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ELASTIC-PLASTIC DYNAMIC ANALYSIS OF AXISYMMETRIC SOLIDS

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STABILITY OF SOME NONLINEAR SYSTEMS

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ABSTRACT

The stability of systems governed by

$$\ddot{x} + f(x) + q(x,\dot{x})\,\dot{x} - \varphi(t)\, r(x) = S(x;t)$$

is studied. Liapunov's Direct Method and a linearization approach have been used in the study of stability of the above system for $\varphi(t)$ L_1 integrable, and periodic, respectively. In the former case a sufficiency region of stability is constructed through the use of a Liapunov function. In the latter case, which is investigated by means of a linearization process, a Hill equation is obtained, whose stability is studied by a method suggested by Malkin. Malkin's method is then modified to obtain, by use of a first approximation, the first stability region in parameter space. A second approximation is also worked out. When the approximations obtained herein for general periodic function are reduced to the special cases of the Mathieu equation and the Hill-3-term equation, the results compare very well with the available numerical results based on the exact solution of each of those equations.

TABLE OF CONTENTS

I. INTRODUCTION

The study of those physical systems which are describable by a set of ordinary differential equations raises two distinct problems. The first problem is to obtain a solution: either in "closed form"(which usually is not possible) or else, approximately. The second problem is to obtain pertinent information about the whole class of solutions. The latter study leads to the qualitative theory of differential equation. A major question in the qualitative theory is that of stability. This question can be answered from a knowledge of the solution, if one has been able to obtain it. Since this is not usually possible, one has to fall back on a qualitative anslysis of the solution.

The systematic analysis in this direction, as far as stability is concerned, starts with the works of Poincaré and Liapunov. The work of Poincaré is not directly applicable to non-autonomous systems (i.e., systems in which the time variable appears explicitly), while the Liapunov approach is applicable to non-autonomous as well as autonomous ones. For this reason Liapunov's approach, rather than that of Poincaré, will be used in this study.

In its early stages of development, stability analysis based on Liapunov's Direct Method was an area of research almost exclusively restricted to Russian investigators. As the importance of this subject matter became evident, more and more researchers from all parts of the world were attracted to work in this area, and, in addition, many Russian works were also translated. At present there are many good books which treat stability analysis of systems by Liapunov's Direct Method. For example, LaSalle and Lefschetz [1][(*)] have a monograph which treats, primarily, autonomous systems; Yoshizawa [2], [3] worked on asymptotic behavior of solutions and periodic solution of systems; Hale [4] worked on periodic systems; Massera [5], Hahn [6], Krasovskii [7], Nemytskii and Stepanov [8] and Malkin [9] treat a variety of systems through Liapunov's Direct Method.

The Question of Stability

The term "stability" itself almost expresses the intuitive concept behind it. Given that a system operates under certain conditions, one can ask, "If we change these conditions slightly, does this change have a slight or a considerable effect upon the operation of the system?" In the former case the system is considered stable and in the latter case it is considered unstable. In mechanics this intuitive concept had found its main early use in the characterizations of the equilibrium of a rigid body. For such a system equilibrium position is said to be stable if the body resumes its original position after it has become subjected to any "sufficiently small" perturbation.

It should be pointed out that stability is not a uniquely defined concept. In fact, there exist many different types of stability, depending on the system, the manner in which the system is utilized, and the particular definition of stability employed.

Out Line of This Work

Although most of the discussion and theorems on stability may be extended to nth order equation (i.e., $\frac{n}{2}$ - degree of freedom systems), this work is mainly concerned with second order equations of the following type:

(*) Number in brackets indicate references listed at the end of this paper.

$$\ddot{x} + f(x) + q(x,\dot{x})\dot{x} - \varphi(t)\ r(x) = S(x;t) \qquad \text{(I-1)}$$

However, the general definitions and theorems which will be given in section II are all for n-dimensional systems.

Equation (I-1) and, special cases of it, have numerous applications in engineering and physics. However, here the basic motivation for investigation was its application in the study of a column which has been idealized so that its mass may be considered to be concentrated at one end of the column and its material properties at the other end. The loading on this column is composed of a time varying vertical load and a time varying vertical base acceleration, plus time varying lateral disturbances which can be either a load, or an acceleration, or both. The differential equation governing the behavior of this column is a special case of (I-1). This problem will be more fully discussed in section VI. In the following sections, however, the differential equation (I-1) will be discussed in a general manner.

In section III the case $\lim_{t\to\infty} \int_0^t |\varphi(\tau)|d\tau < \infty$ and the case $\varphi(t)$ periodic for finite time will be considered. The case $\varphi(t)$ periodic for all $t \geqq t_o$ will be discussed in section IV. In this case, as a special form of (I-1), a Hill equation $[\ddot{x} + (\beta - \mu\ \varphi(t)\ x = 0]$ will be obtained, of which the classical method of solution is discussed in several textbooks [10], [11]. The classical method involves the evaluation of an infinite determinant, which proves to be difficult, unless the Fourier coefficients in the expansion of $\varphi(t)$ decrease very rapidly. Malkin [9], using the Floquet solution, suggested a method for the analysis of the stability of the Hill equation. In section V Malkin's method has been modified to obtain the stability regions in parameter space for the Hill equation. Section VI is devoted to comparisons, applications and some remarks. For the sake of convenience, the details of some of the derivations are given in Appendices A and B.

II. DEFINITIONS AND GENERAL THEOREMS

As was pointed out in the previous section, there are many different definitions of stability, depending on the system and its application. In this section, only those definitions which are needed in the subsequent development of this work will be given. (There are excellent works, like those of LaSalle and Lefschetz [1], Bellman [12] and Zadeh and Desoer [13], which give definitions of various kinds of stability.) No proofs will be given for the theorems in this section, since these theorems are well known; however, references will be given where their proofs are cited.

A. Sufficient Conditions for Stability

Consider a system whose motion is governed by the vector differential equation:

$$\dot{\underset{\sim}{y}} = \underset{\sim}{Y}\ (\underset{\sim}{y};t) \tag{II-1}$$

Let $\underset{\sim}{y} = \underset{\sim}{\xi}\ (t)$ be a particular solution of (II-1). To study the behavior of solutions of (II-1) in the neighborhood of the solutions $\underset{\sim}{\xi}\ (t)$, it is convenient to make the following substitutions:

$$\underset{\sim}{x} = \underset{\sim}{y} - \underset{\sim}{\xi}\ (t) \tag{II-2}$$

The solution $\underset{\sim}{y} = \underset{\sim}{\xi}\ (t)$ or $\underset{\sim}{x} = 0$ is called the unperturbed motion (solution). Now the new variable $\underset{\sim}{x}$ satisfies

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}\ (\underset{\sim}{x};t) \tag{II-3}$$

where

$$\underset{\sim}{f}\ (\underset{\sim}{x};t) = \underset{\sim}{Y}\ (\underset{\sim}{x} + \underset{\sim}{\xi};t) - \underset{\sim}{Y}\ (\underset{\sim}{\xi};t) \tag{II-4}$$

therefore

$$\underset{\sim}{f}\ (\underset{\sim}{o};t) = \underset{\sim}{0} \tag{II-5}$$

In (II-3) $\underset{\sim}{x}$ denotes an n-dimensional vector, and it will be assumed in the sequel that the vector $\underset{\sim}{f}(\underset{\sim}{x};t)$ is a given vector field which is defined and continuous in the product space $E^n \times J$, where E^n is Euclidean n-space and J is the interval $t \geqq 0$. Furthermore, $\underset{\sim}{f}(\underset{\sim}{x};t)$ is of such a nature that the existence and uniqueness of the solutions, as well as their continuous dependence on the initial values, are assured in some region Ω of E^n for all $t \geqq 0$. The region Ω will be an open set containing the point $\underset{\sim}{x} = 0$.

Equation (II-3) is called the equation of perturbed motion. This equation has the trivial solution $\underset{\sim}{x} = 0$ which is called an equilibrium point or a singular point, or a null solution of the differential equation. The majority of the theorems of stability are about the stability or instability of the trivial solution of (II-3). As was shown in the previous paragraph, one can, by a simple substitution, transform the study of the stability of a non-trivial solution to that of a trivial one.

Definition (II-1): [7], [14] The null solution $\underset{\sim}{x} = 0$ of the system (II-3) is said to be stable (at $t = t_o$), provided that for arbitrary positive $\epsilon > 0$ there is a $\delta = \delta(\epsilon, t_o)$ such that, whenever $\| \underset{\sim}{x}(t_o) \| < \delta$, the inequality $\| \underset{\sim}{x}(\underset{\sim}{x}(t_o), t_o;t) \| < \epsilon$ is satisfied for all $t \geqq t_o$. (if δ may be chosen independently of t_o, then the trivial solution is said to be uniformly stable), where $\| \underset{\sim}{x} \| = \mathrm{Sup}(|x_1|, \ldots, |x_n|)$.

Definition (II-2): [7], [14] If the trivial solution of (II-3) is stable and in addition $\lim_{t \to \infty} | x_i(\underset{\sim}{x}(t_o), t_o;t) | = 0$, $i = 1, \ldots, n$, then the null solution $\underset{\sim}{x} = 0$ is said to be asymptotically stable.

Definition (II-3): [1] A scalar function $V(\underset{\sim}{x};t)$ is said to be positive definite in Ω , a neighborhood of the origin $(\underset{\sim}{x} = 0)$, if it satisfies the following conditions:

a) $V(\underset{\sim}{x};t)$ is defined in Ω for all $t \geqq 0$

b) $V(\underset{\sim}{o};t) = 0$ for all $t \geqq 0$

c) $V(\underset{\sim}{x};t)$ dominates a certain positive definite function $W(\underset{\sim}{x})$ in Ω , that is, $V(\underset{\sim}{x};t) \geqq W(\underset{\sim}{x})$ for all $\underset{\sim}{x}$ in Ω and all $t \geqq 0$ (*) .

It will be supposed that the function $V(\underset{\sim}{x};t)$ has continuous first partial derivatives with respect to all variables, in the neighborhood Ω . Therefore, one has

$$\dot{V}(\underset{\sim}{x};t) = \frac{\partial V(\underset{\sim}{x};t)}{\partial t} + \underset{\sim}{f}(\underset{\sim}{x};t) \cdot \text{Grad } V(\underset{\sim}{x};t) \qquad \text{(II-6)}$$

<u>Definition (II-4)</u>: [1] A scalar function $V(\underset{\sim}{x};t)$ is said to be a <u>Liapunov function</u> in Ω , a neighborhood of the origin $(\underset{\sim}{x} = 0)$, if:

a) $V(\underset{\sim}{x};t)$ is positive definite in Ω

b) $\dot{V}(\underset{\sim}{x};t) \leqq 0$ in Ω .

<u>Definition (II-5)</u>: [7], [14] A scalar function $V(\underset{\sim}{x};t)$ is said to permit an infinitesimal upper limit if, for any positive number λ , we can find another number μ such that for all values $t, x_1, \ldots, x_n$ which satisfy

$$t \geqq t_o \ , \ |x_s| \leqq \mu \ , \ s = 1, 2, \ldots, n$$

the following inequality will be satisfied:

$$|V(\underset{\sim}{x};t)| \leqq \lambda \quad .$$

(*) $W(\underset{\sim}{x})$ is said to be a positive definite function if $W(\underset{\sim}{x}) > 0$ for $\underset{\sim}{x} \neq 0$ and $W(\underset{\sim}{o}) = 0$.

In other words, the function V permits an infinitesimal upper limit if it approaches zero as $\sum_{s=1}^{n} x_s^2 \rightarrow 0$ uniformly with respect to t.

Theorem (II-1): [1], [9] If there exists in some neighborhood Ω of the origin of system (II-3) a Liapunov function $V(\underset{\sim}{x};t)$, then the unperturbed motion $\underset{\sim}{x} = 0$ is stable.

Theorem (II-2): [1], [9] If besides satisfying the conditions of Theorem (II-1), the derivative $\dot{V}(\underset{\sim}{x};t)$ is negative definite (i.e., $V(\underset{\sim}{x};t) \leq -U(\underset{\sim}{x})$ where $U(\underset{\sim}{x})$ is a positive definite function) and the function $V(\underset{\sim}{x};t)$ itself permits an infinitesimal upper limit, then the unperturbed motion $\underset{\sim}{x} = 0$ is asymptotically stable.

It should be noted that the above theorems are only sufficient conditions to ensure stability and asymptotic stability.

B. Stability Analysis through Linearization, of Systems with Periodic Input

Consider the system [15], [16], [17],

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}(t)\,\underset{\sim}{x} + \underset{\sim}{f}(\underset{\sim}{x};t) \qquad \text{(II-7)}$$

in which $\underset{\sim}{A}(t)$ is continuous, periodic, of period T, and $\underset{\sim}{f}(\underset{\sim}{x};t)$ is holomorphic in $\underset{\sim}{x}$ for all t in an open set S of the $\underset{\sim}{x}$-space, and its power series expansion in $\underset{\sim}{x}$ starts with terms of degree no less than 2. Further, $\underset{\sim}{f}(\underset{\sim}{x};t)$ is continuous and periodic of the same period as $\underset{\sim}{A}(t)$, i.e., $\underset{\sim}{f}(\underset{\sim}{x};t+T) = \underset{\sim}{f}(\underset{\sim}{x};t)$.

Theorem (II-3): [15] If the characteristic exponents of

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}(t)\,\underset{\sim}{x} \qquad \text{(II-8)}$$

are all less than unity in absolute value, then (II-7) is asymptotically stable at the origin.

Remark (II-1): The above theorem is essentially due to Liapunov [18] and it may be formulated in a somewhat different form [8], [17], namely: If the trivial solution of (II-8) is exponentially-asymptotically stable, so is the trivial solution of (II-7). Exponential-asymptotic stability implies the existence of positive numbers N, ν such that given any $t_o \geqq 0$ and any $\underset{\sim}{x}_o \in R^n$ with $\| \underset{\sim}{x}_o \|_2$ sufficiently small, every solution $\underset{\sim}{u}$ of (II-7) with $\underset{\sim}{u}(t_o) = \underset{\sim}{x}_o$ is defined for $t \geqq t_o$ and satisfies

$$\| \underset{\sim}{u}(t) \|_2 \leqq N \| \underset{\sim}{u}(t_o) \|_2 \, e^{-\nu(t-t_o)}$$

where $\| \underset{\sim}{x} \|_2 = (t_1^2 + \ldots + x_n^2)^{1/2}$.

It should be noted that, if all the characteristic exponents of (II-8) are less than unity in absolute value, then using the Floquet solution one can show that every solution of (II-8) is asymptotically-exponentially stable.

C. Stability in the Presence of Persistent Disturbances that are Bounded in the Mean

When one deals with practical systems, it usually happens that the system will be perturbed not only because of the presence of nonzero initial conditions, but also because of external disturbing actions. Thus the system that one should consider is a modification of (II-3):

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x};t) + \underset{\sim}{R}(\underset{\sim}{x};t) \qquad \text{(II-9)}$$

In general, one may not have completely detailed information of the external disturbances $\underset{\sim}{R}(\underset{\sim}{x};t)$, and they do not necessarily reduce to zero at the point $\underset{\sim}{x} = 0$. This problem has been discussed by many Russians, notably Malkin [9] and Krasovskii [7]. In Malkin's study, he assumes that $R_i(\underset{\sim}{x};t)$, $i = 1, 2, \ldots, n$, are all small for all values of time. In practice, this assumption can be too restrictive, and Krasovskii has generalized it to $R_i(\underset{\sim}{x};t)$, $i = 1, 2, \ldots, n$, which are "bounded in the mean."

The following definitions and Theorem are due to Krasovskii [7], to which the reader is referred for proof and more discussion.

Definition (II-6): [7] $R_i(\underset{\sim}{x};t)$, $i = 1, \ldots, n$ is called bounded in the mean if the intervals of time $t > t_o$ during which the $R_i(\underset{\sim}{x};t)$, $i = 1, \ldots, n$ have large values are extremely short.

Definition (II-7): [7] The null solution $\underset{\sim}{x} = 0$ of the system (II-3) is called stable for persistent disturbances that are bounded in the mean if for every positive ϵ ($\epsilon > 0$) and positive T ($T > 0$), there are two positive numbers δ, η ($\delta > 0$, $\eta > 0$) such that whenever the continuous function $\varphi(t)$ satisfies the relations

$$\int_t^{t+T} \varphi(s)\, dS < \eta , \quad |R_i(\underset{\sim}{x};t)| \leqq \varphi(t) \text{ for } \|\underset{\sim}{x}\|_2 < \epsilon ,$$

every solution $\underset{\sim}{x}(\underset{\sim}{x}_o, t_o;t)$ of equation (II-9) which has initial values, $\|\underset{\sim}{x}_o\|_2 < \delta$ will satisfy $\|\underset{\sim}{x}(\underset{\sim}{x}_o, t_o;t)\| < \epsilon$ for all $t \geqq t_o$.

Definition (II-8): [7] The null solution $\underset{\sim}{x} = 0$ of equation (II-3) is called asymptotically stable uniformly with respect to the time t_o and the coordinates of the initial perturbation $\underset{\sim}{x}_o$ in the region G_δ, if: (a) the solution $\underset{\sim}{x} = 0$ is stable in the sense of Liapunov (definitions (II-1), (II-2)), (b) for every positive number $\eta > 0$, there is a number $T(\eta)$ such that the inequality $\|\underset{\sim}{x}(\underset{\sim}{x}_o, t_o;t)\|_2 < \eta$ is satisfied for

$t \geqq t_o + T(\eta)$, independent of the initial moment of time t_o and the coordinates of the initial perturbation $\underset{\sim}{x}_o$, $\underset{\sim}{x}_o \in G_\delta$.

Theorem (II-4): [7] Suppose the null solution of (II-3), $\underset{\sim}{x} = 0$, is asymptotically stable uniformly in t_o and $\underset{\sim}{x}_o$ in the sense of definition (II-8). Then stability also holds for persistent disturbances that are bounded in the mean.

Remark (II-2): [7] If the functions $f_i(\underset{\sim}{x};t)$, $i = 1, \ldots, n$ are periodic in time t , or if they have the form $f_i(\underset{\sim}{x})$ (that is, are independent of the time t), then uniform stability is an obvious consequence of asymptotic stability. Theorem (II-4) therefore asserts in this case that mere asymptotic stability implies stability for persistent disturbances that are bounded in the mean.

III. STABILITY ANALYSIS THROUGH LIAPUNOV'S DIRECT METHOD

In the following, the stability properties of the null solutions of special cases of equation (I-1) will be studied. A sufficiency region of stability will be constructed, and the question of the control of its size will be discussed.

A. Analysis of Particular Systems

In this section the stability properties of the following system

$$\ddot{x} + f\ (x) + q\ (x,\dot{x})\ \dot{x} - \varphi\ (t)\ r\ (x) = 0 \qquad \text{(III-1)}$$

will be studied under the conditions that $\varphi\ (t)$ is L_1 integrable, i.e.,

$$\lim_{t \to \infty} \int_0^t |\varphi\ (\tau)|\ d\tau < \infty \qquad \text{(III-2)}$$

First, equation (III-1) will be discussed for the special case of $q(x,\dot{x}) = 0$, i.e.,

$$\ddot{x} + f(x) - \varphi(t)\ r(x) = 0 \qquad \text{(III-3)}$$

In all of the following discussions, it will be assumed that $f(x)$, $q(x,\dot{x})$, $\varphi(t)$ and $r(x)$ are all of such a nature that the existence of a unique solution for (III-1) is assured. It is convenient, in order to use definitions and theorems of section II, to write (III-1) and (III-3) in the following forms, respectively.

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ -f(x) - q(x,y)\ y + \varphi(t)\ r(x) \end{Bmatrix} \qquad \text{(III-4)}$$

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ -f(x) + \varphi(t)\ r(x) \end{Bmatrix} \qquad \text{(III-5)}$$

Let $F(x) \equiv \int_0^x f(\xi)\, d\xi$ and let Ω^* be some punctured neighborhood of $x = 0$ for which $F(x) \geqq \frac{1}{2} r^2(x)$ and define Ω to be the following Cartesian product.

$$\Omega = \Omega^* \cup \{x = 0\} \times \{y : |y| \leqq \infty\} \tag{III-6}$$

Theorem (III-1): If for system (III-5) the following are satisfied:

(a) $\int_{t_o}^{t} |\varphi(\tau)|\, d\tau < \infty, \quad t \geqq t_o$

(b) $x\, f(x) > 0$ for $x \neq 0$, $f(o) = 0$

(c) $r(o) = 0$

(d) In some punctured neighborhood Ω^* of $x = 0$, $F(x) \geqq \frac{1}{2} r^2(x)$ where $F(x) \equiv \int_0^x f(\xi)\, d\xi$, then the origin, $x = y = 0$, is a stable trivial solution.

Proof: A Liapunov function will be constructed and through the use of Theorem (II-1) the above theorem will be proved.

From assumptions (b) and (c), it is seen that $x = y = 0$ is a trivial solution of (III-5). Let,

$$V(x,y;t) \equiv \left[\frac{1}{2} y^2 + F(x) \right] e^{-\int_{t_o}^{t} |\varphi(\tau)|\, d\tau} \tag{III-7}$$

Therefore

$$\dot{V}(x,y;t) = -\,|\varphi(t)|\, e^{-\int_{t_o}^{t} |\varphi(\tau)|\, d\tau} \left[\frac{1}{2} y^2 + F(x) \right] + e^{-\int_{t_o}^{t} |\varphi(\tau)|\, d\tau} \left[y\dot{y} + f(x)\, \dot{x} \right]$$

substitution for $\dot{x}$ and $\dot{y}$ from (III-5) into the above expression yields:

$$\dot{V}(x,y;t) = -\left\{|\varphi(t)|\left[\frac{1}{2}y^2 + F(x)\right] - \varphi(t)\, y\, r(x)\right\} e^{-\int_{t_o}^{t}|\varphi(\tau)|d\tau}$$

$$\leqq -\left\{|\varphi(t)|\left[\frac{1}{2}y^2 + F(x)\right] - |\varphi(t)\, y\, r(x)|\right\} e^{-\int_{t_o}^{t}|\varphi(\tau)|d\tau}$$

Therefore

$$\dot{V}(x,y;t) \leqq -\left\{\frac{1}{2}y^2 + F(x) - |y\, r(x)|\right\}|\varphi(t)|e^{-\int_{t_o}^{t}|\varphi(\tau)|d\tau} \qquad \text{(III-8)}$$

As it is seen from (III-8), $\dot{V}(x,y;t) \leqq 0$ for all (x,y) for which

$$R(x,y) \equiv \left\{\frac{1}{2}y^2 + F(x) - |y\, r(x)|\right\} \geqq 0 \qquad \text{(III-9)}$$

$R(x,y)$ can be written in the following form by adding to it and subtracting from it $\frac{1}{2}r^2(x)$.

$$R(x,y) = \frac{1}{2}(|r(x)| - |y|)^2 + F(x) - \frac{1}{2}r^2(x) \qquad \text{(III-10)}$$

Assumption (b) implies that $F(x) > 0$ for all $|x| \neq 0$ and assumption (d), in conjunction with (III-10), implies that $R(x,y) \geqq 0$ for all $(x,y) \in \Omega$, where Ω is defined by (III-6). Therefore $\dot{V}(x,y;t) \leqq 0$ for all $(x,y) \in \Omega$.

As is seen from (III-7), assumption (a) and the above discussion, $V(x,y;t)$ satisfies the following conditions:

(a) $V(x,y;t)$ is defined in the whole of (x,y) space for all $t \geqq t_o$

(b) $V(o,o;t) = 0$

(c) $V(x,y;t) \geqq W(x,y) \equiv \left[\frac{1}{2} y^2 + F(x) \right] e^{-M}$

where $M \equiv \lim_{t \to \infty} \int_{t_o}^{t} |\varphi(\tau)| d\tau$

(d) $\dot{V}(x,y;t) \leqq 0$ for all $(x,y) \in \Omega$

Therefore, according to definition (II-4) $V(x,y;t)$ is a Liapunov function for the system (III-5) and according to Theorem (II-1) the null solution $x = y = 0$ of the system (III-5) is stable.

Theorem (III-2): If for the system (III-4), besides the conditions of Theorem (III-1), the following condition

(e) $q(x,y) \geqq 0$

is also satisfied for all $(x,y) \in \Omega$, then $x = y = 0$ is a stable trivial solution.

Proof: The same Liapunov function as used in Theorem (III-1) may be used here, and following exactly the same manipulations one obtains that in the same neighborhood Ω

$$\dot{V}(x,y,t) \leqq - q(x,y)\, y^2 e^{-M}$$

which according to Theorem (II-1) implies stability of the null solution of the system (III-4), because $q(x,y) \geqq 0$ according to condition (e) above.

B. Stability in the Case of a Periodic Input for Finite Time

Definition (III-1): $\varphi(t)$ is said to be periodic for finite time if:

$$\varphi(t) = \varphi(t + T) \qquad \text{for} \qquad t_o \leq t \leq t_1 - T$$

$$\varphi(t) \equiv 0 \qquad \text{for} \qquad t > t_1$$

where T is the period. From above definition it follows that

$$\lim_{t\to\infty} \int_{t_o}^{t} |\varphi(\tau)|d\tau = \int_{t_o}^{t_1} |\varphi(\tau)|d\tau = M < \infty \qquad \text{(III-11)}$$

Therefore $\varphi(t)$ satisfies the condition (a) of Theorem (III-1). As a result of this, Theorems (III-1) and (III-2) will also be true for the case $\varphi(t)$ periodic for finite time. This seems to be rather an interesting result, because the systems under analysis are non-anticipative, therefore at any time $t = t^* \leqq t_1 - T$ they behave as if $\varphi(t)$ were going to be periodic for all $t \geqq t^*$.

Since in many practical systems $\varphi(t)$ is periodic for finite time, one may use the above method in the study of stability of such systems. However, in applying the above method to particular systems, one should note that if t_1 is too large, even though theoretically the null solutions of the systems are always stable, the region of initial disturbances may become too small to be of any practical interest. This idea will become more clear in sub-section (III-C) where a sufficiency region of stability is constructed.

C. Sufficiency Region of Stability

If the systems whose stability were analyzed in (III-A) were linear, then their stability properties would have been global and would have depended only on the parameters of the systems. For nonlinear systems this fails to be true and stability properties become local properties of the solutions. It is of practical interest to know the extent of the region in which these local stability properties hold. To find the extent of these regions, one should examine the nonlinearities.

It was shown, in Theorem (III-1), in order for the trivial solution of (III-5) to be stable, all (x,y) should belong to a set Ω defined by (III-6). Now a region Ω_I, a neighborhood of the origin, will be determined such that $\left(x(t_o);y(t_o)\right) \in \Omega_I \Rightarrow \left(x(t);y(t)\right) \in \Omega_E$ for all $t \geqq t_o$, where Ω_E is some bounded region which includes the region Ω_I. The region Ω_I is called a Sufficiency Region of Stability.

Determination of Ω_I

Malkin [9], in his discussion of the proof of Theorem (II-1), has shown that if the coordinates of the initial perturbation, for the system (II-3), are such that $V\left(\underset{\sim}{x}(t_o),\ t_o\right) < c$, then the moving surface $V(\underset{\sim}{x},t) = c$ will always lie within the stationary surface $W(\underset{\sim}{x}) = c$, where $V(\underset{\sim}{x};t)$ is a Liapunov function, c is some constant and $W(\underset{\sim}{x})$ is some positive definite function such that $V(\underset{\sim}{x};t) \geqq W(\underset{\sim}{x})$. This implies that if the point $\underset{\sim}{x}$ whose motion is defined by equation (II-3), at some instant of time, falls within the surface $V(\underset{\sim}{x};t) = c$, then it will remain within this surface at all times.

The above information will be used to construct the region Ω_I and Ω_E. Let

$$\Omega_I \equiv \left\{(x_o,y_o) \ : \ V(x_o,y_o;t_o) < c\right\} \qquad \text{(III-12)}$$

Substituting for $V(x_o,y_o;t_o)$ from (III-7) into the above, one obtains

$$\Omega_I = \left\{(x_o,y_o) \ : \ [\ \tfrac{1}{2}\, y_o^2 + F(x_o)] < c\right\} \qquad \text{(III-13)}$$

In the proof of Theorem (III-1), $W(x,y)$ was defined as

$$W(x,y) = [\ \tfrac{1}{2}\, y^2 + F(x)]\ e^{-M} \leqq V(x,y;t) \qquad \text{(III-14)}$$

Therefore

$$W(x,y) = c \Rightarrow \frac{1}{2} y^2 + F(x) = c\, e^M \qquad \text{(III-15)}$$

In general, it is desirable to have the sufficiency region Ω_I as large as possible. This would imply that one should choose c as large a constant as possible. For linear systems, since the stability is complete, c can be chosen arbitrarily. For nonlinear systems, in general, c cannot be chosen arbitrarily large because then (x,y) may assume values for which $\dot{V}(x,y;t) \nleqq 0$. It was shown in the proof of Theorem (III-1), in order that $\dot{V}(x,y;t) \leqq 0$ in Ω , one should require $F(x) \geqq \frac{1}{2} r^2(x)$ in Ω^* . Now let

$$F(x) = \frac{1}{2} r^2(x) \qquad \text{(III-16)}$$

and assume $x_{n1}, x_{n2}, \ldots$ and $x_{p1}, x_{p2}, \ldots$ are the negative and positive roots of (III-16). Let

$$x^* \equiv \min \{x_{p1}, x_{p2}, \ldots, |x_{n1}|, |x_{n2}|, \ldots\} . \qquad \text{(III-17)}$$

Now all $|x| \leqq x^* \Rightarrow \dot{V}(x,y;t) \leqq 0$. Let

$$c \equiv F(x^*)\, e^{-M} \qquad \text{(III-18)}$$

and substitute into (III-15) to obtain:

$$\frac{1}{2} y^2 + F(x) = F(x^*) \qquad \text{(III-19)}$$

Let

$$\Omega_E \equiv \left\{(x,y) : [\, \frac{1}{2} y^2 + F(x)] \leqq F(x^*)\right\} \qquad \text{(III-20)}$$

Since $F(x)$ is monotone increasing (II-20) implies that all

Also let

$$\Omega^*_{PI} \equiv \{(x_o, y_o) : [\tfrac{1}{2} y_o^2 + F(x_o)] < c^*\} \qquad \text{(III-24)}$$

Now one may vary c^* in (III-24) by varying α in (III-22) until Ω^*_{PI} gets as close, from the outside, to Ω_{PI} as is desired, i.e., $\Omega_{PI} \subseteqq \Omega^*_{PI}$, (see Figure 1). Now the ensuing motions are restricted to a set Ω^*_{PE} given by

$$\Omega^*_{PE} = \{(x,y) : [\tfrac{1}{2} y^2 + F(x)] \leqq F(\tfrac{x^*}{\alpha})\} \qquad \text{(III-25)}$$

This is obtained in the same way as Ω_E in (III-20) except that c is replaced by c^*, i.e., x^* is replaced by $\frac{x^*}{\alpha}$. Now $\Omega_{PI} \subseteqq \Omega^*_{PI} \Rightarrow \Omega_{PE} \subseteqq \Omega^*_{PE}$ therefore $(x(t_o), y(t_o)) \in \Omega_{PI} \Rightarrow (x(t), y(t)) \in \Omega^*_{PE}$.

Remark (III-1): Since the region Ω^*_{PI} rather than Ω_{PI} is used as a region for initial perturbations and since Ω_{PI} is imbedded in Ω^*_{PI}, therefore one does not need to know the exact region Ω_{PI} as long as one knows that it is imbedded in Ω^*_{PI}.

Remark (III-2): As is seen from (III-21), the size of the region of initial perturbation Ω_I is directly proportional to e^{-M}. This implies that the smaller the M, the larger Ω_I could be. When $\varphi(t)$ is periodic for finite time, if t_1 is too large then according to (III-11) M may become very large and this in turn implies that Ω_I may become too small to be of any practical interest.

IV. STABILITY ANALYSIS, THROUGH LINEARIZATION, OF SYSTEMS WITH PERIODIC INPUT

In this section the stability properties of

$$\ddot{x} + f(x) + q(x,\dot{x})\dot{x} - \varphi(t)\ r(x) = S(x;t) \qquad \text{(IV-1)}$$

will be studied under the condition that $\varphi(t)$ is periodic of period T and $S(x;t)$ is a persistent disturbance which is bounded in the mean in the sense of definition (II-6). First the equation

$$\ddot{x} + f(x) + q(x,\dot{x})\dot{x} - \varphi(t)\ r(x) = 0 \qquad \text{(IV-2)}$$

will be considered and then the results of its analysis will be extended to (IV-1) by means of Theorem (II-4). As before, it will be assumed that $f(x)$, $q(x,\dot{x})$, $\varphi(t)$ and $r(x)$ are all of such a nature that the existence of a unique solution for (IV-2) is assured. It will be assumed also that $f(x)$, $q(x,\dot{x})$, $r(x)$ and $\varphi(t)$ can be represented as follows:

$$\left.\begin{aligned} f(x) &= kx + \sum_{i=o}^{\infty} k_i\, x^{2+i} \\ r(x) &= cx + \sum_{i=o}^{\infty} c_i\, x^{2+i} \\ q(x,\dot{x}) &= \ell + q^*(x,\dot{x}) \end{aligned}\right\} \qquad \text{(IV-3)}$$

where $\ell > 0$ and $q^*(x,\dot{x})$ is such that its power series expansion in x and $\dot{x}$ starts with terms of degree no less than two.

$$\left.\begin{aligned} &\varphi(t) = \varphi_o + \varphi^*(t) \\ \text{where} \\ &\int_o^T \varphi^*(\tau)\,d\tau = 0\ , \quad \varphi_o \equiv \text{constant} \end{aligned}\right\} \qquad \text{(IV-4)}$$

Substituting (IV-3) and (IV-4) into (IV-2) one obtains:

$$\ddot{x} + [B^* - \mu\varphi^*(t)]\, x + \ell\dot{x} + F(x,\dot{x};t) = 0 \qquad \text{(IV-5)}$$

where $B^* = k - c\varphi_o$, $\mu = c$ and

$$F(x,\dot{x};t) = q^*(x,\dot{x}) + \sum_{i=o}^{\infty} [k_i - c_i\, \varphi(t)]\, x^{2+i} \qquad \text{(IV-6)}$$

Note that $F(x,\dot{x};t)$ is periodic in t, i.e., $F(x,\dot{x};t) = F(x,\dot{x};t+T)$.

Now let (IV-1) and (IV-5) be written as follows:

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ - f(x) - q(x,y)y + \varphi(t)\; r(x) \end{Bmatrix} + \begin{Bmatrix} 0 \\ S(x;t) \end{Bmatrix} \qquad \text{(IV-7)}$$

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ - B^* + \mu\varphi^*(t) & - \ell \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0 \\ F(x,y;t) \end{Bmatrix} \qquad \text{(IV-8)}$$

Now consider the linear part of (IV-8)

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ - B^* + \mu\varphi^*(t) & - \ell \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \qquad \text{(IV-9)}$$

Assuming the characteristic exponents of (IV-9) are all less than unity in absolute value, then according to Theorem (II-3) and Remark (II-1) the trivial solution of (IV-8) is exponentially asymptotically stable. Therefore, according to Theorem (II-4) and Remark (II-2), the trivial solution of (IV-8) is stable in the presence of a persistent disturbance, $S(x;t)$, which is bounded in the mean. Then from the definition (II-7) it follows that the system described by Equation (IV-7) is stable.

Now the task is to study the linear system (IV-9) and to construct a sufficiency region of stability in parameter space (B,μ). For the sake of convenience in manipulation, a transformation of dependent variable

V. STABILITY ANALYSIS OF THE HILL EQUATION

Consider the equation

$$\ddot{x} + [B - \mu\varphi(t)]x = 0 \qquad \text{(V-1)}$$

in which $\varphi(t)$ is periodic of period $\pi^{(*)}$ with zero mean value, i.e., $\int_0^\pi \varphi(\tau)d\tau = 0$. Let $\varphi(t)$ be represented as a Fourier series:

$$\varphi(t) = \sum_{m=1}^{\infty} [a_m \cos 2mt + b_m \sin 2mt] \qquad \text{(V-2)}$$

where

$$a_m = \frac{2}{\pi}\int_0^\pi \varphi(s) \cos 2ms\, ds \ , \quad b_m = \frac{2}{\pi}\int_0^\pi \varphi(s) \sin 2ms\, ds$$

If $a_1 \neq 0$, $b_1 = 0$ and $a_m = b_m = 0$, $m = 2, 3, \ldots$

then (V-1) becomes the Mathieu equation. Another special case of Hill's equation which has received particular attention [19], [20] is that in which

$$a_1 \neq 0 \ , \ a_2 \neq 0 \ , \ b_1 = b_2 = 0 \ \text{and} \ a_m = b_m = 0 \ , \ m = 3, 4, \ldots$$

which is called Hill's 3-term equation.

The classical method of handling (V-1), as is found in most text books [10],[11], is to expand $\varphi(t)$ and $x(t)$ into a Fourier series. Substitution of these expansions in (V-1) yields an infinite system of simultaneous linear homogenous equations whose non-trivial solution can

(*) If $\varphi(t)$ has a period other than π , its period can be changed to π by a simple transformation of the independent variable t .

only be obtained if the corresponding infinite determinant is zero. Whittaker [10], using the condition of absolute convergence of the Fourier coefficients in the expansion of $\varphi(t)$, gave a method for a calculation of this infinite determinant. The computation is very difficult unless the Fourier coefficients in the expansion of $\varphi(t)$ decrease very rapidly. Malkin [9], using the Floquet solution [21],[22],[23]suggested another method for studying the stability of (V-1). His method will be discussed and modified in sub-sections (V-B) and (V-C). As a prerequisite to Malkin's method, certain basic considerations will be discussed in (V-A). It should be noted that the discussion in the next sub-section can be extended for systems of any order, but since (V-1) is of the second order, the discussion will be limited to second order systems.

A. Basic Considerations

Consider

$$\dot{x}_\alpha = \sum_{\beta=1}^{2} P_{\alpha\beta}(t)\, x_\beta \,, \quad \alpha = 1,2 \tag{V-3}$$

where $P_{\alpha\beta}(t)$ is a continuous periodic function of period π , i.e., $P_{\alpha\beta}(t + \pi) = P_{\alpha\beta}(t)$. Let $x_{\alpha\beta}(t)$ be a fundamental system of solutions for (V-3). It is easy to show that $x_{\alpha\beta}(t + \pi)$ is also a solution of (V-3). Since $x_{\alpha\beta}$ is a fundamental system, it follows that all other solutions should be expressible in terms of linear combinations of $x_{\alpha\beta}$. Hence,

$$x_{\alpha\beta}(t + \pi) = \sum_{j=1}^{2} a_{\alpha j}\, x_{j\beta}^{(t)} \,. \tag{V-4}$$

The characteristic equation(*) of (V-4) is ,

$$D(\rho) = \begin{bmatrix} a_{11} - \rho & a_{12} \\ a_{21} & a_{22} - \rho \end{bmatrix} = 0 \quad . \qquad \text{(V-5)}$$

Suppose that the fundamental system being considered is defined by the initial conditions

$$x_{\alpha\beta}(o) = \delta_{\alpha\beta} \qquad \text{(V-6)}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Assuming $t = 0$ in (V-4) and substituting (V-6) into it one obtains

$$x_{\alpha\beta}(\pi) = a_{\alpha\beta} \qquad \text{(V-7)}$$

Substitution of (V-7) in (V-5) yields

$$D(\rho) = \begin{bmatrix} x_{11}(\pi) - \rho & x_{12}(\pi) \\ x_{21}(\pi) & x_{22}(\pi) - \rho \end{bmatrix} = 0 \quad . \qquad \text{(V-8)}$$

Expanding the determinant in (V-8) one obtains

$$\rho^2 + A_1\rho + A_2 = 0 \quad , \qquad \text{(V-9)}$$

where

$$A_1 = -[x_{11}(\pi) + x_{22}(\pi)] \qquad \text{(V-10)}$$

$$A_2 = x_{11}(\pi)\, x_{22}(\pi) - x_{12}(\pi)\, x_{21}(\pi) \quad . \qquad \text{(V-11)}$$

It can be shown, [8],[14] that

$$A_2 = e^{\int_o^{\pi}(\text{trace } P(t))dt} \quad , \qquad \text{(V-12)}$$

(*) The fact that the characteristic equation does not depend upon the selected fundamental system is well known (see Malkin [9]).

where $P(t)$ is the matrix of coefficients $P_{\alpha\beta}(t)$, of (V-3). The fundamental system of solutions of (V-3) as obtained by Floquet's theorem is of the following form:

$$x_{\alpha\beta}(t) = \Psi_{\alpha\beta}(t)\ e^{\lambda_\beta t}\ , \ (\alpha, \beta = 1,2) \qquad , \qquad \text{(V-13)}$$

where

$$\lambda_\beta \equiv \frac{1}{\pi}\ \ell n\ \rho_\beta \qquad , \qquad (\beta = 1,2) \qquad \text{(V-14)}$$

and $\Psi_{\alpha\beta}(t)$ are all periodic functions of period π. The λ_β are distinct and they are called <u>characteristic exponents</u>. The ρ_β are the distinct roots of (V-9) and they can be represented in their general form as

$$\rho_\beta = a_\beta + ib_\beta \equiv r_\beta\ e^{i(\theta_\beta \pm 2n\pi)} \quad , \quad \beta = 1,2 \quad , \qquad \text{(V-15)}$$

where a_β and b_β are real and

$$r_\beta = \text{mod}\ \rho_\beta = \sqrt{a_\beta^2 + b_\beta^2}\ ;\ \theta_\beta = \tan\ \frac{b_\beta}{a_\beta} \quad . \qquad \text{(V-16)}$$

Substituting (V-15) in (V-14) and then putting this result into (V-13) one obtains

$$x_{\alpha\beta}(t) = \Psi_{\alpha\beta}(t)\ e^{i(\theta_\beta \pm 2n\pi)\frac{t}{\pi}}\ e^{(\ell n\ r_\beta)\frac{t}{\pi}} \quad , \qquad \text{(V-17)}$$

$$(\alpha\ , \beta = 1,\ 2)$$

Now the components $x_\alpha(t)$, of the solution vector, are given by

$$x_\alpha(t) = \sum_{\beta=1}^{2} x_{\alpha\beta}(t)\ x_\beta(o) \quad , \quad \alpha = 1,\ 2 \quad , \qquad \text{(V-18)}$$

where $x_\beta(o)$ are the initial conditions. Therefore, as it is seen from (V-17) and (V-18), the solution will be stable(*) if

$$\text{mod } \rho_\beta = r_\beta \leq 1 \qquad , \ \beta = 1, 2 \qquad , \qquad \text{(V-19)}$$

and it will be unstable if any of r_β is greater than unity. Therefore, to determine whether (V-3) is stable or not, it is only necessary to find ρ_β, the roots of (V-9). But this is not simple because the coefficient A_1 of the characteristic equation depends on the fundamental system of solutions [see (V-10)], while the fundamental system of solutions, in turn, depends on the roots of the characteristic equations [see (V-17)]. Therefore, one becomes enmeshed in a kind of vicious circle. A way out is through the use of some method of approximation for the evaluation of the fundamental system of solutions. Fortunately, the fundamental system of solutions is needed only for $t = \pi$, and also the stability conditions are determined by inequalities. This situation permits the use of an approximate method of integration in order to determine A_1 approximately.

B. Malkin's Procedure

It was shown in the previous section that in order to discuss the stability of (V-3) one needs to know mod ρ_β, where the ρ_β were the distinct roots of (V-9). Since (V-9) is a quadratic equation, its roots are

$$\rho_{1,2} = -\left(\frac{A_1}{2}\right) \pm \sqrt{\left(\frac{A_1}{2}\right)^2 - A_2} \qquad \text{(V-20)}$$

The exact value of A_2, as given by (V-12), is known. A_1 is unknown and to determine it one needs to know a fundamental system of solutions of (V-3)

(*) Note that the roots ρ_β should be distinct, i.e., $\rho_1 \neq \rho_2$.

for $t = \pi$ (see (V-10)) . As was pointed out in the previous section, it is not, in general, possible to find such a fundamental system exactly. Malkin [9] has suggested an approximate procedure through which a fundamental system may be computed with any desired degree of accuracy. His procedure will be outlined here, with some of the details omitted.

Consider (V-3) and expand $P_{\alpha\beta}(t)$ into a power series of the following form:

$$P_{\alpha\beta}(t) = q_{\alpha\beta}(t) + \sum_{j=1} \mu^j P^{(j)}_{\alpha\beta}(t) \quad , \tag{V-21}$$

where $q_{\alpha\beta}(t)$ and $P^{(1)}_{\alpha\beta}(t)$, $P^{(2)}_{\alpha\beta}(t)$, ... are all continuous periodic functions of period π , and it is assumed that (V-21) converges in a certain region $|\mu| \leqq E$. Consider also a fundamental system of solutions $x_{\alpha\beta}(t,\mu)$ of (V-3) represented by the following series expansion:

$$x_{\alpha\beta}(t,\mu) = x^{(o)}_{\alpha\beta}(t) + \sum_{j=1} \mu^j x^{(j)}_{\alpha\beta}(t) \tag{V-22}$$

converging for all values of t in the region $|\mu| < E$ and defined by the initial conditions

$$\begin{aligned} x^{(o)}_{\alpha\beta}(0) &= \delta_{\alpha\beta} \\ x^{(1)}_{\alpha\beta}(0) &= x^{(2)}_{\alpha\beta}(0) = \dots = 0 \end{aligned} \tag{V-23}$$

Substituting (V-21) and (V-22) into (V-3) and equating the coefficients of like powers of μ , one obtains a recursive system for the determination of all $x^{(o)}_{\alpha\beta}$, $x^{(1)}_{\alpha\beta}$, ... , namely

$$\dot{x}_{\alpha\beta}^{(o)} = \sum_{i=1}^{n} q_{\alpha i} \, x_{i\beta}^{(o)}$$

$$\dot{x}_{\alpha\beta}^{(1)} = \sum_{i=1}^{n} q_{\alpha i} \, x_{i\beta}^{(1)} + \sum_{j=1}^{n} p_{\alpha j}^{(1)} \, x_{j\beta}^{(o)} \qquad \text{(V-24)}$$

$$\cdots\cdots\cdots$$

$$\dot{x}_{\alpha\beta}^{(r)} = \sum_{i=1}^{n} q_{\alpha i} \, x_{i\beta}^{(r)} + \sum_{j=1}^{n} \sum_{s=1}^{r-1} p_{\alpha j}^{(r-s)} \, x_{j\beta}^{(s)}$$

$$(\alpha, \ \beta = 1, \ \ldots \ n)$$

As is seen, all these nonhomogenous linear systems of equations have the same homogenous part. If the homogenous system of equations

$$\dot{x}_{\alpha\beta}^{(o)} = \sum_{i=1}^{n} q_{\alpha i} \, x_{i\beta}^{(o)} \qquad \text{(V-25)}$$

has a closed integral, then (V-24), in conjunction with the initial conditions (V-23), permits a successive, complete determination of all $x_{\alpha\beta}^{(r)}$. Then, assuming $t = \pi$ in (V-22), and substituting the result in (V-10), one obtains an approximate value for A_1 .

a. <u>Zones of stability for the Hill Equation</u>(*)

Let (V-1) be written as follows:

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -B + \mu\varphi(t) & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \qquad \text{(V-26)}$$

Therefore $\underset{\sim}{P}(t)$, the matrix of coefficients $P_{\alpha\beta}(t)$ of (V-3), is given by

(*) It should be remarked here that these zones are in parameter space.

$$\underset{\sim}{P}(t) = \begin{bmatrix} 0 & 1 \\ -B + \mu\varphi(t) & 0 \end{bmatrix} \quad \text{(V-27)}$$

Hence, the <u>exact value</u> of A_2, as given by (V-12), is equal to <u>unity</u>. Therefore, if $\left(\frac{A_1}{2}\right)^2 < 1$ in (V-20), then $\rho_1 \neq \rho_2$ and $\text{mod}\ \rho = \sqrt{A_2} = 1$, and according to (V-19) the trivial solution would be stable. Hence the stability and instability boundary curves are given by

$$\left(\frac{A_1}{2}\right)^2 = 1 \quad \text{(V-28)}$$

Therefore A_1 is the key quantity in the determination of zones of stability. In the following, the boundaries of these zones will be determined by using Malkin's procedure to evaluate A_1 approximately.

b. <u>Determination of A_1 by First Approximation</u>

Let(*)

$$\underset{\sim}{Q} \equiv q_{\alpha\beta} = \begin{bmatrix} 0 & 1 \\ -B & 0 \end{bmatrix} \quad \text{(V-29)}$$

$$\underset{\sim}{P}^{(1)} \equiv P^{(1)}_{\alpha\beta} = \begin{bmatrix} 0 & 0 \\ \varphi(t) & 0 \end{bmatrix} \quad \text{(V-30)}$$

$$\underset{\sim}{X}(t) \equiv [x_{\alpha\beta}(t)] \quad \text{(V-31)}$$

$$\underset{\sim}{X}^{(o)}(t) \equiv [x^{(o)}_{\alpha\beta}(t)] \quad \text{(V-32)}$$

$$\underset{\sim}{X}^{(1)}(t) \equiv [x^{(1)}_{\alpha\beta}(t)] \quad \text{(V-33)}$$

From (V-27), (V-29), and (V-30),

(*) Matrix notation will be used now and then for the sake of convenience.

$$\underset{\sim}{P}(t) = \underset{\sim}{Q} + \mu P^{(1)} \qquad \text{(V-34)}$$

Therefore, the series (V-21) converges for all values of μ. Now according to the procedure suggested in (V-B), let

$$\underset{\sim}{X}(t) = \underset{\sim}{X}^{(o)}(t) + \mu\, \underset{\sim}{X}^{(1)}(t) \qquad \text{(V-35)}$$

be a fundamental system of solutions defined by the initial conditions:

$$\underset{\sim}{X}^{(o)}(0) = \underset{\sim}{I}$$

$$\underset{\sim}{X}^{(1)}(0) = \underset{\sim}{0} \qquad \text{(V-36)}$$

Substitute (V-34) and (V-35) into (V-26) to obtain the first two systems of (V-24)

$$\dot{\underset{\sim}{X}}^{(o)} = \underset{\sim}{Q}\, \underset{\sim}{X}^{(o)} \qquad \text{(V-37)}$$

$$\dot{\underset{\sim}{X}}^{(1)} = \underset{\sim}{Q}\, \underset{\sim}{X}^{(1)} + \underset{\sim}{P}^{(1)}\, \underset{\sim}{X}^{(o)} \qquad \text{(V-38)}$$

where $\underset{\sim}{Q}$ and $\underset{\sim}{P}^{(1)}$ are defined by (V-29) and (V-30). The solution of the above system yields(*):

$$x_{11}^{(o)}(\pi) + x_{22}^{(o)}(\pi) = 2 \cos \sqrt{B}\, \pi \qquad \text{(V-39)}$$

$$x_{11}^{(1)}(\pi) + x_{22}^{(1)}(\pi) = \frac{\sin \sqrt{B}\pi}{\sqrt{B}} \int_0^{\pi} \varphi(s)\, ds \qquad \text{(V-40)}$$

Therefore, from (V-10), (V-35), (V-39) and (V-40) one obtains

$$A_1 = -\,2 \cos \sqrt{B}\pi - \mu\, [x_{11}^{(1)}(\pi) + x_{22}^{(1)}(\pi)] \qquad \text{(V-41)}$$

(*) See Appendix A for details.

or

$$A_1 = -2\cos\sqrt{B}\pi - \mu\frac{\sin\sqrt{B}\pi}{\sqrt{B}}\int_0^{\pi}\varphi(s)ds \qquad \text{(V-42)}$$

But $\int_0^{\pi}\varphi(s)ds = 0$; therefore,

$$\left(\frac{A_1}{2}\right)^2 = \cos^2\sqrt{B}\pi \qquad \text{(V-43)}$$

but according to (V-28), $\left(\frac{A_1}{2}\right)^2 = 1$ should yield the boundaries of stability and instability zones, that is

$$\cos^2\sqrt{B}\pi = 1 \qquad \text{(V-44)}$$

should define these boundaries. As is seen, (V-44) gives only the points

$$B = n^2 \quad , \quad n = 0, 1, 2, \ldots \qquad \text{(V-45)}$$

as boundaries. It would be very discouraging if the first approximation gives only a sequence of points as the boundaries of stability and instability zones. It would be even more so if one notices that these points are completely independent of the function $\varphi(t)$.

Malkin obtained the expression (V-41) from his first approximation, but apparently he did not notice the fact that

$$\int_0^{\pi}\varphi(s)ds = 0 \Rightarrow x_{11}^{(1)}(\pi) + x_{22}^{(1)}(\pi) = 0 \qquad \text{(V-46)}$$

and introduced another expansion in (V-26), namely

$$B = n^2 + \sum_{i=1} \alpha_i \mu^i \quad n = 0, 1, 2, \ldots \qquad \text{(V-47)}$$

Then, by setting the coefficients of similar powers of μ equal to zero, he obtained systems of equations analogous to system (V-24).

There are certain inherent disadvantages to this method. One disadvantage is that the terms going to infinity must be absent; that is, one must insure the periodicity of solutions by ferreting out the coefficients of secular terms and setting them equal to zero. Another disadvantage is that for each value of $n = n^*$ a different system of differential equations needs to be studied, and the results do not improve on any of the solutions for which $n < n^*$. In the following, Malkin's procedure will be modified by using a value for A_2 consistent with A_1, rather than its exact value from (V-12), which is unity. This, as will become clear later on, yields the boundaries of the stability and instability zones. No use will be made of expansion (V-47), and the difficult problem of ferreting out secular terms does not arise.

C:a. Modified Malkin's Method, First Approximation

Here, instead of using an exact value for A_2, as given by (V-12), one determines A_2 to the same degree of accuracy as A_1. This would then be a more consistent method of approximation. A_2 will be evaluated through (V-11), in which the fundamental system of solutions has been obtained by the method of approximation described in (V-B).

To determine the boundaries of the zones of stability and instability, instead of setting $A_2 = 1$ and $\left(\frac{A_1}{2}\right)^2 < 1$, one sets

$$\left(\frac{A_1}{2}\right)^2 < A_2 \qquad \text{(V-48)}$$

$$A_2 \leqq 1 \qquad \text{(V-49)}$$

The inequalities (V-48) and (V-49) insure that $\rho_1 \mp \rho_2$ and mod ρ_1 = mod ρ_2 $= \sqrt{A_2} \leq 1$ which, according to (V-19), implies stability. Now if the fundamental system of solutions $\underset{\sim}{X}(t)$ in (V-35) [which has been obtained by a first approximation through (V-36), V-37), and (V-38)] is substituted into (V-11), there results(*)

$$A_2 = 1 - \frac{\mu^2}{4B}\left\{\left[\int_o^{\pi} \varphi(s)\cos 2\sqrt{B}S\, ds\right]^2 + \left[\int_o^{\pi} \varphi(s)\sin 2\sqrt{B}S\, ds\right]^2\right\} \tag{V-50}$$

Replacement of $\varphi(s)$ in the above by its Fourier series representation (V-2) yields

$$A_2 = 1 - \frac{\mu^2}{4B}\cdot\left[\left(\sum_{m=1}^{\infty}\frac{a_m\sqrt{B}}{B-m^2}\right)^2 + \left(\sum_{m=1}^{\infty}\frac{mb_m}{B-m^2}\right)^2\right]\sin^2\sqrt{B}\pi \qquad \text{(V-51)}$$

The substitution of (V-43) and (V-51) in the inequality (V-48) yields

$$\frac{\mu^2}{4B}\cdot\left[\left(\sum_{m=1}^{\infty}\frac{a_m\sqrt{B}}{B-m^2}\right)^2 + \left(\sum_{m=1}^{\infty}\frac{mb_m}{B-m^2}\right)^2\right] < 1 \qquad \text{(V-52)}$$

Comparing (V-52) with (V-51) it is seen that the inequality (V-49) is also satisfied. Now from (V-52) one obtains:

$$\boxed{|\mu| < \frac{2\sqrt{B}}{\sqrt{\left(\sum_{m=1}^{\infty}\frac{a_m\sqrt{B}}{B-m^2}\right)^2 + \left(\sum_{m=1}^{\infty}\frac{mbm}{B-m^2}\right)^2}}} \qquad \text{(V-53)}$$

(*) See Appendix A for details.

If $\varphi(t)$ is even, then $\sum_{m=1}^{\infty} \frac{mb_m}{B-m^2} = 0$ and $\sqrt{B}$ cancels from the numerator and denominator of (V-53), but if $\varphi(t)$ is not even, then (V-53) is true only for $\sqrt{B}$ real. Equation (V-53) defines the stability zones in μ, B space. In section VI it will be applied to Mathieu's equation and Hill's-3-term equation, and it will be seen that it compares very well with results available for those equations. In the following, a second approximation is worked out, and, as will be seen in section VI where it is applied to Mathieu's equation, it yields a marked improvement on the first approximation.

C:b. Modified Malkin's Method - Second Approximation

The second approximation proceeds in the same way as the first one except for the fact that three terms of series (V-21) and (V-22) will be considered; therefore

$$\underset{\sim}{P}(t) = \underset{\sim}{Q}(t) + \mu\underset{\sim}{P}^{(1)}(t) + \mu^2\underset{\sim}{P}^{(2)}(t) \tag{V-54}$$

$$\underset{\sim}{X}(t) = \underset{\sim}{X}^{(o)}(t) + \mu\underset{\sim}{X}^{(1)}(t) + \mu^2\underset{\sim}{X}^{(2)}(t) \quad , \tag{V-55}$$

where $X(t)$ is the fundamental system of solutions defined by the initial conditions

$$\begin{aligned} \underset{\sim}{X}^{(o)}(0) &= \underset{\sim}{I} \\ \underset{\sim}{X}^{(1)}(0) = \underset{\sim}{X}^{(2)}(0) &= \underset{\sim}{0} \end{aligned} \tag{V-56}$$

Note that $\underset{\sim}{P}^{(2)}(t)$ is zero because $\underset{\sim}{P}(t)$ is given by (V-27). Substituting (V-54) and (V-55) into (V-26) and setting $\underset{\sim}{P}^{(2)}(t)$ equal to zero and equating the coefficients of like powers of μ, one obtains

$$\left.\begin{aligned}\dot{\underset{\sim}{X}}^{(o)} &= \underset{\sim}{Q}\,\underset{\sim}{X}^{(o)}\\ \dot{\underset{\sim}{X}}^{(1)} &= \underset{\sim}{Q}\,\underset{\sim}{X}^{(1)} + \underset{\sim}{P}^{(1)}\,\underset{\sim}{X}^{(o)}\\ \dot{\underset{\sim}{X}}^{(2)} &= \underset{\sim}{Q}\,\underset{\sim}{X}^{(2)} + \underset{\sim}{P}^{(1)}\,\underset{\sim}{X}^{(1)}\end{aligned}\right\} \quad \text{(V-57)}$$

Note that the first two equations are identical with (V-36) and (V-37) respectively. The solution(*) of (V-57) at $t = \pi$ yields $\underset{\sim}{X}^{(o)}(\pi)$, $\underset{\sim}{X}^{(1)}(\pi)$, and $\underset{\sim}{X}^{(2)}(\pi)$ whereby substituting them into (V-55), one obtains $\underset{\sim}{X}(\pi)$. Now, substitution of $\underset{\sim}{X}(\pi)$ into (V-10) and (V-11) yields A_1 and A_2 . Putting A_1 and A_2 into the inequalities (V-48) and (V-49) one obtains the following inequalities, respectively: (see Appendix B for details)

$$C_1 + C_2\,\mu^2 + C_3\,\mu^3 + C_4\,\mu^4 > 0 \qquad \text{(V-58)}$$

$$S_1\,\mu + S_2\,\mu^2 \leqq 0 \qquad \text{(V-59)}$$

where

$$\left.\begin{aligned}C_1 &= \sin^2\sqrt{B}\pi\\ C_2 &= -\frac{1}{4B}\left[(2A_{sc} - A_s\,A_c)\sin 2\sqrt{B}\pi + \frac{A_c^2 + A_s^2}{2}\cos^2\sqrt{B}\pi\right]\\ C_3 &= \frac{1}{2B\sqrt{B}}\,(A_c\,A_{s2} - A_s\,A_{c2})\\ C_4 &= \frac{1}{16B^2}\left\{\left[\frac{A_c^2 + A_s^2}{2}\sin\sqrt{B}\pi - (2\,A_{sc} - A_s\,A_c)\cos\sqrt{B}\pi\right]^2 - 4\,(A_{c2}^2 + A_{s2}^2)\right\}\end{aligned}\right\} \quad \text{(V-58:a)}$$

(*) See Appendix B for details.

$$\left.\begin{aligned} S_1 &= \frac{1}{2B\sqrt{B}} \left(A_c A_{s2} - A_s A_{c2}\right) \\ S_2 &= \frac{1}{16B^2} \left[\left(\frac{A_c^2 + A_s^2}{2} \right)^2 - 4 \left(A_{c2}^2 + A_{s2}^2\right) + \left(2 A_{sc} - A_c A_s\right)^2 \right] \end{aligned}\right\} \quad \text{(V-59:a)}$$

while

$$\left.\begin{aligned} \frac{A_c^2 + A_s^2}{2} &= \frac{\sin^2 \sqrt{B}\pi}{2} \left(K_1^2 + K_2^2\right) \\ 4 \left(A_{c2}^2 + A_{s2}^2\right) &= \frac{\sin^2 \sqrt{B}\pi}{4} \left(H_1^2 + H_2^2\right) \\ A_c A_{s2} - A_s A_{c2} &= \frac{-\sin^2 \sqrt{B}\pi}{4} \left(H_1 K_2 + H_2 K_1\right) \\ 2 A_{sc} - A_c A_s &= \frac{\pi}{4} Q_3 - \frac{\sin 2\sqrt{B}\pi}{2} \Bigg[\left(K_1 \sin \sqrt{B}\pi + K_2 \cos \sqrt{B}\pi\right)^2 \\ &\quad - \left(\frac{\sin 2\sqrt{B}\pi}{2} Q_1 + \frac{\cos 2\sqrt{B}\pi}{2} Q_2 \right) \Bigg] \end{aligned}\right\} \quad \text{(V-59:b)}$$

where

$$K_1 = \sum_{m=1}^{\infty} \frac{a_m \sqrt{B}}{B - m^2}$$

$$K_2 = \sum_{m=1}^{\infty} \frac{m b_m}{B - m^2}$$

$$H_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{1}{(\sqrt{B} - m)^2 - n^2} \left(\frac{n}{\sqrt{B} - m} a_m b_n + b_m a_n \right) + \frac{1}{(\sqrt{B} + m)^2 - n^2} \left(\frac{n}{\sqrt{B} + m} a_m b_n - b_m a_n \right) \right]$$

$$H_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{1}{(\sqrt{B} - m)^2 - n^2} \left(a_m a_n - \frac{n}{\sqrt{B} - m} b_m b_n \right) + \frac{1}{(\sqrt{B} + m)^2 - n^2} \left(a_m a_n + \frac{n}{\sqrt{B} + m} b_m b_n \right) \right]$$

(V-59:c)

$$Q_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[(T_1 + T_2)\, a_m\, b_n + (T_4 - T_3)\, b_m\, a_n \right]$$

$$Q_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[(T_2 - T_1)\, b_m\, b_n - (T_3 + T_4)\, a_m\, a_n \right]$$

(V-59:c)

$$Q_3 = \sum_{m=1}^{\infty} \frac{\sqrt{B}}{B-m^2} \, (a_m^2 + b_m^2)$$

$$T_1 = \frac{n}{(\sqrt{B}+m)\ [(2\sqrt{B}+m)^2 - n^2]}$$

$$T_2 = \frac{n}{(\sqrt{B}-m)\ [(2\sqrt{B}-m)^2 - n^2]}$$

$$T_3 = \frac{2\sqrt{B}+m}{(\sqrt{B}+m)\ [(2\sqrt{B}+m)^2 - n^2]}$$

$$T_4 = \frac{2\sqrt{B}-m}{(\sqrt{B}-m)\ [(2\sqrt{B}-m)^2 - n^2]}$$

The inequalities (V-58) and (V-59) define the stability zones. To define the boundaries of these zones one should first find the roots of

$$C_1 + C_2\, \mu^2 + C_3\, \mu^3 + C_4\, \mu^4 = 0 \quad , \qquad \text{(V-60)}$$

and then pick the one which satisfies the inequality (V-59). If more than one root satisfies the inequality (V-59), one should pick the root with the least absolute value. If (V-60) does not have any real root or if none of its roots satisfies the inequality (V-59), then consider (V-59) as an equality, that is

$$S_1\, \mu + S_2\, \mu^2 = 0 \quad \Rightarrow \quad \mu = -\frac{S_1}{S_2} \quad . \qquad \text{(V-61)}$$

Now consider this μ if it satisfies the inequality (V-58).

Remark (V-1): To obtain the complete stability region in (μ, B) space, one should replace μ by $-\mu$ in equation (V-1). This yields the following inequalities:

$$C_1 + C_2 \mu^2 - C_3 \mu^3 + C_4 \mu^4 > 0 \qquad \text{(V-62)}$$

$$- S_1 \mu + S_2 \mu^2 \leqq 0 \qquad \text{(V-63)}$$

instead of inequalities (V-58) and (V-59) respectively. The same procedure as explained in the previous paragraph should be applied, using the above inequalities rather than inequalities (V-58) and (V-59) as the basis to obtain the rest of the boundary curves. This means that one should consider the polynomial

$$C_1 + C_r \mu^2 - C_3 \mu^3 + C_4 \mu^4 = 0 \qquad \text{(V-64)}$$

in conjunction with inequality (V-63). It is obvious that for a given set of coefficients, a root of (V-64) satisfying the inequality (V-63) is the same, in magnitude, as its corresponding one obtained from (V-60) except for a change in sign. This means that the stability zones are symmetric about the B axis. This fact leads one to follow a simpler procedure; namely, for different values of B, find the roots of (V-60) which satisfy the inequality (V-59), and then take their absolute values. This yields the stability regions on the upper half of μ,B space. The mirror image of it on the lower half of μ,B space completes the stability zones.

VI. COMPARISON AND APPLICATIONS

In this section, first the general approximate results obtained in section V will be compared with some of the few exact results, which are known, and then Equation (I-1) and its special cases will be applied to a study of the problem of stability of a lumped parameter model of a column.

A:a. Comparison of First Approximation with the Exact Results for the Mathieu Equation

The Mathieu equation is

$$\ddot{x} + [B - \mu \cos 2t]\, x = 0 \quad . \qquad \text{(VI-1)}$$

Therefore it is seen that

$$\varphi(t) = \cos 2t \qquad \text{(VI-2)}$$

Comparison of (VI-2) with relation (V-2) yields:

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \cos^2 2t \, dt = 1$$

$$b_1 = a_m = b_m = 0 \quad , \quad m = 2, 3, \ldots \qquad \text{(VI-3)}$$

Substitution of (VI-3) in (V-53) yields

$$|\, \mu \,| < 2\sqrt{(B-1)^2} \qquad \text{(VI-4)}$$

(VI-4) is plotted in (Fig. 2) where the exact boundaries of stability zones are also plotted by using Ince's table of the elliptic-cylinder functions [24],[25] .

A:b. Comparison of Second Approximations with the Exact Results for the Mathieu Equation

Again $\varphi(t) = \cos 2t$ and $a_1 = 1$, $b_1 = b_m = a_m = 0$, $m = 2, 3, \ldots$.
Now the coefficients C_1, C_2, C_3, C_4, S_1 and S_2 of the inequalities (V-58) and (V-59) will be calculated. Substituting $a_1 = 1$ and $b_1 = a_m = b_m = 0$ for $m = 2, 3, \ldots$ in relations (V-59:c), one obtains

$$K_1 = \frac{\sqrt{B}}{B-1} \quad , \quad K_2 = 0$$

$$Q_1 = 0 \quad , \quad Q_2 = -\frac{B}{(B-1)^2} \quad , \quad Q_3 = \frac{\sqrt{B}}{B-1}$$

$$H_1 = 0 \quad , \quad H_2 = \frac{2}{B-4}$$

Substitution of these results into relations (V-59:b) yields

$$\frac{A_c^2 + A_s^2}{2} = \frac{B \sin^2 \sqrt{B}\pi}{2(B-1)^2}$$

$$4\,(A_{c2}^2 + A_{s2}^2) = \frac{\sin^2 \sqrt{B}\pi}{(B-4)^2}$$

$$A_c\, A_{s2} - A_s\, A_{c2} = -\frac{\sqrt{B}\sin^2 \sqrt{B}\pi}{2(B-1)(B-4)}$$

$$2\,A_{sc} - A_c\, A_s = \frac{\pi\sqrt{B}}{4(B-1)} - \frac{B \sin \sqrt{B}\pi}{4(B-1)^2}$$

Substitution of these in expressions (V-58:a) and (V-59:a) yields the following:

$$C_1 = \sin^2 \sqrt{B}\pi$$

$$C_2 = -\frac{\pi \sin 2\sqrt{B}\pi}{16\sqrt{B}(B-1)}$$

$$C_3 = -\frac{\sin^2 \sqrt{B}\pi}{4B(B-1)(B-4)}$$

$$C_4 = \left(\frac{\sin \sqrt{B}\pi}{8(B-1)^2} - \frac{\pi \cos \sqrt{B}\pi}{16\sqrt{B}(B-1)}\right)^2 - \frac{\sin^2 \sqrt{B}\pi}{16B^2\ (B-4)^2}$$

$$S_1 = -\frac{\sin^2 \sqrt{B}\pi}{4B\ (B-1)(B-4)}$$

$$S_2 = \frac{1}{16}\left\{\sin^2\sqrt{B}\pi\left[\frac{1}{4(B-1)^4} - \frac{1}{B^2(B-4)^2}\right] - \frac{\pi \sin 2\sqrt{B}\pi}{8\sqrt{B}(B-1)^3} + \frac{\pi^2}{16B(B-1)^2}\right\}$$

Now according to Equation (V-60) the polynomial

$$\sin^2\sqrt{B}\pi - \frac{\pi \sin 2\sqrt{B}\pi}{16\sqrt{B}(B-1)}\mu^2 - \frac{\sin^2\sqrt{B}\pi}{4\ B(B-1)(B-4)}\mu^3 + \left[\left(\frac{\sin\sqrt{B}\pi}{8(B-1)^2} - \frac{\pi\cos\sqrt{B}\pi}{16\sqrt{B}(B-1)}\right)^2 - \frac{\sin\sqrt{B}\pi}{16B^2(B-4)^2}\right]\mu^4 = 0 \tag{VI-5}$$

in conjunction with the inequality

$$-\frac{\sin^2\sqrt{B}\pi}{4B(B-1)(B-4)}\mu + \frac{1}{16}\left\{\sin^2\sqrt{B}\pi\left[\frac{1}{4(B-1)^4} - \frac{1}{B^2(B-4)^2}\right] - \frac{\pi\sin 2\sqrt{B}\pi}{8\sqrt{B}(B-1)^3} + \frac{\pi^2}{16B(B-1)^2}\right\}\mu^2 \leqq 0 \tag{VI-6}$$

yields the boundaries between stable and unstable zones.

The roots of the polynomial(VI-5) were calculated for $-2 \leq B \leq 10$ in 0.1 steps on a CDC 6400 digital computer. In each step the root which satisfied the inequality (VI-6) was selected. In cases where more than one root satisfied the inequality, the one with the least absolute value was considered. In cases where the polynomial did not have any real root or if none of its root satisfied the inequality (VI-6), then the inequality

(VI-6) was considered as an equality and its root was chosen if it satisfied the inequality (V-58). After the μ was obtained in this way, then its absolute value $|\mu|$ (see Remark (V-1)) was taken. The results of the second approximation are plated in Figure 2.

A:c. Comparison of First Approximation with an Exact Result for the Hill-3-Term Equation

Klotter and Kotowski [20] have given stability charts for the following equation:

$$\ddot{y} + (\lambda + \gamma_1 \cos \tau + \gamma_2 \cos 2\tau)\, y = 0 \qquad \text{(VI-7)}$$

for the special cases of $\gamma_2 = +.5$ and $\gamma_2 = -.5$.

Let $\tau = 2t$ in (VI-7) to obtain

$$\ddot{y} + (4\lambda + 4\gamma_1 \cos 2t + 4\gamma_2 \cos 4t)\, y = 0 \qquad \text{(VI-8)}$$

or

$$\ddot{y} + [\bar{\lambda} - \bar{\gamma}_1 (-\cos 2t - \frac{\bar{\gamma}_2}{\bar{\gamma}_1} \cos 4t)]\, y = 0 \qquad \text{(VI-9)}$$

where

$$\bar{\lambda} = 4\lambda \quad , \quad \bar{\gamma}_1 = 4\gamma \quad , \quad \bar{\gamma}_2 = 4\gamma_2 \qquad \text{(VI-10)}$$

Comparison of Equation (VI-9) with Equation (V-1) yields:

$$B = \bar{\lambda} \qquad \text{(VI-11)}$$

$$\mu = \bar{\gamma}_1 \qquad \text{(VI-12)}$$

$$\varphi(t) = -\cos 2t - \frac{\bar{\gamma}_2}{\bar{\gamma}_1} \cos 4t \qquad \text{(VI-13)}$$

Comparing Equation (VI-13) with (V-2) one obtains

$$\varphi(t) = a_1 \cos 2t + a_2 \cos 4t \qquad \text{(VI-14)}$$

where $a_1 = -1$, $a_2 = -\frac{\bar{\gamma}_2}{\bar{\gamma}_1}$, $b_1 = b_2 = b_m = a_m = 0$, $m = 3, 4, \ldots$

Now substitution of (VI-13) in (V-53) yields

$$|\mu| < \left| \frac{2\bar{\gamma}_1 (B-1)\ (B-4)}{\bar{\gamma}_1 (B-4) + \bar{\gamma}_2 (B-1)} \right| \qquad \text{(VI-15)}$$

To compare with Klotter and Kotowski's result for the case $\gamma_2 = .5$, let $\bar{\gamma}_2 = 4\gamma_2 = 2$ in (VI-15) to obtain:

$$|\mu| < \left| \frac{2\bar{\gamma}_1 (B-1)\ (B-4)}{\bar{\gamma}_1 (B-4) + 2(B-1)} \right| \qquad \text{(VI-16)}$$

It is not possible to make a plot of the stability zones in μ-B space by using (VI-16) because of the presence of $\bar{\gamma}_1$. This means that an independent μ-B plot for the purpose of comparison is not possible, however, one can do a kind of mathematical comparison as follows: Assume a $\bar{\lambda} = \bar{\lambda}^*$ and read the value of $\bar{\gamma}_1 = \bar{\gamma}_1^*$ from Klotter and Kotowski's plot(†). Now according to (VI-12)

$$\mu = \bar{\gamma}_1^* \ . \qquad \text{(VI-17)}$$

Substitution of $\bar{\lambda}^*$ for B and $\bar{\gamma}_1^*$ for $\bar{\gamma}_1$ in (VI-16) yields a μ^* which may be compared with μ as given by (VI-17). This process is repeated for different values of $\bar{\lambda}$, to obtain a plot of μ versus B . This plot is superimposed on Klotter and Kotowski's plot in Figure 3 for the purpose of comparison. In exactly the same way another plot is

(†) If for $\bar{\lambda} = \bar{\lambda}^*$ there are two values for $\bar{\gamma}_1^*$, read them both and treat them separately.

obtained for the case $\gamma_2 = -.5$. This plot is superimposed on the corresponding Klotter and Kotowski's plot in Figure 4.

Remark (VI-1): Application of the second approximation to the Hill-3-term equation, of the form treated here, is avoided because of its laboriousness.

B. Applications

Equations (I-1) and its special cases have many applications [9], [11], [26] which will not be enumerated here. The main motivation of this work has been the problem of stability of a lumped parameter model of a column. In the following this problem will be discussed.

Stability of a Lumped Parameter Model of a Column

Consider a column which is idealized in such a manner that its mass is concentrated at its top and its material properties concentrated at its bottom (see Figure 5). The loading on the column is composed of an axial time varying load and an axial time varying base acceleration, plus a lateral time varying force and base acceleration. The motion of such a column is governed by the following differential equation:

$$\ddot{x} + \frac{T(x)}{m\ell^2} + \frac{\eta(x,\dot{x})\dot{x}}{m\ell^2} - \left[\frac{mg + P(t)}{m\ell} + \frac{\ddot{v}_b(t)}{\ell} \right] \sin x = \left[\frac{F(t)}{m\ell} - \frac{\ddot{u}_b(t)}{\ell} \right] \cos x \qquad \text{(VI-18)}$$

where

$T(x)$ = The elastic restoring moment

$\eta(x,\dot{x})\dot{x}$ = The retarding moment (damping)

m = The mass of the column

ℓ = The length of the column

$P(t)$ = The axial load on the column

$F(t)$ = The lateral force on the column

$\ddot{v}_b(t)$ = Vertical base acceleration

$\ddot{u}_b(t)$ = Horizontal base acceleration

let

$$f(x) \equiv \frac{T(x)}{m\ell^2}$$

$$q(x,\dot{x}) \equiv \frac{\eta(x,\dot{x})}{m\ell^2}$$

$$g(t) \equiv \frac{ng + P(t)}{m\ell} + \frac{\ddot{v}_b(t)}{\ell} \quad , \quad \text{axial effect}$$

$$h(t) \equiv \frac{F(t)}{m\ell} - \frac{\ddot{u}_b(t)}{\ell} \quad , \quad \text{lateral effect}$$

Substitution of these into (VI-18) yields

$$\ddot{x} + f(x) + q(x,\dot{x})\dot{x} - g(t) \sin x = h(t) \cos x \qquad \text{(VI-19)}$$

The following special cases of (VI-19) will be considered:

(1) $q(x,\dot{x}) = 0$, $h(t) = 0$, $\lim_{t\to\infty} \int_0^t |g(\tau)| d\tau \leq M < \infty$

$$xf(x) > 0 \text{ for } x \neq 0 \text{ and } f(o) = 0$$

Therefore equation (VI-19) reduces to

$$\ddot{x} + f(x) - g(t) \sin x = 0 \qquad \text{(VI-20)}$$

Comparison of this equation with Equation (III-4) yields $r(x) = \sin x$.

Therefore, if there exists a region about $x = 0$ in which

$\int_0^x f(\xi)d\xi \geqq \frac{1}{2}\sin^2 x$, then according to Theorem (III-1) the vertical position of the column is a stable equilibrium position.

(2) $g(t)$ is periodic of period π

$$h(t) \equiv 0$$

$f(x)$, $q(x,\dot{x})$ are representable by relations (IV-3). Expanding $\sin x$ and substituting the series into (VI-19) and considering the linear portion only one obtains:

$$\ddot{x} + kx + \ell\dot{x} - g(t)x = 0 \qquad \text{(VI-21)}$$

Let $g(t) \equiv G + \mu\varphi(t)$ such that $\int_0^\pi \varphi(t)dt = 0$ and also define $B^* \equiv k - G$, then (VI-21) reduces to

$$\ddot{x} + [B^* - \mu\varphi(t)]x + \ell\dot{x} = 0 \qquad \text{(VI-22)}$$

This is the differential equation (IV-11). Using the substitution (IV-12) and (IV-14) one obtains:

$$\ddot{x} + [B - \mu\varphi(t)]x = 0 \qquad \text{(VI-23)}$$

Now if the trivial solution of (VI-23) is stable, then according to (IV-12) the solution of (VI-22) is exponentially asymptotically stable and according to Remark (II-1), the trivial solution of the equation

$$\ddot{x} + f(x) + q(x,\dot{x})x - g(t)\sin x = 0 \qquad \text{(VI-24)}$$

is exponentially asymptotically stable.

(3) Consider the same problem as in (2), except that $h(t) \not\equiv 0$. It is assumed that $h(t)$ is bounded in the mean in the sense of definition (II-6). Comparison of (VI-19) with (IV-1) yields

$$S(x;t) = h(t) \cos x \tag{VI-25}$$

Hence, if $h(t)$ is bounded in the mean, so is $S(x;t)$, because $|\cos x| \leq 1$. Therefore, according to Theorem (II-4) and Remark (II-3), the null solution of the homogenous part of (VI-19), that is the null solution of (VI-24), is stable in the presence of the persistent disturbance $h(t) \cos x$.

C. Concluding Remarks

In section III a sufficiency region of stability Ω_I was constructed through the use of a Liapunov function. It should be noted that if the initial perturbations do not belong to Ω_I, the null solution is not necessarily unstable. The Liapunov stability theorems provide us only with sufficient conditions for stability. That is, if the initial perturbations belong to the region Ω_I, then the null solution of the system is stable. However, this does not imply that if the initial perturbations do not belong to Ω_I then the null solution is unstable. For that matter, one may construct instability region using instability theorems [1],[8].

Another important point to note is the merit of the Liapunov Direct Method. Given the differential equation of a system with its initial conditions, one can decide on the stability of the solution without solving the differential equation. Of course, the method has its shortcoming. It is not always easy to construct a Liapunov function.

In sections IV and V, a method completely different than the Liapunov method was used to study stability. In these sections, previous knowledge of the solution of the linearized differential equation was indispensable to the study of stability. One should also note that the stability regions obtained by the first and second approximations are regions in the space of the parameters of the system - B, μ . It turns out this way because we are essentially obtaining our conclusions from a study of a linear system.

As is seen from Figures 3 and 4, the first approximation as applied to the Hill-3-Term equation gives results which compare relatively well with the exact solution. It should be noted, however, that this comparison is not an independent one. That is, the boundary curves in Figures 3 and 4 are not a direct result of first approximation. A direct application of the first approximation is not possible because Klotter and Kotowski used three independent parameters while here the first approximation is in terms of two independent parameters μ and B .

When applied to the Mathieu equation, it is seen from Figure 2 that the results of second approximation are quite satisfactory.

The merit of these approximations is that they are very simple to use, and may be applied to quite general systems. The first approximation is especially very easy to apply, and in practice one is usually satisfied with the results of such an approximation.

APPENDICES

APPENDIX A

First Approximation & Determination of A_1 and A_2

The equations to be solved are

$$\dot{\underset{\sim}{X}}^{(o)} = \underset{\sim}{Q}\, \underset{\sim}{X}^{(o)} \tag{1}$$

$$\dot{\underset{\sim}{X}}^{(1)} = \underset{\sim}{Q}\, \underset{\sim}{X}^{(1)} + \underset{\sim}{P}^{(1)}\, \underset{\sim}{X}^{(o)} \tag{2}$$

with initial conditions

$$\underset{\sim}{X}^{(o)}(0) = \underset{\sim}{I} \quad , \quad X^{(1)}(0) = \underset{\sim}{0} \tag{3}$$

where

$$\underset{\sim}{X}^{(o)} = [x_{\alpha\beta}^{(o)}] \tag{4}$$

$$\underset{\sim}{X}^{(1)} = [x_{\alpha\beta}^{(1)}] \tag{5}$$

$$\underset{\sim}{Q} = [q_{\alpha\beta}] = \begin{bmatrix} 0 & 1 \\ -B & 0 \end{bmatrix} \tag{6}$$

$$\underset{\sim}{P}^{(1)} = [P_{\alpha\beta}^{(1)}] = \begin{bmatrix} 0 & 0 \\ \varphi(t) & 0 \end{bmatrix} \tag{7}$$

Now the solution of (1) with initial condition (3) is

$$\underset{\sim}{X}^{(o)} = e^{\underset{\sim}{Q}t}\underset{\sim}{I} \tag{8}$$

Let

$$\underset{\sim}{C}(t) \equiv \underset{\sim}{P}^{(1)}\underset{\sim}{X}^{(o)} = \underset{\sim}{P}^{(1)}e^{\underset{\sim}{Q}t}\, \underset{\sim}{I} \tag{9}$$

Substitution of (9) in (2) yields

$$\dot{\underset{\sim}{X}}^{(1)} = \underset{\sim}{Q}\, \underset{\sim}{X}^{(1)} + \underset{\sim}{C}(t) \tag{10}$$

The solution of (10) is

$$\underset{\sim}{X}^{(1)} = e^{\underset{\sim}{Q}(t)}\, \underset{\sim}{X}^{(1)}(o) + \int_o^t e^{\underset{\sim}{Q}(t-s)}\, \underset{\sim}{C}(s)\,ds \qquad (11)$$

Because of initial condition (3), the first term in (11) is zero. Therefore,

$$\underset{\sim}{X}^{(1)} = \int_o^t e^{\underset{\sim}{Q}(t-s)}\, \underset{\sim}{C}(s)\,ds \qquad (12)$$

Hence the first approximate solution is

$$\underset{\sim}{X} = \underset{\sim}{X}^{(o)} + \mu\, \underset{\sim}{X}^{(1)} \qquad (13)$$

It is easy to show that

$$\underset{\sim}{X}^{(o)}(t) = e^{\underset{\sim}{Q}t}\underset{\sim}{I} = \begin{bmatrix} \cos\sqrt{B}t & \frac{\sin\sqrt{B}t}{\sqrt{B}} \\ -\sqrt{B}\sin\sqrt{B}t & \cos\sqrt{B}t \end{bmatrix} \qquad (14)$$

therefore

$$\underset{\sim}{C}(t) = \underset{\sim}{P}^{(1)}\, e^{\underset{\sim}{Q}t}\, \underset{\sim}{I} = \begin{bmatrix} 0 & 0 \\ \varphi(t)\cos\sqrt{B}t & \frac{\varphi(t)\sin\sqrt{B}t}{\sqrt{B}} \end{bmatrix} \qquad (15)$$

Substituting (15) in (12)

$$\underset{\sim}{X}^{(1)}(t) = \int_o^t \begin{bmatrix} \sqrt{B}\cos\sqrt{B}s\,\sin\sqrt{B}(t-s) & \sin\sqrt{B}s\,\sin\sqrt{B}(t-s) \\ B\cos\sqrt{B}s\,\cos\sqrt{B}(t-s) & \sqrt{B}\sin\sqrt{B}s\,\cos\sqrt{B}(t-s) \end{bmatrix} \frac{\varphi(s)}{B}\,ds$$

(16)

Now according to (V-10)

$$A_1 = -[x_{11}(\pi) + x_{22}(\pi)] = -\left[\left(x_{11}^{(o)}(\pi) + x_{22}^{(o)}(\pi)\right) + \mu\left(x_{11}^{(1)}(\pi) + x_{22}^{(1)}(\pi)\right)\right] \quad (17)$$

Substitution of (14) and (16) itno (17) yields

$$A_1 = -2\cos\sqrt{B}\pi - \mu \frac{\sin\sqrt{B}\pi}{\sqrt{B}} \int_0^\pi \varphi(s)ds \quad (18)$$

But $\int_0^\pi \varphi(s)ds = 0$. Therefore,

$$A_1 = -2\cos\sqrt{B}\pi \quad (19)$$

Substitution of (13) into (V-11) yields

$$\begin{aligned} A_2 = &\left[x_{11}^{(o)}(\pi)\, x_{22}^{(o)}(\pi) - x_{12}^{(o)}(\pi)\, x_{21}^{(o)}(\pi)\right] + \mu\left[x_{11}^{(o)}(\pi)\, x_{22}^{(1)}(\pi)\right. \\ &\left. + x_{22}^{(o)}(\pi)\, x_{11}^{(1)}(\pi) - x_{12}^{(o)}(\pi)\, x_{21}^{(1)}(\pi) - x_{12}^{(1)}(\pi)\, x_{21}^{(o)}(\pi)\right] \\ &+ \mu^2\left[x_{11}^{(1)}(\pi)\, x_{22}^{(1)}(\pi) - x_{12}^{(1)}(\pi)\, x_{21}^{(1)}(\pi)\right] . \end{aligned} \quad (20)$$

From (14) it is seen that

$$x_{11}^{(o)}(\pi)\, x_{22}^{(o)}(\pi) - x_{12}^{(o)}(\pi)\, x_{21}^{(o)}(\pi) = 1 \quad . \quad (21)$$

From (16) and (14) it can be shown that

$$x_{11}^{(o)}(\pi)\, x_{22}^{(1)}(\pi) + x_{22}^{(o)}(\pi)\, x_{11}^{(1)}(\pi) - x_{12}^{(o)}(\pi)\, x_{21}^{(1)}(\pi) - x_{12}^{(1)}(\pi)\, x_{21}^{(o)}(\pi) \equiv 0 \; . \quad (22)$$

From (16) it can be shown that

$$x_{11}^{(1)}(\pi)\, x_{22}^{(1)}(\pi) - x_{12}^{(1)}(\pi)\, x_{21}^{(1)}(\pi) = -\frac{1}{4B}\left\{\left[\int_0^{\pi} \varphi(s)\cos 2\sqrt{B}s\,ds\right]^2 + \left[\int_0^{\pi} \varphi(s)\sin 2\sqrt{B}s\,ds\right]^2\right\} \tag{23}$$

Substitution of (21), (22) and (23) in (20) yields

$$A_2 = 1 - \frac{\mu^2}{4B}\left\{\left[\int_0^{\pi} \varphi(s)\cos 2\sqrt{B}s\ ds\right]^2 + \left[\int_0^{\pi} \varphi(s)\sin 2\sqrt{B}s\ ds\right]^2\right\} \tag{24}$$

APPENDIX B

Second Approximation

The equations to be solved are

$$\dot{\underset{\sim}{X}}^{(o)} = \underset{\sim}{Q}\, \underset{\sim}{X}^{(o)} \tag{1}$$

$$\dot{\underset{\sim}{X}}^{(1)} = \underset{\sim}{Q}\, \underset{\sim}{X}^{(1)} + \underset{\sim}{P}^{(1)}\, \underset{\sim}{X}^{(o)} \tag{2}$$

$$\dot{\underset{\sim}{X}}^{(2)} = \underset{\sim}{Q}\, \underset{\sim}{X}^{(2)} + \underset{\sim}{P}^{(1)}\, \underset{\sim}{X}^{(1)} \tag{3}$$

with initial conditions

$$\underset{\sim}{X}^{(o)}(0) = \underset{\sim}{I} \quad , \quad \underset{\sim}{X}^{(1)}(0) = \underset{\sim}{X}^{(2)}(0) = 0 \tag{4}$$

where $\underset{\sim}{P}^{(1)}$ and $\underset{\sim}{Q}$ are given by (V-30) and (V-29).

The solutions of above systems with initial condition (4) are as follows:

$$\underset{\sim}{X}^{(o)}(t) = e^{\underset{\sim}{Q}t}\, \underset{\sim}{I} \tag{5}$$

$$\underset{\sim}{X}^{(1)}(t) = \int_o^t e^{\underset{\sim}{Q}(t-s)}\, \underset{\sim}{P}^{(1)}(s)\, e^{\underset{\sim}{Q}s}\, ds \tag{6}$$

$$\underset{\sim}{X}^{(2)}(t) = \int_o^t \int_o^{s_1} e^{\underset{\sim}{Q}(t-s_1)}\, \underset{\sim}{P}^{(1)}(s_1) e^{\underset{\sim}{Q}(s_1-s_2)}\, \underset{\sim}{P}^{(1)}(s_2) e^{\underset{\sim}{Q}s_2} ds_2\, ds_1, \tag{7}$$

Substituting for $\underset{\sim}{Q}$ and $\underset{\sim}{P}^{(1)}$ and carrying out the matrix operations, one obtains:

$$\underset{\sim}{X}^{(o)}(t) = \frac{1}{\sqrt{B}} \begin{bmatrix} \sqrt{B}\cos\sqrt{B}t & \sin\sqrt{B}t \\ -\sin\sqrt{B}t & \sqrt{B}\cos\sqrt{B}t \end{bmatrix} \tag{8}$$

$$\underset{\sim}{X}^{(1)}(t) = \int_{o}^{t} \begin{bmatrix} \sqrt{B} \cos \sqrt{B}\, s \sin \sqrt{B}(t-s) & \sin \sqrt{B}s \sin \sqrt{B}(t-s) \\ B \cos \sqrt{B}s \cos \sqrt{B}(t-s) & \sqrt{B} \sin \sqrt{B}s \cos \sqrt{B}(t-s) \end{bmatrix} \frac{\varphi)s)}{B}\, ds \tag{9}$$

$$\underset{\sim}{X}^{(2)}(t) =$$

$$\int_{o}^{t}\int_{o}^{s_1} \begin{bmatrix} \sqrt{B}\sin\sqrt{B}(t-s_1)\sin\sqrt{B}(s_1-s_2)\cos\sqrt{B}s_2 & \sin\sqrt{B}(t-s_1)\sin\sqrt{B}(s_1-s_2)\sin\sqrt{B}s_2 \\ B\cos\sqrt{B}(t-s_1)\sin\sqrt{B}(s_1-s_2)\cos\sqrt{B}s_2 & \sqrt{B}\cos\sqrt{B}(t-s_1)\sin\sqrt{B}(s_1-s_2)\sin\sqrt{B}s_2 \end{bmatrix}$$

$$\frac{\varphi(s_1)\varphi(s_2)}{B\sqrt{B}}\, ds_2\, ds_1 \tag{10}$$

where $\varphi(s)$ is given by (V-2).

Setting $t = \pi$ in (8), (9) and (10) and using (V-55), one obtains a second approximate solution for the instant $t = \pi$, namely:

$$\underset{\sim}{X}(\pi) = \underset{\sim}{X}^{(o)}(\pi) + \mu\, \underset{\sim}{X}^{(1)}(\pi) + \mu^2\, \underset{\sim}{X}^{(2)}(\pi) \tag{11}$$

Now

$$A_1 = -\,[x_{11}(\pi) + x_{22}(\pi)] \tag{12}$$

and

$$A_2 = x_{11}(\pi)\, x_{22}(\pi) - x_{12}(\pi)\, x_{21}(\pi) \tag{13}$$

Substitution of (11) into (12) and (13) yields:

$$A_1 = -\,\{[x_{11}^{o}(\pi) + x_{22}^{(o)}(\pi)] + \mu[x_{11}^{(1)}(\pi) + x_{22}^{(1)}(\pi)] + \mu^2[x_{11}^{(2)}(\pi) + x_{22}^{(2)}(\pi)]\} \tag{14}$$

$$A_2 = [x_{11}^{(o)}(\pi)\ x_{22}^{(o)}(\pi) - x_{12}^{(o)}(\pi)\ x_{21}^{(o)}(\pi)] + \mu\ [x_{11}^{(1)}(\pi)\ x_{22}^{(o)}(\pi)$$

$$+ x_{11}^{(o)}(\pi)\ x_{22}^{(1)}(\pi) - x_{21}^{(1)}(\pi)\ x_{12}^{(o)}(\pi) - x_{12}^{(1)}(\pi)\ x_{21}^{(o)}(\pi)]$$

$$+ \mu^2\ \{[x_{11}^{(1)}(\pi)\ x_{22}^{(1)}(\pi) - x_{12}^{(1)}(\pi)\ x_{21}^{(1)}(\pi)]$$

$$+ [x_{11}^{(o)}(\pi)\ x_{22}^{(2)}(\pi) + x_{22}^{(o)}(\pi)\ x_{11}^{(2)}(\pi) - x_{12}^{(o)}(\pi)\ x_{21}^{(2)}(\pi) - x_{21}^{(o)}(\pi) x_{12}^{(2)}(\pi)]\}$$

$$+ \mu^3\ [x_{11}^{(1)}(\pi)\ x_{22}^{(2)}(\pi) + x_{22}^{(1)}(\pi)\ x_{11}^{(2)}(\pi) - x_{12}^{(1)}(\pi)\ x_{21}^{(2)}(\pi) - x_{21}^{(1)}(\pi)\ x_{12}^{(2)}(\pi)]$$

$$+ \mu^4\ [x_{11}^{(2)}(\pi)\ x_{22}^{(2)}(\pi) - x_{12}^{(2)}(\pi)\ x_{21}^{(2)}(\pi)] \qquad (15)$$

Before proceeding further, let us introduce the following notations for the sake of convenience:

$$A_c = \int_o^{\pi} \varphi(s) \cos 2\sqrt{B}s\ ds$$

$$A_s = \int_o^{\pi} \varphi(s) \sin 2\sqrt{B}s\ ds$$

$$A_{c1} = \int_o^{\pi}\int_o^{s_1} \varphi(s_1)\ \varphi(s_2)\ \cos 2\sqrt{B}s_1\ ds_2\ ds_1$$

$$A_{c2} = \int_o^{\pi}\int_o^{s_1} \varphi(s_1)\ \varphi(s_2)\ \cos 2\sqrt{B}\ s_2\ ds_2\ ds_1$$

$$A_{s1} = \int_o^{\pi}\int_o^{s_1} \varphi(s_1)\ \varphi(s_2)\ \sin 2\sqrt{B}\ s_1\ ds_2\ ds_1$$

$$A_{s2} = \int_0^{\pi}\int_0^{s_1} \varphi(s_1)\, \varphi(s_2)\, \sin 2\sqrt{B}s_2\, ds_2\, ds_1$$

$$A_{cc} = \int_0^{\pi}\int_0^{s_1} \varphi(s_1)\, \varphi(s_2)\, \cos 2\sqrt{B}\, s_1\, \cos 2\sqrt{B}\, s_2\, ds_2\, ds_1$$

$$A_{ss} = \int_0^{\pi}\int_0^{s_1} \varphi(s_1)\, \varphi(s_2)\, \sin 2\sqrt{B}s_1\, \sin 2\sqrt{B}s_2\, ds_2\, ds_1$$

$$A_{cs} = \int_0^{\pi}\int_0^{s_1} \varphi(s_1)\, \varphi(s_2)\, \cos 2\sqrt{B}s_1\, \sin 2\sqrt{B}s_2\, ds_2\, ds_1$$

$$A_{sc} = \int_0^{\pi}\int_0^{s_1} \varphi(s_1)\, \varphi(s_2)\, \sin 2\sqrt{B}s_1\, \cos 2\sqrt{B}s_2\, ds_2\, ds_1$$

(16)

It can be shown, by the use of integration by parts, that

$$\left.\begin{aligned} A_{c1} &= -A_{c2} \\ A_{s1} &= -A_{s2} \\ A_{sc} &= A_c A_s - A_{cs} \\ A_{cc} &= \frac{A_c^2}{2}, \qquad A_{ss} = \frac{A_s^2}{2} \end{aligned}\right\} \quad (17)$$

Now from the results of first approximation (Appendix A) one has

$$x_{11}^{(o)}(\pi) + x_{22}^{(o)}(\pi) = 2 \cos \sqrt{B}t \quad (18)$$

$$x_{11}^{(1)}(\pi) + x_{22}^{(1)}(\pi) = 0 \quad (19)$$

$$x_{11}^{(o)}(\pi)\, x_{22}^{(o)}(\pi) - x_{12}^{(o)}(\pi)\, x_{21}^{(o)}(\pi) = 1 \tag{20}$$

$$x_{11}^{(1)}(\pi)\, x_{22}^{(o)}(\pi) + x_{11}^{(o)}(\pi)\, x_{22}^{(1)}(\pi) - x_{21}^{(1)}(\pi)\, x_{12}^{(o)}(\pi) - x_{12}^{(1)}(\pi)\, x_{21}^{(o)}(\pi) = 0 \tag{21}$$

$$x_{11}^{(1)}(\pi)\, x_{22}^{(1)}(\pi) - x_{12}^{(1)}(\pi)\, x_{21}^{(1)}(\pi) = -\frac{A_c^2 + A_s^2}{4B} \tag{22}$$

Setting $t = \pi$ in (8), (9) and (10) and using notation (16) and relations (17), it can be shown that

$$x_{11}^{(2)}(\pi) + x_{22}^{(2)}(\pi) = \frac{\sin\sqrt{B}\pi}{2B}\,(2\,A_{sc} - A_c A_s) + \frac{\cos\sqrt{B}\pi}{2B}\left(\frac{A_c^2 + A_s^2}{2}\right) \tag{23}$$

$$x_{11}^{(o)}(\pi)\, x_{22}^{(2)}(\pi) + x_{22}^{(o)}(\pi)\, x_{11}^{(2)}(\pi) - x_{12}^{(o)}(\pi)\, x_{21}^{(2)}(\pi) - x_{21}^{(o)}(\pi)\, x_{12}^{(2)}(\pi) = \frac{A_c^2 + A_s^2}{4B} \tag{24}$$

$$x_{11}^{(1)}(\pi)\, x_{22}^{(2)}(\pi) + x_{22}^{(1)}(\pi)\, x_{11}^{(2)}(\pi) - x_{12}^{(1)}(\pi)\, x_{21}^{(2)}(\pi) - x_{21}^{(1)}(\pi)\, x_{12}^{(2)}(\pi) = \frac{1}{2B\sqrt{B}}\,(A_c A_{s2} - A_s A_{c2}) \tag{25}$$

$$x_{11}^{(2)}(\pi)\, x_{22}^{(2)}(\pi) - x_{12}^{(2)}(\pi)\, x_{21}^{(2)}(\pi) = \left(\frac{1}{4B}\right)^2 \left[\left(\frac{A_c^2 + A_s^2}{2}\right)^2 - 4(A_{c2}^2 + A_{s2}^2) + (2\,A_{sc} - A_c A_s)^2\right] \tag{26}$$

Substituting the above relations into (14) and (15), one obtains

$$\left(\frac{A_1}{2}\right)^2 = \left\{\cos\sqrt{B}\pi + \frac{\mu^2}{4B}\left[\sin\sqrt{B}\pi\,(2\,A_{sc} - A_c A_s) + \cos\sqrt{B}\pi\,\frac{A_c^2 + A_s^2}{2}\right]\right\}^2 \tag{27}$$

$$A_2 = 1 + \frac{\mu^3}{2B\sqrt{B}}\,(A_c A_{s2} - A_s A_{c2}) + \frac{\mu^4}{16B^2}\left[\left(\frac{A_c^2 + A_s^2}{2}\right)^2 - 4(A_{c2}^2 + A_{s2}^2) + (2\,A_{sc} - A_c A_s)^2\right] \tag{28}$$

Substituting for $\varphi(t)$ its Fourier series representation (V-2) and using relations (16), one obtains

$$\frac{A_c^2 + A_s^2}{2} = \frac{\sin^2 \sqrt{B}\pi}{2} (K_1^2 + K_2^2) \qquad (29)$$

$$4(A_{c2}^2 - A_{s2}^2) = \frac{\sin^2 \sqrt{B}\pi}{4} (H_1^2 + H_2^2) \qquad (30)$$

$$A_c\ A_{s2} - A_s\ A_{c2} = \frac{-\sin^2 \sqrt{B}\pi}{4} (H_1 K_2 + H_2 K_1) \qquad (31)$$

$$2A_{sc} - A_c A_s = \frac{\pi}{4} Q_3 - \frac{\sin 2\sqrt{B}\pi}{2}\Big[(K_1 \sin \sqrt{B}\pi + K_2 \cos \sqrt{B}\pi)^2 - \Big(\frac{\sin 2\sqrt{B}\pi}{2} Q_1 + \frac{\cos 2\sqrt{B}\pi}{2} Q_2\Big)\Big] \qquad (32)$$

where K_1, K_2, H_1, H_2, Q_1, Q_2 and Q_3 are given by relations (V-59:c) . Substituting the above relations into Equations (27) and (28) and then using the inequalities (V-48) and (V-49), one obtains

$$C_1 + C_2\ \mu^2 + C_3\ \mu^3 + C_4\ \mu^4 > 0 \qquad (33)$$

$$S_1\ \mu + S_2\ \mu^2 \leqq 0 \qquad (34)$$

where C_1, C_2, C_3, C_4, S_1 and S_2 are given by relations (V-58:a) and (V-59:a).

LIST OF FIGURES

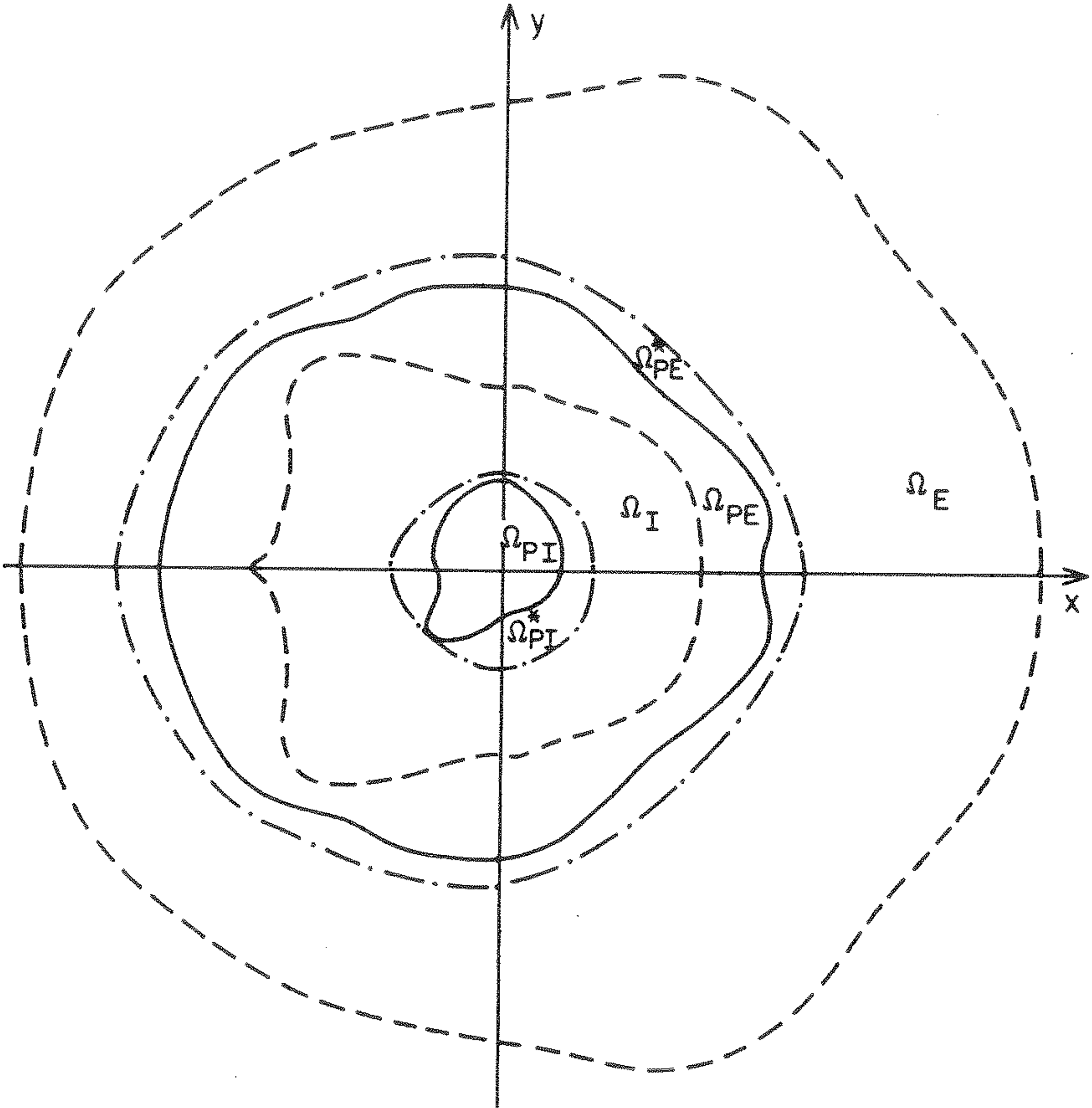

FIG. 1

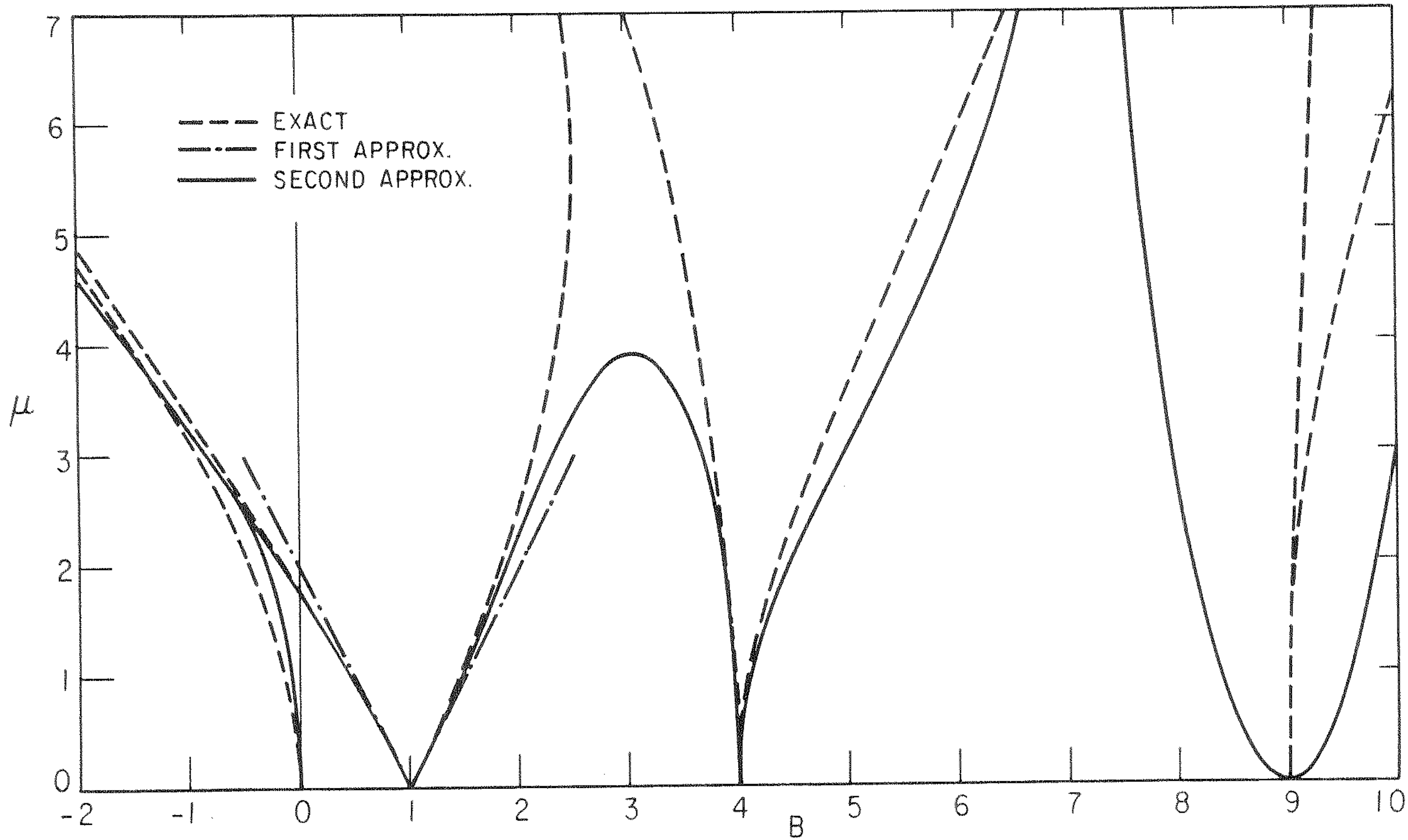

FIG. 2

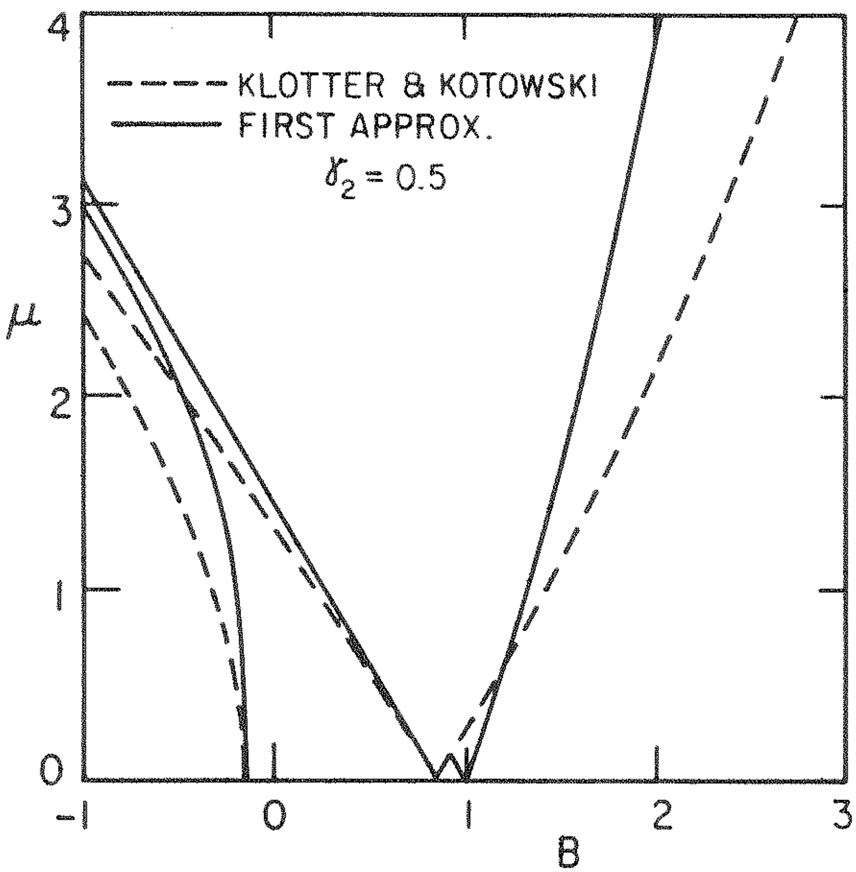
KLOTTER & KOTOWSKI
FIRST APPROX.
$\gamma_2 = 0.5$
μ
B
4
3
2
1
0
-1
0
1
2
3

FIG. 3

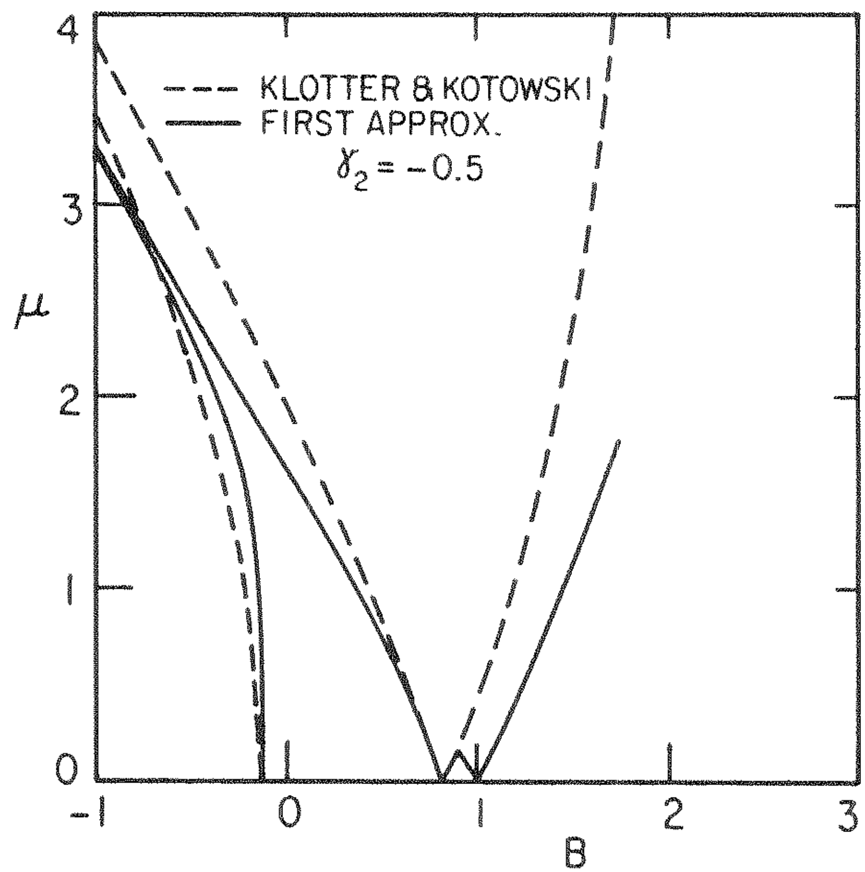
KLOTTER & KOTOWSKI
FIRST APPROX.
$\gamma_2 = -0.5$
μ
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FIG. 4

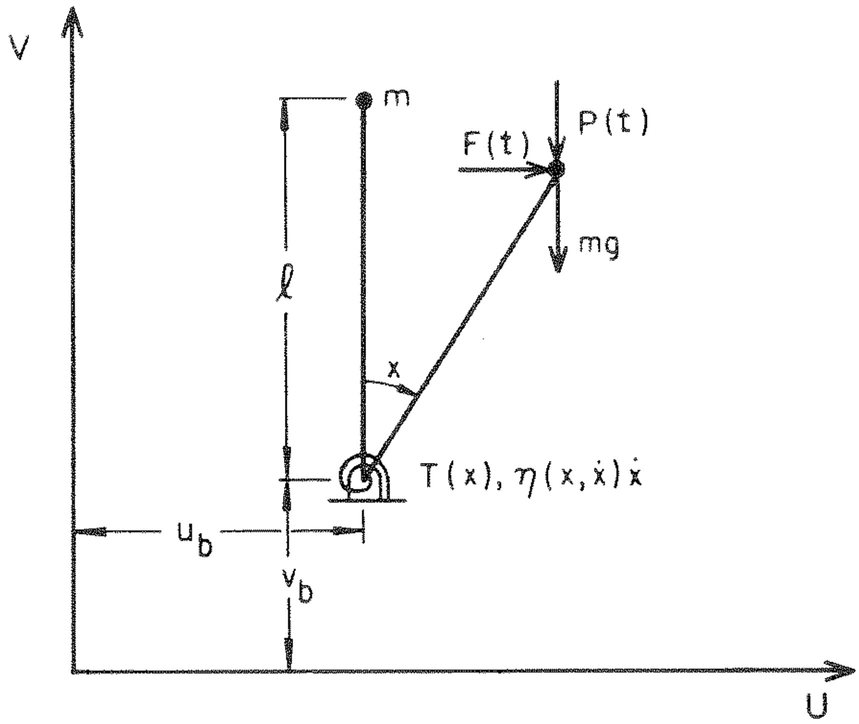

FIG. 5

REFERENCES

[1] J. LaSalle and S. Lefschetz, "Stability by Liapunov's Direct Method with Applications," Mathematics in Science and Engineering, Vol. 4, Academic Press (1961).

[2] Taro Yoshizawa, "Asymptotic Behavior of Solution of a System of Differential Equations," Contributions to Differential Equations, Vol. I, No. 3.

[3] Taro Yoshizawa, "Stable Sets and Periodic Solutions in a Perturbed System," Contributions to Differential Equations.

[4] Jack K. Hale, "Oscillations in Nonlinear Systems," McGraw Hill (1963).

[5] José L. Massera, "Contributions to Stability Theory," Annals of Mathematics, Vol. 64, No. 1, July 1956.

[6] Wolfgang Hahn, "Theory and Application of Liapunov's Direct Method," Prentice-Hall (1963).

[7] N. N. Krasovskii, "Stability of Motion, Applications of Liapunov's Second Method to Differential Systems and Equations with Delay," Stanford University Press (1963)

[8] V. V. Nemytskii and V. V. Stepanov, "Qualitative Theory of Differential Equations," Princeton University Press (1960).

[9] I. G. Malkin, "Theory of Stability of Motion," AEC-tr-3352.

[10] E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis," Fourth Edition, Cambridge University Press.

[11] V. V. Bolotin, "The Dynamic Stability of Elastic Systems," Holden-Day (1964).

[12] R. Bellman, "Stability Theory of Differential Equations," McGraw-Hill (1953).

[13] L. A. Zadeh and C. A. Desoer, "Linear System Theory, the State Space Approach," McGraw-Hill (1963).

[14] Nicholas Minorsky, "Nonlinear Oscillations," D. Van Nostrand Company (1962).

[15] Solomon Lefschetz, "Differential Equations: Geometric Theory," Interscience Publishers (1957).

[16] Lamberto Cesari, "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations," Second Edition (1963), Academic Press.

[17] H. A. Antosiewicz, "Boundedness and Stability," Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press (1961).

[18] M. A. Liapounoff, "Problème Général de la Stabilité de Mouvement," Annals of Mathematics Studies No. 17, Princeton University Press (1949).

[19] F. M. Arscott, "Periodic Differential Equations," The Macmillan Company, New York (1964).

[20] Von Karl Klotter and Gertrund Kotowski, "Über die Stabilität der Lösungen Hillscher Differentialgleichungen mit drei unabhängigen Parametern," ZAMM, Vol. 23, No. 3 (1943).

[21] E. L. Ince, "Ordinary Differential Equations," Longmans, Green (1926).

[22] J. J. Stoker, "Nonlinear Vibrations in Mechanical and Electrical Systems," Interscience Publishers, Inc., New York (1950).

[23] Harold T. Davis, "Introduction to Nonlinear Differential and Integral Equations," Dover Publications, Inc., New York.

[24] E. L. Ince, "Tables of the Elliptic-Cylinder Functions," Roy. Soc. of Edinburgh, Proc. Vol. 52, 1931-32.

[25] E. L. Ince, "Researches into the Characteristic Numbers of the Mathieu Equation," Roy. Soc. of Edinburgh, Proc. Vol. 46, 1925-26.

[26] N. W. McLachlan, "Theory and Application of Mathieu Functions," Oxford University Press (1947).

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13. ABSTRACT

The stability of systems governed by

$$\ddot{x} + f(x) + q(x,\dot{x})\dot{x} - \varphi(t)\ r(x) = S(x;t)$$

is studied. Liapunov's Direct Method and a linearization approach have been used in the study of stability of the above system for $\varphi(t)$ L_1 integrable, and periodic, respectively. In the former case a sufficiency region of stability is constructed through the use of a Liapunov function. In the latter case, which is investigated by means of a linearization process, a Hill equation is obtained, whose stability is studied by a method suggested by Malkin. Malkin's method is then modified to obtain, by use of a first approximation, the first stability region in parameter space. A second approximation is also worked out. When the approximations obtained herein for general periodic function are reduced to the special cases of the Mathieu equation and the Hill-3-term equation, the results compare very well with the available numerical results based on the exact solution of each of those equations.

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