξ) is a contact manifold of dimension $2n-1 \geq 5$, such that ξ is co-oriented and its first Chern class $c_1(\xi) = 0$.

Definition 18.1.

- *(W, λ) is a Liouville filling of (Y, ξ) iff $\mathrm{d}\lambda$ is a symplectic form on W, the Liouville field X_λ, defined by $i_{X_\lambda}\mathrm{d}\lambda = \lambda$, is outward transverse along ∂W, and $(\partial W, \ker \lambda|_{\partial W})$ is contactomorphic to (Y, ξ).*

- *(W, λ, ϕ) is a Weinstein filling of (Y, ξ) iff (W, λ) is a Liouville filling, $\phi : W \to \mathbb{R}$ is a Morse function with maximal level set ∂W, and X_λ is a gradient-like vector field for ϕ.*

- *(W, λ, ϕ) is a flexible Weinstein filling of (Y, ξ) iff (W, λ, ϕ) is a Weinstein filling, and there exist regular values $c_1 < \min \phi < c_2 < \ldots < c_k < \max \phi$ of ϕ, such that there are no critical values in $[c_k, \max \phi]$ and each $\phi^{-1}([c_i, c_{i+1}])$ is a Weinstein cobordism with a single critical point p whose attaching sphere Λ_p is either subcritical or a loose Legendrian in $(\phi^{-1}(c_i), \lambda|_{\phi^{-1}(c_i)})$, see [22, 71]*

For simplicity, we only consider connected fillings in this note.

For grading reasons in symplectic homology, we will only consider Liouville domains satisfying the following property.

Definition 18.2. *A Liouville domain W is topologically simple iff $c_1(W) = 0$ and $\pi_1(\partial W) \to \pi_1(W)$ is injective.*

Proposition 18.3. *Assume (W, λ, ϕ) is a Weinstein filling of (Y, ξ). Then W is topologically simple.*

Proof. Since ϕ has only critical points with index less than $n = \frac{1}{2}\dim W$, the filling W can be constructed from Y by attaching handles with index greater than n; see [22]. This implies that $\pi_2(W, Y) = 0$, since $n \geq 3$. Now injectivity of $\pi_1(Y) \to \pi_1(W)$ follows from the long exact sequence

$$\ldots \to \pi_2(W, Y) \to \pi_1(Y) \to \pi_1(W) \to \ldots$$

By [71, Proposition 2.1], $c_1(\xi) = 0$ implies that $c_1(W) = 0$ □

Given two Liouville domains W, W', we can join them by adding a 1-handle and then extend the Liouville structure to the handle. The resulting Liouville domain $W \natural W'$ is called the boundary connected sum of W and W'. The contact boundary $\partial(W \natural W')$ is the contact connected sum of the contact boundaries $\partial W \# \partial W'$; see [22].

Proposition 18.4. *Let W and W' be two topologically simple Liouville domains. Then the boundary connected sum $W \natural W'$ is topologically simple.*

Proof. Topologically, $W \natural W'$ is constructed by attaching a 1-handle to the disjoint union $W \coprod W'$. Let ι denote the inclusion $W \coprod W' \hookrightarrow W \natural W'$. Then $\iota^* : H^2(W \natural W'; \mathbb{Z}) \to H^2(W; \mathbb{Z}) \oplus H^2(W'; \mathbb{Z})$ is an isomorphism. Note that $c_1(W) = c_1(W') = 0$ implies that $c_1(W \coprod W') = 0$. Since $\iota^* c_1(W \natural W') = c_1(W \coprod W')$, we have $c_1(W \natural W') = 0$.

We have $\pi_1(W \natural W') = \pi_1(W) * \pi_1(W')$ and $\pi_1(\partial(W \natural W')) = \pi_1(\partial W \# \partial W') = \pi_1(\partial W) * \pi_1(\partial W')$ by the van Kampen theorem. Since $\pi_1(\partial W) \to \pi_1(W)$ and $\pi_1(\partial W') \to \pi_1(W')$ are both injective, the induced map on the free product of groups is also injective. Hence $W \natural W'$ is also topologically simple. □

The following definition is due to Lazarev [71, Definition 3.6], and it generalizes the dynamically convex contact structures studied in [2, 24, 57]. A contact form α for the contact structure ξ is called regular, if all Reeb orbits of α are non-degenerate. Let α_1, α_2 be contact forms for the same contact structure ξ. A partial order on the contact forms for a fixed co-oriented contact structure is given by $\alpha_1 \geq \alpha_2$ iff $\alpha_1 = f\alpha_2$ and $f \geq 1$; see [71, §3].

Definition 18.5 ([71, Definition 3.6])**.** *A contact manifold (Y^{2n-1}, ξ) is called asymptotically dynamically convex, if there exist non-increasing regular contact forms $\alpha_1 \geq \alpha_2 \ldots$ for ξ and a sequence of increasing numbers $D_1 < D_2 < \ldots$ going to infinity such that every contractible Reeb orbit γ with period smaller than D_k has the property that $\mu_{CZ}(\gamma) + n - 3 > 0$, where $\mu_{CZ}(\gamma)$ is the Conley-Zehnder index of the non-degenerate orbit γ.*

An important fact is that the property of asymptotically dynamically convex is preserved under subcritical surgeries and flexible surgeries [71, Theorem 3.15, 3.17, 3.18]. In this note, we will use the following special case.

Proposition 18.6 ([71, Theorem 3.15]). *Let (Y_1,ξ_1), (Y_2,ξ_2) be two asymptotically dynamically convex contact manifolds, then the contact connected sum $(Y_1\#Y_2,\xi_1\#\xi_2)$ is also asymptotically dynamically convex.*

Lazarev also showed that there is a large class of asymptotically dynamically convex contact manifolds.

Proposition 18.7 ([71, Corollary 4.1]). *If (Y,ξ) has a flexible Weinstein filling, then (Y,ξ) is asymptotically dynamically convex.*

Symplectic homology and positive symplectic homology

To a Liouville filling (W,λ) of the contact manifold (Y,ξ), one can associate the completion $(\widehat{W},\mathrm{d}\widehat{\lambda}) = (W\cup_Y [1,\infty)_r\times Y, d\widehat{\lambda})$, where $\widehat{\lambda}=\lambda$ on W and $\widehat{\lambda}=r(\lambda|_Y)$ on $[1,\infty)_r\times Y$. For any coefficient ring or field $\boldsymbol{k}$, the symplectic homology $SH_*(W;\boldsymbol{k})$ is defined to be the Hamiltonian Floer homology on $(\widehat{W},\mathrm{d}\widehat{\lambda})$ for the quadratic Hamiltonian H, where $H=r^2$ on the cylindrical end and C^2 small on W. After a C^2 small perturbation of H, the chain complex is generated by the **contractible** Hamiltonian periodic orbits. There are two types of generators: (1) critical points in W, (2) periodic orbits in $[0,\infty)\times Y$. One can choose the perturbation carefully such that there is a one-to-one correspondence between Reeb orbits on Y and pairs of Hamiltonian periodic orbits on $[0,\infty)\times Y$. The differential arises from counting the solutions to the Floer equations; see [21, 97, 71] for details of the construction. The complex generated by the critical points in W is a subcomplex and the positive symplectic homology $SH_*^+(W;\boldsymbol{k})$ is defined to the homology of the quotient complex; see [24]. Moreover, we have the following exact triangle.

Proposition 18.8. *[71, §2.4] For a Liouville domain W with $c_1(W)=0$, there is an exact triangle:*

$$\begin{tikzcd}
H^{n-*}(W;\boldsymbol{k}) \arrow[rr] & & SH_*(W;\boldsymbol{k}) \arrow[dl] \\
& SH_*^+(W;\boldsymbol{k}) \arrow[ul, "{[-1]}"] &
\end{tikzcd}$$

Since there is a correspondence between the chain complex for $SH_*^+(W;\boldsymbol{k})$ and Reeb orbits, one may hope that $SH_*^+(W;\boldsymbol{k})$ is an invariant of the contact boundary. This is not true in general, but if Y is asymptotically dynamically convex, then the positive symplectic homology $SH_*^+(W,\boldsymbol{k})$ is independent of the choice of topologically simple Liouville filling W, hence a contact invariant.

Proposition 18.9 ([71, Proposition 3.8]). *Let (Y,ξ) be an asymptotically dynamically convex contact manifold. If there are two topologically simple Liouville fillings W_1,W_2, then $SH_*^+(W_1;\boldsymbol{k})\cong SH_*^+(W_2;\boldsymbol{k})$.*

For the proof of Theorem 1.6, we also need to recall the ring structure and boundary connected sum formula on symplectic homology.

Proposition 18.10 ([89, Theorem 6.6]). *The pair of pants product $SH_*(W;\boldsymbol{k}) \otimes SH_\star(W;\boldsymbol{k}) \to SH_{*+\star-n}(W;\boldsymbol{k})$ makes $SH_*(W;\boldsymbol{k})$ into a unital ring, with the unit in $SH_n(W)$. The morphism $H^{n-*}(W;\boldsymbol{k}) \to SH_*(W;\boldsymbol{k})$ in Proposition 18.8 is a unital ring morphism, where the multiplication on $H^{n-*}(W;\boldsymbol{k})$ is the usual cup product.*

Proposition 18.11 ([21, Theorem 1.11]). *Let W, W' be two Liouville domains with $c_1(W) = c_1(W') = 0$. Let $\iota : W \coprod W' \to W \natural W'$ denote the inclusion. Then the Viterbo transfer map $\iota^*_{SH} : SH_\star(W\natural W';\boldsymbol{k}) \to SH_\star(W;\boldsymbol{k}) \oplus SH_\star(W';\boldsymbol{k})$ arising from the inclusion ι is an isomorphism.*

18.2 Proof of Theorem 1.6

Proof of Theorem 1.6. We will first show $SH_*(W';\boldsymbol{k}) = 0$ for any field $\boldsymbol{k}$. By the universal coefficient theorem [76], $SH_*(W;\mathbb{Z})$ and $SH_*(W;\boldsymbol{k})$ are related by the following short exact sequence.

$$0 \to SH_*(W;\mathbb{Z}) \otimes_{\mathbb{Z}} \boldsymbol{k} \to SH_*(W;\boldsymbol{k}) \to \operatorname{Tor}\left(SH_{*-1}(W;\mathbb{Z}),\boldsymbol{k}\right) \to 0. \tag{18.1}$$

Hence $SH_*(W;\mathbb{Z}) = 0$ implies that $SH_*(W;\boldsymbol{k}) = 0$ for any field $\boldsymbol{k}$. Then Proposition 18.8 yields an isomorphism

$$SH^+_*(W;\boldsymbol{k}) \cong H^{n+1-*}(W;\boldsymbol{k}). \tag{18.2}$$

Since (Y,ξ) is asymptotically dynamically convex and both W, W' are topologically simple, Proposition 18.9 and (18.2) yield an isomorphism

$$SH^+_*(W';\boldsymbol{k}) \cong SH^+_*(W;\boldsymbol{k}) \cong H^{n+1-*}(W;\boldsymbol{k}). \tag{18.3}$$

Applying Proposition 18.8 to W', we have the following long exact sequence

$$\ldots \to H^{-1}(W';\boldsymbol{k}) \to SH_{n+1}(W';\boldsymbol{k}) \to SH^+_{n+1}(W';\boldsymbol{k}) \to H^0(W';\boldsymbol{k}) \to SH_n(W';\boldsymbol{k}) \to \ldots \tag{18.4}$$

therefore $SH_{n+1}(W';\boldsymbol{k}) \cong \ker\left(SH^+_{n+1}(W';\boldsymbol{k}) \to H^0(W';\boldsymbol{k})\right)$. By (18.3), $SH^+_{n+1}(W';\boldsymbol{k})$ is isomorphic to $H^0(W;\boldsymbol{k}) \cong \boldsymbol{k}$. So there are only two possibilities for $SH_{n+1}(W';\boldsymbol{k})$,

$$SH_{n+1}(W';\boldsymbol{k}) \cong 0 \text{ or } \boldsymbol{k}. \tag{18.5}$$

Next, we will rule out the case of $SH_{n+1}(W';\boldsymbol{k}) = \boldsymbol{k}$. By Proposition C.7, the contact connected sum $Y\#Y$ is also asymptotically dynamically convex. By Proposition 18.4, both $W\natural W$ and $W'\natural W'$ are topologically simple Liouville fillings of $Y\#Y$. By Proposition 18.11, we have an isomorphism $SH_*(W\natural W;\mathbb{Z}) \cong SH_*(W;\mathbb{Z}) \oplus SH_*(W;\mathbb{Z}) = 0$. Therefore the conditions in Theorem 1.6 also hold for $Y\#Y$. Hence the same argument for (18.5) can be applied to $W'\natural W'$, and we conclude that

$$SH_{n+1}(W'\natural W';\boldsymbol{k}) \cong 0 \text{ or } \boldsymbol{k}.$$

On the other hand, by Proposition 18.11, $SH_{n+1}(W'\natural W';\boldsymbol{k})$ is isomorphic to $SH_{n+1}(W';\boldsymbol{k})\oplus SH_{n+1}(W';\boldsymbol{k})$. Therefore the only possibility is

$$SH_{n+1}(W';\boldsymbol{k}) = 0.$$

Consequentially, by the long exact sequence (18.4), $SH^+_{n+1}(W';\boldsymbol{k}) \to H^0(W';\boldsymbol{k})$ is an injective map from $\boldsymbol{k}$ to $\boldsymbol{k}$. Since $\boldsymbol{k}$ is a field, we can conclude $SH^+_{n+1}(W';\boldsymbol{k}) \to H^0(W';\boldsymbol{k})$ is an isomorphism. This is the reason why we want to work with field $\boldsymbol{k}$ instead of $\mathbb{Z}$. Then $H^0(W';\boldsymbol{k}) \to SH_n(W';\boldsymbol{k})$ is zero by (18.4). By Proposition 18.10, the morphism $H^0(W';\boldsymbol{k}) \to SH_n(W';\boldsymbol{k})$ sends the unit in $H^0(W';\boldsymbol{k})$ to the unit of $SH_*(W';\boldsymbol{k})$, therefore the unit of $SH_*(W';\boldsymbol{k})$ is zero. This proves $SH_*(W';\boldsymbol{k}) = 0$ for any field $\boldsymbol{k}$.

Now assume that the integral symplectic homology $SH_*(W';\mathbb{Z})$ does not vanish. Since $H^{n-*}(W;\mathbb{Z})$ is a finitely generated abelian group, $SH^+_*(W';\mathbb{Z})$ is also finitely generated because of (18.3). By (18.4), $SH_*(W';\mathbb{Z})$ is also finitely generated, because both $H^{n-*}(W';\mathbb{Z})$ and $SH^+_*(W';\mathbb{Z})$ are finitely generated. Thus, by the classification of finitely generated abelian groups, $SH_*(W';\mathbb{Z})$ either contains a $\mathbb{Z}$ summand or a $\mathbb{Z}/m$ summand. In either case, there exists a prime p, such that $SH_*(W';\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}/p \neq 0$. By (18.1), this implies $SH_*(W;\mathbb{Z}/p) \neq 0$, in contradiction to $SH_*(W;\boldsymbol{k}) = 0$ for $\boldsymbol{k} = \mathbb{Z}/p$. □

Corollary 18.12. *Under the assumptions of Theorem 1.6, there is an isomorphism $H^*(W;\mathbb{Z}) \cong H^*(W';\mathbb{Z})$.*

Proof. By Theorem 1.6, we have $SH_*(W;\mathbb{Z}) = SH_*(W';\mathbb{Z}) = 0$. We also have $SH^+_*(W;\mathbb{Z}) \cong SH^+_*(W';\mathbb{Z})$ by Proposition 18.9. Then the long exact sequence in Proposition 18.8 yields an isomorphism $H^{n+1-*}(W;\mathbb{Z}) \cong SH^+_*(W;\boldsymbol{k}) \cong SH^+_*(W';\boldsymbol{k}) \cong H^{n+1-*}(W';\mathbb{Z})$. □

The following corollary provides a partial answer to Question 1.7.

Corollary 18.13. *Assume the contact manifold (Y,ξ) has a flexible Weinstein filling W. If W' is a topological simple Liouville filling of Y, then $SH_*(W';\mathbb{Z}) = 0$ and $H^*(W;\mathbb{Z}) \cong H^*(W';\mathbb{Z})$.*

Proof. By Proposition 18.3, W is a topologically simple filling. The contact manifold (Y,ξ) is asymptotically dynamically convex by Proposition 18.7. The flexible Weinstein domain W has vanishing symplectic homology by [15]. Then this corollary follows from Theorem 1.6 and Corollary 18.12. □

Using Theorem 1.6, we can also strengthen [71, Corollary 1.3] to the following statement by the same argument used in [71].

Corollary 18.14. *Let (Y^5,ξ) be the contact structure constructed in [46] for any simply-connected, almost contact 5-manifold (Y^5,J) with $c_1(Y^5,J)=0$. Then all 2-connected Liouville fillings of (Y^5,ξ) are diffeomorphic.*

Moreover, we have the following corollary for cobordisms.

Corollary 18.15. *Suppose (Y,ξ) is simply-connected and has a flexible Weinstein filling. Assume there is a Liouville cobordism E from (Y_0,ξ_0) to (Y,ξ) such that $c_1(E)=0$. Let F be a Liouville filling of (Y_0,ξ_0), such that $c_1(F)=0$ and $\iota_1^*-\iota_2^*: H^1(E)\oplus H^1(F)\to H^1(Y_0)$ is surjective, where $\iota_1,\iota_2: Y_0\to E,F$ are the inclusions. Then we have $SH_*(F;\mathbb{Z})=0$*

Proof. We can glue E and F to obtain a Liouville filling $E\cup_{Y_0}F$ of (Y,ξ). Let $\tau_1: E\to E\cup_{Y_0}F, \tau_2: F\to E\cup_{Y_0}F$ denote the inclusions and consider the Mayer-Vietoris sequence

$$\ldots\longrightarrow H^1(E)\oplus H^1(F)\xrightarrow{\iota_1^*-\iota_2^*}H^1(Y_0)\longrightarrow H^2(E\cup_{Y_0}F)\xrightarrow{\tau_1^*\oplus\tau_2^*}H^2(E)\oplus H^2(F)\longrightarrow\ldots$$

Since $\iota_1^*-\iota_2^*: H^1(E)\oplus H^1(F)\to H^1(Y_0)$ is surjective, $\tau_1^*\oplus\tau_2^*: H^2(E\cup_{Y_0}F)\to H^2(E)\oplus H^2(F)$ is injective. Note that $\tau_1^*(c_1(E\cup_{Y_0}F))=c_1(E)$ and $\tau_2^*(c_1(E\cup_{Y_0}F))=c_1(F)$, therefore $c_1(E)=c_1(F)=0$ implies that $c_1(E\cup_{Y_0}F)=0$. Since $\pi_1(Y)=0$, $E\cup_{Y_0}F$ is a topologically simple Liouville filling of Y. Since Y has a flexible Weinstein filling by assumption, we have $SH_*(E\cup_{Y_0}F;\mathbb{Z})=0$ by Corollary 18.13. Since the Viterbo transfer map $SH_*(E\cup_{Y_0}F;\mathbb{Z})\to SH_*(F;\mathbb{Z})$ is a unital ring map [89, Theorem 9.5], we obtain $SH_*(F;\mathbb{Z})=0$. ☐

Corollary 18.13 also provides the following obstruction to the existence of flexible Weinstein fillings.

Corollary 18.16. *Suppose (Y,ξ) is a contact manifold. If one of the following conditions hold, then there are no flexible Weinstein fillings for (Y,ξ).*

1. *There is a topologically simple Liouville filling W' of Y, such that $SH_*(W';\mathbb{Z})\neq 0$.*
2. *There is a Weinstein filling W' of Y, such that $SH_*(W';\mathbb{Z})\neq 0$.*

Proof. Note that condition 2 implies condition 1 by Proposition 18.3. Suppose condition 1 holds and assume that Y has a flexible Weinstein filling W. Then by Corollary 18.13, $SH_*(W';\mathbb{Z})=0$, which contradicts the condition 1. ☐

18.3 Applications

Brieskorn manifolds have no flexible Weinstein fillings

As an application of the obstruction considered in Corollary 18.16, we show that all the Brieskorn manifolds [69, Definition 2.7] with dimension greater or equal to 5 can not be filled by flexible Weinstein domains. We review some basics of Brieskorn manifolds following

[69]. Let $a = (a_0, \ldots, a_n)$ be an $(n+1)$-tuple of integers, such that $a_i \geq 2$ for all $i = 0, \ldots, n$. The Brieskorn manifold $\Sigma(a)$ is defined as follows:

$$\Sigma(a) := \left\{ z \in \mathbb{C}^{n+1} \middle| z_0^{a_0} + \ldots z_n^{a_n} = 0, \quad \sum_{j=0}^{n} |z_j|^2 = 1 \right\}$$

It has a contact form

$$\alpha_a = \frac{i}{8} \sum_{j=1}^{n} a_j (z_j \mathrm{d}\overline{z}_j - \overline{z}_j \mathrm{d}z_j). \tag{18.6}$$

For ϵ sufficiently small, the Brieskorn manifold $(\Sigma(a), \xi_a := \ker \alpha_a)$ has a Weinstein filling:

$$V_\epsilon(a) = \left\{ z \in \mathbb{C}^{n+1} \middle| z_0^{a_0} + \ldots z_n^{a_n} = \epsilon, \quad \sum_{j=0}^{n} |z_j|^2 \leq 1 \right\} \tag{18.7}$$

with the symplectic form induced from $\mathbb{C}^{n+1}$; see [69, §2]. Moreover, $V_\epsilon(a)$ is homotopy equivalent to $\bigvee^{\prod_{j=0}^{n}(a_j-1)} S^n$, the wedge of $\prod_{j=0}^{n}(a_j - 1)$ spheres [69, Proposition 3.2], $\Sigma(a)$ is $(n-2)$-connected [69, Proposition 3.5].

Theorem 18.17. *The Brieskorn manifolds of dimension ≥ 5 can not be filled by flexible Weinstein domains.*

Proof. We have $H^2(V_\epsilon(a)) = H^2\left(\bigvee^{\prod_{j=0}^{n}(a_j-1)} S^n\right) = 0$, since $n \geq 3$ by assumption. Therefore the first Chern class $c_1(V_\epsilon(a))$ vanishes. Since $c_1(\xi_a) = c_1(V_\epsilon(a))|_{\Sigma(a)}$, we have $c_1(\xi_a) = 0$. By [69, Theorem 1.2], we know $SH_*(V_\epsilon(a); \mathbb{Z}) \neq 0$. This shows that the condition 2 of Corollary 18.16 holds, hence $\Sigma(a)$ can not be filled by flexible Weinstein domains. □

We can also apply Corollary 18.16 to the exotic Weinstein structures constructed in [22]. Cieliebak-Eliashberg [22] proved that for dimension ≥ 6, every almost Weinstein domain (W, J) with $c_1(W, J) = 0$ admits infinitely many non-symplectomorphic Weinstein structures W_k. By the h-principle for flexible Weinstein domain [22], we can put a flexible Weinstein structure on W. W_k is the boundary connected sum of the flexible Weinstein domain W with k copies of the exotic $\mathbb{C}^n_M$ in [82]. Lazarev [71, Remark 1.10] proved that the Cieliebak-Eliashberg-McLean contact manifolds ∂W_k can not be filled by flexible Weinstein domains except for possibly $\dim H^1(\partial W, \mathbb{Z}/2) + 1$ values of $k \in \mathbb{N}$. In fact, when $k \in \mathbb{N}^+$, ∂W_k can not be filled by flexible Weinstein domains. Since $SH_*(W_k; \mathbb{Z}) \cong SH_*(W; \mathbb{Z}) \oplus_{i=1}^{k} SH_*(\mathbb{C}^n_M; \mathbb{Z}) = \oplus_{i=1}^{k} SH_*(\mathbb{C}^n_M; \mathbb{Z}) \neq 0$; see [82], W_k is a Weinstein filling of ∂W_k such that condition 2 of Corollary 18.16 holds. Therefore ∂W_k can not be filled by flexible Weinstein domains for $k \geq 1$.

Not flexibly fillable contact structures on spheres

The contact structures on S^3 is well-understood by the work of Eliashberg [29]. For higher dimensional spheres, Eliashberg [31] constructed exotic contact structures on (S^{2n-1}, J_{std}) for odd n, where J_{std} is the almost contact structure induced by the standard contact structure ξ_{std}. Geiges [46] and Ding-Geiges [28] generalized that to all spheres of odd dimension ≥ 5. Ustilovsky [101] showed that there exist infinitely many different contact structures on (S^{2n-1}, J) for odd $n \geq 3$ and any fixed almost contact structure J. For even n, Uebele [100] showed that there exist infinitely many different contact structures on (S^7, J_{std}), (S^{11}, J_{std}) and (S^{15}, J_{std}). The constructions in [101, 100] use the Brieskorn spheres, which can not be filled by flexible Weinstein domains, see [71, Corollary 1.12] and Theorem 18.17. In contrast, Lazarev [71] constructed infinitely many contact structures on (S^{2n-1}, J_{std}) by taking contact connected sums of the exotic contact structure on (S^{2n-1}, J_{std}) from [46, 28], and all of those exotic contact structures can be filled with flexible Weinstein domains.

In this section, we construct infinitely many different contact structures on (S^{2n-1}, J) for $n \geq 3$, which can not be filled by flexible Weinstein domains. This construction allows fixing any almost contact structure J with the exception of a small class in case $n = 8, 12, 16, \ldots$. More precisely, we can fix any $J \in \mathcal{C}_{2n-1}$ in a subset $\mathcal{C}_{2n-1}$ of almost contact structures on S^{2n-1} as follows. Recall that an almost contact structure J on an oriented closed manifold M is a reduction of the structure group $SO(2n-1)$ to $U(n-1) \times \{\mathrm{id}\}$. If α is a contact form for (Y, ξ), then an almost complex structure J on ξ compatible with $\mathrm{d}\alpha$ induces an almost contact structure on Y. Therefore a co-oriented contact structure induces a unique almost contact structure. The almost contact structures on S^{2n-1} are in one-to-one correspondence with $\pi_{2n-1}(SO(2n)/U(n))$ by [85], and by a result from Massey [77],

$$\pi_{2n-1}(SO(2n)/U(n)) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } n \equiv 0 \mod 4 \\ \mathbb{Z}_{(n-1)!} & \text{if } n \equiv 1 \mod 4 \\ \mathbb{Z} & \text{if } n \equiv 2 \mod 4 \\ \mathbb{Z}_{\frac{(n-1)!}{2}} & \text{if } n \equiv 3 \mod 4 \end{cases} \tag{18.8}$$

Let ρ denote the bijection from the set of almost contact structures on S^{2n-1} to $\pi_{2n-1}(SO(2n)/U(n))$, then $\rho(J_{std}) = 0$. Moreover, ρ is additive with respect to the operation of contact connected sum, i.e. if ξ_1, ξ_2 are two contact structures on S^{2n-1}, and $\xi_1 \# \xi_2$ is their contact connected sum, then $\rho(\xi_1 \# \xi_2) = \rho(\xi_1) + \rho(\xi_2)$; see [100].

Now, we define the set $\mathcal{C}_{2n-1}$ to be all possible almost contact structures on S^{2n-1} for $n \equiv 1, 2, 3 \mod 4$ or $n = 4$. When $n \equiv 0 \mod 4$ and $n \geq 8$, we define $\mathcal{C}_{2n-1} := \rho^{-1}(\mathbb{Z} \times \{0\})$ to be the set of almost contact structures represented by the $\mathbb{Z}$ summand in (18.8).

Theorem 18.18. *For $n \geq 3$ and $J \in \mathcal{C}_{2n-1}$, there exist infinitely many contact structures on (S^{2n-1}, J), such that none of them can be filled by flexible Weinstein domains.*

Proof. Our construction is a modification of the construction in [71, Corollary 1.12]. Hence it is also based on the contact structures in [28, Theorem 2], i.e. for each $J \in \mathcal{C}_{2n-1}$, there is a contact structure on (S^{2n-1}, J) with a flexible Weinstein filling W_J, such that $H^*(W_J, \mathbb{Q}) \neq H^*(D^{2n}, \mathbb{Q})$, i.e. W_J has nontrivial rational cohomology in some degree greater than 0.

Step 1: *There exists a Weinstein domain $M_{J_{std}}$, such that the contact boundary $\partial M_{J_{std}}$ is asymptotically dynamically convex and induces the standard almost contact structure J_{std} on S^{2n-1}. Moreover, $SH_*(M_{J_{std}};\mathbb{Z}) \neq 0$ and $SH^+_*(M_{J_{std}};\mathbb{Q})$ is locally finite.* The Weinstein domain $M_{J_{std}}$ is constructed as follows: Let Σ be the Brieskorn sphere $\Sigma(3,2,\ldots,2)$ and V be its Weinstein filling (18.7). Ustilovsky [101, Lemma 4.1, 4.2] constructed an explicit regular contact form α' on Σ, which is a perturbation of (18.6), such that all the Reeb orbits of α' has positive Conley-Zehnder index and for each integer there are finite many Reeb orbits with that integer as the Conley-Zehnder index[1]. In particular, $\mu_{CZ}(\gamma)+n-3>0$ for all Reeb orbits of α'. Therefore, if we take $\alpha_i = \alpha'$ in definition 18.5, Σ is asymptotically dynamically convex. $SH_*(V;\mathbb{Q})$ is locally finite because the chain complex is locally finite. By [100, Theorem 2.1], Σ is homeomorphic to a sphere but may not be the standard smooth sphere. But there exists $k>0$, such that $\#^k\Sigma$ is the standard smooth sphere. Note that for any almost contact structure J on S^{2n-1}, the connected sum $J\#J$ is in $\mathcal{C}_{2n-1}$. Hence we can choose k, such that $\#^k\Sigma$ is the standard smooth sphere and the almost contact structure on $\#^k\Sigma$ is in $\mathcal{C}_{2n-1}$. Since $\rho(\mathcal{C}_{2n-1})$ is a group and ρ respects the group structures, we can find $J' \in \mathcal{C}_{2n-1}$, such that $\left(\#^k\Sigma\right)\#\partial W_{J'} \simeq (S^{2n-1}, J_{std})$ as almost contact manifold. We define $M_{J_{std}}$ to be the Weinstein filling $\left(\natural^k V\right)\natural W_{J'}$. Since both Σ and $\partial W_{J'}$ are asymptotically dynamically convex, $\partial M_{J_{std}}$ is asymptotically dynamically convex by Proposition C.7. Since $SH_*(V;\mathbb{Q})$ is locally finite, by Proposition 18.11, $SH_*(M_{J_{std}};\mathbb{Q})$ is locally finite. Then $SH^+_*(M_{J_{std}};\mathbb{Q})$ is locally finite following from Proposition 18.8. At last, $SH_*(M_{J_{std}};\mathbb{Z}) = SH_*(\left(\natural^k V\right)\natural W_{J'};\mathbb{Z}) = \oplus^k SH_*(V;\mathbb{Z}) \neq 0$ by [69, Theorem 1.2].

Step 2: For each $J \in \mathcal{C}_{2n-1}$ and each $i \in \mathbb{N}$, we define

$$X_{i,J} := \left(\natural^i W_{J_{std}}\right)\natural W_J \natural M_{J_{std}}$$

Then the contact boundary $\partial X_{i,J}$ induces the almost contact structure J on S^{2n-1}. Since $\partial W_{J_{std}}$, $M_{J_{std}}$ and ∂W_J are asymptotically dynamically convex, so it $\partial X_{i,J}$ by Proposition C.7. By Proposition 18.11, $SH_*(X_{i,J};\mathbb{Q}) = SH_*(M_{J_{std}};\mathbb{Q})$ is locally finite. Then $SH^+_*(X_{i,J};\mathbb{Q})$ is locally finite by Proposition 18.8.

Step 3: *For each $J \in \mathcal{C}_{2n-1}$, $X_{i,J}$ is not contactomorphic to $X_{j,J}$ for $i<j$.* Since $X_{i,J}$ is asymptotically dynamically convex, by Proposition 18.9 it suffices to show that $SH^+_*(X_{i,J};\mathbb{Q}) \not\cong SH^+_*(X_{j,J};\mathbb{Q})$ for $i<j$. Note that $X_{j,J} = (\natural^{j-i}W_{J_{std}})\natural X_{i,J}$. Let ι be the inclusion $\iota : X_{i,J} \to X_{j,J}$, then we have the following commutative diagram of long exact

[1] In [101], only odd n was considered. However, the argument for [101, Lemma 4.1, 4.2] can also be applied to even n.

sequences; see [24],

$$\begin{tikzcd} \ldots \arrow[r] & H^{n-\star}(X_{j,J};\mathbb{Q}) \arrow[r] \arrow[d, "\iota^*"] & SH_\star(X_{j,J};\mathbb{Q}) \arrow[r] \arrow[d, "\iota^*_{SH}"] & SH^+_\star(X_{j,J};\mathbb{Q}) \arrow[r] \arrow[d, "\iota^*_{SH^+}"] & \ldots \\ \ldots \arrow[r] & H^{n-\star}(X_{i,J};\mathbb{Q}) \arrow[r] & SH_\star(X_{i,J};\mathbb{Q}) \arrow[r] & SH^+_\star(X_{i,J};\mathbb{Q}) \arrow[r] & \ldots \end{tikzcd}$$

where $\iota^*_{SH}, \iota^*_{SH^+}$ are the Viterbo transfer maps. By Proposition 18.11, ι^*_{SH} is an isomorphism. By the Mayer-Vietoris sequence, we have

$$H^k(X_{j,J};\mathbb{Q}) = H^k\left((\natural^{j-i}W_{J_{std}})\natural X_{i,J};\mathbb{Q}\right) \cong \left(\oplus^{j-i}H^k(W_{j_{std}};\mathbb{Q})\right) \oplus H^k(X_{i,J}:\mathbb{Q}) \quad \forall k \geq 1. \tag{18.9}$$

Moreover, $\iota^* : H^k(X_{j,J};\mathbb{Q}) \to H^k(X_{i,J};\mathbb{Q})$ is the projection in (18.9). Recall that $H^\star(W_{J_{std}},\mathbb{Q}) \neq H^\star(D^{2n},\mathbb{Q})$, we have ι^* is surjective but not injective. By the four lemma [76], $\iota^*_{SH^+}$ is also surjective. By the five lemma [76], $\iota^*_{SH^+}$ can not be an isomorphism, since ι^* is not an isomorphism. Since $SH^+_\star(X_{j,J};\mathbb{Q})$ and $SH^+_\star(X_{i,J};\mathbb{Q})$ are locally finite, $\iota^*_{SH^+}$ being surjective but not injective implies that there is an integer d, such that $\dim_{\mathbb{Q}} SH^+_d(X_{j,J};\mathbb{Q}) > \dim_{\mathbb{Q}} SH^+_d(X_{i,J};\mathbb{Q})$. Therefore $SH^+_\star(X_{j,J};\mathbb{Q})$ is not isomorphic to $SH^+_\star(X_{i,J};\mathbb{Q})$, $\partial X_{j,J}$ is not contactomorphic to $\partial X_{i,J}$ by Proposition 18.9.

Step 4: Since $SH_*(X_{i,J};\mathbb{Z}) = SH_*((\natural^i W_{J_{std}})\natural W_J \natural M_{J_{std}};\mathbb{Z}) = SH_*(M_{J_{std}};\mathbb{Z}) \neq 0$. Note that we have $c_1(\partial X_{i,J} \simeq S^{2n-1}, J) = 0$, because $H^2(S^{2n-1};\mathbb{Z}) = 0$. So $\partial X_{i,J}$ can not be filled by flexible Weinstein domains by Corollary 18.16. This finishes the proof for Theorem 18.18. □

Remark 18.19. *When $n \geq 4$, a modified version of [71, Theorem 1.14] can also be used to prove Theorem 18.18. The original version uses a family of exotic contact structures on S^{2n-1} . Those exotic structures are the boundary of plumbings of one exotic T^*S^n from [32] and many copies of flexible T^*S^n. When n is odd, one can require the resulted almost contact structure is standard. If n is even, this construction yields a (fixed) nonstandard almost contact structure J_P on S^{2n-1}. But we can take connected sum with one of the contact structures constructed in [28] to modify J_P back to the standard almost contact structure J_{std}. Therefore we can require that the construction in [71, Theorem 1.14] does not change the almost contact structure for all $n \geq 4$.*

Corollary 18.20. *Let (Y, J) be an almost contact manifold admitting an almost Weinstein filling W with $c_1(W) = 0$. Then there exist infinitely many different contact structures on (Y, J), and none of them can be filled by flexible Weinstein domains.*

Proof. By the h-principle for flexible Weinstein domains [22], we can put a flexible Weinstein structure on W. The contact boundary $\partial(W \natural X_{i,J_{std}})$ represents the almost contact structure (Y, J) and is asymptotically dynamically convex by Proposition C.7 and Proposition 18.7. By the same argument used in the proof of Theorem 18.18, $\partial(W \natural X_{i,J_{std}})$ is not contactomorphic to $\partial(W \natural X_{j,J_{std}})$ for $i < j$. Since $SH_*(W \natural X_{i,J_{std}};\mathbb{Z}) \cong SH_*(X_{i,J_{std}};\mathbb{Z}) \neq 0$, $\partial(W \natural X_{i,J_{std}})$ can not be filled by flexible Weinstein domains by Corollary 18.16. □

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Appendix A

Local Lifts, Stabilization and Averaging

A.1 Properties of local lifts

First, we have the following properties of the local action L_ϕ, which follows from directly from Definition 4.2.

Proposition A.1. *Let $F : (\mathcal{X}, \boldsymbol{X}) \supset \mathrm{stab}_x \ltimes \mathcal{U} \to (\mathcal{Y}, \boldsymbol{Y})$ be a fully faithful sc-smooth functor from a local uniformizer $\mathrm{stab}_x \ltimes \mathcal{U}$ around $x \in \mathcal{X}$ such that $F^0 : \mathcal{U} \to \boldsymbol{Y}$ is a local sc-diffeomorphism. Then the following holds.*

1. *Let F_x denote the local diffeomorphism F near x. For $\phi \in \boldsymbol{X}$ with $s(\phi) = x$, then there exits a neighborhood $\mathcal{V} \subset \mathcal{X}$ of x such that*
$$L_{F(\phi)} \circ F_x = F_{\phi(x)} \circ L_\phi \text{ on } \mathcal{V}. \tag{A.1}$$
2. *For composable $\phi, \psi \in \boldsymbol{X}$,there exists a neighborhood $\mathcal{V} \subset \mathcal{X}$ of $s(\psi)$ such that*
$$L_\phi \circ L_\psi = L_{\phi\circ\psi} \text{ on } \mathcal{V}. \tag{A.2}$$
3. *If L_ϕ is defined over $\mathcal{V}$, then there exists a neighborhood $\boldsymbol{V}$ of $\phi \in \boldsymbol{X}$ such that for every $\psi \in \boldsymbol{V}$, L_ψ is defined on $\mathcal{V}$ and*
$$L_\phi = L_\psi \text{ on } \mathcal{V}. \tag{A.3}$$

We first prove a direct corollary of the definition of regularity (Definition 4.13).

Lemma A.2. *Let $(\mathcal{X}, \boldsymbol{X})$ be a regular ep-groupoid. Let x, y be two points in $\mathcal{X}$ and $\mathcal{U}_x$,$\mathcal{U}_y$ two local uniformizers around x resp. y. If there is a sc-smooth map $F : \mathcal{U}_x \to \mathcal{U}_y$ such that*

1. *F is a sc-diffeomorphism from $\mathcal{U}_x$ to an open subset;*

2. *$|F(z)| = |z|$ for every $z \in \mathcal{U}_x$ and $F(x) = y$.*

Then there exists a $\theta \in \mathrm{mor}(x, y)$ and a local uniformizer $\mathcal{V}$ around x, such that $F|_{\mathcal{V}} = L_\theta$.

Proof. Since $|y| = |F(x)| = |x|$, there exists $\theta_0 \in \mathrm{Mor}(x, y)$. Then there is a open neighborhood $\mathcal{V}$ of x such that L_{θ_0} is defined on $\mathcal{V}$ with images in $\mathcal{U}_y$, and $\mathcal{U} := F(\mathcal{V}) \subset \mathcal{U}_y$ is a connected regular uniformizer around y. For every $z \in \mathcal{V}$, we have $|L_{\theta_0}(z)| = |z| = |F(z)|$. Since $L_{\theta_0}(z)$ and $F(z)$ are in the local uniformizer $\mathcal{U}_y$, then for any $z \in \mathcal{V}$, there exists a $\phi_z \in \mathrm{stab}_y$, such that

$$L_{\theta_0}(z) = L_{\phi_z} \circ F(z). \tag{A.4}$$

Then $L_{\phi_{F^{-1}(u)}}(u) = L_{\theta_0} \circ F^{-1}(u)$ for $u \in \mathcal{U}$. In particular by the regular property of $(\mathcal{X}, \boldsymbol{X})$,

$$L_{\phi_{F^{-1}(u)}}(u) = L_\phi(u)$$

for some $\phi \in \mathrm{stab}_y$. Then (A.4) induces:

$$L_{\phi^{-1}\circ\theta_0}(z) = F(z) \quad \forall z \in \mathcal{V}$$

□

A typical scenario of applying this lemma is when we have two equivalences of ep-groupoids $F, G : (\mathcal{W}, \boldsymbol{W}) \to (\mathcal{X}, \boldsymbol{X})$ such that $|F| = |G|$. Then for $w_1, w_2 \in \mathcal{W}$ and $\phi \in \mathrm{mor}(w_1, w_2)$, then there exists $\theta \in \mathrm{mor}(F(w_1), G(w_2))$ such that locally:

$$L_\theta = G_2 \circ L_\phi \circ F_{w_1}^{-1}$$

In the bundle case, we have the following analogue of Lemma A.2 with an identical proof.

Lemma A.3. *Let $(\mathcal{E}, \boldsymbol{E}) \xrightarrow{P} (\mathcal{X}, \boldsymbol{X})$ be a regular strong ep-groupoid bundle. Let x, y be two points in $\mathcal{X}$ and $\mathcal{U}_x$,$\mathcal{U}_y$ two local uniformizers around x resp. y. If there is a sc-smooth strong bundle map $F : P^{-1}(\mathcal{U}_x) \to P^{-1}(\mathcal{U}_y)$ such that*

1. *F is a sc-diffeomorphism from $P^{-1}(\mathcal{U}_x)$ to an open subset;*

2. *$|F(v)| = |v|$ for every $v \in P^{-1}(\mathcal{U}_x)$ and $F(x, 0) = (y, 0)$.*

Then there exists a $\theta \in \mathrm{mor}(x, y)$ and a local uniformizer $\mathcal{V}$ around x, such that $F|_{P^{-1}(\mathcal{V})} = R_\theta$.

To prove Proposition 4.33, we need to argue that the local representation $\rho_{g,x}$ in Definition 4.27 can be chosen as a family of embeddings and the inverses of $\rho_{g,x}$ can be defined on a uniform open set, i.e. we have the following.

Proposition A.4. *Let $\rho_{g,x} : U \times (\mathrm{stab}_x \ltimes \mathcal{U}) \to (\mathcal{X}, \boldsymbol{X})$ be a local representation of a sc-smooth group action near g, x on a connected regular uniformizer $\mathrm{stab}_x \ltimes \mathcal{U}$. Then we have*

1. *for each $h \in U$, $\rho_{g,x}(h,\cdot) : \mathrm{stab}_x \ltimes \mathcal{U} \to (\mathcal{X}, \boldsymbol{X})$ is a fully faithful functor and on object space and $\rho_{g,x}(h,\cdot)$ is a sc-diffeomorphism onto an open subset;*

2. *there exists a neighborhood $U' \subset U$ of g, such that $\cap_{h\in U'}\rho_{g,x}(h,\mathcal{U})$ contains an open set $\mathcal{V} \subset \mathcal{X}$, which contains $\{\rho_{g,x}(h,x)|h \in U'\}$.*

As a corollary of Proposition A.4, $\rho_{g,x}(h,\cdot)^{-1}$ is defined on $\mathcal{V}$ for all $h \in U'$ and contains x in its image.

Proof. For the first assertion, for each $h \in G$, $\mathfrak{P}(h) : Z \to Z$ is an isomorphism of polyfolds. As a local representation of such map, $\rho_{g,x}(h,\cdot) : \mathrm{stab}_x \ltimes \mathcal{U} \to (\mathcal{X}, \boldsymbol{X})$ is fully faithful and local diffeomorphism on objects. To show it is a sc-diffeomorphism on objects, we only need to show it is injective on objects. Assume otherwise, i.e. there exist $h \in U, y, z \in \mathcal{U}$, such that $\rho_{g,x}(h,y) = \rho_{g,x}(h,z)$. Since $\rho_{g,x}(h,\cdot)$ is local diffeomorphism, we have $\rho_{g,x}(h,\cdot)|_z^{-1} \circ \rho_{g,x}(h,\cdot)$ is well defined in a neighborhood of y, where $\rho_{g,x}(h,\cdot)|_z^{-1}$ is the inverse of $\rho_{g,x}(h,\cdot)$ near z. By Lemma A.2, $\rho_{g,x}(h,\cdot)|_z^{-1} \circ \rho_{g,x}(h,\cdot) = L_\phi$ near y for some $\phi \in \mathrm{Mor}(y,z)$. Since $\mathcal{U}$ is a local uniformizer, $L_\phi = L_\psi$ for $\psi \in \mathrm{stab}_x$ by Corollary 4.5. Such ψ is unique in $\mathrm{stab}_x^{\mathrm{eff}}$ because if we have another ψ' such that $L_\psi = L_{\psi'}$ near y, then $L_{\psi^{-1}\circ\psi'}$ is identity near y. By the regular property (Definition 4.11), $L_{\psi^{-1}\circ\psi'} = \mathrm{id}$ on $\mathcal{U}'$, i.e. ψ and ψ' induce the same element in $\mathrm{stab}_x^{\mathrm{eff}}$. Next we consider a set S:

$$S := \{p \in \mathcal{U}|\rho_{g,x}(h,\cdot) = \rho_{g,x}(h,\cdot) \circ L_\psi \text{ in a neighborhood of } p\}.$$

It is clear that S is open. We claim S is also closed. Suppose we have $\lim_i x_i = \overline{x}$ for $x_i \in S$ and $\overline{x} \in \mathcal{U}$. Since $\rho_{g,x}(h,x_i) = \rho_{g,x}(h,L_\psi(x_i))$, we have $\rho_{g,x}(h,\overline{x}) = \rho_{g,x}(h,L_\psi(\overline{x}))$. Therefore by the same argument before, there exists a unique $\eta \in \mathrm{stab}_x^{\mathrm{eff}}$ such that $\rho_{g,x}(h,\cdot)|_{L_\psi(\overline{x})}^{-1} \circ \rho_{g,x}(h,\cdot) = L_\eta$ near $\overline{x}$. Since $\rho_{g,x}(h,\cdot)|_{L_\psi(\overline{x})}^{-1} = \rho_{g,x}(h,\cdot)|_{L_\psi(x_i)}^{-1}$ for $i \gg 0$, we have $\rho_{g,x}(h,\cdot)|_{L_\psi(x_i)}^{-1} \circ \rho_{g,x}(h,\cdot) = L_\eta$ near x_i for $i \gg 0$. By assumption $\rho_{g,x}(h,\cdot)|_{L_\psi(x_i)}^{-1} \circ \rho_{g,x}(h,\cdot) = L_\psi$ near x_i. We must have $\eta = \psi$ in $\mathrm{stab}_x^{\mathrm{eff}}$. Therefore S is closed. Since $\mathcal{U}$ is connected and S is not empty by assumption, we have $S = \mathcal{U}$. Since $L_\psi \neq \mathrm{id}$ on any open subset $\mathcal{U}$, there exists $y_i \to x$ such that $L_\psi(y_i) \neq y_i$. This contradicts $\rho_{g,x}(h,\cdot)$ is a local diffeomorphism near x.

To prove the second assertion, we first pick a local uniformizer $\mathcal{W}$ around $y := \rho_{g,x}(g,x)$. Then there exist open neighborhood $U' \subset U$ of g and $\mathcal{U}' \subset \mathcal{U}$ of x such that $\rho_{g,x}(h,\mathcal{U}') \subset \mathcal{W}$ for every $h \in U'$. Since a polyfold is a metrizable space, by the proof of Lemma 3.31, we may choose the metric d to be G-invariant. Because open set $|\mathcal{U}'| \subset Z$ contains a r-ball around $|x|$. Then we can choose the neighborhood $U' \subset U$ of g small enough, such that the following two conditions hold:

- U' is connected;
- $\cup_{h\in U'}\rho(h,|x|)$ has diameter smaller than $\frac{r}{2}$.

Since $\rho(h,B_r(|x|))$ is a r-ball centered at $\rho(h,|x|)$, we have $B_{\frac{r}{2}}(|y|) \subset \cap_{h\in U'}\rho(h,B_r(|x|))$. Let $\mathcal{V} \subset \mathcal{W}$ be the neighborhood of y such that $|\mathcal{V}| = B_{\frac{r}{2}}(|y|)$. We claim that $\mathcal{V} \subset \rho_{g,x}(h,\mathcal{U}')$ for every $h \in U'$. Since on object level we have $|\mathcal{V}| \subset \rho(h, |\mathcal{U}'|)$ and $\rho_{g,x}(h, \mathcal{U}') \subset \mathcal{W}$, for every $z \in \mathcal{V}$ there exists a $\phi \in \operatorname{stab}_y$, such that $L_\phi(z) \in \rho_{g,x}(h, w)$ for some $w \in \mathcal{U}'$. Since $\rho_{g,x}(h, \cdot)$ is a fully faithful functor, we have $\operatorname{stab}_w \simeq \operatorname{stab}_{L_\phi(z)}$. Therefore the stab_x-orbit of w in $\mathcal{U}'$ has size $|\operatorname{stab}_x|/|\operatorname{stab}_w|$ and the stab_y-orbit of $L_\phi(z)$ in $\mathcal{V}$ has size $|\operatorname{stab}_y|/|\operatorname{stab}_{L_\phi(z)}| = |\operatorname{stab}_x|/|\operatorname{stab}_w|$. Because $\rho_{g,x}(h, \cdot)$ maps the stab_x-orbit of w into the stab_y-orbit of $L_\phi(z)$ and $\rho_{g,x}(h, \cdot)$ is injective by the first assertion, we must have $z \in \rho_{g,x}(h, \mathcal{U})$ as z is in the stab_y-orbit of $L_\phi(z)$. Therefore $\mathcal{V} \subset \rho_{g,x}(h, \mathcal{U})$ for every $h \in U'$. □

Proposition 4.33. *Let Z be a regular polyfold with a G-action $(\rho, \mathfrak{P})$ and $(\mathcal{X}, \boldsymbol{X})$ a polyfold structure of $Z = |\mathcal{X}|$. Then for $x, y \in \mathcal{X}$ and $g \in G$, the isotropy* stab_x *acts on $L(x, y, g)$ by precomposing the local sc-diffeomorphism L_ϕ for $\phi \in \operatorname{stab}_x$ and this action is transitive. The isotropy* stab_y *also acts on $L(x, y, g)$ by post-composing L_ϕ for $\phi \in \operatorname{stab}_y$, this action is also transitive.*

Proof. We first recall the definition of local lifts from Subsection 4.3. For $x, y \in \mathcal{X}$ and $g \in G$ such that $\rho(g, |x|) = |y|$. We have two equivalent polyfold structures $(\mathcal{X}_a, \boldsymbol{X}_a)$,$(\mathcal{X}_b, \boldsymbol{X}_b)$ and a local uniformizer $\mathcal{U}_a \subset \mathcal{X}_a$ around x_a, such that $|x_a| = x$ and the action can be locally represented by

$$\rho_{g,x_a} : U \times (\operatorname{stab}_{x_a} \ltimes \mathcal{U}_a) \to (\mathcal{X}_b, \boldsymbol{X}_b),$$

where $U \subset G$ is a neighborhood of g. Since $(\mathcal{X}_a, \boldsymbol{X}_a), (\mathcal{X}_b, \boldsymbol{X}_b)$ and $(\mathcal{X}, \boldsymbol{X})$ are equivalent polyfold structures, we have the following diagram of equivalences as in (4.4) (we suppress the morphism space from now on):

$$\mathcal{X} \xleftarrow{F_a} \mathcal{W}_a \xrightarrow{G_a} \mathcal{X}_a \xrightarrow{\rho_{g,x_a}(\cdot,\cdot)} \mathcal{X}_b \xleftarrow{F_b} \mathcal{W}_b \xrightarrow{G_b} \mathcal{X}$$

A local lift Γ at (g, x, y) is constructed by first finding $w_a \in \mathcal{W}_a, w_b \in \mathcal{W}_b, \phi, \psi \in \boldsymbol{X}, \delta \in \boldsymbol{X}_a, \eta \in \boldsymbol{X}_b$, such that $F_a(w_a) = \phi(x)$, $\delta(G_a(w_a)) = x_a$, $\eta(\rho_{g,x_a}(g, x_a)) = F_b(w_b)$ and $G_b(w_b) = \psi(y)$. Let F_{a,w_a} resp. F_{b,w_b} denote the local diffeomorphisms F_a resp. F_b near w_a resp. w_b. We define for z close to x and h close to g,

$$\Gamma(h, z) := L_{\psi^{-1}} \circ G_b \circ F_{b,w_b}^{-1} \circ L_\eta \circ \rho_{g,x_a}(h, \cdot) \circ L_\delta \circ G_a \circ F_{a,w_a}^{-1} \circ L_\phi(z). \tag{A.5}$$

Case one - different choices of $w_a, w_b, \phi, \psi, \delta, \eta$: Assume we have two sets of different choices inducing two local lifts Γ, Γ'.

$$
\begin{tikzcd}
\mathcal{X} & \mathcal{W}_a \arrow[l, "F_a"'] \arrow[r, "G_a"] & \mathcal{X}_a \arrow[r, "\rho_{g,x_a}(g,\cdot)"] & \mathcal{X}_b & \mathcal{W}_b \arrow[l, "F_b"'] \arrow[r, "G_b"] & \mathcal{X}
\end{tikzcd}
$$

$$
\begin{tikzcd}
x \arrow[d, "\phi"'] & & x_a \arrow[r] & \rho_{g,x_a}(g, x_a) \arrow[d, "\eta"] & & y \arrow[d, "\psi"] \\
F_a(w_a) & w_a \arrow[l] \arrow[r] \arrow[dd, "\theta_a"'] & G_a(w_a) \arrow[u, "\delta"] & F_b(w_b) & w_b \arrow[l] \arrow[r] \arrow[dd, "\theta_b"] & G_b(w_b) \\
x \arrow[d, "\phi'"'] & & x_a \arrow[r] & \rho_{g,x_a}(g, x_a) \arrow[d, "\eta'"] & & y \arrow[d, "\psi'"] \\
F_a(w'_a) & w'_a \arrow[l] \arrow[r] & G_a(w'_a) \arrow[u, "\delta'"] & F_b(w'_b) & w'_b \arrow[l] \arrow[r] & G_b(w'_b)
\end{tikzcd}
$$

Since F_a is an equivalence, F_a induces a homeomorphism on the orbit spaces, so we can find $\theta_a \in \boldsymbol{W}_a$ such that $\theta_a(w_a) = w'_a$. Moreover $F_b : \mathrm{mor}(w_b, w'_b) \to \mathrm{mor}(F_b(w_b), F_b(w_b)')$ is a bijection. Thus there exists $\theta_b \in \mathrm{mor}(w_b, w'_b)$, such that

$$F_b(\theta_b) = \eta' \circ \rho_{g,x_a}(g, (\delta' \circ G_a(\theta_a) \circ \delta^{-1})) \circ \eta^{-1}. \tag{A.6}$$

Here $\delta' \circ G_a(\theta_a) \circ \delta^{-1} \in \mathrm{mor}(x_a, x_a)$ and $\rho_{g,x_a}(g, \cdot)$ is the functor $\mathrm{stab}_{x_a} \ltimes \mathcal{U}_b \to (\mathcal{X}_b, \boldsymbol{X}_b)$, hence it can acts on a morphism. The following computation is made for z close for x and h close to g, or equivalently one can treat them as germs.

$$
\begin{aligned}
\Gamma(h, z) \quad &\overset{(A.5)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_\eta \circ \rho_{g,x_a}(h, \cdot) \circ L_\delta \circ G_a \circ F^{-1}_{a,w_a} \circ L_\phi(z) \\
&\overset{(A.1)+(A.2)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_\eta \circ \rho_{g,x_a}(h, \cdot) \circ L_\delta \circ G_a \circ L_{\theta_a^{-1}} \circ F^{-1}_{a,w'_a} \circ L_{F_a(\theta_a)\circ\phi}(z) \\
&\overset{(A.1)+(A.2)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_\eta \circ \rho_{g,x_a}(h, \cdot) \circ L_{\delta\circ G_a(\theta_a^{-1})\circ\delta'^{-1}} \circ L_{\delta'} \circ G_a \circ F^{-1}_{a,w'_a} \circ L_{F_a(\theta_a)\circ\phi}(z) \\
&\overset{(A.1)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_{\eta\circ\rho_{g,x_a}(h,\delta\circ G_a(\theta_a^{-1})\circ\delta'^{-1})} \circ \rho_{g,x_a}(h, \cdot) \circ L_{\delta'} \circ G_a \circ F^{-1}_{a,w'_a} \circ L_{F_a(\theta_a)\circ\phi}(z) \\
&\overset{(A.3)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_{\eta\circ\rho_{g,x_a}(g,\delta\circ(G_a(\theta_a^{-1})\circ\delta'^{-1})} \circ \rho_{g,x_a}(h, \cdot) \circ L_{\delta'} \circ G_a \circ F^{-1}_{a,w'_a} \circ L_{F_a(\theta_a)\circ\phi}(z) \\
&\overset{(A.6)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_{F_b(\theta_b^{-1})\circ\eta'} \circ \rho_{g,x_a}(h, \cdot) \circ L_{\delta'} \circ G_a \circ F^{-1}_{a,w'_a} \circ L_{F_a(\theta_a)\circ\phi}(z) \\
&\overset{(A.2)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_{F_b(\theta_b^{-1})} \circ L_{\eta'} \circ \rho_{g,x_a}(h, \cdot) \circ L_{\delta'} \circ G_a \circ F^{-1}_{a,w'_a} \circ L_{F_a(\theta_a)\circ\phi}(z) \\
&\overset{(A.1)}{=} L_{\psi^{-1}} \circ G_b \circ L_{\theta_b^{-1}} \circ F^{-1}_{b,w'_b} \circ L_{\eta'} \circ \rho_{g,x_a}(h, \cdot) \circ L_{\delta'} \circ G_a \circ F^{-1}_{a,w'_a} \circ L_{F_a(\theta_a)}(z) \circ \phi \\
&\overset{(A.1)}{=} L_{\psi^{-1}\circ G_b(\theta_b^{-1})} \circ G_b \circ F^{-1}_{b,w'_b} \circ L_{\eta'} \circ \rho_{g,x_a}(h, \cdot) \circ L_{\delta'} \circ G_a \circ F^{-1}_{a,w'_a} \circ L_{F_a(\theta_a)\circ\phi}(z) \\
&= L_{\psi^{-1}\circ G_b(\theta_b^{-1})\circ\psi'} \circ \Gamma'(h, \cdot) \circ L_{\phi'^{-1}\circ F_a(\theta_a)\circ\phi}(z).
\end{aligned}
$$

So far, we prove that:

$$\Gamma = L_\beta \circ \Gamma' \circ L_\alpha$$

for $\alpha = \phi'^{-1} \circ F_a(\theta_a) \circ \phi \in \mathrm{stab}_x, \beta = \psi^{-1} \circ G_b(\theta_b^{-1}) \circ \psi' \in \mathrm{stab}_y$. By the similar arguments as above, one can move L_β inside to get

$$\Gamma = \Gamma' \circ L_{\beta'} \circ L_\alpha$$

for

$$\beta' = \phi^{-1} \circ F_a(G_a^{-1}(\delta^{-1} \circ \rho_{g,x_a}(g, \eta^{-1} \circ F_b(G_b^{-1}(\psi \circ \beta \circ \psi^{-1})) \circ \eta) \circ \delta)) \circ \phi \in \mathrm{stab}_x\,.$$

Therefore different choices of $w_a, w_b, \phi, \psi, \delta, \eta$ give different germs of local lifts up to pre-composing the stab_x-action. And post-composing with stab_y-action can be rewritten as precomposing the stab_x action. The transitivity for the pre-composition action is equivalent to the transitivity for the post-composition action.

Case two-different choices of equivalent polyfold structures $\mathcal{W}_a$ and $\mathcal{W}_b$: If we have two ep-groupoids $\mathcal{W}_a, \mathcal{W}_a'$ along with equivalences $F_a : \mathcal{W}_a \to \mathcal{X}$, $G_a : \mathcal{W}_a \to \mathcal{X}_a$, $F_a' : \mathcal{W}_a' \to \mathcal{X}$ and $G_a' : \mathcal{W}_a' \to \mathcal{X}_a$. Let $\mathcal{W}_a \times_{\mathcal{X}} \mathcal{W}_a'$ denote the weak fiber product [65, Definition 10.3] of F_a and F_a', then by [65, Theorem 10.2], we have two equivalences $\pi : \mathcal{W}_a \times_{\mathcal{X}} \mathcal{W}_a' \to \mathcal{W}_a$ and $\pi' : \mathcal{W}_a \times_{\mathcal{X}} \mathcal{W}_a' \to \mathcal{W}_a'$. Therefore, we have the following (not necessarily commutative) diagram:

$$\begin{tikzcd}
 & \mathcal{W}_a \arrow[dl, "F_a"'] \arrow[dr, "G_a"] & & & & \\
\mathcal{X} & \mathcal{W}_a \times_{\mathcal{X}} \mathcal{W}_a' \arrow[u, "\pi"] \arrow[d, "\pi'"'] & \mathcal{X}_a \arrow[r, "\rho_{g,x_a}(g,\cdot)"] & \mathcal{X}_b & \mathcal{W}_b \arrow[l, "F_b"'] \arrow[r, "G_b"] & \mathcal{X} \\
 & \mathcal{W}_a' \arrow[ul, "F_a'"] \arrow[ur, "G_a'"'] & & & &
\end{tikzcd}$$

We can choose $w \in \mathcal{W}_a \times_{\mathcal{X}} \mathcal{W}_a'$ such that $|w| = |x|$ in Z, let $w_a := \pi(w) \in \mathcal{W}_a, w_a' := \pi'(w) \in \mathcal{W}_a'$ Then we can find $\phi, \phi' \in \boldsymbol{X}$, $\delta, \delta' \in \boldsymbol{X}_a$, $\eta \in \boldsymbol{X}_b$, $w_b \in \mathcal{W}_b$ and $\psi \in \boldsymbol{X}$, such that we have the following diagram:

$$\begin{tikzcd}
F_a(w_a) & w_a \arrow[l, "F_a"'] \arrow[r, "G_a"] & G_a(w_a) \arrow[d, "\delta"] & & & \\
x \arrow[u, "\phi"] \arrow[d, "\phi'"'] & w \arrow[u, "\pi"] \arrow[d, "\pi'"'] & x_a \arrow[r, "\rho_{g,x_a}(g,\cdot)"] & \rho_{g,x_a}(g, x_a) \arrow[d, "\eta"] & & y \arrow[d, "\psi"] \\
F_a'(w_a') & w_a' \arrow[l, "F_a'"'] \arrow[r, "G_a'"] & G_a'(w_a') \arrow[u, "\delta'"] & F_b(w_b) & w_b \arrow[l, "F_b"'] \arrow[r, "G_b"] & G_b(w_b),
\end{tikzcd}$$

which define two different local lifts Γ and Γ'. We only need to prove Γ and Γ' are related by a pre-composition and post-composition of the stab_x-action.

By Lemma A.2, There exists $\theta_1 \in \mathrm{Mor}(F_a(w_a), F'_a(w'_a))$, $\theta_2 \in \mathrm{Mor}(G_a(w_a), G'_a(w'_a))$, such that

$$\begin{aligned} L_{\theta_1} &= F'_a \circ \pi' \circ \pi_w^{-1} \circ F^{-1}_{a,w_a} && (A.7)\\ L_{\theta_2} &= G'_a \circ \pi' \circ \pi_w^{-1} \circ G^{-1}_{a,w_a} && (A.8)\end{aligned}$$

If we write

$$\theta_3 := F_b^{-1}(\eta \circ \rho_{g,x_a}(g, \delta' \circ \theta_2 \circ \delta^{-1}) \circ \eta^{-1}) \in \mathrm{stab}_{w_b} \tag{A.9}$$

Then

$$\begin{aligned}
\Gamma(h,z) &= L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_\eta \circ \rho_{g,x_a}(h,\cdot) \circ L_\delta \circ G_a \circ F^{-1}_{a,w_a} \circ L_\phi(z)\\
&\overset{(A.7)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_\eta \circ \rho_{g,x_a}(h,\cdot) \circ L_\delta \circ G_b \circ \pi \circ \pi'^{-1}_w \circ F'^{-1}_{b,w'_b} \circ L_{\theta_1\circ\phi}(z)\\
&\overset{(A.8)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_\eta \circ \rho_{g,x_a}(h,\cdot) \circ L_\delta \circ L_{\theta_2^{-1}} \circ G'_a \circ F'^{-1}_{a,w'_a} \circ L_{\theta_1\circ\phi}(z)\\
&\overset{(A.1)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_\eta \circ L_{\rho_{g,x_a}(h,\delta\circ\theta_2^{-1}\circ\delta'^{-1})} \circ \rho_{g,x_a}(h,\cdot) \circ L_{\delta'} \circ G'_a \circ F'^{-1}_{a,w'_a} \circ L_{\theta_1\circ\phi}(z)\\
&\overset{(A.3)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_\eta \circ L_{\rho_{g,x_a}(g,\delta\circ\theta_2^{-1}\circ\delta'^{-1})} \circ \rho_{g,x_a}(h,\cdot) \circ L_{\delta'} \circ G'_a \circ F'^{-1}_{a,w'_a} \circ L_{\theta_1\circ\phi}(z)\\
&\overset{(A.9)}{=} L_{\psi^{-1}} \circ G_b \circ F^{-1}_{b,w_b} \circ L_{F_b(\theta_3^{-1})\circ\eta} \circ \rho_{g,x_a}(h,\cdot) \circ L_{\delta'} \circ G'_a \circ F'^{-1}_{a,w'_a} \circ L_{\theta_1\circ\phi}(z)\\
&= L_{\psi^{-1}} \circ G_b \circ L_{\theta_3^{-1}} \circ F^{-1}_{b,w_b} \circ L_\eta \circ \rho_{g,x_a}(h,\cdot) \circ L_{\delta'} \circ G'_a \circ F'^{-1}_{a,w'_a} \circ L_{\theta_1\circ\phi}(z)\\
&= L_{\psi^{-1}\circ G_b(\theta_3^{-1})} \circ G_b \circ F^{-1}_{b,w_b} \circ L_\eta \circ \rho_{g,x_a}(h,\cdot) \circ L_{\delta'} \circ G'_a \circ F'^{-1}_{a,w'_a} \circ L_{\theta_1\circ\phi}(z)\\
&= L_{\psi^{-1}\circ G_b(\theta_3^{-1})\circ\psi} \circ \Gamma' \circ L_{\phi'^{-1}\circ\theta_1\circ\phi}
\end{aligned}$$

We can move $L_{\psi^{-1}\circ G_2(\theta_3^{-1})\circ\psi}$ inside just like before, thus Γ and Γ' are in the same stab_x-orbit. Similarly, different polyfold structures for $\mathcal{W}_b$ also give local lifts in the same orbit.

Case three-different local representations of the actions: Suppose we have polyfold structures $\mathcal{X}_a, \mathcal{X}'_a, \mathcal{X}_b, \mathcal{X}'_b$ and two connected regular local uniformizers $\mathcal{U}_a, \mathcal{U}'_a$ around $x_a \in \mathcal{X}_a$ and $x'_a \in \mathcal{X}'_a$ respectively, such that the action is locally represented by both $\rho_{g,x_a} : U \times (\mathrm{stab}_{x_a} \ltimes \mathcal{U}_a) \to \mathcal{X}_b$ and $\rho'_{g,x'_a} : U \times (\mathrm{stab}_{x'_a} \ltimes \mathcal{U}'_a) \to \mathcal{X}'_b$. A priori, the polyfold structures $\mathcal{W}_a, \mathcal{W}'_a$ might be different. However, since they are equivalent, there is a common refinement. Similarly for $\mathcal{W}_b, \mathcal{W}'_b$, there is a common refinement. Since we have shown the case of changing $\mathcal{W}_a, \mathcal{W}_b$. Therefore we can assume the following diagram of equivalences:

$$\begin{tikzcd}
 & & \mathcal{X}_a \arrow[r, "\rho_{g,x_a}"] & \mathcal{X}_b & & \\
\mathcal{Z} & \mathcal{W}_a \arrow[l, "F_a"'] \arrow[ur, "G_a"] \arrow[dr, "G'_a"'] & & & \mathcal{W}_b \arrow[ul, "F_b"'] \arrow[r, "G_b"] \arrow[dl, "F'_b"] & \mathcal{X} \\
 & & \mathcal{X}'_a \arrow[r, "\rho'_{g,x'_a}"] & \mathcal{X}'_b & &
\end{tikzcd}$$

To define the local lifts for the two representations, we fix elements $w_a \in \mathcal{W}_a, w_b \in \mathcal{W}_b$, and morphisms $\phi \in \mathrm{Mor}(x, F_a(w_a))$, $\psi \in \mathrm{Mor}(y, G_b(w_b))$, $\delta \in \mathrm{Mor}(G_a(w_a), x_a)$, $\delta' \in \mathrm{Mor}(G'_a(w_a), x'_a)$, $\eta \in \mathrm{Mor}(\rho_{g,x_a}(g, x_a), F_b(w_b))$, $\eta' \in \mathrm{Mor}(\rho'_{g,x'_a}(g, x'_a), F'_b(w_b))$, such that we have the following diagram:

$$\begin{tikzcd}
 & & x_a \arrow[r] & \rho_{g,x_a}(g,x_a) \arrow[d, "\eta"] & & \\
x \arrow[d, "\phi"'] & & G_a(w_a) \arrow[u, "\delta"] & F_b(w_b) & & y \arrow[d, "\psi"] \\
F_a(w_a) & w_a \arrow[l, "F_a"'] \arrow[ur, "G_a"] \arrow[dr, "G'_a"'] & & & w_b \arrow[ul, "F_b"'] \arrow[dl, "F'_b"] \arrow[r, "G_b"] & G_b(w_b) \\
 & & G'_a(w_a) \arrow[d, "\delta'"'] & F'_b(w_b) & & \\
 & & x'_a \arrow[r] & \rho'_{g,x'_a}(g,x'_a) \arrow[u, "\eta'"'] & &
\end{tikzcd}$$

For every h close to g, we define the local diffeomorphism around w_a:

$$\mu_h := G'^{-1}_{a,w_a} \circ L_{\delta'^{-1}} \circ \rho'_{g,x'_a}(h,\cdot)^{-1} \circ L_{\eta'^{-1}} \circ F'_b \circ F^{-1}_{b,w_b} \circ L_\eta \circ \rho_{g,x_a}(h,\cdot) \circ L_\delta \circ G_a.$$

By Proposition A.4, there is neighborhood $U' \subset U$ of g and regular local uniformizers $\mathcal{V}, \mathcal{O}$ of w_a, such that for every $h \in U'$, μ_h is a diffeomorphism from $\mathcal{V}$ with images in $\mathcal{W}$. We can assume $\mathcal{V}$ is a connected local uniformizer around w_a. By Lemma A.2, we can find $\theta_g \in \mathrm{stab}_{w_a}$, such that near w_a we have

$$L_{\theta_g} = \mu_g.$$

We claim $\mu_g = L_{\theta_g}$ on $\mathcal{V}$. To see this, we define

$$S := \{p \in \mathcal{V} | \mu_g = L_{\theta_g} \text{ in a neighborhood of } p\}.$$

Then S is open and non-empty by definition. Since $\mathcal{V}$ is connected, it suffices to prove S is closed. Let $p_i \in S$ such that $\lim_i p_i = p_\infty \in \mathcal{V}$. Then by Lemma A.2, $\mu_g = L_\lambda$ for $\lambda \in \mathrm{mor}(p_\infty, \mu_g(p_\infty))$. By Corollary 4.5, $L_\lambda = L_{\theta'_g}$ on $\mathcal{W}$ for $\theta'_g \in \mathrm{stab}_{w_a}$. Since $L_{\theta'_g} = L_{\theta_g}$ on a neighborhood of p_i for $i \gg 0$, the regular property of $\mathcal{W}$ implies that $\theta'_g = \theta_g$ in $\mathrm{stab}^{\mathrm{eff}}_{w_a}$. Hence p_∞ is also in S and S is closed.

Similarly for every $h \in U'$, we can find $\theta_h \in \mathrm{stab}_{w_a}$, such that

$$L_{\theta_h} = \mu_h \text{ on } \mathcal{V}.$$

If there is a sequence of h_i converging to g, such that $L_{\theta_{h_i}} \neq L_{\theta_g}$ as germs at w_a. Since $|\mathrm{stab}_{w_a}| < \infty$, we can pick a subsequence such that $L_{\theta_{h_i}} = L_\theta$ for $\theta \in \mathrm{stab}_{w_a}$. Since $L_{\theta_{h_i}} \neq L_{\theta_g}$, there is an object $p \in \mathcal{V}$, such that $L_{\theta^{-1}} \circ L_{\theta_g}(p) \neq p$. Then

$$\mu_{h_i}(p) = L_\theta(p) \neq L_{\theta_g}(p) = \mu_g(p). \tag{A.10}$$

Note that $|\mu_{h_i}(p)| = |\mu_g(p)|$, $\mu_{h_i}(p)$ only has finite possibilities. Then (A.10) contradicts that μ_h is continuous in h. Therefore $\mu_h = L_{\theta_g}$ for $h \in U'' \subset U$, where U'' is a neighborhood of g. Therefore the two local lifts Γ, Γ' are related as follows.

$$\begin{aligned}
\Gamma(h,z) &= L_{\psi^{-1}} \circ G_b \circ F_b^{-1} \circ L_\eta \circ \rho_{g,x_a}(h,\cdot) \circ L_\delta \circ G_a \circ F_a^{-1} \circ L_\phi(z) \\
&= L_{\psi^{-1}} \circ G_b \circ F_b'^{-1} \circ L_{\eta'} \circ \rho'_{g,x'_a}(h,\cdot) \circ L_{\delta'} \circ G'_a \circ L_{\theta_g} \circ F_a^{-1} \circ L_\phi(z) \\
&= \Gamma'(h,\cdot) \circ L_{\phi^{-1}\circ F_a^{-1}(\theta_g)\circ\phi}(z).
\end{aligned}$$

Therefore, we have the group action by stab_x is transitive. $\square$

As a corollary of Proposition 4.33, we have the following.

Corollary A.5. *Let $\mathcal{U}$ be a regular local uniformizer of x, such that every $\Gamma \in L(x,y,g)$ defined over $\mathcal{U}$, then the restriction to g:*

$$\begin{gathered}
L(x,y,g) \to \mathrm{Map}_{\mathrm{sc}^\infty}(\mathcal{U},\mathcal{X}) \\
\Gamma(\cdot,\cdot) \to \Gamma(g,\cdot)
\end{gathered}$$

is injective.

Proof. Assume otherwise that two different Γ_1,Γ_2 satisfy $\Gamma_1(g,\cdot)=\Gamma_2(g,\cdot)$. By Proposition 4.33, $\Gamma_1 = \Gamma_2 \circ L_\phi$ as germs for $\phi \in \mathrm{stab}_x$. Therefore $\Gamma_1(g,\cdot) = \Gamma_2(g,\cdot) = \Gamma_2(g, L_\phi(\cdot))$ as germs near x. Since $\Gamma_1(g,\cdot),\Gamma_2(g,\cdot)$ are local sc-diffeomorphisms by Proposition 4.31, we have $L_\phi = \mathrm{id}$ near x. This contradicts $\Gamma_1 \neq \Gamma_2$. $\square$

Proposition 4.35. *Let Z be a regular polyfold with a G-action $(\rho,\mathfrak{P})$ and $(\mathcal{X},\boldsymbol{X})$ a polyfold structure of $Z=|\mathcal{X}|$. Then for $x,y\in\mathcal{X}$ and $g\in G$, (x,y,g) has the following structures.*

1. *For every $\phi \in \mathrm{mor}(y,z)$, $\Gamma \mapsto L_\phi \circ \Gamma$ defines a bijection from $L(x,y,g)$ to $L(x,z,g)$. And $\Gamma \mapsto \Gamma \circ L_\phi$ defines a bijection from $L(z,x,g)$ to $L(y,x,g)$.*

2. *There is a well-defined multiplication $\circ : L(y,z,g) \times L(x,y,h) \to L(x,z,gh)$ with the property that if $\Gamma_1 \in L(y,z,g)$ and $\Gamma_2 \in L(x,y,h)$, then*

$$\Gamma_1 \circ \Gamma_2(\epsilon gh, z) = \Gamma_1(\epsilon g, \Gamma_2(h,z)) = \Gamma_1(g, \Gamma_2(g^{-1}\epsilon gh, z)) \tag{A.11}$$

   for ϵ in a neighborhood of $\mathrm{id}\in G$ and z in a neighborhood of x.

3. *There is a unique identity element $\Gamma_{\mathrm{id}} \in L(x,x,\mathrm{id})$, such that $\Gamma_{\mathrm{id}}(\mathrm{id},z)=z$ for z in an open neighborhood of x. This identity is both left and right identity in the multiplication structure.*

4. *There is an (both right and left) inverse map $L(x,y,g) \to L(y,x,g^{-1})$ with respect to the multiplication and identity structures above.*

5. *For $x \in \mathcal{X}, g \in G$, there exists an open neighborhood $\mathcal{U} \times \mathcal{O} \times V$ of (x, y, g) in $\mathcal{X} \times \mathcal{C} \times G$ such that all the local lifts in $L(x, y, g)$ is defined on $V \times \mathcal{U}$ with image in $\mathcal{O}$. Moreover, for any $(x', g') \in \mathcal{U} \times V$ and $y' \in \mathcal{O}$ such that $\rho(g', |x'|) = |y'|$, every element $\Gamma' \in L(x', y', g')$ is represented by the restriction of a unique element $\Gamma \in L(x, y, g)$ to a neighborhood of (g', x') as germ.*

Proof of Proposition 4.35. For property 1, the map is defined as

$$\begin{aligned} L(x,y,g) &\to L(\phi(x), \psi(x), g) \\ \Gamma &\to L_\psi \circ \Gamma \circ L_{\phi^{-1}} \end{aligned}$$

It has an inverse:

$$\begin{aligned} L(\phi(x), \psi(y), g) &\to L(x,y,g) \\ \Gamma &\to L_{\psi^{-1}} \circ \Gamma \circ L_\phi \end{aligned}$$

For property 2, let $\rho(h, |x|) = |y|$ and $\rho(g, |y|) = |z|$, fix two three local representations of the actions:

$$\begin{aligned} \rho_a &: U \times (\mathrm{stab}_{x_a} \ltimes \mathcal{U}_a) && \to \mathcal{X}_b \\ \rho_c &: V \times (\mathrm{stab}_{x_c} \ltimes \mathcal{U}_c) && \to \mathcal{X}_d \\ \rho'_a &: W \times (\mathrm{stab}_{x'_a} \ltimes \mathcal{U}'_a) && \to \mathcal{X}'_d \end{aligned}$$

for open neighborhoods $U \ni h$, $V \ni g$ and $W \ni gh$, and $\mathcal{U}_a, \mathcal{U}_c, \mathcal{U}'_a$ are local uniformizers in $\mathcal{X}_a, \mathcal{X}_c, \mathcal{X}'_a$ around x_a, x_c, x'_a. We have the following diagram of equivalences:

$$\begin{tikzcd}
 & & \mathcal{X}_a \arrow[r, "\rho_1(h,\cdot)"] & \mathcal{X}_b & \mathcal{W}_b \arrow[l, "F_b"'] \arrow[r, "G_b"] & \mathcal{X} & \mathcal{W}_c \arrow[l, "F_c"'] \arrow[r, "G_c"] & \mathcal{X}_c \arrow[r, "\rho_c(g,\cdot)"] & \mathcal{X}_d & & \\
\mathcal{X} & \mathcal{W}_a \arrow[l, "F_a"'] \arrow[ur, "G_a"] \arrow[dr, "G'_a"'] & & & & & & & & \mathcal{W}_d \arrow[ul, "F_d"'] \arrow[dl, "F'_d"] \arrow[r, "G_d"] & \mathcal{X} \\
 & & \mathcal{X}'_a \arrow[rrrrrr, "\rho'_a(gh,\cdot)"] & & & & & & \mathcal{X}'_d & &
\end{tikzcd} \tag{A.12}$$

Let $\Gamma_1 \in L(y, z, g), \Gamma_2 \in L(x, y, h)$. By (A.3) and (A.5), for ϵ close to id and u close to x we have

$$\Gamma_1(\epsilon g, \Gamma_2(h, u)) = \Gamma_1(g, \Gamma_2(g^{-1}\epsilon g h, u)).$$

Using the same argument in the case three of the proof of Proposition 4.33 to the diagram (A.12), we can find $\Gamma \in L(x, z, gh)$ such that

$$\Gamma_1(\epsilon g, \Gamma_2(h, u)) = \Gamma_1(g, \Gamma_2(g^{-1}\epsilon g h, u)) = \Gamma(\epsilon g h, u).$$

We define Γ to be $\Gamma_1 \circ \Gamma_2$. To verify that the definition does not depend on the specific digram, note that the map in Corollary A.5 transfers the product structure defined by any diagram (A.12) to the composition of maps. By injectivity result in Corollary A.5 and that the composition of maps is a fixed structure, the multiplication structure is well-defined.

For property 3, note that $\mathfrak{P}(\mathrm{id}, \cdot)$ is an equivalence of polyfold structure. The local representation $\rho_{\mathrm{id},x_a} : U \times (\mathrm{stab}_{x_a} \ltimes \mathcal{U}_a) \to \mathcal{X}_b$ has the property that $\rho_{\mathrm{id},x_a}(\mathrm{id}, \cdot)$ is an embedding of local uniformizer and induces the inclusion of $|\mathcal{U}| \to Z$ on the orbit space. Then by the argument in proof of the quotient map in Theorem 1.1 or [65, Section 17.1], we can build a new polyfold structure $\mathcal{X}'_a$ containing $\mathcal{U}$ as a connected component[1], such that there is an equivalence $\rho_0 : \mathcal{X}'_a \to \mathcal{X}_b$ covers the identity map on Z extending $\rho_{\mathrm{id},x_a}(\mathrm{id}, \cdot)$. Then we have the following diagram:

$$\begin{tikzcd}
\mathcal{X} & \mathcal{W}_a \arrow[l, "F_a"'] \arrow[r, "G_a"] & \mathcal{X}'_a \arrow[r, "\rho_{\mathrm{id},x_a}(\mathrm{id},\cdot)"] & \mathcal{X}_b & \mathcal{W}_a \arrow[l, "\rho_0 \circ G_a"'] \arrow[r, "F_a"] & \mathcal{X}
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[d, "\phi"] & & x_a \arrow[r] & \rho_{\mathrm{id},x_a}(\mathrm{id}, x_a) \arrow[d, "\eta"] & & x \arrow[d, "\phi"] \\
F_1(w_a) & w_a \arrow[l] \arrow[r] & G_a(w_a) \arrow[u, "\delta"] & \rho_0 \circ G_a(w_a) & w_a \arrow[l] \arrow[r] & F_a(w_a).
\end{tikzcd}$$

Since $\rho_0 \circ G_a(w) = \rho_{\mathrm{id},x_a}(\mathrm{id}, G_a(w_a)) = \rho_{\mathrm{id},x_a}(\mathrm{id}, x_a)$, we can choose $\eta = \rho_{\mathrm{id},x_a}(\mathrm{id}, \delta^{-1})$. Then the local lift from this digram is

$$\begin{aligned}
\Gamma(\mathrm{id}, z) &= L_{\phi^{-1}} \circ F_a \circ G_a^{-1} \circ \rho_0^{-1} \circ L_{\rho_{\mathrm{id},x_a}(\mathrm{id},\delta^{-1})} \circ \rho_{\mathrm{id},x_a}(\mathrm{id}, \cdot) \circ L_\delta \circ G_a \circ F_a^{-1} \circ L_\phi(z) \\
&\overset{(\mathrm{A.1})}{=} L_{\phi^{-1}} \circ F_a \circ G_a^{-1} \circ \rho_0^{-1} \circ \rho_{\mathrm{id},x_a}(\mathrm{id}, \cdot) \circ G_a \circ F_a^{-1} \circ L_\phi(z) \\
&= z
\end{aligned}$$

By Corollary A.5, the identity element is unique and is both left and right identity.

As another consequence of Corollary A.5, the inverse of an element is unique and is both left and right inverse. To prove the existence, for every $\Gamma_1 \in L(x, y, g)$, we pick some $\Gamma_2 \in L(y, x, g^{-1})$. Then $\Gamma_2 \circ \Gamma_1 \in L(x, x, \mathrm{id})$. By proposition 4.33, there exists $\phi \in \mathrm{stab}_x$ such that $L_\phi \circ \Gamma_2 \circ \Gamma_1 = \Gamma_{\mathrm{id}}$, thus $L_\phi \circ \Gamma_2$ is an inverse to Γ_1.

For property 5, pick regular local uniformizers $\mathcal{U}, \mathcal{O}$ around x and y and an open set V of g such that every $\Gamma \in L(x, y, g)$ is defined on $V \times \mathcal{U}$ with image contained in $\mathcal{O}$. For $\Gamma \in L(x, y, g)$, $x' \in \mathcal{U}$, $y' \in \mathcal{O}$ and $g' \in g$ such that $|\Gamma(g', x')| = |y'|$, by Theorem 4.4 there exists a $\phi \in \mathrm{stab}_y$, such that $L_\phi(\Gamma(g', x')) = y'$. By the Proposition 4.33, $L_\phi \circ \Gamma = \Gamma \circ L_{\phi'} \in L(x, y, g)$. By definition the restriction $\Gamma \circ L_{\phi'}$ defines an element in $L(x', y', g')$. Then by Proposition 4.33, every element in $L(x', y', g')$ is $\Gamma \circ L_{\phi'}$ precomposing with $L_{\psi'}$ for $\psi' \in \mathrm{stab}_{x'}$. By Corollary 4.5, $L_{\psi'}$ is the restriction of L_δ for some $\delta \in \mathrm{stab}_x$. Therefore any element in $L(x', y', g')$ comes from a restriction of elements of $L(x, y, g)$. The regular property of the polyfold implies that this restriction is injective, i.e. if $\Gamma_1, \Gamma_2 \in L(x, y, g)$ restricted to same element in $L(x', y', g')$, then $\Gamma_1 = \Gamma_2$. This is because $\Gamma_1 = \Gamma_2 \circ L_\phi$ and if $\Gamma_1 = \Gamma_2 \in L(x', y', g')$, we must have that $L_\phi = \mathrm{id}$ near x'. Then by Definition 4.11, $L_\phi = \mathrm{id}$ on the local uniformizer, which means $\Gamma_1 = \Gamma_2$. $\square$

[1]Explicitly, $\mathcal{X}'_a$ can be chosen as $\mathcal{U} \cup \mathcal{X}_b$.

Proposition 4.16. *Let $(\mathcal{X}, \boldsymbol{X})$ be an effective groupoid. Assume for every $x \in \mathcal{X}$, there exists a local uniformizer $\mathcal{U}$ around x, such that for any connected uniformizer $\mathcal{V} \subset \mathcal{U}$ around x, we have $\mathcal{V} \backslash \cup_{\phi \neq \mathrm{id} \in \mathrm{stab}_x} \mathrm{Fix}(\phi)$ is connected, where $\mathrm{Fix}(\phi)$ is the fixed set of L_ϕ. Then $(\mathcal{X}, \boldsymbol{X})$ is regular.*

Proof. Since the polyfold is effective, the first property of regularity is satisfied for any local uniformizer by Theorem 4.4. To proved the second property, we show that $\mathcal{U}$ is a regular local uniformizer. Consider any connected uniformizer $\mathcal{V} \subset \mathcal{U}$ and map $\Phi : \mathcal{V} \to \mathrm{stab}_x$, for $\phi \in \mathrm{stab}_x$ we define

$$S_\phi := \{z | z \in \mathcal{V} \backslash \cup_{\phi \neq \mathrm{id}} \mathrm{Fix}(\phi), \Phi(z) = \phi\}.$$

We claim S_ϕ is open, assume otherwise, there is point $p \in S_\phi$, and $p_i \in S_\psi$ converging to p for $\psi \neq \phi$. Then

$$\lim_{i \to \infty} L_{\Phi(p_i)}(p_i) = \lim_{i \to \infty} L_\psi(p_i) = L_\psi(p) \neq L_\phi(p) = L_{\Phi(p)}(p)$$

contradicts the continuity of $L_{\Phi(z)}(z)$. Since $\mathcal{V} \backslash \cup_{\phi \neq \mathrm{id}} \mathrm{Fix}(\phi)$ is connected, hence there is a $\phi \in \mathrm{stab}_x$, such that $\mathcal{V} \backslash \cup_{\phi \neq \mathrm{id}} \mathrm{Fix}(\phi) = S_\phi$. That is

$$L_{\Phi(z)}(z) = L_\phi(z) \text{ for } z \in \mathcal{V} \backslash \cup_{\phi \neq \mathrm{id}} \mathrm{Fix}(\phi)$$

By the first property of regularity, $\mathcal{V} \backslash \cup_{\phi \neq \mathrm{id}} \mathrm{Fix}(\phi)$ is dense $\mathcal{V}$, thus

$$L_{\Phi(z)}(z) = L_\phi(z) \quad \forall z \in \mathcal{V}.$$

□

We mention here some other consequences of the regular property of polyfolds.

Proposition A.6. *If $(\mathcal{X}, \boldsymbol{X})$ is an effective and regular ep-groupoid and we have two equivalences $F, G : (\mathcal{W}, \boldsymbol{W}) \to (\mathcal{X}, \boldsymbol{X})$, such that $|F| = |G|$. Then F and G are naturally equivalent.*

Proof. By Lemma A.2, for any $w \in \mathcal{W}$, there exist a morphism $\phi \in \mathrm{Mor}(F(w), G(w))$, such that $L_\phi = G \circ F^{-1}$ near $F(w)$. Since $\mathcal{X}$ is reduced, ϕ is unique. Then the assignment $w \to \phi$ is an sc^∞ natural equivalence between F and G, the sc-smoothness follows from the étale property. □

The following theorem was proven in [65], if we assume the ep-groupoids are regular.

Theorem A.7 ([65, Theorem 10.7])**.** *Let $(\mathcal{X}, \boldsymbol{X})$ and $(\mathcal{Y}, \boldsymbol{Y})$ be two regular ep-groupoids, if $\mathfrak{f}$ and $\mathfrak{g}$ are two generalized isomorphism from $(\mathcal{X}, \boldsymbol{X})$ to $(\mathcal{Y}, \boldsymbol{Y})$ such that $|\mathfrak{f}| = |\mathfrak{g}|$, then $\mathfrak{f} = \mathfrak{g}$.*

A.2 Stabilization on the fixed locus

In this appendix, all sc-Banach spaces are sc-Hilbert spaces. Therefore we have sc-smooth bump functions and partition of unity by [65, Theorem 7.4]. Let V^λ be a fixed irreducible representation of G, we call a G-M-polyfold bundle $\mathcal{E} \to \mathcal{X}$ a λ-M-polyfold bundle iff each fiber of $\mathcal{E}$ is $V^\lambda \otimes \mathbb{H}$ for some sc-Hilbert space $\mathbb{H}$. Therefore the bundles N^λ, W^λ in Definition 6.9 are λ-M-polyfold bundles. An M-polyfold subbundle of $\mathcal{E}$ is an M-polyfold bundle $\mathcal{F}$ together with a sc-smooth inclusion $\mathcal{F} \hookrightarrow \mathcal{E}$.

Proof of Proposition 6.15

Proposition 6.15. *Under the tubular neighborhood assumption (Definition 6.9), assume $s^{-1}(0) \cap Z^G$ is a compact manifold. Then there is a G-invariant finite-dimensional trivial subbundle $\overline{W}^\lambda \subset W^\lambda$ over $s^{-1}(0) \cap Z^G$ with constant rank, such that $\overline{W}^\lambda$ covers the cokernel of $\mathrm{D}^\lambda s$ over $s^{-1}(0) \cap Z^G$.*

The following propositions are needed for the proof.

Proposition A.8. *Let $p : \mathcal{E} \to \mathcal{X}$ be a λ-M-polyfold bundle and $F \subset \mathcal{E}$ is a finite-dimensional G-invariant subbundle with constant rank. Then there is a complement M-polyfold subbundle $F^\perp \subset \mathcal{E}$, such that $F^\perp$ is G-invariant and $F^\perp \oplus F = \mathcal{E}$*

Proof. Pick an open cover $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ of $\mathcal{X}$, such that over each U_α, $\mathcal{E}$ has trivialization modeled on the image of bundle retraction $\Pi : U_\alpha \times \mathbb{E}_\alpha \to U_\alpha \times \mathbb{E}_\alpha$ and $\phi_\alpha : p^{-1}(U) \to \operatorname{im} \Pi$ is the local trivialization, for some sc-Hilbert space $\mathbb{E}_\alpha$. Using a partition of unity $\{f_\alpha\}$ subordinated to $\{U_\alpha\}$, then there is a sc^∞ map $g(\cdot,\cdot) : \mathcal{E} \oplus \mathcal{E} \to \mathbb{R}$ by

$$(x, e_1, e_2) \to \sum f_\alpha(x) \langle \phi_\alpha e_1, \phi_\alpha e_2 \rangle_0.$$

Then g defines a metric on $\mathcal{E}$. By averaging over G, we can assume g is G-invariant on $\mathcal{E}$. Using the metric g, we can define a map $\pi_F : \mathcal{E} \to \mathcal{E}$ to be the orthogonal projection from $\mathcal{E}$ to F. Since g and F are both G-invariant, π_F is a G-equivariant map. We claim that π_F is sc-smooth. To see that, let $x \in \mathcal{X}$, we can trivialize F locally to find n sc^∞ sections $s_1, \ldots, s_n$ of F near x, such that $s_1, \ldots, s_n$ are pointwise linearly independent and span F. Since $F \subset \mathcal{E}$ is a subbundle, in particular, the inclusion $F \subset \mathcal{E}$ is sc-smooth, $s_1, \ldots, s_n$ can be treated as sc^∞ sections of $\mathcal{E}$. Using GramSchmidt process, we can assume $s_1, \ldots, s_n$ form an orthonormal basis. Therefore the projection to F near x can be written as

$$\pi_F(y, e) = \sum_{i=1}^n g(y, e, s_i(y)) s_i(y).$$

Therefore π_F is sc-smooth. Since π_F is a G-equivariant projection, $\mathrm{id} - \pi_F$ is a G-equivariant projection from $\mathcal{E}$ to $\ker \pi_F$. Composing the bundle chart maps with $\mathrm{id} - \pi_F$, we can get bundle charts for $\ker \pi_F$. It is not hard to prove that $\ker \pi_F$ is a G-invariant M-polyfold subbundle of $\mathcal{E}$. Using the projections π_F and $\mathrm{id} - \pi_F$, we see that $F^\perp := \ker \pi_F$ is the complement of F in $\mathcal{E}$. A G-invariant subbundle $F^\perp$ of the λ M-polyfold bundle $\mathcal{E}$ is automatically a λ M-polyfold bundle $\square$

Proposition A.9. *Let M be a compact n-dimensional manifold and $W_\lambda \to M$ be an infinite-dimensional λ M-polyfold bundle. For any triangulation on M, there exists a refined triangulation, such that if we have a trivial G-invariant subbundle $F \subset W_\lambda$ of rank $k \dim R_\lambda$ over a neighborhood of n n-simplex $\triangle$, then we can extend F from $\triangle$ to M as a trivial G-invariant subbundle of the same rank*[2].

To prove Proposition A.9, we need the following proposition. Recall that an irreducible representation R_λ of rank m, there exist $g_1, \ldots, g_m \in G$, such that for any $v \neq 0 \in R_\lambda$, $\{g_1 v, \ldots, g_m v\}$ form a basis of R_λ. We fix a such set $\{g_1, \ldots, g_m\}$ in the following proposition.

Proposition A.10. *Given a triangulation on a compact n-dimensional manifold M and W_λ is a λ M-polyfold bundle over M. For $(x, w) \in W_\lambda$, we use $I_{(x,w)} \subset (W_\lambda)_x$ to denote the G-invariant subspace spanned by $w \in (W_\lambda)_x$ in the fiber $(W_\lambda)_x$. Assume over an open neighborhood U of a k-simplex $\triangle^k$, the bundle is modeled on a bundle retraction $\Pi : U \times \mathbb{F} \to U \times \mathbb{F}$ and there are elements $\rho_1, \ldots, \rho_{n+1} \in \mathbb{F}_\infty$, such that $I_{\Pi(u,\rho_1)}, \ldots, I_{\Pi(u,\rho_{n+1})}$ do not have nontrivial intersections with each other for all $v \in U$. If there exists a nowhere zero sc^∞ section $s : V \to W_\lambda$ over an open neighborhood V of $\partial \triangle^k$, then*

1. *if $k < n$, there is a nowhere zero section $\tilde{s}$ defined over a neighborhood O of $\triangle^k$, such that $s = \tilde{s}$ on V' for a neighborhood $V' \subset V \cap O$ of $\partial \triangle^k$;*

2. *if $k = n$, there is a nowhere zero section section $\tilde{s}$ defined over a neighborhood O of $\triangle^n$, such that $s = \tilde{s}$ on $O\triangle^n \cup V'$ for a neighborhood $V' \subset V \cap O$ of $\partial \triangle^n$;*

Proof. We will prove the claim for $k = n$ as the remaining case is similar and simpler. Since $I_{\Pi(u,\rho_1)}, \ldots, I_{\Pi(u,\rho_{n+1})}$ do not have nontrivial intersections with each other for all $v \in U$, $g_1\Pi(u,\rho_1), \ldots, g_m\Pi(u,\rho_1), g_1\Pi(u,\rho_2) \ldots, g_m\Pi(u,\rho_{n+1})$ are linearly independent for all $u \in U$. Therefore they span a G-invariant subbundle $F \subset \mathcal{W}_\lambda$ of constant rank. By Proposition A.8, we have a decomposition

$$W_\lambda = F \oplus F^\perp$$

over U. The projection to F is denoted by π_F. Let O denote the open neighborhood $\triangle^n \cup V$ of $\triangle^n$ and $\triangle^n_2 \subset \triangle^n_1 \subset \triangle^n$ be a two smaller n-simplexes with $\triangle^n \backslash \subset \triangle^n_2$.Using a smooth bump function, we can extend s to O of $\triangle^n$ such that $s = \tilde{s}$ outside $\triangle^n_1$. We also call the extended section s, since we will only consider the extended section from now on. We can find a smooth C^0 small perturbation p on O in the finite dimensional bundle F, such that p vanish on $O \backslash \triangle^n_1$. Moreover $\pi_F s + p$ is a transverse section of F near the $\partial \triangle^n_2$. By dimension reason, transverse here implies $\pi_F s + p \neq 0$ near $\partial \triangle^n_2$. Since $\dim F > n$, $(\pi_F s + p)$ can be

[2] But it may change the bundle outside the simplex.

extended smoothly to a section of F from $O\backslash\triangle_2^n$ to O and is nowhere zero on $\triangle_2^n$. Denote the extension by s_F. Then

$$\tilde{s} := s(x) + p(x) - \pi_F(s(x) + p(x)) + s_F(x) = s(x) - \pi_F(s(x)) + s_F(x)$$

satisfies the conditions. This is because $s_F(x) = \pi_F(s)$ on $O\backslash\triangle_2^n$ then $s = \tilde{s}$ on $O\backslash\triangle_2^n$. Therefore $\tilde{s} \neq 0$ on $O\backslash\triangle_2^n$. Since $\pi_F(\tilde{s}) = s_F(x)$, $\tilde{s} \neq 0$ on $\triangle_2^n$. $\square$

Proof of Proposition A.9. Given a triangulation, we can always find a refinement of triangulation such that the conditions in Proposition A.10 hold because W_λ is infinite dimensional. Since F is a trivial λ bundle, we can find sections $s_1, \ldots, s_n$ of F, such that $I_{s_1}, \ldots, I_{s_n}$ generate F. We extend s_1 first, pick all the 0-simplexes in M not contained in $\triangle^n$, we can assign nonzero vectors to them, then by Proposition A.10, we can extend s_1 to neighborhoods of all the 1-simplexes not contained in $\triangle_n$. Therefore after applying Proposition A.10 enough times, s_1 can be extended to a nonzero section in W_λ. By Proposition A.8, there is a complement to I_{s_1}, denoted by $I_{s_1}^\perp$. Then the projection of $s_2, \ldots, s_n$ to $I_{s_1}^\perp$ are linearly independent over $\triangle^n$. Then we can extend the projection of s_2 in $I_{s_1}^\perp$ by the same argument. Repeat it n times, we find the extension F. $\square$

Proof of Proposition 6.15. Following the argument in the proof of [59, Theorem 5.21], for every point x in $M = s^{-1}(0) \cap \mathcal{Z}^G$, there exist a neighborhood $U \subset M$ of x and sections $V_1, \ldots, V_k$ of W_λ over U, such that $I_{V_1}, \ldots, I_{V_k}$ have trivial intersection with each other and covers the cokernel of $\mathrm{D}^\lambda s$. We can find a finite cover of M, such that each open set in the cover has this property. Then we can find a triangulation such that any simplex is contained in some open set in the cover and the triangulation satisfies Proposition A.9. We order the top-dimensional simplex by $\triangle_1, \ldots, \triangle_l$. N denotes of the sum of the numbers of the vector fields over every open set in that open cover, or $N \gg 0$.Therefore for each top simplex $\triangle_i$, we have a G-invariant trivial bundle F_i covers the cokernel of $\mathrm{D}^\lambda s$ on $\triangle_i$. By Proposition A.9, we can find trivial subbundle $\tilde{F_1}$ extending F_1 hence covers the cokernel over $\triangle_1$. By Proposition A.8, there is complement M-polyfold bundle $\tilde{F_1}^\perp$. By Proposition A.9 and infinite dimensionality of $\tilde{F_1}^\perp$, we can find a trivial bundle $R_\lambda \otimes \underline{\mathbb{R}^{2N}} \subset \tilde{F_1}^\perp$ over M. Let $\pi_{\mathbb{R}^{2N}}$ denote the composition of two projections $W_\lambda \to F_1^\perp$ and $F_1^\perp \to R_\lambda \otimes \underline{\mathbb{R}^{2N}}$.

Similarly, we could have $\tilde{F_2}$ over $\triangle_2$. However, if $\tilde{F_1} \cap \tilde{F_2} \neq 0$, the rank of $\tilde{F_1} + \tilde{F_2}$ may be varying. To avoid this, we start with trivial bundle F_2 which covers the cokernel over $\triangle_2$. Then for any small $\epsilon > 0$, there exists G-equivariant bundle map $\tau : F_2|_{\triangle_2} \to R_\lambda \otimes \underline{\mathbb{R}^{2N}}|_{\triangle_2}$, such that $|\tau| < \epsilon$ and $\pi_{\mathbb{R}^{2N}}|_{F_2} + \tau$ is injective. For ϵ small enough, $\mathrm{im}((\mathrm{id} + \tau)|_{F_2})$ still covers the cokernel over $\triangle_2$ and is G-invariant, then $\pi_{\tilde{F_1}^\perp} \mathrm{im}((\mathrm{id} + \tau)|_{F_2})$ is still a trivial subbundle over $\triangle_2$ of the same dimension. We can extend $\pi_{\tilde{F_1}^\perp} \mathrm{im}((\mathrm{id} + \tau)|_{F_2})$ to M in $\tilde{F_1}^\perp$, the extended bundle is denoted by $\tilde{F_2}$, then $\tilde{F_1} + \tilde{F_2}$ is a subbundle and covers the cokernel over $\triangle_1 \cup \triangle_2$. Then applying the same argument to $\triangle_3, \ldots, \triangle_l$, we can find the required $\overline{W}_\lambda$. $\square$

Proof of Proposition 6.16

Proposition 6.16. *Under the tubular neighborhood assumption (Definition 6.9), then we have the following.*

1. *Suppose that* $s|_{Z^G} : Z^G \to W^G$ *is transverse to* 0. *Let* $\overline{W}^\lambda$ *be a trivial bundle asserted in Proposition 6.15, then* $\overline{N}^\lambda := (\mathrm{D}^\lambda s)^{-1}(\overline{W}^\lambda)$ *is smooth subbundle in* N^λ_∞ *over* $s^{-1}(0) \cap Z^G$.
2. $\operatorname{ind} \mathrm{D}^\lambda s$ *is locally constant on* Z^G_∞ *and* $\operatorname{rank} \overline{N}^\lambda_x - \operatorname{rank} \overline{W}^\lambda_x = \operatorname{ind} \mathrm{D}^\lambda s_x$ *for* $x \in s^{-1}(0) \cap Z^G$.
3. *For every* $k \in \mathbb{N}$, $\{x \in s^{-1}(0) \cap Z^G | \dim \operatorname{Coker} \mathrm{D}^\lambda s_x \leq k\}$ *is an open subset of* Z^G_∞.

Proof. (3) follows from [37, Corollary 3.3]. For (1), first note that by (3) there are finitely many $\overline{W}^\lambda$. Then $\iota + s : \pi^*(\oplus_{\lambda\in\Lambda} \overline{W}^\lambda) \to \pi^* W$, where ι is the inclusion. Then by construction $\iota + s$ is equivariant and transverse on $M := s|^{-1}_{Z^G}(0)$. Then $(\iota + s)^{-1}(0)$ is G-manifold containing M as a submanifold, then $\overline{N}^\lambda$ is identified with the λ-direction of the normal bundle of M in $(\iota + s)^{-1}(0)$. (2) follows from (1). □

A.3 Averaging in polyfolds

In this section, we prove that averaging argument works for polyfolds. The following Lemma follows from a classical argument e.g. [94, Theorem 9.42], where the integration in Banach space is defined using Riemann sum, e.g. see [50].

Lemma A.11. *Let* M *be a compact manifold with a volume form. Let* $U \subset \mathbb{R}^m_+ \times E$ *and* $V \subset \mathbb{R}^n_+ \times F$ *be two open subsets for Banach spaces* E, F. *Assume we have a map* $f : M \times U \to V$, *then*

1. *if* f *is continuous, then*
$$\overline{f} : U \to \mathbb{R}^n_+ \times \mathbb{F}, \quad x \mapsto \int_M f(p, x)\mathrm{d}p$$
is continuous;
2. *if* f *is* C^1, *then* $\overline{f}$ *is* C^1 *and the differential is*
$$\mathrm{D}\overline{f}_x : \mathbb{R}^m \times E \to \mathbb{R}^n \times F, \quad v \mapsto \int_M \mathrm{D}f_{(p,x)} v\mathrm{d}p.$$

Proposition A.12. *Let* $\mathscr{L}(E, F)$ *be the space of bounded linear maps between Banach spaces* E, F. *Given a compact maniold* M *and a map* $D : M \to \mathscr{L}(E, F)$ *such that* $T : M \times E \to F, (p, e) \mapsto (p, D(p)e)$ *is contious. Then* $\sup_{p\in M} ||D(p)|| < \infty$.

Proof. For $p \in M$, since T is continous, for every $\epsilon > 0$, there exits a $\delta > 0$ and a neighborhood $U \subset M$ of p such that $T|_{U \times B_\delta(0)} \subset B_\epsilon(0)$. Hence for every $q \in U$, $||D(q)|| \leq \frac{\epsilon}{\delta}$. Since M is compact, we have $\sup_{p \in M} ||D(p)|| < \infty$. □

Lemma A.13. *Let M be a compact manifold with a volume form. Let $U \subset \mathbb{R}^m_+ \times \mathbb{E}$ and $V \subset \mathbb{R}^n_+ \times \mathbb{F}$ be two open subsets. Assume we have a sc^1 map $f : M \times U \to V$, then*

$$\overline{f} : U \to \mathbb{R}^n_+ \times \mathbb{F}, \quad x \mapsto \int_M f(p,x)\mathrm{d}p$$

is a sc^1 map.

Proof. By (1) of Lemma A.11, $\overline{f}$ is a sc^0 map. By [65, Proposition 1.5], for every $k \geq 1$ the induced map $f : M \times U_k \to R^n \times \mathbb{F}_{k-1}$ is C^1 a and the tange map $Tf : T(M \times U)_{k-1} \to TV_{k-1}, \quad ((p,x),(v,u)) \mapsto (f(p,x), \mathrm{D}f_{(p,x)}(v,u))$ is continuous. Then by (2) of Lemma A.11, $\overline{f} : U_k \to \mathbb{R}^n \times \mathbb{F}_{k-1}$ is rC^1 and $\overline{Tf} : TU \to \mathbb{R}^n_+ \times \mathbb{F}$ is sc^0. By [65, Proposition 1.5], to show $\overline{f}$ is sc^1 it suffices to show that for $x \in U_k$, the linear map

$$v \in R^m \times \mathbb{E}_{k-1} \mapsto \int_M \mathrm{D}f_{(p,x)} v \mathrm{d}p$$

defines an element in $\mathscr{L}(\mathbb{R}^m \times \mathbb{E}_{k-1}, \mathbb{R}^n \times \mathbb{F}_{k-1})$. This follows from Proposition A.12. □

By induction, we have the following corollary of Lemma A.13.

Corollary A.14. *Let M be a compact manifold with a volume form. Let $U \subset \mathbb{R}^m_+ \times \mathbb{E}$ and $V \subset \mathbb{R}^n_+ \times \mathbb{F}$ be two open subsets. Assume we have a sc^k map $f : M \times U \to V$ for $0 \leq k \leq \infty$, then*

$$\overline{f} : U \to \mathbb{R}^n_+ \times \mathbb{F}, \quad x \mapsto \int_M f(p,x)\mathrm{d}p$$

is a sc^k map.

The following two corollaries are direct consequences of Lemma A.13.

Corollary A.15. *Let G acts on a polyfold Z by $(\rho, \mathfrak{P})$. Assume we have two G-invariant neighborhoods $V \subset U$ and a sc-smooth function $f : Z \to [0,1]$ such that $f|_V = 1$ and* $\operatorname{supp} f \subset U$. *Then $\overline{f} := \frac{1}{\mathrm{vol}(G)} \int_G (\mathfrak{P}(g,\cdot)^* f)\mathrm{d}g$ is a sc-smooth fuction on Z such that $\overline{f}|_V = 1$ and* $\operatorname{supp} \overline{f} \subset U$.

The following corollary is used in the proof of Theorem 6.17.

Corollary A.16. *Let $\mathcal{N} \to \mathcal{X}$ and $\mathcal{E} \to \mathcal{X}$ be two G-M-polyfold bundles. Suppose G is equipped with a bi-invaraint volume form. Assume we have map $f : \mathcal{N} \to \mathcal{E}$ be a sc-smooth bundle map then*

$$\overline{f} : \mathcal{N} \to \mathcal{E}, \quad v \mapsto \int_G g \cdot f(g^{-1}v)\mathrm{d}g$$

is G-equivariant sc-smooth bundle map.

Appendix B

De Rham theory on M-polyfolds and Polyfolds

This appendix is to review some technical details of the constructions in [60, 59, 63], and provide some details on the some other aspects on de Rham theory used in this paper. Given a M-polyfold $\mathcal{X}$, define $\Omega^k(\mathcal{X}^{i,j})(i>j)$ as set of sc^∞ maps $\oplus_k T\mathcal{X}^j|_{\mathcal{X}^i}\to\mathbb{R}$ which is linear in each argument and skew symmetric. Since we have sc^∞ embedding $T\mathcal{X}^{j_1}|_{\mathcal{X}_{i_1}}\to T\mathcal{X}^{j_2}|_{\mathcal{X}^{i_2}}$ for $i_1>i_2, j_1>j_2$, we have $\Omega^*(\mathcal{X}^{i_2,j_2})\to\Omega^*(\mathcal{X}^{i_1,j_1})$, hence we have a direct system over (i,j), the direct limit is denoted by $\Omega^*_\infty(\mathcal{X})$. Functorality follows from chain rule.

Proposition B.1. *$X:\mathcal{X}^{k_1}\to T\mathcal{X}^{k_2}$, $Y:\mathcal{X}^{j_1}\to T\mathcal{X}^{j_2}$ are two sc^∞ vector fields($k_1>k_2$,$j_1>j_2$), then $[X,Y]$ can be defined as a sc^∞ vector field $\mathcal{X}^{k_1+j_1}\to T\mathcal{X}$. Such that for any sc^∞ map $f:\mathcal{X}\to\mathbb{R}$, $[Dfx,DfY]=Df[X,Y]$, and Jacobi identity holds for this Lie bracket in the sense that $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$ as a vector from $\mathcal{X}^{k_1+j_1+i_1+2}$ to $T\mathcal{X}$.*

Proof. First of all, $X(Y)$ will be sc^∞ map from $\mathcal{X}^{k_1}(\mathcal{X}^{j_1})$ to $T\mathcal{X}$. Then DX will be an sc^∞ map from $T\mathcal{X}^{k_1}$ to $TT\mathcal{X}$, and Y can be understood as sc^∞ map $\mathcal{X}^{j_1+k_1}\to T\mathcal{X}^{k_1}$, hence $DX\circ Y$ will be an sc^∞ map from $\mathcal{X}^{j_1+k_1}$ to $TT\mathcal{X}$, the projection to the tangent direction of the tangent direction of $\mathcal{X}$ is again sc^∞, which is denoted by $DX\cdot Y$. Similarly we have $DY\cdot X$ is also sc^∞ from $\mathcal{X}^{j_1+k_1}$ to $T\mathcal{X}$, then $[X,Y]$ is defined as $DY\cdot X-DX\cdot Y$. It's not hard to see in finite dimensional case, this coincides with normal definition. Functoriality follows from chain rule. And Jacobi identity comes from $X:\mathcal{X}^{k_1+i}\to T\mathcal{X}$ is C^i map on each level and normal Jacobi identity, and shift of 2 comes from the fact we need to use the Hessian of C^2 map is symmetric. □

We can define the exterior differential on $\Omega^k(\mathcal{X}^{i,j})$ taking image in $\Omega^{k+1}(\mathcal{X}^{i+1,i})$ as follows: $\omega:\oplus_k T\mathcal{X}^j|_{\mathcal{X}^i}\to\mathbb{R}$ is sc^∞, so is $D\omega: T\oplus_k T\mathcal{X}^j_{\mathcal{X}^i}\to\mathbb{R}$, plug in 0 in the tangent directions of the fiber of $\oplus_k T\mathcal{X}^j_{\mathcal{X}^i}$, we get a sc^∞ map, still denoted by $D\omega$: $\oplus_k T\mathcal{X}^{j+1}|_{\mathcal{X}^{i+1}}\oplus T\mathcal{X}^i\to\mathbb{R}$. It is easy to see it is linear in each argument, and skew symmetric for the first k arguments. Since $i>j$, hence we can restrict $D\omega$ to $\oplus_{k+1}T\mathcal{X}^i_{\mathcal{X}^{i+1}}$, then skew symmetrize it will give us $d\omega$ in the space as we claimed. This d will be compatible with the direct system, hence induce a map on $\Omega^*_\infty(\mathcal{X})$. The following proposition is the justification for formula in section 1.

Proposition B.2. *Given $\omega \in \Omega^k(\mathcal{X}^{i,j})$, pick a point $(x_0, x_1, \ldots, x_k) \in \oplus_{k+1} T\mathcal{X}^i|_{\mathcal{X}^{2i+j}}$ over $p \in \mathcal{X}^{2i+j}$, and extend it to sc^∞ vector fields $X_0, \ldots, X_k$ of $T\mathcal{X}^j|_{\mathcal{X}^i}$ locally. Then at p:*

$$\begin{array}{rcl} d\omega(x_0,\ldots,x_k) & = & \sum(-1)^i D(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k))\cdot X_i \\ & + & \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i\ldots\hat{X}_j,\ldots,X_k) \end{array}$$

In particular, the local definition and global definition above coincide in $\Omega^*_\infty(\mathcal{X})$

Proof. By chain rule $D(\omega(X_0,\ldots,X_{n-1}))\cdot X_n = D\omega(X_0,\ldots,X_{n-1})\cdot X_n + \omega(DX_0\cdot X_n,\ldots,X_{n-1}) + \ldots + \omega(X_0,\ldots DX_{n-1}X_n)$. Skew-symmetrize both side, we can get the identity. □

Use the local definition, and Jacobi-identity, one can show that $d^2 = 0$ on $\Omega^*_\infty(\mathcal{X})$. And by chain rule $f^*d = df^*$ is f is an sc^∞ between two M-polyfolds.

Proposition B.3. *$\Omega^*_\infty(\mathcal{X})$ is a G^* algebra.*

Proof. We will use the sign convention in [53], i.e. $\xi \in g$ will induce a vector field $\xi_M = \frac{d}{dt}|_{t+0}\exp(-t\xi)$ from $\mathcal{X}^1$ to $T\mathcal{X}$. Since for any element $g \in G$, $dg : T\mathcal{X}^j|_{\mathcal{X}^i} \to T\mathcal{X}^j|_{\mathcal{X}^i}$[1] is sc^∞, hence the pull back of form in $\Omega^*(\mathcal{X}^{i,j})$ is still in $\Omega^*(\mathcal{X}^{i,j})$, this is compatible with the inclusion for different i. Therefore pull back is well defined on $\Omega^*_\infty(\mathcal{X})$. And the G action can be defined as $\rho(g)\omega\omega = g^{-1*}\omega$. Then there is an sc^∞ map $\rho(g)\omega : G \times \oplus^k T\mathcal{X}^j|_{\mathcal{X}^i} \to \mathbb{R}$, Then we define $L_\xi\omega = D\rho(g)\omega_{id,\omega}(\xi, 0) \in \Omega^k(\mathcal{X}^{i+1,j+1})$. We can also write this as $L_\xi\omega = \frac{d}{dt}|_{t=0}\rho(\exp(t\xi))\omega$[2]. $\iota_x i$ is defined by the interior multiplication by ξ_M

$\rho(a)L_\xi\rho(a^{-1}) = L_{Ad_a\xi}$ follows from chain rule, $\rho(a)\iota_\xi\rho(a^{-1}) = \iota_{Ad_a\xi}$ follows from, $Da(\xi_M) = (Ad_a\xi)_M$ and $\rho(a)d\rho(a^{-1}) = d$ follows from functoriality of pull back.

$[d,d] = 0$ is equivalent to $d^2 = 0$. $[\iota_i, \iota_j] = 0$ follows form skew symmetric property. $[L_i, L_j] = c^k_{ij}L_k, [L_i, \iota_j] = c^k_{ij}\iota_k, [d, L_i] = 0$ will follow from the Cartan formula $L_i = [d, \iota_i]$, and the local formula for d we have established[3].

So it suffices to prove Cartan formula. After writing out the local formula, we only need to prove $\frac{d}{dt}|_{t=0}\exp(t\xi)_*X = [\xi_M, X]$, for $X : \mathcal{X}^i \to T\mathcal{X}^j$. Since both sides are well defined, hence only need to prove they are equal in some level of the domain, so we shift the index of the domain to gain some classical differentiability. X will be C^1 from $\mathcal{X}^{i+1}$ to $T\mathcal{X}^j$ so is ξ_M, and $\rho(\exp t\xi)$ will C^2 on $(-\epsilon,\epsilon) \times \mathcal{X}^2 \to \mathcal{X}$, use these classical differentiability we can show $\frac{d}{dt}|_{t=0}\exp(t\xi)_*X = [\xi_M, X]$ holds after restricting to space with higher index. □

Proposition B.4. *$\Omega^*(\mathcal{X}, \tau_i)$ is G^* algebra.*

[1] The base direction is obvious, and fiber direction follows from $sc-$smoothness of $dg : T\mathcal{X}^j \to T\mathcal{X}^j$ and $sc-$smoothness of $T\mathcal{X}^j|_{\mathcal{X}^i} \subset T\mathcal{X}^j$

[2] The only requirements for this derivative is that it should be linear, and when the path is constant, it's zero, which is obviosly true here

[3] These equality will make sense after a suitable index shift, but they will induce equality in $\Omega^*_\infty(\mathcal{X})$

Proof. Since every element $\alpha \in \Omega^*(\mathcal{X}, \tau_i)$ is represented by the following data: A open cover $\{R_a\}$ of $\mathcal{X}_\infty$, with each R_a is an open set of $\mathcal{X}_i$, and for each R_a, there is a $\alpha_a \in \Omega^*(R_a^{i,j})$ for some i, j, and for any point $x \in R_a \cap R_n \cap \mathcal{X}_\infty$, there is a open set $R_x \subset R_a$ and R_b, such that $\alpha_a = \alpha_b \in \Omega^*(R_x^{k,l})$ for some k, l. Since the argument in the previous proposition is local, thus can be applied to $\Omega^*(\mathcal{X}, \tau_i)$. □

In the proof of Theorem 8.20, one need to average a form ω in $\Omega^*_\infty(\mathcal{X})$ over the compact group G. Thus the following proposition is needed:

Proposition B.5. *If $\omega \in \Omega^*(\mathcal{X}^{i,j})$, then $\frac{1}{Vol(G)} \int_G g^*\omega dg \in \Omega^*(\mathcal{X}^{i+1,j+1})$ and also G invariant, where the measure is determined by a bi-invariant metric.*

Proof. The G invariance is obvious, and average of ω will be the shifted space, comes from a general fact: given sc^∞ $\theta : M \times \mathcal{X} \to \mathbb{R}$, where M is a compact space, then $\int_M \theta : \mathcal{X}^1 \to \mathbb{R}$ is sc^∞. This is because $\theta : M \times \mathcal{X}^k \to \mathbb{R}$ is C^k, hence the $\int_M \theta : \mathcal{X}^k \to \mathbb{R}$ is again C^k, by Proposition 2.12, it's sc^∞ from $\mathcal{X}^1 \to \mathbb{R}$. □

Corollary B.6. *The average process above also gives map $\Omega^*(\mathcal{X}, \tau_i) \to \Omega^*(\mathcal{X}, \tau_i)^G$*

Proof. Let $\alpha \in \Omega^*(\mathcal{X}, \tau_i)$, since G is compact, for any $x \in \mathcal{X}_\infty$, $G \cdot x$ is compact, then there is G invariant open neighborhood U_x of $G \cdot x$ in $\mathcal{X}_\infty$, such that α is defined as $\Omega^*(U_x^{i,j})$, then we can apply the averaging process as before. All these local description gives rise to the averaged element in $\Omega^*(\mathcal{X}, \tau_i)^G$ □

Proposition B.7. *Mayer-Vietrious sequence holds for $\Omega^*_\infty(\mathcal{X})$, $\Omega^*(\mathcal{X}, \tau_i)$.*

Proof. Since we have sc^∞ partition of functions. □

Proposition B.8. *Given a closed oriented manifold M, and a M-polyfold $\mathcal{X}$, $H^*(\mathcal{X} \times M, \tau_i) \simeq H^*(\mathcal{X}, \tau_i) \otimes H^*(M)$*

Proof. It was proven in [8] that for abelian group G, paracompact space X and compact space Y, then there is an isomorphism $\check{H}^*(X \times Y, G) = \check{H}^*(X, \check{H}^*(Y, G))$. Combining with $H^*(\mathcal{X}, \tau_i) = \check{H}^*(\mathcal{X}_\infty, \tau_i, \mathbb{R})$, we prove the Künneth formula. □

Since any two local action $\Gamma_1, \Gamma_2 \in L(x, y, g)$ are differed by precompose of the $Stab_x$ action, since a differential form on polyfold is invariant under such action, there is a well-defined version of pull back by the G action. And all the conclusion holds for polyfold by same argument.

Appendix C

Convergence and Künneth Formula

C.1 Convergence

We now return to the convergence results used in chapter 12. In this appendix, we will see another reason of why the transversality of fiber product is essential, besides it is a natural requirement from polyfold point of view. That is the convergence results, especially Lemma 12.13, is more or less equivalent to the transversality of the fiber products.

Thom class.

We follow the construction of Thom class in [13]: let $\pi : E \to M$ be an oriented vector bundle with a metric over an oriented manifold. The fiber F, the base manifold M and the total space E are oriented in the manner of $[M][F] = [E]$. $S(E)$ denotes the sphere bundle of E, then we can a form ψ(angular form) on $S(E)$, such that the integration over each fiber is 1, and $\mathrm{d}\psi = -\pi^*e$, where e is the Euler class of the sphere bundle [13]. Then we pick smooth functions $\rho_n : \mathbb{R}^+ \to \mathbb{R}$, such that ρ_n is increasing, supported in $[0, \frac{1}{n}]$ and is -1 near 0:

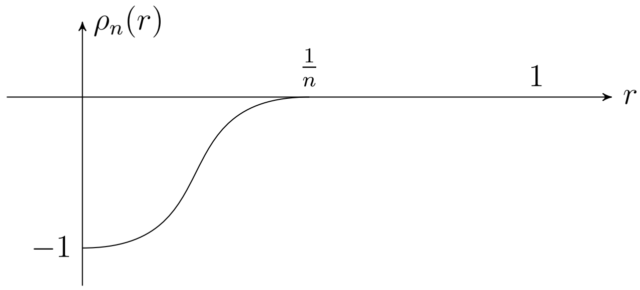

Figure C.1: Graph of ρ_n

$\mathrm{d}(\rho_n\psi)$ defines a form on $\mathbb{R}^+ \times S(E)$, and it is π^*e on an open neighborhood of $\{0\} \times S(E)$, thus $\mathrm{d}(\rho_n\psi)$ is a lift of some form on E, i.e. $\mathrm{d}(\rho_n\psi) = p^*\delta^n$ for $\delta^n \in \Omega^*(E)$, where p is the natural map $\mathbb{R}^+ \times S(E) \to E$. δ^n is proven to be the Thom class in [13]. The next lemma asserts δ^n actually represent the zero section not only in homological sense, but also in a stronger sense of current. Let δ_M denote the Dirac current of the zero section, i.e. $\delta_M(\alpha) = \int_M i^*\alpha$, for $\alpha \in \Omega^*(E)$, where $i : M \to E$ is the zero section.

Lemma 12.1. *$\delta^n \to \delta_M$ in the sense of current, i.e. $\forall \alpha \in \Omega^*(E)$*

$$\lim_{n\to\infty} \int_E \alpha \wedge \delta^n \to \delta_M(\alpha)$$

Proof. Let $F \simeq \mathbb{R}^n$ be a fiber of the bundle, since δ^n is compactly supported, then the integration over a fiber is

$$\int_F \delta^n = \int_{F-\{0\}} \delta^n = \int_{(0,\infty)\times S^{n-1}} p^*\delta^n = \int_{[0,\infty)\times S^{n-1}} p^*\delta^n = \int_{[0,\infty)\times S^{n-1}} \mathrm{d}(\rho_n\psi) = -\int_{\{0\}\times S^{n-1}} \psi = 1$$

Let $\alpha \in \Omega^*(E)$, since $\int_F \delta^n = 1$ for any fiber F. then

$$\int_E \pi^* i^*\alpha \wedge \delta^n = \int_M \int_F \pi^* i^*\alpha \wedge \delta^n = \int_M i^*\alpha$$

Therefore, it is enough to show

$$\lim_{n\to\infty} \int_E (\alpha - \pi^* i^*\alpha) \wedge \delta^n = 0$$

We will prove this by partition of unity: Let $\{U_i\}$ be an open cover of M, and $\{p_i\}$ be a partition of unity subordinated to the open cover, and assume that we fix trivialization over each U_i. Then over $\pi^{-1}(U_i)$:

$$\pi^* p_i(\alpha - \pi^* i^*\alpha) = \sum f^{I,J}\mathrm{d}x^I \wedge \mathrm{d}y^J$$

where x are the coordinates in U_i, and y are the coordinates in the fiber direction, I, J are sets of indexes. Since α and $\pi^* i^*\alpha$ are the same when restricted to the zero section, therefore $\lim\limits_{r\to 0} f^{I,\emptyset} = 0$ where r is the radius coordinate in the fiber direction. So we have:

$$\begin{aligned} \lim_{n\to\infty} \int_{\pi^{-1}(U)} f^{I,\emptyset}\mathrm{d}x^I \wedge \delta^n &= \lim_{n\to\infty} \int_{\mathbb{R}^+\times S^{n-1}\times U} f^{I,\emptyset}\mathrm{d}x^I \wedge \mathrm{d}\rho_n \wedge \psi - f^{I,\emptyset}\mathrm{d}x^I \wedge \rho_n \pi^* e \\ &= \lim_{n\to\infty} \int_0^{\frac{1}{n}} \int_{S(E)|_U} \pm f^{I,\emptyset}\mathrm{d}\rho_n \wedge \psi \wedge \mathrm{d}x^I \pm \rho_n f^{I,\emptyset}\pi^* e \wedge \mathrm{d}x^I \end{aligned}$$

Since $|\rho_n|$ is supported in $[0, \frac{1}{n}]$ and bounded by 1, $\displaystyle\int_0^{\frac{1}{n}} |\mathrm{d}\rho_n| = 1$, $\lim\limits_{r\to 0} f^{I,\emptyset} = 0$, and ψ is bounded on $S(E)$, we have

$$\lim_{n\to\infty} \int_{\pi^{-1}(U)} f^{I,\emptyset}\mathrm{d}x^I \wedge \delta^n = 0$$

When the cardinality $|J|$ of J is greater than 1, $\mathrm{d}y^I = Cr^{|J|}\mathrm{d}\theta^J + Dr^{|J|-1}\mathrm{d}r \wedge \mathrm{d}\theta^{J-1}$, where $\mathrm{d}\theta^J, \mathrm{d}\theta^{J-1}$ is a form on the sphere, and C, D are bounded functions. Because $\mathrm{d}\rho_n$ is purely in $\mathrm{d}r$ direction, then:

$$\begin{aligned}\lim_{n\to\infty}\int_{\pi^{-1}(U)} f^{I,J}\mathrm{d}x^I \wedge \mathrm{d}y^J \wedge \delta^n \;=\; & \lim_{n\to\infty}\int_0^{\frac{1}{n}}\int_{S(E)|_U} f^{I,J}Cr^{|J|}\mathrm{d}x^I \wedge \mathrm{d}\theta^J \wedge \mathrm{d}\rho_n \wedge \psi \\ & - \lim_{n\to\infty}\int_0^{\frac{1}{n}}\int_{S(E)|_U} f^{I,J}Cr^{|J|}\psi \wedge \mathrm{d}x^I \wedge \mathrm{d}\theta^J \wedge \rho_n\pi^* e \\ & - \lim_{n\to\infty}\int_0^{\frac{1}{n}}\int_{S(E)|_U} f^{I,J}Dr^{|J|-1} \wedge \psi \wedge \mathrm{d}x^I \wedge \mathrm{d}r \wedge \mathrm{d}\theta^{J-1} \wedge \rho_n\pi^* e\end{aligned}$$

Because $f^{I,J}, C$ are bounded, $\mathrm{d}\theta^J$ is bounded on $S(E)$, $\int_0^{\frac{1}{n}} |\mathrm{d}\rho_n| = 1$ and $\lim_{r\to 0} r^{|J|} = 0$, thus the first term limits to zero. Since everything in the last two terms are uniformly bounded, and ρ_n is supported in $[0, \frac{1}{n}]$, thus the last two terms limit to zero. Thus we have:

$$\lim_{n\to\infty}\int_{\pi^{-1}(U_i)} \pi^* p_i(\alpha_i - \pi^* i^* \alpha) \wedge \delta^n = 0$$

Therefore

$$\lim_{n\to\infty}\int_E (\alpha_i - \pi^* i^* \alpha) \wedge \delta^n = \lim_{n\to\infty}\sum_i \int_{\pi^{-1}(U_i)} \pi^* p_i \wedge (\alpha_i - \pi^* i^* \alpha) \wedge \delta^n = 0$$

□

Next, we will show that Lemma 12.1 is preserved under pullback, when transversality conditions are met.

Lemma C.1. *Let M be a compact manifold with boundary and corner, $E \to B$ be a vector bundle over a closed manifold B. If $f : M \to E$ is transverse to B, and we orient $f^{-1}(B)$ by $[f^{-1}(B)]f^*[E] = [TM|_{f^{-1}(B)}]$, then for $\alpha \in \Omega^*(C)$*

$$\lim_{n\to\infty}\int_M \alpha \wedge f^*\delta^n = \int_{f^{-1}(B)} \alpha|_{f^{-1}(B)}$$

Proof. We fix a tubular neighborhood $\pi : N \to f^{-1}(B)$. For n big enough, $f^*\delta^n$ is the Thom class of $f^{-1}(B)$, i.e. $f^*\delta^n$ has integration 1 along each fiber. This is because the fiber F of $f^{-1}(B)$ is diffeomorphic to a submanifold homotopic to a fiber in $E \to B$ though the map f. Since δ^n is closed and has a small enough support. Stokes' theorem implies $\int_F f^*\delta_n = \int_{f(F)} \delta^n = \int_{\text{fiber of } E} \delta^n = 1$. Then for the same reason as in the proof of Lemma 12.1, we only need to prove:

$$\lim_{n\to\infty}\int_N (\alpha - \pi^* i^* \alpha) \wedge f^*\delta^n = 0$$

.

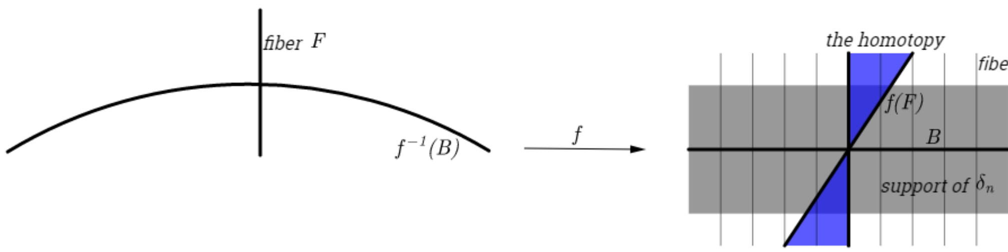

Figure C.2: Pullback of Thom class

Pick a point $x \in f^{-1}(B)$, then by implicit function theorem, we can find a local chart of x in M:

$$\phi : \mathbb{R}^k_+ \times \mathbb{R}^n \to M, \phi(0) = x$$

And local trivialization of $E \to B$ over $f(x)$:

$$\psi : \mathbb{R}^i \times \mathbb{R}^j \to E, \psi(0,0) = (f(x), 0)$$

Such that

$$\psi^{-1} \circ f \circ \phi(x_1, \ldots, x_k, y_1, \ldots, y_{n-j}, z_{n-j+1}, \ldots, z_n) = (f_1, \ldots, f_i, z_{n-j+1}, \ldots, z_n)$$

where $f_1, \ldots, f_i$ are functions of x_*, y_*, z_*. We replace the z coordinates by spherical coordinates. With such coordinates, the pullback of $\mathrm{d}(\rho_n\psi)$ through f is $\mathrm{d}(\rho_n\tilde{\psi})$, where $\tilde{\psi}$ is defined on $\mathbb{R}^k_+ \times \mathbb{R}_{n-j} \times S^{j-1} \times \mathbb{R}_+$ and uniformly bounded. Then the proof of Lemma 12.1 can be applied line by line. $\square$

Following the discussion in section 12.1, we pick representatives $\{\theta_{i,a}\}$ of a basis of $H^*(C_i)$ in $\Omega^*(C_i)$ to get quasi-isomorphic embedding:

$$H^*(C_i) \to \Omega^*(C_i)$$

and the dual basis is denoted by $\{\theta^*_{i,a}\}$, such that $\{\theta^*_{i,a}\}$ are in the image of the chosen embedding $H^*(C_i) \to \Omega^*(C_i)$ and $(-1)^{\dim C_i |\theta^b_i|} \int_{C_i} \theta^*_{i,a} \wedge \theta_{i,b} = \delta_{ab}$. Then by Proposition 12.2, the Thom class $\delta^n_i = \mathrm{d}(\rho_n\psi_i)$ of $\Delta_i \subset C_i \times C_i$ and $\sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta^*_{i,a}$ both represent the Poincaré dual of the diagonal Δ_i, thus they are cohomologous in $\Omega^*(C_i \times C_i)$. Therefore we can find f^n_i, such that $\mathrm{d}f^n_i = \delta^n_i - \sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta^*_{i,a}$, and

$$f^n_i - f^m_i = (\rho_n - \rho_m)\psi_i \tag{C.1}$$

Thus the support of $f_i^n - f_i^m$ converges to a measure zero set, to show the convergence results, i.e. Lemma 12.5 12.13, we need to show f_i^n is uniformly bounded. It is not necessarily true in $C_i \times C_i$, but it is true after we use polar coordinates near the diagonal Δ_i. To apply polar coordinate in an intrinsic way, we recall the definition of blow up of real submanifolds from [83, Chapter 5].

Definition C.2 (Blowing up)**.** *Let $p : E \to M$ be vector bundle over a manifold, then the blow up of E along M is the following manifold.*

$$E_{bl} = \{(v, e) \in E \times S(E) | p(v) = p(e), \exists a \geq 0, \text{ such that } ae = v\}$$

where $S(E)$ is the sphere bundle $(E \backslash \{0_M\})/\mathbb{R}^+$, and 0_M is the zero section of $E \to M$.

Then one can define a blow-up of a submanifold $N \subset M$, by blowing up the normal bundle, after we identify a tubular neighborhood of the submanifold N with the normal bundle. Moreover, the blow up of the submanifold N can be described intrinsically as follows:

$$Bl_N M = (M \backslash N) \cup S(TM/TN|_N)$$

where $S(TM/TN|_N)$ is the sphere bundle of the quotient bundle $TM/TN|_N$ over N, the smooth structure can be given using an auxiliary tubular neighborhood, but it is independent of the tubular neighborhood [83]. The natural map $Bl_N M \to M$ is smooth and is a diffeomorphism up to measure zero sets. The Thom class $\delta_i^n = \mathrm{d}(\rho_n \psi_i)$ can be pulled back to $Bl_{\Delta_i} C_i \times C_i$, and the primitive $\rho_n \psi_i$ is uniformly bounded on $Bl_{\Delta_i} C_i \times C_i$.

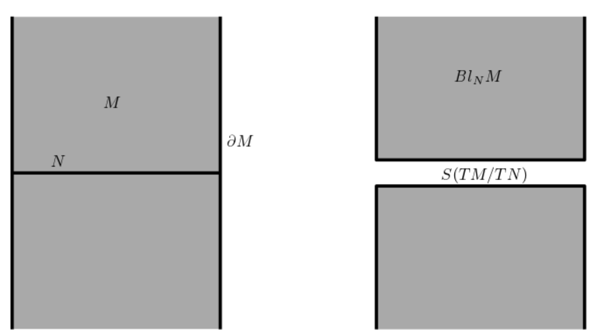

Figure C.3: Blow up one submanifold

Using this intrinsic description, when $M \times N \to C \times C$ is transverse to the diagonal Δ_C, there is a natural map $Bl_{M \times_C N} M \times N \to Bl_{\Delta_C} C \times C$ induced by $M \times N \to C \times C$. And we have the following commutative diagram of smooth maps:

$$\begin{tikzcd}
Bl_{M \times_C N} M \times N \arrow[r] \arrow[d] & Bl_{\Delta_C} C \times C \arrow[d] \\
M \times N \arrow[r] & C \times C
\end{tikzcd}$$

If we have two submanifolds N_1, N_2 of M, such that N_1 is transverse to N_2, then we can blow up N_1, N_2, it is shown in [83] the order of blowing up does not influence the diffeomorphism type, the resulting blow-up is denoted by $Bl_{N_1,N_2}M$. Similarly, if we have a sequence of submanifolds $N_1, N_2, \ldots, N_k$, such that $(\cap_{\alpha\in A} N_\alpha)$ is transverse to N_β for $\beta \notin A$, then we can blow up all $N_1, \ldots, N_k$. The diffeomorphism type does not depend on the order, and let $Bl_{N_1,\ldots,N_k}M$ denote the blow-up.

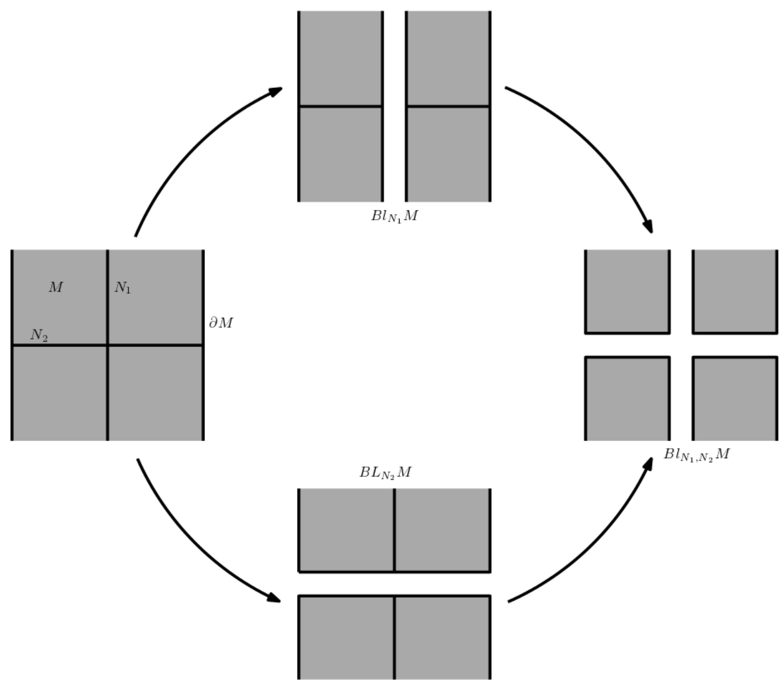

Figure C.4: Blow up two submanifolds

In the setting of flow category, the fiber product $\mathcal{M}_{i_0,i_1} \times_{i_1} \mathcal{M}_{i_1,i_2} \times_{i_2} \ldots \times_{i_n} \mathcal{M}_{i_n,i_{n+1}}$ is cut out transversely in $\mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,\iota_2} \times \ldots \times \mathcal{M}_{i_n,i_{n+1}}$, then $N_j = \mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,i_2} \times \ldots \times \mathcal{M}_{i_{j-1},i_j} \times_{i_j} \mathcal{M}_{i_j,i_{j+1}} \times \ldots \times \mathcal{M}_{i_n,i_{n+1}}$ are submanifolds in the product $\mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,i_2} \times \ldots \times \mathcal{M}_{i_n,i_{n+1}}$, such that $(\cap_{\alpha\in A} N_\alpha)$ is transverse to N_β for $\beta \notin A$. Then we have blow up $Bl_n = Bl_{N_1,\ldots,N_n}\mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,i_2} \times \ldots \times \mathcal{M}_{i_n,i_{n+1}}$. And we have similar commutative diagrams of smooth maps:

$$\begin{tikzcd}
Bl_n \arrow[r] \arrow[d] & Bl_{\Delta_i} C_i \times C_i \arrow[d] \\
\mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,i_2} \times \ldots \times \mathcal{M}_{i_n,i_{n+1}} \arrow[r] & C_i \times C_i
\end{tikzcd} \tag{C.2}$$

The next two lemmas are about the convergences in the definition of d_{BC} and the proof of $d_{BC}^2 = 0$. The definition of $\mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\alpha, f^{n_1}_{s+i_1}, \ldots, f^{n_r}_{s+i_r}, \gamma]$ is (12.5).

Lemma 12.5. $\lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^n_{s+i_1}, \dots, f^n_{s+i_r}, \gamma]$ *exists.*

Proof. Since $\mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^{n_1}_{s+i_1}, \dots, f^{n_r}_{s+i_r}, \gamma]$ is an integration over $\mathcal{M}^{s,k}_{i_1,\dots,i_r}$, and $\bigcup_j \mathcal{M}^{s,k}_{i_1,\dots,\overline{i_j},\dots,i_r}$ is a measure zero set in $\mathcal{M}^{s,k}_{i_1,\dots,i_r}$, thus we can restrict to $\mathcal{M}^{s,k}_{i_1,\dots,i_r} - \bigcup_j \mathcal{M}^{s,k}_{i_1,\dots,\overline{i_j},\dots,i_r}$ to get the same integral.

We have a blow-up $Bl_r\mathcal{M}^{s,k}_{i_1,\dots,i_r}$, by blowing up all $\mathcal{M}^{s,k}_{i_1,\dots,\overline{i_j},\dots,i_r}, 1 \le j \le r$. The primitives f^n_i can be lifted to $Bl_{\Delta_i} C_i \times C_i$, and $t_{*,i} \times s_{i,*}$ can be lifted to the blow-ups by (C.2). Thus we can define $Bl_r\mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^{n_1}_{s+i_1}, \dots, f^{n_r}_{s+i_r}, \gamma]$ to be the integration of wedge product of pullbacks of $\alpha, f^{n_1}_{s+i_1}, \dots, f^{n_r}_{s+i_r}, \gamma$ to $Bl_r\mathcal{M}^{s,k}_{i_1,\dots,i_r}$. Because $Bl_r\mathcal{M}^{s,k}_{i_1,\dots,i_r}$ and $\mathcal{M}^{s,k}_{i_1,\dots,i_r} - \bigcup_j \mathcal{M}^{s,k}_{i_1,\dots,\overline{i_j},\dots,i_r}$ are also differed by a measure zero set, by commutative diagram (C.2), we have

$$Bl_r\mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^n_{s+i_1}, \dots, f^n_{s+i_r}, \gamma] = \mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^n_{s+i_1}, \dots, f^n_{s+i_r}, \gamma]$$

Since

$$\begin{aligned} &Bl_r\mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^n_{s+i_1}, \dots, f^n_{s+i_r}, \gamma] - Bl_r\mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^m_{s+i_1}, \dots, f^m_{s+i_r}, \gamma] \\ =&\sum_{p=1}^r Bl_r\mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^m_{s+i_1}, \dots, f^m_{s+i_{p-1}}, f^n_{s+i_p} - f^m_{s+i_p}, f^n_{s+i_{p+1}}, \dots f^n_{s+i_r}, \gamma] \end{aligned} \tag{C.3}$$

$f^n_{s+i_j}, \forall n$ are uniformly bounded over $Bl_{\Delta_{s+i_j}} C_{s+i_j} \times C_{s+i_j}$ and the support of $f^n_{s+i_j} - f^m_{s+i_j}$ converges to a measure zero set in $Bl_{\Delta_{s+i_j}} C_{s+i_j} \times C_{s+i_j}$ when $n, m \to \infty$. By commutative diagram (C.2), the pullbacks of $f^n_{s+i_j}$ to $Bl_r\mathcal{M}^{s,k}_{i_1,\dots,i_r}$ have the same properties. Thus (C.3) implies the convergence. □

Lemma 12.13. *For an oriented flow category $\mathcal{C}$,*

$$\lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^n_{s+i_1}, \dots, \delta^n_{s+i_p}, \dots, f^n_{s+i_r}, \gamma] = (-1)^* \lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\dots,i_{p-1},\overline{i_p},i_{p+1},\dots,i_r}[\alpha, f^n_{s+i_1}, \dots, f^n_{s+i_r}, \gamma]$$

where $* = (a + m_{s,s+i_p})c_{s+i_p}$.

Proof. $\lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\dots,i_{p-1},\overline{i_p},i_{p+1},\dots,i_r}[\alpha, f^n_{s+i_1}, \dots, f^n_{s+i_r}, \gamma]$ exists by the same argument used in the proof of Lemma 12.5.

To prove the limit on the left-hand-side exists, we can blow up everything except for $\mathcal{M}^{s,k}_{i_1,\dots\overline{i_p},\dots,i_r}$ to get Bl_{r-1}. Assume the pull back of $\delta^n_{s+i_p}$ is supported in tubular neighborhood U of $\mathcal{M}^{s,k}_{i_1,\dots\overline{i_p},\dots,i_r}$ in $\mathcal{M}^{s,k}_{i_1,\dots,i_r}$, then U can be lifted to the blow-up Bl_{r-1} to get $Bl_{r-1}U$ (c.f. figure 7). For simplicity, we suppress wedge and pullback notations, then

$$\lim_{n\to\infty} \int_{\mathcal{M}^{s,k}_{i_1,\dots,i_r}} \alpha f^n_{s+i_1} \dots \delta^n_{s+i_p} \dots f^n_{s+i_r}\gamma = \lim_{n\to\infty} \int_{Bl_{r-1}U} \alpha f^n_{s+i_1} \dots \delta^n_{s+i_p} \dots f^n_{s+i_r}\gamma$$

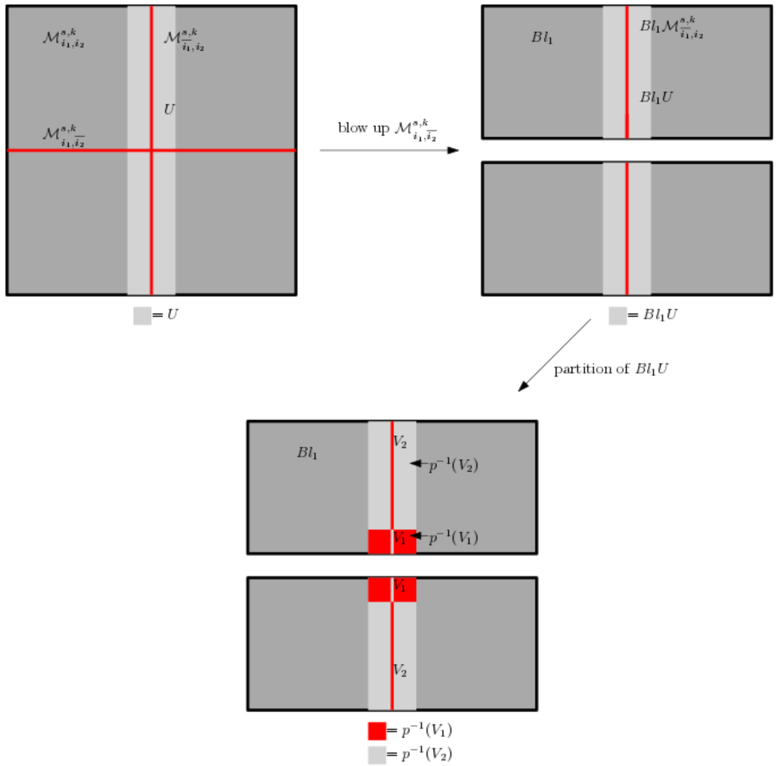

Figure C.5: $r = 2, p = 1$ case

Let $Bl_{r-1}\mathcal{M}^{s,k}_{i_1,\dots\overline{i_p},\dots,i_r}$ denote the lift of $\mathcal{M}^{s,k}_{i_1,\dots\overline{i_p},\dots,i_r}$ in Bl_{r-1}, then $Bl_{r-1}U$ is still a tubular neighborhood of $Bl_{r-1}\mathcal{M}^{s,k}_{i_1,\dots\overline{i_p},\dots,i_r}$ and $p : Bl_{r-1}U \to Bl_{r-1}\mathcal{M}^{s,k}_{i_1,\dots\overline{i_p},\dots,i_r}$ denote the projection of the tubular neighborhood. Then we can divide $Bl_{r-1}\mathcal{M}^{s,k}_{i_1,\dots\overline{i_p},\dots,i_r}$ into two part V_1, V_2, such that V_1 is a small open set containing the blow-up domain, and V_2 is the complement. Then $p^{-1}(V_1)$ and $p^{-1}(V_2)$ are partitions of $Bl_{r-1}U$ (c.f. figure 7). Using the same local coordinate in Lemma C.1, and if we integrate the radical coordinate first, because $f^n_{s+i_1}, \dots, f^n_{s+i_{p-1}}, f^n_{s+i_{p+1}}, \dots, f^n_{s+i_r}$ are uniformly bounded over Bl_{r-1}. , we have

$$|\int_{p^{-1}(V_1)} \alpha f^n_{s+i_1} \dots \delta^n_{s+i_p} \dots f^n_{s+i_r}\gamma| \le K \operatorname{vol}(V_1) \tag{C.4}$$

where K is a constant. Over $p^{-1}(V_2)$, the pullback of $f^n_{s+i_1}, \dots, f^n_{s+i_{p-1}}, f^n_{s+i_{p+1}}, \dots, f^n_{s+i_r}$ do not change for n large enough, because $p^{-1}(V_2)$ stays away from the blown-up area. Thus the only thing varies over $p^{-1}(V_2)$ is $\delta^n_{s+i_p}$. The by Lemma C.1 and orientation relations in Definition 11.16, we can conclude that

$$\lim_{n\to\infty}\int_{p^{-1}(V_2)} \alpha f^n_{s+i_1} \dots \delta^n_{s+i_p} \dots f^n_{s+i_r}\gamma = \lim_{n\to\infty}(-1)^{(a+m_{s,s+i_p})c_{s+i_p}} \int_{V_2} \alpha f^n_{s+i_1} \dots f^n_{s+i_{p-1}} f^n_{s+i_{p+1}} \dots f^n_{s+i_r}\gamma \tag{C.5}$$

By (C.4) and (C.5), because V_1 can be arbitrarily small, we have

$$\lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\alpha, f^n_{s+i_1}, \ldots, \delta^n_{s+i_p}, \ldots, f^n_{s+i_r}, \gamma] \text{ exists.}$$

Since $f^n_{s+i_1}, \ldots, f^n_{s+i_{p-1}}, f^n_{s+i_{p+1}}, \ldots, f^n_{s+i_r}$ are uniformly bounded over $Bl_{r-1}\mathcal{M}^{s,k}_{i_1,\ldots,\overline{i_p},\ldots,i_r}$,

$$|\int_{V_1} \alpha f^n_{s+i_1} \ldots f^n_{s+i_{i_{p-1}}} f^n_{s+i_{p+1}} \ldots f^n_{s+i_r}\gamma| < K' \operatorname{vol}(V_1) \tag{C.6}$$

Since $Bl_{r-1}\mathcal{M}^{s,k}_{i_1,\ldots,\overline{i_p},\ldots,i_r}$ and $\mathcal{M}^{s,k}_{i_1,\ldots,\overline{i_p},\ldots,i_r}$ are differed by a measure zero set, we have

$$\begin{aligned}\int_{\mathcal{M}^{s,k}_{i_1,\ldots,\overline{i_p},\ldots,i_r}} \alpha f^n_{s+i_1} \ldots f^n_{s+i_{p-1}} f^n_{s+i_{p+1}} \ldots f^n_{s+i_r}\gamma &= \int_{Bl_{r-1}\mathcal{M}^{s,k}_{i_1,\ldots,\overline{i_p},\ldots,i_r}} \alpha f^n_{s+i_1} \ldots f^n_{s+i_{p-1}} f^n_{s+i_{p+1}} \ldots f^n_{s+i_r}\gamma \\ &= \int_{V_1\cup V_2} \alpha f^n_{s+i_1} \ldots f^n_{s+i_{p-1}} f^n_{s+i_{p+1}} \ldots f^n_{s+i_r}\gamma \qquad \text{(C.7)}\end{aligned}$$

Therefore by (C.4), (C.5), (C.6) and (C.7), we have

$$\left|\lim_{n\to\infty}\left(\int_{M^{s,k}_{i_1,\ldots,i_r}} \alpha f^n_{s+i_1} \ldots \delta^n_{s+i_p} \ldots f^n_{s+i_r}\gamma - (-1)^{(a+m_{s,s+i_p})c_{s+i_p}} \int_{\mathcal{M}^{s,k}_{i_1,\ldots,\overline{i_p},\ldots,i_r}} \alpha f^n_{s+i_1} \ldots f^n_{s+i_r}\gamma\right)\right| \leq (K+K') \operatorname{vol}(V_1)$$

Since V_1 can be arbitrarily small, thus

$$\lim_{n\to\infty}\int_{M^{s,k}_{i_1,\ldots,i_r}} \alpha f^n_{s+i_1} \ldots \delta^n_{s+i_p} \ldots f^n_{s+i_r}\gamma = \lim_{n\to\infty}(-1)^{(a+m_{s,s+i_p})c_{s+i_p}} \int_{\mathcal{M}^{s,k}_{i_1,\ldots,\overline{i_p},\ldots,i_r}} \alpha f^n_{s+i_1} \ldots f^n_{s+i_r}\gamma$$

□

C.2 Künneth formula

In this part, we will prove the Künneth formula in Proposition 15.3. To prove the proposition, we will first introduce a twisted version of Morse-Bott differential, again we assume for simplicity each $\mathcal{M}_{i,j}.C_i$ have only one component, then the twisted differential is defined to be

$$\widehat{d_{BC}} := \prod_{k\geq 1} \widehat{d}_k$$

where $\widehat{d}_k$ is defined as

$$\begin{aligned}\left\langle \widehat{d}_k[\alpha], [\gamma]\right\rangle_{s+k} &= (-1)^{|\alpha|(c_s+1)+(|\alpha|+m_{s,s+k})(c_{s+k}+1)} \left\langle d_k[\alpha], [\gamma]\right\rangle_{s+k} \\ &= (-1)^{|\alpha|(c_s+1)+|d_k\alpha|(c_{s+k}+1)} \left\langle d_k[\alpha], [\gamma]\right\rangle_{s+k}\end{aligned}$$

for $\alpha \in h(\mathcal{C}, s), \gamma \in h(\mathcal{C}, s+k)$.

Then

$$\left\langle \widehat{d_{BC}}^2[\alpha], [\gamma] \right\rangle_{s+k} = (-1)^{|\alpha|(c_s+1)+(|\alpha|+m_{s,s+k}+1)(c_{s+k}+1)} \left\langle d_{BC}^2[\alpha], [\gamma] \right\rangle_{s+k} = 0$$

That is $(BC, \widehat{d_{BC}})$ is also a cochain complex. Moreover, there is an isomorphism $\rho : BC \to BC$, such that

$$\rho(\alpha) = (-1)^{|\alpha|(c_s+1)}\alpha$$

for $\alpha \in h(\mathcal{C}, s)$. It is direct to check that ρ is a cochain isomorphism between (BC, d_{BC}) and $(BC, \widehat{d_{BC}})$, hence they are quasi-isomorphic.

Remark C.3. *In fact, the theory on flow morphism and flow homotopy in chapter 12 works for the twisted version also. But we prefer d_{BC} over $\widehat{d_{BC}}$, because d_{BC} follows a consistent sign rule in remark 12.4. Moreover, the sign rule for d_{BC} can be generalized to A_∞ structures.*

It turns out the Künneth formula is easier (with cleaner signs) to prove using $\widehat{d_{BC}}$.

Lemma 15.3. $H^*(\mathcal{C} \times B, \widehat{d_{BC}}) = H^*(\mathcal{C}, \widehat{d_{BC}}) \otimes H^*(B)$

Sketch of the proof. Let $\{\theta_{i,a}\}$ be representatives of basis for $H^*(C_i)$ and $\{\xi_a\}$ be representatives of bases for $H^*(B)$ with dual basis $\{\xi_a^*\}$, then $\{\pi_{C_i}^*\theta_{i,a} \wedge \pi_B^*\xi_b\}$ are representatives of basis for $H^*(C_i \times B)$. Then the dual bases are $\{(-1)^{|\theta_{i,a}^*|\cdot|\xi_b|}\pi_{C_i}^*\theta_{i,a}^* \wedge \pi_B^*\xi_b^*\}$. When the δ_i^n is the Thom class of Δ_i and δ^n is the Thom class for Δ_B, then $\pi_{C_i\times C_i}^*\delta_i^n \wedge \pi_{B\times B}^*\delta^n$ is the Thom class of $\Delta_{C_i\times B}$. Let f_i^n, f^n be the primitives over $C_i \times C_i$ and $B \times B$, such that:

$$\begin{aligned} \mathrm{d}f_i^n &= \delta_i^n - \sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta_{i,a}^* \\ \mathrm{d}f^n &= \delta^n - \sum_a \pi_1^*\xi_a \wedge \pi_2^*\xi_a^* \end{aligned}$$

Then

$$\begin{aligned} & \pi_{C_i\times C_i}^*\delta_i^n \wedge \pi_{B\times B}^*\delta^n - \sum_{a,b} \pi_1^*\left(\pi_{C_i}^*\theta_{i,a} \wedge \pi_B^*\xi_b\right) \wedge \pi_2^*\left((-1)^{|\theta_{i,a}^*|\cdot|\xi_b|}\pi_{C_i}^*\theta_{i,a}^* \wedge \pi_B^*\xi_b^*\right) \\ = \; & \pi_{C_i\times C_i}^*\delta_i^n \wedge \pi_{B\times B}^*\delta^n - \pi_{C_i\times C_i}^*\left(\sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta_{i,a}^*\right) \wedge \pi_{B\times B}^*\left(\sum_b \pi_1^*\xi_b \wedge \wedge \pi_2^*\xi_b^*\right) \\ = \; & \pi_{C_i\times C_i}^* df_i^n \wedge \pi_{B\times B}^*\delta^n + \pi_{C_i\times C_i}^*\left(\sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta_{i,a}^*\right) \wedge df^n \end{aligned}$$

Thus $f_{C_i\times B}^n = \pi_{C_i\times C_i}^* f_i^n \wedge \pi_{B\times B}^*\delta^n + (-1)^{c_i}\pi_{C_i\times C_i}^*\left(\sum_a \pi_1^*\theta_i^a \wedge \pi_2^*\theta_i^{a*}\right) \wedge f^n$ could play the rule of the primitive of $\delta - \sum \theta \wedge \theta^*$ over $C_i \times B \times C_i \times B$. Although that is different from the construction used in section 12, the convergence results i.e. Lemma 12.5,12.13, hold. Thus all the constructions from chapter 12 can be applied using $f_{C_i\times B}^n$.

We claim $\widehat{d_{\mathcal{C}\times B}}[\pi_C^*\alpha\wedge\pi_B^*\beta] = (-1)^{|\beta|}\pi_C^*\widehat{d_{\mathcal{C}}}[\alpha]\wedge\pi_B^*[\beta]$. Obviously, if we only have $\pi^*_{C_i\times C_i}f_i^n\wedge \pi^*_{B\times B}\delta^n$ in $f^n_{C_i\times B}$, then the claim is true. With the extra term $(-1)^{c_i}\pi^*_{C_i\times C_i}\left(\sum_a \pi_1^*\theta_{i,a}\wedge\pi_2^*\theta^*_{i,a}\right)\wedge f^n$, we have

$$\begin{aligned}
& \widehat{d^k_{\mathcal{C}\times B}}[\pi^*_{C_s}\alpha\wedge\pi_B^*\beta] \\
= \quad & (-1)^{|\beta|}\widehat{d}_k[\alpha]\wedge\pi_B^*[\beta] \pm \sum_i \widehat{d}_i\circ\widehat{d_{k-i}}[\alpha]\sum_b\left(\lim_{n\to\infty}\int_{B\times B}\pi_1^*\beta\wedge f^n\wedge\pi_2^*\xi\right)\xi_b^* \\
& \pm\sum_{i,j}\widehat{d}_i\circ\widehat{d_{j-i}}\circ\widehat{d_{k-j}}[\alpha]\sum_b\left(\lim_{n\to\infty}\int_{B\times B\times B}\pi_1^*\beta\wedge(\pi_1\times\pi_2)^*f^n\wedge(\pi_2\times\pi_3)^*f^n\wedge\pi_3^*\xi_b\right)\xi_b^* \\
& \pm\dots
\end{aligned}$$

Since $\widehat{d}^2=0$, $\widehat{d_{\mathcal{C}\times B}}[\pi_C^*\alpha\wedge\pi_B^*\beta] = (-1)^{|\beta|}\pi_C^*\widehat{d_{\mathcal{C}}}[\alpha]\wedge\pi_B^*[\beta]$. $\square$