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Stochastic nonzero-sum duopoly games with economic applications

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Stochastic nonzero-sum duopoly games with economic applications

A dissertation submitted in partial satisfaction
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by

Liangchen Li

Committee in charge:

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The Dissertation of Liangchen Li is approved.

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Stochastic nonzero-sum duopoly games with economic applications

To my parents,

for their endless love and support.

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Curriculum Vitæ

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Education

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Publications and Preprints

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Abstract

Stochastic nonzero-sum duopoly games with economic applications

by

Liangchen Li

We study a class of stochastic duopoly games inspired by the two time-scale feature of many markets. The firms convert their short-term "local" advantage driven by exogenous infinitesimal shocks into a more durable gain through long-term market dominance. As an extension of existing literature, we consider two asymmetric players each of whom adopts timing strategies to increase her profitability and possibly bring negative externality to the rival. In turn, this leads us to more general settings of nonzero-sum games. Characterizing Nash equilibrium as a fixed-point of each player's best-response to her rival, we construct threshold-type Feedback Nash Equilibrium via best-response iteration. Our main contribution is explicitly constructing equilibria for types of duopoly games that represent a wide range of industries. Motivated by the competition among sectors of power generators, we consider a duopoly of producers with finite options to increase their production capacity. We study nonzero-sum games in which two players compete for market dominance via switching controls. We also study mixed switching and impulses games inspired by the vertical competition among the producers and consumers of a commodity. Our analysis quantifies the dynamic competition effects and brings economic insights.

Contents

Chapter 1

Introduction

Guidance and attributions: The topic of my thesis concentrates on the study of nonzero-sum duopoly games in stochastic environments. We consider the competition between asymmetric players with various types of controls, motivated by well-discussed economic applications. Chapter 1 introduces the problem we studied and provides an overview of this thesis. Rigorous formulation and building blocks that help us solving the problem are stated in Chapter 2. Three types of solvable games are discussed respectively. The content of Chapter 3 is the result of a collaboration with René Aïd and Mike Ludkovski, and has appeared as [6]. The content of Chapter 4 is of a submitted paper [56] that I had with Mike Ludkovski. Chapter 5 summarizes the ongoing work [4] collaborated with René Aïd, Luciano Campi and Mike Ludkovski. They are reproduced here with permissions.

1.1 Background

We consider firms in a competitive market who aim to maximize their profit while being exposed to exogenous stochastic shocks. The uncertainty may come from uncertain market development, exogenous demand/supply shocks, unpredictable costs, etc. In turn, one main problem faced by administrators of the firms is to make decisions reacting to the stochastic environment, e.g. building additional production capacity, investing in R&D to be on the cutting edge, switching on/off production to maintain solid profit and so on, with the anticipation that they would gain more revenue in the foreseen future. Such problems have been extensively studied as stochastic optimal control problems since the late 1950s. Existing research has considered a variety of approaches to the choices faced by the firm, including singular control by Steg [73] in which firms repeatedly make investments of arbitrary size to increase their respective capital stocks; timing control by Grenadier [39] in which firms determine the optimal time to make investment; impulse control by Baccarin [11] in which agents make timing decisions to shift the underlying process to a desired level, and two-sided optimal switching control by Hamadène and Jeanblanc [41] in which a power station decides to switch between operating and closed modes.

Furthermore, firms' profits are affected not only by her choices but also by decisions of other participants of the competitive market. In the electricity market, investor of coal-fired plants and renewable power plants compete to provide base-load power generation, while the competitiveness of non-emissive energies highly relies on a substantial price for carbon emission. Smartphone producers like Apple and Samsung compete for increasing market shares via investing in high technology devices and advertising, while the global demand of smartphones grows in a stochastic way. Another representative example could be the auto market of fuel cars and hybrid cars. As the price of crude increases considerably, one may expect that costumers will switch to hybrid cars, thus the producers have to adjust their capacity accordingly. To the extent that the profits of the firm are affected by decisions of others, it is important to consider the strategic interaction across firms. Assuming that firms take into account the other firms' reactions to their own actions and they know their rivals think the same way, their decisions can be treated as a *dynamic game*.

Such dynamic competition under uncertainty provides a natural generalization from the classical one-agent optimization problems and hence is applicable in a large variety of applied settings. One classical example of dynamic games would be that producers of substitutable goods (e.g. steel, electricity, cars, etc.) compete *à la Cournot* on the product market. To wit, the producers compete on the amount of product they produce whose price is determined by their aggregate output. As one producer invests to increase her capacity, she will lower the good price which brings negative externality to the others and herself. Therefore, in such a competitive market, every single producer must take the strategic interaction across all producers into consideration when making decisions. The general problem of capacity expansion under uncertainty has been extensively studied as a stochastic optimal control problem and offers a natural link to the theory of real options. Single-agent models for multi-stage capacity expansion were initiated in Dixit [32] and Var-Ilan et al. [12]. See also the very recent work Aïd et al. [5] using a continuous-control framework. Early pioneering works to mix concepts from both the real options frameworks and the game theory were Smets [72], who first introduced the effect of competition in the real option literature, and Williams [77], who provided the first rigorous derivation of a Nash equilibrium in a real option framework. See also the books [40, 25]. The "standard" real option game features two symmetric firms competing to invest in a non-exclusive underlying project over the infinite (continuous) time horizon. The competition is of the leader/follower type: at the time of the first investment, one firm becomes the leader; the follower is then able to invest at a later date [33, 46, 63]. We refer to the survey by Azevedo and Paxson [10] who present a catalog of more than fifty articles dealing with variants of this setup. A simpler version is the preemption game introduced by Grenadier [39], where two identical firms compete to be the first to initiate a new project. To our knowledge the first paper to explicitly consider competitive capacity expansion was Bashyam [13]. Another notable contribution is Huisman and Kort [48] who allow for joint optimization of the timing and project size, demonstrating that the first mover over-builds to delay entry of the other firm. More examples include: (i) Boyer et al. [17] model capacity-building investments, via irreversible addition of production units, in a homogeneous product duopoly facing uncertain demand growth. They demonstrate equilibrium paths of the investment game may include episodes during which firms invest at different times, a preemption pattern, and episodes in which firms invest simultaneously, a tacit collusion pattern; (ii) Huisman and Kort [47] study a dynamic duopoly in which firms compete in the adoption of new technologies to provide a framework where firms take into account technological progress in making their investment decisions; (iii) Huberts et al. [45] examine a dynamic incumbent-entrant framework with stochastic evolution of the inverse demand and find incumbent invests earlier than the entrant in equilibrium while the size of investment plays an important role of the game.

1.2 Stochastic Nonzero-sum Duopoly Games

This thesis focuses on studying *nonzero-sum* games, in which the interacting firms' aggregate gains and losses can be less than or more than zero (cf. Morgenstern and Neumann [60]). Comparing to zero-sum games in which each firm's gain or loss is exactly balanced by the losses or gains of the other firms, nonzero-sum games draw our interest because: (i) nonzero-sum games are more flexible to better fit the motivating economic examples, since the aggregate revenue of the market may change as one firm makes investment. For instance, additional capacity may decrease the overall revenue through a lower product price, while new technique developed by one firm will bring extra welfare to the whole industry; (ii) technique-wise solving nonzero-sum games is more complicated while zero-sum games are usually solved via minimax theorem (cf. Aumann and Hart [9, Chapter 2]). Though much less than studies on zero-sum games, such features have drawn many studies in stochastic nonzero-sum games. Some recent related works are: Aïd et al. [2] who consider nonzero-sum games of two players with impulse controls, Hamadene and Zhang [42] and Attard [8] who study a two player nonzero-sum game on stopping times, De Angelis et al. [30] and Martyr and Moriarty [59] who construct threshold-type Nash equilibrium for nonzero-sum stopping games, while Riedel and Steg [69] develop a subgame-perfect equilibrium of stochastic timing games in mixed strategies.

1.2.1 Markets with Two Time-scales Feature

We concentrate on a specific class of dynamic games which are inspired by the two time-scales feature of many markets. Indeed, dynamic competition is often driven by infinitesimal shocks that determine the rapidly fluctuating short-run market conditions. These fluctuations yield “local” advantage to firms, e.g. high carbon emission price benefits non-emissive power plants, a material gas price makes the producer of hybrid cars more competitive and so on. After a sustained period of advantageous market conditions we expect the respective firm to become dominant. To wit, the firms convert such short-term effect into a more durable gain through market dominance, e.g. longer-term capacity gains, technological edge, advertising to increase market shares, etc. Thus, the two time-scales link the immediate competitive advantage and the long-run market organization. The novelty of our setup is to fully integrate this well-known idea within a non-cooperative game model, by considering a “microeconomic” stochastic factor (X_t) that drives market conditions, e.g. carbon emission price, aluminum price for aluminum producers and automakers, crude price, domestic R&D costs, etc, vis-a-vis the “macroeconomic” market regime (M_t) that determines market power and relative profits of the firms, for instance the numbers of coal-fired power plants and renewable power plants built reflect not only the electricity market organization but also profitability of each generation sector. One can also consider (M_t) as a direct index of the proportion of market shares the firms are taking, or an index of their R&D level.

1.2.2 Competition of Asymmetric Players

We consider a duopoly of two firms, dubbed player $i, j \in \{1, 2\}, i \neq j$, that naturally mirrors the two-sided nature of the up/down market conditions represented by the one-dimensional X. Players compete *continuously* by exercising discrete controls. They collect continuous-time profit at their respective rates π^i's driven by both the "microeconomic" market condition X and the "macroeconomic" market regime M. Meanwhile, they may take actions (we also say they exercise controls) at any time as the local market condition develops, e.g. building additional capacity, switching to expanding mode, increasing advertising investments and so on, which yield instantaneous "one-shot" effect on the market regime M. This assumption creates a feedback effect between X and M: e.g. as X varies, one player gets more motivated to enhance her market dominance, eventually triggering her to act and moves M_t towards her preferred direction. From the narrative of two time-scale feature, it follows that the long-run market organization is then fully controlled by the players, while the local market fluctuation is fully exogenous.[1]

These players' actions yield sunk-cost K. For simplicity, we assume such acting cost is lump-sum and adjusted to reflect both the instant cost occurred at the acting time and any operating cost in the future. Consequently, the integrated total profit of the players is given by their future cash flow received over an infinite time horizon, minus all

[1] An extension to this setting, in which the local market fluctuation X is partially controlled by these players, is represented in Chapter 5.

the acting costs. Taking the time value of money and the risk of future uncertainty into consideration, we assume these two players aim to optimize the net present value (NPV) of their expected integrated total profit, at an exogenous discounting rate $r > 0$. In the context of game theory, such quantities are accordingly defined to be the *game payoffs* these players receive (see Definition 2.3).

Distinct to most of the literature that considers games of identical players (though see Takashima et al. [75], Aïd et al. [2], De Angelis et al. [30]), one of our contributions is that we consider *asymmetric* players. In terms of the profit rates, they may have different preference on the micro/macro market. For instance, higher carbon emission price and higher proportion of renewable powers in the base-load power generation imply a higher profit rate π of the corresponding energy generators, while the coal-fired power plants investors desire the opposite. Moreover, impact yielded by these players' actions is supposed to be distinct. As an example, new technology developed by one smartphone producer enhances her own dominance in the industry described by the process M. To wit, innovation made by different producer leads to a different macro market regime M_t, whereas both producers may get benefit from the macro market development due to increased aggregate demand of phones. In turn, this asymmetry leads us to a more general nonzero-sum game, and combines dynamic competition with (possibly) cooperation or immediate preemption, e.g. the coal-fired power generators might build capacity, when emission is cheap, to prevent her rival from expanding in the future.

1.2.3 Equilibrium: Emerging Macro Market

Let $\boldsymbol{\alpha}^i$ denote a game *strategy* of player i, which specifies when and how she drives the macro market regime M (i.e. exercises a control). A *strategy profile* $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$ then fully specifies these two players' action in the game. Recall that the long-term market organization M is considered to be fully controlled by these players, while the short-term market condition X captures the exogenous stochasticity. Given the players' strategy profile, one is able to infer the dynamic of M from the dynamic of X. In particular, the discrete nature of the macro regime M, following from the setting that the players make discrete/lumpy actions, allows a high degree of analytic tractability which brings us insights of the competitive market. For instance, we find that a static uptrend of the carbon emission price X grants long-term market advantage to the renewable power generators, and may exclude the coal-fired power generators from building more capacity, i.e. convergent M in the long-run (see Section 4.4.2). Such inference motivates us to explicitly construct equilibria of the duopoly game which allow structural insights into the strategic interaction between the players, the short-term fluctuations in X, and the emerging M.

1.2.3.1 Nash Equilibrium: Fixed-point of Best-response

In the context of game theory, solving a game often amounts to identifying one (or all) possible equilibrium(s) of the game. We utilize the standard concept of Nash equilibrium to describe the optimal behavior. Letting $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$ be a strategy profile of these two players, we denote the corresponding game payoff received by player i by $J^i(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$. Here we use the term strategy profile to emphasize the joint dependence of the players' game payoffs upon both of their strategies, as opposed to the single-agent control problems. The strategy profile $(\boldsymbol{\alpha}^{\mathbf{1,*}}, \boldsymbol{\alpha}^{\mathbf{2,*}})$ is said to be a *Nash equilibrium* of the duopoly game if for any strategy $\boldsymbol{\alpha}^i$ of player i such that $(\boldsymbol{\alpha}^i, \boldsymbol{\alpha}^{j,*})$ is *admissible*

$$J^i(\boldsymbol{\alpha}^i, \boldsymbol{\alpha}^{j,*}) \le J^i(\boldsymbol{\alpha}^{i,*}, \boldsymbol{\alpha}^{j,*}), \qquad i \in \{1, 2\}, \qquad j \neq i. \tag{1.1}$$

The corresponding game payoff $V^i := J^i(\boldsymbol{\alpha}^{i,*}, \boldsymbol{\alpha}^{j,*})$ is then named the equilibrium payoff of player $i \in \{1, 2\}$. We would like to make a note here that above definition of Nash equilibrium is informal because it strongly depends upon the definition of *admissibility* (i.e. the types of controls the players are allowed to exercise). In Chapter 2, we rigorously formulate admissible strategies and accordingly define a Nash equilibrium in Definition 2.4.

Notice that the Nash equilibrium criterion (1.1) characterizes equilibrium strategies as a fixed point of each player's *best-response* to her rival's strategy. Specifically, given an arbitrary rival's strategy $\boldsymbol{\alpha}^j$ define the resulting best-response payoff of player i

$$\widetilde{V}^i(\boldsymbol{\alpha}^j) := \sup_{\{\boldsymbol{\alpha}^i : (\boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j) \in \mathcal{A}\}} J^i(\boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j), \tag{1.2}$$

where we use $\mathcal{A}$ to denote the set of all admissible strategy profiles. Under some restrictive assumptions, one can guarantee the single-agent control problem (1.2) to be solved uniquely, and the corresponding unique maximizer $\widetilde{\boldsymbol{\alpha}}^i$ defines a so-called "best-response map" accordingly. In turn, if such maps can be well-defined in this way, a strategy profile is a Nash equilibrium if and only if it is a fixed-point of the best-responses. Unlike most approaches studied in the field of game theory which look at the equations in equilibrium directly, the fixed-point characterization inspires us to determine a Nash equilibrium of the game via *best-response iterating*. To wit, we first derive the players' best-response to the rival's strategies, which boils down to solving a (constrained) single-agent optimizing problem. Then we attempt to obtain an equilibrium by finding the best-response of the players iteratively. Consequently, equilibrium payoffs (if are attained) satisfy:

$$V^i = \widetilde{V}^i(\boldsymbol{\alpha}^{j,*}), \quad i \in \{1, 2\}, \qquad j \neq i, \tag{1.3}$$

and the strategy profile $(\boldsymbol{\alpha}^{\mathbf{1},*}, \boldsymbol{\alpha}^{\mathbf{2},*})$ is a Nash equilibrium of the game.

On one hand, the fixed-point characterization allows us to break down the problem into single-agent optimizing sub-problems which we can attack by using classical methods (introduced in Chapter 2). On the other hand, it allows us to concentrate on a specific class of strategies instead of the whole set of admissible strategies. The latter is either not feasible to enumerate in general, or makes the control problem (1.2) not solvable. Nevertheless, by looking at a specific class of strategies which is closed under the best-response maps (i.e. given the rival's strategy one player's best-response is in the same class), we are able to obtain a Nash equilibrium among all admissible strategy profiles. Unfortunately, such an iterative approach is not enough to directly justify neither existence nor uniqueness of a Nash equilibrium. For related discussion, we refer to De Angelis et al. [30] who show existence of a unique Nash equilibrium for a nonzero-sum duopoly game, and our discussion of multiple Nash equilibria in Section 3.2.2 and Section 4.4.1.

In terms of the optimality, we would like to mention that Nash equilibrium is not the only choice. Though not being discussed in this thesis, Pareto efficiency is another type of game equilibrium, very popular in economics literature and in operations research application. In words, it is a strategy profile from which there is no strategy that makes *every player* at least as well off and *at least one player* strictly better off. As a recent reference, Carmona in [22] discusses such equilibria for stochastic differential games with N players, as well as mean-field games.

1.2.3.2 Acting Order: Endogenous or Pre-determined

It is well-known that the order of players' actions is essential in determining the solution of a game. Comparing to existing literature (e.g. Siddiqui and Takashima [71] who consider optimal expansion timing between symmetric firms with pre-determined investment order), we contribute by studying *endogenous equilibria* in which the acting order is stochastically determined. As discussed, we expect fluctuations of the micro market condition X grant local advantages to one player and motivate her to act, thus the players' acting order in an equilibrium is determined from their dynamic cooperation and preemption. Note that it is perfectly possible that both players are incentivised to preempt under some specific market condition. One way to address this is to assume preemptive priority of one player, e.g. coal-fired power plants are easier to build so that corresponding generators are prior to preempt. Grasselli et al. [38] propose an infinitesimal coordination game in which firms attempt to invest over an infinitesimal round. See also Steg and Thijssen [74], and Riedel and Steg [69] who construct mixed strategies that the players decide to preempt immediately or at a specific rate.

We also consider other types of game solutions. Inspired by Grasselli et al. [38], we study the situation that one player is granted a priority option, which guarantees her to act first without concerning that the rival may preempt. We find that the player assigned to be the leader intends to act later (than that in a more competitive game) to reap more game payoffs. Moreover, we consider a cooperative solution, which can be viewed a regulator who controls both players and optimizes aggregate profit (a related problem was treated in Aïd et al. [1]). We observe that such a regulator will let the players act later, even present one player from acting, which corroborate that competition leads to preemption and over-investment. Comparing these scenarios with endogenous competitive equilibria, we obtain an "apples-to-apples" quantification for the value of being the leader, and the cost of competition.

Stackelberg competition (cf. Von Stackerberg [76]) is another classical and well-discussed strategic game in which the predetermined leader acts first and the follower acts sequentially. The follower observes the leader's action and makes decisions accordingly, whereas she must have no means of committing to a future non-Stackelberg follower action. A priority game as mentioned above in which each player acts once and one player has one priority is of Stackelberg-type, but a game that each player acts twice and one player has one priority is not. Note that such a designated acting order allows these players to signal to each other. Following such an idea, we discuss the selection among multiple equilibria in Section 3.2.2. We also mimic a Stackelberg competition by constructing a two-regime switching game (see detailed illustration in Section 4.2.4).

1.2.3.3 Extension: Partially Controlled Local Market Condition X

In above narrative, we assume that the players take actions to change the macro market regime M while the local market condition X is a purely exogenous stochastic process. In fact, one may speculate that the macro economic affects the micro market fluctuation (cf. Joëts et al. [50]). Namely, the dynamic of X may depend on the current regime M_t, which creates a mixed feedback effect between X and M. On the other hand, the players may directly impose shocks to the local market, e.g. demand/supply shocks, discounts, etc. Consequently, given a strategy profile of the players, not only the dynamic of M but the dynamic of X as well will emerge from the strategic interaction. In Chapter 5, we represent such a mixed duopoly game.

1.3 Tractable Games from Economic Applications

Inspired by well-discussed economic applications, we now introduce three types of duopoly games that can fit in the generic setup introduced in preceding sections. These games are mainly distinguished in terms of the choices (i.e. controls to be exercised) faced by the players and the strategic interaction between the micro/macro market X and M. Explicit game equilibria and corresponding numerical case studies are presented in the following chapters respectively.

1.3.1 Capacity Expansion: Finite Controls

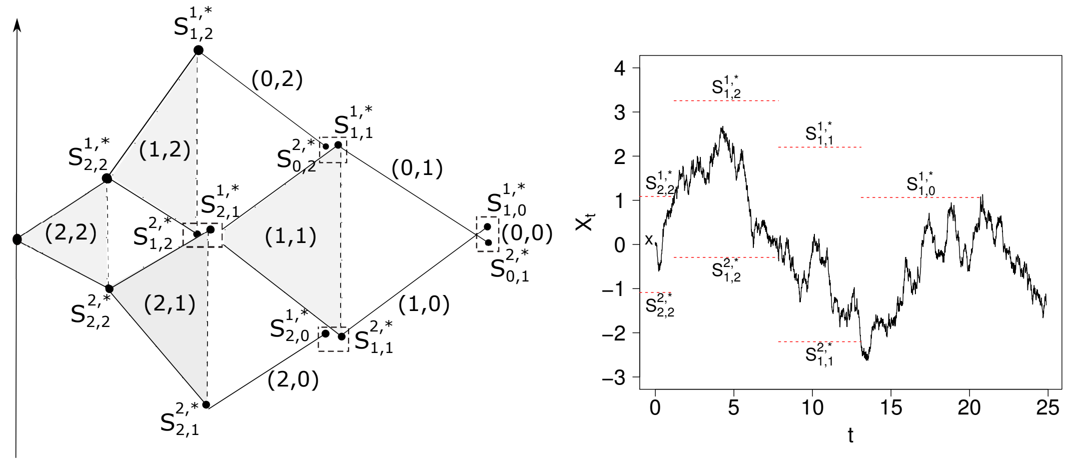
The need to reduce carbon emission to achieve the 2 Celsius degree target puts under pressure power systems of many countries. Lowering the carbon content of electricity requires the development of competitive non-emissive energies for base-load generation. The most immediately viable alternative to provide dispatchable base-load power would be nuclear power plants. But, as shown in the 2005 and 2010 editions of the Projected Cost of Electricity Generation by the International Energy Agency, the relative competitiveness of nuclear power compared to coal-fired generation strongly depends on the existence of a material price for carbon emission. Indeed, a carbon price of 30 USD/tCO2 would definitively make nuclear power plants much more economical than coal-fired plants for electricity base-load generation. Unfortunately for the nuclear industry, as Figure 1.1 shows, the carbon price of the European Union Emission Trading System (EU-ETS) has fallen to a low of 5 €/tCO2 since mid-2012, and has not recovered since then to a value high enough to sustain emission reduction based on economic efficiency. Nevertheless, ongoing political developments, market design changes and technological advances might change this situation and benefit the nuclear producers. A crucial dilemma thus arises for the nuclear industry: either wait for a significant rise in the carbon price at the risk of base-load generation being preemptively taken by coal-fired plants, or intervene now at the cost of enduring short-term losses.

In line with the above narrative, we consider a duopoly game takes place between two players, representing sectors of electricity generators. Producer 1 invests in nuclear power plants with unit expansion cost K^1, while producer 2 invests in coal-fired plants with expansion cost K^2. We consider that those costs include the Operation & Maintenance costs since once the decision is made to invest, they become sunk costs. These investment costs are so massive that projects can be considered as a one-shot decision.

Figure 1.1: Price (in euros per ton of CO_2) of the one year-ahead emission allowance on the EU-ETS. Source: TheIce.

To give an order of magnitude, the Hinkley Point Project of two nuclear power plants being built in the UK carries a cost of approximate 15 billion USD, and the cost of a 1 GW-capacity supercritical coal-fired plant is approximately 1 billion USD. Moreover, given the enormous sunk costs and plant lifetime of 40+ years, investments are viewed as irreversible. We focus on the carbon price X_t as the main state variable. Higher X_t benefits nuclear producers, while lower X_t benefits coal-fired plants. To reflect the significant uncertainties associated with the carbon price (see again Figure 1.1 which can be viewed as a historical trajectory of X_t), we work in a continuous-time stochastic setting. Thus, firms' investment strategies correspond to stopping times related to X_t. The game aspect of the model arises from the negative externality of capacity expansion. Namely, the competitive price is driven by the aggregate capacity of the producers, so that when one of the firms expands, electricity prices decline, hurting her competitor. This creates a preemptive motive for the investors and converts our framework into a non-zero-sum duopolistic game of timing.

Beyond the two profit-maximizing investors, we also aim to understand the role of



(a) Stages

(b) Sample Trajectory of X

Figure 1.2: $S^{i,*}_{n_1,n_2}$ denotes the equilibrium threshold of firm i at stage (n_1, n_2) (Left: a) Sketch of the various stage thresholds as a function of (n_1, n_2). (Right: b) A sample trajectory of X with $X_0 = 0, \vec{M}_0 = (2, 2)$. The corresponding macro market evolution is (2, 2) → (1, 2) → (1, 1) → (0, 1) → (0, 0) with expansions at the first hitting times of the corresponding thresholds.

the third-party regulator, or government in the game outcome. Carbon emission markets remain highly politicized, with a fluid market design. For instance, we can mention initiatives to prevent carbon price collapse, such as the Stability Reserve Mechanism in the ETS, and the United Kingdom carbon price floor of approximately 18 GBP/tCO2 institutionalized since 2016. France is following the same path. Thus, the establishment of a high and steady value for carbon strongly depends on the political will and ability of each state. Our purpose is thus to analyze the effect of such commitment on the market equilibrium. In particular, we are interested in the deviation of this equilibrium compared to the decision a benevolent planner would do.

To sum-up, we consider a duopoly of two distinct producers, each of whom has options to irreversibly increases her current production capacity Q by paying a fixed lump-sum capital K, so as to generate more revenue. Consequently, since the producers' profitabilities are contingent upon number of options they exercised, the macro market regime in this case can be modeled in terms of *numbers of options left* for each producer, denoted by $\vec{M}_t = (N_t^1, N_t^2)$. Such capacity expansion is modeled in terms of timing strategies characterized through threshold rules and solved through dynamic programming-like arguments. The framework necessitates to specify the initial *finite number of possible expansions*. This can be justified by assuming a fixed demand curve, so that one can infer the maximum additional capacity that is economically feasible. In other words, financially we work backwards, starting from potential *end-game* capacities (i.e. stages where no more investment will take place) to determine the maximum number of initial options needed (see a related discussion in [17, Sec 2.5]). As a illustration, Figure 1.2a shows a schematic for all the different thresholds starting at $\vec{M}_0 = (2, 2)$, namely each firm has two options to expand her capacity. To better visualize the game evolution, a simulated state trajectory is presented in Figure 1.2b with the firms' thresholds $S^{i,*}_{n_1,n_2}$ denoting the equilibrium threshold of firm i at stage (n_1, n_2). As the state process X hits her threshold $S^{1,*}_{2,2}$ (the first time X_t exceeds $S^{1,*}_{2,2}$), firm 1 invests to expand her production capacity by exercising one option, which in turn moves the macro market to the regime $(1, 2)$, namely one option left for firm 1 and two options left for firm 2. Sequentially, the firms exercise their options at the hitting times of their thresholds and the resulting macro market evolution is (2, 2) $\rightarrow$ (1, 2) $\rightarrow$ (1, 1) $\rightarrow$ (0, 1) $\rightarrow$ (0, 0).

1.3.2 Optimal Switching: Infinite Controls

The problem of determining an optimal sequence of stopping times to switch between several regimes (or modes), e.g. entry/exiting from a market, starting/shutting down a production, different investing modes, etc, is called optimal switching problem. The theory of optimal switching, as a special case of impulse control, was extensively studied in the past decades, as an important subject both in mathematics and economics. Optimal two-regime switching problems (or starting and stopping problems) were introduced into the study of real options by Brennan and Schwarz [20], and by Dixit [31] to analyze production facility problems. An example of multiple switching problems is Ludkovski [57] who applies to energy tolling agreement. Mathematically, Brekke and Økesendal [19] apply a verification approach for solving the variational inequality associated with the switching problem, which is generalized by Pham and Ly Vath [58] by using a viscosity solution approach, Pham [66] and Pham et al. [67] studies smooth-fit principle in the context of optimal multi-regime switching problem via dynamic programing principle and systems of variational inequalities, Hu and Tang [44] study a multi-dimensional backward stochastic differential equation (BSDE) with oblique reflection which arises in the study of optimal switching problem, while Bayraktar and Egami [14] construct the optimal value functions by utilizing the dynamic programming principle and construct explicit solutions using the smallest concave majorant method. See also Carmona and Ludkovski [23] who study operational flexibility of energy assets and propose a method of numerical solutions relying on Monte Carlo regression; Johnson and Zervos [51] who derive an explicit characterization of optimal tactics with relaxed smoothness requirements of payoffs.

Characterized by system states (i.e. the macro regime M) and the evolution of state variables (i.e. the micro market fluctuation X), such switching problems are naturally linked to the market with two time-scale feature. Most relevant are the multi-mode models [57, 66, 67, 44] mentioned above. Moreover, considering the players dynamically react to actions of their rivals by strategically switching the system states (i.e. moving the market regime) leads us to a *switching game*, which merges the single-agent switching models and the nonzero-sum stopping games [42, 8, 59, 69] accordingly. The overall model then links exogenous stochastic shocks with the endogenized players' decisions to obtain the dynamic equilibrium for the macro market organization. In Figure 1.3, we sketch the emerging equilibrium associated to one of our case studies as an illustration. Player 1 increases M_t by $+1$ whenever (X_t) hits her thresholds (the dashed lines in the bottom panel) from below, while P2 decreases M_t by -1 whenever (X_t) hits her thresholds from above. The top panel shows the resulting macro stage (M_t^*) along one realized trajectory of the local market fluctuations (X_t).

Our methodological interest in this model stems from three different directions. First, it extends our work on multi-stage capacity expansion games discussed in Chapter 3. In that version, the number of controls available to the players was a priori restricted; here we consider the more plausible situation of an infinitely-repeated game. One economic motivation is the capacity expansion problem under a growing stochastic environment (e.g. demand) X. This yields a non-stationary model but the ultimate number of aggregate investments is unbounded, and must be modeled by a switching game. Second, we are interested in a *stationary* switching game, where the market undergoes cyclical shocks (in the sense of X being a recurrent Markov process). We wish to find the endogenous dynamic equilibrium that will mirror this cyclicality through the strategically adjusted market regime. Describing such a recurrent stochastic investment-timing com-

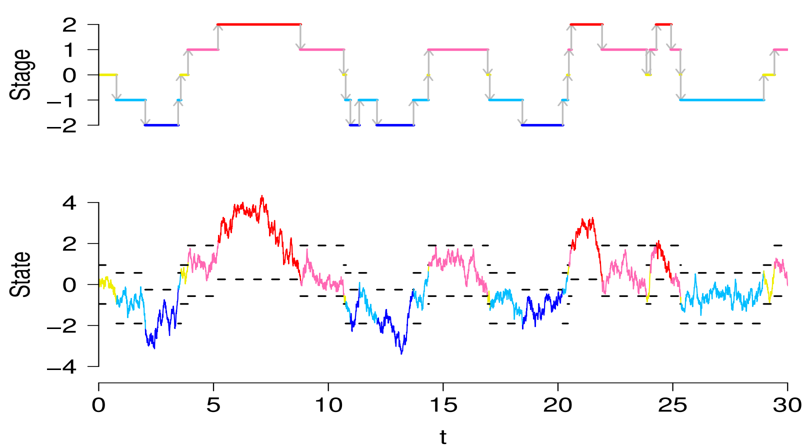


Figure 1.3: A trajectory of X and equilibrium M^* starting at $X_0 = 0$, $M_0^* = 0$. Here X is an Ornstein-Uhlenbeck process and $\mathcal{M} = \{-2, -1, 0, 1, 2\}$. The equilibrium strategies are of *threshold-type*; the dashed lines in the bottom plot indicate the respective switching thresholds.

petition naturally links to switching duopoly games. Third, our model is motivated by the desire for tractability while allowing for dynamic cross-effects due to competition and stochastic shocks. The repeated stationary nature of the competition allows to remove the time-variable but still maintain the stochastic dynamics. In particular, we leverage the related analytical results about the variational inequalities satisfied by the value functions [19, 58] and the construction of threshold-type strategies. We also make extensive use of the finite-control approximation inspired by Bayraktar and Egami [14] to establish the dynamic programming principle. As a result, the equilibrium structure is intuitive (summarized through switching thresholds) yet brings novel insights.

1.3.3 Mixed Optimal Switching and Impulse Controls

Under a typical setting of stochastic impulse control problem, the controller receives continuous and instantaneous reward/cost according to the underlying diffusion process. By exercising impulse controls which results costs bounded from below, she is able to move the underlying process by a certain amount. Therefore, optimality of impulse control problems involves both the choice of optimal sequence of intervening times and the choice of optimal impulse amounts in every time instant. Such stochastic impulse control problem has attracted a growing interest of many researchers over the last decades, and has been widely studied in inventory control [43], exchange rate problem [21], dividend payout problems [49] and portfolio optimization with transaction costs [61]. We refer to the work of Korn [53] which surveys the applications of impulse control in mathematical finance. The appropriate mathematical framework to cover these problem is in Bensoussan and Lions [15]. The controlled underlying process is described as an Itô diffusion in many economic and financial applications, solving which generally exploits a study of related Hamilton-Jacobi-Bellman (HJB) equations and quasi-variational inequalities.

Alternatively, Egami [34] shows a new mathematical characterization of the value function through the smallest concave majorant viewing the impulse control problem as a sequence of optimal stopping problems.

Surprisingly, there is a lack of literature in the field of stochastic impulse games. In very recent work, Aïd et al. [2] extend the single-agent optimization to a general nonzero-sum impulse duopoly game and provide a verification theorem for the value functions and the players' optimal strategies; Ferrari and Koch [36] model pollution control problem between a firm and a regulator as a stochastic impulse nonzero-sum game. See also a related work Aïd et al. [3] who present a policy iteration algorithm to tackle nonzero-sum stochastic impulse games. As an extension to the existing literature, we consider a duopoly of two players with different types of controls. In particular, we assume Player 1 can directly shift the underlying process X to her desired level (i.e. impulse controls), while Player 2 can affect X through switching the macro market regime M (unlike games we discuss in the preceding subsections, only Player 2 may exercise controls on the macro market). Their expected future profits depend on the *jointly controlled* process X, leading us to a nonzero-sum duopoly game.

Our motivation comes from the vertical competition among producer P1 and consumer P2 of a commodity. The producer extracts the commodity and sell it for a price X. The consumer buys the commodity and converts it into a final good with price P. This situation could represent a range of industries, e.g. extraction of crude oil, which is then consumed by refineries and chemical industries into final consumer goods. Or the production of aluminum that is converted by automaker into vehicles. Indeed, producers directly influence the supply yielding shocks to X, while consumers can be in either austerity or expansion mode that determines the drift of X. An illustration is sketched in Figure. 1.4. This cyclic behavior continues ad infinitum, yielding a stationary distribution for the pair (X_t, μ_t). Note that in Expand/Reduce regime, the consumer uses her

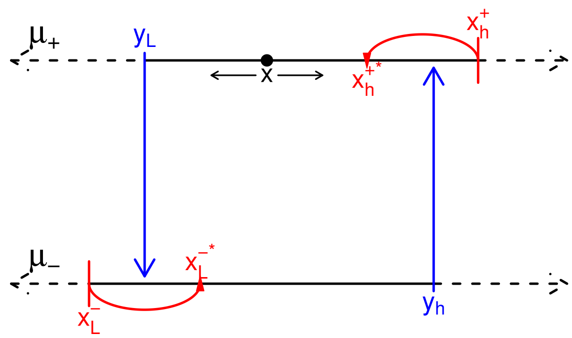


Figure 1.4: An illustration of the vertical competition between the producer and the consumer. The blue arrows represent drift-switching controls exercised by the consumer at levels $y_{l,h}$, while the red curved arrows represent impulse controls exercised by the producer at levels $x^{\pm}_{l,h}$.

switching control to keep X_t from going too high or too low, and the producer acts as a "back-up", explicitly forcing prices from becoming extreme.

1.4 Overview of the Thesis

We study nonzero-sum duopoly games in the market with two time-scale feature: a "microeconomic" stochastic factor (X_t) and the "macroeconomic" market regime (M_t). Compared to existing literature, we consider *asymmetric* players whose strategies are completely *endogenous*. Explicit Markovian Nash equilibria are constructed in three specific games motivated by economic applications, which bring novel insights. The rest of this thesis is organized as follows.

In Chapter 2, we provide rigorous formulation of the duopoly game in a recursive manner, introduce techniques used to tackle optimal stopping problems, and compute fundamental functions and quantities related to the underlying diffusion process which are building blocks to construct and analyze game equilibria.

In Chapter 3, we consider competitive capacity investment for a duopoly of two distinct producers, who are exposed to stochastically fluctuating costs and interact through aggregate supply. Capacity expansion is irreversible and modeled in terms of timing strategies characterized through threshold rules. Section 3.1 formalizes the nonzero-sum timing game we are led to, describing the transitions among the discrete investment stages (i.e. the macro market regime). Working in a continuous-time diffusion framework, we characterize and analyze the resulting Nash equilibrium and game values in Section 3.2. Our analysis quantifies the dynamic competition effects and yields insight into dynamic preemption and over-investment in a general asymmetric setting. A case-study considering the impact of fluctuating emission costs on power producers investing in nuclear and coal-fired plants is also presented in Section 3.3.

In Chapter 4, we study nonzero-sum stochastic switching games, in which two players compete for market dominance through controlling (via timing options) the discrete-state market regime M. Switching decisions are driven by a continuous stochastic factor X that modulates instantaneous revenue rates and switching costs. This generates a competitive feedback between the short-term fluctuations due to X and the medium-term advantages based on M. In Section 4.2 we construct threshold-type Feedback Nash Equilibria which characterize stationary strategies describing long-run dynamic equilibrium market organization. Section 4.3 describes two sequential approximation schemes linking the switching equilibrium to (i) constrained optimal switching; (ii) multi-stage timing games. We provide illustrations using an Ornstein-Uhlenbeck X that leads to a recurrent equilibrium M^* in Section 4.4.1 and a Geometric Brownian Motion X that makes M^* eventually "absorbed" as one player eventually gains permanent advantage in Section 4.4.2. Explicit computations and comparative statics regarding the emergent macroscopic market equilibrium are also provided.

In Chapter 5, we study vertical impulse competition between the producer and consumer of a commodity. The two players receive continuous and instantaneous profit based on their *jointly controlled* process X, leading us to a nonzero-sum game. As formulated in Section 5.1, both players can exercise their influence on the commodity price X, although their actions of distinct types, specifically impulse control by the producer and switching-drift control by the consumer. This extends the framework that X is fully exogenous. Section 5.2 discusses our heuristics on possible structure of the resulting equilibrium, and characterizes the best-responses of the players through solving a coupled system of quasi-inequalities. In Section 5.3, we provide some numerical examples in which explicit equilibria are obtained. Further research based on the preliminary results achieved is discussed in Section 5.4.

Overall, we provide a rigorous and generic formulation of nonzero-sum duopoly games with closed-loop strategies, which can be fitted by numerous economic examples. The exogenous stochastic shocks reflecting the micro local market condition are modeled by general diffusion processes possessing strong Markov property. Case studies related to specific widely used processes, e.g Brownian motions, Geometric Brownian motions, and Ornstein-Uhlenbeck (OU) processes, are also represented. In order to solve optimal control problems rising from determining the game solutions, we implement not only the verification approach via variational inequalities, but also a direct solution method via the smallest concave majorant. Employing tatonnement, i.e. best-response iteration, rather than looking into the equilibrium equations directly, we construct explicit threshold-type equilibria. High tractability of such equilibria allows us to analyze the dynamics of the short-term market X and the long-term market M emerging in the equilibrium, and brings us structural insights of competitive markets, e.g the long-run behavior of the micro/macro market, the pattern of transitions of the macro market regime, the strategic interaction between X and M, etc. Numerical studies of the game solutions are also implemented, which allow us to quantify several features of competitive markets: loss due to competition, impact of local market dynamics on competitive behavior, impact of the acting cost and so on.

Chapter 2

Formulation and Building Blocks

In this chapter, we provide a generic formulation of duopoly games in Section 2.1. Particularly, we model the exogenous risk factor by an Itô diffusion process X and the macro market regime by an endogenous discrete-state process M, define the admissibility of these players strategy profiles in a recursive manner, and construct explicit Markovian Nash equilibrium via best-response iteration. Section 2.2 provides two classical approaches: via the smallest concave majorant and via solving variational inequalities (VIs), to tackle single-agent control problems, which are involved in searching for the players' best-response. In Section 2.3, we calculate fundamental solutions to ordinary differential equations (ODEs), expected first passage time, and first hitting probabilities related to the underlying process X. These results are critical in the search of a Nash equilibrium and the analysis of the market organization in the emerging equilibrium.

2.1 Generic Game Formulation

2.1.1 Exogenous Factor X and Endogenous Regime M

To capture the local fluctuating market condition, we introduce an *exogenous* diffusion process $(X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the stochastic differential equation (SDE)

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t, \tag{2.1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}$. Denote by $\mathcal{D} := (\underline{d}, \bar{d})$, with $-\infty \leq \underline{d} < \bar{d} \leq +\infty$, the domain of (X_t) and $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the natural filtration generated by (X_t). The coefficients $\mu : \mathcal{D} \to \mathbb{R}$ and $\sigma : \mathcal{D} \to \mathbb{R}_+$ are assumed to ensure a unique strong solution to (2.1). Moreover, let $\tau_x := \inf\{t > 0 : X_t = x\}$ denote the hitting time of the one-point set $\{x\}$, we assume that X is *regular* in $\mathcal{D}$, i.e. for any $x \in \mathcal{D} \setminus \partial\mathcal{D}$ and $y \in \mathcal{D}$,

$$\mathbb{P}[\tau_y < \infty \,|\, X_0 = x] > 0,$$

which informally means that starting at any x, X will reach any y with positive probability, and the the boundaries are *natural*, i.e. for any $t > 0$ and $y \in \mathcal{D} \setminus \partial\mathcal{D}$

$$\lim_{x \downarrow \underline{d}} \mathbb{P}[\tau_y < t \,|\, X_0 = x] = 0, \qquad \lim_{x \uparrow \bar{d}} \mathbb{P}[\tau_y < t \,|\, X_0 = x] = 0,$$

which means $\underline{d}, \bar{d}$ can neither be reached in finite time nor be a starting point of the process; see [16, Ch. 2] and [68, Ch. VII] for detailed exposition.

On the other hand, the macro market regime is fully controlled by the players through actions. Suppose that the two players exercise discrete/lumpy controls, by paying a cost K^i, to move the macro market toward their preferred direction (and possibly carry negative externality to the rival in the meantime). The macro market regime is then described by a discrete-state *endogenous* process (M_t) with domain $\mathcal{M}$. To each regime m there is an associated *action set* $C^i_m \subseteq \mathcal{M}$ that determines the potential new regimes that player i can drive M into. In a general setting, players are allowed to act on M_t in multiple ways and the corresponding player must determine the optimal new regime. In this thesis, we focus on the simple case where $C^i_m = \{m'\}$ is singleton, or $C^i_m = \emptyset$ (player i will not change market regime away from m).

Remark 2.1 *As stated in Section 1.2.3.3, the process (X_t), in a more general setting, may be partially controlled by the players, e.g one may take the coefficients μ, σ in (2.1) to depend on M_t, or the players may move X_t by exercising impulse controls. Such an extension requires refining our definition of the action set C^i_m's to include both the level of M to move into and the desired amount to move the underlying process X. We discuss this extension in Chapter 5.*

2.1.2 Admissible Strategies and Game Payoff

To define game strategies, we need to introduce some technical constructs needed to precise closed-loop equilibrium. Informally, closed-loop strategies are based on the history of (X_t) and the history of players' past actions. Note that due to the possibility that one player may act immediately following her rival, (M_t) is not sufficient on its own for this purpose.

In particular, we postulate the players adopt *timing strategies* and denote a strategy of player i by $\boldsymbol{\alpha}^i := \{\tau^i(n) : n \geq 1\}$ where τ^i's are certain stopping times. Admissibility of $\tau^i(n)$ is defined recursively, based on the initial state (x, m). Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by (X_t). Set $\sigma_0 = 0$, $X_0 = x$, $\tilde{M}_0 = m$. For $n \geq 1$, we require $\tau^i(n)$

to be $\tilde{\mathcal{F}}^{(n)}$-adapted and set

$$\tilde{\mathcal{F}}_t^{(n)} = \mathcal{F}_t \bigvee \sigma\big\{(\sigma_k, P_k, \tilde{M}_k), k < n\big\}, \tag{2.2a}$$

$$\sigma_n = \tau^1(n) \wedge \tau^2(n), \tag{2.2b}$$

$$P_n = 1 \cdot \mathbb{1}_{\{\tau^1(n)<\tau^2(n)\}} + 2 \cdot \mathbb{1}_{\{\tau^1(n)>\tau^2(n)\}} + \mathcal{H}_n \cdot \mathbb{1}_{\{\tau^1(n)=\tau^2(n)\}}, \tag{2.2c}$$

$$\tilde{M}_n = C^1_{\tilde{M}_{n-1}} \cdot \mathbb{1}_{\{P_n=1\}} + C^2_{\tilde{M}_{n-1}} \cdot \mathbb{1}_{\{P_n=2\}}. \tag{2.2d}$$

The meaning of $n = 1, \dots,$ is the counter for the overall "round" of the game, with σ_n recording the corresponding n-th acting time, P_n the identity of the player who exercises the n-th control, and $\tilde{M}_n$ the macro market regime after n total controls are exercised. Note that $C^i_{\tilde{M}_{n-1}}$ denotes the regime, to which player i would change the macro market from $\tilde{M}_{n-1}$.

In (2.2c) we address scenarios that both players intend to intervene at the same time by letting $\mathcal{H}_n$ denote the identity of the resulting leader. As a simple example, $\mathcal{H}_n \equiv 1$ if Player 1 has the instantaneous priority to intervene. In general, resolving $\mathcal{H}_n$ requires consideration of auxiliary discrete-stage game [38, 69] that happens instantaneously at $\underline{\tau}_m$ on the event $\{\tau^1(n) = \tau^2(n)\}$; the latter could involve mixed strategies, i.e. there is an additional random variable ω_t that determines the value of $\mathcal{H}_n$. This is another reason why we must explicitly augment the history of (P_k) to the history of (X_t) in (2.2a).

Definition 2.2 *(Admissible Strategies) The set of admissible closed-loop strategies $\mathcal{A}$ is $\boldsymbol{\alpha}^i := \{\tau^i(n) : n \geq 1\}$ where $\tau^i(n)$ is adapted to $(\tilde{\mathcal{F}}_t^{(n)})$ with σ_n, P_n, $\tilde{M}_n$ constructed in* (2.2)*, and satisfying*

- *no-acting regimes:* $\tau^i(n) = +\infty$ *if* $C^i_{\tilde{M}_{n-1}} = \emptyset, i \in \{1,2\}, \quad \forall n \geq 1;$
- *ordered in time:* $\tau^i(n) \geq \sigma_{n-1}, i \in \{1,2\}, \quad \forall n \geq 1;$

- *defined for all times:* $\lim_{n\to\infty} \sigma_n = +\infty$.

Note that strategies in $\mathcal{A}$ are of Closed Loop Perfect State (CLPS) type, see a detailed exposition in [22, Ch. 3]. The three admissibility conditions state that a player will not act at the regime where her action set is empty and also rule out a clustering of actions in finite time. The latter restriction $\lim_{n\to\infty} \sigma_n = +\infty$ is mild, as it would be sub-optimal to make infinite controls, as soon as there are some intervening costs. Note that Definition 2.2 is joint over the profile $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$ and also depends on the initial condition. In the sequel we suppress this dependence for lighter notation.

Let us revisit the construction (2.2) with $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$ denoting these players' strategies. Given the strategy profile $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$, the evolution of (M_t) is admitted as

$$M_t := \tilde{M}_{\eta(t)}, \quad \text{with} \quad \eta(t) = \max\{n \geq 0 : \sigma_n \leq t\}. \tag{2.3}$$

It is entirely possible and feasible that one player acts immediately, $\tau^i(n) = \sigma_{n-1}$, in which case $\sigma_n = \sigma_{n-1}$, hence (M_t) formally undergoes multiple changes simultaneously. Furthermore, we describe the sequence of acting times realized by *each player*, denoted by σ^i_k, $i \in \{1, 2\}$, $k \geq 1$ as

$$\sigma^i_k := \sigma_{\eta(i,k)}, \quad \text{with} \quad \eta(i,k) = \min\{n \geq 1 : \sum_{l=1}^{n} \mathbb{1}_{\{P_l = i\}} = k\}. \tag{2.4}$$

As explained in the preceding chapter, the game payoffs J^i's received by these players are the total net present value (NPV) of future profits, namely the expected future cashflow discounted at an exogenous, constant interest rate $r > 0$, minus the discounted lump-sum costs K^i paid at each action epoch.

Definition 2.3 *(Game Payoffs) Given a strategy profile* $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$*, the NPV of future* *profits received by player i is*

$$J_m^i(x;\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2) := \mathbb{E}\Bigg[-\sum_{n=1}^{\infty}\mathbb{1}_{\{P_n=i\}}e^{-r\sigma_n}\cdot K^i\big(X_{\sigma_n},\tilde{M}_{n-1}\big) + \int_0^{\infty} e^{-rt}\pi^i\big(X_t,\tilde{M}_{\eta(t)}\big)\,dt \,\Bigg|\, X_0=x, M_0=m\Bigg], \qquad (2.5)$$

where σ_n^i denotes the n-th acting time and P_n denotes the identity of the player who exercises the n-th control.

Note that the intervening costs K^i's are deterministic functions to be defined in specific games and we use $(\tilde{M}_{\eta(t)})$ defined in (2.3) to emphasize the discrete nature of the macro market regime (M_t).

2.1.3 Threshold-type Markov Nash Equilibrium

From Definition 2.3, game payoffs received by these players are functions of the initial local market condition X_0 and the macro regime M_0. Accordingly, we refine our definition of the optimal behavior in the game as a Markov Nash Equilibrium (MNE).

Definition 2.4 *(Markov Nash Equilibrium) Let $X_0 = x, M_0 = m$. The strategy profile $(\boldsymbol{\alpha}^{\mathbf{1,*}}, \boldsymbol{\alpha}^{\mathbf{2,*}}) \in \mathcal{A}$ is said to be a Markov Nash equilibrium of the duopoly game if for $\forall x \in \mathcal{D}, \forall m \in \mathcal{M}$ and strategy $\boldsymbol{\alpha}^i$ of player i such that $(\boldsymbol{\alpha}^i, \boldsymbol{\alpha}^{j,*})$ is admissible*

$$J_m^i(x;\,\boldsymbol{\alpha}^i,\boldsymbol{\alpha}^{j,*}) \le J_m^i(x;\,\boldsymbol{\alpha}^{i,*},\boldsymbol{\alpha}^{j,*}), \qquad i\in\{1,2\}, \qquad j\ne i, \qquad (2.6)$$

with $V_m^i(x) := J_m^i(x;\,\boldsymbol{\alpha}^{i,*},\boldsymbol{\alpha}^{j,*})$ denoting the corresponding equilibrium payoff.

In Section 1.2.3.1, we demonstrate that a Nash equilibrium can be characterized as a fixed-point of the best-response maps among these two players, which allows us to focus on a special class of strategies and explicitly construct an equilibrium. To do so, two key properties of the class are needed. First, given a strategy from this class, the corresponding single-agent optimal control problem should be uniquely solved so that we are able to construct a best-response map. Second, this class of strategies should ideally be closed under the the best-response map. In this thesis, we introduce the class of strategies, which are *stationary* and of *threshold-type*, as follows and exploit it to construct explicit Nash equilibria for specific games.

The time-stationary Markovian strategies, also known as Feedback Perfect State (FPS) type defined in [22, Ch. 3] depend only on the current X_t and $\tilde{M}_{\eta(t)}$. Following the idea of a similar construction in [2], we define a strategy of player $i \in \{1, 2\}$ by $\boldsymbol{\alpha}^i := (\Gamma^i_m)_{m\in\mathcal{M}}$, where Γ^i_m's are fixed subsets of $\mathcal{D}$. Specifically, when $M_t = m$, player i adopts the (feedback) acting region Γ^i_m: player i exercises a control at the first hitting time τ^i_m of (X_t) to Γ^i_m (with the convention that the hitting time of an empty set is ∞). Moreover, for a simpler asymmetry in their profit rates, we assume that Player 1 is in favor of high X_t, while Player 2 prefers the opposite; it is therefore natural to assume that P1 acts when X becomes high enough and P2 acts when X becomes low enough.

Definition 2.5 *(Threshold-type Strategies) Let $\boldsymbol{s}^i := (s^i_m)_{m\in\mathcal{M}}$ be a vector which characterizes subsets of $\mathcal{D}$ for the acting regions Γ^i_m of player $i \in \{1, 2\}$ according to*

$$\Gamma^1_m \equiv \Gamma^1_m(\boldsymbol{s}^1) := [s^1_m, \overline{d}), \qquad \text{and} \qquad \Gamma^2_m \equiv \Gamma^2_m(\boldsymbol{s}^2) := (\underline{d}, s^2_m]. \tag{2.7}$$

A strategy associated to $(\Gamma^i_m)_{m\in\mathcal{M}}$ is called of threshold-type and denoted by $\boldsymbol{s}^i$.

Note that such mononicity assumption could be extended. In Chapter 5, we consider a game in which players' profit rates are both quadratic in X_t, nevertheless explicit threshold-type Nash equilibria are attainable.

One notable merit of such threshold-type strategies is the time homogeneity, which combining with the Dynamic Programming Principle (DPP) provides us effective approaches to explore the players' best-response. In particular, given a *threshold-type* strategy $\boldsymbol{\alpha}^j \equiv \boldsymbol{s}^j$ of player j, the best-response value function of player i, $\widetilde{V}^i(\cdot;\boldsymbol{s}^j)$ defined in (4.6), solves a system of coupled stopping problems (see [14]). Namely letting $\mathcal{I}_m := \tau_m^1 \wedge \tau_m^2$ (with τ_m^j pre-specified hitting times according to $\boldsymbol{s}^j$), we expect that

$$\begin{aligned}\widetilde{V}_m^i(x\,;\boldsymbol{s}^j) = \sup_{\tau_m^i\in\mathcal{T}} \mathbb{E}_x \Bigg[\int_0^{\mathcal{I}_m} & e^{-rt}\pi^i(X_t,m)dt \\ &+ e^{-r\mathcal{I}_m}\mathbb{1}_{\{\tau_m^1>\tau_m^2\}}\Big(\widetilde{V}_{m''}^i(X_{\tau_m^2}\,;\boldsymbol{s}^j) - \mathbb{1}_{\{i=2\}}K^i(X_{\tau_m^2},m)\Big) \\ &+ e^{-r\mathcal{I}_m}\mathbb{1}_{\{\tau_m^1<\tau_m^2\}}\Big(\widetilde{V}_{m'}^i(X_{\tau_m^1}\,;\boldsymbol{s}^j) - \mathbb{1}_{\{i=1\}}K^i(X_{\tau_m^1},m)\Big)\Bigg], \end{aligned} \tag{2.8}$$

for $i \in \{1,2\}$, $j \neq i$, $\forall x \in \mathcal{D}$ and all $m \in \mathcal{M}$. We use the shorthand notation $\mathbb{E}_x[\cdot] := \mathbb{E}[\cdot|X_0 = x]$, and the subscript in $\widetilde{V}_m^i$ to indicate the conditioning on $M_0 = m$, which is unchanged until $\mathcal{I}_m$. Intuitively, at regime m player i implements a timing strategy to exercise her control at τ_m^i, and a realization of these two stopping times yields a "*leader*", who acts first and changes the market regime into her desired regime (m' for Player 1 and m'' for Player 2). Note that the case $\{\tau^1 = \tau^2\}$ requires ad hoc discussion, which we provide in the following chapters respectively.

Another favorable aspect of such a strategy class is that we reduce dimension of the players' acting regions by characterizing them through threshold vectors. Moreover, in specific games considered in this thesis, we are able to either directly demonstrate the best-response of a threshold-type strategy is of threshold-type, or justify this statement via verification arguments. In turn, the best-response of player i solving the system of problems (2.8) is characterized by a threshold vector denoted by $\tilde{\boldsymbol{s}}^i(\boldsymbol{s}^j)$. Furthermore, a Markov Nash equilibrium of the game $(\boldsymbol{s}^{1,*}, \boldsymbol{s}^{2,*})$ is indicated as a fixed-point of the threshold map

$$\boldsymbol{s}^{i,*} = \tilde{\boldsymbol{s}}^i(\boldsymbol{s}^{j,*}), \qquad i \in \{1,2\}, \qquad j \neq i. \tag{2.9}$$

2.2 Building Block: Method of Solution

In fact approaching the coupled system (2.8) requires handling the solutions of optimal stopping problems with an exit constraint. As an essential building block, the set of optimal stopping problems is narrowed to considering

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} h\left(X_\tau\right) \right\}, \quad \text{and} \tag{2.10a}$$

$$V_R(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ \mathbb{1}_{\{\tau < \tau_R\}} e^{-r\tau} h(X_\tau) + \mathbb{1}_{\{\tau > \tau_R\}} e^{-r\tau_R} l(X_{\tau_R}) \right\}, \tag{2.10b}$$

where τ_R is restricted to be an *exit time* associated to a given interval $R := (a, \bar{d})$ or $R := (\underline{d}, a)$ with $a \in \operatorname{int}(\mathcal{D})$ (accordingly a *hitting time* associated to a threshold-type acting region $(\underline{d}, a]$ or $[a, \bar{d})$), $h(\cdot)$ is the *first-mover* payoff, and $l(\cdot)$ corresponds to the resulting *second-mover* payoff.

2.2.1 Smallest Concave Majorant

One technique we implement is of smallest concave majorants (see e.g. [29]), which has been used for zero-sum games in [64], [35] and [54], and for nonzero-sum games in recent works [30, 7]. A key advantage of the method, which characterizes the value functions via the smallest concave majorant associated to transformed first-mover payoff h, is that it directly determines the value function, as well as the structure of the optimal stopping region, which allows us to explicit construct a MNE.

The method proceeds by standardizing (X_t) via the transformations:

$$\psi(x) := \frac{F}{G}(x), \qquad \varphi(x) := \frac{G}{F}(x) \tag{2.11}$$

where F and G are respectively the fundamental *increasing* and *decreasing* solutions to the diffusion ODE:

$$(\mathcal{L} - r)u(x) = 0, \qquad x \in \mathcal{D}, \tag{2.12}$$

where $\mathcal{L} = b(x)\frac{d}{dx} + \frac{\sigma^2(x)}{2}\frac{d^2}{dx^2}$ is the infinitesimal generator of X. These linearly independent solutions are positive, continuous, strictly monotone and convex and admit the representations (see [70, vol.II, p.292])

$$\mathbb{E}_x\left\{e^{-r\tau_a}\mathbb{1}_{\{\tau_a<\infty\}}\right\} = \begin{cases} \frac{F(x)}{F(a)}, & \text{if } x \le a, \\ \frac{G(x)}{G(a)}, & \text{if } x \ge a. \end{cases} \tag{2.13}$$

Moreover for $\underline{d}$ and $\bar{d}$ natural boundary points one also has (see [16], Sec.2)

$$\lim_{x\downarrow\underline{d}} F(x) = 0, \qquad \lim_{x\downarrow\underline{d}} G(x) = +\infty, \qquad \lim_{x\uparrow\bar{d}} F(x) = +\infty, \qquad \lim_{x\uparrow\bar{d}} G(x) = 0. \tag{2.14}$$

In Section 2.3.1, we derive such fundamental solutions associated to three well-studied diffusion processes.

It follows from properties of F and G that ψ (resp. φ) : $\mathcal{D} \longmapsto \mathbb{R}^+$ is positive, strictly *increasing* (resp. *decreasing*), continuous, and twice differentiable on $\mathcal{D}$. Define the ψ-transform operator Ψ as:

$$\Psi h(y) := \begin{cases} \frac{h}{G} \circ \psi^{-1}(y), & \text{if } y > 0, \\ \lim\limits_{x \downarrow \underline{d}} \dfrac{h(x)}{G(x)}, & \text{if } y = 0, \end{cases} \tag{2.15}$$

and similarly the φ-transform operator Φ by

$$\Phi h(z) := \begin{cases} \frac{h}{F} \circ \varphi^{-1}(z), & \text{if } z > 0, \\ \lim\limits_{x \uparrow \bar{d}} \dfrac{h(x)}{F(x)}, & \text{if } z = 0. \end{cases} \tag{2.16}$$

Applying the operator Ψ (Φ resp.) transforms the optimal stopping problem from x coordinate to the $y = \psi(x)$ ($z = \varphi(x)$ resp.) coordinate.

Recalling the optimal stopping problem (2.10b), if x were to be in the region R, since X is regular in $\mathcal{D}$, it reaches the R-boundary a with positive probability. Therefore, one can consider X as a living on the restricted domain $\bar{R} := [a,\, \bar{d})$ or $(\underline{d},\, a]$, where the boundary at a is *absorbing*, i.e. the process X is stopped when it reaches the level a. Accordingly, we impose a boundary condition on the first-mover payoff h by defining

$$\hat{h}_R(x) := \begin{cases} h(x), & \text{if } x \in R, \\ l(x), & \text{if } x = a. \end{cases} \tag{2.17}$$

Thanks to the work of [29] and [30], if x were to be in the region $R = (a, \bar{d})$ (resp. $R = (\underline{d},\, a)$), the value function $V_R(x)$ is associated to the smallest concave majorant of transformed payoff $\Psi\hat{h}_R(y)$ (resp. $\Phi\hat{h}_R(z)$) over $\psi(\bar{R}) = [\psi(a),\, +\infty)$ (resp. $\varphi(\bar{R})$), denoted by $\mathcal{W}\Psi_{\bar{R}}\hat{h}(y)$ (resp. $\mathcal{W}\Phi_{\bar{R}}\hat{h}(z)$). If x were to be in the exit region $\mathcal{D} \setminus R$ (i.e $\tau_R = 0$), the value function $V_R(x)$ is equivalent to the instant second-mover payoff $l(x)$. To recap, we state the following proposition for the case $R = (a,\, \bar{d})$.

Proposition 2.6 *The value function $V_R(x)$ defined in* (2.10b) *is*

$$V_R(x) = \begin{cases} G(x) \cdot \left[\mathcal{W}\Psi_{\bar{R}}\hat{h} \circ \psi(x) \right], & \text{if } x \in R, \\ l(x), & \text{if } x \in \mathcal{D} \setminus R. \end{cases} \tag{2.18}$$

Moreover, if $l(a) \geq h(a)$, then $\lim_{x \searrow a} V_R(x) = l(a)$ and an optimal stopping rule τ^* can be defined as

$$\Gamma := \{x \in \bar{R} : V_R(x) = h(x)\} \qquad \text{and} \qquad \tau^* := \inf\{t \geq 0 : X_t \in \Gamma\}. \tag{2.19}$$

Note that the unconstrained value function $V(x)$ defined in (2.10a) corresponds to the special case that $R = \mathcal{D}$ and $\hat{h} \equiv h$.

Definition 2.7 *Let $\mathcal{L}$ be the infinitesimal generator of the state process X_t Let $\mathcal{H}$ be the class of real valued functions $h \in \mathcal{C}^2(\mathcal{D})$ such that*

$$\limsup_{x \to \underline{d}} \left| \frac{h(x)}{G(x)} \right| = 0 = \limsup_{x \to \bar{d}} \left| \frac{h(x)}{F(x)} \right|, \tag{2.20}$$

$$\text{and} \qquad \mathbb{E}_x \left[\int_0^\infty e^{-rt} \left| (\mathcal{L} - r)h\left(X_t\right) \right| dt \right] < \infty, \tag{2.21}$$

for all $x \in \mathcal{D}$. We denote by $\mathcal{H}_{inc}$ (resp. $\mathcal{H}_{dec}$) the set of all $h \in \mathcal{H}$ such that $x \mapsto (\mathcal{L} - r)h(x)$ is strictly positive (resp. negative) on $(\underline{d}, b_h)$ and strictly negative (resp. positive) on $(b_h, \bar{d})$ for some $b_h \in \mathcal{D}$.

In order to obtain threshold-type equilibrium, one would expect the optimal stopping rule τ^* (2.19) to be of threshold-type. To do so, some regularity of the underlying payoff functions is required. Similar to De Angelis et al. [30], we introduce two classes of functions $\mathcal{H}_{\text{inc}}$ and $\mathcal{H}_{\text{dec}}$ in Definition 2.7 and apply operators Ψ and Φ to payoffs in these classes. The following lemma states key properties of the resulting transformed payoff functions. Proof of the first statement can be done by multiple approaches and we give one as follows; for the rest of the statements we refer to [30, Lemma 3.1].

Lemma 2.8 *Let $h \in \mathcal{H}_{inc}$ (resp. $\mathcal{H}_{dec}$) and set $\hat{y} := \psi(b_h)$ (resp. $\hat{z} := \varphi(b_h)$). Then the transformed function $\hat{H} := \Psi h$ (resp. Φh):*

(i) is convex on $[0, \hat{y})$ (resp. $[0, \hat{z})$) and concave on $(\hat{y}, +\infty)$ (resp. $(\hat{z}, +\infty)$),

(ii) satisfies $\hat{H}(0+) = 0$ and $\hat{H}'(0+) = -\infty$,

(iii) has a unique global minimum at some $\bar{y} \in [0, \hat{y})$ (resp. $\bar{z} \in [0, \hat{z})$) and $\lim_{y\to\infty} \hat{H}(y) = +\infty$, hence it is monotonic increasing on $(\hat{y}, +\infty)$ (resp. $(\hat{z}, +\infty)$).

Proof: We take up the ψ-transform $\Psi h(y)$ as an example; the proof for φ-transform function can be done following the same scheme. The continuity and differentiability of $\Psi h(y)$ follow directly from those of h, G and ψ, and it is equivalent to show that

$$(\Psi h)''(y) = \frac{2}{\sigma^2(x)G(x)\left(\psi'(x)\right)^2}\left(\mathcal{L} - r\right)h(x), \qquad x = \psi^{-1}(y). \tag{2.22}$$

By definition of the operator Ψ (2.15),

$$(\Psi h)''(y) = \left(\frac{1}{\psi'(x)}(\frac{h}{G})'(x)\right)' = \frac{1}{\left(\psi'(x)\right)^2}\left[(\frac{h}{G})''(x) - \frac{\psi''(x)}{\psi'(x)}(\frac{h}{G})'(x)\right]. \tag{2.23}$$

On the other hand, by direct differentiation (and dropping the x-argument for typographical convenience)

$$\begin{aligned}
\left(\mathcal{L} - r\right)h &= \left(\mathcal{L} - r\right)\left(\frac{h}{G}\cdot G\right) \\
&= b(x)(\frac{h}{G})'G + \frac{\sigma^2(x)}{2}\left[(\frac{h}{G})''G + 2(\frac{h}{G})'G'\right] + \frac{h}{G}\left(\mathcal{L} - r\right)G \\
&= \frac{\sigma^2(x)G}{2}\left[(\frac{h}{G})'' + \left(\frac{2b(x)}{\sigma^2(x)} + 2\frac{G'}{G}\right)\cdot(\frac{h}{G})'\right].
\end{aligned} \tag{2.24}$$

Meanwhile,

$$
\begin{aligned}
(\mathcal{L}-r)\,F &= (\mathcal{L}-r)\left(G\frac{F}{G}\right) = (\mathcal{L}-r)\,(G\psi) \\
&= b(x)\psi' G + \frac{\sigma^2(x)G}{2}\psi'' + \sigma^2(x)\psi' G' + \psi\,(\mathcal{L}-r)\,G \\
&= b(x)\psi' G + \frac{\sigma^2(x)G}{2}\psi'' + \sigma^2(x)\psi' G' = 0. \qquad (2.25)
\end{aligned}
$$

Equations (2.24) and (2.25) follow from the fact that F and G are solutions to the ODE (2.12), and equation (2.25) yields that

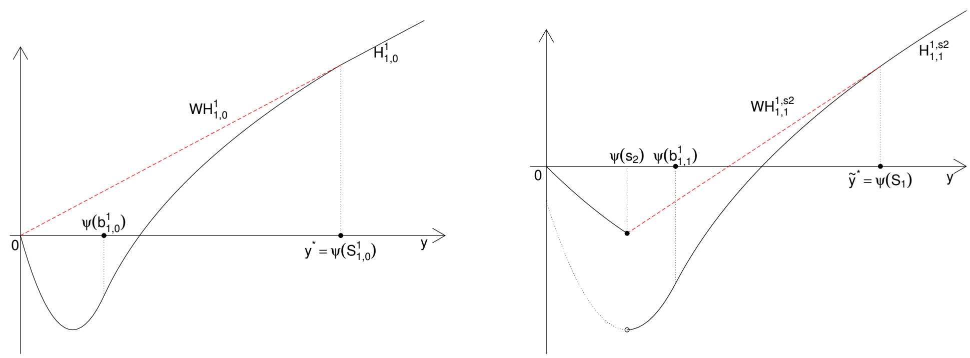
$$
-\frac{\psi''(x)}{\psi'(x)} = \frac{2b(x)}{\sigma^2(x)} + 2\frac{G'(x)}{G(x)}. \qquad (2.26)
$$

Substituting (2.26) into (2.24) and comparing with (2.23), we obtain (2.22) and complete the proof. In the case $h \in \mathcal{H}_{\text{dec}}$ and using φ-transformed $(\Phi h)\,(y)$ it follows by similar arguments that

$$
(\Phi h)''\,(z) = \frac{2}{\sigma^2(x)G(x)\,(\varphi'(x))^2}\,(\mathcal{L}-r)\,h(x), \qquad x = \varphi^{-1}(z). \qquad (2.27)
$$

Note that since $\varphi' < 0$, the interval $(0,\, b_h)$ on x coordinate corresponds to $(\varphi(b_h),\, +\infty)$ on $z = \varphi(x)$ coordinate, which completes the proof accordingly. ■

If the first-mover payoff h is in $\mathcal{H}_{\text{inc}}$ (resp. in $\mathcal{H}_{\text{dec}}$), it follows from Lemma 2.8 that the transformed $\hat{H}$ is convex and then concave. Consequently, its smallest concave majorant is a straight line which is tangent to $\hat{H}$ at a unique point, and then coincides with $\hat{H}$ as sketched in Figure 2.1a (see a similar work by Leung and Li [55]). This construction reduces to determining the tangency point of $\hat{H}$, corresponding to the transformed threshold. However, given τ_R being a hitting time of $(\underline{d}, a]$ (resp. $[a, \bar{d})$), the behavior at a is crucial for the existence of an optimal stopping rule τ^* defined in (2.19). We claim



(a) (b)

Figure 2.1: ψ corresponds to transformation defined in (2.11), and b denotes the reflection point where transformed payoff H switches concavity. (Left: a) $H^1_{1,0}$ and its smallest concave majorant $\left(\mathcal{W}H^1_{1,0}\right)$, sketched according to Lemma 2.8 with uppercase S denotes the optimal investment threshold. (Right: b) The transformed payoff $H^{1,s_2}_{1,1}$ and its smallest concave majorant over $(\psi(s_2), +\infty)$with s_2 denote given thresholds of firm 2 and S_1 denotes the best-response threshold of firm 1.

that such τ^* is a hitting time to a threshold-type acting region $(\tilde{a}, \bar{d}]$ (resp. $(\underline{d}, \tilde{a}]$), if the agent is not incentivised to preempt, namely $l(a) \geq h(a)$. See a sketch of the transformed payoff and its smallest concave majorant in Figure 2.1b and related details in Section 3.4.1.

On the contrary, the agent will try to *preempt* right before τ_R if she foresees second-mover payoff less than being the first-mover at a. In the content of a duopoly game, this phenomenon can be interpreted as one player's best-response is to preempt if her rival behaves aggressively. Such a preemptive response leads us to lack of optimal τ^*. We formulate this observation in Remark 2.9.

Remark 2.9 *(Preemptive Best-response) If $l(a) < h(a)$, the payoff $\hat{h}$ has a negative jump at the boundary a, and therefore the value function is also discontinuous there: $\lim_{x \searrow a} V_R(x) > l(a)$ (see a case study sketched in Figure 3.1a). This down-jump rules out* (2.19) *since in fact one ought to stop before reaching $a \in \Gamma$. In that case, there is no optimal stopping time; however for any $\varepsilon > 0$, an ε-optimal rule can be defined as the* *first hitting time of* $\Gamma^{\varepsilon} = \{x \in \bar{R} : V_R(x) \leq h(x) + \varepsilon\}$.

2.2.2 Variational Inequalities

A typical solution approach for optimal stopping problems driven by diffusion processes involves studies of variational inequalities (VIs). Such an approach has been widely implemented in solving optimal switching problems [19, 58, 66, 67], optimal impulse control problems [15, 18, 11], and nonzero-sum games [2].

Let V be the value function associated to the control problem (2.10a). Then it solves the following variational inequality

$$\max\{\mathcal{L}V - rV, h - V\} = 0. \tag{2.28}$$

Let V_R be the value functions associated to the control problem with constraint (2.10b). Then it solves the following variational inequality

$$\ell - V_R = 0, \qquad \text{in } \mathcal{D} \setminus R, \tag{2.29a}$$

$$\max\{\mathcal{L}V_R - rV_R, h - V_R\} = 0, \qquad \text{in } R. \tag{2.29b}$$

We refer to the books [65, 62] which provide verification theorems of above arguments. Furthermore, solving these VIs can be reduced to looking for solutions of the ODE (2.12) subject to certain free boundary and smooth pasting regularities. Assuming that h is smooth we know that the optimal stopping problem defined in (2.10b) leads to the following free-boundary problem:

$$(\mathcal{L}-r)V_R = 0, \qquad \text{in } \bar{R}\setminus\Gamma, \tag{2.30a}$$

$$V_R = h, \qquad \text{in } \Gamma, \tag{2.30b}$$

$$V_R = \ell, \qquad \text{in } \mathcal{D}\setminus R, \tag{2.30c}$$

$$\frac{\partial V_R}{\partial x} = \frac{\partial h}{\partial x}, \qquad \text{at } \partial\Gamma, \tag{2.30d}$$

where $\Gamma := \{x \in \bar{R} : V_R(x) = h(x)\}$ is the optimal stopping region and $\tau^* := \inf\{t \geq 0 : X_t \in \Gamma\}$. Analogously, the unconstrained value function $V(x)$ defined in (2.10a) corresponds to the special case that $R = \mathcal{D}$.

This method requires a priori assumptions about the shape of the stopping region and may lead to analytic challenges accordingly. Specifically, the method in general is as follows: One speculates forms of the stopping region and the associated value function, then determines the value function by using appropriate boundary conditions and verifies optimality of the candidate. Recall that for the sake of threshold-type equilibria we expect the optimizer to the problem (2.10b) to be a hitting time of an acting region characterized by a threshold $\tilde{a}$. Taking $R = (a, \bar{d})$ as an example, we conjecture V_R is of the form

$$V_R(x) = \begin{cases} h(x), & x \leq \tilde{a}, \\ \omega F(x) + \nu G(x), & a < x < \tilde{a}, \\ \ell(x), & x \leq a, \end{cases} \tag{2.31}$$

with the smooth pasting and boundary conditions:

$$\begin{cases} \omega F(\tilde{a}) + \nu G(\tilde{a}) = h(\tilde{a}), & (\mathcal{C}^0\text{-pasting at } \tilde{a}) \\ \omega F(a) + \nu G(a) = \ell(a), & (\mathcal{C}^0\text{-pasting at } a) \\ \omega F_x(\tilde{a}) + \nu G_x(\tilde{a}) = h_x(\tilde{a}), & (\mathcal{C}^1\text{-pasting at } \tilde{a}) \end{cases} \tag{2.32}$$

where F and G are fundamental solutions to the ODE (2.12). If such a function can verified to be a solution to the VIs (2.29), we conclude V_R is the value function associated to the control problem (2.10b) and the optimal stopping rule is the hitting time to $[\tilde{a}, \bar{d})$.

Back to the system (2.8), we assume the best-response of player i to be the hitting time of a threshold-type region with threshold $\widetilde{s}^i_m(s^j_m)$ and her corresponding game payoffs to be in form (2.31) at each regime $m \in \mathcal{M}$. Combining the smooth pasting and boundary conditions (2.32), we obtain a coupled non-linear system of thresholds and coefficients that characterize her value functions. In turn, we parameterize the corresponding coupled optimal stopping problem and boil it down into solving the non-linear system. If the value functions associated to a solution of the system solves a system of Quasi-variational Inequalities (QVIs) derived from (2.8), we can conclude, via a verification approach similar to the classical uncoupled problems, that the best-response of player i is of threshold-type with $\widetilde{\boldsymbol{s}}^i(\boldsymbol{s}^j)$ (we refer to the closely related work [2]).

2.3 Building Block: Elementary Computations

2.3.1 Fundamental Solutions to ODE

From the preceding subsection, it follows that the fundamental solutions F and G to the ODE (2.12) is critical to determine the best-response of the players and furthermore explicitly construct MNEs of the duopoly game. In this thesis, we consider three classical diffusion processes that are widely implemented to model the stochastic risk factor.

Suppose that (X_t) is a *Brownian motion* with a drift term, i.e. the strong solution to the SDE

$$dX_t = \mu dt + \sigma dW_t, \tag{2.33}$$

with $\mathcal{D} = \mathbb{R}$, $\sigma > 0$. The fundamental solutions for the ODE

$$\mu u'(x) + \frac{1}{2}\sigma^2 u''(x) - ru(x) = 0, \tag{2.34}$$

are

$$F_{BM}(x) := e^{\theta_+ x}, \qquad G_{BM}(x) := e^{\theta_- x}, \tag{2.35}$$

where θ_+ and θ_- are the positive and negative roots of the quadratic equation $\frac{1}{2}\sigma^2\theta^2 + \mu\theta - r = 0$.

Suppose that (X_t) is a *Geometric Brownian motion* (GBM), i.e. the strong solution to

$$dX_t = \mu X_t\, dt + \sigma X_t\, dW_t, \tag{2.36}$$

with $\mathcal{D} = (0, \infty)$, $\sigma > 0$. The fundamental solutions for the ODE

$$\mu x u'(x) + \frac{1}{2}\sigma^2 x^2 u''(x) - ru(x) = 0, \tag{2.37}$$

are:

$$F_{GBM}(x) := x^{\eta_+}, \qquad G_{GBM}(x) := x^{\eta_-}, \tag{2.38}$$

where η_+ and η_- are the positive and negative roots of the quadratic equation $\frac{\sigma^2}{2}\eta(\eta - 1) + \mu\eta - r = 0$.

Suppose that (X_t) is an *Ornstein-Uhlenbeck* (OU) process, i.e. the strong solution to

$$dX_t = \mu(\theta - X_t)dt + \sigma dW_t, \tag{2.39}$$

with $\mathcal{D} = \mathbb{R}$, μ, $\sigma > 0$ and $\theta \in \mathbb{R}$. The fundamental solutions for the ODE

$$\mu(\theta - x)u'(x) + \frac{1}{2}\sigma^2 u''(x) - ru(x) = 0, \tag{2.40}$$

are:

$$\begin{aligned} F_{OU}(x) &:= \int_0^\infty u^{\frac{r}{\mu}-1} e^{\sqrt{\frac{2\mu}{\sigma^2}}(x-\theta)u - \frac{u^2}{2}} du, \\ G_{OU}(x) &:= \int_0^\infty u^{\frac{r}{\mu}-1} e^{-\sqrt{\frac{2\mu}{\sigma^2}}(x-\theta)u - \frac{u^2}{2}} du. \end{aligned} \tag{2.41}$$

Notice that direct differentiation yields that $F'(x) > 0$, $F''(x) > 0$, $G'(x) < 0$, $G''(x) > 0$. It follows that both $F(x)$ and $G(x)$ are strictly positive and convex, and $F(x)$ is strictly increasing while $G(x)$ is strictly decreasing. One can also easily check their limits at the corresponding natural boundary points of the domain $\mathcal{D}$ satisfy the condition (2.14).

2.3.2 First Passage Times and Hitting Probabilities

The macro market evolution M^* emerging in equilibrium is a time inhomogeneous *non-Markovian* process with discrete state space $\mathcal{M}$. Thanks to the stationary nature of the threshold-type equilibria, the behavior of M^* is highly tractable. As a quick glimpse, we analyze the long-run market organization via a jump chain $\check{M}$ defined on an extended regime space M^* traverses (see detailed discussion in Section 4.2.3). Since these players act when the process (X_t) hits their acting thresholds, the expected first passage times associated to those thresholds are linked to the average sojourn times of the chain $\check{M}$. In effect, the threshold characterization of the players' strategies highlights the importance of first passage times and hitting probabilities in describing the long-run market organization. Therefore we demonstrate these essential computations related to the underlying process X in this subsection for later use.

2.3.2.1 First Passage Times

Let us first consider the one-sided passage time $\tau(x;s) := \inf\{t \geq 0 : X_t^x = s\}$. We condition on the exit time τ being finite, denoting

$$\delta_s(x) = \mathbb{E}[\tau(x;s)\mathbb{1}_{\{\tau(x;s)<\infty\}}]. \tag{2.42}$$

Then, we implement the well-known result by Darling and Siegert in [28] about the Laplace transform of $\tau(x;s)$,

$$\mathbb{E}_x\left[e^{-\rho\tau(x;s)}\mathbb{1}_{\{\tau(x;s)<\infty\}}\right] = \begin{cases} \frac{F(x;\rho)}{F(s;\rho)}, & \text{if } x \leq s, \\ \frac{G(x;\rho)}{G(s;\rho)}, & \text{if } x \geq s, \end{cases} \tag{2.43}$$

where $F(\cdot;\rho)$ and $G(\cdot;\rho)$ are solutions to $(\mathcal{L}-\rho)u=0$ and we emphasize their dependence on the Laplace parameter ρ, to compute $\delta_s(x)$

$$\delta_s(x) = -\frac{\partial}{\partial\rho}\mathbb{E}_x\left[e^{-\rho\tau(x;s)}\mathbb{1}_{\{\tau(x;s)<\infty\}}\right]\Bigg|_{\rho=0}, \tag{2.44}$$

Example 2.10 *(Brownian motion). Suppose that the drift term is not trivial, i.e.* $\mu \neq 0$. *Taking* $\mu > 0$ *as an example, from* (2.44) *we obtain*

$$\delta_s(x) = \mathbb{E}\Big[\tau(x;s)\mathbb{1}_{\{\tau(x;s)<\infty\}}\Big] = \begin{cases} -\frac{x-s}{\mu}e^{\frac{2\mu}{\sigma^2}(x-s)}, & \text{if } x \leq s, \\ \frac{x-s}{\mu} & \text{if } x \geq s. \end{cases}$$

(Geometric Brownian motion). Suppose that $\mu - \frac{1}{2}\sigma^2 > 0$ *so that for* $s < x$ *the expected one-sided first passage time is infinite, i.e.* $\mathbb{E}[\tau(x;s)] = \infty$. *Nevertheless, from* (2.44) *we compute*

$$\delta_s(x) = \mathbb{E}\Big[\tau(x;s)\mathbb{1}_{\{\tau(x;s)<\infty\}}\Big] = \frac{1}{\mu-\frac{1}{2}\sigma^2}\cdot\ln\left(\frac{x}{s}\right)\cdot\left(\frac{x}{s}\right)^{1-\frac{2\mu}{\sigma}}.$$

(Ornstein-Uhlenbeck process.) Following from (2.44), *the expected first passage time* $\delta_s(x)$ *to a level* s *is admitted as*

$$\delta_s(x) = \frac{\sqrt{2\pi}}{\mu}\left\{\left[\int_{(x-\theta)\sqrt{\frac{2\mu}{\sigma^2}}}^{(s-\theta)\sqrt{\frac{2\mu}{\sigma^2}}}\Phi\left(z\right)e^{\frac{1}{2}z^2}dz\right]1_{\{s\geq x\}} + \left[\int_{(\theta-x)\sqrt{\frac{2\mu}{\sigma^2}}}^{(\theta-s)\sqrt{\frac{2\mu}{\sigma^2}}}\Phi\left(z\right)e^{\frac{1}{2}z^2}dz\right]1_{\{s<x\}}\right\}, \tag{2.45}$$

where Φ *is the standard Gaussian cumulative distribution function.*

Now let us look into the two-sided first passage time defined as follows

$$\tau(x;a,b) := \inf\{t \geq 0 \,:\, X_t^x \leq a \text{ or } X_t^x \geq b\}, \qquad (a,b) \supset x,$$

and accordingly its expectation $\delta_{ab}(x) := \mathbb{E}\big[\tau(x;a,b)\big]$. Applying Dynkin's formula, it is well known that $\delta_{ab}(\cdot)$ solves the ordinary differential equation

$$\mathcal{L}\delta + 1 = 0, \qquad \text{with} \qquad \delta_{ab}(a) = \delta_{ab}(b) = 0.$$

In addition, we would like to mention that Darling and Siegert in [28] show that the expected exit time from an interval $x \in (a,b)$, $\delta_{ab}(x)$ can then be obtained via

$$\delta_{ab}(x) = \frac{\delta_a(x)\delta_b(a) + \delta_b(x)\delta_a(b) - \delta_a(b)\delta_b(a)}{\delta_b(a) + \delta_a(b)}. \tag{2.46}$$

Example 2.11 *(Brownian motion.) The expected exit time $\delta_{ab}(\cdot)$ as a solution to*

$$\mu\delta'_{ab}(x) + \frac{1}{2}\sigma^2\delta''_{ab}(x) + 1 = 0,$$

is in the form

$$\delta_{ab}(x) = -c_1 \cdot \frac{\sigma^2}{2\mu} \cdot e^{-\frac{2\mu}{\sigma^2}x} - \frac{x}{\mu} + c_2,$$

where c_1, c_2 are constants such that the boundary conditions $\delta_{ab}(a) = \delta_{ab}(b) = 0$ are fulfilled.

(Geometric Brownian motion.) The expected exit time $\delta_{ab}(\cdot)$ is a solution to

$$\mu x\delta'_{ab}(x) + \frac{1}{2}\sigma^2x^2\delta''_{ab}(x) + 1 = 0, \qquad x \in (a,b), \quad \textit{and} \quad \delta_{ab}(a) = \delta_{ab}(b) = 0.$$

Solving that we obtain

$$\delta_{ab}(x) = \left(\frac{1}{2}\sigma^2 - \mu\right)^{-1} \left\{ \ln\left(\frac{x}{a}\right) + \ln(\frac{a}{b}) \frac{\left(x^{1-2\mu/\sigma^2} - a^{1-2\mu/\sigma^2}\right)}{b^{1-2\mu/\sigma^2} - a^{1-2\mu/\sigma^2}} \right\}, \qquad x \in (a,b).$$

2.3.2.2 First Hitting Probabilities

Given an interval $(a,b) \supset x$, we are interested in the probabilities that the underlying process X starting from $X_0 = x$ hits one of the two boundaries rather than the other. Recalling the threshold characterization of the players' strategies, one can naturally interpret these probabilities as the likelihood that one player acts faster than her rival.

According to Revuz and Yor in [68, Ch VII.3], these first hitting probabilities can be evaluated via the scale functions $S(\cdot)$:

$$\mathbb{P}\big[X^x_{\tau(x;a,b)} = b\big] = \frac{S(x) - S(a)}{S(b) - S(a)}, \qquad \mathbb{P}\big[X^x_{\tau(x;a,b)} = a\big] = \frac{S(b) - S(x)}{S(b) - S(a)}. \tag{2.47}$$

Recall that S is the general solution to the ODE $\mathcal{L}S = 0$ that is available in closed-form for linear diffusions.

Example 2.12 *(Brownian motion). The scale function $S(\cdot)$ solves*

$$\mu S'(x) + \frac{1}{2}\sigma^2 S''(x) = 0, \qquad \Rightarrow \quad S_{BM}(x) = e^{-\frac{2\mu}{\sigma^2}x}, \quad x \in \mathbb{R}.$$

(Geometric Brownian motion). The scale function $S(\cdot)$ solves

$$\mu x S'(x) + \frac{1}{2}\sigma^2 x^2 S''(x) = 0, \qquad \Rightarrow \quad S_{GBM}(x) = x^{1-2\mu/\sigma^2}, \quad x \in \mathbb{R}_+.$$

(Ornstein-Uhlenbeck Process). The scale function $S(\cdot)$ solves

$$\mu(\theta - x)S'(x) + \frac{1}{2}\sigma^2 S''(x) = 0, \qquad \Rightarrow \quad S_{OU}(x) = \int_{-\infty}^{x} e^{\frac{\mu}{\sigma^2}(z-\theta)^2} dz, \quad x \in \mathbb{R}.$$

Note that since (X_t) is continuous in $\mathcal{D}$ the probability that (X_t^x) hits a threshold $s < x$ can be evaluated as the limiting probability as follows

$$\mathbb{P}(X^x \text{ hits } s) = \lim_{u \uparrow \overline{d}} \mathbb{P}\big[X^x_{\tau(x;s,u)} = u\big].$$

Similarly, one can handle the threshold $s > x$.

Chapter 3

Capacity Expansion Games

In this chapter, we consider competitive capacity investment for a duopoly of two distinct producers. The producers are exposed to stochastically fluctuating costs and interact through aggregate supply. Capacity expansion is irreversible and modeled in terms of timing strategies characterized through threshold rules. Because the impact of changing costs on the producers is asymmetric, we are led to a nonzero-sum timing game describing the transitions among the discrete investment stages. Working in a continuous-time diffusion framework, we characterize and analyze the resulting Nash equilibrium and game values. Importantly, depending on the competition strength, we find that both threshold-type and preemptive equilibria may arise. Our analysis quantifies the dynamic competition effects and yields insight into dynamic preemption and over-investment in a general asymmetric setting. A case-study related to the motivating economic example considering the impact of fluctuating emission costs on power producers investing in nuclear and coal-fired plants is also presented.

3.1 Problem Formulation

We consider a duopoly of two producers, dubbed firm 1 and firm 2. Each firm has options to irreversibly increase her current production capacity $Q^i(t)$ by paying a fixed lump-sum capital K^i, so as to generate more revenue. However, because the firms compete on the same market, expansion decisions of one firm carry negative externality (via lower market prices $P(t)$) for both of them, which leads to a nonzero-sum duopoly game.

3.1.1 Relative Cost X and Game Stage M

In this chapter, to capture market uncertainty, we introduce the *relative cost* between the production expenses of the two firms as a one-dimensional diffusion process $(X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the Îto stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \tag{3.1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}$. Denote by $\mathcal{D} := (\underline{d}, \bar{d})$, with $-\infty \leq \underline{d} < \bar{d} \leq +\infty$, the domain of X_t and $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the natural filtration generated by X_t. The coefficients $b : \mathcal{D} \to \mathbb{R}$ and $\sigma : \mathcal{D} \to \mathbb{R}_{++}$ are assumed to be Lipschitz so as to ensure a unique strong solution to (3.1).

To describe dynamic capacity expansion, we decompose the overall model into stages. Let $\vec{M}_t \in \{(N_t^1, N_t^2)_{t\geq 0} : N_t^i = 0, 1, \ldots, N_0^i\}$, where N_t^i counts how many expansion options remain for firm i, denote the game stage at date t. Irreversible investment implies that starting at $\vec{M}_0 = (N_0^1, N_0^2)$, each coordinate of $\vec{M}_t$ is *piecewise constant* and *non-increasing*. We postulate that firm capacities are fully determined by the investment game stage $Q^i(t) = Q^i(\vec{M}_t)$, and market price is solely a function of aggregate supply $Q(t) :=$

$Q^1(t) + Q^2(t)$ (here equated with aggregate production capacity). This is equivalent to assuming constant (or at least deterministic) demand, which is not far from the truth for base-load electricity generation where revenue is determined by fixed long-term contracts.

Remark 3.1 *Since we will rely on dynamic programming-like arguments, the framework necessitates to specify the initial number of possible expansions (N_0^1, N_0^2). This can be justified by assuming a fixed demand curve, so that one can infer the maximum additional capacity that is economically feasible. In other words, financially we work backwards, starting from potential* end-game *capacities (i.e. stages where no more investment will take place) to determine the maximum number of initial options needed (see a related discussion in [17, Sec 2.5]). Given the long-run electricity demand forecasts, power generators can in aggregate determine how much capacity could be added, with the competition centered about who and when will expand (but not how far). If a total of $\Delta Q \bar{N}$ extra capacity is required, one can set $\vec{M}_0 = (\bar{N}, \bar{N})$.*

It follows that price is a function of game stage, $P(t) = P(\vec{M}_t)$. Consequently $Q(\vec{M}_s) \geq Q(\vec{M}_t)$, $P(\vec{M}_s) \leq P(\vec{M}_t)$ for $s \geq t$ are *piecewise constant* as well. For typographical convenience, we henceforth use subscripts (n_1, n_2) to index above stage quantities, e.g. $Q^i_{n_1,n_2} \equiv Q^i(\vec{n})$. As a simple example, one may take $Q^i_{n_1,n_2} = \underline{q}^i + (\Delta Q)(N_0^i - n_i)$ at a stage $(n_1,\, n_2)$, where ΔQ is the size of each expansion, and $\underline{q}^i$'s are the initial capacities. For the clearing prices, a typical setting is a linear inverse-demand curve:

$$P_{n_1,n_2} = D(1 - \eta[Q^1_{n_1,n_2} + Q^2_{n_1,n_2}]), \tag{3.2}$$

where $\eta > 0$ is the demand elasticity and D is a price multiplier.

When X_t is large, firm 1 has the comparative advantage in production, while when X_t is close to $\underline{d}$ firm 2 has the advantage. Since X_t lives on the real line, this monotonicity assumption is rather natural. Specifically we shall assume that production costs of firm 1 decrease (linearly) in X_t, while the production costs of firm 2 increase (linearly) in X_t, which leads to profit rates of the form

$$\begin{aligned} \pi^1_{n_1,n_2}(X_t) &= (P_{n_1,n_2} - C^1 + \rho^1 X_t)Q^1_{n_1,n_2}, \\ \pi^2_{n_1,n_2}(X_t) &= (P_{n_1,n_2} - C^2 - \rho^2 X_t)Q^2_{n_1,n_2}, \end{aligned} \tag{3.3}$$

at stage $(n_1,\, n_2)$, where $C^i, \rho^i > 0$ are the firm-specific fixed production cost, and sensitivity of relative costs to X, respectively. The *monotonicity* of $\pi^i(\cdot)$ in x will be important for the analytic derivations in the sequel.

Remark 3.2 *In energy markets literature, market uncertainty is usually captured by stochastic electricity prices (see e.g. [37]). In the context of supply-demand equilibrium this can be interpreted as stochastic demand [17]. Our setting could similarly accommodate a general stochastic power price of the form $P(\sum_i Q^i, X_t)$, assuming that the factor X_t is also negatively affecting one of the firm's costs. For example if X_t represents oil prices, then power prices are positively linked to X and nuclear production cost is independent of X_t, giving overall positive sensitivity $\rho^1 > 0$ of π^1 to x. In contrast, an oil-fired competitor has costs denominated in X_t and hence has overall a negative exposure $-\rho^2 < 0$ to oil.*

3.1.2 Game Policies and Game Payoffs

By expanding capacity, firms shift the game to a subsequent stage and henceforth change the profit rates they receive. From the definition of game stages $\vec{M}$, it follows that the firms' acting sets at the stage (n_1, n_2) are

$$C^1_{(n_1,n_2)} = \begin{cases} (n_1 - 1, n_2), & \text{if } n_1 > 0, \\ \emptyset, & \text{if } n_1 = 0, \end{cases} \qquad C^2_{(n_1,n_2)} = \begin{cases} (n_1, n_2 - 1), & \text{if } n_2 > 0, \\ \emptyset, & \text{if } n_2 = 0. \end{cases} \tag{3.4}$$

Following from Definition 2.2, we recursively determine admissible strategy profiles $\mathcal{A} := \{(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)\}$ of the firms.

The time-homogeneity of X_t and the feedback form of prices in terms of $\vec{M}$ is the natural motivation to restrict attention to time-homogenous investment strategies. Thus, we postulate actions of the firms to be of time-stationary Feedback Perfect State (FPS) or Markov type, namely the strategy set of firm i is

$$\mathcal{A}^i = \left\{ \boldsymbol{\alpha}^i := \alpha^i(X_t, \vec{M}_t) \right\}, \qquad i = 1, 2. \tag{3.5}$$

Since the market price declines as aggregate capacity rises, investment by firm i will take place only once X_t moves sufficiently towards her preferred direction. Accordingly, we model capacity investment in terms of timing strategies. Due to the piecewise-constant feature of $\vec{M}_t$, it is sufficient to consider the $\mathbb{F}$-stopping times τ^i for the expansion epochs at each stage $(n_1,\, n_2)$. As a result, the strategy sets can be represented as:

$$\mathcal{A}^1 = \left\{ \boldsymbol{\alpha}^1 := \left(\tau^1_{n_1,n_2}\right) \,\middle|\, n_1 > 0, \forall n_2 \right\}, \quad \mathcal{A}^2 = \left\{ \boldsymbol{\alpha}^2 := \left(\tau^2_{n_1,n_2}\right) \,\middle|\, n_2 > 0, \forall n_1 \right\}, \tag{3.6}$$

maintaining the structure of (3.5).

Meanwhile, we assume that the firms evaluate their decisions based on the total net present value of future profits (NPV), namely the expected future cashflow discounted at an exogenous, constant interest rate $r > 0$, minus the discounted lump-sum costs K^i paid at each expansion epoch. Therefore, the game payoffs they received are constructed through Definition 2.3. Given a strategy profile $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$ and $X_0 = x, \vec{M}_0 = (n_1,\, n_2)$, the NPV of firm i is

$$J^i_{n_1,n_2}(x; \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2) := \mathbb{E}\left[\int_0^\infty e^{-rs}\pi^i_{N^1_s,N^2_s}(X_s)\,ds - \sum_{k=1}^{n_1} K^i \cdot e^{-r\sigma^i_k}\,\middle|\, X_0 = x, \vec{M}_0 = (n_1, n_2)\right], \tag{3.7}$$

where σ^i_k is the k-th investment time of player i. For brevity, we denote $J^i_{n_1,n_2}(x; \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$ by $J^i_{n_1,n_2}(x)$ henceforth.

This, combining with the definition of $\mathcal{A}$ in (3.6), leads to a recursive formulation of equilibrium expected profits in analogue to dynamic programming. Let us consider the *interior* stages where both firms have at least one expansion option left, namely $\vec{M}_0 = (n_1, n_2)$ where $n_1, n_2 > 0$. Thanks to (3.5) and the strong Markov property of X, the NPV of firm 1 can be decomposed as (letting $\underline{\tau} := \tau^1_{n_1,n_2} \wedge \tau^2_{n_1,n_2}$ and $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$)

$$\begin{aligned} J^1_{n_1,n_2}(x, \boldsymbol{\alpha}) = \mathbb{E}_x \Bigg[\int_0^{\underline{\tau}} & e^{-rs}\pi^1_{n_1,n_2}(X_s)\,ds \\ & + \Big\{ e^{-r\underline{\tau}}\mathbb{1}_{\{\tau^1_{n_1,n_2} < \tau^2_{n_1,n_2}\}}\Big(J^1_{n_1-1,n_2}\Big(X_{\tau^1_{n_1,n_2}}, \boldsymbol{\alpha}\Big) - K^1_{n_1}\Big) \\ & + e^{-r\underline{\tau}}\mathbb{1}_{\{\tau^1_{n_1,n_2} > \tau^2_{n_1,n_2}\}} J^1_{n_1,n_2-1}\Big(X_{\tau^2_{n_1,n_2}}, \boldsymbol{\alpha}\Big) \\ & + e^{-r\underline{\tau}}\mathbb{1}_{\{\tau^1_{n_1,n_2} = \tau^2_{n_1,n_2}\}}\Big(J^1_{n_1-1,n_2-1}\Big(X_{\tau^1_{n_1,n_2}}, \boldsymbol{\alpha}\Big) - K^1_{n_1}\Big)\Big\}\Bigg]. \end{aligned} \tag{3.8}$$

We use the shorthand notation $\mathbb{E}_x\{\cdot\} := \mathbb{E}\{\cdot | X_0 = x\}$ and the subscript of $J^i_{n_1,n_2}$ to indicate the conditioning on $\vec{M}_0 = (n_1, n_2)$. In the *boundary* stages $(n_1, 0)$, or similarly $(0, n_2)$, one firm has no options left, which can be represented via e.g. $\tau^2_{n_1,0} = +\infty$ in (3.8), removing the last two cases/terms. For future use we introduce the static discounted future cashflows D^i's. The latter capture the situation where capacities are forever fixed, which is associated with the first term in (3.8):

$$\begin{aligned} D^i_{n_1,n_2}(x) &:= \mathbb{E}_x\left[\int_0^\infty e^{-rs}\pi^i_{n_1,n_2}(X_s)\, ds\right] \\ &= Q^i_{n_1,n_2} \times \left\{\frac{P_{n_1,n_2}-C_i}{r} + (-1)^{i+1}\rho_i \int_0^\infty e^{-rt}\mathbb{E}\left[X_t \mid X_0 = x\right]\, dt\right\}. \end{aligned} \tag{3.9}$$

In order to characterize corresponding smallest concave majorant, some regularity of the underlying payoff functions are required. We henceforth assume that all $D^1_{n_1,n_2}$ are *increasing*, and their differences $D^1_{n_1-1,n_2} - D^1_{n_1,n_2} - K^1_{n_1}$ are contained in the class $\mathcal{H}_{\text{inc}}$, while all $D^2_{n_1,n_2}$ are *decreasing* and $D^2_{n_1,n_2-1} - D^2_{n_1,n_2} - K^2_{n_2}$ are contained in the class $\mathcal{H}_{\text{dec}}$ (see Definition 2.7). For the sake of concise exposition, we further concentrate on the case where the discounted cashflows $D^i_{n_1,n_2}(x)$ are *affine* in x. This essentially corresponds to the expectation $\mathbb{E}_x[X_t]$ being affine in x,

$$\mathbb{E}_x[X_t] := x \cdot A(t) + B(t). \tag{3.10}$$

Substituting into (3.9) leads to:

$$\begin{aligned} D^i_{n_1,n_2}(x) &= Q^i_{n_1,n_2} \times \left[\frac{P_{n_1,n_2}-C_i}{r} + (-1)^{i+1}\rho^i \int_0^\infty e^{-rs}\left\{A(s)x + B(s)\right\} ds\right] \\ &= \zeta^i_{n_1,n_2} + (-1)^{i+1}\frac{\rho^i Q^i_{n_1,n_2}}{\delta}\cdot x, \end{aligned} \tag{3.11}$$

where ζ^i and δ are defined via

$$\zeta^i_{n_1,n_2} := Q^i_{n_1,n_2} \times \left[\frac{P_{n_1,n_2}-C_i}{r} + (-1)^{i+1}\rho^i \int_0^\infty e^{-rs} B(s) ds\right], \qquad \delta := \frac{1}{\int_0^\infty e^{-rs}A(s)ds}.$$

Example 3.3 *Under a GBM model* (2.36)*, we have* $\mathbb{E}_x[X_t] = xe^{\mu t}$*, and consequently* D^i*'s are of the form* (3.11) *with* $\int_0^\infty e^{-rs}B(s)ds = 0$, $\delta = r - \mu$.

Under an OU model (2.39)*, we have* $\mathbb{E}_x[X_t] = xe^{-\mu t} + \theta\,(1 - e^{-\mu t})$*, and consequently* D^i*'s are of the form* (3.11) *with* $\int_0^\infty e^{-rs}B(s)ds = \frac{\mu\theta}{r(r+\mu)}$, $\delta = r + \mu$.

3.1.3 Game Equilibrium

Because decisions of one firm affect the other through the joint dependence of J^i's on $\vec{M}_t$, capacity expansion becomes a non-zero-sum stochastic game driven by the state variable X_t and endogenous game stage $\vec{M}_t$. To describe optimal behavior in this game we rely on the standard concept of Markov *Nash equilibrium*.

Let $V^i_{n_1,n_2}(x) := J^i_{n_1,n_2}(x, \boldsymbol{\alpha}^*)$ denote the equilibrium game values at game stage $(n_1,\, n_2)$. Recursive construction of J^i's revealed by equation (3.8) and the time homogeneity of the state process X_t motivate dynamic programming methods that characterize $V^i_{n_1,\,n_2}$ by looking at the single-stage timing game defined by $\tau^i_{n_1,n_2}$'s. The resulting stopping time (if it exists) which yields the single-stage equilibrium is in turn part of the dynamic equilibrium strategy of firm i at stage $(n_1,\, n_2)$, and so denoted by $\tau^{i,*}_{n_1,n_2}$. Indeed, fixing $\tau^{-i,*}_{n_1,n_2}$, the stopping time $\tau^{i,*}_{n_1,n_2}$ is the maximizer of the RHS in (3.8). This can be seen most simply in the boundary stages, where one of the strategy sets is empty and (3.8) reduces to the traditional situation of a single-agent optimization. In our context, this optimization is an optimal stopping problem for X_t via $\tau^1_{n_1,0}$ (resp. τ^2_{0,n_2}):

$$V^1_{n_1,0}(x) = D^1_{n_1,0}(x) + \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} \left[V^1_{n_1-1,0}(X_\tau) - D^1_{n_1,0}(X_\tau) - K^1_{n_1} \right] \right\}, \tag{3.12}$$

$$V^2_{0,n_2}(x) = D^2_{0,n_2}(x) + \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} \left[V^2_{0,n_2-1}(X_\tau) - D^2_{0,n_2}(X_\tau) - K^2_{n_2} \right] \right\}, \tag{3.13}$$

where $\mathcal{T} := \mathcal{T}_{[0,+\infty)}$ denotes the collection of all $\mathbb{F}$-stopping times with values in $[0, +\infty)$, and D^i's are from (3.9). For the firm who has no remaining expansion options, her game value is obtained as

$$V^1_{0,n_2}(x) = D^1_{0,n_2}(x) + \mathbb{E}_x\left[e^{-r\tau^{2,*}_{0,n_2}}\left\{V^1_{0,n_2-1}\left(X_{\tau^{2,*}_{0,n_2}}\right) - D^1_{0,n_2}\left(X_{\tau^{2,*}_{0,n_2}}\right)\right\}\right], \tag{3.14}$$

$$V^2_{n_1,0}(x) = D^2_{n_1,0}(x) + \mathbb{E}_x\left[e^{-r\tau^{1,*}_{n_1,0}}\left\{V^2_{n_1-1,0}\left(X_{\tau^{1,*}_{n_1,0}}\right) - D^2_{n_1,0}\left(X_{\tau^{1,*}_{n_1,0}}\right)\right\}\right]. \tag{3.15}$$

Notice that these game values are determined by their rivals' game strategies $\tau^{2,*}_{0,n_2}$ or $\tau^{1,*}_{n_1,0}$ respectively, thus there is no more any optimization.

In the interior game stages (n_1, n_2), by the Nash equilibrium criterion (2.6) the equilibrium strategy of each firm is the **best-response** to her rival's action, and we denote the resulting NPVs by $\widetilde{V}^i(\,\cdot\,;\tau^{-i})$. Namely, based on (3.8), given the rival's game policy as $\tau^2_{n_1,n_2}$ (resp. $\tau^1_{n_1,n_2}$), firm 1 (resp. firm 2) solves the optimal stopping problem:

$$\begin{aligned}
&\widetilde{V}^1_{n_1,n_2}(x;\tau^2_{n_1,n_2}) - D^1_{n_1,n_2}(x)\\
&= \sup_{\tau\in\mathcal{T}}\mathbb{E}_x\Big[\mathbb{1}_{\{\tau>\tau^2_{n_1,n_2}\}}e^{-r\tau^2_{n_1,n_2}}\left\{V^1_{n_1,n_2-1}\left(X_{\tau^2_{n_1,n_2}}\right) - D^1_{n_1,n_2}\left(X_{\tau^2_{n_1,n_2}}\right)\right\}\\
&\qquad\qquad +\mathbb{1}_{\{\tau<\tau^2_{n_1,n_2}\}}e^{-r\tau}\left\{V^1_{n_1-1,n_2}\left(X_\tau\right) - D^1_{n_1,n_2}\left(X_\tau\right) - K^1_{n_1}\right\}\\
&\qquad\qquad +\mathbb{1}_{\{\tau=\tau^2_{n_1,n_2}\}}e^{-r\tau}\left\{V^1_{n_1-1,n_2-1}\left(X_\tau\right) - D^1_{n_1,n_2}\left(X_\tau\right) - K^1_{n_1}\right\}\Big],
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\widetilde{V}^2_{n_1,n_2}(x;\tau^1_{n_1,n_2}) - D^2_{n_1,n_2}(x)\\
&= \sup_{\tau\in\mathcal{T}}\mathbb{E}_x\Big[\mathbb{1}_{\{\tau>\tau^1_{n_1,n_2}\}}e^{-r\tau^1_{n_1,n_2}}\left\{V^2_{n_1-1,n_2}\left(X_{\tau^1_{n_1,n_2}}\right) - D^2_{n_1,n_2}\left(X_{\tau^2_{n_1,n_2}}\right)\right\}\\
&\qquad\qquad +\mathbb{1}_{\{\tau<\tau^1_{n_1,n_2}\}}e^{-r\tau}\left\{V^2_{n_1,n_2-1}\left(X_\tau\right) - D^2_{n_1,n_2}\left(X_\tau\right) - K^2_{n_2}\right\}\\
&\qquad\qquad +\mathbb{1}_{\{\tau=\tau^1_{n_1,n_2}\}}e^{-r\tau}\left\{V^2_{n_1-1,n_2-1}\left(X_\tau\right) - D^2_{n_1,n_2}\left(X_\tau\right) - K^2_{n_2}\right\}\Big].
\end{aligned} \tag{3.17}$$

Observe that simultaneous investment can be ruled out since on the event $\{\tau = \tau^2_{n_1,n_2}\}$ it is strictly dominated by the strategy of first waiting $\tau > \tau^2_{n_1,n_2}$ and then optimally investing as a follower: $V^1_{n_1,n_2-1} \geq V^1_{n_1-1,n_2-1} - K^1_{n_1}$. Assuming that the suprema above are attained, we obtain the best-response policy $\tilde{\tau}^i_{n_1,n_2} = \tilde{\tau}^i_{n_1,n_2}(\tau^{-i}_{n_1,n_2})$ that maximizes (3.16)-

(3.17), where we emphasize the dependence on the rival's strategy. The condition for a Nash equilibrium at (n_1, n_2) (as defined in Definition 2.4) is then characterized as a fixed point of the best-response strategies: $\tau^{1,*}_{n_1,n_2} = \tilde{\tau}^1_{n_1,n_2}(\tau^{2,*}_{n_1,n_2})$ and $\tau^{2,*}_{n_1,n_2} = \tilde{\tau}^2_{n_1,n_2}(\tau^{1,*}_{n_1,n_2})$. By induction on the discrete stages (n_1, n_2), we may patch these local equilibria to construct a global one:

$$\mathcal{A}^* = \left\{ \left(\boldsymbol{\alpha}^{\mathbf{1},*}, \boldsymbol{\alpha}^{\mathbf{2},*}\right) : \begin{array}{l} \boldsymbol{\alpha}^{\mathbf{1},*} = \left(\tilde{\tau}^1_{n_1,n_2}\left(\tau^{2,*}_{n_1,n_2}\right),\, n_1 > 0, \forall n_2\right) \\ \boldsymbol{\alpha}^{\mathbf{2},*} = \left(\tilde{\tau}^2_{n_1,n_2}\left(\tau^{1,*}_{n_1,n_2}\right),\, n_2 > 0, \forall n_1\right) \end{array} \right\}. \tag{3.18}$$

3.2 Constructing Equilibria

In this section, we will specify game strategies and game values of each firm at each game stage (n_1, n_2) by dynamic programming. The boundary game stages, at which only one firm has expansion option(s), can be solved directly as in (3.12)-(3.13), allowing us to determine (3.14)-(3.15) accordingly. For the interior game stages at which both firms have expansion options, we first derive their best-response to the rival's action and then obtain the equilibrium strategies via the *Nash equilibrium* fixed-point characterization.

3.2.1 Equilibria at Boundary Stages

In the scenarios where one firm has expansion options while her rival does not, deriving game values and policies boils down to solving a series of single-agent optimization problems (3.12)-(3.13). Note that because capacities are forever constant after $\vec{M}_t = (0, 0)$, $V^i_{0,0}(x) \equiv D^i_{0,0}(x)$ for $i = 1, 2$, which serves as an inductive starting point to solve the multiple-stopping problems associated to boundary stages. Solutions to optimization problems at stage (1, 0) and (0, 1) are classical and stated in Section 3.4.1 for completeness. We then extend inductively to the general boundary case $(n_1, 0)$ and $(0, n_2)$, with a full proof in Section 3.4.2.

Theorem 3.4 ***(Boundary Cases)*** *The game value of firm 1 at stage* $(n_1,\, 0)$ *and the game value for firm 2 at stage* $(0,\, n_2)$ *for* $n_1,\, n_2 \geq 1$ *are admitted as:*

$$V^1_{n_1,0}(x) = \begin{cases} D^1_{n_1,0}(x) + \frac{F(x)}{F(S^{1,*}_{n_1,0})} \cdot h^1_{n_1,0}\left(S^{1,*}_{n_1,0}\right), & \text{if } x \in (\underline{d},\, S^{1,*}_{n_1,0}), \\ V^1_{n_1-1,0}(x) - K^1_{n_1}, & \text{if } x \in [S^{1,*}_{n_1,0},\, \bar{d}), \end{cases} \tag{3.19}$$

$$V^2_{0,n_2}(x) = \begin{cases} V^2_{0,n_2-1}(x) - K^2_{n_2}, & \text{if } x \in (\underline{d},\, S^{2,*}_{0,n_2}], \\ D^2_{0,n_2}(x) + \frac{G(x)}{G(S^{2,*}_{0,n_2})} \cdot h^2_{0,n_2}(S^{2,*}_{0,n_2}), & \text{if } x \in (S^{2,*}_{0,n_2},\, \bar{d}), \end{cases} \tag{3.20}$$

with first-mover payoff functions

$$\begin{aligned} h^1_{n_1,0}(x) &= V^1_{n_1-1,0}(x) - D^1_{n_1,0}(x) - K^1_{n_1}, \\ h^2_{0,n_2}(x) &= V^2_{0,n_2-1}(x) - D^2_{0,n_2}(x) - K^2_{n_2}. \end{aligned}$$

Their corresponding policies are characterized by threshold-type stopping times

$$\begin{aligned} \tau^{1,*}_{n_1,0} &= \inf\{t \geq 0 :\ X^x_t \geq S^{1,*}_{n_1,0}\} \\ \tau^{2,*}_{0,n_2} &= \inf\{t \geq 0 :\ X^x_t \leq S^{2,*}_{0,n_2}\}, \end{aligned}$$

where the series of optimal stopping levels $S^{1,*}_{n_1,0}$, $S^{2,*}_{0,n_2}$ *satisfy the equations*

$$F(S^{1,*}_{n_1,0}) = \frac{h^1_{n_1,0}}{\left(h^1_{n_1,0}\right)'}(S^{1,*}_{n_1,0}) \times F'(S^{1,*}_{n_1,0}), \tag{3.21}$$

$$G(S^{2,*}_{0,n_2}) = \frac{h^2_{0,n_2}}{\left(h^2_{0,n_2}\right)'}(S^{2,*}_{0,n_2}) \times G'(S^{2,*}_{0,n_2}). \tag{3.22}$$

Remark 3.5 *We do not assume any order of the threshold sequences* $\left(S^{1,*}_{n_1,0}\right)_{n_1 \geq 1}$ *or*

$\left(S^{2,*}_{0,n_2}\right)_{n_2\geq 1}$. *If firm 1's thresholds are not increasing (resp. decreasing for firm 2), she would simultaneously exercise multiple expansion options if* X_t *moves in her preferred direction. This might happen for example if the market prices are non-convex in* $\vec{M}$.

We have shown that the optimal game policies of the "follower" who is the only firm with expansion options left are of threshold-type. Her "leader" rival's game values are then accordingly determined from (3.14)-(3.15):

Corollary 3.6 *(Leader Game Value) The game values of firm 1 at stage* $(0, n_2)$ *and game values of firm 2 at stage* $(n_1, 0)$ *for* n_1, $n_2 \geq 1$ *are admitted as:*

$$V^1_{0,n_2}(x) = \begin{cases} V^1_{0,n_2-1}(x), & \text{if } x \in (\underline{d},\, S^{2,*}_{0,n_2}], \\ D^1_{0,n_2}(x) + G(x)\cdot \left[\frac{V^1_{0,n_2-1}-D^1_{0,n_2}}{G}\right]\left(S^{2,*}_{0,n_2}\right), & \text{if } x \in (S^{2,*}_{0,n_2},\, \bar{d}), \end{cases} \tag{3.23}$$

$$V^2_{n_1,0}(x) = \begin{cases} D^2_{n_1,0}(x) + F(x)\cdot \left[\frac{V^2_{n_1-1,0}-D^2_{n_1,0}}{F}\right]\left(S^{1,*}_{n_1,0}\right), & \text{if } x \in (\underline{d},\, S^{1,*}_{n_1,0}), \\ V^2_{n_1-1,0}(x), & \text{if } x \in [S^{1,*}_{n_1,0},\, \bar{d}). \end{cases} \tag{3.24}$$

3.2.2 Equilibria at Interior Stage (1, 1)

In the scenarios that each firm has available expansion options, i.e. at stages $(n_1,\, n_2)$ with n_1, $n_2 > 0$, the firms interact through the negative externality of expansion on the electricity price. In the given context of Nash equilibrium, we will obtain equilibrium policies as the fixed-point of the firms' best-response to each other's strategy. We first derive the solution at stage $(1, 1)$, then extend to an arbitrary interior stage.

Based on (3.16) with $n_1 = n_2 = 1$, firm 1's first-mover payoff is admitted as $h^1_{1,1}(x) = V^1_{0,1}(x) - D^1_{1,1}(x) - K^1_1$, and her second-mover payoff is $l^1_{1,1}(x) = V^1_{1,0}(x) - D^1_{1,1}(x)$. Following preceding results for boundary stages, one can easily verify that $l^1_{1,1}(x) > h^1_{1,1}(x)$ for x small enough, and $l^1_{1,1}(x) < h^1_{1,1}(x)$ for x large. Assuming that $h - l$ is strictly monotone, we accordingly define a "leadership" point $L^1_{1,1}$ where first-mover payoff of firm 1 equals her second-mover payoff:

$$L^1_{1,1} := \inf\{x \in \mathcal{D} \,:\, h^1_{1,1}(x) > l^1_{1,1}(x)\}. \tag{3.25}$$

The meaning of the leadership point arises from the competitive aspect: when $x \le L^1_{1,1}$, firm 1 does not compete to be first, since she is in fact (instantaneously) better-off being a second mover. On the other hand, when $x > L^1_{1,1}$, firm 1 would prefer to be a leader than a follower. Similar considerations lead to the leadership threshold of firm 2:

$$L^2_{1,1} := \sup\{x \in \mathcal{D} \,:\, h^2_{1,1}(x) > l^2_{1,1}(x)\}. \tag{3.26}$$

Recall that the game strategy of firm 2 at stage $(1,1)$ is defined as a $\mathbb{F}$-stopping time $\tau^2_{1,1}$. Since the first-mover payoff of firm 2 is greater than her second-mover payoff if and only if the level of X_t is low, it is reasonable to assume that $\tau^2_{1,1}$ is of threshold-type:

$$\tau^2_{1,1} = \inf\{t \ge 0 : \, X_t \le s_2\}, \tag{3.27}$$

i.e. expansion of firm 2 takes place once X_t drops below s_2.

Depending on relationship between $h^1_{1,1}(s_2)$ and $l^1_{1,1}(s_2)$, the payoff of firm 1 would experience a jump up/down at the exercise threshold of firm 2. In particular, in the case that $s_2 < L^1_{1,1}$, i.e. $h^1_{1,1} < l^1_{1,1}$ at $x = s_2$, firm 1 actually benefits from having firm 2 invest at s_2, accordingly is not incentivized to preempt when firm 2 intends to invest. She now solves the optimal stopping problem following (3.16):

$$\widetilde{V}^1_{1,1}(x,s_2) - D^1_{1,1}(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \left[\mathbb{1}_{\{\tau<\tau^2_{1,1}\}} e^{-r\tau} \left\{ h^1_{1,1}(X_\tau) \right\} + \mathbb{1}_{\{\tau>\tau^2_{1,1}\}} e^{-r\tau^2_{1,1}} \left\{ l^1_{1,1}(X_{\tau^2_{1,1}}) \right\} \right]. \tag{3.28}$$

Proposition 3.7 *(threshold-type best-response of firm 1 at stage* $(1,1)$*) If* $s_2 < L^1_{1,1}$*, the best-response of firm 1 associated to* $\tau^2_{1,1} = \tau_{s_2}$ *specified in* (3.27) *is the stopping time given by*

$$\tau^1_{1,1}(s_2) = \inf\{t \geq 0 :\ X_t \geq S^1_{1,1}(s_2)\}, \tag{3.29}$$

where the optimal stopping level $S^1_{1,1}(s_2) := S_1 > s_2$ *is a function of* s_2*, characterized by the following equation:*

$$\begin{aligned} &\left[\left(h^1_{1,1} \vee l^1_{1,1}\right)(s_2)G(S_1) - h^1_{1,1}(S_1)G(s_2)\right] F'(S_1) \\ &\qquad + \left[h^1_{1,1}(S_1)F(s_2) - \left(h^1_{1,1} \vee l^1_{1,1}\right)(s_2)F(S_1)\right] G'(S_1) \\ &= \left(h^1_{1,1}\right)'(S_1)\left[G(S_1)F(s_2) - G(s_2)F(S_1)\right]. \end{aligned} \tag{3.30}$$

Consequently, the optimal stopping problem (3.28) *admits the value function*

$$\widetilde{V}^1_{1,1}(x,s_2) = \begin{cases} V^1_{1,0}(x), & \textit{if } x \in (\underline{d},\ s_2), \\ D^1_{1,1}(x) + \tilde{\omega}^1_{1,1}F(x) + \tilde{\nu}^1_{1,1}G(x), & \textit{if } x \in (s_2,\ S_1), \\ V^1_{0,1}(x) - K^1_1, & \textit{if } x \in (S_1,\ \bar{d}), \end{cases} \tag{3.31}$$

where $\tilde{\omega}^1_{1,1} := \tilde{\omega}^1_{1,1}(s_2)$ *and* $\tilde{\nu}^1_{1,1} := \tilde{\nu}^1_{1,1}(s_2)$ *are defined as*

$$\tilde{\omega}^1_{1,1} = \frac{h^1_{1,1}(S_1)G(s_2) - \left(h^1_{1,1} \vee l^1_{1,1}\right)(s_2)G(S_1)}{F(S_1)G(s_2) - F(s_2)G(S_1)}, \tag{3.32a}$$

$$\tilde{\nu}^1_{1,1} = \frac{\left(h^1_{1,1} \vee l^1_{1,1}\right)(s_2)F(S_1) - h^1_{1,1}(S_1)F(s_2)}{F(S_1)G(s_2) - F(s_2)G(S_1)}. \tag{3.32b}$$

Conversely, in the case that $s_2 > L^1_{1,1}$, i.e. if $h^1_{1,1} > l^1_{1,1}$ at $x = s_2$, then firm 1 is better off preemptively exercising *right before* firm 2, since her first-mover payoff is higher than her second-mover one. Recalling the definition of the leadership points L^i, we see that firm 1 is incentivized to preempt immediately $\tau^1_{1,1} = 0$ when the state process is in $(L^1_{1,1}, s_2]$ (see also [52]). On $(s_2, \bar{d})$, firm 1 again solves an optimal stopping problem following (3.16).

Proposition 3.8 *(preemptive best-response of firm 1) If* $s_2 > L^1_{1,1}$*, the best-response of firm 1 is*

$$\tau^{1,e}_{1,1}(s_2) = \inf\{t \geq 0 : \; L^1_{1,1} < X_t \leq (s_2+) \; \textit{or} \; X_t \geq S^{1,e}_{1,1}(s_2)\}, \tag{3.33}$$

where the optimal stopping level $S^{1,e}_{1,1}(s_2) := S^e_1 \geq s_2$ *is a solution to* (3.30)*.*

Note that the infinitesimal preemption of firm 1 corresponds to "stopping at s_2+" which can be considered as a limit of ε-optimal strategies. This is because the value function on $(s_2, \bar{d})$ is admitted in terms of concave majorant which yields $\lim_{x \searrow s_2} \widetilde{V}^1_{1,1}(x, s_2) - D^1_{1,1}(s_2) = h^1_{1,1}(s_2) > l^1_{1,1}(s_2)$. Therefore, stopping at s_2 is too late and firm 1 prefers to preempt right before s_2. Proof of this proposition is in Section 3.4.3, and very similar steps for the best-response of firm 2 are stated in Section 3.4.4. As expected, the best-response of firm 2 depends on the relationship between threshold s_1 of firm 1 and $L^2_{1,1}$.

To determine Nash equilibria of these firms' strategies, we start by deriving the best-response of firm 2 corresponding to $\tau^{1,e}_{1,1}(s_2)$ defined in (3.33). Since the state process X_t is assumed to be regular in $\mathcal{D}$, from firm 2 perspective, $\tau^{1,e}_{1,1}(s_2)$ is indifferent from the situation that firm 1 invest at $\tau^1_{1,1} = \inf\{t \geq 0 : X_t > L^1_{1,1}\}$. From Section 3.4.4, if $L^1_{1,1} \geq L^2_{1,1}$, the corresponding best-response of firm 2 is a threshold-type stopping time of threshold $S^2 \leq L^1_{1,1}$, which leads us to a threshold-type equilibrium (if it exists) following Proposition 3.7. Otherwise, if $L^1_{1,1} < L^2_{1,1}$, the corresponding best-response of firm 2 is admitted as

$$\tau^{2,e,*}_{n_1,n_2} = \inf\{t \geq 0 : L^1_{1,1} \leq X_t < L^2_{1,1} \text{ or } X_t < S^{2,e,*}_{1,1}\}, \tag{3.34}$$

where $S^{2,e,*}_{1,1} := S^{2,e}_{1,1}(L^1_{1,1})$. Note that firm 1 (resp. firm 2) is not incentivized to invest when $X_t = L^1_{1,1}$ (resp. $X_t = L^2_{1,1}$). Back to firm 1, since $L^2_{1,1} > L^1_{1,1}$, her best-response to $\tau^{2,e*}_{n_1,n_2}$ is then admitted by Proposition 3.8 as:

$$\tau^{1,e,*}_{1,1} = \inf\{t \geq 0 : L^1_{1,1} < X_t \leq L^2_{1,1} \text{ or } X_t \geq S^{1,e,*}_{1,1}\}, \tag{3.35}$$

where $S^{1,e,*}_{1,1} := S^{1,e}_{1,1}(L^2_{1,1})$. To summarize, when $L^1_{1,1} < L^2_{1,1}$ we always have the *preemptive equilibrium* defined by $\left(\tau^{1,e,*}_{1,1}, \tau^{2,e,*}_{1,1}\right)$. Under that equilibrium, one or more firms invest immediately when $L^1_{1,1} < x < L^2_{1,1}$, otherwise the investment happens either at the thresholds $S^{i,e,*}_{1,1}$ or at the leadership points $L^i_{1,1}$. It remains to specify the outcome of the first situation, $x \in (L^1_{1,1}, L^2_{1,1})$. This is similar to an infinitesimal coordination game which admits multiple solutions. One approach proposed by [38] involves instantaneous mixed strategies and leads to the following proposition (see proof in Section 3.4.5).

Proposition 3.9 ***(coordination game at stage*** $(1,1)$***)*** *Let* $(p_1(x), p_2(x))$ *be a mixed strategy profile, with* $p_i(x)$ *denoting the probability that firm* i *attempts to invest at* $X_t = x$ *over an infinitesimal round, played repeatedly. There are three equilibrium strategies:*

(i) $(p_1^*(x),\, p_2^*(x)) = (0, 1)$;

(ii) $(p_1^*(x),\, p_2^*(x)) = (1, 0)$;

(iii) $(p_1^*(x),\, p_2^*(x)) = \left(\frac{V^1_{0,1} - V^1_{1,0} - K^1_1}{V^1_{0,1} - D^1_{0,0}}(x),\ \frac{V^2_{1,0} - V^2_{0,1} - K^2_1}{V^2_{1,0} - D^2_{0,0}}(x)\right)$.

Note that there is a positive probability that the firms will invest simultaneously if they implement the third equilibrium, and firm 1 is more likely to invest when X_t is close to $L^2_{1,1}$ while firm 2 is more likely to invest when X_t is close to $L^1_{1,1}$. Moreover, the third equilibrium coincides with the first/second equilibrium when $x = L^1_{1,1}$ ($x = L^2_{1,1}$, resp.).

Choices (i) and (ii) above can be interpreted via a *preemptive priority* that predetermines the winner of the instantaneous competition. For example, in our original economic example, a coal-fired plant is easier to build than a nuclear power plant, so one may assume that firm 2 has a preemptive priority, i.e. the coordination equilibrium selected is of type (i) above. Under that assumption firm 1 receives her second-mover value when $L^1_{1,1} \le X_t < L^2_{1,1}$, which yields an upward jump in her resulting game value at

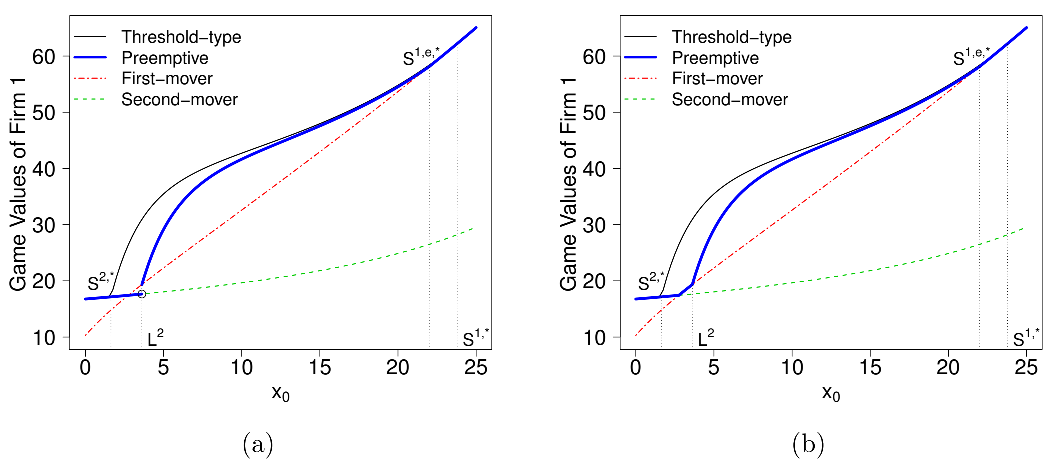


Figure 3.1: Game values of the nuclear industry investor (cf. Section 3.3.4 for details) for both threshold-type equilibrium and preemptive equilibrium at stage $(2,2)$ and $\mu = 0.23$. $S^{\cdot,*}$ denote equilibrium thresholds of these two power generators, and L^2 denotes the leadership threshold of the coal-fired investor as in (3.26) (Left: a) Firm 2 has a preemptive priority. (Right: b) Firm 1 has a preemptive priority.

$x = L^2_{1,1}$. In the converse scenario that firm 1 has a preemptive priority, she receives her first-mover value on $[L^1_{1,1}, L^2_{1,1})$ and the resulting game value is continuous at $x = L^2_{1,1}$. Figure 3.1 illustrates these choices.

Returning to Nash equilibria involving threshold-type strategies, the fixed-point characterization (3.18) boils down to solving the following system of equations:

$$\begin{cases} \left[l^1_{1,1}(S_2)G(S_1) - h^1_{1,1}(S_1)G(S_2)\right] F'(S_1) + \left[h^1_{1,1}(S_1)F(S_2) - l^1_{1,1}(S_2)F(S_1)\right] G'(S_1) \\ \qquad = \left(h^1_{1,1}\right)'(S_1)\left[G(S_1)F(S_2) - G(S_2)F(S_1)\right], \\ \left[h^2_{1,1}(S_2)G(S_1) - l^2_{1,1}(S_1)G(S_2)\right] F'(S_2) + \left[l^2_{1,1}(S_1)F(S_2) - h^2_{1,1}(S_2)F(S_1)\right] G'(S_2) \\ \qquad = \left(h^2_{1,1}\right)'(S_2)\left[G(S_1)F(S_2) - G(S_2)F(S_1)\right], \end{cases} \tag{3.36}$$

for $(S_1, S_2) \in [L^2_{1,1}, \bar{d}) \times (\underline{d}, L^1_{1,1}]$, and solutions to this system correspond to pairs of investment thresholds at stage $(1,1)$. To discuss the existence of such equilibria, we state the following corollary characterizing the best-response curves.

Corollary 3.10 ***(best-response curves)***

(i) For $s_2 \le L^1_{1,1}$ (resp. $s_1 \ge L^2_{1,1}$), the best-response function $s_2 \longmapsto S^1_{1,1}(s_2)$ (resp. $s_1 \longmapsto S^2_{1,1}(s_1)$) is continuous.

(ii) As $s_2 \downarrow \underline{d}$ (resp. $s_1 \uparrow \overline{d}$), the best-response of firm 1 (resp. firm 2) converges to a finite threshold $S^{1,P,*}_{1,1}$ (resp. $S^{2,P,*}_{1,1}$).

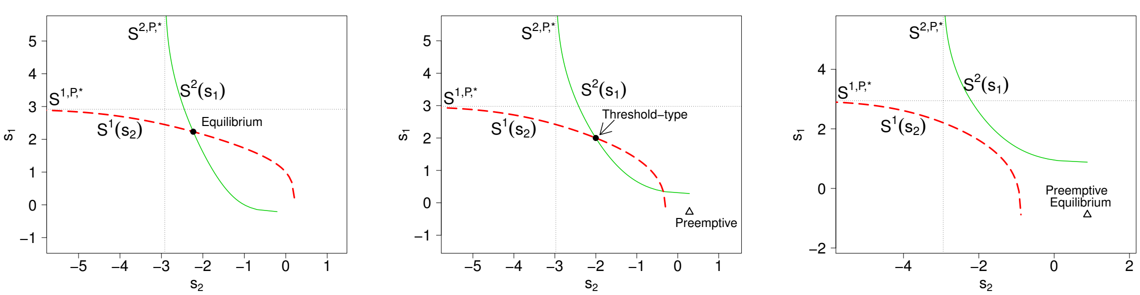
The first statement is a simple application of the implicit function theorem. An interpretation of $S^{i,P,*}$ is provided in Section 3.2.4.1. Existence of solutions to the system (3.36) then corresponds to existence of crossing points of these best-response curves. Depending on the relation between $L^1_{1,1}$ and $L^2_{1,1}$, there are three scenarios of the best-response curves, sketched in Figure 3.2.

Scenario I: $L^1_{1,1} > L^2_{1,1}$. In this case there is guaranteed at least one crossing point of the best-response curves, which corresponds to a threshold-type equilibrium at stage $(1,1)$ (Figure 3.2a). Only threshold-type equilibria exist in this scenario, matching the setting studied by [30, Section 3.1].

Scenario II: $L^1_{1,1} < L^2_{1,1}$ and the best-response curve cross (see Figure 3.2b which has 2 crossings). Consequently, both threshold-type equilibria and a preemptive equilibrium characterized by (3.34)-(3.35) exist.

Scenario III: $L^1_{1,1} < L^2_{1,1}$ and no crossing points between the best-response curves (Figure 3.2c), which implies that only a preemptive equilibrium exists.

If $L^1_{1,1} < L^2_{1,1}$ (i.e. beyond of Scenario I), existence of threshold-type equilibria is not guaranteed. From Corollary 3.10, one sufficient condition for existence is that $S^1_{1,1}(L^1_{1,1}) > L^2_{1,1}$ and $S^2_{1,1}(L^2_{1,1}) < L^1_{1,1}$. The latter condition does not actually hold in the numerical



(a) Scenario I (b) Scenario II (c) Scenario III

Figure 3.2: Equilibrium scenarios for the best-response curveswith lowercase s denote given thresholds of one firm and uppercase S denote the other firm's best-response threshold. The red dashed lines represent best-response of firm 1, while the green solid lines represent best-response of firm 2. The dotted lines represent the limiting thresholds $S^{1,P,*}_{1,1}$, $S^{2,P,*}_{1,1}$ discussed in Section 3.2.4.1. Note that for $s_2 > L^1_{1,1}$ and $s_1 < L^2_{1,1}$ there is no threshold-type best-response and the "Δ" marks a preemptive equilibrium which corresponds to $(L^2_{1,1},\, L^1_{1,1})$ in this numerical example.

example sketched in Figure 3.2b. Numerical examples in Section 3.3 suggest that Scenario III occurs under high volatility σ.

Remark 3.11 *In Scenarios I & II there are multiple Nash MPEs, so equilibrium selection is an important issue. From monotonicity of payoff functions, typically a higher threshold of firm 1 and a lower threshold of firm 2 yield higher game values to both firms. This assumption combining with the sequential nature of investment decisions with a flavor of a Stackelberg competition preferences the* latest *equilibrium, i.e. selecting the highest threshold $S^1_{1,1}$ and the corresponding lowest threshold $S^2_{1,1}$. To understand the logic for this preference, consider two equilibria termed the later (higher threshold of firm 1 and lower threshold of firm 2) and the earlier. Now consider firm 1 currently at her early threshold $S^{1,erl}_{1,1}$ and contemplating whether to expand now, i.e. pick the early equilibrium, or wait. Conditional on firm 2 implementing $S^{2,lat}_{1,1}$, best-response optimality implies that*

$$\widetilde{V}^1_{1,1}(S^{1,erl}_{1,1};\tau^{2,lat}_{1,1}) = J^1_{1,1}(S^{1,erl}_{1,1};\tau^{1,lat}_{1,1},\tau^{2,lat}_{1,1}) \geq J^1_{1,1}(S^{1,erl}_{1,1};\tau^{1,erl}_{1,1},\tau^{2,lat}_{1,1}) = V^1_{0,1}(S^{1,erl}_{1,1}) - K^1_1.$$

So under that assumption, firm 1 can extract higher expected NPV by waiting and not investing immediately. Of course, there is a risk that the assumption is false, firm 2 will implement the early threshold $S^{2,erl}_{1,1}$, whereby firm 1 will lose by waiting. However, by symmetry, when in the future, *X_t were to reach $S^{2,erl}_{1,1}$, firm 2 would face the same dilemma, and (by then knowing that firm 1 did not invest in the past) would also prefer to wait, in the hope of realizing the later equilibrium. It follows that the sequential nature of decisions encourages maximization of game values – each firm can rationally assume that* in the future *her rival will refrain from the earlier equilibrium, and hence rationally commit to waiting right now, and not expanding early. In effect, a firm can credibly signal to her rival that she is implementing the later equilibrium, yielding a higher game value* *to both.*

Note that the above argument works when firms make decisions sequentially, but does not work for simultaneous actions where threat of preemption takes precedence. Namely, the Stackelberg logic cannot rule out preemptive equilibria. For example, in Scenario II, when X_t hits $L^2_{1,1}$, firm 1 has no time to signal that she prefers a threshold-type equilibrium, as she faces the immediate threat of firm 2 investing which would at once generate a loss in her (firm 1) NPV.

3.2.3 Equilibria at General Stage (n_1, n_2)

To generalize to further interior stages (n_1, n_2) we assume that for all $n'_1 < n_1, n'_2 < n_2$, there is a threshold-type equilibrium (which has been selected, if necessary, among available choices) at stage (n'_1, n'_2). Under this assumption we can inductively apply the concave majorant method.

To fix ideas, consider stage $(2, 1)$; we use $(S^{1,*}_{1,1},\ S^{2,*}_{1,1})$ to denote investment thresholds of the *threshold-type* equilibrium strategies adopted at stage $(1, 1)$. Then given firm 2's strategy with threshold $s_2 < L^1_{2,1}$, firm 1 solves:

$$\widetilde{V}^1_{2,1}(x, s_2) - D^1_{2,1}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[\mathbb{1}_{\{\tau < \tau^2_{2,1}\}} e^{-r\tau} \left\{ h^1_{2,1}(X_\tau) \right\} + \mathbb{1}_{\{\tau > \tau^2_{2,1}\}} e^{-r\tau^2_{2,1}} \left\{ l^1_{2,1}(X_{\tau^2_{2,1}}) \right\} \right], \tag{3.37}$$

with $h^1_{2,1}(x) = V^1_{1,1}(x) - D^1_{2,1}(x) - K^1_2$, where $V^1_{1,1}$ is the equilibrium game value received by firm 1 at stage $(1, 1)$. Since $S^{2,*}_{1,1} < S^{1,*}_{1,1}$ from Proposition 3.7, from (3.31) the corresponding first-mover payoff is derived as

$$h^1_{2,1}(x) = \begin{cases} D^1_{1,1}(x) - D^1_{2,1}(x) - K^1_2 + V^1_{1,0}(x) - D^1_{1,1}(x), & \text{if } x \le S^{2,*}_{1,1}, \\ D^1_{1,1}(x) - D^1_{2,1}(x) - K^1_2 + \omega^1_{1,1}F(x) + \nu^1_{1,1}G(x), & \text{if } S^{2,*}_{1,1} < x \le S^{1,*}_{1,1}, \\ D^1_{1,1}(x) - D^1_{2,1}(x) - K^1_2 + h^1_{1,1}(x), & \text{if } x > S^{1,*}_{1,1}. \end{cases} \tag{3.38}$$

Since F and G terms do not contribute to $(\mathcal{L} - r)\, h^1_{2,1}$, and $(\mathcal{L} - r)\, h^1_{1,1}(x) < 0$ for $x > S^{1,*}_{1,1}$, we conclude that $h^1_{2,1}(x)$ is in the class $\mathcal{H}_{\text{inc}}$. Similarly, one can check that the first-mover payoff of firm 2, $h^2_{2,1}(x)$ is in the class $\mathcal{H}_{\text{dec}}$. Consequently, Proposition 2.6 and Lemma 2.8 allow us to apply similar arguments as Proposition 3.7 and Appendix 3.4.4 to derive threshold-type best-response of these firms. Similar arguments yield

Theorem 3.12 *Let $h^i_{n_1,n_2}$ and $l^i_{n_1,n_2}$ be the first-mover payoffs and second-mover payoffs associated to optimal stopping problems* (3.16)-(3.17)*, for $i = 1, 2$. The threshold-type equilibrium policies implemented by the firms at stage $(n_1,\, n_2)$ are the stopping times*

$$\tau^{1,*}_{n_1,n_2} = \inf\{t \ge 0 :\ X_t \ge S^{1,*}_{n_1,n_2}\},$$
$$\tau^{2,*}_{n_1,n_2} = \inf\{t \ge 0 :\ X_t \le S^{2,*}_{n_1,n_2}\},$$

where $\left(S^{1,*}_{n_1,n_2},\, S^{2,*}_{n_1,n_2}\right)$ is a solution to the system of equations

$$\begin{cases} \left[l^1_{n_1,n_2}(S_2)G(S_1) - h^1_{n_1,n_2}(S_1)G(S_2)\right] F'(S_1) + \left[h^1_{n_1,n_2}(S_1)F(S_2) - l^1_{n_1,n_2}(S_2)F(S_1)\right] G'(S_1) \\ \qquad = \left(h^1_{n_1,n_2}\right)'(S_1)\left[G(S_1)F(S_2) - G(S_2)F(S_1)\right], \\ \left[h^2_{n_1,n_2}(S_2)G(S_1) - l^2_{n_1,n_2}(S_1)G(S_2)\right] F'(S_2) + \left[l^2_{n_1,n_2}(S_1)F(S_2) - h^2_{n_1,n_2}(S_2)F(S_1)\right] G'(S_2) \\ \qquad = \left(h^2_{n_1,n_2}\right)'(S_2)\left[G(S_1)F(S_2) - G(S_2)F(S_1)\right]. \end{cases} \tag{3.39}$$

Consequently, the equilibrium game values are

$$V^1_{n_1,n_2}(x) = \begin{cases} V^1_{n_1,n_2-1}(x), & \text{if } x \in (\underline{d},\, S^{2,*}_{n_1,n_2}], \\ D^1_{n_1,n_2}(x) + \omega^1_{n_1,n_2}F(x) + \nu^1_{n_1,n_2}G(x), & \text{if } x \in (S^{2,*}_{n_1,n_2},\, S^{1,*}_{n_1,n_2}), \\ V^1_{n_1-1,n_2}(x) - K^1_{n_1}, & \text{if } x \in [S^{1,*}_{n_1,n_2},\, \bar{d}), \end{cases} \tag{3.40}$$

$$V^2_{n_1,n_2}(x) = \begin{cases} V^2_{n_1,n_2-1}(x) - K^2_{n_2}, & \text{if } x \in (\underline{d},\, S^{2,*}_{n_1,n_2}], \\ D^2_{n_1,n_2}(x) + \omega^2_{n_1,n_2}F(x) + \nu^2_{n_1,n_2}G(x), & \text{if } x \in (S^{2,*}_{n_1,n_2},\, S^{1,*}_{n_1,n_2}), \\ V^2_{n_1-1,n_2}(x), & \text{if } x \in [S^{1,*}_{n_1,n_2},\, \bar{d}), \end{cases} \tag{3.41}$$

where

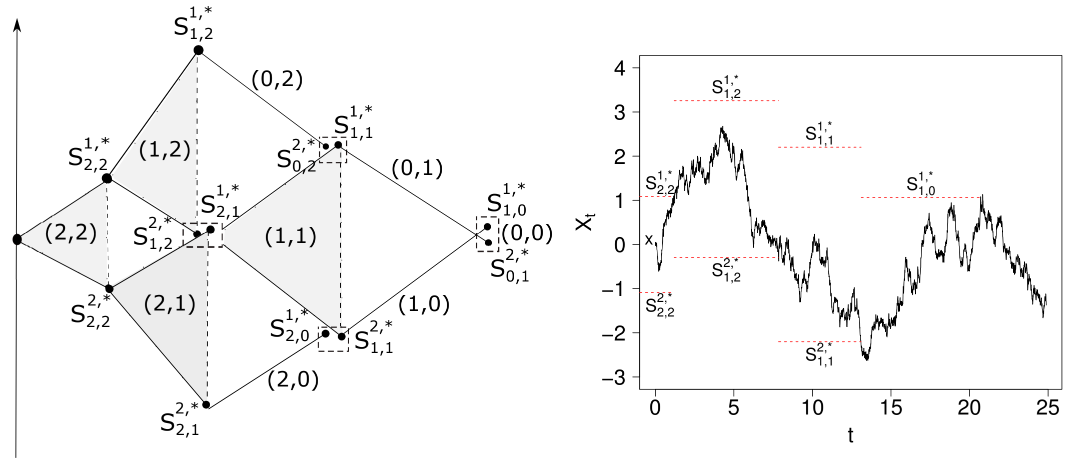
$$\omega^1_{n_1,n_2} = \frac{h^1_{n_1,n_2}(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - l^1_{n_1,n_2}(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}{F(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - F(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}, \tag{3.42}$$

$$\nu^1_{n_1,n_2} = \frac{l^1_{n_1,n_2}(S^{2,*}_{n_1,n_2})F(S^{1,*}_{n_1,n_2}) - h^1_{n_1,n_2}(S^{1,*}_{n_1,n_2})F(S^{2,*}_{n_1,n_2})}{F(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - F(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}, \tag{3.43}$$

$$\omega^2_{n_1,n_2} = \frac{l^2_{n_1,n_2}(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - h^2_{n_1,n_2}(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}{F(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - F(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}, \tag{3.44}$$

$$\nu^2_{n_1,n_2} = \frac{h^2_{n_1,n_2}(S^{2,*}_{n_1,n_2})F(S^{1,*}_{n_1,n_2}) - l^2_{n_1,n_2}(S^{1,*}_{n_1,n_2})F(S^{2,*}_{n_1,n_2})}{F(S^{1,*}_{n_1,n_2})G(S^{2,*}_{n_1,n_2}) - F(S^{2,*}_{n_1,n_2})G(S^{1,*}_{n_1,n_2})}. \tag{3.45}$$

To recap, the overall dynamic expansion game proceeds in discrete stages. At each interior stage, there are two thresholds $S^{i,*}_{n_1,n_2}$, which determine the investment level of firm $i = 1, 2$. Figure 3.3a shows a schematic for all the different thresholds starting at $\vec{M}_0 = (2, 2)$. To better visualize the game evolution, a simulated state trajectory is presented in Figure 3.3b with the firms' thresholds for the case $\Delta Q^i = 0.25$ (in which interior stage equilibria correspond to Scenario II and we assume the firms implement the latest threshold-type equilibrium). The firms' equilibrium policies determine a two-sided exit region for each interior game stage and one-sided exit region for the boundary cases.



(a) Stages

(b) Sample Trajectory of X

Figure 3.3: $S^{i,*}_{n_1,n_2}$ denotes the equilibrium threshold of firm i at stage (n_1, n_2) (Left: a) Sketch of the various stage thresholds as a function of (n_1, n_2). (Right: b) A sample trajectory of X with $X_0 = 0, \vec{M}_0 = (2, 2)$. The corresponding macro market evolution is (2, 2) → (1, 2) → (1, 1) → (0, 1) → (0, 0) with expansions at the first hitting times of the corresponding thresholds.

As the state process X hits one of the firms' expansion threshold, the game jumps to the subsequent stage and yields a new exit region.

3.2.4 Predetermined Priority and Central Planner

3.2.4.1 Predetermined Expansion Priority

In a competitive situation, the threat of the rival investing first causes the firms to act preemptively. As a result, competition leads to loss of value compared to a first-best strategy without any rivalry. To quantify this loss, we compare the derived equilibrium game values to the setting where the order of investment is pre-assigned. In the latter model, one firm is granted a *priority* option [38] meaning that she is allowed to single-handedly optimize her investment level without worrying about preemption. After the pre-assigned leader invests, the rival obtains a chance to invest as well. Thus, the priority option removes the preemption threat, but still maintains the multi-stage competition aspect.

With multiple investment options one may consider a combination of several priority options; to fix ideas we focus on the simplest situation where each firm starts with one expansion option $\vec{M}_0 = (1,1)$, and therefore priority grants *leadership* status, making the rival a follower. Assuming that the priority option is given to firm 1, her decision now reduces to solving the optimal stopping problem:

$$V_{1,1}^{1,P}(x) - D_{1,1}^1(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} h_{1,1}^1 \left(X_\tau^x \right) \right\}, \tag{3.46}$$

where the payoff function is specified in (3.64)—after her investment the game will be in stage $(0,1)$ with the associated game value $V_{0,1}^1$.

Proposition 3.13 *(Policy and value function with priority option) The value function associated to the optimal stopping problem* (3.46) *is:*

$$V_{1,1}^{1,P}(x) = \begin{cases} D_{1,1}^1(x) + \frac{F(x)}{F(S_{1,1}^{1,P,*})} \cdot h_{1,1}^1 \left(S_{1,1}^{1,P,*} \right), & \text{if } x \in (\underline{d},\, S_{1,1}^{1,P,*}), \\ D_{0,1}^1(x) - K^1, & \text{if } x \in [S_{1,1}^{1,P,*},\, \bar{d}). \end{cases} \tag{3.47}$$

The corresponding investing policy is $\tau_P^{1,*} = \inf\{t \geq 0 : X_t^x \geq S_{1,1}^{1,P,*}\}$, *where the optimal stopping level* $S_{1,1}^{1,P,*}$ *solves* $F(S_{1,1}^{1,P,*}) \times \left(h_{1,1}^1\right)' (S_{1,1}^{1,P,*}) = h_{1,1}^1(S_{1,1}^{1,P,*}) \times F'(S_{1,1}^{1,P,*})$.

The proof matches that of Proposition 3.14, and hence is omitted. An important property is that the optimal priority threshold $S_{1,1}^{1,P,*}$ is no less than the leader's threshold $S_{1,1}^{1,*}$, which implies that competition causes preemption: if $X_0 \in (S_{1,1}^{1,*}, S_{1,1}^{1,P,*})$ then firm 1 chooses to invest now even though without competition she would be better off to wait until X rises up to $S_{1,1}^{1,P,*}$.

Note that pre-assigning firm 1 as the first-mover is mathematically equivalent to taking $s_2 \to \underline{d}$, i.e. best-response when firm 2 never invests. It follows that $S_{1,1}^{1,P,*} > S^1(s_2)$

for $\forall s_2$, i.e. $S^{1,P,*}$ is the limiting value of the best-response curve $\lim_{s_2 \searrow \underline{d}} S^1(s_2)$, see the earlier Figure 3.2.

3.2.4.2 Central Planner

A different perspective on competition is offered by considering the difference between the primary non-cooperative setting and its cooperative analogue. The latter can be thought of as a central planner (or state-controlled holding company) that jointly optimizes the aggregate expected profits. Since there is no more rivalry, this "monopoly" model reduces to a classical sequential real option problem; a related problem was treated in [1].

Treatment of the cooperative investment problem is analogous to the problems considered after we aggregate the profit rates via

$$\pi^M_{n_1,n_2}(x) := \pi^1_{n_1,n_2}(x) + \pi^2_{n_1,n_2}(x). \tag{3.48}$$

If π^i's are linear in x, then so is π^M and hence the solution structure remains the same. In particular, in states $(1,0),(0,1)$ we have investment thresholds $S^{M,*}_{1,0}, S^{M,*}_{0,1}$. To handle the investment decision in stage $(1,1)$ and beyond, we can view it as optimizing the *two-sided* stopping time $\tau^{1,M}_{1,1} \wedge \tau^{2,M}_{1,1}$, where $\tau^{1,M}_{1,1}$ is the time to invest in firm 1-expansion, while $\tau^{2,M}_{1,1}$ is the time to invest in firm 2:

$$\begin{aligned} V^M_{1,1}(x) = D^1_{1,1}(x) + D^2_{1,1}(x) + \sup_{\tau^{1,M}_{1,1},\tau^{2,M}_{1,1} \in \mathcal{T}} \mathbb{E}_x \Big[& \mathbb{1}_{\{\tau^{1,M}_{1,1} < \tau^{2,M}_{1,1}\}} e^{-r\tau^{1,M}_{1,1}} \left\{ h^{1,M}_{1,1}\left(X_{\tau^{1,M}_{1,1}}\right)\right\} \\ & + \mathbb{1}_{\{\tau^{1,M}_{1,1} > \tau^{2,M}_{1,1}\}} e^{-r\tau^{2,M}_{1,1}} \left\{ h^{2,M}_{1,1}\left(X_{\tau^{2,M}_{1,1}}\right)\right\}\Big]. \end{aligned} \tag{3.49}$$

Section 3.4.6 presents the resulting solution for $V^M_{1,1}$ and the optimal investment thresholds $S^{i,M,*}_{1,1}$ that define $\tau^{i,M}_{1,1}$. Since the cooperative solution is first-best, $V^M_{1,1} \geq$

$V^1_{1,1} + V^2_{1,1}$, see Figure 3.4b.

3.3 Numerical Examples

We assume in all following numerical examples that when available, the (latest) threshold-type equilibria are selected at each stage. Economically this means that the firms are not very aggressive and refrain from preemptive equilibrium strategies.

3.3.1 Dynamic Preemption and Over-investment for 1-shot Expansions

In this section, we use a symmetric example to compare competitive investment strategies to their counterparts where competition is constrained (priority option) or firms co-operate. To focus on the preemption effect, we assume that each firm possesses only one option to expand her capacity. The firm parameters are identical, except that one prefers positive X_t and the other negative X_t.

Parameter	Meaning	Value
θ	mean-reversion level	0
μ	mean-reversion rate	0.06
σ	volatility	0.70
r	interest rate	0.03
ρ_i	cost sensitivity	± 1.60
$Q^i_{1,1}$	initial capacity of firm i	1.00
$Q^i_{0,0}$	expanded capacity of firm i	1.50
K^i	expansion cost	5
x_0	initial state of X_t	0

Table 3.1: Numerical setting for Section 3.3.1.

The relative cost X_t is a mean-reverting OU process with zero mean-reversion level $\theta = 0$. As a consequence of these choices, all the equilibrium thresholds will be symmetric about $x = 0$. The price model is:

$$P_{n_1,n_2} = 30\big(1 - 0.17(Q^1_{n_1,n_2} + Q^2_{n_1,n_2})\big).$$

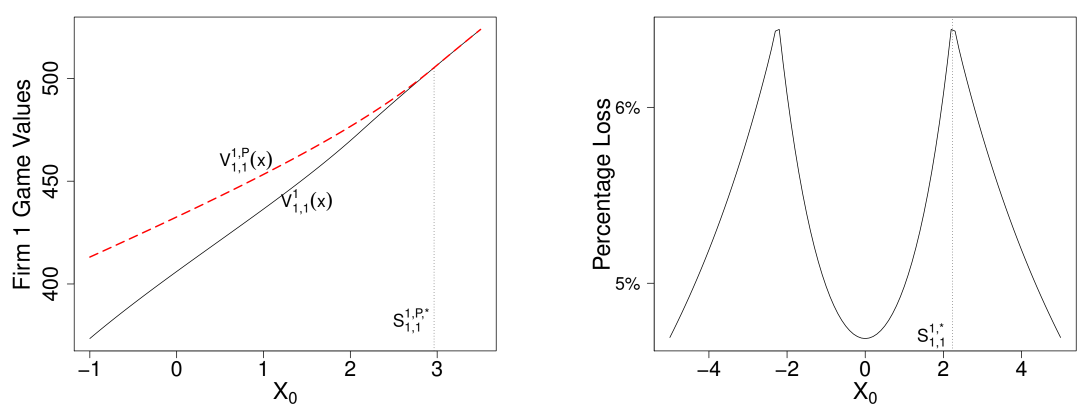
Starting with stage $(1,1)$ we compare three competition models: (i) non-cooperative game, where both firms compete to become the "leader" by investing first, the follower then has a chance to invest second; (ii) priority case where firm 1 is pre-determined to be the leader and hence can optimize her threshold $S^{1,P,*}_{1,1}$ without worrying about threat of preemption; (iii) cooperative game or the central planner model where the aggregate profit of the two firms is optimized. Thanks to the symmetry present in the example, in the competitive model the thresholds are symmetric about zero $S^{1,*}_{1,1} = -S^{2,*}_{1,1}$; also the second-investment thresholds are the same in case (i) and (ii) since the follower does not care if there was an initial priority option or not.

	Non-cooperative	Predetermined Leader	Central Planner
First-stage Policy	$S^{1,*}_{1,1} = 2.1822$	$S^{1,P,*}_{1,1} = 2.964$	$S^{1,M,*}_{1,1} = 2.886$
Second-stage Policy	$S^{1,*}_{1,0} = -0.0387$	—	$S^{M}_{1,0} = 10.5209$
Expected time of the first investment	$m_{S^{2,*}_{1,1},S^{1,*}_{1,1}}(0)$ =16.125	$m_{S^{1,*}_P}(0) = 107.448$	$m_{S^{1,M,*}_{1,1},S^{2,M,*}_{1,1}}(0)$ =39.372
Expected time of the second investment	$m_{S^{2,*}_{0,1}}(S^{1,*}_{1,1})$ =18.943	$m_{S^{2,*}_{0,1}}(S^{1,P,*}_{1,1}) = 23.269$	$m_{S^{2,M,*}_{0,1}}(S^{1,M,*}_{1,1})$ $= 7.280 \times 10^8$

Table 3.2: Equilibrium thresholds of firm 1 and expected times of sequential investments. By symmetry, equilibrium thresholds of firm 2 are the same values with opposite signs at each game stage. Stage $(1,1)$ equilibrium corresponds to Scenario I and is therefore unique.

With parameter values stated in Table 3.1, the equilibrium thresholds of firm 1 associated to each competition model are presented in Table 3.2. For example, her game strategy at stage $(1,1)$ is: $\tau^{1,*}_{1,1} = \inf\{t \geq 0 : X_t \geq S^{1,*}_{1,1} = 2.1822\}$, and so forth. We remark that under these parameters, stage $(1,1)$ yields Scenario I and the resulting threshold-type equilibrium is unique.

We observe that firm 1's threshold in a competitive market, $S^{1,*}_{1,1} = 2.1822$, is lower than her thresholds corresponding to other situations, i.e. competition leads to earlier expansion. With a priority option, firm 1 would not invest until $X_t \geq 2.964$, and under central planner, firm 1 would not invest until $X_t \geq 2.886$. In other words, when the state process X_t is in $(S^{1,*}_{1,1}, S^{1,P,*}_{1,1}) = (2.182, 2.964)$, the firm over-invests immediately, rather waiting for her first-best (i.e. non-competitive) threshold. Figure 3.4 quantifies the resulting impacts on expected profits, which decline due to the above pre-emption effect that reduces the value of the timing flexibility. The left panel compares $V^1_{1,1}$ to $V^{1,P}_{1,1}$—note that the two are equal for $x \geq S^{1,P,*}_{1,1}$. The right panel shows $\big(1 - \frac{V^1_{1,1}(x)+V^2_{1,1}(x)}{V^M_{1,1}(x)}\big)$ which is the difference between the net profit of the central planner and the sum of two competitive firms' net profit. Cooperation increases profits, and the above ratio quantifies the aggregate loss caused by competitive preemption. This loss is maximized when the initial state $X_0 = x$ is equal to the expansion thresholds $S^{i,*}_{1,1}$, whereby one of the firms overinvests immediately.



(a) Game values of firm 1

(b) Percentage loss

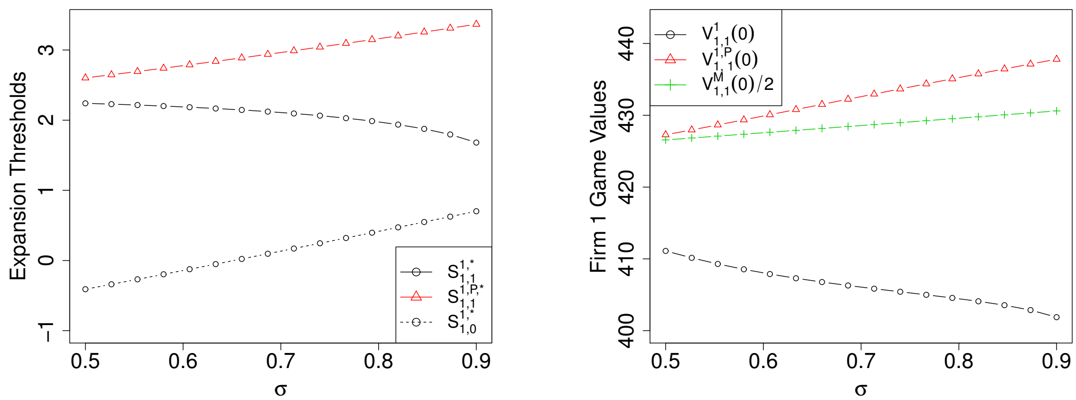
Figure 3.4: Impact of competition on game values. (Left: a) Equilibrium game value of firm 1 predetermined as the leader $V^{1,P}_{1,1}$ (red dashed curve), versus firm 1 game value in a competitive market $V^1_{1,1}$ (solid black). (Right: b) Percentage loss $1 - (V^1_{1,1}(x_0) + V^2_{1,1}(x_0))/V^M_{1,1}(x_0)$ in the firms' aggregate profit due to competition.

To convert the above thresholds into a more economic context, we compute the average timing of an investment. For example, the first investment takes place at $\tau^1_{1,1} \wedge \tau^2_{1,1}$. The respective expected value can be obtained by viewing this quantity as the first exit time from an interval $(a,b) \supset x$, $\tau_{ab} = \inf\{t \geq 0 : X^x_t \leq a \text{ or } X^x_t \geq b\}$. Denote its expectation as $m(x;a,b) := \mathbb{E}_x[\tau_{ab}]$ (see detailed computation in Section 2.3.2). Then the expected time of the first investment in a competitive market, the priority case, or the central planner are $m(0;S^{2,*}_{1,1},S^{1,*}_{1,1})$, $m(0;-\infty,S^{1,P,*}_{1,1})$ and $m(0;S^{2,M,*}_{1,1},S^{1,M,*}_{1,1})$, respectively. We also will consider the time between the first and the second investments (i.e. between the leader and follower times).

Table 3.2 shows that there is in fact a very significant wedge between average investment under competition, and average investment by the central planner. With an initial state $X_0 = 0$, the expected time to *finish* expansion in a competitive market, $m_{S^{2,*}_{1,1},S^{1,*}_{1,1}}(0)+m_{S^{2,*}_{0,1}}(S^{1,*}_{1,1}) = 35.068$, is much shorter compared to the priority/cooperative analogues, so overall capacity build-up is hastened throughout the game, not just due to first-stage preemption. Indeed, because the first investment occurs sooner, the leader gets less time to enjoy her competitive advantage, i.e. lowered $m_{S^{2,*}_{0,1}}(S^{1,*}_{1,1})$, which is another way to explain the harmful impact of competition on industry profitability. An interesting observation is that the expected time of the second investment for a central planner (7.280×10^8) is so long that it is almost equivalent that the central planner will invest only once. Therefore, competition can alter not only the timing of investment, but even the long-run market organization.

3.3.2 Effects of Market Fluctuation

We next discuss the effect of market fluctuations which can be parameterized by the volatility σ of the OU process (2.39). Higher volatility of the relative costs X_t implies more fluctuations in market conditions.



(a) (b)

Figure 3.5: Effect of cost volatility σ. (Left: a) Equilibrium thresholds of firm 1 $S^{1,*}_{1,1}, S^{1,*}_{1,0}, S^{1,P,*}_{1,1}$ as the volatility σ of (X_t) varies. (Right: b) Respective equilibrium game values of firm 1 at $X_0 = 0$ versus σ.

Figure 3.5 shows that as the volatility σ increases, the expansion threshold at stage $(1, 0)$ $S^{1,*}_{1,0}$ of firm 1 increases, while her stage $(1, 1)$-threshold $S^{1,*}_{1,1}$ decreases. With a priority option, the corresponding threshold of firm 1 is positively related to σ. In Figure 3.5b, the equilibrium expected profit of firm 1 increases if she gets first-mover priority, which coincides with the intuition that more market fluctuations lead to higher average revenue. In particular, with higher σ, the pre-determined leader can wait longer until the state process moves to her preferred direction and then reap higher rewards. On the contrary, in a competitive market, game values *decline* as σ increases. This discrepancy highlights the effects of competition. Namely, in the face of a more volatile market, firms become more aggressive and expand capacity much sooner, to the extent that their expected net profits drop. We observe that for σ large, the preemptive equilibrium (3.34)-(3.35) becomes the only available game strategy the firms can adopt (i.e. we are in scenario III from Section 3.2.2). The financial interpretation is that under high profit volatility, firms wish to delay their expansion in order to be certain that X will not quickly move against them. Consequently, they are more concerned about pre-emption by the competitor which is another way for future gains to dissipate. As a result, the competition effect gets stronger and eventually takes over, ruling out threshold-type equilibria.

3.3.3 Case Study: Impact of Multi-part Investments

As discussed, investments in generation capacity are done on a very large-scale with multi-billion dollar commitments. These massive single-shot decisions carry a lot of risk, so more flexible technologies might be preferable (see also [26] and [48]). We interpret flexibility as the ability to split a large investment into smaller ones, for example by sequentially installing several small plants. In this section we present a numerical example to discuss the respective effect of expansion size and the number of expansion options. This analysis also links the sequential, discrete-stage model herein to a continuous control formulation where capacity is added incrementally in infinitesimal amounts.

We maintain the symmetric parameter setting with the OU process X_t from the previous section. Capacity expansion is modeled by $Q^i_{n_1,n_2} = \bar{q} - (\Delta Q)n_i$, where $\bar{q}$ is the *terminal* capacity to be reached, and ΔQ is the unit investment. We now compare the previous single-expansion situation that used $\Delta Q = 0.5$ and $n_i \in \{0, 1\}$, $\bar{q} = 1.5$, with a two-stage expansion for firm i, modeled by $\Delta Q^i = 0.25$ and $n_i \in \{0, 1, 2\}$. The expansion lump-sum costs K^i are proportional to the expansion size ΔQ^i, allowing a direct ceteris paribus comparison. We remark that with added flexibility, the game in interior stages now features Scenario II with multiple threshold-type equilibria.

3.3.3.1 Single-Firm Increased Flexibility

We first consider the case that only firm 1 is allowed to split her project, namely $(\Delta Q^1, \Delta Q^2) = (0.25, 0.5)$. The resulting best-response curves are sketched in Figure 3.6

	firm 1	firm 2
Stage $(2,1)$	$S^{1,*}_{2,1} = 1.133$	$S^{2,*}_{2,1} = -2.1312$
Stage $(1,1)$	$S^{1,*}_{1,1} = 3.323$	$S^{2,*}_{1,1} = -1.2083$
Stage $(2,0)$	$S^{1,*}_{2,0} = -1.043$	—
Stage $(1,0)$	$S^{1,*}_{2,0} = 1.064$	—
Stage $(0,1)$	—	$S^{2,*}_{2,0} = 0.0387$

Table 3.3: Investment thresholds for the case $\Delta Q^1 = 0.25, \Delta Q^2 = 0.5$. Interior stage equilibria correspond to Scenario II with multiple threshold-type equilibria; according to Remark 3.11 we always pick the latest one.

and the equilibrium expansion thresholds are summarized in Table 3.3. Compared to the case $(\Delta Q^1,\, \Delta Q^2) = (0.5, 0.5)$ in Table 3.2, thanks to increased flexibility firm 1 will begin adding capacity sooner, and is much more likely now to invest first: $\mathbb{P}_0\left(\tau^{1,*}_{2,1} < \tau^{2,*}_{2,1}\right) = 0.6853$; recall that in the base case that probability was 50/50.

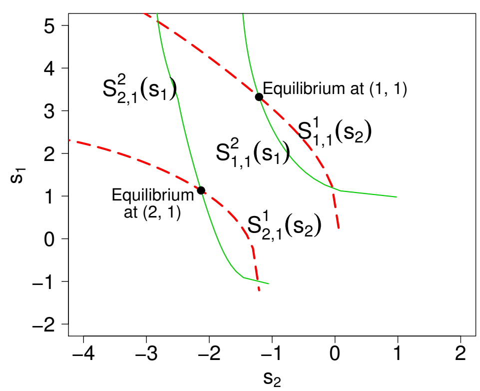
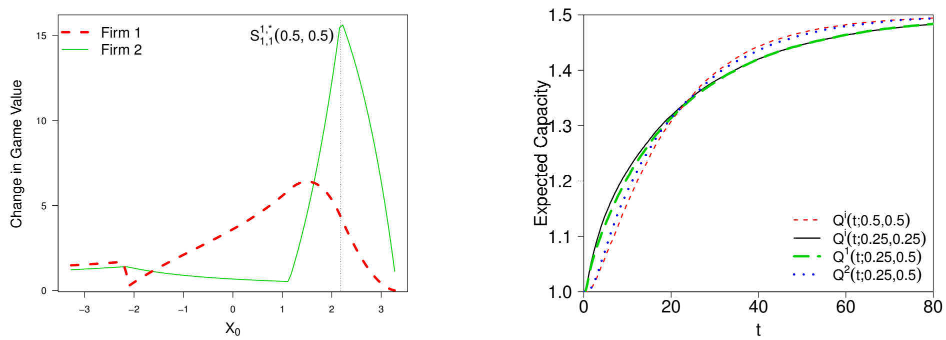


Figure 3.6: Best-response curves in the case $\left(\Delta Q^1,\, \Delta Q^2\right) = (0.25, 0.5)$ with lowercase s denote given thresholds of one firm and uppercase S denote the other firm's best-response threshold. Interior stage equilibria correspond to Scenario II with multiple threshold-type equilibria; the latest ones are highlighted in the plot.

As expected, additional flexibility increases the game value of firm 1, see the red dashed line in Figure 3.7a. The extra profit is maximized when the initial X_0 is between $S^{1,*}_{2,1}(0.25, 0.5)$ and $S^{1,*}_{1,1}(0.5, 0.5)$. Surprisingly, additional flexibility for firm 1 also increases game value of firm 2. This can be partly understood by supposing that $X_0 = S^{1,*}_{1,1}(0.5, 0.5)$, in which situation under $\Delta Q^1 = 0.5$ firm 1 will expand her capacity to 1.5 immediately, putting firm 2 into the undesirable "follower" state; with $\Delta Q^1 = 0.25$, the expansion is only to $Q^1_{1,2} = 1.25$, reducing the negative impact on firm 2. As a result, in this numerical example, *both* firms benefit from one of them gaining additional flexibility.



(a) $V^i_{2,1}(x; 0.25, 0.5) - V^i_{1,1}(x; 0.5, 0.5)$ (b) Expected Capacity

Figure 3.7: (Left: a) Impact of firm's 1 increased flexibility on game values received by each firm. The two "cusps" of the red dashed line are due to the game values not being smooth at the thresholds $S^{2,*}_{2,1}(0.25, 0.5)$ and $S^{2,*}_{1,1}(0.5, 0.5)$. (Right: b) The expected capacity $\mathbb{E}_0[Q^i(t)]$ of each firm starting with $X_0 = 0$.

3.3.3.2 Marco Market Organization: Expected Capacity

Another question we are interested in is the expected capacity $\mathbb{E}_x[Q^i(t)]$ of each firm at time t, or equivalently the distribution of $\vec{M}_t = (N^1(t), N^2(t))$. The exact answer depends on $\mathbb{P}(\tau^i \le t)$ and requires computing the running maximum of an OU process which is not available in closed form. For our purposes we accordingly use Monte Carlo simulation to estimate the expected capacity of firm 1 in the cases $(\Delta Q^1, \Delta Q^2) = (0.25, 0.25)$, $(0.25, 0.5)$ and $(0.5, 0.5)$, assuming that $X_0 = 0$.

To compute $\mathbb{E}_x[Q^i(t)]$, we employ a Monte Carlo method based on the Euler scheme with $\Delta t = 1/120$ and 10000 simulated trajectories of the state process X. The estimated capacities at time t for each case are presented in Figure 3.7b, from which we observe that added flexibility allows firms to smooth out their investment profiles over time, installing more capacity early on, and less (on average) later. Comparing the curves for $(\Delta Q^1,\ \Delta Q^2) = (0.25, 0.25)$ against those of $(\Delta Q^1,\ \Delta Q^2) = (0.5, 0.5)$ we see that smaller project size ΔQ makes aggregate capacity grow slower.

3.3.4 Case Study: Political Will for Meaningful Carbon Prices

We return to the motivating economic example where firm 1 is the nuclear power generator and firm 2 is a coal-fired plant investor. With X_t representing the carbon emission price, higher X_t implies higher net profit made by the nuclear investors (who are carbon-neutral), while lower profit is made by the CO_2-emitting coal-fired plant. As mentioned in Remark 3.2, we can also interpret the firms' sensitivity to the carbon price via the correlation between CO2 allowance price and electricity prices

Parameter	Value	Unit
Private discount rate r	10%	
Public discount rate r_{Public}	3%	
Nuclear expansion cost K^1	1400	USD/MWe
Coal expansion cost K^2	850	USD/MWe
Revenue rate $P_{1,1}$	24	USD/MWh
Revenue rate $P_{1,0}$	22	USD/MWh
Revenue rate $P_{0,1}$	22	USD/MWh
Revenue rate $P_{0,0}$	10	USD/MWh
Cost Sensitivity ρ	0.25	
Long-run carbon price θ	30	USD/tCO2
Political will μ	$[0.1, 0.25]$	
Initial carbon price X_0	5	USD/tCO2

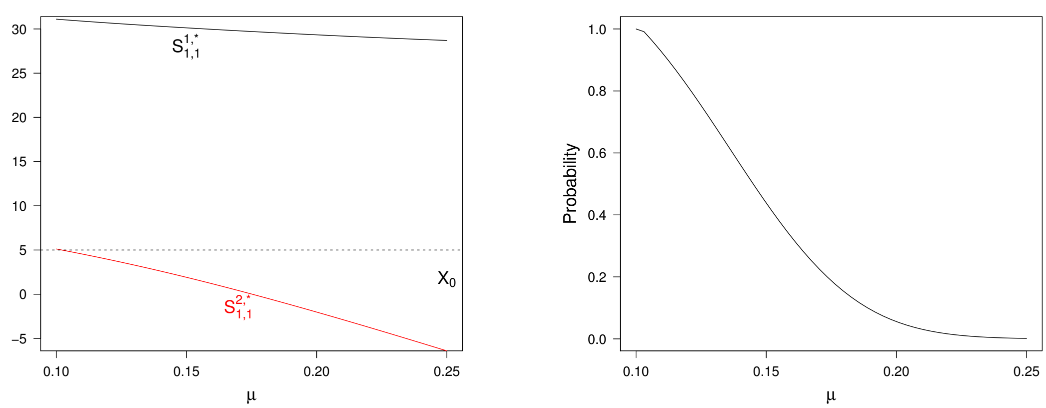
Table 3.4: Parameter values for Section 3.3.4.

As in the previous example, we model X_t as a mean-reverting OU diffusion. Such dynamics are interpreted in terms of the government policy to target a carbon price of $\$\theta$ per ton of CO_2. Market conditions generate fluctuations around this *long-run* average price, and the mean-reversion parameter μ represents the strength of the political will to keep prices around θ. Specifically, in light of recent experiences around the world, policy makers have tried to impose significant carbon prices ($\theta = 30$), while the actual prices have been rather low ($X_0 = 5$). The mean-reversion rate μ in (2.39) determines the expected time to reach the carbon price target, with the time-scale proportional to μ^{-1}.

In the short-run, market conditions are favorable for the coal-fired plants, reflected in the fact that their investment costs are lower, $K^2 < K^1$. In the long-run, the carbon price will rise and erode this favorable situation. For the social planner, the nuclear investment is therefore preferable (and can be justified through a lower social discount factor r_{Public}). However, private investors have much larger discounting $r = 10\%$. Therefore, depending on the political will, coal-fired investment might still be made in the near future. To sharpen this conflict, we assume that the leveraged costs of electricity generation (LCOE) and the nominal electricity prices are such that at most one investment is profitable. Thus, starting at stage $\vec{M}_0 = (1, 1)$, stages $(1, 0)$ and $(0, 1)$ are both absorbing. Consequently, the two firms are competing to make the first and only expansion (i.e. become the "leader" in this asymmetric single-shot setting). Namely, the coal-fired investor might want to preempt the base-load market before the carbon price makes her less competitive. Knowing this, one wonders whether the "green" nuclear power plant generator will hasten her own investment.

The nominal levels of prices P_{n_1,n_2} are designed in the following way. Noting r, the discount rate, the LCOE for player i is $p_i := \frac{r \cdot K_i}{N}$, where N is the number of hours per year to get a price in USD/MWh. In words, LCOE is the price level for which the net present value of building a new plant is zero. We have $p_2 \leq p_1$ and take $P_{1,1} > \max(p_1, p_2)$, but $P_{0,0} \ll \min(p_1, p_2)$. Thus, nominal prices after a first investment are set in such a way that once one player has invested, a second investment will lead to a nominal price much lower than the LCOE's of both players, making it unlikely that the price plus the carbon premium will rise above investment levels again. The intermediate nominal prices $P_{0,1}, P_{1,0}$ are right around p_i's, so that investment is possible, but is conditional on a favorable carbon price (low enough for the coal-fired investor, high-enough for the nuclear investor).

It turns out that with the above parameters, stage $(1,1)$ leads to a unique non-preemptive equilibrium (scenario I). To explain the long-term structure of the market, we consider the *end stage* $\lim_{t\to\infty} \vec{M}_t$. With the parameter settings given, it is only profitable to make (exactly) one investment, so that $\lim_{t\to\infty} \vec{M}_t \in \{(1,0),(0,1)\}$. Figure 3.8 plots the probability $Prob_{0,1} = \mathbb{P}_{x_0}(\lim_{t\to\infty} \vec{M}_t = (0,1))$ that the coal-fired producer is the one to build. We see that this quantity is highly sensitive to μ. If μ is too low, the competition will "choose" to preemptively build coal-fired plants ($S^{2,*}_{1,1} > X_0 = 5$), while the public decision-makers will still be struggling to establish a high and steady value of carbon price. As μ rises, the investment threshold of the coal-fired investor $S^{2,*}_{1,1}$ falls as



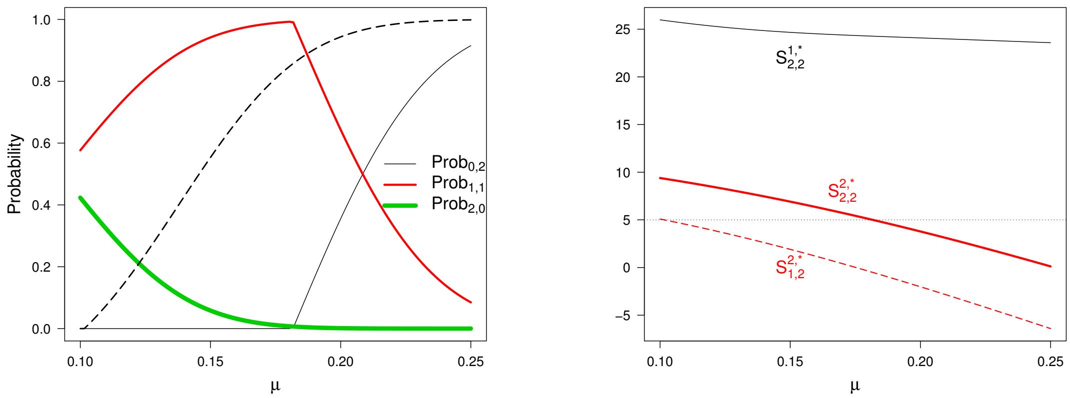
(a) (b)

Figure 3.8: Game equilibrium at stage $(1,1)$ for the nuclear-coal generation capacity market. Equilibrium is based on Scenario I with a unique threshold-type equilibrium. (Left: a) Investment thresholds $S^{1,*}_{1,1}, S^{2,*}_{1,1}$ as the mean-reversion rate μ varies. For convenience we also indicate the level of initial carbon price X_0. (Right: b) Probability that the coal-fired investor invests first $Prob_{1,0}$ as μ varies for the given $X_0 = 5$. For small μ the coal-fired producer is guaranteed to invest first; for large μ nuclear producer 1 is almost guaranteed to invest first, $Prob_{1,0} \simeq 0$.

she anticipates lower future profits, and hence demands larger short-term gains (possible only if carbon price is minimal) as compensation. Of course, with strong political will, carbon prices are unlikely to fall from $X_0 = 5$, so that the likelihood of coal-fired investor making an investment becomes negligible. This confirms the strong impact of policy-making on power plant investments. At the same time, the investment threshold of the nuclear investor is insensitive to μ, because nuclear capacity is not added until $X_t \simeq \theta$, whereby the mean-reversion rate is less relevant.

We next consider a multi-stage situation, whereby each of the two producers can build up to *two* equal-sized smaller-scale plants. We again assume that expansion costs are proportional to plant size, and also that the overall market demand economically supports aggregate capacity up to two plants. Specifically, we assume that with a single small plant, market price will decline to $P_{2,1} = P_{1,2} = 23$ and with any two small plants, $P_{2,0} = P_{0,2} = P_{1,1} = 22$, matching the large-plant setting in Table 3.4. Beyond that, investment becomes impractical, i.e. $P_{1,0}$ and $P_{0,1}$ are too low to ever be profitable. Therefore, starting at stage $(2, 2)$, either (i) the nuclear producer builds 2 small nuclear plants; (ii) the coal-fired investor builds two small coal-fired plants; or (iii) each firm builds one plant apiece. The probabilities of the respective outcomes are labeled $Prob_{2,0}, Prob_{1,1}, Prob_{0,2}$, with $Prob_{n_1,n_2} := P(\lim_{t\to\infty} \vec{M}_t = (n_1, n_2))$. In contrast to the original large-scale investment competition, the initial competitive market at stage $(2, 2)$ corresponds to Scenario II (unless μ is close to 0.1) supporting both a threshold- and preemptive-type equilibria. This occurs because smaller scale investments make nuclear investment profitable at lower carbon prices, sharpening the competition to install capacity first (algebraically it turns out that with given parameter values $L^1_{2,2} < L^2_{2,2}$). Stages $(2, 1)$ and $(1, 2)$ still correspond to Scenario I with a unique non-preemptive equilibrium.

We first assume the threshold-type equilibrium is selected. Figure 3.9 shows that the coal-fired investor will increase her investment threshold $S^{2,small,*}_{2,2}$ for a smaller project



(a) (b)

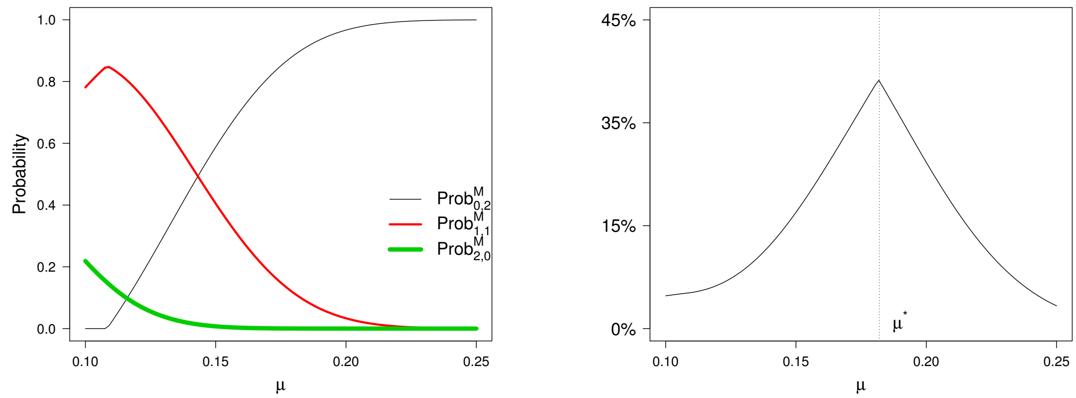
Figure 3.9: (Left: a) Distribution of the terminal stages $Prob_{n_1,n_2} := \mathbb{P}(\lim_{t\to\infty}\vec{M}_t = (n_1,n_2))$ under $\Delta Q^i = 0.25$. The solid lines represent probabilities of terminal stages in a competitive market. The dashed line represents the probability $Prob^e_{0,2} = \mathbb{P}(\lim_{t\to\infty}\vec{M}_t = (0,2)|\vec{M}_0 = (1,2))$ that two nuclear plants are built if a small nuclear plant is built at $X_0 = 5$ preemptively. (Right: b) Investment thresholds $S^{1,*}_{2,2}, S^{2,*}_{2,2}$ and $S^{2,*}_{1,2}$, as μ varies. For convenience we also indicate the level of initial carbon price $X_0 = 5$.

compared to the preceding single large plant $S^{2,large,*}_{1,1}$. As a result, for $\mu < \mu^* = 0.181$, the coal-fired investor is going to build one small plant at once and wait for a moment that the carbon price drops to expand her existing plant. Moreover, even for large μ, the coal-fired investor still has a good chance to build one small plant (leading to terminal stage $(1,1)$), which means that only a very strong policy can guide the market to exclusively "green" power plants.

From Figure 3.9b we also observe that the investment threshold of coal-fired investors at stage $(1,2)$ is significantly lower relative to the threshold at stage $(2,2)$. This is the opposite effect from what was observed in Section 3.3.3, due to the different relationship between stages and prices. Thus, one way for the public decision-makers to guide the industry could be via preempting the base-load market by a small "green" power plant at the initial time (e.g. built with government subsidies). In turn, the lowered electricity price reduces anticipated future profits of the coal-fired investor, and makes them less likely to ever invest (see also [13]). In Figure 3.9a, as the dashed line shows, public decision-maker's preemption sharply increases the probability that two (small) nuclear power plants will be built. Another alternative for policy makers is to grant a priority option to nuclear investors at the first stage $(2, 2)$. Distinguished from the preemption case, nuclear investors with a priority can simply wait until a high-enough carbon price to make their investment. Since the long-run carbon price is taken to be $\theta = 30$, once it is high, it will likely remain high. Consequently, a single priority option is enough to guide the market to the $(0, 2)$ terminal stage, since the coal producer becomes very unlikely to invest in the $(1, 2)$-stage.

Figure 3.10a shows that the social planner (equivalent to a cooperative game, or a generator monopoly) is likely to build two nuclear plants, consistent with the idea that "green" generation is more profitable in the long-run. Significantly, Figure 3.10b illustrates that the percentage loss caused by competition can be as high as 40% at moderate levels of μ (when a small coal-fired plant is built instantly). This confirms the anecdotal evidence of very significant losses incurred by producers in newly deregulated markets, and the accompanying capacity over-investment (due to the preemptive race to build first). It also illustrates the dramatic impact that the short-term driven competition can have on the long-term market organization; here over-investment drastically alters the mix of power plants likely to be built, hurting long-run profits of *both* producers. We also observe that such losses are reduced to almost zero for μ large enough, which again corroborates the strong impact of public policy.

Coming back to the equilibrium type at stage $(2, 2)$, suppose instead that the investors are aggressive and implement the preemptive equilibrium strategy. Since a coal-fired power plant is cheaper to build, it is natural to assume that the coal-fired investor possesses preemptive priority. For $\mu < \mu^* = 0.181$, this makes no difference: an aggressive coal investor will behave exactly the same as before because she will build a small plant



(a) (b)

Figure 3.10: (Left: a) Distribution of the terminal stage $\lim_{t\to\infty} \vec{M}_t$ under co-operative game model. (Right: b) Percentage loss caused by competition: $\left[1 - (V^1_{2,2} + V^2_{2,2})/V^M_{2,2}\right](X_0)$.

at once and there is no preemptive equilibrium at stage $(2, 1)$. For larger μ, it turns out that $L^2_{2,2} < X_0 = 5$, which prevents firm 2 from immediate investment, as the NPV of an expansion is negative. Meanwhile, the nuclear investor will choose to preempt right before the carbon emission price drops down to $L^2_{2,2}$ (see resulting game value of firm 1 in Figure 3.1a). Consequently, exclusively "green" power plants are more likely to be established.

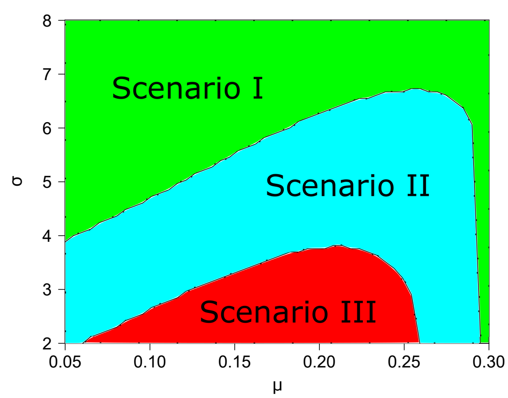


Figure 3.11: Resulting equilibrium types at stage $(2, 2)$ in the small-scale power plants case, as the political will μ and carbon volatility σ vary.

Finally, we end this section by illustrating which equilibrium scenario takes place during stage $(2,2)$ as the political will μ and carbon market fluctuation σ vary. As Figure 3.11 shows, we observe that only a preemptive equilibrium exists (Scenario III) under low carbon volatility σ. Also, the impact of μ is non-monotone: when μ is very small or very large, the competition between industries is less preemptive (as one industry is clearly ahead in the short-term) and hence threshold-type equilibria exist. However, for intermediate values of μ, the equal-strength competition raises benefits of aggressive investment and generates preemptive equilibria. The impact of σ is harder to explain and is ultimately linked to its recursive effect on the leadership thresholds $L^i_{2,2}$ of the two industries.

3.4 Propositions and Proofs

3.4.1 Optimization at Stage (1, 0) and (0, 1)

In the game stage $(1,0)$ firm 2 has already invested and firm 1 now optimizes her expected discounted profits. Substituting (3.11) into (3.12) for $n_1 = 1$, firm 1 solves the optimal stopping problem:

$$V^1_{1,0}(x) - D^1_{1,0}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} h^1_{1,0}(X_\tau) \right\}, \tag{3.50}$$

where the first-mover payoff is:

$$h^1_{1,0}(x) = D^1_{0,0}(x) - D^1_{1,0}(x) - K^1_1 = \frac{\rho^1 \Delta Q^1_1}{\delta} \cdot x - \left(K^1_1 + \zeta^1_{1,0} - \zeta^1_{0,0} \right),$$

and we set ΔQ^1_1 as the expansion size of firm 1 when she has one option left. The payoff $h^1_{1,0}$ is linear and increasing in x, similar to a Call option payoff. Thus, this optimal stopping problem can be considered as an analogue to pricing a perpetual American Call. In order to solve this problem, we apply the operator Ψ (2.15) and then Proposition 2.6 with $R = \mathcal{D}$, see Section 3.4.1.1 for the details.

Proposition 3.14 *(firm 1 at stage (1, 0)) The value function associated to the optimal stopping problem* (3.50) *is admitted as:*

$$V^1_{1,0}(x) = \begin{cases} D^1_{1,0}(x) + \frac{F(x)}{F(S^{1,*}_{1,0})} \cdot h^1_{1,0}\left(S^{1,*}_{1,0}\right), & \text{if } x \in (\underline{d},\, S^{1,*}_{1,0}), \\ D^1_{0,0}(x) - K^1_1, & \text{if } x \in [S^{1,*}_{1,0},\, \bar{d}). \end{cases} \tag{3.51}$$

The corresponding policy is characterized by a threshold-type stopping time

$$\tau^{1,*}_{1,0} = \inf\{t \geq 0 :\, X^x_t \geq S^{1,*}_{1,0}\}, \tag{3.52}$$

where the expansion threshold $S^{1,*}_{1,0}$ satisfies the equation

$$F(S^{1,*}_{1,0}) = \frac{h^1_{1,0}}{\left(h^1_{1,0}\right)'}(S^{1,*}_{1,0}) \times F'(S^{1,*}_{1,0}). \tag{3.53}$$

In the converse scenario, at stage $(0, 1)$ firm 2 possesses the only expansion option. Substituting (3.11) into (3.13) for $n_2 = 1$, she solves the following optimal stopping problem:

$$V^2_{0,1}(x) - D^2_{0,1}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left\{ e^{-r\tau} h^2_{0,1}(X_\tau) \right\}. \tag{3.54}$$

The first-mover payoff is derived as:

$$h^2_{0,1}(x) = D^2_{0,0}(x) - D^2_{0,1}(x) - K^2_1 = -\frac{\rho^2 \Delta Q^2_1}{\delta} \cdot x - \left(K^2_1 + \zeta^2_{0,1} - \zeta^2_{0,0}\right),$$

where we set ΔQ_1^2 as the expansion size of firm 2 when she has one option left. Since $h_{0,1}^2$ is decreasing and linear in x, the single-agent optimizing problem can be considered as an analog to the perpetual American Put. The following Proposition readily follows, see Section 3.4.1.2.

Proposition 3.15 *(firm 2 at stage (0, 1)) The value function associated to the optimal stopping problem* (3.54) *is admitted as:*

$$V_{0,1}^2(x) = \begin{cases} D_{0,0}^2(x) - K_1^2, & \text{if } x \in (\underline{d},\, S_{0,1}^{2,*}], \\ D_{0,1}^2(x) + \frac{G(x)}{G(S_{0,1}^{2,*})} \cdot h_{0,1}^2(S_{0,1}^{2,*}), & \text{if } x \in (S_{0,1}^{2,*},\, \bar{d}). \end{cases} \tag{3.55}$$

The corresponding policy is characterized by a threshold-type stopping time

$$\tau_{0,1}^{2,*} = \inf\{t \geq 0 : \, X_t^x \leq S_{0,1}^{2,*}\}, \tag{3.56}$$

where the expansion threshold $S_{0,1}^{2,*}$ satisfies the equation

$$G(S_{0,1}^{2,*}) = \frac{h_{0,1}^2}{\left(h_{0,1}^2\right)'}(S_{0,1}^{2,*}) \times G'(S_{0,1}^{2,*}). \tag{3.57}$$

Example 3.16 *Under a GBM model* (2.36)*, the game value of firm 1 at stage* $(1,0)$ *is derived as:*

$$V_{1,0}^1(x) = \begin{cases} D_{1,0}^1(x) + \frac{K_1^1 + \zeta_{1,0}^1 - \zeta_{0,0}^1}{\eta_+ - 1} \left(\frac{x}{S_{1,0}^{1,*}}\right)^{\eta_+}, & \text{if } x \in (0,\, S_{1,0}^{1,*}), \\ D_{0,0}^1(x) - K_1^1, & \text{if } x \in [S_{1,0}^{1,*},\, +\infty), \end{cases}$$

where $S_{1,0}^{1,*} = \frac{\delta\left(K_1^1 + \zeta_{1,0}^1 - \zeta_{0,0}^1\right)\eta_+}{\rho^1 \Delta Q_1^1(\eta_+ - 1)}$ *is the expansion threshold of firm 1 at stage (1, 0). The* *game value of firm 2 at stage (0, 1) is derived as:*

$$V^2_{0,1}(x) = \begin{cases} D^2_{0,0}(x) - K^2_1, & \textit{if } x \in (0,\ S^{2,*}_{0,1}], \\ D^2_{0,1}(x) + \frac{\left(K^2_1+\zeta^2_{0,1}-\zeta^2_{0,0}\right)}{1-\eta_-}\left(\frac{x}{S^{2,*}_{0,1}}\right)^{\eta_-}, & \textit{if } x \in (S^{2,*}_{0,1},\ +\infty), \end{cases}$$

where $S^{2,*}_{0,1} = \frac{\delta\left(K^2_1+\zeta^2_{0,1}-\zeta^2_{0,0}\right)\eta_-}{\rho^2 \Delta Q^2_1(\eta_- -1)}$ *is the expansion threshold of firm 2.*

Under an OU model (2.39)*, there is no explicit formula. The thresholds and game values can be obtained by plugging* F (2.41) *and* G (2.41) *into preceding propositions and solving the resulting equations numerically.*

3.4.1.1 Proof of Proposition 3.14

Proof: This is a canonical single agent optimal stopping problem. In this thesis, we prove it following the work of [55]. Recall that the optimal stopping problem (3.50) corresponds to the case $R = \mathcal{D}$ discussed in Proposition 2.6. Applying operator Ψ to $h^1_{1,0}$, we obtain $H^1_{1,0}(y) := \Psi h^1_{1,0}(y)$, which is continuous and twice differentiable on $\psi(\mathcal{D}) = (0, +\infty)$. Meanwhile, denoting the smallest concave majorant of $H^1_{1,0}$ over $\mathbb{R}^+$ by $\mathcal{W}H^1_{1,0}(y)$, and referring to Proposition 2.6, we obtain

$$V^1_{1,0}(x) - D^1_{1,0}(x) = G(x) \cdot \left[\mathcal{W}H^1_{1,0} \circ \psi(x)\right], \qquad x \in \mathcal{D}. \tag{3.58}$$

Since $h^1_{1,0}$ is a linear increasing function which is in the class $\mathcal{H}_{\text{inc}}$, the transformed payoff $y \mapsto H^1_{1,0}(y)$ possesses properties stated in Lemma (2.8), namely it is convex on $[0, \psi\left(b^1_{1,0}\right))$ and concave on $(\psi\left(b^1_{1,0}\right), +\infty)$. Therefore, we conclude that there exists a unique number $y^* > \psi(b^1_{1,0})$, such that the smallest concave majorant $\mathcal{W}H^1_{1,0}$ is a straight line from the origin tangent to $H^1_{1,0}$ at $\left(y^*, H^1_{1,0}(y^*)\right)$ on $[0, y^*)$, and then coincides with

$H^1_{1,0}$ on $[y^*, +\infty)$ (see Figure 2.1a):

$$\left(\mathcal{W}H^1_{1,0}\right)(y) = \begin{cases} y\frac{H^1_{1,0}(y^*)}{y^*}, & \text{if } y < y^*, \\ H^1_{1,0}(y), & \text{if } y \geq y^*. \end{cases} \tag{3.59}$$

Define $S^{1,*}_{1,0} := \psi^{-1}(y^*)$. By direct differentiation, we obtain

$$\frac{dH^1_{1,0}(y)}{dy}\bigg|_{y=\psi(S^{1,*}_{1,0})} = \frac{\left(h^1_{1,0}\right)'(S^{1,*}_{1,0})G(S^{1,*}_{1,0}) - h^1_{1,0}(S^{1,*}_{1,0})G'(S^{1,*}_{1,0})}{F'(S^{1,*}_{1,0})G(S^{1,*}_{1,0}) - F(S^{1,*}_{1,0})G'(S^{1,*}_{1,0})}.$$

To match the first derivative at the tangent point, it must hold that

$$\frac{H^1_{1,0}(y^*)}{y^*} = (H^1_{1,0})'(y^*), \tag{3.60}$$

where by (2.11) the LHS is admitted as

$$\frac{H^1_{1,0}(y^*)}{y*} = \frac{H^1_{1,0}(\psi(S^{1,*}_{1,0}))}{\psi(S^{1,*}_{1,0})} = \frac{h^1_{1,0}(S^{1,*}_{1,0})}{F(S^{1,*}_{1,0})}. \tag{3.61}$$

Consequently, we can rewrite condition (3.60) in terms of $S^{1,*}_{1,0}$, and simplify it to (3.53). Substituting (3.61) into (3.59), we get

$$\mathcal{W}H^1_{1,0} \circ \psi(x) = \begin{cases} \psi(x)\frac{H^1_{1,0}(y^*)}{y^*} = \frac{F(x)}{G(x)}\frac{h^1_{1,0}(S^{1,*}_{1,0})}{F(S^{1,*}_{1,0})}, & \text{if } x \in (\underline{d},\, S^{1,*}_{1,0}), \\ H^1_{1,0}(\psi(x)) = \frac{h^1_{1,0}(x)}{G(x)}, & \text{if } x \in [S^{1,*}_{1,0},\, \bar{d}). \end{cases}$$

Combining above with (3.58) we obtain the expression for the value function $V^1_{1,0}(x)$ in (3.51). This also yields the structure of the optimal stopping region as (3.52) and the smooth pasting condition at the threshold $S^{1,*}_{1,0}$ via (3.60). ■

3.4.1.2 Proof of Proposition 3.15

Proof: The optimal stopping problem (3.54) corresponds to the special case $R = \mathcal{D}$. Applying Φ operator and referring to Proposition 2.6 yields that

$$V_{0,1}^2(x) - D_{0,1}^2(x) = F(x) \cdot \left[\mathcal{W}H_{0,1}^2 \circ \varphi(x)\right], \qquad x \in \mathcal{D}, \tag{3.62}$$

where $H_{0,1}^2(z) := \Phi h_{0,1}^2(z) = \frac{h_{0,1}^2}{F} \circ \varphi^{-1}(z)$ is continuous and twice differentiable on $\varphi(\mathcal{D}) = (0, +\infty)$, and $\mathcal{W}H_{0,1}^2$ is its smallest concave majorant in the z-coordinate. Since $h_{0,1}^2$ is in $\mathcal{H}_{\text{dec}}$, Lemma 2.8 implies that $z \mapsto H_{0,1}^2(z)$ possesses the same shape as $y \mapsto H_{1,0}^1(y)$ sketched in Figure 2.1a, and consequently its smallest concave majorant $\mathcal{W}H_{0,1}^2$ has the same shape as $\mathcal{W}H_{1,0}^1$. Similar arguments as in the proof of Proposition 3.14 yield that

$$\mathcal{W}H_{0,1}^2 \circ \varphi(x) = \begin{cases} \varphi(x) \frac{H_{0,1}^2(z^*)}{z^*} = \frac{G(x)}{F(x)} \frac{h_{0,1}^2(S_{0,1}^{2,*})}{G(S_{0,1}^{2,*})}, & \text{if } x \in (S_{0,1}^{2,*},\, \bar{d}), \\ H_{0,1}^2(\varphi(x)) = \frac{h_{0,1}^2(x)}{F(x)}, & \text{if } x \in (\underline{d},\, S_{0,1}^{2,*}], \end{cases}$$

where $S_{0,1}^{2,*}$ is obtained by matching the first derivative at $z^* := \varphi\left(S_{0,1}^{2,*}\right)$

$$\frac{H_{0,1}^2\left(z^*\right)}{z^*} = \left(H_{0,1}^2\right)'\left(z^*\right).$$

Finally, the value function $V_{0,1}^2(x)$, as well as the stopping region (3.56) stated in (3.55) is obtained by (3.62). ■

3.4.2 Proof of Theorem 3.4

We use induction to prove the result for the case where only firm 1 has expansion options. Suppose that at stage $(n_1 - 1,\, 0)$ firm 1 implements game strategy characterized by threshold $S_{n_1-1,0}^{1,*}$ and receives game value $V_{n_1-1,0}^1$. Following (3.12), at stage $(n_1,\, 0)$

firm 1 solves an optimal stopping problem with first-mover payoff:

$$\begin{aligned} h^1_{n_1,0}(x) &= V^1_{n_1-1,0}(x) - D^1_{n_1,0}(x) - K^1_{n_1} \\ &= \begin{cases} D^1_{n_1-1,0}(x) - D^1_{n_1,0}(x) - K^1_{n_1} + \frac{h^1_{n_1-1,0}\left(S^1_{n_1-1,0}\right)}{F\left(S^1_{n_1-1,0}\right)} F(x), & \text{if } x < S^{1,*}_{n_1-1,0}, \\ D^1_{n_1-1,0}(x) - D^1_{n_1,0}(x) - K^1_{n_1} + h^1_{n_1-1,0}(x), & \text{if } x \geq S^{1,*}_{n_1-1,0}, \end{cases} \end{aligned}$$

where $\Delta Q^1_{n_1}$ is the expansion size of firm 1 when she has n_1 options and $h^1_{n_1-1,0}(x) = V^1_{n_1-2,0}(x) - D^1_{n_1-1,0}(x) - K^1_{n_1-1}$ is her first-mover payoff at stage $(n_1-1,0)$ and is contained in the class $\mathcal{H}_{\text{inc}}$. Also note that $h^1_{n_1,0}$ is smooth at the point $S^{1,*}_{n_1-,0}$ following the smooth pasting condition. This problem corresponds again the case $R = \mathcal{D}$ in Proposition 2.6 and therefore yields a value function

$$V^1_{n_1,0}(x) - D^1_{n_1,0}(x) = G(x) \cdot \left[\mathcal{W}H^1_{n_1,0} \circ \psi(x)\right], \tag{3.63}$$

where $H^1_{n_1,0}(y) := \Psi h^1_{n_1,0}(y)$ and $\mathcal{W}H^1_{n_1,0}(y)$ is its smallest concave majorant over $\mathbb{R}^+$. Since F is a solution to the ODE (2.12), the F term in $h^1_{n_1,0}$ does not contribute to $(\mathcal{L} - r)\, h^1_{n_1,0}$. From the assumption that all increasing linear functions are in $\mathcal{H}_{\text{inc}}$ and $(\mathcal{L} - r)\, h^1_{n_1-1,0}(x) < 0$ for $x \geq S^{1,*}_{n_1-1,0}$, we then conclude that $h^1_{n_1,0}(x)$ is in the class $\mathcal{H}_{\text{inc}}$ and there exists $b^1_{n_1,0}$, such that $H^1_{n_1,0}(y)$ is convex over $(0, \psi\left(b^1_{n_1,0}\right))$ and concave over $(\psi\left(b^1_{n_1,0}\right), +\infty)$, cf. Lemma 2.8. Repeating the proof of Proposition 3.14 then gives the game value and strategy of firm 1 stated in Theorem 3.4. Identical arguments work for firm 2, using the Φ-transform.

3.4.3 Proof of Proposition 3.7 and 3.8

From the definition of $\tau^2_{1,1}$ in (3.27), the optimization problem (3.28) corresponds to the case $R = \left(s_2,\, \overline{d}\right)$ in Proposition 2.6. The first-mover payoff is:

$$\begin{aligned} h^1_{1,1}(x) &= V^1_{0,1}(x) - D^1_{1,1}(x) - K^1_1 \\ &= \begin{cases} D^1_{0,0}(x) - D^1_{1,1}(x) - K^1_1, & \text{if } x \in (\underline{d},\, S^{2,*}_{0,1}), \\ D^1_{0,1}(x) - D^1_{1,1}(x) - K^1_1 + G(x)\cdot\left[\frac{D^1_{0,0}-D^1_{0,1}}{G}\right]\left(S^{2,*}_{0,1}\right), & \text{if } x \in (S^{2,*}_{0,1},\, \bar{d}). \end{cases} \end{aligned} \tag{3.64}$$

Given the strategy of firm 2 stated in (3.27), we define

$$\hat{h}^{1,s_2}_{1,1}(x) := 1_{(s_2,\bar{d})}(x) h^1_{1,1}(x) + 1_{(x=s_2)} l^1_{1,1}(x). \tag{3.65}$$

Applying the operator Ψ defined in (2.15), we denote the transformed function by $H^{1,s_2}_{1,1}(y) := \Psi \hat{h}^{1,s_2}_{1,1}(y)$, and its smallest concave majorant over $[\psi(s_2),\, +\infty)$ by $\mathcal{W}H^{1,s_2}_{1,1}$. Following Proposition 2.6, the corresponding value function is admitted as

$$\widetilde{V}^1_{1,1}(x, s_2) - D^1_{1,1}(x) = G(x)\cdot\left[\mathcal{W}H^{1,s_2}_{1,1} \circ \psi(x)\right], \qquad s_2 < x < \bar{d}. \tag{3.66}$$

Let us first consider the case $s_2 < L^1_{1,1}$. Since G is a solution to the ODE (2.12), we conclude that $h^1_{1,1}$ is in the class $\mathcal{H}_{\text{inc}}$. Following Lemma 2.8, there exists a fixed point $b^1_{1,1}$ such that $y \mapsto \Psi h^1_{1,1}(y)$ is convex on $\left(0,\, \psi(b^1_{1,1})\right)$ and concave on $\left(\psi(b^1_{1,1}),\, +\infty\right)$. Consequently (see Figure 2.1b), there exists a unique $\tilde{y}^* > \psi(b^1_{1,1})$, such that the smallest concave majorant $\mathcal{W}H^{1,s_2}_{1,1}(y)$ is a straight line from $\left(\psi(s_2),\, \Psi l^1_{1,1}(\psi(s_2))\right)$, tangent to

$H_{1,1}^{1,s_2}(y)$ at $\left(\tilde{y}^*,\, \Psi h_{1,1}^1(\tilde{y}^*)\right)$ and then coincides with $H_{1,1}^{1,s_2}(y)$:

$$\mathcal{W}H_{1,1}^{1,s_2}(y) = \begin{cases} \Psi l_{1,1}^1(\psi(s_2)) + (y - \psi(s_2))\left(\Psi l_{1,1}^1\right)'(\tilde{y}^*), & \text{if } y \in [\psi(s_2),\, \tilde{y}^*), \\ \Psi h_{1,1}^1(y), & \text{if } y \geq \psi(\tilde{y}^*), \end{cases} \tag{3.67}$$

which yields the optimal stopping region characterized in (3.29). To match the first derivative at the tangent point, it must hold that

$$\frac{\Psi h_{1,1}^1(\tilde{y}^*) - \Psi l_{1,1}^1(\psi(s_2))}{\tilde{y}^* - \psi(s_2)} = \left(\Psi h_{1,1}^1\right)'(\tilde{y}^*). \tag{3.68}$$

Define $S_1 := \psi^{-1}(\tilde{y}^*)$. Substituting (3.65) and $\psi = \frac{F}{G}$ into the LHS of (3.68), we obtain:

$$\frac{\Psi h_{1,1}^1(\tilde{y}^*) - \Psi l_{1,1}^1(\psi(s_2))}{\tilde{y}^* - \psi(s_2)} = \frac{\frac{h_{1,1}^1(S_1)}{G(S_1)} - \frac{l_{1,1}^1(s_2)}{G(s_2)}}{\frac{F(S_1)}{G(S_1)} - \frac{F(s_2)}{G(s_2)}} = \frac{h_{1,1}^1(S_1)G(s_2) - l_{1,1}^1(s_2)G(S_1)}{F(S_1)G(s_2) - F(s_2)G(S_1)} := \tilde{\omega}_{1,1}^1. \tag{3.69}$$

Differentiating the RHS directly, it follows that

$$\frac{dH_{1,1}^1(y)}{dy}\Big|_{\psi^{-1}(\tilde{y}^*)=S_1} = \frac{\left(h_{1,1}^1\right)'(S_1)G(S_1) - h_{1,1}^1(S_1)G'(S_1)}{F'(S_1)G(S_1) - F(S_1)G'(S_1)},$$

hence we can rewrite condition (3.68) in terms of S_1 and simplify it to equation (3.30). Finally, for $x \in (s_2,\, S_1)$, $\left(H_{1,1}^{1,s_2}\right)'(\tilde{y}^*) = \tilde{\omega}_{1,1}^1$ implies that

$$W_{1,1}^{1,s_2}(\psi(x)) = \Psi l_{1,1}^1(\psi(s_2)) + (\psi(x) - \psi(s_2))\tilde{\omega}_{1,1}^1 \triangleq \tilde{\omega}_{1,1}^1\psi(x) + \tilde{\nu}_{1,1}^1, \qquad s_2 \leq x < S_1,$$

where $\tilde{\omega}_{1,1}^1$ and $\tilde{\nu}_{1,1}^1$ can be verified to match (3.32). From Proposition 2.6, the value function is then admitted as

$$\widetilde{V}^1_{1,1}(x,s_2)-D^1_{1,1}(x)=\begin{cases}G(x)W^{1,s_2}_{1,1}(\psi(x))=\tilde{\omega}^1_{1,1}F(x)+\tilde{\nu}^1_{1,1}G(x), & \text{if } s_2<x<S_1,\\ h^1_{1,1}(x), & \text{if } x\ge S_1,\end{cases}$$

which coincides with (3.31), and completes the proof of Proposition 3.7.

Next, suppose that $s_2 > L^1_{1,1}$. For $L^1_{1,1} < x \le s_2$, firm 1 will try to preempt her rival since her corresponding first-mover payoff is higher than her second-mover payoff. For $x > s_2$, the value function of firm 1 is again admitted as (3.66) according to Proposition 2.6. However, since there is a *negative* jump at $y = \psi(s_2)$ in $H^{1,s_2}_{1,1}(y)$, the smallest concave majorant $\mathcal{W}H^{1,s_2}_{1,1}(y)$ is now a straight line from $\big(\psi(s_2),\, \Psi h^1_{1,1}(\psi(s_2))\big)$, tangent to $H^{1,s_2}_{1,1}(y)$ at $\big(\tilde{y}^*,\, \Psi h^1_{1,1}(\tilde{y}^*)\big)$ and then coincides with $H^{1,s_2}_{1,1}(y)$:

$$W^{1,s_2}_{1,1}(y)=\begin{cases}\Psi h^1_{1,1}(\psi(s_2))+(y-\psi(s_2))\left(\Psi h^1_{1,1}\right)'(\tilde{y}^*), & \text{if } y\in(\psi(s_2),\,\tilde{y}^*),\\ \Psi h^1_{1,1}(y), & \text{if } y\ge\psi(\tilde{y}^*).\end{cases} \tag{3.70}$$

And the first derivative is matched at the tangent point

$$\frac{\Psi h^1_{1,1}(\tilde{y}^*)-\Psi h^1_{1,1}(\psi(s_2))}{\tilde{y}^*-\psi(s_2)}=\left(\Psi h^1_{1,1}\right)'(\tilde{y}^*). \tag{3.71}$$

Repeating the preceding steps then yields the threshold S^e_1. Note that if $s_2 \ge b^1_{1,1}$, equation (3.71) yields $S^e_1 = s_2$. Meanwhile, following from (3.66), $\lim\limits_{x\searrow s_2}\widetilde{V}^1_{1,1}(x,s_2) - D^1_{1,1}(s_2) = h^1_{1,1}(s_2) > l^1_{1,1}(s_2)$, which implies stopping at s_2 is too late, and firm 1 would prefer to preempt right before s_2. Therefore, with an ε-optimal stopping rule defined as

$$\Gamma^\varepsilon := \{x\in(s_2,\bar{d}) :\, \widetilde{V}^1_{1,1}(x,s_2)-D^1_{1,1}(x)\le\hat{h}(x)+\varepsilon\} \quad \text{and} \quad \tau^\varepsilon := \inf\{t\ge 0 :\, X_t\in\Gamma^\varepsilon\}, \tag{3.72}$$

the best-response of firm 1 in this situation is $\lim\limits_{\varepsilon \searrow 0} \tau^{\varepsilon}$ which corresponds to (3.33).

3.4.4 Best-response of Firm 2 at stage (1, 1)

Similar to previous discussion, we start with the assumption that firm 1's policy is of threshold type: $\tau^1_{1,1} = \inf\{t \geq 0 :\ X_t \geq s_1\}$. For $s_1 > L^2_{1,1}$, firm 2 solves the optimal stopping problem:

$$\widetilde{V}^2_{1,1}(x, s_1) - D^2_{1,1}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[\mathbb{1}_{\{\tau < \tau^1_{1,1}\}} e^{-r\tau} \left\{ h^2_{1,1}(X_\tau) \right\} + \mathbb{1}_{\{\tau > \tau^1_{1,1}\}} e^{-r\tau^1_{1,1}} \left\{ l^2_{1,1}(X_{\tau^1_{1,1}}) \right\} \right]. \tag{3.73}$$

The resulting *threshold-type* best-response of firm 2 is

$$\tau^2_{1,1}(s_1) = \inf\{t \geq 0 :\ X_t \leq S^2_{1,1}(s_1)\},$$

where the optimal stopping level is characterized as the solution to:

$$\begin{aligned}
&\left[h^2_{1,1}(S_2)G(s_1) - \left(h^2_{1,1} \vee l^2_{1,1}\right)(s_1)G(S_2)\right] F'(S_2) \\
&\qquad + \left[\left(h^2_{1,1} \vee l^2_{1,1}\right)(s_1)F(S_2) - h^2_{1,1}(S_2)F(s_1)\right] G'(S_2) \\
&= \left(h^2_{1,1}\right)'(S_2)\left[G(s_1)F(S_2) - G(S_2)F(s_1)\right].
\end{aligned} \tag{3.74}$$

Consequently, the optimal stopping problem (3.73) admits the value function

$$\widetilde{V}^2_{1,1}(x, s_1) = \begin{cases} V^2_{1,0}(x) - K^2_1, & \text{if } x < S_2(s_1), \\ D^2_{1,1}(x) + \tilde{\omega}^2_{1,1}F(x) + \tilde{\nu}^2_{1,1}G(x), & \text{if } x \in [S_2(s_1),\ s_]), \\ V^2_{0,1}(x), & \text{if } x > s_1, \end{cases} \tag{3.75}$$

where $\tilde{\omega}^2_{1,1} := \tilde{\omega}^2_{1,1}(s_1)$ and $\tilde{\nu}^2_{1,1} := \tilde{\nu}^2_{1,1}(s_1)$ are defined as

$$\tilde{\omega}^2_{1,1} = \frac{\left(h^2_{1,1} \vee l^2_{1,1}\right)(s_1)G(S_2) - h^2_{1,1}(S_2)G(s_1)}{F(s_1)G(S_2) - F(S_2)G(s_1)}, \tag{3.76a}$$

$$\tilde{\nu}^2_{1,1} = \frac{h^2_{1,1}(S_2)F(s_1) - \left(h^2_{1,1} \vee l^2_{1,1}\right)(s_1)F(S_2)}{F(s_1)G(S_2) - F(S_2)G(s_1)}. \tag{3.76b}$$

For $s_1 < L^2_{1,1}$, firm 2 is incentivized to preempt when $s_1 \leq X_t < L^2_{1,1}$ or right before X_t hits s_1. To wit, the *preemptive* best-response of firm 2 is a "stopping time" admitted as

$$\tau^{2,e}_{1,1}(s_1) = \inf\{t \geq 0 : \; (s_1-) \leq X_t < L^2_{1,1} \text{ or } X_t \leq S^{2,e}_{1,1}(s_1)\}, \tag{3.77}$$

where the optimal stopping level $S^{2,e}_{1,1} := S^e_2 \leq s_1$ is a solution to (3.74).

The proof is a symmetric repetition of the proof of Proposition 3.7 and 3.8, except the fact that the first-mover payoff of firm 2, $h^2_{1,1}(x)$, is in the class $\mathcal{H}_{\text{dec}}$, and $z = \varphi(x)$ coordinate has opposite direction from y coordinate.

3.4.5 Proof of Proposition 3.9

Proof: The coordination game is used to model instantaneous competition without imposing simultaneous action. It is played over infinitely many rounds, each of which lasts an infinitesimal amount of time. If at any round at least one firm invests, the game stops. Otherwise, we move on to the next round. Firm strategies are assumed to be fixed, i.e. stationary, across rounds; namely firm i attempts to invest with probability $p_i(x) \in [0,1]$. Given the strategy profile $(p_1(x), p_2(x))$, the outcome of a given round is that firm 1 invests first with probability $p_1(x)(1-p_2(x))$, firm 2 invests first with probability $(1-p_1(x))p_2(x)$, and both firms invest simultaneously with probability $p_1(x)p_2(x)$. The fourth outcome is that nobody invests and we continue to the next round. Over infinitely many rounds, the game will eventually terminate and the final outcome will be

$$\begin{aligned}
P_{0,1}(x) &= \frac{p_1(x)(1-p_2(x))}{p_1(x)+p_2(x)-p_1(x)p_2(x)} \qquad \text{(firm 1 invests first)},\\
P_{1,0}(x) &= \frac{p_2(x)(1-p_1(x))}{p_1(x)+p_2(x)-p_1(x)p_2(x)} \qquad \text{(firm 2 invests first)},\\
P_{0,0}(x) &= \frac{p_1(x)p_2(x)}{p_1(x)+p_2(x)-p_1(x)p_2(x)} \qquad \text{(simultaneous investment)}.
\end{aligned}$$

Consequently, the NPV of firm 1 is

$$\begin{aligned}
V_{1,1}^1(x) &= P_{0,1}(x)(V_{0,1}^1(x)-K^1) + P_{1,0}(x)V_{1,0}^1(x) + P_{0,0}(x)(D_{0,0}^1(x)-K^1)\\
&= \frac{p_1(x)(V_{0,1}^1(x)-K^1) + p_2(x)V_{1,0}^1(x) - (V_{1,0}^1(x)+V_{0,1}^1(x)-D_{0,0}^1(x))p_1(x)p_2(x)}{p_1(x)+p_2(x)-p_1(x)p_2(x)}.
\end{aligned} \tag{3.78}$$

Differentiating $w.r.t.$ $p_1(x)$, we get

$$\frac{\partial V_{1,1}^1}{\partial p_1}(x) = \frac{p_2(x)(V_{0,1}^1(x)-V_{1,0}^1(x)) - (V_{0,1}^1(x)-D_{0,0}^1(x))p_2^2(x)}{(p_1(x)+p_2(x)-p_1(x)p_2(x))^2},$$

which is free of $p_1(x)$. Finally, we obtain the best-response strategy for firm 1 as

$$\begin{cases}
\text{if } p_2(x) > \frac{V_{0,1}^1-V_{1,0}^1-K^1}{V_{0,1}^1-D_{0,0}^1}(x), & \text{then } p_1^*(x)=0,\\
\text{if } p_2(x) < \frac{V_{0,1}^1-V_{1,0}^1-K^1}{V_{0,1}^1-D_{0,0}^1}(x), & p_1^*(x)=1,\\
\text{if } p_2(x) = \frac{V_{0,1}^1-V_{1,0}^1-K^1}{V_{0,1}^1-D_{0,0}^1}(x), & p_1^*(x)\in(0,1) \quad \text{is free.}
\end{cases}$$

Since in this scenario we have $V_{0,1}^1(x)-K^1 > V_{1,0}^1 > D_{0,0}^1 - K^1$, combining similar results obtained for firm 2, we obtain the three stated equilibrium strategies. ∎

3.4.6 Central Planner Cooperative Monopoly

Denote by $D^M_{n_1,n_2}(x) \triangleq \left(\zeta^1_{n_1,n_2} + \zeta^2_{n_1,n_2}\right) + \frac{\rho^1 Q^1_{n_1,n_2} - \rho^2 Q^2_{n_1,n_2}}{\delta} \cdot x$ the aggregate expected profits for the central planner in stage (n_1, n_2). In stage $(1, 0)$ the payoff is

$$h^M_{1,0}(x) = D^M_{0,0}(X^x_\tau) - D^M_{1,0}(X^x_\tau) - K_1 = \frac{\rho^1 \Delta Q^1_1}{\delta} \cdot x - K^M_{1,0}$$

and using Proposition 3.14 the value function is

$$V^M_{1,0}(x) = \begin{cases} D^M_{1,0}(x) + \frac{F(x)}{F(S^M_{1,0})} \cdot h^M_{1,0}\left(S^M_{1,0}\right), & \text{if } x \le S^M_{1,0}; \\ D^M_{0,0}(x) - K^1, & \text{if } x > S^M_{1,0}, \end{cases} \tag{3.79}$$

where the optimal stopping level $S^M_{1,0}$ satisfies (3.53) after substituting $h^1_{1,0}$ by $h^M_{1,0}$. In stage $(0, 1)$ the payoff is $h^M_{0,1}(x) = -\frac{\rho^2 \Delta Q^2_1}{\delta} \cdot x - K^M_{0,1}$. Using Proposition 3.15 the corresponding value function is:

$$V^M_{0,1}(x) = \begin{cases} D^M_{0,0} - K^2, & \text{if } x \le S^M_{0,1}; \\ D^M_{0,1}(x) + \frac{G(x)}{G(S^M_{0,1})} \cdot h^M_{0,1}(S^M_{0,1}), & \text{if } x > S^M_{0,1}, \end{cases} \tag{3.80}$$

where the optimal stopping level $S^M_{0,1}$ satisfies equation (3.57), substituting $h^2_{0,1}$ by $h^M_{0,1}$.

In stage $(1, 1)$ the payoffs become $h^{1,M}_{1,1}(x) = V^M_{0,1}(x) - D^M_{1,1}(x) - K^1$ and $h^{2,M}_{1,1}(x) = V^M_{1,0}(x) - D^M_{1,1}(x) - K^2$, which can be easily verified to belong to $\mathcal{H}_{\text{inc}}$ and $\mathcal{H}_{\text{dec}}$, respectively. With $\tau^{2,M,*}_{1,1}$ fixed (resp. $\tau^{1,M,*}_{1,1}$), the optimal problem (3.49) converts to an optimal stopping problem with an exit level (3.28), where the function $h^{2,M}_{1,1}$ (resp. $h^{1,M}_{1,1}$) acts as the second-mover payoff. Consequently, the *first-stage policy* of the monopoly is the paired stopping time given by

$$\tau^{1,M,*}_{1,1} = \inf\{t \ge 0 : X_t \ge S^{1,M,*}_{1,1}\}, \qquad \tau^{2,M,*}_{1,1} = \inf\{t \ge 0 : X_t \le S^{2,M,*}_{1,1}\}. \tag{3.81}$$

The overall value function of the central planner in stage (1,1) is:

$$V_{1,1}^{M}(x) = \begin{cases} V_{1,0}^{M}(x), & \text{if } x \in (\underline{d},\, S_{1,1}^{2,M,*}) \\ D_{1,1}^{M}(x) + \omega_M F(x) + \nu_M G(x), & \text{if } x \in (S_{1,1}^{2,M,*},\, S_{1,1}^{1,M,*}) \\ V_{0,1}^{M}(x), & \text{if } x \in (S_{1,1}^{1,M,*},\, \bar{d}) \end{cases} \tag{3.82}$$

where

$$\begin{aligned} \omega_M &= \frac{h_{1,1}^{1,M,*}(S_{1,1}^{1,M,*})G(S_{1,1}^{2,M,*}) - h_{1,1}^{2,M,*}(S_{1,1}^{2,M,*})G(S_{1,1}^{1,M,*})}{F(S_{1,1}^{1,M,*})G(S_{1,1}^{2,M,*}) - F(S_{1,1}^{2,M,*})G(S_{1,1}^{1,M,*})}, \\ \nu_M &= \frac{h_{1,1}^{2,M,*}(S_{1,1}^{2,M,*})F(S_{1,1}^{1,M,*}) - h_{1,1}^{1,M,*}(S_{1,1}^{1,M,*})F(S_{1,1}^{2,M,*})}{F(S_{1,1}^{1,M,*})G(S_{1,1}^{2,M,*}) - F(S_{1,1}^{2,M,*})G(S_{1,1}^{1,M,*})}. \end{aligned}$$

The thresholds $(S_{1,1}^{1,M,*},\, S_{1,1}^{2,M,*})$ solve the system of equations (compare to (3.39))

$$\begin{cases} \left[h_{1,1}^{2,M}(S_2)G(S_1) - h_{1,1}^{1,M}(S_1)G(S_2)\right] F'(S_1) + \left[h_{1,1}^{1,M}(S_1)F(S_2) - h_{1,1}^{2,M}(S_2)F(S_1)\right] G'(S_1) \\ \qquad = \left(h_{1,1}^{1,M}\right)'(S_1)\left[G(S_1)F(S_2) - G(S_2)F(S_1)\right]; \\ \left[h_{1,1}^{2,M}(S_2)G(S_1) - h_{1,1}^{1,M}(S_1)G(S_2)\right] F'(S_2) + \left[h_{1,1}^{1,M}(S_1)F(S_2) - h_{1,1}^{2,M}(S_2)F(S_1)\right] G'(S_2) \\ \qquad = \left(h_{1,1}^{2,M}\right)'(S_2)\left[G(S_1)F(S_2) - G(S_2)F(S_1)\right]. \end{cases} \tag{3.83}$$

The extension to general (n_1, n_2) stage is analogous.

Chapter 4

Stochastic Switching Games

We study nonzero-sum stochastic switching games, which extends our work on multi-stage capacity expansion games discussed in Chapter 3. Rather than a priori restricted number of controls available to the players, here we consider the situation of an infinitely-repeated game. Two players compete for market dominance through controlling (via timing options) the discrete-state market regime M. Switching decisions are driven by a continuous stochastic factor X that modulates instantaneous revenue rates and switching costs. We construct threshold-type Feedback Nash Equilibria which characterize stationary strategies describing long-run dynamic equilibrium market organization. Two sequential approximation schemes link the switching equilibrium to (i) constrained optimal switching; (ii) multi-stage timing games. We provide illustrations using an Ornstein-Uhlenbeck X that leads to a recurrent equilibrium M^* and a Geometric Brownian Motion X that makes M^* eventually "absorbed" as one player eventually gains permanent advantage. Explicit computations and comparative statics regarding the emergent macroscopic market equilibrium are also provided.

4.1 Problem Formulation

We consider two firms, dubbed player $i, j \in \{1, 2\}, i \neq j$, competing on the same market. As discussed, we introduce an exogenous diffusion process $(X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to capture the local fluctuating market condition, which satisfies the following SDE

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t, \tag{4.1}$$

with domain $\mathcal{D} := (\underline{d}, \bar{d})$. We refer Section 2.1.1 for detailed regularity of X.

The macro market regime is described by a discrete-state process (M_t) and represents the relative market dominance of each player. The domain of (M_t) is a finite set $\mathcal{M}$; for simplicity we consider integer-valued M_t and $\mathcal{M} = \{\underline{m}, \underline{m}+1, \ldots, \overline{m}\}$. The players exercise switching-type controls to *enhance* their market dominance; thus, Player 1 can *increase* M_t by +1, and Player 2 can *decrease* M_t by -1. To exercise a switch, player i must pay a cost $K^i(X_t, M_t)$. Note that a switch by Player 1, followed by a switch by Player 2 completely neutralize each other and bring the market to its original state. The interpretation of M_t as a relative dominance can be motivated by taking $M_t = M_t^1 - M_t^2$, where $M_t^i \in \mathbb{N}$ represents the production capacity, or technology level of firm i. Thus, players repeatedly make competing investments to increase their capacity; investments by Player 1 raise M_t and those by Player 2 lower it.

To match the intuition about the role of (M_t), we postulate that: (i) Player 1 (resp. P2) is dominant when $M_t > 0$ (resp. $M_t < 0$); (ii) Player 1 (resp. P2) prefers *higher* (resp. *lower*) X_t. The last assumption creates a positive feedback effect between X and M: as X rises, Player 1 gets more motivated to enhance her market dominance, eventually triggering her to act and make M_t higher too; when X falls sufficiently Player

2 gains short-term advantage and moves M_t towards her preferred negative direction.

By way of illustration, we consider the following two representative examples:

Example 4.1 (Mean-reverting Advantage) *Local market fluctuations are mean-reverting, modeled by an Ornstein-Uhlenbeck process*

$$dX_t = \mu(\theta - X_t)dt + \sigma dW_t,$$

with $\mathcal{D} = \mathbb{R}$, $\mu, \sigma > 0$ and $\theta \in \mathbb{R}$. Thus, the long-run market is stationary and market organization is expected to undergo a cyclical behavior as X stochastically oscillates around θ. The players receive constant profit rates based on deterministic profit ladders π^i_m that are independent of X_t with

$$\pi^1_m < \pi^1_{m+1} \qquad \text{and} \qquad \pi^2_m > \pi^2_{m+1} \quad \forall m.$$

Thus, Player 1 maximizes her revenue when M_t is high and Player 2 when M_t is low. In complement, the present market conditions X_t affect the switching or investment costs. Thus, when X_t is high/low, K^1 is low/high (K^2 is high/low). Economically this could be interpreted as X representing exchange rate, with dollar-denominated investment costs both for the domestic firm P1 and foreign firm P2. For concreteness, we suppose the switching costs are exponential in X_t:

$$K^i(x, m) = c^i(m) + \alpha^i(m)e^{\beta^i(m)\cdot x}, \qquad i = 1, 2,$$

where $c^i(m), \alpha^i(m) > 0$, $\beta^1(m) < 0$ and $\beta^2(m) > 0$.

Example 4.2 (Long-run Advantage) *In the second example we suppose that in the long-run one player will possess the competitive advantage and become dominant. How-*

ever, in the medium-term fluctuations X creates uncertainty in M. This is captured by using a Geometric Brownian Motion (GBM) for X

$$dX_t = \mu X_t\, dt + \sigma X_t\, dW_t, \tag{4.2}$$

with $\mathcal{D} = (0, +\infty)$, $\mu \in \mathbb{R}$ and $\sigma > 0$. The players receive profit rates according to predetermined profit ladders π^i_m as well; for the sake of diversity we use linear switching costs,

$$K^i(x, m) = \left[c^i(m) + \beta^i(m) \cdot x\right]_+, \qquad i = 1, 2$$

where $\beta^1(m) < 0$ and $\beta^2(m) > 0$.

In line with the discrete nature of M we postulate the players adopt timing strategies, denoted by $\boldsymbol{\alpha}^i := \{\tau^i(n) : n \geq 1\}, i \in \{1, 2\}$ where τ^i are certain stopping times. Admissibility of $\boldsymbol{\alpha}^i$'s is defined recursively as introduced in Definition 2.2, where the players' acting sets at a regime m are as follows

$$C^1_m = \begin{cases} m+1, & \text{if } m < \bar{m}, \\ \emptyset, & \text{if } m = \bar{m}, \end{cases} \qquad C^2_m = \begin{cases} m-1, & \text{if } m > \underline{m}, \\ \emptyset, & \text{if } m = \underline{m}. \end{cases} \tag{4.3}$$

We suppose the players aim to maximize their expected future (discounted) profits on $[0, \infty)$ defined through revenue rates π^i's that are driven by (X_t, M_t). The integrated total profit, i.e. the *game payoff*, is then given by $\int_0^\infty e^{-rs} \pi^i(X_s, M_s) ds$ minus the net present value of switching costs. Given a strategy profile $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$, the NPV of future profits received by player i is

$$\begin{aligned} J^i_m(x;\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2) &:= \mathbb{E}\Bigg[-\sum_{n=1}^{\infty}\mathbb{1}_{\{P_n=i\}}e^{-r\sigma_n}\cdot K^i\big(X_{\sigma_n},\tilde{M}_{n-1}\big) \\ &\qquad\qquad + \int_0^\infty e^{-rt}\pi^i\big(X_t,\tilde{M}_{\eta(t)}\big)dt\,\Bigg|\,X_0=x,M_0=m\Bigg], \\ &= \mathbb{E}\Bigg[-\sum_k e^{-r\sigma^i_k}K^i\big(X_{\sigma^i_k},\tilde{M}_{\eta(i,k)-1}\big) \\ &\qquad\qquad + \int_0^\infty e^{-rt}\pi^i\big(X_t,M_t\big)dt\,\Bigg|\,X_0=x,M_0=m\Bigg] \end{aligned} \tag{4.4}$$

where $r>0$ is the constant discount rate. Let us also introduce the static discounted future cashflows

$$D^i_m(x) := \mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i(X_t,m)dt\,\middle|\,X_0=x\right], \tag{4.5}$$

which are assumed to satisfy the growth condition $D^i_m(x)\le C(1+|x|)$ for $i\in\{1,2\}$ and all $m\in\mathcal{M}$. Because switching costs are non-negative, game payoffs are also of linear growth since they are dominated by D^i's, in particular $J^1_m(x)\le D^1_{\overline{m}}(x)$ while $J^2_m(x)\le D^2_{\underline{m}}(x)$.

The Nash equilibrium criterion stated in Definition 2.4 characterizes equilibrium strategies as a fixed point of each player's *best-response* to her rival's strategy. Specifically, given an arbitrary rival's strategy $\boldsymbol{\alpha}^j$ define the resulting best-response payoff of player i

$$\widetilde{V}^i_m(x\,;\boldsymbol{\alpha}^j) := \sup_{\{\boldsymbol{\alpha}^i:(\boldsymbol{\alpha}^i,\boldsymbol{\alpha}^j)\in\mathcal{A}\}} J^i_m(x;\boldsymbol{\alpha}^i,\boldsymbol{\alpha}^j), \qquad x\in\mathcal{D}, m\in\mathcal{M}. \tag{4.6}$$

Because (taking Player 1 as an example) game payoffs satisfy $D^1_{\overline{m}}(x)\ge\widetilde{V}^1_m(x;\boldsymbol{\alpha}^2)\ge J^1_m(x;\bar{\boldsymbol{\alpha}}^1,\boldsymbol{\alpha}^2)\ge D^1_{\underline{m}}(x)$, such best-response values are always well-defined. Equilibrium payoffs then satisfy:

$$V_m^i(x) = \widetilde{V}_m^i(x\,;\boldsymbol{\alpha}^{j,*}), \quad i \in \{1,2\}, \qquad j \neq i. \tag{4.7}$$

4.2 Constructing Equilibria

We now focus on the special class of *stationary* and *threshold-type* strategies as introduced in Section 2.1, which allow us to explicitly construct a MNE (see Definition 2.4). To do so, two key properties are needed. First, one must show that this class of strategies is closed under the best-response map (4.6). Second, a verification theorem is needed to show that the resulting fixed point of (4.7), defined through a system of equations, is indeed a MNE of the game. The programme starts in Section 4.2.1 where we define threshold-type strategies and then characterize the best-response to such strategies as a solution to a system of coupled optimal stopping problems. Next, in Section 4.2.2 we state the verification theorem which provides a system of nonlinear equations for the equilibrium threshold vectors $\boldsymbol{s}^{1,*}, \boldsymbol{s}^{2,*}$. Lastly in Section 4.2.3 we study the emerging equilibrium macro state M^*.

4.2.1 Stationary and Threshold-type Strategies

Recall that Player 1 is in favor of high X_t and large M_t, while Player 2 prefers the opposite; it is therefore natural to assume that P1 switches up when X becomes high enough and P2 switches down when X becomes low enough. Following the idea of a similar construction in [2], we define a strategy of player $i \in \{1,2\}$ by $\boldsymbol{\alpha}^i := (\Gamma_m^i)_{m\in\mathcal{M}}$, where Γ_m^i's are *threshold-type* subsets of $\mathcal{D}$ introduced in Definition 2.5. Given a strategy profile $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$, a sequence of switches is uniquely determined as follows:

— when $M_t = m$, player i adopts the (feedback) switching region Γ_m^i: player i exercises a switch (changes M_t^i by ± 1) at the first hitting time τ_m^i of (X_t) to Γ_m^i (with the convention that the hitting time of an empty set is ∞);

— if both players want to switch, Player 1 has the priority.

Admissibility of the strategy profile $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$ in Definition 2.2 now reduces to

— $\Gamma_{\overline{m}}^1 = \Gamma_{\underline{m}}^2 = \emptyset$ $(M_t \in \mathcal{M})$

— $\Gamma_m^1 \cap \Gamma_{m+1}^2 = \emptyset$ for $m < \overline{m}$ and $\Gamma_{m-1}^1 \cap \Gamma_m^2 = \emptyset$ for $m > \underline{m}$. This rules out simultaneous switching loops; for instance if there were an $x \in \Gamma_m^1 \cap \Gamma_{m+1}^2$ then starting in regime m, we would have that P1 switches up to $m+1$, but them immediately P2 switches back down to m, generating an infinite sequence of instantaneous switches.

Relying on the resulting Markov structure of threshold-type strategies, we revisit the formal game evolution which can now be constructed using independent auxiliary copies $\tilde{X}^{(n)}, n = 1, \dots,$ of the strong Markov X. Below, X^x denotes the X-process started at $X_0 = x$. Let $x \in \mathcal{D}$, $m \in \mathcal{M}$, and a strategy profile $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2) \in \mathcal{A}$. Set $\sigma_0 = 0$, $X_0 = x$ and $\tilde{M}_0 = m$. For $n \geq 0$, define

$$\tilde{X}_t^{(n)} = X_{\sigma_n + t}^x, \qquad \text{for } t \geq 0, \tag{4.8a}$$

$$\tilde{\tau}^{i,n} = \inf\{s \geq 0 \,:\, \tilde{X}_s^{(n)} \in \Gamma_{\tilde{M}_n}^i\}, \qquad i \in \{1, 2\}, \tag{4.8b}$$

$$\sigma_{n+1} = \sigma_n + \tilde{\tau}^{1,n} \wedge \tilde{\tau}^{2,n}, \tag{4.8c}$$

$$P_{n+1} = 1 \cdot \mathbb{1}_{\{\tilde{\tau}^{1,n} < \tilde{\tau}^{2,n}\}} + 2 \cdot \mathbb{1}_{\{\tilde{\tau}^{1,n} > \tilde{\tau}^{2,n}\}} + \mathcal{H}_{n+1} \mathbb{1}_{\{\tilde{\tau}^{1,n} = \tilde{\tau}^{2,n}\}}, \tag{4.8d}$$

$$\tilde{M}_{n+1} = \tilde{M}_n + 1 \cdot \mathbb{1}_{\{P_{n+1}=1\}} - 1 \cdot \mathbb{1}_{\{P_{n+1}=2\}}. \tag{4.8e}$$

Then the evolution of (M_t) and the sequence of switching times of each player $(\sigma_k^i)_{k \geq 1}$ are obtained as in (2.3) and (2.4). The strong Markov property of X implies that each

$\tilde{X}^{(n)}$ can be considered as a fresh (independent) copy of X starting at $\tilde{X}_0^{(n)} = X^x_{\sigma_n}$. Consequently, given these players' strategies, the pair (X_t, M_t) is Markovian.

In Figure 4.1 we sketch the emerging equilibrium based on threshold-type strategies associated to one of our case studies. The players make a switch whenever the process (X_t) hits the threshold $s^{1,*}_m$ from below or $s^{2,*}_m$ from above when at stage m, see the dashed lines in the bottom plot. The switching times σ^i_k are described through the respective hitting times. The top panel shows the resulting macro stage (M^*_t) driven by σ^i_k's along one realized trajectory of the local market fluctuations (X_t). These players are "at equal strength" in the beginning, $M^*_0 = 0$; as (X_t) drops, it enters Player 2's switching region first $(\tau^2(1) < \tau^1(1))$ leading her to exercise a switch and change $M^*_{\sigma^2_1} = -1$. The players then recursively wait for (X_t) to hit either the threshold $s^{1,*}_{-1}$ or $s^{2,*}_{-1}$ $(\tau^1(2) \wedge \tau^2(2))$, to make further switches.

Note that in the above definition we require Γ^i_m to be connected, so that they are fully characterized by their boundary s^i_m. In turn, threshold-type strategies allow to move from looking at the unstructured (in the sense of optimization) switching strategies

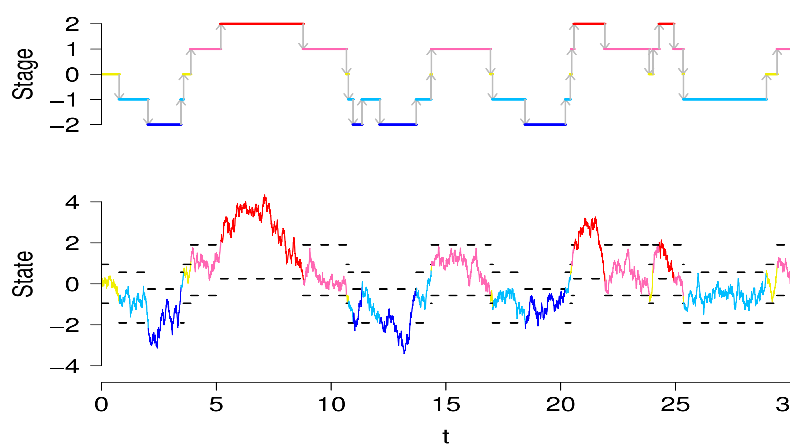


Figure 4.1: A trajectory of X and equilibrium M^* starting at $X_0 = 0$, $M^*_0 = 0$. Here X is an Ornstein-Uhlenbeck process and $\mathcal{M} = \{-2, -1, 0, 1, 2\}$. The equilibrium strategies are of *threshold-type*; the dashed lines in the bottom plot indicate the respective switching thresholds.

defined by general Γ^i_m to searching for equilibria parametrized by the $|\mathcal{M}|$-vectors $\boldsymbol{s}^1, \boldsymbol{s}^2$. In particular, this reduces the search for MNE to a $2|\mathcal{M}|$-dimensional setting where numerical resolution becomes possible. Towards this goal, the main aim in this section is to constructively find such threshold equilibria.

As discussed, given a *threshold-type* strategy $\boldsymbol{\alpha}^j \equiv \boldsymbol{s}^j$ of player j, we expect the best-response strategy of player i to be consistently of threshold-type (see Corollary 4.8). The Dynamic Programming Principle (DPP) implies that her corresponding value function, $\widetilde{V}^i(\cdot; \boldsymbol{s}^j)$ defined in (4.6), solves a system of coupled stopping problems (2.8). To approach the coupled system, we first consider the corresponding generic local constrained optimal stopping problem (which uncouples (2.8) by removing $\widetilde{V}^i_{m-1}, \widetilde{V}^i_{m+1}$ from the right-hand-side) and then the game equilibrium that is characterized as the best response to $\boldsymbol{s}^{j,*}$. See [14] for a related analysis of unconstrained optimal switching problems.

Remark 4.3 *(Boundary Stages) Recall that admissible strategies defined in Definition 2.2 imply Player 1 (resp. Player 2) cannot make any switches at stage $\overline{m}$ (resp. at stage $\underline{m}$). In terms of threshold-type strategies this is equivalent to simply taking $s^1_{\overline{m}} = \overline{d}$ and $s^2_{\underline{m}} = \underline{d}$ which can be viewed as a constraint on possible admissible controls.*

To find the best-response of player i, we consider a *local* optimal stopping problem of the form

$$\widetilde{v}^i(x; \tau^j) = \sup_{\tau^i \in \mathcal{T}} \mathbb{E}_x \left[\mathbb{1}_{\{\tau^i < \tau^j\}} e^{-r\tau^i} h^i(X_{\tau^i}) + \mathbb{1}_{\{\tau^i > \tau^j\}} e^{-r\tau^j} l^i(X_{\tau^j}) + \mathbb{1}_{\{\tau^i = \tau^j\}} e^{-r\tau^i} g^i(X_{\tau^i}) \right], \tag{4.9}$$

where τ^j is a given stopping time, $h^i(\cdot)$ is the *leader* payoff from switching before τ^j, and $l^i(\cdot)$ is the *follower* payoff from switching after τ^j. $g^i(\cdot)$ denotes the payoff of player i when both players want to switch simultaneously. In our setting, $g^1 = h^1$, while $g^2 = l^2$ due to the priority of Player 1.

In order to obtain threshold-type equilibrium, one would expect the optimizer to (4.9), $\tilde{\tau}^i$ to be of threshold-type, given τ^j is of threshold-type. However, as discussed at length in Section 3.2.2 this is not always true. If player j behaves aggressively, player i would try to preempt right before, leading to lack of optimal τ^i.

Assumption 4.4 *(i) The exogenous stopping time τ^j is of threshold-type,*

$$\tau^j := \inf\{t \geq 0 \,:\, X_t \in \Gamma^j\}, j \in \{1,2\}, \quad \textit{with } \Gamma^1 := [s^1, \overline{d}) \textit{ and } \Gamma^2 := (\underline{d}, s^2].$$

(ii) $h^1 \in \mathcal{H}_{inc}$ and $h^2 \in \mathcal{H}_{dec}$ introduced in Definition 2.7.

(iii) player i is not incentivized to preempt at s^j, i.e. $h^i(s^j) < l^i(s^j)$.

Under the above assumptions, it is known that the solution of (4.9) is of threshold-type. Specifically, this can be established using the smallest concave majorant method, see e.g. [29, 30]. Let us remark that Assumption 4.4 (iii) is essential for this result and would be hard to check in the sequel. Nevertheless, if the rest of Assumption 4.4 is fulfilled, there exists *uniquely* a *preemptive* best-response, see Section 3.2.2.

Proposition 4.5 *Suppose that all conditions of Assumption 4.4 are satisfied. Let F, G be the solutions to $(\mathcal{L} - r)u = 0$, where $\mathcal{L}$ is the infinitesimal generator of X. Set*

$$W(x_1, x_2) := F'(x_1)G(x_2) - F(x_1)G'(x_2) \tag{4.10}$$

$$\mathcal{W}(x_1, x_2) := F(x_1)G(x_2) - F(x_2)G(x_1). \tag{4.11}$$

Then the value function of (4.9) *is admitted as*

$$\widetilde{v}^i(x;\tau^j) = \begin{cases} h^i(x), & \textit{for } x \in \Gamma^i, \\ l^i(x), & \textit{for } x \in \Gamma^j, \\ \widetilde{\omega}^i F(x) + \widetilde{\nu}^i G(x), & \textit{for } x \in \mathcal{D} \setminus (\Gamma^i \cup \Gamma^j), \end{cases}$$

where the optimal stopping region $\Gamma^i = \Gamma(\tilde{s}^i)$ *is of threshold-type and defined uniquely through the threshold* $\tilde{s}^i := \tilde{s}^i(s^j)$ *(with* $\tilde{s}^1 > s^2$ *and* $\tilde{s}^2 < s^1$*) that satisfies*

$$h^i(\tilde{s}^i)W(\tilde{s}^i, s^j) - l^i(s^j)W(\tilde{s}^i, \tilde{s}^i) - (h^i)'(\tilde{s}^i)\mathcal{W}(\tilde{s}^i, s^j) = 0. \tag{4.12}$$

The coefficients $\widetilde{\omega}^i := \widetilde{\omega}^i(\tilde{s}^i, s^j)$ *and* $\widetilde{\nu}^i := \widetilde{\nu}^i(\tilde{s}^i, s^j)$ *are defined as*

$$\widetilde{\omega}^i = \frac{h^i(\tilde{s}^i)G(s^j) - l^i(s^j)G(\tilde{s}^i)}{\mathcal{W}(\tilde{s}^i, s^j)}, \qquad \widetilde{\nu}^i = \frac{l^i(s^j)F(\tilde{s}^i) - h^i(\tilde{s}^i)F(s^j)}{\mathcal{W}(\tilde{s}^i, s^j)}. \tag{4.13}$$

Moreover, the coefficient $\widetilde{\omega}^i$ in Proposition 4.5 corresponds to the slope of the straight line segment and $\widetilde{\nu}^i$ corresponds to the y-intercept (see Figure 2.1b). From the fact that being a follower is assumed to sub-optimal, it follows that $\widetilde{\omega}^1, \widetilde{\nu}^2 \geq 0$ and $\widetilde{\nu}^1, \widetilde{\omega}^2 \leq 0$.

Remark 4.6 *The above proposition subsumes the case where only one player is able to act. In this situation we may simply take* $s^1 = \overline{d}$ *or* $s^2 = \underline{d}$*, and player* i *then effectively solves a standard optimal stopping problem as a special case of* (4.9)*. See related discussion in Section 3.2.1. These cases arise in the boundary stages* $\underline{m}, \overline{m}$ *associated to* (2.8).

4.2.2 Best-response Verification Theorem

By construction of MNEs in Definition 2.4, game payoffs and threshold-type strategies associated to an equilibrium necessarily solve the local optimizing problems stated in (2.8). Moreover, they necessarily are fixed-points to the following pair of optimizing problems at each regime m:

$$\begin{cases} V^1_m(x) = & \sup_{\tau^1_m \in \mathcal{T}} \mathbb{E}_x \Bigg[\int_0^{\mathcal{I}_m} e^{-rt} \pi^1_m(X_t) dt + e^{-r\mathcal{I}_m} \mathbb{1}_{\{\tau^1_m > \tau^{2,*}_m\}} \Big(V^1_{m-1}(X_{\tau^{2,*}_m}) \Big) \\ & \qquad + e^{-r\mathcal{I}_m} \mathbb{1}_{\{\tau^1_m \le \tau^{2,*}_m\}} \Big(V^1_{m+1}(X_{\tau^1_m}) - K^1_m(X_{\tau^1_m}) \Big) \Bigg], \\ V^2_m(x) = & \sup_{\tau^2_m \in \mathcal{T}} \mathbb{E}_x \Bigg[\int_0^{\mathcal{I}_m} e^{-rt} \pi^2_m(X_t) dt + e^{-r\mathcal{I}_m} \mathbb{1}_{\{\tau^{1,*}_m > \tau^2_m\}} \Big(V^2_{m-1}(X_{\tau^2_m}) - K^2_m(X_{\tau^2_m}) \Big) \\ & \qquad + e^{-r\mathcal{I}_m} \mathbb{1}_{\{\tau^{1,*}_m \le \tau^2_m\}} \Big(V^2_{m+1}(X_{\tau^{1,*}_m}) \Big) \Bigg], \end{cases} \tag{4.14}$$

where $\tau^{1,*}_m, \tau^{2,*}_m$ are the stopping times associated to the thresholds $s^{1,*}_m, s^{2,*}_m$. Comparing to the generic problem in (4.9) and subtracting $D^i_m(x) = \mathbb{E}_x \left[\int_0^\infty e^{-rt} \pi^i_m(X_s) dt \right]$, we then wish to set

$$\begin{cases} h^1_m(x) := V^1_{m+1}(x) - D^1_m(x) - K^1_m(x), \\ l^1_m(x) := V^1_{m-1}(x) - D^1_m(x), \end{cases} \qquad \begin{cases} h^2_m(x) := V^2_{m-1}(x) - D^2_m(x) - K^2_m(x), \\ l^2_m(x) := V^2_{m+1}(x) - D^2_m(x). \end{cases} \tag{4.15}$$

Plugging above into (4.12) and (4.13) for all m and combining, we obtain a coupled nonlinear system of $s^i_m, \omega^i_m, \nu^i_m$ in (4.18), whose solutions are expected to be a MNE of the switching game. First for each $m \notin \{\underline{m}, \overline{m}\}$ there are 6 equations:

$$\begin{cases} \left\{V^1_{m+1} - D^1_m - K^1_m\right\}(s^{1,*}_m) \cdot W(s^{1,*}_m, s^{2,*}_m) - \left\{V^1_{m-1} - D^1_m\right\}(s^{2,*}_m) \cdot W(s^{1,*}_m, s^{1,*}_m) \\ \qquad - \left\{V^1_{m+1} - D^1_m - K^1_m\right\}'(s^{1,*}_m) \cdot \mathcal{W}(s^{1,*}_m, s^{2,*}_m) = 0, \\ \left\{V^1_{m+1} - D^1_m - K^1_m\right\}(s^{1,*}_m) \cdot G(s^{2,*}_m) - \left\{V^1_{m-1} - D^1_m\right\}(s^{2,*}_m) \cdot G(s^{1,*}_m) \\ \qquad -\omega^1_m \cdot \mathcal{W}(s^{1,*}_m, s^{2,*}_m) = 0, \\ \left\{V^1_{m-1} - D^1_m\right\}(s^{2,*}_m) \cdot F(s^{1,*}_m) - \left\{V^1_{m+1} - D^1_m - K^1_m\right\}(s^{1,*}_m) \cdot F(s^{2,*}_m) \\ \qquad -\nu^1_m \cdot \mathcal{W}(s^{1,*}_m, s^{2,*}_m) = 0, \end{cases} \tag{4.16a}$$

$$\begin{cases} \left\{V^2_{m-1} - D^2_m - K^2_m\right\}(s^{2,*}_m) \cdot W(s^{2,*}_m, s^{1,*}_m) - \left\{V^2_{m+1} - D^2_m\right\}(s^{1,*}_m) \cdot W(s^{2,*}_m, s^{2,*}_m) \\ \qquad - \left\{V^2_{m-1} - D^2_m - K^2_m\right\}'(s^{2,*}_m) \cdot \mathcal{W}(s^{1,*}_m, s^{2,*}_m) = 0, \\ \left\{V^2_{m-1} - D^2_m - K^2_m\right\}(s^{2,*}_m) \cdot G(s^{1,*}_m) - \left\{V^2_{m+1} - D^2_m\right\}(s^{1,*}_m) \cdot G(s^{2,*}_m) \\ \qquad -\omega^2_m \cdot \mathcal{W}(s^{2,*}_m, s^{1,*}_m) = 0, \\ \left\{V^2_{m+1} - D^2_m\right\}(s^{1,*}_m) \cdot F(s^{2,*}_m) - \left\{V^2_{m-1} - D^2_m - K^2_m\right\}(s^{2,*}_m) \cdot F(s^{1,*}_m) \\ \qquad -\nu^2_m \cdot \mathcal{W}(s^{2,*}_m, s^{1,*}_m) = 0, \end{cases} \tag{4.16b}$$

where F and G are the solutions to the ODE (2.12), and $W(\cdot,\cdot), \mathcal{W}(\cdot,\cdot)$ are from (4.10). In boundary regimes $m = \underline{m}$ and $m = \overline{m}$ we have the following systems of 3 equations:

$$\begin{cases} s^{1,*}_{\overline{m}} = \overline{d}, \omega^1_{\overline{m}} = 0,\ \nu^1_{\underline{m}} = 0, \\ \left\{V^1_{\overline{m}-1} - D^1_{\overline{m}}\right\}(s^{2,*}_{\overline{m}}) - \nu^1_{\overline{m}} \cdot G(s^{2,*}_{\overline{m}}) = 0, \\ \left\{V^1_{\underline{m}+1} - D^1_{\underline{m}} - K^1_{\underline{m}}\right\}(s^{1,*}_{\underline{m}}) \cdot F'(s^{1,*}_{\underline{m}}) - \left\{V^1_{\underline{m}+1} - D^1_{\underline{m}} - K^1_{\underline{m}}\right\}'(s^{1,*}_{\underline{m}}) \cdot F(s^{1,*}_{\underline{m}}) = 0, \\ \left\{V^1_{\underline{m}+1} - D^1_{\underline{m}} - K^1_{\underline{m}}\right\}(s^{1,*}_{\underline{m}}) - \omega^1_{\underline{m}} \cdot F(s^{1,*}_{\underline{m}}) = 0, \end{cases} \tag{4.17a}$$

$$\begin{cases} s^{2,*}_{\underline{m}} = \underline{d},\ \omega^2_{\overline{m}} = 0,\ \nu^2_{\underline{m}} = 0, \\ \left\{V^2_{\overline{m}-1} - D^2_{\overline{m}} - K^2_{\overline{m}}\right\}(s^{2,*}_{\overline{m}}) \cdot G'(s^{2,*}_{\overline{m}}) - \left\{V^2_{\overline{m}+1} - D^2_{\overline{m}} - K^2_{\overline{m}}\right\}'(s^{2,*}_{\overline{m}}) \cdot G(s^{2,*}_{\overline{m}}) = 0, \\ \left\{V^2_{\overline{m}-1} - D^2_{\overline{m}} - K^2_{\overline{m}}\right\}(s^{2,*}_{\overline{m}}) - \nu^2_{\overline{m}} \cdot G(s^{2,*}_{\overline{m}}) = 0, \\ \left\{V^2_{\underline{m}+1} - D^2_{\underline{m}}\right\}(s^{1,*}_{\underline{m}}) - \omega^2_{\underline{m}} \cdot F(s^{1,*}_{\underline{m}}) = 0. \end{cases} \tag{4.17b}$$

We now propose a verification theorem which confirms that this is indeed the case. Our proof in Section 4.5.1 follows the methods in [2] who considered nonzero-sum games with impulse controls.

Theorem 4.7 (Verification Theorem) *Let* $\Gamma^{1,*}_m := [s^{1,*}_m, \overline{d}), \Gamma^{2,*}_m := (\underline{d}, s^{2,*}_m],\ s^{1,*}_m > s^{2,*}_m$

and $\omega^1_m \geq 0, \omega^2_m \leq 0, \nu^1_m \leq 0, \nu^2_m \geq 0$. *Define*

$$V^1_m(x) = \begin{cases} V^1_{m+1}(x) - K^1_m(x), & \text{for } x \in \Gamma^{1,*}_m, \\ V^1_{m-1}(x), & \text{for } x \in \Gamma^{2,*}_m, \\ D^1_m(x) + \omega^1_m F(x) + \nu^1_m G(x), & \text{for } x \in \mathcal{D} \setminus \left(\Gamma^{1,*}_m \cup \Gamma^{2,*}_m\right), \end{cases} \tag{4.18a}$$

$$V^2_m(x) = \begin{cases} V^2_{m+1}(x), & \text{for } x \in \Gamma^{1,*}_m, \\ V^2_{m-1}(x) - K^2_m(x), & \text{for } x \in \Gamma^{2,*}_m, \\ D^2_m(x) + \omega^2_m F(x) + \nu^2_m G(x), & \text{for } x \in \mathcal{D} \setminus \left(\Gamma^{1,*}_m \cup \Gamma^{2,*}_m\right). \end{cases} \tag{4.18b}$$

Assume that (cf. Assumption 4.4)

– $D^1_{m+1} - D^1_m - K^1_m \in \mathcal{H}_{inc}$ *for* $m < \overline{m}$, *and* $D^2_{m-1} - D^2_m - K^2_m \in \mathcal{H}_{dec}$ *for* $m > \underline{m}$;

– $V^1_{m-1}(s^{2,*}) \geq V^1_{m+1}(s^{2,*}_m) - K^1_m(s^{2,*}_m)$, *for* $m > \underline{m}$, *and* $V^2_{m+1}(s^{1,*}) \geq V^2_{m-1}(s^{1,*}_m) - K^2_m(s^{1,*}_m)$, *for* $m < \overline{m}$;

– *thresholds* $s^{i,*}_m$ *and coefficients* ω^i_m, ν^i_m, $i \in \{1,2\}$, $m \in \mathcal{M}$ *satisfy a system of non-linear equation stated in* (4.16) - (4.17).

Then, $(\boldsymbol{s}^{1,*}, \boldsymbol{s}^{2,*}) := (\Gamma^{1,*}_m, \Gamma^{2,*}_m)_{m\in\mathcal{M}}$ *is a Markov Nash Equilibrium, and* V^i*'s in* (4.18) *are the corresponding equilibrium payoffs.*

We slightly abuse the notation in (4.18) as $V^1_{\overline{m}+1}$ and $V^2_{\underline{m}+1}$ do not exist. However, since in fact $s^{1,*}_{\overline{m}} = \overline{d}$ and $s^{2,*}_{\underline{m}} = \underline{d}$, so that $\Gamma^{1,*}_{\overline{m}} = \Gamma^{2,*}_{\underline{m}} = \emptyset$, the respective equations in (4.16) - (4.17) are indeed well-defined.

The proof of Theorem 4.7 can be repeated to obtain an analogous verification theorem for the system of equations corresponding to the best-response value function $\widetilde{V}^i_m(x\,; \boldsymbol{s}^j)$ as defined in (2.8) for any threshold-type rival strategy $\boldsymbol{s}^j$:

Corollary 4.8 *Let $\boldsymbol{s}^2$ be the fixed switching thresholds of P2 and $\widetilde{V}^1_\cdot(\cdot;\boldsymbol{s}^2)$ be constructed as in* (4.18a)*. Suppose that*

— $D^1_{m+1} - D^1_m - K^1_m \in \mathcal{H}_{inc}$ *for* $m < \overline{m}$*;*

— $\widetilde{V}^1_{m-1}(s^2_m;\boldsymbol{s}^2) \geq \widetilde{V}^1_{m+1}(s^2_m;\boldsymbol{s}^2) - K^1_m(s^2_m)$*, for* $m > \underline{m}$*;*

— $(\tilde{\boldsymbol{s}}^1, \boldsymbol{s}^2, \tilde{\boldsymbol{\omega}}^1, \tilde{\boldsymbol{\nu}}^1)$ *is a solution to* (4.16a) *&* (4.17a).

Then $\tilde{\boldsymbol{s}}^1 \equiv \tilde{\boldsymbol{s}}^1(\boldsymbol{s}^2)$ are the best-response thresholds, and $\widetilde{V}^1_m(x)$ are the corresponding best-response value function of P1.

Theorem 4.7 provides a direct approach to find a MNE of the switching game via solving the system of equations for the threshold vectors $\boldsymbol{s}^i$ and equilibrium payoffs defined through $\boldsymbol{\omega}^i$ and $\boldsymbol{\nu}^i$. Unfortunately, because this is a large system of equations (namely there are $6|\mathcal{M}-1|$ equations in total), the latter is non-trivial even numerically. In particular, most standard root-finding algorithms require a reasonable initial guess. In our experience, providing such a guess is not easy, so that the high-dimensional optimization algorithm frequently does not converge. Thus, in Section 4.3 we propose two approaches to obtain threshold vectors and game payoffs *close* to those in equilibrium.

Remark 4.9 *As discussed in Section 2.1.1, players are allowed to act on M_t in multiple ways in a more general setting. When C^i_m has multiple elements, the corresponding player must choose* how to switch, *not just* when. *In the latter case we need to specify the respective switching costs, i.e. to consider $K^i(m,m')$ which defines the cost of switching from m to m'. Such an extension can be handled by replacing the leader payoff in* (4.15) *with* $h^1_m(x) = \max_{m'\in C^1_m}[V^1_{m'}(x) - D^1_m(x) - K^1(x,m,m')]$ *and the follower payoff with* $\ell^1_m(x) = V^1_{m'}(x) - D^1_m(x)$*, where* $m' = \arg\max\{m'' \in C^2_m : V^2_{m'}(s^2_m) - D^2_m - K^2(s^2_m, m, m')\}$*. The above* max-*terms resemble the intervention operators in impulse control.*

4.2.3 Equilibrium Macro Dynamics

The macro market evolution M^* emerging in equilibrium is a time inhomogeneous *non-Markovian* process with discrete state space $\mathcal{M}$. Thanks to the stationary nature of the threshold-type strategies, the behavior of M^* is highly tractable and is the subject of this subsection.

Recall that in (4.8e) we define the sequence of regimes M^* traverses, i.e. $\tilde{M}^*_n \equiv M^*_{\sigma_n}$. According to (4.8e), $\tilde{M}^*_n$ has memory: the next transition of $\tilde{M}^*_n$ is affected by the last transition. For example, if $\tilde{M}^*_n = +1$ and the previous regime was $\tilde{M}^*_{n-1} = +2$, this implies that the latest switch was due to Player 2, and hence we begin the sojourn in regime $+1$ at location $s^{2,*}_{+2}$, i.e. $\tilde{X}^{(n)}_0 = X^x_{\sigma_n} = s^{2,*}_{+2}$, while if the previous state was $\tilde{M}^*_{n-1} = 0$ then it was Player 1 who switched last and we begin the sojourn at $s^{1,*}_0$, i.e. $\tilde{X}^{(n)}_0 = X^x_{\sigma_n} = s^{1,*}_0$.

To capture this 1-step memory we define the extended state space

$$E := \{\underline{m}^-, (\underline{m}+1)^-, (\underline{m}+1)^+, \cdots, m^-, m^+, \cdots, (\overline{m}-1)^-, (\overline{m}-1)^+, \overline{m}^+\} \cup \{\underline{m}^a, \overline{m}^a\}, \tag{4.19}$$

where the superscript "+" corresponds to the previous transition being made by Player 1 ("up move in M") and "−" corresponds to Player 2 making a "down move in M". We discuss the last two states $\underline{m}^a$, $\overline{m}^a$ below.

Instead of M^*_t we now define its extended jump chain $\check{M}_n$ that takes values in E and represents $(\tilde{M}^*_{n-1}, \tilde{M}^*_n)$. Note that $\check{M}_0$ is undefined, as we need to know the previous transition to know the state of $\check{M}$. Let us use Figure 1.3 to explain how $\check{M}$ behaves. The macro market starts at $X_0 = 0$ and $M^*_0 = 0$, while $\check{M}^*$ starts when (X_t) hits $s^{2,*}_0$ with $\check{M}^*_1 = (-1)^-$. The first sojourn begins at $s^{2,*}_0$ and ends when (X_t) hits $s^{2,*}_{-1}$, leading us to $\check{M}^*_2 = (-2)^-$, and so forth.

We proceed to compute the qualitative behavior of M^* via $\check{M}_n$. In the case that X is recurrent, the nature of threshold strategies implies that M^* will also have recurrent dynamics. To quantify the dynamic macro equilibrium we then compute the long-run distribution of M^* on $\mathcal{M}$. The latter is summarized via the transition probabilities of $\check{M}_n^*$ and the sojourn times ξ_m of $\check{M}_n^*$.

In the case when X is transient, M^* should be transient too. Specifically, we should encounter the situation that $\tilde{\tau}^{1,n} \wedge \tilde{\tau}^{2,n} = +\infty$ (see (4.8c)), so that no more switches take place and M^* remains constant forever or "absorbed". Under the assumption that X is continuous and regular, this phenomenon can only occur at the boundary states of $\mathcal{M}$, whereby one player is a priori restricted from switching. This yields a one-sided switching region and hence the possibility of a scenario that $M_t^* \equiv \overline{m}$ (or $\underline{m}$), for all t conditional on $M_0^* = \overline{m}$, i.e. that X never hits $s^2_{\overline{m}}$ starting at $s^1_{\overline{m}-1}$ (or $s^1_{\underline{m}}$ starting at $s^2_{\underline{m}+1}$). Note that given $X_t = x, M_t^* = \overline{m}$ (recall that the pair (X_t, M_t^*) is Markovian) one can not determine whether M^* is absorbed or not. This is handled via *taboo probabilities* [27, Ch. Taboo Probabilities] which are taken into account by adding the two "absorbing" states $\{\underline{m}^a, \overline{m}^a\}$ to E. Probabilistically, when switching up from $(\overline{m}-1)^{\pm}$, potential absorption can be captured by nature tossing a coin to decide whether the new state of $\check{M}$ is $\overline{m}^+$ or $\overline{m}^a$.

Returning to the case of recurrent X, let $\vec{\Pi}$ denote the invariant distribution of $\check{M}^*$, solved from $\vec{\Pi}\boldsymbol{P} = \vec{\Pi}$, where $\boldsymbol{P}$ is the transition probability matrix of $\check{M}^*$. Furthermore, let $\vec{\xi}$ be the vector of expected sojourn times at each state of $\check{M}$, defined as

$$\xi_{m^-} := \mathbb{E}\big[\tilde{\tau}^{1,n} \wedge \tilde{\tau}^{2,n} \,|\, \check{M}_n^* = m^-\big], \qquad \xi_{m^+} := \mathbb{E}\big[\tilde{\tau}^{1,n} \wedge \tilde{\tau}^{2,n} \,|\, \check{M}_n^* = m^+\big], \tag{4.20}$$

where the threshold hitting times $\tilde{\tau}^{i,n}$ are defined in (4.8b). It follows that the long-run proportion of time that M^* spends at regime m (recall that $M_t^* = m$ is captured by

$\check{M}^*_{\eta(t)} = m^{\pm}$) is given by:

$$\rho_m = \frac{\Pi_{m^+}\xi_{m^+} + \Pi_{m^-}\xi_{m^-}}{\sum_{j\in\mathcal{M}}\{\Pi_{j^+}\xi_{j^+} + \Pi_{j^-}\xi_{j^-}\}}, \qquad \text{for all } m \in \mathcal{M}. \tag{4.21}$$

Now let us consider X to be *non-recurrent* so that one or both of the boundary regimes are absorbing, w.l.o.g $\overline{m}^+$ for example. In the long-run we then trivially have $\lim_{t\to\infty} M^*_t = \overline{m}$ and the quantities of interest in this situation are the expected number of controls exercised by player i before M^* gets absorbed, i.e.

$$\mathbb{N}^i_m(x) := \lim_{T\to\infty} \mathbb{E}_x\Big[\sum_k \mathbb{1}_{\{\sigma^i_k \le T\}} \,\Big|\, M^*_0 = m\Big], \quad i \in \{1,2\}, \tag{4.22}$$

and the expected time until absorption,

$$\mathbb{T}_m(x) := \mathbb{E}_x\Big[\min\big\{t \ge 0 : \check{M}^*_{\eta(t)} \in \{\underline{m}^a, \overline{m}^a\}\big\} \,\Big|\, M^*_0 = m\Big]. \tag{4.23}$$

Analytic evaluation of these quantities is given in Section 4.5.3 which also provides expressions for the transition matrix $\boldsymbol{P}$ of $\check{M}^*$ and sojourn times $\vec{\xi}$. Computations specific to the OU Example 4.1 and the GBM Example 4.2 processes are also discussed.

4.2.4 Stackelberg Switching

We emphasize that the order of switches is never pre-determined and so the identity of the n-th switcher, P_n, is resolved endogenously based on game evolution and the realization of (X_t). A variant of the switching game would be to pre-specify the identity of the player making the next switch, but not its timing, akin to a Stackelberg equilibrium where the leader and follower roles are fixed but timing strategy remains. The latter situation also arises organically if we restrict $M_t \in \{-1, +1\}$ which implies that players will alternate in their actions: $... \le \sigma_k^1 \le \sigma_k^2 \le \sigma_{k+1}^1 \le \sigma_{k+1}^2 \le ...$ Indeed, at any given stage only one firm can control (M_t) so no consideration of simultaneous competition is needed (See [17]).

It is instructive to consider a stationary threshold-type equilibrium in this setting, which reduces to characterizing the two thresholds $s_{-1}^{1,*}$ and $s_{+1}^{2,*}$. Furthermore, if their profit rates and switching costs depend on the local market environment (X_t) symmetrically around 0 and (X_t) is a process symmetric around 0 (like the OU process), we may search for a symmetric equilibrium with $s_{-1}^{1,*} = -s_{+1}^{2,*} =: \check{s}$ and $V_\cdot^1(x) = V_\cdot^2(-x)$ for any $x \in \mathcal{D}$. In turn this reduces finding the MNE to solving a *single* nonlinear equation in $\check{s}$, providing some insight into the respective structure.

Examining Theorem 4.7 for Player 1, the system of equations is simplified to

$$V_{-1}^1(x) = \begin{cases} V_{+1}^1(x) - K_{-1}^1(x), & x \ge \check{s}, \\ D_{-1}^1(x) + \omega_{-1}^1 F(x), & x < \check{s}, \end{cases}$$
$$V_{+1}^1(x) = \begin{cases} D_{+1}^1(x) + \nu_{+1}^1 G(x), & x > -\check{s}, \\ V_{-1}^1(x), & x \le -\check{s}, \end{cases}$$

where $\check{s}, \omega_{-1}^1, \nu_{+1}^1$ satisfy the following system (compare to (4.17))

$$\left(V_{+1}^1 - D_{-1}^1 - K_{-1}^1\right)(\check{s}) \cdot F'(\check{s}) - \left(V_{+1}^1 - D_{-1}^1 - K_{-1}^1\right)'(\check{s}) \cdot F(\check{s}) = 0, \tag{4.24a}$$
$$\left(V_{+1}^1 - D_{-1}^1 - K_{-1}^1\right)(\check{s}) - \omega_{-1}^1 F(\check{s}) = 0, \tag{4.24b}$$
$$\left(V_{-1}^1 - D_{+1}^1\right)(-\check{s}) - \nu_{+1}^1 G(-\check{s}) = 0. \tag{4.24c}$$

Note that the last two equations specify $\omega_{-1}^1, \nu_{+1}^1$ in terms of $V_{\pm 1}^1(\pm\check{s})$. One can now show that this system admits at least one solution.

Corollary 4.10 *Suppose that profit rates and switching costs are continuous and depend on the local market environment* (X_t) *symmetrically about 0, and* (X_t) *is a process symmetric around 0. Then there exists a threshold-type MNE for the switching game with* $\mathcal{M} = \{-1, +1\}$.

Proof: See Section 4.5.4. ■

4.3 Sequential Approach to MNEs

To approximate the system of nonlinear equations (4.16) - (4.17) proposed in Section 4.7 we provide two sequential approaches. The first approach is through best-response iterations among threshold-type strategies, while the other inducts on equilibrium in finite-switch strategies. The latter links multi-stage timing game equilibrium discussed in Chapter 3 to the switching equilibrium. The resulting threshold vectors $\boldsymbol{s}^i$ can be used as initial guesses in a root-finding algorithm.

4.3.1 Constructing MNE by Best-response Iteration

Given the rival's strategy, determining the best-response of one player is similar to a single-agent optimal switching problem, which has been studied in [14, 24]. Let us assume that Player 2 implements a threshold-type strategy $\boldsymbol{s}^2$ as in Definition 2.5. The best-response of Player 1 is then expected to be characterized through (2.8), which is a system of *coupled* optimal stopping problems.

We then decouple this system, in particular to apply Proposition 4.5 that provides the best-response threshold and game payoff of Player 1 once the leader/follower payoffs are fully specified. To do so, we consider auxiliary problems where the number of actions/switches available to Player 1 is bounded. Namely, Player 1 is constrained to ever use at most $N^1(\geq 1)$ controls. Her corresponding set of strategies is defined as

$$\mathcal{A}^{1,(N^1)} := \left\{(\boldsymbol{\alpha}^1, \boldsymbol{s}^2) \in \mathcal{A} : \tau^1(n) = +\infty, n > \eta(1, N^1)\right\}, \tag{4.25}$$

where $\tau^1(n)$ is the stopping rule Player 1 adopts at the n-th "round" of the game and $\eta(1, N^1)$ defined in (2.4) denotes the round at which Player 1 exercises her N^1-th switch. Note that now the stopping sets are allowed to explicitly depend on the remaining number of controls left (equivalent to number of switches already used plus an initial constraint). The best-response of Player 1 with N^1 controls is then admitted as

$$\widetilde{V}_m^{1,(N^1)}(x\,; \boldsymbol{s}^2) := \sup_{\boldsymbol{\alpha}^{1,(N^1)} \in \mathcal{A}^{1,(N^1)}} J_m^1(x; \boldsymbol{\alpha}^{1,(N^1)}, \boldsymbol{s}^2), \qquad \forall x \in \mathcal{D}, \tag{4.26}$$

for all $m \in \mathcal{M}$. When Player 1 has zero controls $N^1 = 0$, her payoff at any stage m is fully determined by $\boldsymbol{s}^2$, for instance at regime $\underline{m}+1$

$$\widetilde{V}_{\underline{m}+1}^{1,(0)}(x\,; \boldsymbol{s}^2) = \mathbb{E}_x\left[\int_0^{\tau^2_{\underline{m}+1}} e^{-rt}\pi^1_{\underline{m}+1}(X_t)\,dt\right] + \mathbb{E}_x\left[e^{-r\tau^2_{\underline{m}+1}}\right] \cdot D^1_{\underline{m}}(s^2_{\underline{m}+1}), \tag{4.27}$$

where the last term is the NPV of fixed-market-state cashflows defined in (4.5).

Proposition 4.11 *Given a threshold-type strategy $\boldsymbol{s}^j$ of player j, the best-response game payoffs of player i with finite controls converge as $N^i \to \infty$, i.e. $\forall x \in \mathcal{D}$,*

$$\widetilde{V}_m^{i,(N^i)}(x\,; \boldsymbol{s}^j) \nearrow \widetilde{V}_m^i(x\,; \boldsymbol{s}^j), \quad \textit{for all } m \in \mathcal{M} \qquad \textit{as } N^i \nearrow \infty.$$

Proof of Proposition 4.11 is inspired by [14] and stated in Section 4.5.2. Moreover, strong Markov property of X and Dynamic Programming Principle (DPP) imply that

$$\widetilde{V}_m^{1,(N^1)}(x\,;\boldsymbol{s}^2) = \sup_{\tau^1(1)\in\mathcal{T}} \mathbb{E}_x\Bigg[\int_0^{\mathcal{I}_m} e^{-rt}\pi_m^1(X_t)dt + e^{-r\mathcal{I}_m}\mathbb{1}_{\{\tau^1(1)>\tau_m^2\}}\cdot \widetilde{V}_{m-1}^{1,(N^1)}(X_{\tau_m^2}\,;\boldsymbol{s}^2) + e^{-r\mathcal{I}_m}\mathbb{1}_{\{\tau^1(1)\le\tau_m^2\}}\Big(\widetilde{V}_{m+1}^{1,(N^1-1)}(X_{\tau^1(1)}\,;\boldsymbol{s}^2) - K_m^1(X_{\tau^1(1)})\Big)\Bigg], \tag{4.28}$$

for all $m\in\mathcal{M}$, $\forall x\in\mathcal{D}$, with τ_m^2 the first hitting time of $\Gamma_m^2 = (\underline{d}, s_m^2]$, and dependence of $\tau^1(1)$ on N^1 omitted for brevity. We refer to [14, 24] who proved that DPP holds in this problem and our analysis of finite-control stopping games in Chapter 3.

Notice that game payoffs (4.27) can be treated as starting points to implement a backward Dynamic Programming scheme to solve the finite-control optimal stopping problem introduced in (4.28). Suppose that $\widetilde{V}_{m-1}^{1,(N^1)}(\cdot\,;\boldsymbol{s}^j)$ and $\widetilde{V}_{m+1}^{1,(N^1-1)}(\cdot\,;\boldsymbol{s}^j)$ are determined, and Assumption 4.4 holds. We denote

$$\begin{aligned}
\widetilde{v}^{1,N^1}(x\,;\tau_m^2) &:= \widetilde{V}_m^{1,(N^1)}(x\,;\boldsymbol{s}^j) - D_m^1(x),\\
h^{1,N^1}(x) &:= \widetilde{V}_{m+1}^{1,(N^1-1)}(x\,;\boldsymbol{s}^j) - D_m^1(x) - K_m^1(x),\\
l^{1,N^1}(x) &:= \widetilde{V}_{m-1}^{1,(N^1)}(x\,;\boldsymbol{s}^j) - D_m^1(x)
\end{aligned}$$

and apply Proposition 4.5 with leader/follower payoffs h^{1,N^1}, l^{1,N^1} to obtain best-response game payoff $\widetilde{V}_m^{1,(N^1)}(x\,;\boldsymbol{s}^2)$, which is parameterized by $\widetilde{\omega}_m^{1,(N^1)}$,$\widetilde{\nu}_m^{1,(N^1)}$, $\tilde{s}_m^{1,(N^1)}$. Thanks to Proposition 4.11 we know $\widetilde{V}_m^{1,(N^1)}(x\,;\boldsymbol{s}^2)$ converges, thus expect $\tilde{s}_m^{1,(N^1)} \to \tilde{s}_m^1$ would converge as well as $N^1\to\infty$. Thus, for N^1 large, we may use $\tilde{\boldsymbol{s}}^{1,(N^1)}$ to define a time-stationary strategy that is a proxy for the best response.

Building upon the preceding convergence result, we propose the following algorithm to determine a threshold-type Markov Nash equilibrium. Essentially, we apply the tâtonnement approach, alternating in finding the best-response strategies of the two players, expecting to converge to an associated best-response fixed point. These alternating best-responses are indexed by "rounds" $a = 1, 2, \ldots, A$. At odd rounds, Player 1 solves for her best response ($i = 1, j = 2$); at even rounds, Player 2 solves for her best response ($i = 2, j = 1$):

①: Set the strategy of player j to be of threshold-type as $\boldsymbol{s}^{j,a}$:

- For $a = 1$, set $\boldsymbol{s}^{2,1}$ as the monopoly thresholds of P2, i.e. when P1 is not allowed to switch ($N^1 = 0$ case). The thresholds $\boldsymbol{s}^{2,1}$ can then be obtained by solving a single-agent optimal switching problem.
- For $a > 1$ set $\boldsymbol{s}^{j,a} = \widetilde{\boldsymbol{s}}^{j,a-1}$.

②: Solve for $\widetilde{\boldsymbol{s}}^{i,a}$ and value function $\widetilde{V}_m^{i,(N)}(\cdot\,; \boldsymbol{s}^{j,a})$ for all $m \in \mathcal{M}$:

- Solve optimal stopping problems when player i is allowed at most n switches and player j applies $\boldsymbol{s}^{j,a}$ using Proposition 4.5, iteratively for $n = 1, \ldots, N$.
- Record $\widetilde{V}_m^{i,(N)}(\cdot)$ and the approximate best-response strategy $\widetilde{\boldsymbol{s}}^{i,a}(\boldsymbol{s}^{j,a-1}) \simeq \tilde{\boldsymbol{s}}^{i,(N)}$

③: Change the roles of i and j (alternate which player is solving for the best response)

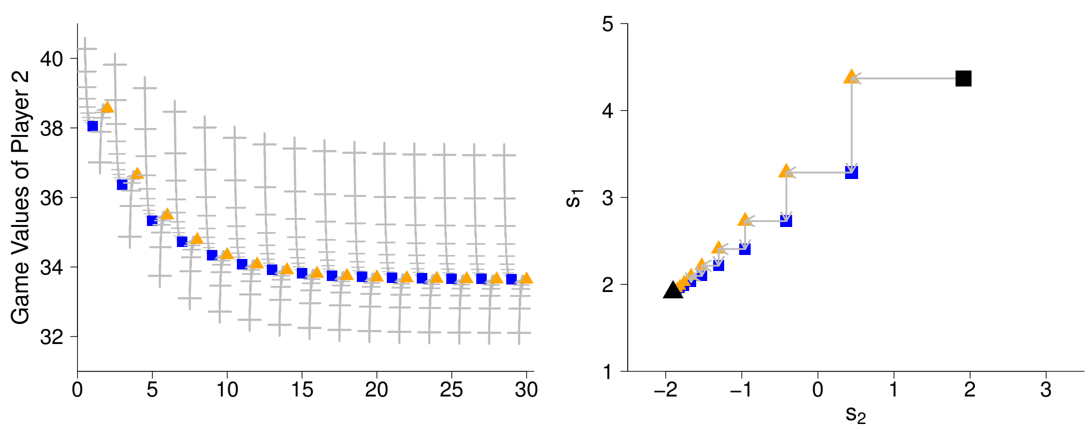
④: Repeat steps ① - ③ as $a = 1, \ldots,$ until the maximum change in $|\widetilde{\boldsymbol{s}}^{i,(N),a} - \widetilde{\boldsymbol{s}}^{i,(N),a-2}|$, $i \in \{1, 2\}$ are both less than a predetermined tolerance level Tol (or simply for A rounds).

Figure 4.2 illustrates the above best-response induction in one of our case-studies. In each round we iterate to find the best response assuming player i has up to N^i switches. During the odd rounds $a = 1, 3, \ldots$ Player 2 implements the stationary strategy $\boldsymbol{s}^{2,a}$ and her game values (gray '+') decrease as the number of Player 1's controls $N^1 = 1, \ldots, 30$

increases. In contrast, during the even iterations, Player 2 game values $\widetilde{V}_m^{2,(N^2)}$ converge upwards as $N^2 = 1, \ldots, 30$. The corresponding thresholds $s_m^{i,a}$ are shown on the right panel. We observe that both game values and thresholds converge after 30 inner iterations over N^i, and over $A = 30$ outer tatonnement rounds (a total of $30 \times 30 \times 2$ optimal stopping problems solved via Proposition 4.5). In particular, we may take $s_m^{i,(N),A}$ as an approximation of a best response fixed-point and hence of the equilibrium $s_m^{i,*}$.

4.3.2 Constructing MNEs by Equilibrium Induction

Another approach to construct an (approximate) threshold-type MNE of the switching game is to take limits in a finite-control game of timing. This links to our work in the preceding chapter. Suppose that both players are constrained to *finite* control strategies with respective bounds n^1, n^2 on total allowed number of switches. Specifically



(a) Approximations of Game Values (b) Approximations of Thresholds

Figure 4.2: Finding fixed point of best-response maps via the tatonnement process over a with $\mathcal{M} = \{-1, 0, +1\}$. Squares represent rounds $a = 1, 3, \ldots$ where player's 2 strategy is fixed. Triangles represent even rounds $a = 2, 4, \ldots$ where player's 1 strategy is fixed. (Left): Game values of Player 2 with $M_0 = 0$ and $X_0 = 0$ indexed according to $\tilde{V}^{2,N_2}(x; s^{1,a})$ with $N_2 = 1, 2, \ldots, 30$. (Right): Thresholds $s_0^{i,a}$ as a function of a at $m = 0$ and $N = 30$. The enlarged square represents the first round $a = 1$ and the enlarged triangle represents the last $a = 30$ round which appears to be close to a fixed point.

we consider strategies of the form

$$\boldsymbol{\alpha}^{i,(n^1,n^2)} := \left(\Gamma_m^{i,(k^1,k^2)}\right)_{m\in\mathcal{M}}^{k^1\le n^1,k^2\le n^2}, \qquad \text{with } \Gamma_m^{i,(0,k^j)} \equiv \emptyset, \tag{4.29}$$

where $k^i \le n^i$ denotes the number of controls remaining for player i, and index stages of this game as

$$\begin{aligned}(M_t, N_t^1, N_t^2) :=& \{\text{macro market regime},\\ &\#\text{ controls remaining for P1},\\ &\#\text{ controls remaining for P2}\},\end{aligned}$$

with $M_t \in \mathcal{M}$, and N_t^i is a non-increasing piecewise-constant process on $\mathbb{N}$ with $N_0^i = k^i$ for $i \in \{1,2\}$. Duopoly games of this type were studied in Chapter 3, in which we determine local equilibria at each game stage by backward dynamic programming and patch them to construct a global one.

At sub-stage (m, k^1, k^2), the local equilibrium is characterized as a fixed point of these players' best-response based on Proposition 4.5. Taking Player 1 as an example again, her leader and follower payoffs are related to her equilibrium game payoffs at adjacent stages which are known when implementing backward dynamic programming:

$$\begin{cases} h_m^{1,(k^1,k^2)}(x) & := V_{m+1}^{1,(k^1-1,k^2)}(x) - D_m^1(x) - K_m^1(x), \\ l_m^{1,(k^1,k^2)}(x) & := V_{m-1}^{1,(k^1,k^2-1)}(x) - D_m^1(x), \end{cases} \tag{4.30}$$

and their equilibrium strategies $(\tau_m^{1,(k^1,k^2),*}, \tau_m^{2,(k^1,k^2),*})$ and game payoffs solve a pair of optimal stopping problems:

$$\begin{cases} V_m^{1,(k^1,k^2)}(x) - D_m^1(x) = \sup_{\tau_m^{1,(k^1,k^2)} \in \mathcal{T}} \mathbb{E}_x \Big[\mathbb{1}_{\{\tau_m^{1,(k^1,k^2)} < \tau_m^{2,(k^1,k^2),*}\}} \cdot h_m^{1,(k^1,k^2)}(X_{\tau_m^{1,(k^1,k^2)}}) \\ \qquad + \mathbb{1}_{\{\tau_m^{1,(k^1,k^2)} > \tau_m^{2,(k^1,k^2),*}\}} \cdot l_m^{1,(k^1,k^2)}(X_{\tau_m^{2,(k^1,k^2),*}}) \Big], \\ V_m^{2,(k^1,k^2)}(x) - D_m^2(x) = \sup_{\tau_m^{2,(k^1,k^2)} \in \mathcal{T}} \mathbb{E}_x \Big[\mathbb{1}_{\{\tau_m^{1,(k^1,k^2),*} < \tau_m^{2,(k^1,k^2)}\}} \cdot l_m^{2,(k^1,k^2)}(X_{\tau_m^{1,(k^1,k^2),*}}) \\ \qquad + \mathbb{1}_{\{\tau_m^{1,(k^1,k^2),*} > \tau_m^{2,(k^1,k^2)}\}} \cdot h_m^{2,(k^1,k^2)}(X_{\tau_m^{2,(k^1,k^2)}}) \Big]. \end{cases} \tag{4.31}$$

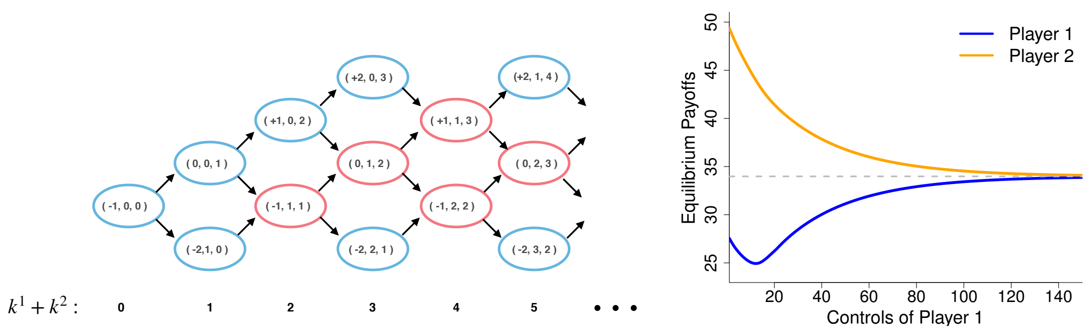
Note that simultaneous switches can be ruled out since on the event $\{\tau_m^{i,(k^1,k^2)} = \tau_m^{j,(k^1,k^2),*}\}$, stopping by Player 1 is strictly dominated by the strategy of first waiting, and then optimally switching as follower. In Chapter 3 we show that the local equilibrium exists under some regularity conditions on D^i's and K^i's, however uniqueness cannot be guaranteed. Moreover, such a local equilibrium is not always of threshold-type, as *preemptive* equilibria may emerge.

In the example sketched in Figures 4.3a, we implement a *forward* scheme to generate a sequence of equilibria starting at sub-stage $(m, k^1, k^2) = (-1, 0, 0)$ where the payoffs are $V_{-1}^{i,(0,0)}(x) = D_{-1}^i(x)$. With this known, we can solve for the local equilibria at stages $(0, 0, 1)$ and $(-2, 1, 0)$ utilizing (4.30). Iterating, we find local equilibria for all triplets (m, k^1, k^2) shown in the Figure (Throughout, we make the ansatz that local equilibria are all of threshold-type at any sub-stage (m, k^1, k^2)). These triplets can be characterized as $k^2 = k^1 + \Delta_m$, where the auxiliary parameter Δ_m is the difference between the number of switches available to the players at regime m. For instance $\Delta_{-1} = 0$ in Figure 4.3a, so that the players are equally endowed whenever they are at regime $M_t = -1$, cf. the sub-stages $(-1, 1, 1), (-1, 2, 2), \ldots$. The sub-stages (m, k^1, k^2) that are not reachable from $(-1, 0, 0)$ are omitted and in this instance we need not consider the respective local equilibria.

Using the terminal game stage $(-1, 0, 0)$ and continuing up to $k^1 \leq N$, the above forward scheme iteratively yields a sequence of equilibrium thresholds $s_m^{i,(n,n+\Delta_m)}$ and game coefficients $(\omega_m^{i,(n,n+\Delta_m)}, \nu^{i,(n,n+\Delta_m)})$. The resulting game payoffs are shown in Figure 4.3b. As mentioned, the parameter Δ_m influences all the equilibria in Figure 4.3a. For example, in the presented scheme, the game will eventually end with $M_t = -1$ for t large enough. Nevertheless, as N increases, we expect that this effect vanishes, so that the limits are independent of Δ_m:

$$\begin{pmatrix} s_m^{i,(n,n+\Delta_m)} \\ \omega_m^{i,(n,n+\Delta_m)} \\ \nu^{i,(n,n+\Delta_m)} \end{pmatrix} \xrightarrow{\text{as } n \nearrow \infty} \begin{pmatrix} s_m^{i,*} \\ \omega_m^{i,*} \\ \nu_m^{i,*} \end{pmatrix}, \qquad i \in \{1, 2\}, m \in \mathcal{M}. \tag{4.32}$$

This convergence can be observed in Figure 4.3 where the underlying symmetries imply



(a) Induction on local timing equilibria

(b) Eqm Payoffs $V_0^{i,(n,n+1)}(X_0)$

Figure 4.3: *Left*: A schematic diagram illustrating induction on local timing equilibria of Section 4.3.2, starting at $(-1, 0, 0)$ and with $\underline{m} = -2$ and $\overline{m} = +2$, which leads to $\Delta_{-2} = -1, \Delta_{-1} = 0, \ldots, \Delta_{+2} = 3$ in (4.32). The diagram illustrates the reachable stages (m, k^1, k^2) relative to $(M_0, 0, 0)$ and using the "forward" dynamic programming scheme. Blue circles denote single-agent optimization sub-stages that correspond to optimal stopping problems, while red circles denote interior stages where local timing equilibrium is determined according to (4.31). Boundary stages are those where $k^1 = 0$ or $k^2 = 0$ or $m \in \{\underline{m}, \overline{m}\}$. Stages not reachable from $(-1, 0, 0)$ are omitted. *Right:* Equilibrium payoffs $V_{M_0}^{i,(N_0^1,N_0^2)}(X_0)$ with $X_0 = 0$, $M_0 = 0$ indexed by N_0^1. Player 2 is given one extra control, $N_0^2 = N_0^1 + 1 \Leftrightarrow \Delta_0 = -1$. The dashed line denotes the limiting payoff $V_0^i(X_0)$ in the original infinite-control game.

$V_0^{1,(n,n+1)}(0) = V_0^{2,(n+1,n)}(0)$. Thus, we may interpret the top curve in Figure 4.3b as the game payoff in the finite-stage setup when the player has *one more* switch than her rival, and the bottom curve as her game payoff when she has *one fewer* switch relative to the rival. As $n \to \infty$, the relative benefit vanishes and both $V_0^{i,(n,n\pm1)}(x)$ approach $V_0^i(x)$.

Two issues arise with the above scheme. First, the associated equilibrium payoffs $V^{i,(N_0^1,N_0^2)}$ are not monotone in terms of N_0^1 or N_0^2. For instance, higher N_0^1 benefits P1, while higher N_0^2 harms her since her rival now has more flexibility. Changing both N^i's simultaneously leads to ambiguous results: in Figure 4.3b P1's payoff decreases first, then increases in terms of $N_0^1 = N_0^2 - 1$. Thus convergence in (4.32) is hard to prove. Second, the local timing game might generate multiple threshold-type equilibria as explained in Chapter 3. As a result, equilibrium selection becomes important when inducting on N_0^i's.

Remark 4.12 *Setting $\Delta_m = \underline{m}$ (resp. $\Delta_m = \overline{m}$) is equivalent to granting P2 (resp. P1) infinite number of allowed switches, while her rival is restricted to finite number of controls. This of course confers an ultimate advantage to the privileged player who will ultimately "win out" the competition. For example, taking $(M_0, N_0^1, N_0^2) = (0, 4, 2)$ in the running example means that P2 only has 2 switches, while P1 has four, so she will ultimately succeed in driving to the best possible regime $\lim_{t\to\infty} M_t = +2$ and will never require more than 4 switches anyway (recall that $\overline{m} = +2$). Thus this setting resembles the auxiliary game discussed in Section 4.3.1, except that both players are now dynamically optimizing their thresholds.*

4.4 Numerical Examples

4.4.1 Case Study: Mean-reverting Market Advantage

Continuing Example 4.1, we describe the local market fluctuation (X_t) by an Ornstein-Uhlenbeck (OU) process mean-reverting to $\theta = 0$:

$$dX_t = -\mu X_t dt + \sigma dW_t, \tag{4.33}$$

with $\mu = 0.15, \sigma = 1.5$, $\mathcal{D} = \mathbb{R}$ (i.e. natural boundaries $\underline{d} = -\infty$ and $\overline{d} = +\infty$). This implies that the stationary distribution of X is Gaussian, $\mathcal{N}(0, 7.5)$. For the discounting rate we take $r = 10\%$. The profit rates π^i_m are constant and listed in Table 4.1. Note that π^i's are monotone but *concave* in terms of the regime m. A motivating economic context is the advertising competition between two firms. They can make an advertising campaign by paying K^i, with the cost dependent on the exchange rate (X_t). The effect of advertising (i.e. exercising a change in M) is to enhance one firm's dominance in the market, bringing her higher profit rates. Due to diminishing returns to scale, improvement in the profit rate decreases as the firm captures more and more market share, so that π^i_m is concave in m.

m	-3	-2	-1	0	$+1$	$+2$	$+3$
π^1_m	0.0	1.5	2.8	4.0	5.1	5.9	6.0
π^2_m	6.0	5.9	5.1	4.0	2.8	1.5	0.0

Table 4.1: Profit rate ladders π^i_m for Section 4.4.1.

In Table 4.1 our intent is that the profit ladder extends to the right and to the left forever, however the above concavity makes it uneconomical to reach extreme levels of dominance. Thus, we progressively enlarge the number of market regimes considered: $M_t \in \{-1, 0, 1\}$ (Case I), $M_t \in \{-2, -1, 0, 1, 2\}$ (Case II), and $M_t \in \{-3, \dots, 3\}$ (Case

III). Below we also consider an asymmetric situation with $M_t \in \{-1, 0, 1, 2\}$.

The switching costs K^i's are affected by X_t:

$$K^i_m(x) := c_i \cdot (1 + e^{(-1)^i \beta_i x}), \qquad i \in \{1, 2\}, \tag{4.34}$$

where $c_i = 0.5$, $\beta_i = 0.5$, $i \in \{1, 2\}$. Thus, Player 1 can make cheap switches to dominate the market when $x \gg 0$ and Player 2 has the advantage when $x \ll 0$. For simplification, K^i's are not directly affected by (M_t). By construction, the profit rates and all other parameters are symmetric (about zero), so in equilibrium we expect players to act symmetrically as X fluctuates from positive to negative and vice versa.

Best-response induction associated to this case study with $\mathcal{M} = \{-1, 0, 1\}$ was the one sketched in Figure 4.2 and explained in Section 4.3.1. We also implement the equilibrium induction (see Section 4.3.2) for both Case I & II and observe that players behave aggressively when they have more controls than their rivals in the finite-control scenario. As sketched in Figure 4.3, Player 2 will have the "last word" and $\lim_{t\to\infty} M^*_t = -1$. Consequently, she can behave more aggressively, be the leader more frequently, and reap higher payoff already in the medium-term.

4.4.1.1 Equilibrium Thresholds

Table 4.2 lists the computed equilibrium thresholds for the three cases. Recall that in Case I M is restricted to be in $\{-1, 0, 1\}$, so Player 1 (P2) is not allowed to act when $M_t = +1$ ($M_t = -1$, respectively), hence there is no $s^{1,*}_1$ or $s^{2,*}_{-1}$. Thus, there are 4 total thresholds to be computed, and 12 equations in the system (4.16). Due to the symmetric parameter setting, thresholds of Player 1 are symmetric to thresholds of Player 2 around 0, so in principle the equilibrium is fully characterized by the pair $s^{1,*}_0, s^{1,*}_{-1}$. Similarly, in Case II there are 8 thresholds (4 unique ones) and 24 equations, and in Case III there are 12 thresholds and 36 equations.

Regime	π^1_m	π^2_m	Case I		Case II		Case III	
			Player 1	Player 2	Player 1	Player 2	Player 1	Player 2
$+3$	6.0	0.0					–	3.47853
$+2$	5.9	1.5			–	0.25184	13.16216	0.25184
$+1$	5.1	2.8	–	-0.56688	1.90352	-0.56688	1.90352	-0.56688
0	4.0	4.0	0.94861	-0.94861	0.94861	-0.94861	0.94861	-0.94861
-1	2.8	5.1	0.56688	–	0.56688	-1.90352	0.56688	-1.90352
-2	1.5	5.9			-0.25184	–	-0.25184	-13.16216
-3	0.0	6.0					-3.47853	–

Table 4.2: Equilibrium thresholds $s^{i,*}_m$ for Cases I, II & III of Section 4.4.1.

A major finding is that the players implement the *same* thresholds at each interior regime in *all* cases. For example, $s^{1,*}_0 = 0.94681$ in all three Cases I/II/III. Therefore, when in regime 0, Player 1 "does not see" whether stage +2 is reachable or not, and only makes her decision based on $\pi^i_{\pm 1}$. This phenomenon is un-intuitive in the following two aspects. On the one hand, Player 1 adopts the same equilibrium threshold $s^{1,*}_0 = 0.94861$ despite the fact that she can further exercise switches to enhance her dominance at state +1 in Cases II & III. The latter would be expected to make the switch from 0 to +1 more valuable and therefore make P1 more aggressive in regime 0. As we see, this intuition, while valid in single-agent contexts, fails in the constructed equilibrium. On the other hand, players are also myopic about the multi-step threat from the switches of the other player. For example, P2 implements the same threshold $s^{2,*}_{+1} = -0.56688$, though in Cases II & III she is facing the threat that P1 may switch the market to an even more disadvantageous regime.

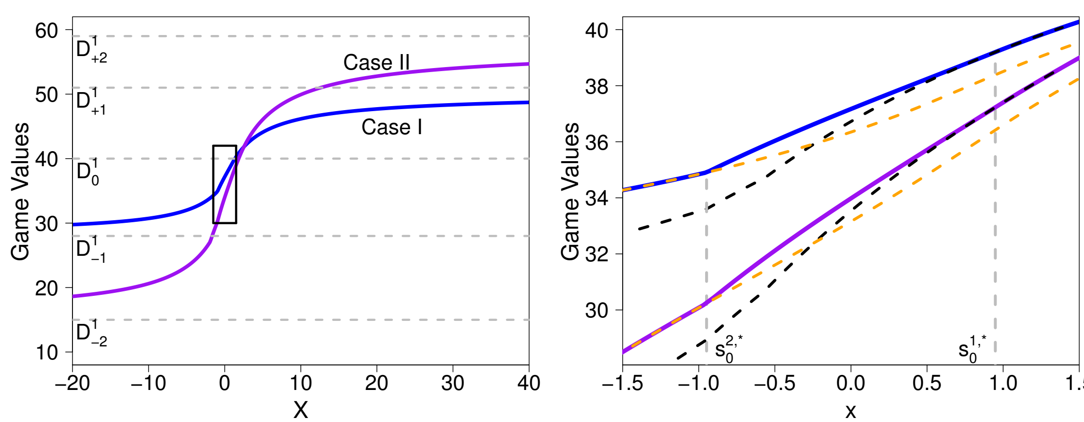
Figure 4.4a plots equilibrium payoffs of Player 1 $x \longmapsto V^1_0(x)$ (constructed as (4.18)) when they are at equal strength $M_0 = 0$ for Case I & II. As expected, V^1_m's are continuous, increasing and bounded: $V^{1,(I)}_0$ is bounded by D^1_{-1} and D^1_{+1}, while $V^{1,(II)}_0$ is bounded by D^1_{-2} and D^1_{+2}. Note that when these players are at equal strength locally (i.e. $X_0 = 0$)

Player 1 has lower equilibrium payoff in Case II, which can be interpreted as an influence of heavier competition between them.

The right panel of Figure 4.4b illustrates the nature of the variational inequalities for V^1_m. Dashed lines represent leader payoff $V^1_{+1}(x) - K^1_0$ from switching (M_t) to $(+1)$ and the follower payoff $V^1_{-1}(x)$. We have that $V^1_0(x)$ coincides with $V^1_{-1}(x)$ for $x < s^{2,*}_m$ (stopping region of P2), and smooth-pastes to $V^1_{+1}(x) - K^1_0(x)$ at the switching threshold $s^{1,*}_0$ of P1. As explained, this switching threshold is the same in Case I and II even though all the game payoffs (in particular V^1_0 and $V^1_{\pm 1}$) change.

4.4.1.2 Macroscopic Market Structure in Equilibrium

Due to the mean-reverting property of the OU process in (4.33), the resulting M^* is recurrent. In particular, at the extreme regimes $\check{M}$ is guaranteed to move back towards zero. Table 4.3 presents the resulting long-run proportion of time M^* spends at



(a) Equilibrium payoff $V^1_0(x)$ of Player 1 (b) Zoomed-In view of $V^1_0(x)$

Figure 4.4: Equilibrium payoff $V^1_0(x)$ of Player 1 for Case I and Case II. *Left panel*: the dashed levels indicate D^1_m; $D^1_{\pm 1}$ are the asymptotes of V^1_m for Case I, and $D^1_{\pm 2}$ are the asymptotes of V^1_m in Case II. The box denotes the region corresponding to the zoomed-in right panel. *Right*: Dashed curves denote the leader payoffs $V^1_{+1}(x) - K^1_0(x)$ and the follower payoffs $V^1_{-1}(x)$ for each Case. The same resulting thresholds $s^{i,*}_0$, $i \in \{1, 2\}$ are adopted in *both* Cases.

each regime, ρ_m given by (4.21). The table also lists the transition matrix $\mathbf{P}$ of the extended jump chain $\check{M}$, and mean sojourn times $\vec{\xi}$ for Case I, whence $\check{M}$ takes values in $\{-1^-, 0^-, 0^+, +1^+\}$. Note that due to the limited number of regimes, the invariant distribution of $\check{M}$ is uniform, i.e. the original jump chain $\tilde{M}$ spends half the time at regime 0 and 25% of its time at regimes ± 1. Of course, the corresponding sojourn times are not equal, so the long-run distribution of M^* is more complex.

Regime	π^1_m	π^2_m	Case I	Case II	Case III
+3	6.0	0.0			0.00002
+2	5.9	1.5		0.34476	0.34474
+1	5.1	2.8	0.47186	0.12711	0.12711
0	4.0	4.0	0.05628	0.05628	0.05628
−1	2.8	5.1	0.47186	0.12711	0.12711
−2	1.5	5.9		0.34476	0.34474
−3	0.0	6.0			0.00002

$$
\boldsymbol{P} = \begin{array}{c} \\ (-1)^- \\ (0)^+ \\ (0)^- \\ (+1)^+ \end{array} \begin{array}{c} \begin{array}{cccc} (-1)^- & (0)^+ & (0)^- & (+1)^+ \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.205 & 0 & 0 & 0.795 \\ 0.795 & 0 & 0 & 0.205 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}, \quad \vec{\xi} = \begin{array}{c} \begin{array}{cccc} (-1)^- & (0)^+ & (0)^- & (+1)^+ \end{array} \\ (\,4.431 \quad 0.264 \quad 0.264 \quad 4.431\,) \end{array}.
$$

Table 4.3: Equilibrium stationary distribution ρ_m of M^* for Cases I, II & III. *Bottom:* Dynamics of $\check{M}^*$ in Case I.

From Table 4.2, we observe that the thresholds $s_0^{i,*}$ are quite low, so that M^* does not spend much time in regime 0 and the market is typically not at “equal strength”. In Case I, only one level of market dominance is possible and so we observe rapid switches from “equal strength” to “P1 dominant” or “P2 dominant”, each of which occurs around $\rho^{(I)}_{\pm 1} = 47\%$ of the time. In Case II, because $s_0^{i,*}$ remain the same, we have the same $\rho^{(II)}_0$, so the market continues to be dominated (but now by different degrees) by one player around 47% of time. Thus, the long-run distribution of regimes $\{+1, +2\}$ in Case II can be considered as “splitting” of those 47% of regime +1 in Case I. Moreover, when one player dominates the market, she will max-out her dominance most of the time ($\rho_{\pm 2}^{(II)} = 34\%$ out of 47%).

A second finding is that one can effectively endogenize the domain $\mathcal{M}$ of M. Recall that concavity of profit rates π_m^i in terms of m reduces players' incentive to make further switches if the game stage is already advantageous. On the contrary, the rival becomes more incentivized to switch M back towards 0. In the presented example, we make the marginal gain in profit rates minimal when going from +2 to +3 (and -2 to -3 for Player 2, respectively). As a result, in Case III there is very little incentive for P1 to switch from +2 to +3, reflected in the very high equilibrium threshold $s_{+2}^{2,*} = 13.1621594$. Because this threshold is far above the mean-reverting level $\theta = 0$, it follows that these players are not likely to enhance their dominance up to the maximum level and regimes ± 3 will take place extremely rarely; according to Table 4.3, M^* spends less than 0.001% of time in those extreme regimes. Consequently, from a financial perspective it is reasonable to simply restrict M to be in $\{-2, -1, 0, +1, 2\}$, since effectively $\rho_m^{(III)} \simeq \rho_m^{(II)}$ for all m.

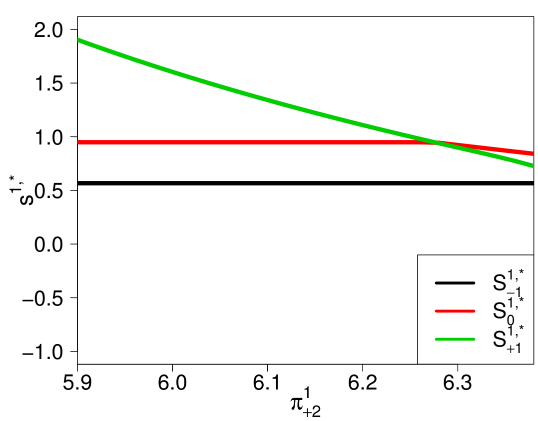
4.4.1.3 Effect of Profit Ladder

To isolate the effect of the profit rates π_m^i, we construct threshold-type equilibria with M restricted to $\mathcal{M}^{(IV)} = \{-1, 0, +1, +2\}$, and vary profit rate π_2^1 of Player 1 at stage +2 (all other profit rates remain as in Table 4.1). Resulting equilibrium thresholds of these players are sketched in Figure 4.5 by solid lines. As expected, when π_{+2}^1 increases, there is more benefit to being in regime +2 and as a result, Player 1 is more willing to make a switch up from +1. Consequently, she implements lower switching thresholds and $s_{+1}^{1,*}$ is decreasing in π_{+2}^1. On the contrary, she does not change her threshold $s_0^{1,*}$ at regime 0, confirming the myopic nature of equilibrium thresholds discussed in the previous section. However, eventually $\pi_{+2}^1 - \pi_{+1}^1$ is large enough (or alternatively $s_{+1}^{1,*}$ is low enough) to trigger simultaneous switches ($s_0^{1,*} > s_{+1}^{1,*}$), so that P1 will pass directly from regime 0 to regime 2 (recall that we assume that this incurs two switching costs, linearly added). In the latter situation, she switches sooner already in regime 0, see the extreme right of Figure 4.5a, where $s_0^{1,*}$ starts changing, as soon as $s_{+1}^{1,*} < s_0^{1,*}$.

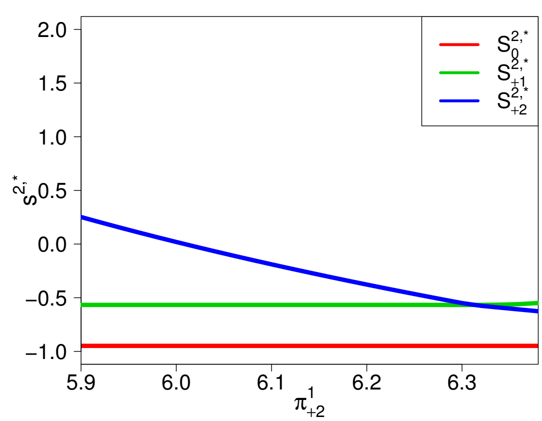
Turning attention to Player 2, her switching threshold $s_0^{2,*}$ in regime 0 is never affected by π_{+2}^1. Moreover, while her profit rate in regime +2 is unaffected, more aggressive behavior of P1 who switches into $M_t = +2$ more frequently, causes her to respond by lowering $s_{+2}^{2,*}$. Additionally, in the situation where P1 goes straight from 0 to +2 ($\pi_{+2}^1 \geq 6.25$), P2 increases $s_{+1}^{2,*}$, adjusting his strategy in response to a more aggressive strategy of P1 which reduces his anticipated gain from switching M from +1 to 0. These observations illustrate the complex feedback effects between thresholds in different market states and the underlying π_m^i's.

4.4.1.4 Effect of Switching Costs

Another essential parameter is the switching cost K. To study the effect of K, we vary the overall level of switching costs in $K^i(x) = c_i \cdot (1 + e^{(-1)^i \beta_i})$. Specifically, we try



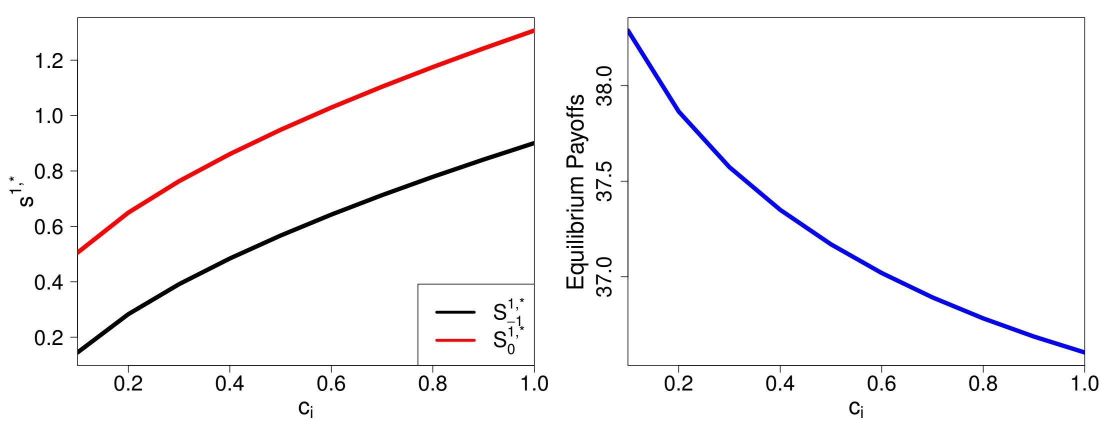
(a) Thresholds of Player 1



(b) Thresholds of Player 2

Figure 4.5: Equilibrium switching thresholds $s_m^{i,*}$ (P1 on the left, P2 on the right) as the profit rate π_{+2}^1 of P1 varies in Case IV. We use the same type of line for each regime m across the two panels.

$c_i \in [0.1, 1]$, i.e. from 20% to 200% relative to the baseline $c_i = 0.5$ used in preceding setup, with all other parameters unchanged. The resulting equilibrium thresholds of Player 1 and her equilibrium payoff at the "equal strength", $V_0^1(0)$ are sketched in Figure 4.6 for Case I where $\mathcal{M} = \{-1, 0, +1\}$. Observe that as c_i decreases (from the right to the left in Figure 4.6) P1 adopts lower thresholds while P2 adopts higher thresholds by symmetry, which means they switch the macro market environment more frequently. For instance, the expected sojourn time of $\tilde{M}^*$ at regime $(+1)^+$ drops from 6.522 at $c_i = 1.0$ to 1.887 at $c_i = 0.1$, while the expected sojourn time at regime $(0)^+$ drops from 0.422 at $c_i = 1.0$ to 0.105 at $c_i = 0.1$. Recall that $K_0^i(x)$ is a function of x, so that lowering the threshold is equivalent to paying more. While lower (single) switching cost induces more frequent switching, the overall cost of switching still declines, so that equilibrium payoffs increase as c_i declines.



(a) Switching thresholds of Player 1 (b) Equilibrium payoff of Player 1

Figure 4.6: *Left*: Equilibrium thresholds adopted by P1 as c_i varies in $[0.1, 1]$. Thresholds of P2 are anti-symmetric about 0. *Right*: Equilibrium payoff at "equal-strength", i.e. $V_0^1(0) = V_0^2(0)$ due to symmetry.

4.4.1.5 Multiple Threshold-type Equilibria

As mentioned, existence of *multiple* threshold-type equilibria is highly likely. According to Theorem 4.7, any suitable solution to the system of non-linear equations is an MNE of the switching game. This situation arises in Case II above and is "detected" by selecting different local equilibria during the equilibrium induction of Section 4.3.2. Specifically, in some sub-stages there are two different threshold-type equilibria in the local stopping game, which can be interpreted as "Sooner" (players behave more aggressively and switch quickly once (X_t) deviates from zero) and "Later" (players are more relaxed and s^i_m are larger in absolute value). This phenomenon was already discussed for stopping games in Section 3.2.2. Then during equilibrium induction we consistently choose (i) later equilibria (this is what was done and reported above in Tables 4.2-4.3); (ii) sooner equilibria. This generates two different sequences of $\boldsymbol{s}^{i,*}_m$, which ultimately yield two different solutions to the nonlinear system, reported in Table 4.4.

		-2	-1	0	$+1$	$+2$	$V^i_0(0)$
Later	$s^{1,*,L}_m$	-0.25184	0.56688	0.95861	1.90352	$-$	33.98
	$s^{2,*,L}_m$	$-$	-1.90352	-0.94861	-0.56688	0.25184	
	ρ^L_m	0.34476	0.12711	0.05628	0.12711	0.34476	
Sooner	$s^{1,*,S}_m$	0.17983	-0.19139	1.38933	1.02891	$-$	33.65
	$s^{2,*,S}_m$	$-$	-1.02891	-1.38933	0.19139	-0.17983	
	ρ^S_m	0.41143	0	0.17714	0	0.41143	

Table 4.4: Equilibrium thresholds $s^{i,*}_m$ and long-run distribution $\vec{\rho}$ of (M^*_t) associated to two distinct equilibria in Case II

We note that in the "Sooner" equilibrium which was described previously, players effectively skip regimes ± 1 as $s^{1,*}_0 > s^{1,*}_{+1}$ and $s^{2,*}_0 < s^{2,*}_{-1}$. For example, starting at $M^*_0 = 0$ and $X_0 = 0$, P1 will not switch up until $X_t = 1.389 = s^{1,*,S}_0$, but then directly go to $M^* = +2$ because $1.389 > 1.029 = s^{1,*,S}_{+1}$. As a result, the alternative stationary distribution $\vec{\rho}^S$ of $M^{S,*}$ is only supported on $\{-2, 0, 2\}$ (interestingly, $\rho^S_0 > \rho^L_0$ so in the aggressive equilibrium the market is more frequently at "equal strength"). Such multiple instantaneous switches are indicative of their aggressiveness—once (X_t) moves in their preferred direction, players attempt to extract maximum dominance by switching into regime ± 2. In line with previous analysis, the Sooner equilibrium carries *lower* equilibrium payoffs, as players are penalized for aggressive interventions that leads to "wasted" effort, e.g. $V_0^{i,S}(0) = 33.65 < V_0^{i,L}(0) = 33.98$.

Remark 4.13 *In Case I there seems to be a unique equilibrium, which we conjecture is due to having only a single interior regime where players compete simultaneously. Thus, with $\mathcal{M} = \{-1, 0, 1\}$ we always observe a unique local threshold-type equilibrium during either of the finite-control inductions. It remains an open problem to establish more precise conditions regarding equilibrium uniqueness in the infinite-control switching game. Similarly, we do not have the machinery to check whether* further *threshold-type equilibria exist in Case II.*

4.4.2 Case Study: Long-run Advantage

Returning to Example 4.2, we now consider local market fluctuations X to follow a Geometric Brownian motion (2.36) with drift $\mu = 0.08$, volatility $\sigma = 0.25$, and discounting rate $r = 10\%$. Because $\mu - \frac{1}{2}\sigma^2 > 0$, $\lim_{t\to\infty} X_t = +\infty$ a.s., and so in the long-run Player 1 will dominate the market since she will eventually have the advantage in terms of X.

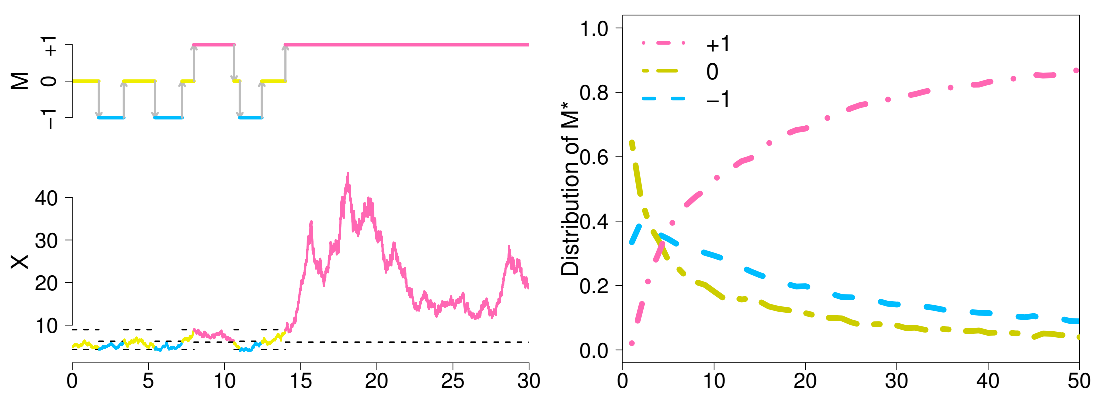
The profit rates π_m^i are constant and given by

$$\pi_{-1}^1 = 0; \quad \pi_0^1 = 3; \quad \pi_{+1}^1 = 5;$$
$$\pi_{-1}^2 = 5; \quad \pi_0^2 = 3; \quad \pi_{+1}^2 = 0.$$

The switching costs K^i_m's are again independent of m and driven by X:

$$K^1(x) := (10 - x)_+, \qquad K^2(x) := (-2 + x)_+. \tag{4.35}$$

This case study can be interpreted as competition between an energy producer using a renewable resource (Player 1) and a producer using exhaustible resources (Player 2). The competition is in terms of generating capacity, with M_t denoting the *relative* production capacity. Here X_t represents the marginal cost of exhaustibility which connects to the relative cost of increasing capacity. We expect that $X_t \to +\infty$ ("peak oil"); as non-renewable resources are depleted, P2 becomes noncompetitive. In the long run, P1 will therefore dominate, however there is no upper bound on how many times the competing investments in new capacity will take place. Thus, the market will first go through a transient phase where both producers compete, and then will eventually enter the high-X regime where the renewable P1 dominates and (endogenously) never relinquishes her advantage.



(a) Sample Trajectory (b) Distribution of M^*_t

Figure 4.7: *Left*: A trajectory of (X_t, M^*_t) for the GBM example, starting from $X_0 = 5$, $M^*_0 = 0$. *Right*: Distribution of $M^*_t \in \{-1, 0, 1\}$ as a function of t.

4.4.2.1 Macroscopic Market Structure in Equilibrium

	Expansion Thresholds s^i_m			Ave Plants Built
	-1	0	$+1$	
Player 1	5.9796	8.9594	-	3.8151
Player 2	-	4.1296	5.9574	2.8151

Table 4.5: Equilibrium in the GBM case study. Average Plants Built refers to the expected number of switches by player i starting at $X_0 = 5, M_0 = 0$.

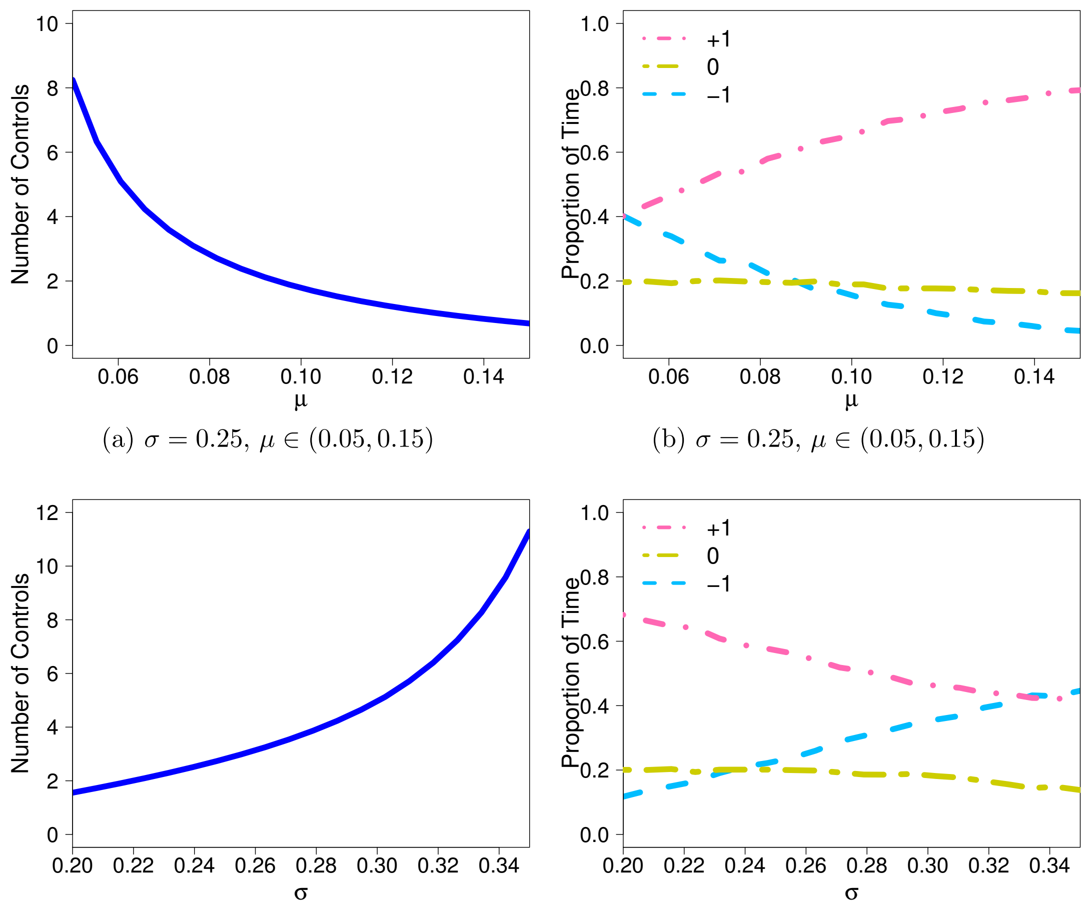
Table 4.5 shows the resulting equilibrium thresholds, and Figure 4.7a plots a trajectory of (X_t) and (M^*_t) starting at $X_0 = 5$, $M^*_0 = 0$. The right panel Figure 4.7b shows the distribution of M^* via $t \mapsto \mathbb{P}(M^*_t = m)$. We observe that Player 2 is likely to make the first expansion ($\mathbb{P}(M^*_t = -1)$ increases for low t), while in the medium-term Player 1 becomes more and more likely to be dominant. In line with $X_t \to +\infty$ (due to $\mu - \sigma^2/2 > 0$) we have $\mathbb{P}(M^*_t = +1) \to 1$ as t grows. The probability of absorption for $\check{M}^*$ when moving up from the states $(0)^\pm$ is (see Section 4.5.3.1)

$$P_{(+1)^a} = \lim_{u \uparrow \infty} \frac{(s^{1,*}_0)^{1-\frac{2\mu}{\sigma^2}} - (s^{2,*}_1)^{1-\frac{2\mu}{\sigma^2}}}{u^{1-\frac{2\mu}{\sigma^2}} - (s^{2,*}_{+1})^{1-\frac{2\mu}{\sigma^2}}} = 1 - \left(\frac{s^{1,*}_0}{s^{2,*}_1}\right)^{1-\frac{2\mu}{\sigma^2}} = 0.4709,$$

leading to the transition probability matrix $\boldsymbol{P}$ of $\check{M}^*$ as

$$\boldsymbol{P} = \begin{array}{c} \\ (-1)^- \\ (0)^+ \\ (0)^- \\ (+1)^+ \\ (+1)^a \end{array} \begin{array}{c} \begin{array}{ccccc} (-1)^- & (0)^+ & (0)^- & (+1)^+ & (+1)^a \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0.374 & 0 & 0 & 0.331 & 0.295 \\ 0.379 & 0 & 0 & 0.329 & 0.292 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{4.36}$$

Note that in the scenario plotted in Figure 4.7a, $M^*_t = +1$ after $t = 15$ which can be interpreted as "absorption". The theoretical average time until absorption (defined in (4.23)) is $\mathbb{T}_0(5) = 30.775$. In the Figure we also note that P1 makes 5 switches up and P2 makes 4 switches down. Recall that on the infinite time horizon P1 will always make one more switch since $M_t^* = +1$ eventually. The last column of Table 4.5 shows the *average* number of expansions implemented by each producer, $\mathbb{N}_0^i(5)$ defined in (4.22).



(a) $\sigma = 0.25$, $\mu \in (0.05, 0.15)$ (b) $\sigma = 0.25$, $\mu \in (0.05, 0.15)$

(c) $\mu = 0.08$, $\sigma \in (0.20, 0.35)$ (d) $\mu = 0.08$, $\sigma \in (0.20, 0.35)$

Figure 4.8: *Left*: Average total number of switches exercised by P2. *Right*: Estimated proportion of time that (M_t^*) spends in regime m in the next $\bar{T} = 30$ years, $\rho_m(\bar{T})$ from (4.37), for each $m \in \mathcal{M}$. In all cases we take $X_0 = 5$, $M_0^* = 0$.

4.4.2.2 Effect of the Drift μ and Volatility σ

We examine the effect of the drift μ and volatility σ in (2.36) on the equilibrium strategies and the macro market regime M^*. To do so, we evaluate the expected number of expansions carried out by Player 2 conditional on $M_0 = 0$, $\mathbb{N}_0^2(5)$ (which always satisfies $\mathbb{N}_0^2(5) = \mathbb{N}_0^1(5) - 1$). Additionally, we also compute the proportion of time that M^* spends at each regime $m \in \{-1, 0, 1\}$ in the next $\bar{T} = 30$ years, $\rho_m(\bar{T})$:

$$\rho_m(\bar{T}) := \mathbb{E}\left[\frac{1}{\bar{T}} \int_0^{\bar{T}} \mathbb{1}_{\{M_t^* = m\}} dt \,\middle|\, M_0^* = 0 \right]. \tag{4.37}$$

Figure 4.8 shows the results as we vary μ from 0.05 to 0.15 with fixed $\sigma = 0.25$, or in complement vary $\sigma \in [0.20, 0.25]$ with fixed $\mu = 0.08$. As expected, higher μ increases the tendency of (X_t) to go to $+\infty$ and hence enforces the dominance of Player 1; thus Player 2 expands less. A similar effect holds as σ falls —with less fluctuations there are fewer opportunities for P2. As a result, the overall number of switches, which can be viewed as the "observable competition", decreases as μ increases or σ decreases. A related effect is observed in Figure 4.8b: the dominance of P1, $\rho_{+1}(\bar{T})$, increases as μ rises or σ falls. For μ low (σ high), the transition to long-run advantage takes place more slowly, so the players are more even-handed in the medium term on $[0, \bar{T})$. Note that higher volatility hurts Player 1, intensifying the medium-term competition and causing both players to expend a lot of capital on repeated expansion. Finally, we remark that the proportion of time M^* spends in regime 0, $\rho_0(\bar{T})$ is quite stable with respect to different combinations of μ and σ.

4.5 Computations and Proofs

4.5.1 Proof of Theorem 4.7

Proof: To begin with, we argue that by construction of V^i's in (4.18) we have:

1. $V^i_m \in C^2(\mathcal{D} \setminus (\boldsymbol{s}^i \cup \boldsymbol{s}^j)) \cap C^1(\mathcal{D} \setminus \boldsymbol{s}^j) \cap C(\mathcal{D})$, for $\forall m \in \mathcal{M}$, $i \in \{1,2\}$, $i \neq j$;

2. V^i_m is at most linear growth, i.e.

$$|V^i_m(x)| \leq C(1+|x|), \text{ for } \forall x \in \mathcal{D}; \tag{4.38}$$

3. V^i's satisfy the following system of variational inequalities (VIs) for $m < \overline{m}$:

$$V^1_{m+1} - K^1_m - V^1_m \leq 0, \qquad \text{in } \mathcal{D}, \tag{4.39a}$$

$$V^1_{m-1} - V^1_m = 0, \qquad \text{in } \Gamma^{2,*}_m, \tag{4.39b}$$

$$V^2_{m+1} - V^2_m = 0, \qquad \text{in } \Gamma^{1,*}_m, \tag{4.39c}$$

$$\max\left\{(\mathcal{L} - r)\, V^1_m + \pi^1_m, V^1_{m+1} - K^1_m - V^1_m\right\} = 0, \qquad \text{in } \mathcal{D} \setminus \Gamma^{2,*}_m, \tag{4.39d}$$

and for $m > \underline{m}$,

$$V^2_{m-1} - K^2_m - V^2_m \leq 0, \qquad \text{in } \mathcal{D}, \tag{4.40a}$$

$$V^1_{m-1} - V^1_m = 0, \qquad \text{in } \Gamma^{2,*}_m, \tag{4.40b}$$

$$V^2_{m+1} - V^2_m = 0, \qquad \text{in } \Gamma^{1,*}_m, \tag{4.40c}$$

$$\max\left\{(\mathcal{L} - r)\, V^2_m + \pi^2_m, V^2_{m-1} - K^2_m - V^2_m\right\} = 0, \qquad \text{in } \mathcal{D} \setminus \Gamma^{1,*}_m. \tag{4.40d}$$

The smoothness of V^i follows directly from the regularity of $F(\cdot)$ and $G(\cdot)$ and the piecewise construction. The second statement follows from the linear growth assumption imposed on D^i's in (4.5) and signs of coefficients ω^i, ν^i. Note that this is a natural property of correct equilibrium payoffs since the best-response game payoff of player i satisfies

$$\min_{m \in \mathcal{M}} D^i_m(x) \le \widetilde{V}^i_m(x; \boldsymbol{\alpha}^j) \le \max_{m \in \mathcal{M}} D^i_m(x), \tag{4.41}$$

and a MNE is characterized as a fixed-point of best-responses. The key is the last assertion; we show (4.39), with (4.40) then following analogously. Comparing the system (4.16) for fixed m with (4.12) and (4.13), one can see that $V^1_m - D^1_m$ is indeed the solution to the optimal stopping problem (4.9) with

$$\begin{aligned} h^1_m(x) &:= V^1_{m+1}(x) - D^1_m(x) - K^1_m(x), \\ l^1_m(x) &:= V^1_{m-1}(x) - D^1_m(x), \end{aligned}$$

which in turn brings the restriction on signs of $\boldsymbol{\omega}^i, \boldsymbol{\nu}^i$. Taking P1 as an example, ω^1_m corresponds to the slope of the straight line segment of its transformed smallest concave majorant which ought be positive for $m < \overline{m}$ and equal to zero for $m = \overline{m}$. Similarly, ν^1_m corresponds to the y-intercept of that line segment which ought to be negative for $m > \underline{m}$ and equal to zero for $m = \underline{m}$. Furthermore, the signs of the derivatives of $F(\cdot), G(\cdot)$ imply that $V^1_m - D^1_m$ is increasing. The assumption $V^1_{m-1}(s^{2,*}) \ge V^1_{m+1}(s^{2,*}_m) - K^1_m(s^{2,*}_m)$, for $m > \underline{m}$ plus the smallest concave majorant characterization then yields

$$\begin{aligned} V^1_m - D^1_m &= l^1_m \ge h^1_m = V^1_{m+1} - D^1_m - K^1_m, && \text{in } \Gamma^{2,*}_m, \\ V^1_m - D^1_m &= h^1_m = V^1_{m+1} - D^1_m - K^1_m, && \text{in } \Gamma^{1,*}_m, \\ V^1_m - D^1_m &\ge h^1_m = V^1_{m+1} - D^1_m - K^1_m, && \text{in } \mathcal{D} \setminus \left(\Gamma^{1,*}_m \cup \Gamma^{2,*}_m\right), \end{aligned}$$

which shows (4.39a). (4.39b) and (4.39c) are obtained directly from the construction of V^i's, reflecting the payoff in x-states where the rival switches immediately. Lastly, to check (4.39d), recall that the discounted cash flows in (4.5) satisfy $(\mathcal{L}-r)\,D^i_m = -\pi^i_m$, and by their definition $(\mathcal{L}-r)\,F = (\mathcal{L}-r)\,G = 0$. For $x \in \mathcal{D} \setminus (\Gamma^{2,*}_m \cup \Gamma^{1,*}_m)$ ("the no-action region") we have by (4.18) that $V^1_m(x) = D^1_m(x) + \omega^1_m F(x) + \nu^1_m G(x)$, so applying the operator $(\mathcal{L}-r)$ we get

$$(\mathcal{L}-r)\,V^1_m + \pi^1_m = (\mathcal{L}-r)\,D^1_m + \pi^1_m = -\pi^1_m + \pi^1_m = 0, \qquad x \in \mathcal{D} \setminus (\Gamma^{2,*}_m \cup \Gamma^{1,*}_m).$$

For $x \in \Gamma^{1,*}_m \setminus \Gamma^{1,*}_{m+1}$, we have $V^1_m(x) = V^1_{m+1}(x) - K^1_m(x) = D^1_{m+1}(x) + \omega^1_{m+1}F(x) + \nu^1_{m+1}G(x) - K^1_m$ so that

$$\begin{aligned}(\mathcal{L}-r)\,V^1_m + \pi^1_m &= (\mathcal{L}-r)\left(V^1_{m+1} - K^1_m\right) + \pi^1_m \\ &= (\mathcal{L}-r)\,D^1_{m+1} + (\mathcal{L}-r)\left(-K^1_m\right) + \pi^1_m \qquad (4.42)\\ &= (\mathcal{L}-r)\left(D^1_{m+1} - D^1_m - K^1_m\right) < 0, \qquad (4.43)\end{aligned}$$

where the last inequality (4.43) is due to $D^1_{m+1} - D^1_m - K^1_m \in \mathcal{H}_{\text{inc}}$. Similar arguments apply to $x \in \Gamma^{1,*}_{m+1} \setminus \Gamma^{1,*}_{m+2}$ where two simultaneous switches by P1 will take place; by induction we conclude that $(\mathcal{L}-r)\,V^1_m + \pi^1_m < 0$ for $x \in \Gamma^{1,*}_m$, establishing (4.39d).

We now prove $(\boldsymbol{s}^{1,*}, \boldsymbol{s}^{2,*})$ is a Nash equilibrium. To do so, we first consider the point of view of P1, letting $\boldsymbol{\alpha}^1 = \{\tau^1(n) : n \geq 1\}$ be her arbitrary strategy satisfying $(\boldsymbol{\alpha}^1, \boldsymbol{s}^{2,*}) \in \mathcal{A}$, and $(\sigma_n)_{n\geq 0}$ be the sequence of resulting switching times defined in (2.2), with $X_0 = x$, $\tilde{M}_0 = m$. As a first step, we use induction to establish that

$$V_m^1(x) \geq \mathbb{E}\Big[\int_0^{\sigma_n} e^{-rt}\pi^1(X_t^x, \tilde{M}_{\eta(t)})dt - \sum_{k=1}^{n} \mathbb{1}_{\{P_k=1\}} e^{-r\sigma_k} \cdot K^1\big(X_{\sigma_k}, \tilde{M}_{k-1}\big) + e^{-r\sigma_n} V^1_{\tilde{M}_n}(X^x_{\sigma_n})\Big] \quad \forall n \geq 1. \tag{4.44}$$

For $n = 1$, since $\sigma_1 = \tau^1(1) \wedge \tau_m^{2,*}$, applying Itô's formula to the process $e^{-rt}V_m^1(X_t^x)$ over the interval $[0, \sigma_1]$ and taking expectations yields

$$\begin{aligned}
V_m^1(x) &= \mathbb{E}\Big[-\int_0^{\sigma_1} e^{-rt}\,(\mathcal{L} - r)\,V_m^1\,(X_t^x)\,dt + e^{-r\sigma_1}V_m^1(X^x_{\sigma_1})\Big] \\
&\geq \mathbb{E}\Big[\int_0^{\sigma_1} e^{-rt}\pi^1(X_t^x, \tilde{M}_0)dt + e^{-r\sigma_1}V_m^1(X^x_{\sigma_1})\Big] &(4.45a)\\
&\geq \mathbb{E}\Big[\int_0^{\sigma_1} e^{-rt}\pi^1(X_t^x, \tilde{M}_0)dt + e^{-r\sigma_1}\{-K^1(X^x_{\sigma_1}, \tilde{M}_0)\cdot \mathbb{1}_{\{\sigma_1=\tau^1(1)\}} + V^1_{\tilde{M}_1}(X^x_{\sigma_1})\}\Big], &(4.45b)
\end{aligned}$$

where the inequality (4.45a) follows from (4.39d) and the fact that $\sigma_1 \leq \tau_m^{2,*}$, and the inequality (4.45b) is due to (4.39a) and (4.39b):

$$\mathbb{E}\Big[V_m^1(X^x_{\sigma_1})\Big] \geq \mathbb{E}\Big[\mathbb{1}_{\{\sigma_1=\tau^1(1)\}}\Big\{V^1_{m+1}(X^x_{\sigma_1}) - K^1(X^x_{\sigma_1}, \tilde{M}_0)\Big\} + \mathbb{1}_{\{\sigma_1=\tau_m^{2,*}\}}V^1_{m-1}(X^x_{\sigma_1})\Big]. \tag{4.46}$$

Next we show (4.44) for $n = 2$. By construction, we have $\sigma_2 = \tau^1(2) \wedge (\sigma_1 + \tau^{2,*}_{\tilde{M}_1})$. Consider the second-round sub-game started at initial state $X^x_{\sigma_1}$; applying Itô's formula to the process $e^{-rt}V^1_{m+1}\big(X_t^{X^x_{\sigma_1}}\big)$ over the interval $[0, \sigma_2 - \sigma_1]$ and taking expectation conditional on $\tilde{\mathcal{F}}^{(2)}_{\sigma_1}$, cf. (2.2a), we obtain

$$V^1_{m+1}(X^x_{\sigma_1}) \geq \mathbb{E}\Big[\int_0^{\sigma_2-\sigma_1} e^{-rt}\pi^1\big(X_t^{X^x_{\sigma_1}}, m+1\big)dt + e^{-r(\sigma_2-\sigma_1)}V^1_{\tilde{M}_2}\big(X^x_{\sigma_2}\big) \\ - \mathbb{1}_{\{\sigma_2=\tau^1(2)\}}e^{-r(\sigma_2-\sigma_1)}\cdot K^1(X^x_{\sigma_2}, m+1)\Big|\,\tilde{\mathcal{F}}^{(2)}_{\sigma_1}\Big],$$

analogously to (4.45b) and replacing $X^{X^x_{\sigma_1}}_{\sigma_2-\sigma_1}$ by $X^x_{\sigma_2}$ based on the strong Markov property of (X_t). Furthermore, using $\int_{\sigma_1}^{\sigma_2} e^{-rt}\pi^1(X^x_t, m{+}1)dt = e^{-r\sigma_1}\int_0^{\sigma_2-\sigma_1} e^{-rt}\pi^1\big(X_s^{X^x_{\sigma_1}}, m{+}1\big)ds$ we have

$$\begin{aligned} &\mathbb{E}\Big[\mathbb{1}_{\{\sigma_1=\tau^1(1)\}}e^{-r\sigma_1}V^1_{m+1}(X^x_{\sigma_1})\Big] \\ \geq\ &\mathbb{E}\Bigg[\mathbb{1}_{\{\sigma_1=\tau^1(1)\}}\cdot\Big[\int_{\sigma_1}^{\sigma_2} e^{-rt}\pi^1\big(X^x_t, m+1\big)dt + e^{-r\sigma_2}V^1_{\tilde{M}_2}\big(X^x_{\sigma_2}\big)\Big] \\ &\qquad - \mathbb{1}_{\{\sigma_2=\tau^1(2)\}}\mathbb{1}_{\{\sigma_1=\tau^1(1)\}}e^{-r\sigma_2}\cdot K^1(X^x_{\sigma_2}, m+1)\Bigg]. \end{aligned} \tag{4.47}$$

Similarly, we have

$$\begin{aligned} &\mathbb{E}\Big[\mathbb{1}_{\{\sigma_1=\tau^{2,*}_m\}}e^{-r\sigma_1}V^1_{m-1}(X^x_{\sigma_1})\Big] \\ \geq\ &\mathbb{E}\Bigg[\mathbb{1}_{\{\sigma_1=\tau^{2,*}_m\}}\cdot\Big[\int_{\sigma_1}^{\sigma_2} e^{-rt}\pi^1\big(X^x_t, m-1\big)dt + e^{-r\sigma_2}V^1_{\tilde{M}_2}\big(X^x_{\sigma_2}\big)\Big] \\ &\qquad - \mathbb{1}_{\{\sigma_2=\tau^1(2)\}}\mathbb{1}_{\{\sigma_1=\tau^{2,*}_m\}}e^{-r\sigma_2}\cdot K^1(X^x_{\sigma_2}, m-1)\Bigg]. \end{aligned} \tag{4.48}$$

Substituting (4.47) - (4.48) into (4.45b), we obtain

$$
\begin{aligned}
V_m^1(x) \geq \mathbb{E}\Big[& \int_0^{\sigma_1} e^{-rt}\pi^1(X_t^x, \tilde{M}_0)dt + \int_{\sigma_1}^{\sigma_2} e^{-rt}\pi^1(X_t^x, m+1)dt \\
& + e^{-r\sigma_2} V^1_{\tilde{M}_2}(X^x_{\sigma_2}) - \mathbb{1}_{\{\sigma_1 = \tau^1(1)\}} e^{-r\sigma_1} K^1(X^x_{\sigma_1}, \tilde{M}_0) \\
& - \mathbb{1}_{\{\sigma_2 = \tau^1(2)\}} \mathbb{1}_{\{\sigma_1 = \tau^1(1)\}} e^{-r\sigma_2} \cdot K^1(X^x_{\sigma_2}, m+1) \\
& - \mathbb{1}_{\{\sigma_2 = \tau^1(2)\}} \mathbb{1}_{\{\sigma_1 = \tau_m^{2,*}\}} e^{-r\sigma_2} \cdot K^1(X^x_{\sigma_2}, m-1)\Big] \\
= \mathbb{E}\Big[& \int_0^{\sigma_2} e^{-rt}\pi^1(X_t^x, \tilde{M}_{\eta(t)})dt \\
& \quad - \sum_{k=1}^{2} \mathbb{1}_{\{P_k = 1\}} e^{-r\sigma_k} \cdot K^1\big(X_{\sigma_k}, \tilde{M}_{k-1}\big) + e^{-r\sigma_2} V^1_{\tilde{M}_2}(X^x_{\sigma_2})\Big].
\end{aligned}
$$

Iterating this argument for $n = 3, \ldots,$ establishes (4.44). Let us remark that the above works without any modifications in the boundary regimes $m \in \{\underline{m}, \overline{m}\}$, where $\tau^1(n)$ or $\tau_m^{2,*}$ are set to be infinite. Since V_m^1 is at most of linear growth from (4.38) and admissibility of $(\boldsymbol{\alpha}^1, \boldsymbol{s}^{2,*})$ requires $\lim_{n\to\infty} \sigma_n = +\infty$, dominated convergence theorem implies

$$
\begin{aligned}
V_m^1(x) &\geq \mathbb{E}\Big[\int_0^{\infty} e^{-rt}\pi^1(X_t^x, \tilde{M}_{\eta(t)})dt - \sum_{k=1} \mathbb{1}_{\{P_k=1\}} e^{-r\sigma_k} \cdot K^1\big(X_{\sigma_k}, \tilde{M}_{k-1}\big)\Big] \\
&= J_m^1(x; \boldsymbol{\alpha}^1, \boldsymbol{s}^{2,*}).
\end{aligned} \tag{4.49}
$$

Similarly for P2 we obtain that

$$
V_m^2(x) \geq J_m^2(x; \boldsymbol{s}^{1,*}, \boldsymbol{\alpha}^2), \text{ for } \forall (\boldsymbol{s}^{1,*}, \boldsymbol{\alpha}^2) \in \mathcal{A}.
$$

Last but not least, one can verify that replacing $\boldsymbol{\alpha}^1$ by $\boldsymbol{s}^{1,*}$ in above argument leads to $\sigma_1 = \tau_m^{1,*} \wedge \tau_m^{2,*}$ so that $(\mathcal{L} - r) V_m^1(X_t^x) = -\pi^1(X_t^x, \tilde{M}_0)$ on $[0, \sigma_1)$ and $V_m^1(X_t^x) = V_{m+1}^1(X_t^x) - K_m^1(X_t^x)$ at $\sigma_1 = \tau_m^{1,*}$ and $V_m^1(X_t^x) = V_{m-1}^1(X_t^x)$ at $\sigma_1 = \tau_m^{2,*}$. These turn inequalities in (4.45a) and (4.45b) into equalities, and inductively yield

$$V_m^1(x) = J_m^1(x; \boldsymbol{s}^{1,*}, \boldsymbol{s}^{2,*}),$$

which, combining with (4.49), completes the proof. ■

4.5.2 Proof of Proposition 4.11

Let $X_0 = x \in \mathcal{D}$, $M_0 = m \in \mathcal{M}$, and fix $\boldsymbol{s}^j$. The best-response of player i with $N^i \geq 1$ controls is

$$\widetilde{V}_m^{i,(N^i)}(x\,; \boldsymbol{s}^j) = \sup_{\boldsymbol{\alpha}^{i,(N^i)} \in \mathcal{A}^{i,(N^i)}} J_m^i(x\,; \boldsymbol{\alpha}^{i,(N^i)}, \boldsymbol{s}^j), \tag{4.50}$$

where

$$\mathcal{A}^{i,(N^i)} := \left\{(\boldsymbol{\alpha}^i, \boldsymbol{s}^j) \in \mathcal{A} : \tau^i(n) = +\infty, n > \eta(i, N^i)\right\}, \tag{4.51}$$

with $\eta(i, N^i)$ defined in (2.4) denotes the round at which player i exercises her N^i-th switch. Since $\mathcal{A}^{i,(N^i)} \subseteq \mathcal{A}^{i,(N^i+1)}$ we have that $N^i \mapsto \widetilde{V}_m^{i,(N^i)}(x\,; \boldsymbol{s}^j)$ is *non-decreasing.* Moreover, since $\widetilde{V}_m^{i,(N^i)}(x\,; \boldsymbol{s}^j)$ is bounded from above by $\max_m D_m^i(x)$, $\lim_{n\to\infty} \widetilde{V}_m^{i,(N^i)}(x\,; \boldsymbol{s}^j)$ is well-defined. It remains to show that this limit is $\widetilde{V}_m^i(x\,; \boldsymbol{s}^j)$.

Because $\mathcal{A}^{i,(N^i)} \subseteq \{\boldsymbol{\alpha}^i : (\boldsymbol{\alpha}^i, \boldsymbol{s}^j) \in \mathcal{A}\}$, we trivially obtain

$$\lim_{N^i\to\infty} \widetilde{V}_m^{i,(N^i)}(x\,; \boldsymbol{s}^j) \leq \widetilde{V}_m^i(x\,; \boldsymbol{s}^j). \tag{4.52}$$

To obtain the opposite inequality, for any $\varepsilon > 0$, let $\boldsymbol{\alpha}_\varepsilon^i := \{\tau_\varepsilon^i(n) : n \geq 1\}$ (which depends on x) be a ε-optimal strategy satisfying $(\boldsymbol{\alpha}^i_\varepsilon, \boldsymbol{s}^j) \in \mathcal{A}$ and

$$J^i_m(x\,; \boldsymbol{\alpha}^i_\varepsilon, \boldsymbol{s}^j) \geq \widetilde{V}^i_m(x\,; \boldsymbol{s}^j) - \varepsilon. \tag{4.53}$$

Now for a fixed $N^i \geq 1$ we define the respective truncated N^i-finite strategy $\boldsymbol{\alpha}^{i,(N^i)}_\varepsilon := \{\tau^{i,(N^i)}_\varepsilon(n) : n \geq 1\}$ as

$$\tau^{i,(N^i)}_\varepsilon(n) = \begin{cases} \tau^i_\varepsilon(n), & n \leq \eta(i, N^i) \\ +\infty, & o.w. \end{cases} \tag{4.54}$$

Thus, the truncated strategy stops switching completely after the first N^i switches. Denote by $M^{(N^i)}_t$ the resulting macro regime and by $(\sigma^{i,(N^i)}_k)_{k \leq N^i}$ the sequence of switching times of player i, cf. (2.4), based on $(\boldsymbol{\alpha}^{i,(N^i)}_\varepsilon, \boldsymbol{s}^j)$, which we compare against the corresponding $M^{(\infty)}_t$ and $(\sigma^{i,(\infty)}_k)_{k\geq 1}$ based on the non-truncated $(\boldsymbol{\alpha}^i_\varepsilon, \boldsymbol{s}^j)$. By the construction of the truncation,

$$\sigma^{i,(N^i)}_k = \sigma^i_k, \ \text{ for } k \leq N^i, \qquad M^{(N^i)}_t = M_t, \ \text{ for } t \leq \sigma^i_{N^i},$$

and the two cashflows completely match up to $\sigma^{i,(\infty)}_{N^i}$. In the truncated version, thereafter only the other player i applies her controls. Since $\sigma^{i,(\infty)}_{N^i} \to \infty$ as $N^i \to \infty$ from admissibility of $\boldsymbol{\alpha}^i_\varepsilon$, it follows that there exists $N^\varepsilon > 1$ s.t. for $\forall N > N^\varepsilon$

$$\mathbb{E}_{x,m}\left[\int_{\sigma^i_N}^\infty e^{-rt} |\pi^i\big(X_t, M^{(\infty)}_t\big)|\, dt\right] < \varepsilon; \tag{4.55a}$$

$$\mathbb{E}_{x,m}\left[\int_{\sigma^i_N}^\infty e^{-rt} |\pi^i(X_t, M^{(N)}_t)|\, dt\right] < \varepsilon; \tag{4.55b}$$

$$\mathbb{E}_{x,m}\left[\sum_{k=N+1}^\infty e^{-r\sigma^{i,(\infty)}_k} K^i\left(X_{\sigma^{i,(\infty)}_k}, \tilde{M}^{(\infty)}_{\eta(i,k)-1}\right)\right] < \varepsilon. \tag{4.55c}$$

For the second bound we use the fact that M has a finite state space so that

$$|\pi^i(X_t, M_t^{(N)})| \leq \max_m |\pi^i(X_t, m)|$$

which still satisfies the growth condition. Using (4.55) and (2.5) we have for $N > N^\varepsilon$

$$\begin{aligned}
&\left|J_m^i(x\,; \boldsymbol{\alpha}_\varepsilon^{i,(N)}, \boldsymbol{s}^j) - J_m^i(x\,; \boldsymbol{\alpha}_\varepsilon^i, \boldsymbol{s}^j)\right| \\
&\leq \mathbb{E}_{x,m}\Big[\int_{\sigma_N^{i,\infty}}^\infty e^{-rt}\big(|\pi^i\left(X_t, M_t^{(\infty)}\right)| + |\pi^i(X_t, M_t^{(N)})|\big)dt \\
&\qquad\qquad + \sum_{k=N+1}^\infty e^{-r\sigma_k^{i,(\infty)}} K^i\left(X_{\sigma_k^{i,(\infty)}}, \tilde{M}_{\eta(i,k)-1}^{(\infty)}\right)\Big] \leq 3\varepsilon. \qquad (4.56)
\end{aligned}$$

By Fatou's lemma and (4.56) we obtain

$$\begin{aligned}
&\liminf_{N\to\infty} J_m^i(x\,; \boldsymbol{\alpha}_\varepsilon^{i,(N)}, \boldsymbol{s}^j) \\
&= \liminf_{N\to\infty} \mathbb{E}_{x,m}\left[\int_0^\infty e^{-rt}\pi^i(X_t, M_t^{(N)})dt - \sum_{k=1}^N K^i\left(X_{\sigma_k^{i,(N)}}, \tilde{M}_{\eta(i,k)-1}^{(N)}\right) \cdot e^{-r\sigma_k^{i,(N)}}\right] \\
&\geq J_m^i(x\,; \boldsymbol{\alpha}_\varepsilon^i, \boldsymbol{s}^j) - 3\varepsilon. \qquad (4.57)
\end{aligned}$$

In turn, from (4.53), we get

$$\liminf_{N\to\infty} \widetilde{V}_m^{i,(N)}(x\,; \boldsymbol{s}^j) \geq \liminf_{N\to\infty} J_m^i(x\,; \boldsymbol{\alpha}_\varepsilon^{i,(N)}, \boldsymbol{s}^j) \geq \widetilde{V}_m^i(x\,; \boldsymbol{s}^j) - 4\varepsilon, \qquad (4.58)$$

which along with (4.52) and letting $\varepsilon \downarrow 0$ completes the proof.

4.5.3 Dynamics of $\tilde{M}^*$ in Threshold-type Equilibrium

In this subsection, we present computational details related to the macro market equilibrium described in Section 4.2.3. While the computations are largely classical, we state them for the completeness and the reader's convenience. For ease of presentation we consider the case where $s_m^{i,*}$'s are in strictly ascending/descending order in terms of m, so that all transitions of M^* are by ± 1.

4.5.3.1 Transition probabilities of $\check{M}^*$ in interior states

Conditional on $\check{M}^*_{n-1} \in \{m^-, m^+\}$ and $m \notin \partial\mathcal{M}$, we have that ($\tilde{X}^{(n)}$ being defined in (4.8a))

$$\check{M}^*_n = \begin{cases} (m+1)^+, & \text{if } \tilde{X}_t^{(n)} \text{ hits } s_m^{1,*} \text{ before } s_m^{2,*}, \\ (m-1)^-, & \text{if } \tilde{X}_t^{(n)} \text{ hits } s_m^{2,*} \text{ before } s_m^{1,*}, \end{cases} \tag{4.59}$$

with the starting position $\tilde{X}_0^{(n)} = s_{m-1}^{1,*}$ if $\check{M}^*_n = m^+$ and $\tilde{X}_0^{(n)} = s_{m+1}^{2,*}$ if $\check{M}^*_n = m^-$. Let us use X^x to denote a generic copy of X started at $X_0 = x$ and consider the two-sided passage times

$$\tau(x; a, b) := \inf\{t \geq 0 \,:\, X_t^x \leq a \text{ or } X_t^x \geq b\}, \qquad (a, b) \supset x.$$

Thus we have:

$$\begin{aligned} \boldsymbol{P}_{m^+,(m+1)^+} &= \mathbb{P}\big[X^{s_{m-1}^{1,*}}_{\tau(s_{m-1}^{1,*}; s_m^{2,*}, s_m^{1,*})} = s_m^{1,*}\big], & \boldsymbol{P}_{m^+,(m-1)^-} &= \mathbb{P}\big[X^{s_{m-1}^{1,*}}_{\tau(s_{m-1}^{1,*}; s_m^{2,*}, s_m^{1,*})} = s_m^{2,*}\big], \\ \boldsymbol{P}_{m^-,(m+1)^+} &= \mathbb{P}\big[X^{s_{m+1}^{2,*}}_{\tau(s_{m+1}^{2,*}; s_m^{2,*}, s_m^{1,*})} = s_m^{1,*}\big], & \boldsymbol{P}_{m^-,(m-1)^-} &= \mathbb{P}\big[X^{s_{m+1}^{2,*}}_{\tau(s_{m+1}^{2,*}; s_m^{2,*}, s_m^{1,*})} = s_m^{2,*}\big]. \end{aligned} \tag{4.60}$$

Evaluation of (4.60) via the scale function $S(\cdot)$ of (X_t) is stated in Section 2.3.2.2.

4.5.3.2 Transition probabilities of $\check{M}^*$ in boundary regimes

Recall that at regimes $\underline{m}^-, \overline{m}^+$ only one player can switch. For a recurrent X, she is guaranteed to do so eventually and we simply have

$$\boldsymbol{P}_{\underline{m}^-,(\underline{m}+1)^+} = \boldsymbol{P}_{\overline{m}^+,(\overline{m}-1)^-} = 1. \tag{4.61}$$

When X is transient, one player will be permanently dominant in the long-run and at least one of the following absorbing probabilities

$$\begin{aligned} P_{\underline{m}^a} &:= \mathbb{P}\big[X_t^{s^{2,*}_{\underline{m}+1}} \le s^{1,*}_{\underline{m}} \forall t\big] = \lim_{d \downarrow \underline{d}} \mathbb{P}\big[X^{s^{2,*}_{\underline{m}+1}}_{\tau(s^{2,*}_{\underline{m}+1}; d, s^{1,*}_{\underline{m}})} = d\big], \\ P_{\overline{m}^a} &:= \lim_{u \uparrow \overline{d}} \mathbb{P}\big[X^{s^{1,*}_{\overline{m}-1}}_{\tau(s^{1,*}_{\overline{m}-1}; s^{2,*}_{\overline{m}}, u)} = u\big], \end{aligned} \tag{4.62}$$

are strictly positive. Namely, when M^* enters a boundary regime, there is a positive probability that M^* will stay constant henceforth. To address this, we use the states $\{\underline{m}^a, \overline{m}^a\}$ of the extended $\check{M}$ that are entered from the regime adjacent to the corresponding boundary. For instance, three transitions are possible from $\check{M}^*_{n-1} \in \{(\overline{m}-1)^-, (\overline{m}-1)^+\}$:

$$\check{M}^*_n = \begin{cases} \text{up to} \quad \overline{m}^a, & \text{if } \tilde{X}^{(n)}_t \text{ hits } s^{1,*}_{\overline{m}-1} \text{ before } s^{2,*}_{\overline{m}-1} \text{ and } \check{M}^* \text{ gets absorbed}, \\ \text{up to} \quad \overline{m}^+, & \text{if } \tilde{X}^{(n)}_t \text{ hits } s^{1,*}_{\overline{m}-1} \text{ before } s^{2,*}_{\overline{m}-1} \text{ and } \check{M}^* \text{ is not absorbed}, \\ \text{down to} \quad (\overline{m}-2)^-, & \text{if } \tilde{X}^{(n)}_t \text{ hits } s^{2,*}_{\overline{m}-1} \text{ before } s^{1,*}_{\overline{m}-1}\ . \end{cases} \tag{4.63}$$

Probabilistically, we may interpret absorption as an independent "coin toss" at the transition out of $(\overline{m}-1)^{\pm}$, so that using (4.62)

$$\boldsymbol{P}_{(\overline{m}-1)^+,\overline{m}^a} = \mathbb{P}\big[X^{s^{1,*}_{\overline{m}-2}}_{\tau(s^{1,*}_{\overline{m}-2};s^{2,*}_{\overline{m}-1},s^{1,*}_{\overline{m}-1})} = s^{1,*}_{\overline{m}-1}\big] \times P_{\overline{m}^a}, \tag{4.64}$$

$$\boldsymbol{P}_{(\overline{m}-1)^+,\overline{m}^+} = \mathbb{P}\big[X^{s^{1,*}_{\overline{m}-2}}_{\tau(s^{1,*}_{\overline{m}-2};s^{2,*}_{\overline{m}-1},s^{1,*}_{\overline{m}-1})} = s^{1,*}_{\overline{m}-1}\big] \times (1-P_{\overline{m}^a}), \tag{4.65}$$

$$\boldsymbol{P}_{(\overline{m}-1)^+,(\overline{m}-2)^-} = 1 - \boldsymbol{P}_{(\overline{m}-1)^+,\overline{m}^a} - \boldsymbol{P}_{(\overline{m}-1)^+,\overline{m}^+}. \tag{4.66}$$

Similar computations are used for $\boldsymbol{P}_{(\overline{m}-1)^-,\cdot}, \boldsymbol{P}_{(\underline{m}+1)^-,\cdot}, \boldsymbol{P}_{(\underline{m}-1)^+,\cdot}$.

4.5.3.3 Average sojourn times of $\check{M}^*$

The expected sojourn times $\vec{\xi}$ of M^* in (4.20), or equivalently expected inter-arrival times between jumps of $\check{M}^*$ correspond to the mean two-sided exit time, $\delta_{ab}(x) := \mathbb{E}\big[\tau(x;a,b)\big], x \in (a,b)$, namely

$$\xi_{m^-} := \mathbb{E}\big[\tau(s^{2,*}_{m+1}; s^{2,*}_m, s^{1,*}_m)\big], \quad \xi_{m^+} := \mathbb{E}\big[\tau(s^{1,*}_{m-1}; s^{2,*}_m, s^{1,*}_m)\big], \tag{4.67}$$

for $m \notin \partial\mathcal{M}$. We refer Section 2.3.2.1 for detailed evaluation.

4.5.3.4 One-sided exit times and sojourn times in boundary regimes

To compute mean sojourn times $\xi_{\underline{m}^-}, \xi_{\overline{m}^+}$ we make use of the one-sided passage times

$$\tau(x;s) := \inf\{t \geq 0 \,:\, X^x_t = s\}.$$

If the corresponding absorbing probability (4.62) is zero, we have

$$\xi_{\underline{m}^-} := \mathbb{E}\big[\tau(s^{2,*}_{\underline{m}+1}; s^{1,*}_{\underline{m}})\big], \qquad \xi_{\overline{m}^+} := \mathbb{E}\big[\tau(s^{1,*}_{\overline{m}-1}; s^{2,*}_{\overline{m}})\big].$$

Otherwise, we condition on the exit time τ being finite, denoting $\delta_s(x) = \mathbb{E}[\tau(x;s)\mathbb{1}_{\{\tau(x;s)<\infty\}}]$. Then, e.g.

$$\xi_{\overline{m}^+} = \mathbb{E}\big[\tau(s^{1,*}_{\overline{m}-1}; s^{2,*}_{\overline{m}})\big|\tau(s^{1,*}_{\overline{m}-1}; s^{2,*}_{\overline{m}}) < \infty\big] = \frac{1}{1-P_{\overline{m}^a}}\delta_{s^{2,*}_{\overline{m}}}(s^{1,*}_{\overline{m}-1}). \tag{4.68}$$

Computation of $\delta_s(x)$ is stated in Section 2.3.2.1.

Geometric Brownian motion. GBM is non-recurrent; suppose that $\mu - \frac{1}{2}\sigma^2 > 0$ so that $\overline{m}^a$ is the absorbing regime. Then from (2.44) we compute

$$\delta_{s^{2,*}_{\overline{m}}}(s^{1,*}_{\overline{m}-1}) = \mathbb{E}\Big[\tau(s^{1,*}_{\overline{m}-1}; s^{2,*}_{\overline{m}})\mathbb{1}_{\{\tau(s^{1,*}_{\overline{m}-1};s^{2,*}_{\overline{m}})<\infty\}}\Big] = \frac{1}{\mu-\frac{1}{2}\sigma^2}\cdot\ln\left(\frac{s^{1,*}_{\overline{m}-1}}{s^{2,*}_{\overline{m}}}\right)\cdot\left(\frac{s^{1,*}_{\overline{m}-1}}{s^{2,*}_{\overline{m}}}\right)^{1-\frac{2\mu}{\sigma}}.$$

4.5.3.5 Expected number of switches until absorption under non-recurrent (X_t)

Without loss of generality, let us assume that $\overline{m}$ is the absorbing regime, so that $\lim_{t\to\infty} M^*_t = \overline{m}$. Define

$$\begin{aligned}
\upsilon^{up}_e :=& \mathbb{E}\Big[\#\textit{up-moves} \text{ before } \check{M}^* \text{ hits } \overline{m}^a \,|\, \check{M}^*_0 = e\Big], \qquad e \in E\setminus\{\overline{m}^a\},\\
\upsilon^{dn}_e :=& \mathbb{E}\Big[\#\textit{down-moves} \text{ before } \check{M}^* \text{ hits } \overline{m}^a \,|\, \check{M}^*_0 = e\Big], \qquad e \in E\setminus\{\overline{m}^a\},
\end{aligned}$$

where E is the state space of $\check{M}$ from (4.19) and $\boldsymbol{P}$ is the transition matrix of $\check{M}^*$. Let $\boldsymbol{P}_{-a}$ be the sub-matrix with the row and column corresponding to $\overline{m}^a$ removed. Define $\vec{\upsilon}^{up} := [\upsilon^{up}_{\underline{m}^-}, \cdots, \upsilon^{up}_{\overline{m}^+}]^T$, $\vec{\upsilon}^{dn} := [\upsilon^{dn}_{\underline{m}^-}, \cdots, \upsilon^{dn}_{\overline{m}^+}]^T$, $\vec{P}^{up} := [P^{up}_{\underline{m}^-}, \cdots, P^{up}_{\overline{m}^+}]^T$ and

$\vec{P}^{dn} := [P^{dn}_{\underline{m}^-}, \cdots, P^{dn}_{\overline{m}^+}]^T$, with

$$\begin{cases} P^{up}_{m^\pm} := \boldsymbol{P}_{m^\pm,(m+1)^+}, \text{ for } m < \overline{m}-1, & P^{up}_{\overline{m}^+} := 0, \\ P^{dn}_{m^\pm} := \boldsymbol{P}_{m^\pm,(m-1)^-}, \text{ for } m > \underline{m}, & P^{dn}_{\underline{m}^-} := 0, \end{cases}$$

and $P^{up}_{(\overline{m}-1)^\pm} := \boldsymbol{P}_{(\overline{m}-1)^\pm,\overline{m}^+} + \boldsymbol{P}_{(\overline{m}-1)^\pm,\overline{m}^a}$. Then we obtain

$$\vec{v}^{up} = (\boldsymbol{I} - \boldsymbol{P_{-a}})^{-1}\vec{P}^{up} \quad \text{and} \qquad \vec{v}^{dn} = (\boldsymbol{I} - \boldsymbol{P_{-a}})^{-1}\vec{P}^{dn}, \tag{4.69}$$

and after taking care of the initial condition $X_0 = x$ which leads to a non-standard first transition probability, obtain the expected number of switches defined in (4.22)

$$\begin{aligned} \mathbb{N}^1_m(x) &= \mathbb{P}_{x,m}\big[\check{M}^*_1 = (m+1)^+\big] \times \big(v^{up}_{(m+1)^+} + 1\big) + \mathbb{P}_{x,m}\big[\check{M}^*_1 = (m-1)^-\big] \times v^{up}_{(m-1)^-}, \\ \mathbb{N}^2_m(x) &= \mathbb{P}_{x,m}\big[\check{M}^*_1 = (m+1)^+\big] \times v^{dn}_{(m+1)^+} + \mathbb{P}_{x,m}\big[\check{M}^*_1 = (m-1)^-\big] \times \big(v^{dn}_{(m-1)^-} + 1\big). \end{aligned} \tag{4.70}$$

In general $\mathbb{P}_{x,m}\big[\check{M}^*_1 = (m+1)^+\big] = \mathbb{P}(X^x_{\tau(x;s^{2,*}_m,s^{1,*}_m)} = s^{1,*}_m)$; however one must also consider the situation when $M^*_0 = \overline{m}-1$, so that $\check{M}^*_1 = \overline{m}^a$ becomes possible, and also $M^*_0 = \overline{m}$, in which case one must assign $\check{M}_0 = \overline{m}^a$ or $\check{M}_0 = \overline{m}^+$ according to the probability $P_{\overline{m}^a}(x) := \lim_{u\uparrow\overline{d}} \mathbb{P}\big[X^x_{\tau(x;s^{2,*}_{\overline{m}},u)} = u\big]$.

4.5.3.6 Non-recurrent (X_t): expected time until absorption

To begin with, we need the expected number of visits to each non-absorbing regime. Define

$$\mathcal{V}_{e_1,e_2} := \mathbb{E}\left[\#\textit{visits} \text{ to } e_2 \text{ before } \check{M}^* \text{ reaches } \overline{m}^a \,\middle|\, \check{M}^*_0 = e_1\right], \qquad \text{for all } e_1, e_2 \in E \setminus \{\overline{m}^a\},$$

and let $\boldsymbol{\mathcal{V}}$ denote the matrix of $\mathcal{V}_{e_1,e_2}$ with rows $\vec{\mathcal{V}}_{e_1,\cdot} := \left[\mathcal{V}_{e_1,\underline{m}^-},\ldots,\mathcal{V}_{e_1,\overline{m}^+}\right]$, for $e_1 \in E \setminus \{\overline{m}^a\}$. Then from standard Markov chain arguments,

$$\boldsymbol{\mathcal{V}} = (\boldsymbol{I} - \boldsymbol{P_{-a}})^{-1},$$

where $\boldsymbol{P}_{-a}$ is the transient transition sub-matrix defined in the preceding subsection. Multiplying by the respective sojourn times ξ_m, the expected absorption time starting from an arbitrary regime $e \in E \setminus \{\overline{m}^a\}$ is $\widetilde{\mathbb{T}}_e := \vec{\mathcal{V}}_{e,\cdot} \cdot \vec{\xi}_{-a}$, where $\vec{\xi}_{-a}$ is the vector of expected sojourn times excluding $\xi_{\overline{m}^a}$. Finally, the expected time until M^* gets absorbed, as defined in (4.23), is admitted as (cf. (4.70))

$$\begin{aligned}\mathbb{T}_m(x) = \mathbb{E}_x\left[\tau(x; s_m^{2,*}, s_m^{1,*})\right] &+ \mathbb{P}_{x,m}\left[\check{M}_1^* = (m+1)^+\right] \times \widetilde{\mathbb{T}}_{(m+1)^+} \\ &+ \mathbb{P}_{x,m}\left[\check{M}_1^* = (m-1)^-\right] \times \widetilde{\mathbb{T}}_{(m-1)^-}.\end{aligned}$$

Again further adjustments are needed when $M_0^* = \overline{m} - 1$ or $M_0^* = \overline{m}$ as discussed in the preceding subsection.

4.5.4 Proof of Corollary 4.10

Proof: From (4.24b) and (4.24c), we write $V_{+1}^1(x)$ explicitly for $x \geq \check{s}$ by substituting in the respective expressions for ν_{+1}^1 and ω_{-1}^1:

$$\begin{aligned}V_{+1}^1(x) &= D_{+1}^1(x) + \nu_{+1}^1 G(x) \\ &= D_{+1}^1(x) + \frac{V_{-1}^1(-\check{s}) - D_{+1}^1(-\check{s})}{G(-\check{s})} G(x) \\ &= D_{+1}^1(x) + \frac{D_{-1}^1(-\check{s}) + \omega_{-1}^1 F(-\check{s}) - D_{+1}^1(-\check{s})}{G(-\check{s})} G(x) \\ &= D_{+1}^1(x) + \frac{G(x)}{G(-\check{s})}\Big[(V_{+1}^1 - D_{-1}^1 - K_{-1}^1)(\check{s})\frac{F(-\check{s})}{F(\check{s})} + D_{-1}^1(-\check{s}) - D_{+1}^1(-\check{s})\Big].\end{aligned}$$

The above gives an equation relating $V^1_{+1}(x)$ to $V^1_{+1}(\check{s})$; therefore, if one defines

$$Q(s) := \frac{D^1_{+1}(s) - \frac{G(s)}{G(-s)}\Big[\big(D^1_{-1}(s) + K^1_{-1}(s)\big)\frac{F(-s)}{F(s)} + D^1_{+1}(-s) - D^1_{-1}(-s)\Big]}{1 - \frac{G(s)}{G(-s)}\frac{F(-s)}{F(s)}}, \tag{4.71}$$

and let $\check{s}$ be a solution to the system (4.24), then it holds $Q(\check{s}) = V^1_{+1}(\check{s})$. Similarly, after differentiating with respect to x (guaranteed by Corollary 4.10 which requires smoothness of $D^1_m(\cdot)$ and $K^1_m(\cdot)$), one can define

$$q(s) := D^{1'}_{+1}(s) + \frac{G'(s)}{G(-s)}\Big[(Q(s) - D^1_{-1}(s) - K^1_{-1}(s))\frac{F(-s)}{F(s)} + D^1_{-1}(-s) - D^1_{+1}(-s)\Big], \tag{4.72}$$

and conclude $q(\check{s}) = V^{1'}_{+1}(\check{s})$. Then replacing $V^1_{+1}(x)$ by $Q(x)$ and $(V^1_{+1})'(x)$ by $q(x)$ in (4.24a) we obtain that solving the system (4.24) is equivalent to finding the root(s) of

$$\mathcal{Z}(s) := \big[Q(s) - D^1_{-1}(s) - K^1_{-1}(s)\big]F'(s) - \big[q(s) - (D^1_{-1})'(s) - (K^1_{-1})'(s)\big]F(s) = 0, \tag{4.73}$$

Since $\check{s} > s^{2,*}_{+1} = -\check{s} \implies \check{s} > 0$ (otherwise the switching regions would overlap), we seek positive solutions to (4.73). We shall show that $\mathcal{Z}(0) < 0$ and $\mathcal{Z}(s) > 0$ for s large enough, which by continuity (as each term in (4.73) is continuous) implies the existence of a root.

On the one hand, the numerator of $Q(s)$ at $s = 0$ is admitted as

$$D^1_{+1}(0) - \frac{G(0)}{G(0)}\Big[\big(D^1_{-1}(0) + K^1_{-1}(0)\big)\frac{F(0)}{F(0)} + D^1_{+1}(0) - D^1_{-1}(0)\Big] = -K^1_{-1}(0) < 0,$$

while the denominator $1 - \frac{G(s)}{G(-s)}\frac{F(-s)}{F(s)} = 1 - \big(\frac{F(-s)}{F(s)}\big)^2$ is strictly positive ($F(\cdot)$ is increasing) for $s > 0$ and tends to zero as $s \downarrow 0$, so that $\lim_{s\downarrow 0} Q(s) = -\infty$. Furthermore,

$$\begin{aligned}\lim_{s\downarrow 0} q(s) &= (D^1_{+1})'(0) + \frac{G'(0)}{G(0)}\Big[(\lim_{s\downarrow 0} Q(s) - D^1_{-1}(0) - K^1_{-1}(0))\frac{F(0)}{F(0)} + D^1_{-1}(0) - D^1_{+1}(0)\Big] \\ &= +\infty,\end{aligned}$$

since $G(\cdot)$ is positive and decreasing ($G'(\cdot) < 0$) while all other terms beyond $\lim_{s\downarrow 0} Q(s)$ are finite. Putting everything together,

$$\begin{aligned}\lim_{s\downarrow 0} \mathcal{Z}(s) &= \big[\lim_{s\downarrow 0} Q(s) - D^1_{-1}(0) - K^1_{-1}(0)\big]F'(0) - \big[\lim_{s\downarrow 0} q(s) - (D^1_{-1} - K^1_{-1})'(0)\big]F(0) \\ &= -\infty,\end{aligned}$$

since $F(\cdot)$ is positive and increasing, and all other terms are finite.

On the other hand, for s large enough and using the property of F and G at natural boundary points (2.14), we have $Q(s) \approx D^1_{+1}(s), q(s) \approx (D^1_{+1})'(s)$ asymptotically as $s \uparrow \bar{d}$ and hence

$$\begin{aligned}\mathcal{Z}(s) &\approx \big[D^1_{+1}(s) - D^1_{-1}(s) - K^1_{-1}(\bar{s})\big]F'(s) - \big[D^{1'}_{+1}(s) - D^{1'}_{-1}(s) - K^{1'}_{-1}(s)\big]F(s), \\ &= \Big[-\big(\frac{D^1_{+1}(s) - D^1_{-1}(s) - K^1_{-1}(s)}{F(s)}\big)'\Big]\cdot F^2(s) > 0.\end{aligned}$$

The last inequality follows from $\Delta D := D^1_{+1} - D^1_{-1} - K^1_{-1} \in \mathcal{H}_{\text{inc}}$ (cf. Definition 2.7), thus

$$\begin{cases}\limsup_{s\uparrow\bar{d}} \dfrac{\Delta D(s)}{F(s)} = 0, \\ \Delta D(s) > 0, \text{ for } s \text{ large,}\end{cases} \implies \left(\frac{\Delta D(s)}{F(s)}\right)' < 0 \quad \text{as} \quad s \uparrow \bar{d}.$$

■

Chapter 5

Vertical Impulse Competition

In this chapter, we consider the dynamic competition between the producer and the consumer of a commodity. As an extension to our work stated in Chapter 3 and Chapter 4, these two players possess distinct types of controls and are allowed to intervene the local market condition. In particular, the producer is able to move X_t to her desired level by exercising impulse control, while the consumer can change the macro market regime (e.g. the overall demand level) which would increase/decrease the commodity price on average. The process X is then (partially) jointly controlled by the players rather than being fully exogenous. Moreover, the macro market regime M, though fully endogenously determined, affects not only the players' profitabilities but also the underlying commodity price X, which leads us to a more complex feedback effect between these two processes. By hypothesizing structure of the dynamic competition in equilibrium, we obtain reasonable threshold-type Nash equilibria by iteratively applying the best-response maps. The preliminary results represented here are based on the ongoing work [4] without rigorous mathematical proof, which will be completed in future research.

5.1 Problem Formulation

We consider vertical competition among producers, Player 1, and consumers, Player 2, of a commodity. The producer extracts the commodity at cost c_q and sells it for a price X. The consumer buys the commodity and converts it into a final good that has price P. This situation could represent a range of industries, for example extraction of raw oil, which is then consumed by refineries and chemical industries into final consumer goods. Or the production of aluminum that is converted by automakers into vehicles. We focus on the role of the commodity price X that intrinsically creates competition between the two players (which should be thought of as representative agents of the respective industry sectors). Indeed, producers prefer high X, while consumers prefer low X. This competition is dynamic and manifests itself through strategic price effects actuated by the two industries. Therefore, X is (partially) jointly controlled by the producer/consumer, leading to game-theoretic impacts.

On the production side, the producer needs X_t to be high enough to make a profit margin, and can directly influence the supply (think of OPEC). Such *production capacity shocks* lead to disinvestment/investment shocks summarized by $N_t := \sum_{s \leq t} \xi_s$, where ξ_s denotes the amount by which the producer shifts the price X_s. On the consumption side, consumers can be in austerity or expansion mode that lowers demand for the commodity and influences the *drift* of X_t (think building more fuel efficient cars when oil prices become high, or substituting aluminum with other materials when aluminum is too expensive). Thus, we assume that the drift $\mu_t \in \{\mu_-, \mu_+\}$ of (X_t) fluctuates over time according to consumer behavior, with $\mu_+ > 0 > \mu_-$. Because such consumer shifts are slow and expensive, the drift term is persistent (i.e. piecewise constant in time) and changing it incurs heavy switching costs. In fact, it follows from above narrative that the drift term reflects the macro market. We then model the macro market regime by a process

M taking values in $\{\mu_+, \mu_-\}$ and name them as Reduce/Expand regime. Consequently, the price follows the dynamics

$$dX_t = -M_t dt + \sigma dW_t + dN_t, \tag{5.1}$$

where $N_t := \sum_{s \le t} \xi_s$ summarize impulses of the producer and the Brownian motion (W_t) captures exogenous price shocks due to speculators, geopolitical events, etc. We note that both players can push X in either direction, although their actions of distinct types, namely impulse control by the producer and switching-drift control by the consumer. However, the consumer is the only player who may exercise switching controls to change M. Notice also that the commodity price follows a Brownian motion with drift if no impulse exercised by the producer.

5.1.1 Instantaneous Profit Rates

The price X of the commodity also influences the *volume* of trade. This is captured by the demand function $Q(X)$. A similar phenomenon plays out in the final-good market: the goods price P leads to sales volume $Q_A(P)$. Since the consumer is in effect the intermediary between the commodity and the goods market, she will pass some of her price shocks to $P \equiv P(X)$.

Based on the above discussion, the instantaneous profit rate of the producer is

$$\pi^1(x) := (x - c_q)Q(x) \tag{5.2}$$

while the instantaneous profit rate of the consumer is

$$\pi^2(x) := Q_A(P)P - \alpha Q(x)(x + c_d) \tag{5.3}$$

where α is the proportion of commodity consumed by the consumer (e.g. percentage of overall crude oil used to produce gasoline) and c_d is the processing cost from input commodity to final good. We shall consider linear inverse demand

$$Q(X) = d_0 - d_1 X,$$

which leads to a quadratic profit rate of the producer

$$\pi^1(x) = -d_1 x^2 + (d_0 + c_q d_1)x - c_q d_0. \tag{5.4}$$

If we further assume that $P(X) = p_0 + p_1 X$ (the price of the final good is linearly proportional to the commodity price), and $Q_A(P) = d_0' - d_1' P$ (final good demand is linearly decreasing in its price P), the profit rate of the consumer becomes:

$$\begin{aligned}\pi^2(x) &= Q_A(P(x))P(x) - \alpha Q(x)(x + c_d)\\ &= p_0(d_0' - d_1' p_0) - \alpha d_0 c_d + (p_1(d_0' - 2d_1' p_0) - \alpha(d_0 - d_1 c_d))x + (\alpha d_1 - d_1' p_1^2)x^2\\ &=: \gamma_0 + \gamma_1 x + \gamma_2 x^2. \end{aligned}\tag{5.5}$$

To ensure that the consumer profit is concave (downward parabola) in x is equivalent to assuming that the output market is more elastic to the price x than the intermediary market: $\alpha d_1 < d_1' p_1^2 \Leftrightarrow \gamma_2 < 0$.

To sum up, the profit rates of both producer and consumer are concave and quadratic in x, implying that each player has their *preferred* commodity levels $\bar{X}_q, \bar{X}_d$ that maximize their profit rates:

$$\bar{X}_q = \frac{d_0 + c_q d_1}{2d_1} \qquad \bar{X}_d = \frac{\gamma_1}{-2\gamma_2}.$$

Typically, we expect that $\bar{X}_d < \bar{X}_q$, so that the preferred commodity price by the consumer is lower than that of the producer. In turn this implies that the stochastic fluctuations coming from (W_t) can generate three different market conditions:

$X_t < \bar{X}_d$	abnormally low prices
$\bar{X}_d \le X_t \le \bar{X}_q$	vertical competition
$X_t > \bar{X}_q$	very high prices

In the first case, both players wish to raise X_t; in the last setup both wish to lower it. In the most interesting and relevant intermediate case, they compete against each other.

5.1.2 Admissible Strategies and Game Payoff

In line with above motivating economic narrative, we postulate the players adopt *timing strategies*:

$$\boldsymbol{\alpha}^1 = \{\big(\tau^1(n), \xi(n)\big) \,:\, n \ge 1\}, \qquad \boldsymbol{\alpha}^2 = \{\tau^2(n) \,:\, n \ge 1\}, \tag{5.6}$$

where $\xi(n)$'s denote the amounts Player 1 will intervene X by exercising impulse controls. To define admissibility recursively, we require all $\tau^i(n)$'s and $\xi(n)$ to be adapted to $\tilde{\mathcal{F}}^{(n)}$ defined in (2.2) with a minor change in (2.2d)

$$\tilde{M}_n = \tilde{M}_{n-1} \cdot \mathbb{1}_{\{P_n=1\}} + C^2_{\tilde{M}_{n-1}} \cdot \mathbb{1}_{\{P_n=2\}}, \tag{5.7}$$

where the consumer's action sets are

$$C^2_{\mu^-} := \{\mu^+\}, \qquad C^2_{\mu^+} := \{\mu^-\}.$$

Note that action sets of the producer, which is in effect considered as the set of impulses, are *not empty* though she is not able to change the macro market regime. Consequently, admissibility of these players' strategy profiles follows from Definition 2.2.

The objective functionals of the players consist of integrated profit rates $\pi^{\cdot}(x)$, discounted at constant rate $r > 0$ and subtracting the control costs. We take the investment cost function of the producer to be $K^1(\xi) = \kappa_0 + \kappa_1|\xi|$ and of the consumer as $K^2(\mu) = h_0 > 0$. Given a strategy profile $(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2)$ and $X_0 = x, M_0 = \mu_0$, the producer's game payoff is given by:

$$J^1_{\pm}(x; \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2) := \mathbb{E}_x\Big[\int_0^{\infty} e^{-r\,t}\big(X_t - c_q\big)Q(X_t)dt - \sum_{n=1}^{+\infty} \mathbb{1}_{\{P_n=1\}} e^{-r\,\sigma_n} \cdot K^1\big(\xi(n)\big)\Big], \quad (5.8)$$

and similarly the representative consumer's game payoff is:

$$J^2_{\pm}(x; \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2) := \mathbb{E}_x\Big[\int_0^{\infty} e^{-r\,t}\big(\gamma_0 + \gamma_1 X_t + \gamma_2 X_t^2\big)dt - \sum_{n=1}^{+\infty} \mathbb{1}_{\{P_n=2\}} e^{-r\,\sigma_n} \cdot K^2(\tilde{M}_{n-1})\Big], \quad (5.9)$$

where we use the subscript to denote the initial drift μ_0 being positive/negative. Recall the construct (2.2) that n is the counter of the overall "round" of the game and σ_n records the corresponding n-th acting time (either of the producer or the consumer). (X_t) is jointly controlled by these two players as defined in (5.1) with $M_t = \tilde{M}_{\eta(t)}$ and $N_t = \sum_k \xi(k) \mathbb{1}_{\{\sigma^1_k \le t\}}$, where σ^1_k indicates the producer's k-th intervention times and $\xi(k)$ denotes the corresponding impulse.

Lastly, we mention the scenario that no player exercises any control which leads us to the micro market condition modeled by a Brownian motion with a static drift term

$\mu_\pm$, i.e.

$$d\widetilde{X}_t = -\mu_\pm dt + \sigma dW_t,$$

and the corresponding static discounted future cashflows received by these players

$$D^i_\pm(x) := \mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i(\widetilde{X}_t)dt \,\middle|\, \widetilde{X}_0 = x\right]. \tag{5.10}$$

We compute them here for future usage. For the producer, we have

$$D^1_\pm(x) = Ax^2 + B_\pm x + C_\pm, \tag{5.11}$$

where

$$A = -\frac{d_1}{r}, \quad B_\pm = \frac{1}{r}\Big(d_0 + \frac{2\,\mu_\pm\,d_1}{r} + c_q\,d_1\Big), \quad C_\pm = \frac{1}{r}\big(- \mu_\pm B_\pm + A\sigma^2 - c_q d_0\big).$$

Similarly for the consumer, we obtain

$$D^2_\pm(x) = Ex^2 + F_\pm x + G_\pm, \tag{5.12}$$

where

$$E = \frac{\gamma_2}{r}, \quad F_\pm = \frac{1}{r}\Big(\gamma_1 - 2\mu_\pm\frac{\gamma_2}{r}\Big), \quad G_\pm = \frac{1}{r}\big(\gamma_0 + \sigma^2\frac{\gamma_2}{r} - \mu_\pm F_\pm\big).$$

5.2 Constructing Equilibria

We aim to construct explicit equilibria for the vertical competition between the producer and the consumer. In particular, we shall aim to construct threshold-type Feedback

Nash equilibria which allow structural insights of the competition and the resulting two time-scale market dynamics. To begin with, we conjecture the shape of players' acting regions, illustrate the resulting possible competitive dynamics and characterize players' game values via variational inequalities in Section 5.2.1. Best-response of the players are discussed in Section 5.2.2 and Section 5.2.3, which allows us to determine Nash equilibria by implementing tatonnement, i.e. iterating best-response. Reasonable Nash equilibria constructed are also discussed in Section 5.3.

5.2.1 Heuristics on Threshold-type Equilibrium

We concentrate on a specific class of strategies which are *stationary* and *of threshold-type*. Similar to Definition 2.5, these players exercise their controls (impulse or switch-type) at the first hitting times of their action regions $\Gamma^i_\cdot$'s which are characterized by thresholds. For lighter sub-/superscripts, we now use letter x to denote a threshold of the producer, and letter v to denote her game values. For the consumer, we use letter y to denote a threshold and letter w to denote her game values.

The action regions are expected to be as follows. The impulse continuation region $\left(\Gamma^1_\cdot\right)^c = (x_\ell, x_h)$ is two-sided: the producer will act whenever X_t reaches x_h from below or drops to x_ℓ from above. Note that these thresholds $x^\pm_\ell, x^\pm_h$ are μ-dependent. Such a conjecture follows naturally from the producer's quadratic profit rate π^1 in (5.4). When the producer intervenes, she will bring X_t to her impulse level $x^{\pm *}_{lh}$ so that the impulse amount is always $\xi^\pm = x^{\pm *}_{lh} - x^\pm_{lh}$. When $M_t = \mu_+$, we have the intuition that if X_t is very low, the consumer has an interest in switching from a decreasing to an increasing demand, i.e. when $X_t \le y_\ell$ for some endogenous threshold y_ℓ the consumer wants to switch to μ_-. Similarly, when $M_t = \mu_-$, the consumer would want to switch to μ_+ only when $X_t \ge y_h$ for another endogenous threshold y_h. To wit, the consumer's switching regions are $\Gamma^2_- = [y_h, +\infty)$ and $\Gamma^2_+ = (-\infty, y_\ell]$

The natural ordering we expect is that for the producer

$$x^\pm_\ell < x^{\pm*}_\ell < \bar{X}_q < x^{\pm*}_h < x^\pm_h \tag{5.13}$$

and for the consumer

$$y_\ell < \bar{X}_d < y_h \tag{5.14}$$

so that when acting both players try to move X towards their preferred levels. However, the precise ordering between the impulse thresholds $x^\pm$ and the switching thresholds y's is not clear a priori and will emerge as part of the overall equilibrium construction.

5.2.1.1 Illustrating Competitive Dynamics

To understand the market evolution under competition of the producer and consumer, we focus on the case where both players are active. The producer's strategy is summarized via a 2×4 matrix $\mathcal{C}^1$ which lists the thresholds $x^\pm_\ell, x^\pm_h$ and the target levels $x^{\pm*}_\ell, x^{\pm*}_h$. Thus, the no-intervention regions are $[x^\pm_\ell, x^\pm_h]$ and impulse amounts are $x^{\pm*}_h - x^\pm_h, x^{\pm*}_\ell - x^\pm_\ell$:

$$\mathcal{C}^1 = \begin{bmatrix} x^+_\ell, & x^{+*}_\ell, & x^+_h, & x^{+*}_h \\ x^-_\ell, & x^{-*}_\ell, & x^-_h, & x^{-*}_h \end{bmatrix}. \tag{5.15}$$

The consumer has two switching thresholds y_ℓ, y_h satisfying the following order among the thresholds

$$x^\pm_\ell < y_\ell < y_h < x^\pm_h. \tag{5.16}$$

Note that in the Expand regime (drift μ_-), we assume that $x_h^- > y_h$. Therefore, coming from below, X_t hits y_h first, causing the consumer to switch into the Reduce regime with drift μ_+. When $y_h \leq X_t < x_h^-$, the consumer switches first as well. To define the players' behavior when $X_t \geq x_h^-$ (i.e. the situation both players want to intervene), we assume that *the consumer has the priority* for simplicity. As a result, the impulse threshold x_h^- is not effective, i.e. will never get triggered along a controlled path of (X_t). Similar argument could be applied to claim that x_ℓ^+ is *not effective* either if $x_\ell^+ < y_\ell$. In Fig. 5.1, we provide an illustration to describe the vertical competition among the two players.

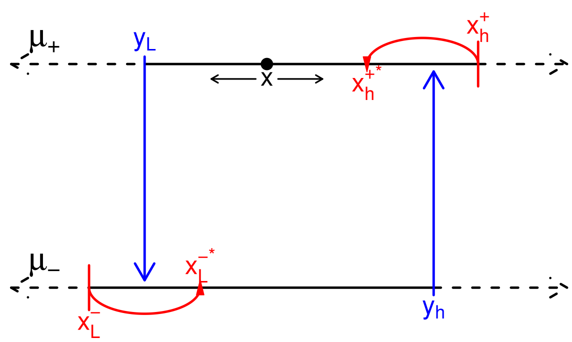


Figure 5.1: An illustration of the vertical competition between the producer and the consumer. The blue arrows represent drift-switching controls exercised by the consumer at levels $y_{l,h}$, while the red curved arrows represent impulse controls exercised by the producer at levels $x_{l,h}^{\pm}$.

To illustrate equilibrium dynamics, Figure 5.2 shows a sample trajectory of (X_t) with producer strategy $\mathcal{C}^1 = \begin{bmatrix} 1.0, & 1.3, & 2.0, & 1.7 \\ 1.0, & 1.3, & 2.0, & 1.7 \end{bmatrix}$ and consumer strategy $(y_\ell, y_h) = (1.2, 1.8)$. The effective thresholds are (y_ℓ, x_h^+) when $M_t = \mu_+$, or (x_ℓ^-, y_h) when $M_t = \mu_-$. In other words, in the Reduce regime, $(X_t)_{t>0}$ will be between $[1.2, 2.0]$, in the Expand regime it will be between $[1.0, 1.8]$, and overall we expect it will be bounded between $[1.0, 2.0]$. Extreme commodity price at the beginning of the game X_0 is still possible. Nevertheless, the game is well-defined following from above discussion. For instance, if

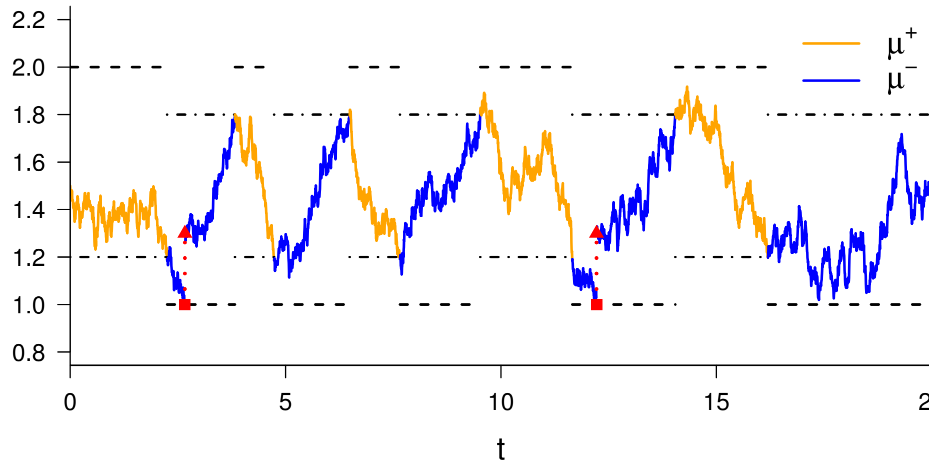


Figure 5.2: A sample path of the controlled market price (X_t) under competitive equilibrium. Observe that $X_t \in [1, 2]$ for all t

$M_0 = \mu_+, X_0 = 2.2$, the consumer (with the priority) will switch first the drift M_0 to μ_-, then the producer will move the price to $x_h^{-*} = 1.7$.

In Figure 5.2, we start in the Reduce regime with $X_0 = 1.5$ and $M_0 = \mu^+$. On this trajectory, (X_t) moves down (due to negative drift) and so the consumer switches to a positive drift to draw the price up. Nevertheless, the price keeps decreasing and hits $x_\ell^- = 1.0$, whereby the producer intervenes and pushes it to $x_\ell^{-*} = 1.3$. Prices then continue to rise up to $y_h = 1.8$ at which point the consumer switches again and starts pushing them back down (supposedly she wishes to keep them somewhere around 1.5). This cyclic behavior continues ad infinitum, yielding a stationary distribution for the pair (X_t, M_t). Note that in Expand/Reduce regime, the consumer uses her switching control to keep X_t from going too high or too low, and the producer acts as a "back-up", explicitly forcing prices from becoming extreme. The resulting mean-reversion behavior (due to M_t essentially alternating between μ_-, μ_+ as X_t moves up and down) is clearly evident. The additional interventions by the producer further make the domain of (X_t) to be bounded.

5.2.1.2 Heuristics on Game Values

We develop first some heuristics about possible Nash equilibria on the producer side by analyzing the value functions $v^+(x)$ and $v^-(x)$ resp. corresponding to a positive and negative drift. Ignoring for the moment the consumer, the value functions $v^\pm$ of the producer satisfy the quasi-variational inequality:

$$\sup\Big\{-rv^\pm - \mu_\pm v_x^\pm + \frac{1}{2}\sigma^2 v_{xx}^\pm + \pi^1(x); \sup_\xi \big\{v^\pm(x+\xi) - v^\pm(x) - K^1(\xi)\big\}\Big\} = 0. \quad (5.17)$$

Note that as written, the two variational inequalities are automonous, hence uncoupled from each other. The game coupling comes from the additional *boundary* condition that when the consumer switches, the producer payoffs are unaffected:

$$v^+(y_r) = v^-(y_r), \quad r \in \{\ell, h\}. \quad (5.18)$$

The general solution of the ODE

$$-rv - \mu_\pm v_x + \frac{1}{2}\sigma^2 v_{xx} + \pi^1(x) = 0$$

is of the form $v^\pm(x) = D^1_\pm(x) + u^\pm(x)$ where $D^1_\pm(x)$ is the static discounted cashflow functions and $u^\pm$ satisfies the homogenous ODE

$$-ru - \mu_\pm u_x + \frac{1}{2}\sigma^2 u_{xx} = 0,$$

i.e. the fundamental solutions computed in Section 2.3.1. Letting $\theta_1^\pm > 0$ and $\theta_2^\pm < 0$ be the two real roots of the quadratic $-r - \mu_\pm z + \frac{1}{2}\sigma^2 z^2 = 0$, we use the ansatz that

$$v^\pm(x) = D^1_\pm(x) + \lambda_1^\pm e^{\theta_1^\pm x} + \lambda_2^\pm e^{\theta_2^\pm x}, \quad (5.19)$$

for the solution of each equality in the continuation regions. When applying the impulse ξ, which are treated temporarily as unknowns, at the boundary $x_\ell^\pm$ the producer brings the system back to the point $x_\ell^{\pm *} := x_\ell^\pm + \xi_\ell^\pm$, resp. $x_h^{\pm *}$. Making the hypothesis that the value function is continuous at $x_{\ell h}^\pm$ we have:

$$v^\pm(x_\ell^\pm) = v^\pm(x_\ell^{\pm *}) - \kappa_0 - \kappa_1(x_\ell^{\pm *} - x_\ell^\pm), \tag{5.20}$$

$$v^\pm(x_h^\pm) = v^\pm(x_h^{\pm *}) - \kappa_0 - \kappa_1(x_h^\pm - x_h^{\pm *}). \tag{5.21}$$

And making the hypothesis that the value function is differentiable at the borders of the intervention region, we have:

$$v_x^\pm(x_\ell^\pm) = v_x^\pm(x_\ell^{\pm *}) + \kappa_1, \tag{5.22}$$

$$v_x^\pm(x_h^\pm) = v_x^\pm(x_h^{\pm *}) - \kappa_1. \tag{5.23}$$

Finally, the optimal investment impulse ξ is given by the first order condition at $x_{\ell h}^{\pm *}$

$$v_x^\pm(x_\ell^\pm + \xi_\ell^\pm) = \kappa_1, \tag{5.24}$$

$$v_x^\pm(x_h^\pm + \xi_h^\pm) = -\kappa_1. \tag{5.25}$$

Hence on producer's side the parameters to be determined are: $\lambda_1^\pm, \lambda_2^\pm$, the thresholds $x_\ell^\pm, x_h^\pm$ and the targets $x_\ell^{\pm *}, x_h^{\pm *}$ ($12 = 4 + 4 + 4$ parameters).

For the consumer, there are two quasi-variational inequalities satisfied by her value functions $w^\pm(x)$ depending on the sign of μ:

$$\sup\Big\{ - rw^+ - \mu_+ w_x^+ + \frac{1}{2}\sigma^2 w_{xx}^+ + \pi^2(x); w^-(x) - w^+(x) - K^2 \Big\} = 0,$$
$$\sup\Big\{ - rw^- - \mu_- w_x^- + \frac{1}{2}\sigma^2 w_{xx}^- + \pi^2(x); w^+(x) - w^-(x) - K^2 \Big\} = 0,$$

where $\pi^2(x)$ is as in (5.5). Our ansatz of the solution in the continuation region is as follows

$$w^{\pm}(x) = D^2_{\pm}(x) + \nu_1^{\pm} e^{\theta_1^{\pm} x} + \nu_2^{\pm} e^{\theta_2^{\pm} x}, \tag{5.26}$$

where $D^2_{\pm}(x)$ is the static discounted cashflow, and $\theta_1^{\pm}, \theta_2^{\pm}$ are as before. When $M_t = \mu_-$, the consumer will switch to μ_+ if $X_t \geq y_h$ and similarly when $M_t = \mu_+$ she will switch to μ_- if $X_t \leq y_\ell$. These switches translate to

$$w^-(y) = w^+(y_h) - h_0, \quad y \geq y_h, \tag{5.27}$$

$$w^+(y) = w^-(y_\ell) - h_0, \quad y \leq y_\ell. \tag{5.28}$$

together with the smooth pasting C^1 regularity

$$w^+_x(y_r) = w^-_x(y_r), \quad r \in \{\ell, h\}. \tag{5.29}$$

Hence, on consumer's side we have 6 parameters: $\nu_1^{\pm}, \nu_2^{\pm}$ and the two thresholds y_ℓ, y_h. Like in (5.18) when the producer impulses nothing happens to consumer payoff:

$$w^{\pm}(x_r^{\pm *}) = w^{\pm}(x_r^{\pm}), \quad r \in \{\ell, h\}, \tag{5.30}$$

notice that equations (5.18) and (5.30) also correspond to C^0 smooth fit conditions of equilibrium payoffs of each player at his/her competitor's thresholds. This is due to the fact that there is no cross-term intervention cost in our model.

To sum up, we have $18 = 12 + 6$ parameters to determine. By substitution of the parametric formulae for $v^{\pm}$ and $w^{\pm}$ in the equations (5.20) to (5.25) (for the producer), (5.27) to (5.29) (for the consumer) and (5.18)-(5.30) (cross effects) above, we obtain two coupled systems of 22 equations in total, which are more than unknowns. Nevertheless, as we discussed in the previous subsection, the order of players' thresholds makes some of their thresholds ineffective. We can in turn get rid of the corresponding boundary conditions.

5.2.2 Consumer Best Response

Fixing $x_r^{\pm}$, the consumer faces a two-state switching control problem on the bounded domain $(x_\ell^{\pm}, x_h^{\pm})$. To proceed with analysis of the different sub-cases that are possible, we start with the case where the consumer is completely inactive. In that case, she simply collects the payoff based on the strategy $(x_{\ell,h}^{\pm})$, and can be considered as a "follower". Assuming that the consumer never makes any switching, her corresponding value functions, denoted by $\omega_0^{\pm}$, are of the form

$$\omega_0^{\pm}(x) = D_{\pm}^2(x) + \nu_{1,0}^{\pm} e^{\theta_1^{\pm} x} + \nu_{2,0}^{\pm} e^{\theta_2^{\pm} x} \tag{5.31}$$

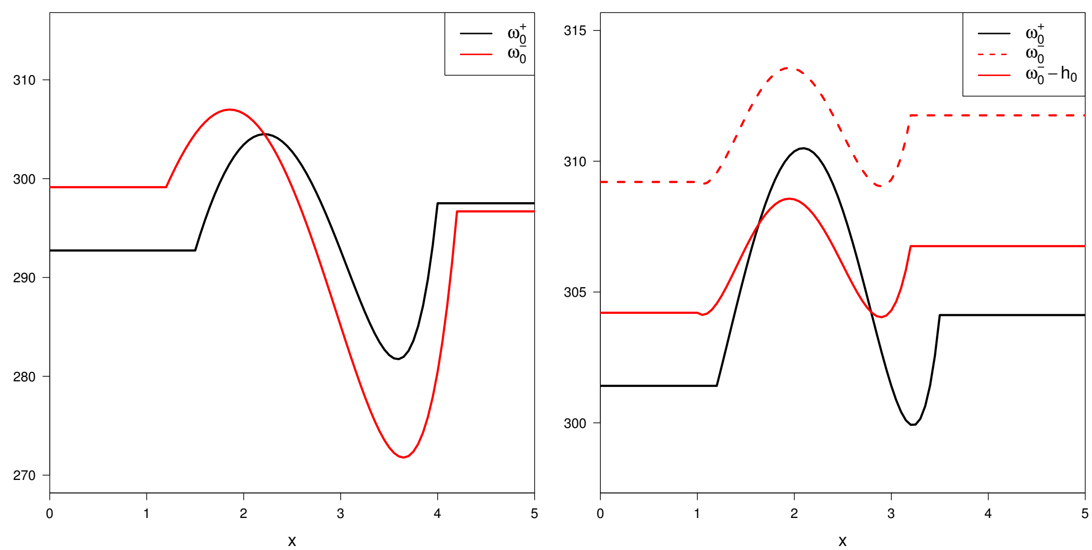
on $[x_\ell^{\pm}, x_h^{\pm}]$, with boundary conditions

$$\omega_0^{\pm}(x_r^{\pm}) = \omega_0^{\pm}(x_r^{\pm *}), \qquad r \in \{\ell, h\}. \tag{5.32}$$

For $x > x_h^{\pm}$ we take $\omega_0^{\pm}(x) = \omega_0^{\pm}(x_h^{\pm *})$ and similarly for $x < x_\ell^{\pm}$. In turn, from (5.32) we can solve for the coefficients $\nu_{1,0}^{\pm}, \nu_{2,0}^{\pm}$ via the following uncoupled linear system:

$$\begin{cases} \nu_{1,0}^{\pm} \cdot \left[e^{\theta_1^{\pm} x_\ell^{\pm}} - e^{\theta_1^{\pm} x_\ell^{\pm *}} \right] + \nu_{2,0}^{\pm} \cdot \left[e^{\theta_2^{\pm} x_\ell^{\pm}} - e^{\theta_2^{\pm} x_\ell^{\pm *}} \right] = D_{\pm}^2(x_\ell^{\pm *}) - D_{\pm}^2(x_\ell^{\pm}), \\ \nu_{1,0}^{\pm} \cdot \left[e^{\theta_1^{\pm} x_h^{\pm}} - e^{\theta_1^{\pm} x_h^{\pm *}} \right] + \nu_{2,0}^{\pm} \cdot \left[e^{\theta_2^{\pm} x_h^{\pm}} - e^{\theta_2^{\pm} x_h^{\pm *}} \right] = D_{\pm}^2(x_h^{\pm *}) - D_{\pm}^2(x_h^{\pm}). \end{cases} \tag{5.33}$$

Fig. 5.3a illustrates the shape of $\omega_0^{\pm}(x)$. In the left panel, we have monotonicity between ω^+ and ω^-: the consumer is incentivised to switch to μ^- when X_t is low and to μ^+ when X_t is high. In that situation, we expect that a threshold-type strategy is a best response. In contrast, on the right panel two other cases are illustrated. First, we see that it is possible that $\omega^+(\cdot) \ll \omega^-(\cdot)$, in other words the consumer has a strong preference to one regime over the other. In that case, the Expand regime could be absorbing, i.e. it is *optimal* to never switch to μ_+. In the plot this would happen if h_0 is low (dashed line), whereby $\omega^-(x) > \omega^+(x) - h_0$ and it is optimal to switch to μ_- at any x (therefore μ_+ would never be observed in the resulting game evolution). At the same time, we see that if h_0 is moderate (the solid line), then the region where $\omega_0^-(x) > \omega_0^+(x) - h_0$ is *disconnected*, so it is likely that a two-threshold switching strategy is an optimal response. This illustrates the fact that a threshold switching strategy might not be optimal in all potential situations.



(a) $\mathcal{C}^1 = \begin{bmatrix} 1.5, & 3.0, & 4.0, & 2.8 \\ 1.2, & 2.5, & 4.2, & 2.6 \end{bmatrix}$ (b) $\mathcal{C}^1 = \begin{bmatrix} 1.2, & 3.0, & 3.5, & 2.8 \\ 1.0, & 2.8, & 3.2, & 2.4 \end{bmatrix}$

Figure 5.3: No-Switch payoffs $\omega_0^{\pm}(x)$ of the consumer given the producer's strategy $\mathcal{C}^1$.

5.2.2.1 Consumer's single-switch best-response

In the scenario when the payoff in the Expand regime is higher for any price x, she will never switch to the Reduce regime. We then expect the consumer's corresponding best-response to be either a single-switch strategy (to the preferred regime) or no-switch (if already there), cf. Figure 5.3b. Economically, this corresponds to $y_h > x_h^{\pm}$ so that as the price rises, the producer impulses X down, and the consumer is not intervening to increase her demand. As a result, the consumer never switches (except perhaps the first time from positive to negative drift) and $\lim_t M_t = \mu_-$. This can be observed when demand switching is very expensive, so that producer has full market power and is able to keep prices consistently high. The consumer is forced to be in the Expand regime forever but is not able to influence X any further.

Suppose that the consumer prefers Reduce regime ($M_t = \mu_+$) and adopts threshold-type strategies. We represent her thresholds as

$$y_\ell = -\infty, \qquad x_\ell^{\pm} < y_h < x_h^{\pm},$$

given the producer's strategy $\mathcal{C}^1$, and posit that her best-response is of the form

$$\omega^+(x) = \omega_0^+(x); \tag{5.34a}$$

$$\omega^-(x) = \begin{cases} \omega_0^+(x) - h_0, & x \geq y_h, \\ D_-^2(x) + \lambda_1^- e^{\theta_1^- x} + \lambda_2^- e^{\theta_2^- x}, & x_\ell^- < x < y_h, \\ \omega^-(x_\ell^{-*}), & x \leq x_\ell^-, \end{cases} \tag{5.34b}$$

with the smooth pasting and boundary conditions:

$$\begin{cases} D_-^2(x_\ell^-) + \lambda_1^- e^{\theta_1^- x_\ell^-} + \lambda_2^- e^{\theta_2^- x_\ell^-} = D_-^2(x_\ell^{-*}) + \lambda_1^- e^{\theta_1^- x_\ell^{-*}} + \lambda_2^- e^{\theta_2^- x_\ell^{-*}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad (\mathcal{C}^0\text{-pasting at } x_\ell^-) \\ D_-^2(y_h) + \lambda_1^- e^{\theta_1^- y_h} + \lambda_2^- e^{\theta_2^- y_h} = D_+^2(y_h) + \lambda_{1,0}^+ e^{\theta_1^+ y_h} + \lambda_{2,0}^+ e^{\theta_2^+ y_h} - h_0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (\mathcal{C}^0\text{-pasting at } y_h) \\ D_{-,x}^2(y_h) + \lambda_1^- \theta_1^- e^{\theta_1^- y_h} + \lambda_2^- \theta_2^- e^{\theta_2^- y_h} = D_{+,x}^2(y_h) + \lambda_{1,0}^+ \theta_1^+ e^{\theta_1^+ y_h} + \lambda_{2,0}^+ \theta_2^+ e^{\theta_2^+ y_h}, \\ \qquad\qquad\qquad\qquad\qquad\qquad (\mathcal{C}^1\text{-pasting at } y_h) \end{cases} \tag{5.35}$$

where $\lambda_{1,0}^+, \lambda_{2,0}^+$ are coefficients of the consumer's payoff associated to the no-switch strategy in (5.31). The system (5.35) is to be solved for the three unknowns $y_h, \lambda_{1,2}^-$. It can be re-written as first solving for $\lambda_{1,2}^-$ from the linear system

$$\begin{bmatrix} e^{\theta_1^- y_h} & e^{\theta_2^- y_h} \\ e^{\theta_1^- x_\ell^-} - e^{\theta_1^- x_\ell^{-*}} & e^{\theta_2^- x_\ell^-} - e^{\theta_2^- x_\ell^{-*}} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1^- \\ \lambda_2^- \end{bmatrix} = \begin{bmatrix} \omega_0^+(y_h) - D_-^2(y_h) - h_0 \\ D_-^2(x_\ell^{-*}) - D_-^2(x_\ell^-) \end{bmatrix} \tag{5.36}$$

and then determining y_h from the smooth pasting C^1-regularity

$$\omega_x^-(y_h) = \omega_{0,x}^+(y_h). \tag{5.37}$$

The overarching plan is that such ω^- should be a solution to the variational inequality

$$\sup\Big\{ - r\omega^- - \mu_- \omega_x^- + \frac{1}{2}\sigma^2 \omega_{xx}^- + \pi^2;\ \omega_0^+ - h_0 - \omega^- \Big\} = 0. \tag{5.38}$$

Note that while the above equation for ω^- depends on ω_0^+, the equation for ω_0^+ is autonomous —the system of equations becomes decoupled because the two regimes of M

no longer communicate.

5.2.2.2 Double-Switch best-response of the consumer

Finally, we consider the main case where the consumer adopts threshold-type switches at both regimes, i.e. the ordering in (5.16) holds. Given the producer's strategy $\mathcal{C}^1$, we make the ansatz that the consumer's best-response value is of the following form

$$\omega^+(x) = \begin{cases} \omega^+(x_h^{+*}), & x \geq x_h^+, \\ D_+^2(x) + \lambda_1^+ e^{\theta_1^+ x} + \lambda_2^+ e^{\theta_2^+ x}, & y_\ell < x < x_h^+, \\ \omega^-(x) - h_0, & x \leq y_\ell, \end{cases} \tag{5.39a}$$

$$\omega^-(x) = \begin{cases} \omega^+(x) - h_0, & x \geq y_h, \\ D_-^2(x) + \lambda_1^- e^{\theta_1^- x} + \lambda_2^- e^{\theta_2^- x}, & x_\ell^- < x < y_h, \\ \omega^-(x_\ell^{-*}), & x \leq x_\ell^-, \end{cases} \tag{5.39b}$$

with the smooth pasting and boundary conditions:

$$\begin{cases} D^2_+(x_h^+) + \lambda_1^+ e^{\theta_1^+ x_h^+} + \lambda_2^+ e^{\theta_2^+ x_h^+} = D^2_+(x_h^{+*}) + \lambda_1^+ e^{\theta_1^+ x_h^{+*}} + \lambda_2^+ e^{\theta_2^+ x_h^{+*}}, \qquad (\mathcal{C}^0 \text{ at } x_h^+) \\ D^2_-(x_\ell^-) + \lambda_1^- e^{\theta_1^- x_\ell^-} + \lambda_2^- e^{\theta_2^- x_\ell^-} = D^2_-(x_\ell^{-*}) + \lambda_1^- e^{\theta_1^- x_\ell^{-*}} + \lambda_2^- e^{\theta_2^- x_\ell^{-*}}, \qquad (\mathcal{C}^0 \text{ at } x_\ell^-) \\ D^2_+(y_\ell) + \lambda_1^+ e^{\theta_1^+ y_\ell} + \lambda_2^+ e^{\theta_2^+ y_\ell} = D^2_-(y_\ell) + \lambda_1^- e^{\theta_1^- y_\ell} + \lambda_2^- e^{\theta_2^- y_\ell} - h_0, \qquad (\mathcal{C}^0 \text{ at } y_\ell) \\ D^2_-(y_h) + \lambda_1^- e^{\theta_1^- y_h} + \lambda_2^- e^{\theta_2^- y_h} = D^2_+(y_h) + \lambda_1^+ e^{\theta_1^+ y_h} + \lambda_2^+ e^{\theta_2^+ y_h} - h_0, \qquad (\mathcal{C}^0 \text{ at } y_h) \\ D^2_{+,x}(y_\ell) + \lambda_1^+ \theta_1^+ e^{\theta_1^+ y_\ell} + \lambda_2^+ \theta_2^+ e^{\theta_2^+ y_\ell} = D^2_{-,x}(y_\ell) + \lambda_1^- \theta_1^- e^{\theta_1^- y_\ell} + \lambda_2^- \theta_2^- e^{\theta_2^- y_\ell}, \\ \qquad\qquad\qquad\qquad\qquad\qquad (\mathcal{C}^1\text{-pasting at } y_\ell) \\ D^2_{-,x}(y_h) + \lambda_1^- \theta_1^- e^{\theta_1^- y_h} + \lambda_2^- \theta_2^- e^{\theta_2^- y_h} = D^2_{,x}(y_h) + \lambda_1^+ \theta_1^+ e^{\theta_1^+ y_h} + \lambda_2^+ \theta_2^+ e^{\theta_2^+ y_h}. \\ \qquad\qquad\qquad\qquad\qquad\qquad (\mathcal{C}^1\text{-pasting at } y_h) \end{cases} \tag{5.40}$$

The six equations can be split into a linear system for the coefficients λ's and the smooth-pasting conditions determining the two switching thresholds y_r (viewed as free boundaries)

$$\omega_x^+(y_r) = \omega_x^-(y_r), \qquad r \in \{\ell, h\}. \tag{5.41}$$

The $\omega^\pm$ are supposed to be a solution to the coupled variational inequalities

$$\begin{aligned} &\sup\Big\{ - r\omega^+ - \mu_+ \omega_x^+ + \frac{1}{2}\sigma^2 \omega_{xx}^+ + \pi^2; \max\{\omega^- - h_0, \omega_0^+\} - \omega^+ \Big\} = 0, \\ &\sup\Big\{ - r\omega^- - \mu_- \omega_x^- + \frac{1}{2}\sigma^2 \omega_{xx}^- + \pi^2; \max\{\omega^+ - h_0, \omega_0^-\} - \omega^- \Big\} = 0, \end{aligned}$$

which would then indeed lead to the consumer's best-response by a verification argument.

5.2.3 Producer Best Response

We now consider the best-response of the producer, given the consumer's switching strategy denoted by $\mathcal{C}^2 := [y_\ell, y_h]$. For simplification, we assume pre-specified impulse amounts (also with a fixed cost denoted specifically by K_q), e.g. $\xi = \pm 0.2$, and $x^{\pm *}_{\ell h} = x^{\pm}_{\ell h} + \xi$. This simplification reduces the dimension of searching solutions to the QVIs (5.17) by eliminating the first-order conditions (5.24)-(5.25), and leads us to the simplified QVIs as follows

$$\sup\Big\{ -rv^{\pm} - \mu_{\pm} v^{\pm}_x + \frac{1}{2}\sigma^2 v^{\pm}_{xx} + \pi^1(x); v^{\pm}(x+\xi) - v^{\pm}(x) - K^1(\xi)\Big\} = 0. \tag{5.42}$$

5.2.3.1 Monopoly Best-response

To begin with, we determine the monopoly-like strategy of the producer assuming the consumer adopts a no-switch strategy. In that case M_t is constant throughout. Intuitively, this is the case when the thresholds $y_r \to \infty, r \in \{\ell, h\}$. We obtain two *uncoupled* QVIs to solve following from (5.42). Assuming the producer adopts threshold-type impulse strategies, her expected payoff is of the form:

$$v^{\pm}(x) = \begin{cases} v^{\pm}(x^{\pm *}_h) - K_q, & x \geq x^{\pm}_h, \\ D^1_{\pm}(x) + \nu^{\pm}_1 e^{\theta^{\pm}_1 x} + \nu^{\pm}_2 e^{\theta^{\pm}_2 x}, & x^{\pm}_\ell < x < x^{\pm}_h, \\ v^{\pm}(x^{\pm *}_\ell) - K_q, & x \leq x^{\pm}_\ell, \end{cases} \tag{5.43}$$

with the smooth pasting and boundary conditions:

$$\begin{cases} D^1_{\pm}(x^{\pm}_h) + \nu^{\pm}_1 e^{\theta^{\pm}_1 x^{\pm}_h} + \nu^{\pm}_2 e^{\theta^{\pm}_2 x^{\pm}_h} = D^1_{\pm}(x^{\pm *}_h) + \nu^{\pm}_1 e^{\theta^{\pm}_1 x^{\pm *}_h} + \nu^{\pm}_2 e^{\theta^{\pm}_2 x^{\pm *}_h} - K_q, \\ \qquad (\mathcal{C}^0 \text{ at } x^{\pm}_h) \\ D^1_{\pm}(x^{\pm}_\ell) + \nu^{\pm}_1 e^{\theta^{\pm}_1 x^{\pm}_\ell} + \nu^{\pm}_2 e^{\theta^{\pm}_2 x^{\pm}_\ell} = D^1_{\pm}(x^{\pm *}_\ell) + \nu^{\pm}_1 e^{\theta^{\pm}_1 x^{\pm *}_\ell} + \nu^{\pm}_2 e^{\theta^{\pm}_2 x^{\pm *}_\ell} - K_q, \\ \qquad (\mathcal{C}^0 \text{ at } x^{\pm}_\ell) \\ D^1 \pm, x(x^{\pm}_h) + \nu^{\pm}_1 \theta^{\pm}_1 e^{\theta^{\pm}_1 x^{\pm}_h} + \nu^{\pm}_2 \theta^{\pm}_2 e^{\theta^{\pm}_2 x^{\pm}_h} = D^1_{\pm,x}(x^{\pm *}_h) + \nu^{\pm}_1 \theta^{\pm}_1 e^{\theta^{\pm}_1 x^{\pm *}_h} + \nu^{\pm}_2 \theta^{\pm}_2 e^{\theta^{\pm}_2 x^{\pm *}_h}, \\ \qquad (\mathcal{C}^1\text{-pasting at } x^{\pm}_h) \\ D^1_{\pm,x}(x^{\pm}_\ell) + \nu^{\pm}_1 \theta^{\pm}_1 e^{\theta^{\pm}_1 x^{\pm}_\ell} + \nu^{\pm}_2 \theta^{\pm}_2 e^{\theta^{\pm}_2 x^{\pm}_\ell} = D^1_{\pm,x}(x^{\pm *}_\ell) + \nu^{\pm}_1 \theta^{\pm}_1 e^{\theta^{\pm}_1 x^{\pm *}_\ell} + \nu^{\pm}_2 \theta^{\pm}_2 e^{\theta^{\pm}_2 x^{\pm *}_\ell}. \\ \qquad (\mathcal{C}^1\text{-pasting at } x^{\pm}_\ell) \end{cases} \tag{5.44}$$

These are 2 linear systems of 2 equations each in the coefficients $\nu^{\pm}_{1,2}$ plus two nonlinear equations each in $x^{\pm}_{l,h}$.

In Fig. 5.4 we sketch the producer's monopoly payoffs when she exercises two-sided impulse controls (along arrows) and pushes the commodity price to $x^{\pm *}_r$ (dashed lines) automonously. These two payoffs cross at $\bar{X}_q$ since we take symmetric $\mu_{\pm}$ in this example. Notice that we observe the orders $x^-_h < x^+_h$ and $x^-_\ell < x^+_\ell$. An interpretation could be that when the price drift is negative $(-\mu_+)$ the producer is more tolerant toward the high values X could take before intervening as, apart from her intervention, she can also rely on a negative drift which could keep X away from high values on average. A similar reasoning would justify $x^-_\ell < x^+_\ell$. One also notice that the producer, though not able to switch the drift, would benefit from the consumer's switching with $y_\ell < \bar{X}_q$ and $y_h > \bar{X}_q$.

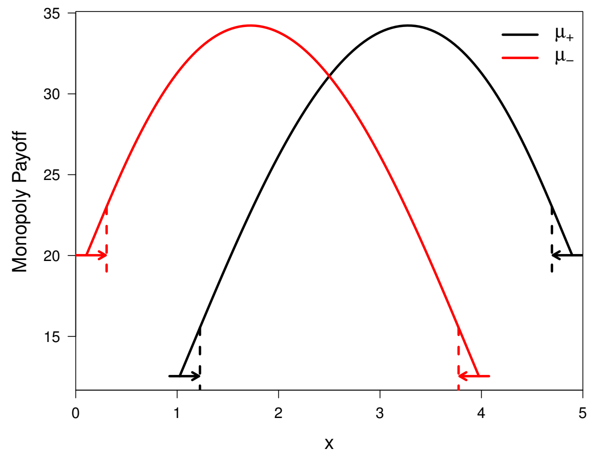


Figure 5.4: Monopoly payoffs of the producer. The arrows indicate directions of her impulse controls and the dashed lines denote the price level she would push to.

5.2.3.2 Non-preemptive Response

We now consider the main case supposing the ordering (5.16) holds. To obtain the producer best-response it suffices to identify the two active impulse thresholds $[x_h^+, x_\ell^-]$. Our ansatz is

$$v^+(x) = \begin{cases} v^+(x_h^{+*}) - K_q, & x \geq x_h^+, \\ D_+^1(x) + \nu_1^+ e^{\theta_1^+ x} + \nu_2^+ e^{\theta_2^+ x}, & y_\ell < x < x_h^+, \\ v^-(x), & x \leq y_\ell, \end{cases} \tag{5.45a}$$

$$v^-(x) = \begin{cases} v^+(x), & x \geq y_h, \\ D_-^1(x) + \nu_1^- e^{\theta_1^- x} + \nu_2^- e^{\theta_2^- x}, & x_\ell^- < x < y_h, \\ v^-(x_\ell^{-*}) - K_q, & x \leq x_\ell^-, \end{cases} \tag{5.45b}$$

with the smooth pasting $\mathcal{C}^1$ regularity and boundary conditions:

$$\begin{cases} D^1_+(y_\ell) + \nu_1^+ e^{\theta_1^+ y_\ell} + \nu_2^+ e^{\theta_2^+ y_\ell} = D^1_-(y_\ell) + \nu_1^- e^{\theta_1^- y_\ell} + \nu_2^- e^{\theta_2^- y_\ell}, \qquad (\mathcal{C}^0 \text{ at } y_\ell) \\ D^1_-(y_h) + \nu_1^- e^{\theta_1^- y_h} + \nu_2^- e^{\theta_2^- y_h} = D^1_+(y_h) + \nu_1^+ e^{\theta_1^+ y_h} + \nu_2^+ e^{\theta_2^+ y_h}, \qquad (\mathcal{C}^0 \text{ at } y_h) \\ D^1_+(x_h^+) + \nu_1^+ e^{\theta_1^+ x_h^+} + \nu_2^+ e^{\theta_2^+ x_h^+} = D^1_+(x_h^{+*}) + \nu_1^+ e^{\theta_1^+ x_h^{+*}} + \nu_2^+ e^{\theta_2^+ x_h^{+*}} - K_q, \\ \qquad\qquad (\mathcal{C}^0 \text{ at } x_h^+) \\ D^1_-(x_\ell^-) + \nu_1^- e^{\theta_1^- x_\ell^-} + \nu_2^- e^{\theta_2^- x_\ell^-} = D^1_-(x_\ell^{-*}) + \nu_1^- e^{\theta_1^- x_\ell^{-*}} + \nu_2^- e^{\theta_2^- x_\ell^{-*}} - K_q, \\ \qquad\qquad (\mathcal{C}^0 \text{ at } x_\ell^-) \\ D^1_{+,x}(x_h^+) + \nu_1^+ \theta_1^+ e^{\theta_1^+ x_h^+} + \nu_2^+ \theta_2^+ e^{\theta_2^+ x_h^+} = D^1_{+,x}(x_h^{+*}) + \nu_1^+ \theta_1^+ e^{\theta_1^+ x_h^{+*}} + \nu_2^+ \theta_2^+ e^{\theta_2^+ x_h^{+*}}, \\ \qquad\qquad (\mathcal{C}^1\text{-pasting in } x_h^+) \\ D^1_{-,x}(x_\ell^-) + \nu_1^- \theta_1^- e^{\theta_1^- x_\ell^-} + \nu_2^- \theta_2^- e^{\theta_2^- x_\ell^-} = D^1_{-,x}(x_\ell^{-*}) + \nu_1^- \theta_1^- e^{\theta_1^- x_\ell^{-*}} + \nu_2^- \theta_2^- e^{\theta_2^- x_\ell^{-*}}. \\ \qquad\qquad (\mathcal{C}^1\text{-pasting in } x_\ell^-). \end{cases} \tag{5.46}$$

Unlike the monopoly (i.e. single-agent) setting, here the equations are coupled. The coefficients $\nu_{1,2}^{\pm}$ are the solution to a linear system and the thresholds x_h^+, x_ℓ^- are determined by the C^1 smooth-pasting at x_h^{+*} and x_ℓ^{-*}. Lastly, to justify above $v^{\pm}$ and corresponding $\mathcal{C}^1$ are the producer's best-response payoffs and strategy, one need to show they solve the following QVIs (5.42).

Note that, though hidden in the system of equations above, the producer's thresholds $[x_\ell^+, x_h^-]$ are essential to define her best-response. On one hand, taking $M_t = \mu_-$ as an example, the players' behavior need to be defined when $X_t > x_h^-$ since both of them want to exercise a control, hence the payoff v^- needs to be defined for $x > x_h^-$ as well. On the other hand, the consumer's best-response are highly contingent upon the thresholds

$[x_\ell^+, x_h^-]$. For the sake of simplicity, we assume that *the consumer possesses the priority* to intervene when they both want to act. Consequently, if the producer decides not to prevent the consumer from switching, it is equivalent for her to set $x_\ell^+ = -\infty$ and $x_h^- = +\infty$. The producer's strategy associated to the payoffs (5.45) is in effect

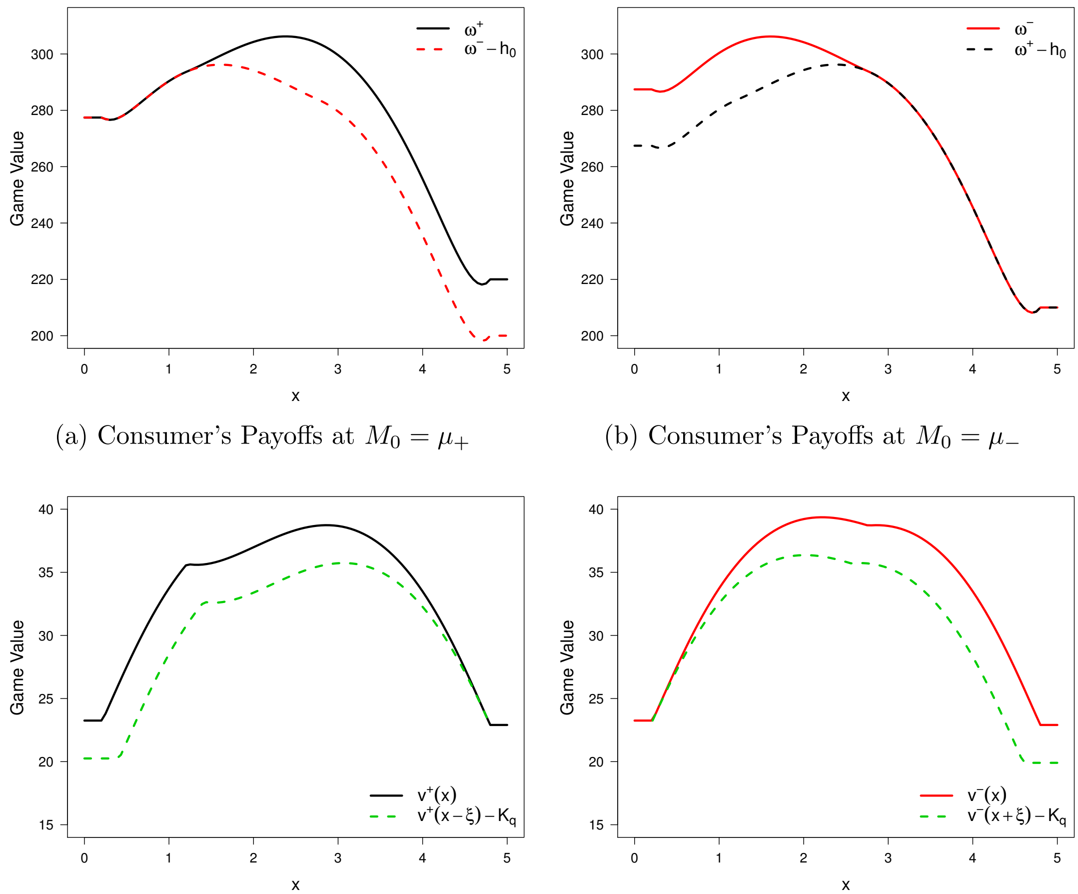
$$\mathcal{C}^1 = \begin{bmatrix} -\infty, & -, & x_h^+, & x_h^{+*} \\ x_\ell^-, & x_\ell^{-*}, & +\infty, & - \end{bmatrix}.$$

Preemptive Response In the case when the static discounted future profits of the producer satisfy $D_-^1(x) \geq D_+^1(x)$ for any x, one possible strategy for her is to *preempt* in order to prevent the consumer from switching the drift to μ_+. Such a strategy can be realized by taking $x_h^- < y_h$. However, imposing this constraint might not solve the system (5.44). In the latter situation the best response is to impulse X right before it hits y_h, i.e. $x_h^- = y_h -$.

In general, we need to manually verify whether $x_h^- > y_h$ (the main case) or $x_h = y_h -$ (the pre-emptive case) whenever consider the producer best-response. The two situations lead to different boundary conditions at the upper threshold, and hence cannot be directly compared.

5.3 Numerical Examples

In this section, we represent some numerical examples in which reasonable threshold-type Nash equilibria are obtained. The results are preliminary and future discussion is expected in our ongoing work.



(a) Consumer's Payoffs at $M_0 = \mu_+$

(b) Consumer's Payoffs at $M_0 = \mu_-$

(c) Producer's Payoffs at $M_0 = \mu_+$

(d) Producer's Payoffs at $M_0 = \mu_-$

Figure 5.5: Equilibrium payoffs of the producer and consumer.

5.3.1 Double-Switch + Non-preemptive Impulse

The corresponding parameter values are $d_0 = 10/3, d_1 = 2/3, p_0 = 1.044, p_1 = 1, d'_0 = 23, d'_1 = 11/3, c_q = c_d = 0, \alpha = 1.0, r = 0.1, \mu_- = -0.1, \mu_+ = 0.1, \sigma = 0.25, h_0 = 10$, with fixed impulse amounts $\xi \equiv 0.2$ and impulse cost $K_q = 3$, which yield $\bar{X}_d = 2.0, \bar{X}_q = 2.5$. To construct an *interior, non-preemptive* equilibrium satisfying the ordering (5.16) we employ tatonnement, i.e. iteratively apply the best-response maps. The resulting equilibrium is

$$\mathcal{C}^{1*} = \begin{bmatrix} -\infty, & -, & 4.7991, & 4.5991 \\ 0.21281, & 0.41281, & +\infty, & - \end{bmatrix}, \qquad \mathcal{C}^{2*} = [1.2134, 2.7632]. \tag{5.47}$$

In Fig. 5.5 we verify that the ansatz (5.39) and (5.45) associated to above strategies solve corresponding QVIs.

In the equilibrium (see Figure 5.6), the commodity price X^* increases on average due to the positive drift in the Expand regime ($M_t^* = \mu_-$). When it exceeds $y_h = 2.76$, the consumer switches to the Reduce regime which makes the price decrease on average. The producer benefits from such switches, thus decides to act as a "back-up", i.e. impulse up when X^* is extremely low (≤ 0.21), rather than preempting to prevent the consumer from switching. Similarly, in the Reduce regime, the price decreases on average due to the negative drift and triggers the consumer to switch to the Expand regime when it drops below $y_\ell = 1.21$. The producer will not intervene until the price becomes extremely high (≥ 4.80). In turn, the players' strategy profile yields a stationary distribution for the

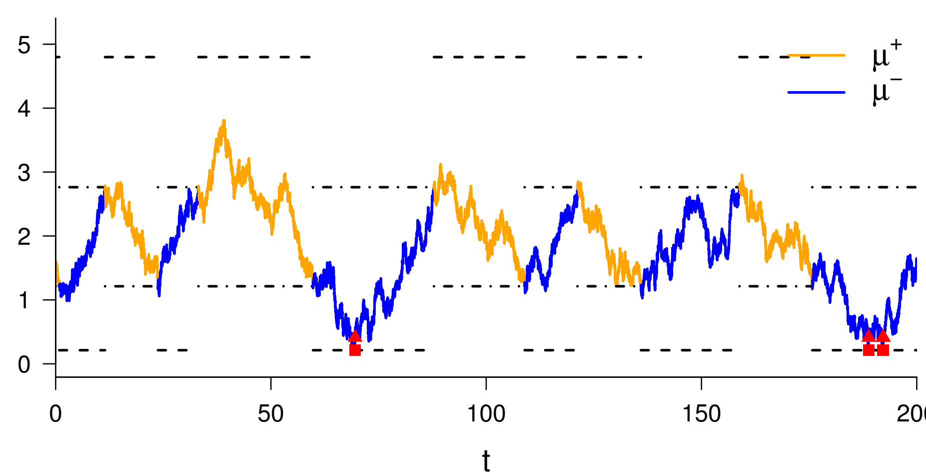


Figure 5.6: A sample path of the controlled market price (X_t) under a Double-Switch equilibrium

pair (X_t^*, M_t^*). The macro market M^* would switch between the Expand regime and the Reduce regime back and forth, while the jointly controlled price X^*, bounded in the range $[x_\ell^-, x_h^+]$, fluctuates at a *mean-reverting* pattern (due to the changes of its drift).

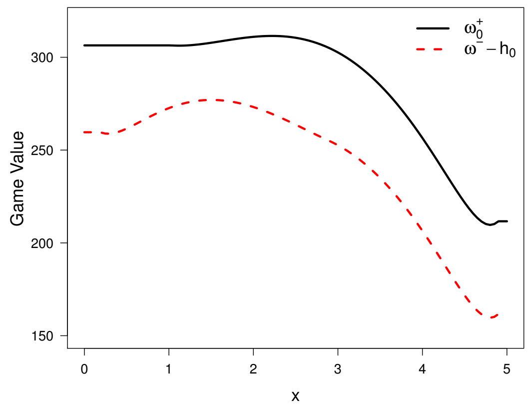
5.3.2 Single-Switch + Monopoly/One-sided Impulse

There is another type of equilibrium in which the consumer decides to switch from one regime to the other only. For instance, suppose that the consumer chooses to switch to the Reduce regime (μ_+, negative drift) only, then the producer's best-response is expected to be acting like a monopoly at the Reduce regime and exercise one-sided impulse at the Expand regime (assuming that she is not incentivised to preempt).

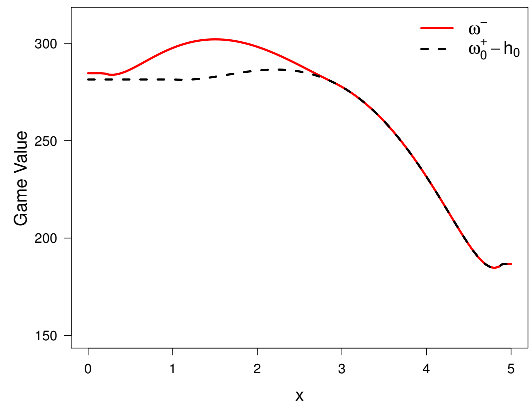
To construct such an equilibrium, we increase the switching cost $h_0 = 25$ and keep all other parameters unchanged. We start with assuming the consumer is passive and solving for the monopoly impulse thresholds $x_{\ell h}^{(0)}$ of the producer. Next, we fix that impulse strategy and solve for $y_{\ell h}^{(1)}$ switching thresholds, which in effect imply a single-switch strategy (i.e. $y_\ell^{(1)} = -\infty$). Then we revert back to the producer and find her best response to $x_{\ell h}^{(1)}$, and so forth. The resulting equilibrium is

$$\mathcal{C}^{1*}_{single} = \begin{bmatrix} 1.0250, & 1.2250, & 4.8954, & 4.7954 \\ 0.1608, & 0.3608, & +\infty, & - \end{bmatrix}, \qquad \mathcal{C}^{2*}_{single} = [-\infty, 2.8869], \tag{5.48}$$

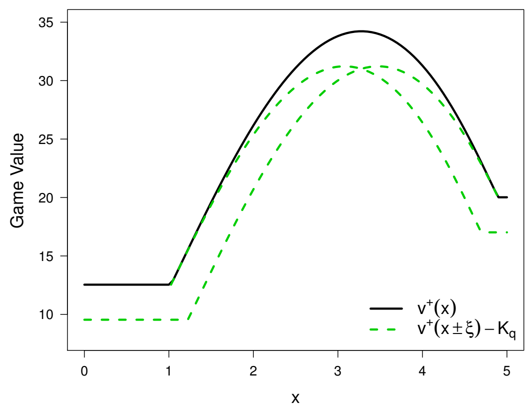
where we use the subscript to denote that this strategy profile corresponds to a single-switch case. In turn, when there exists a heavy switching cost, the consumer would only switch off from the Expand regime (μ_-) and decrease the commodity price. Instead of switching to the Expand regime, she relies on the producer who would impulse the commodity price up when X is too low. In Fig. 5.7 we verify that the ansatz (5.39) and (5.45) associated to above strategies solve corresponding QVIs.



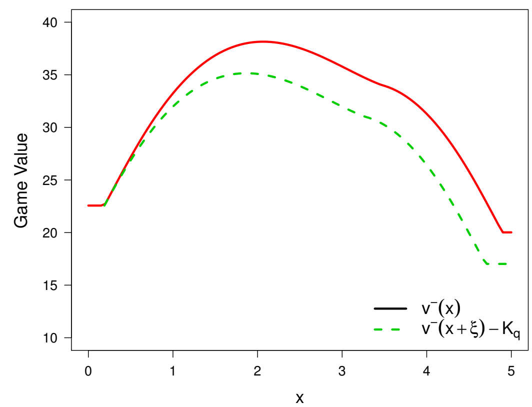
(a) Consumer's Payoffs at $M_0 = \mu_+$



(b) Consumer's Payoffs at $M_0 = \mu_-$



(c) Producer's Payoffs at $M_0 = \mu_+$



(d) Producer's Payoffs at $M_0 = \mu_-$

Figure 5.7: Equilibrium payoffs of the producer and consumer.

In the single-switch equilibrium (see Figure 5.8), the players act similarly to the double-switch equilibrium when the macro market is in the Expand regime ($M_t^* = \mu_-$). Namely, the commodity price X^* increases on average due to the positive drift. When it exceeds $y_h = 3.46$, the consumer switches to the Reduce regime which makes the price decrease on average. The producer acts as a "back-up" by impulsing up when X^* is extremely low (≤ 0.19). However, in the Reduce regime, the heavy switching cost makes it not optimal for the consumer to switch to the Expand regime. The price drops on average due to the negative drift and the producer acts to prevent it from being too low (≤ 1.03). In turn, the jointly controlled commodity price X^* *decreases* in the long-run,

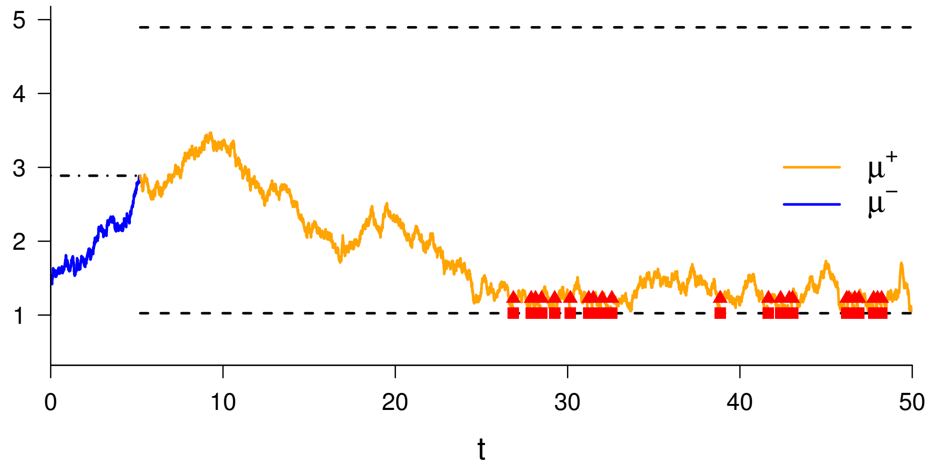


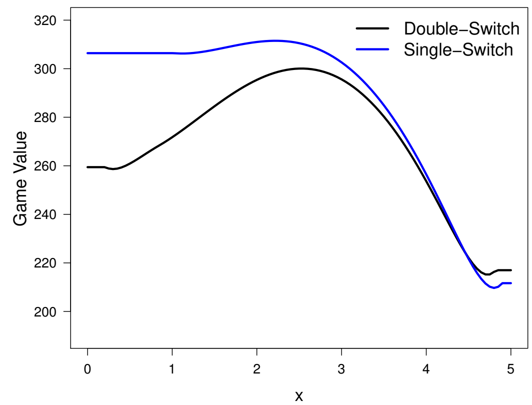
Figure 5.8: A sample path of the controlled market price (X_t) under a Single-Switch equilibrium

and the producer would intervene more frequently to maintain a sound price.

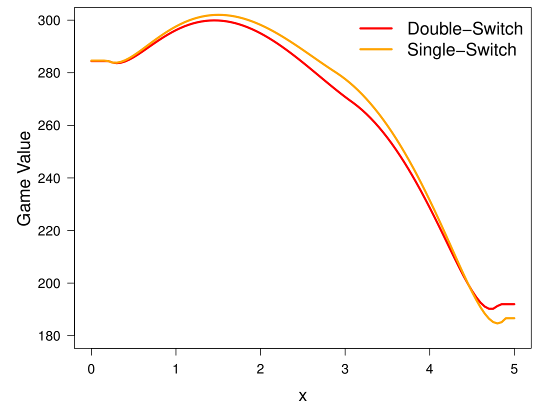
One interesting observation is that there also exists a double-switch equilibrium with the following strategy profile:

$$\mathcal{C}^{1*}_{double} = \begin{bmatrix} -\infty, & -, & 4.8182, & 4.6182 \\ 0.2099, & 0.4099, & +\infty, & - \end{bmatrix}, \qquad \mathcal{C}^{2*}_{double} = [0.8487, 3.0788], \tag{5.49}$$

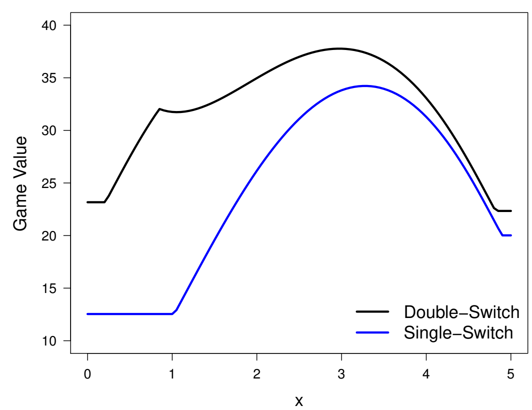
which is obtained by starting from a double-switch strategy of the consumer and iteratively apply the best-response maps. One direct observation is that the producer does not move the price X up at the Reduce regime (μ_+) so that the consumer has to switch to the Expand regime. Equilibrium selection is not trivial in this situation, since one can observe that the consumer collects more payoff in the double-switch equilibrium when X_t is high but prefers the opposite when X_t is low, though the producer consistently favors the double-switch equilibrium.



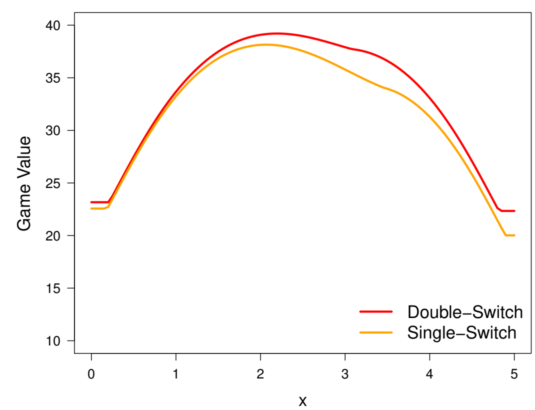
(a) Consumer's Payoffs at $M_0 = \mu_+$



(b) Consumer's Payoffs at $M_0 = \mu_-$



(c) Producer's Payoffs at $M_0 = \mu_+$

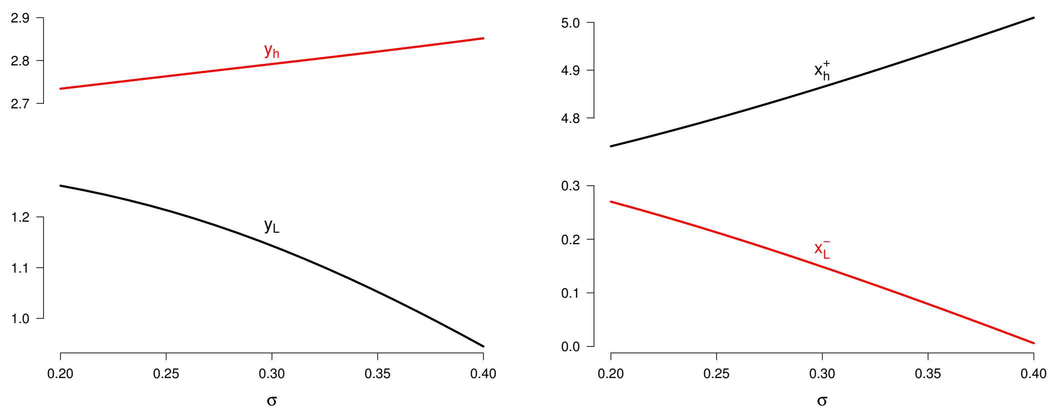


(d) Producer's Payoffs at $M_0 = \mu_-$

Figure 5.9: Equilibrium payoffs in two types of equilibria.

5.3.3 Effect of Market Fluctuation

In this subsection, we study how the volatility of the commodity price may affect the players' decisions in *double-switch* equilibria constructed. Recall that the parameters used in preceding sections are $r = 0.1, \bar{X}_d = 2.0, \bar{X}_q = 2.5, \mu_- = -0.1, \mu_+ = 0.1, \sigma = 0.25, h_0 = 10$, with fixed impulse amounts $\xi \equiv 0.2$ and impulse cost $K_q = 3$. We now vary σ from 0.2 to 0.4. As σ increases, both the consumer and the producer decide to wait longer, which is also observed when their acting costs h_0, K_q increase.



(a) Consumer's Threholds as $\sigma \in$ [0.2, 0.4] (b) Producer's Thresholds as $\sigma \in$ [0.2, 0.4]

Figure 5.10: Double-Switch equilibrium thresholds of the players as σ variates.

5.4 Future Research

The work represented in this chapter is still ongoing. The assumption of fixed impulse amount should be extended. In fact, with linear impulse costs, such amount can be optimally determined via (5.24)-(5.25). Meanwhile, one essential task to be done is to provide rigorous game formulation and mathematical proofs of the best-response maps we constructed in Section 5.2. Though analytical inference for equilibria of the general competition is hard to approach, we are interested in solving some limiting cases, e.g. free actions by the consumer $h_0 = 0$ which should lead to $y_\ell = y_h$.

We shall aim to analyze dynamics of the pair (X_t, M_t) in the emerging equilibrium. In this model, the commodity price is partially controlled by the players which brings difficulty for us to obtain analytic results about the short-/long-term market condition. Nevertheless, explicit constructed threshold-type equilibria allow us to investigate the resulting competition via numerical approach (e.g. Monte Carlo methods).

Another notable extension we aim to do is an economic case study. We would like to consider a simplified vision of the crude oil market and its refined products. The model would be calibrated in the vicinity of the current world consumption of oil and gasoline. Switching from the Expand regime to the Reduce regime is considered as replacing gasoline cars by electricity cars which lowers the yearly grow rate of gasoline consumption. We expect that this calibrated case study would bring us insights about the future regime of the oil market (will it be switched into the Reduce regime?), the existence of vertically integrated companies (BP, Shell, etc.), and so on.

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