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Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints

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Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2013

The dissertation of David Lipshutz is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

2013

DEDICATION

To my parents, Nancy and Rob.

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Chapters 1 through 5 and Appendices A through F are based on the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solution for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

VITA

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ABSTRACT OF THE DISSERTATION

Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints

by

David Lipshutz

Doctor of Philosophy in Mathematics

University of California, San Diego, 2013

Professor Ruth J. Williams, Chair

Deterministic dynamical system models with delayed feedback and state constraints arise in a variety of applications in science and engineering. Under certain conditions oscillatory behavior has been observed and it is of interest to know when there are periodic solutions. Here we consider one-dimensional delay differential equations with non-negativity constraints as prototypes for such models. We obtain sufficient conditions for the existence of slowly oscillating periodic solutions of such equations when the delay/lag interval is long and the dynamics depend only on the current and lagged state. Under further assumptions, including possibly longer delay/lag intervals and restricting the dynamics to depend only on the lagged state, we prove uniqueness and a strong form of asymptotic stability for such solutions.

Chapter 1

Introduction

Dynamical system models with delay in the dynamics arise in a variety of applications in science and engineering. Examples include Internet rate control models where the finiteness of transmission speeds leads to a delay in receipt of congestion signals or prices [22, 23, 24, 25, 26, 27], neuronal models where the spatial distribution of neurons can result in a propagation delay [2, 9], epidemiological models where incubation periods result in delayed transmission of disease [5], and biochemical models of gene regulation where transcription and translation processes can lead to a delay in signaling effects [1, 4, 11, 15, 21]. Oscillatory (especially periodic) behavior can be important for the functioning of such systems [11, 21]. Furthermore, often the quantities of interest in such systems are non-negative. For instance, rates and prices in Internet models, proportions of a population that are infected, and concentrations of ions or molecules are all non-negative. In a delay differential equation model for such systems, sometimes the drift function may naturally constrain all components to be non-negative, but sometimes (e.g., because of the delay), the dynamics need to be modified when one of the components of the current state becomes zero, to prevent that component from becoming negative. This can be thought of as imposing a regulating control at the boundary, which creates a discontinuity in the right hand side of the differential equation.

While there is a considerable mathematical literature on oscillatory solutions of unconstrained delay differential equations (see e.g., [8]), there is limited mathematical literature studying oscillatory solutions for constrained delay differential equations with discontinuous dynamics at the boundary. Some examples tied to specific applications include a biochemical application studied in Mather et al. [15], where a simple genetic circuit model exhibits oscillatory behavior that is potentially critical to biological functioning; and an Internet rate control model in which the existence of oscillatory behavior is shown numerically to arise from an unstable equilibrium solution [16]. Even a one-dimensional delay differential equation with a non-negativity constraint is an interesting non-linear system whose natural state descriptor is infinite-dimensional because of the need to track position over the delay/lag period. The behavior of the constrained system can be quite different from that of the unconstrained analogue. For example, as we will show, in the case of dynamics that are linear in the unconstrained context, the additional non-negativity constraint can turn an equation with unbounded oscillatory solutions into one with bounded periodic solutions.

As a first step towards studying oscillatory solutions of constrained delay differential equations, we provide sufficient conditions for existence, uniqueness, and stability of periodic solutions for prototypical one-dimensional delay equations with non-negativity constraints of the form

$$x(t) = x(0) + \int_0^t f(x_s)ds + y(t), \ t \geq 0, \tag{1.1}$$

where x is a continuous function on $[-\tau, \infty)$ and takes values in the non-negative real numbers, $\tau \in (0, \infty)$ is the fixed length of the delay interval, $x_t = \{x(t+s) : -\tau \leq s \leq 0\}$ tracks the history of $x(\cdot)$ over the delay interval, f is a real-valued continuous function defined on these continuous path segments, and y ensures $x(t)$ remains non-negative for all $t \geq 0$ (y is a continuous, non-decreasing function that can increase only at time t if $x(t)$ is zero — see Definition 2.1 for further details). Given f, we refer to (1.1) as a delay differential equation with reflection (at the boundary).

In this work, we focus on slowly oscillating periodic solutions. Here, "slowly oscillating" refers to the fact that the solution oscillates about an equilibrium point and the time spent above/below the equilibrium point per oscillation is greater than the length of the delay interval (see Definition 3.2 for a precise definition and

Figure 5.3 for an example of such a solution). For our results on existence of slowly oscillating periodic solutions, we restrict f to be a function that depends only on the current and delayed state, i.e.,

$$f(x_t) = g(x(t), x(t-\tau)),\ t \geq 0, \tag{1.2}$$

where g is a real-valued locally Lipschitz continuous function on the non-negative quadrant in two-dimensional Euclidean space satisfying a negative feedback type condition. Our existence results are inspired by the prior work of Nussbaum [18], Hadeler and Tomiuk [9], Mallet-Paret and Nussbaum [12], and, in particular, Atay [2], on the existence of slowly oscillating periodic solutions for unconstrained delay differential equations. Our assumptions on g are similar to those used in [2], although we allow somewhat relaxed boundedness assumptions on g since our non-negativity constraint a priori prevents unbounded excursions in the negative direction. For uniqueness and stability, we further restrict the function f to depend only on the delayed state, i.e.,

$$f(x_t) = h(x(t-\tau)),\ t \geq 0, \tag{1.3}$$

where h is a real-valued continuously differentiable function on the non-negative real numbers satisfying a negative feedback condition. Our conditions on h and our proof of uniqueness and stability are inspired by an approach introduced by Xie [30, 31] to prove the uniqueness and stability of slowly oscillating periodic solutions of the unconstrained system. However, due to the discontinuous dynamics at the boundary, methods used in the unconstrained case for studying uniqueness and stability do not readily apply in the constrained setting. Therefore we develop new techniques for understanding perturbations of solutions in the constrained environment, which may be of independent interest.

The paper is organized as follows. A precise definition of a solution to a delay differential equation with reflection is given in Section 2. Here our non-negativity constraint is described and its relation to the one-dimensional Skorokhod problem is explained (a formulation of the one-dimensional Skorokhod problem is detailed in Appendix A). We also explain a parallel formulation using delay differential equations with discontinuous dynamics. The main results of the paper are stated in Section 3. Sufficient conditions are provided for existence of slowly oscillating periodic solutions of (1.1) and further restrictions implying uniqueness and stability are described. Applications of the results to biochemical and Internet rate control models are presented in Section 3.4.

The proof of existence of slowly oscillating periodic solutions is presented in Section 4. By linearizing g about an equilibrium point, we are able to describe conditions under which solutions of the constrained delay differential equation oscillate about the equilibrium point and the equilibrium solution is locally unstable. A version of Browder's fixed point theorem implying the existence of a non-ejective fixed point is used to show the existence of a non-constant fixed point of a certain mapping on a path space, which corresponds to a slowly oscillating periodic solution of (1.1). This follows a similar approach used in prior works for analogous unconstrained systems (see e.g., [2, 9, 12, 18]). The main difference in our case is the presence of the non-negativity constraint, which prevents unbounded oscillations, and therefore allows for a less restrictive class of functions g.

The proof of stability and uniqueness of slowly oscillating periodic solutions is presented in Section 5. We show that if the delay interval length is sufficiently large we can construct an approximate Poincaré map associated with a slowly oscillating periodic solution of (1.1). If $h'(s)$ approaches zero sufficiently fast as s approaches infinity, then the operator norm of the derivative is less than one, which is used to prove that the associated slowly oscillating periodic solution is asymptotically stable (It would be sufficient to prove a weaker condition that the spectral norm is less than one; however this would not improve our stability result.) Uniqueness of the associated slowly oscillating periodic solution then follows from its asymptotic stability and an application of theorems for fixed point indices. This follows a similar approach used to prove analogous results for unconstrained systems [30, 31]. Our conditions on h are similar to those imposed in [30], though ours are more restrictive because the lower boundary prevents us from using a single transformation to incorporate a linear dependence on the current state. However, the presence of the lower boundary does allows us to relax conditions on the asymptotic rate that the derivative h' approaches zero at infinity.

In order to construct an approximate Poincaré map associated with a slowly oscillating periodic solution, a linear variational equation for solutions of delay differential equations with reflection is derived in Appendix C. Solutions of the linear variational equation in the constrained setting can differ considerably from those in the unconstrained setting. In the unconstrained system, a differentiability condition on f ensures that solutions of the linear variational equation are continuously differentiable for sufficiently large times. However, adapting that approach here is complicated by the presence of a lower boundary and indeed solutions of the linear variational equation in the constrained setting are not necessarily continuous. To derive the linear variational equation in our constrained setting, results on the derivative of the one-dimensional Skorokhod reflection map are used. These results were first presented by Mandelbaum and Massey [13] and generalized first by Whitt [28] and then by Mandelbaum and Ramanan [14] (a specific formulation of their results is presented in Appendix B).

We shall use the following notation throughout this paper. Let $\mathbb{N}$ denote the set of positive integers $\{1, 2, \dots\}$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, let $\mathbb{R}^n$ denote n-dimensional Euclidean space and let $\mathbb{R}^n_+ = \{v \in \mathbb{R}^n : v_i \geq 0$ for $i = 1, \dots, n\}$ denote the closed non-negative orthant in $\mathbb{R}^n$. Given $v \in \mathbb{R}^n$, let $|v|$ denote the Euclidean norm of v. When $n = 1$, we suppress the n and write $\mathbb{R}$ for the real numbers and $\mathbb{R}_+$ for the non-negative real numbers. For $r, s \in \mathbb{R}$, let $r^+ = \max(r, 0)$, $r^- = \max(-r, 0)$ and let $r \vee s = \max(r, s)$, $r \wedge s = \min(r, s)$. For a subset $\mathbb{A}$ of the real numbers, let $1_{\mathbb{A}}$ denote the indicator function of $\mathbb{A}$ defined on the real numbers by

$$1_{\mathbb{A}}(r) = \begin{cases} 1, & \text{if } r \in \mathbb{A}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\tau \in (0, \infty)$ denote a constant *delay*. For an interval of the form $I = [-\tau, t]$, $[-\tau, \infty)$, $[0, t]$, or $[0, \infty)$, where $t \geq 0$, we will be concerned with the following spaces of functions from I into the real numbers. We let $\mathcal{D}_I$ denote the set of functions from I into $\mathbb{R}$ that have finite left and right limits at each finite value in I (at the left endpoint of I we only require a finite right limit and at a finite right endpoint we only require a finite left limit). We let $\mathcal{C}_I$ denote the set of continuous functions from I into $\mathbb{R}$ and we let $\mathcal{C}_I^+$ denote the subset of continuous function from the interval I into $\mathbb{R}_+$. We endow $\mathcal{C}_I$ and its further subsets with the topology of uniform convergence on compact intervals in I. For a function $x \in \mathcal{D}_I$ and a compact interval J in I, we define

$$\|x\|_J \equiv \sup_{t \in J} |x(t)|.$$

Note that $\mathcal{D}_{[-\tau,0]}$ and $\mathcal{C}_{[-\tau,0]}$ are Banach space under $\|\cdot\|_{[-\tau,0]}$.

For a function $x \in \mathcal{D}_I$, we say x is non-decreasing (resp. non-increasing) on I if $x(s) \leq x(t)$ (resp. $x(s) \geq x(t)$) for each $s \leq t$ in I. We say that x is increasing (resp. decreasing) on I if $x(s) < x(t)$ (resp. $x(s) > x(t)$) for each $s < t$ in I. For a function $x \in \mathcal{D}_{[-\tau,\infty)}$ and $t \geq 0$, let $x_t \in \mathcal{D}_{[-\tau,0]}$ denote the τ-length function defined by

$$x_t(s) \equiv x(t+s), \; -\tau \leq s \leq 0.$$

We let $\tilde{x} \in \mathcal{D}_{[0,\infty)}$ denote the restriction of x to the interval $[0, \infty)$. For a function $x \in \mathcal{D}_I$, let $x^+, x^- \in \mathcal{D}_I$ denote the functions defined by $x^+(t) = \max(x(t), 0)$ and $x^-(t) = \max(-x(t), 0)$ for each $t \in I$. For a function $x \in \mathcal{D}_{[0,\infty)}$, let $\overline{x} \in \mathcal{D}_{[0,\infty)}$ denote the upper-envelope function defined by

$$\overline{x}(t) \equiv \sup_{0 \leq s \leq t} x(s), \; t \geq 0.$$

We let $L^1(\mathbb{R}_+)$ denote the space of Lebesgue integrable functions on $\mathbb{R}_+$ with finite L^1-norm

$$\|x\|_{L^1(\mathbb{R}_+)} \equiv \int_0^\infty |x(s)| ds.$$

Given two Banach spaces X and Y, we let $\mathcal{L}(X, Y)$ denote the space of bounded linear operators from X into Y. We shall use $\|\cdot\|$ to denote the norm on X or Y, depending on the argument, which will be clear from context. For $L \in \mathcal{L}(X, Y)$, we let $\|L\| \equiv \sup\{\|Lx\| : x \in X, \|x\| = 1\}$ denote the operator norm of L. For an open subset U of X and a function $f : U \to Y$, we say $f(x) = o(x)$ if $\lim_{x \to 0} \|f(x)\| / \|x\| = 0$.

This chapter is based on the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

Chapter 2

Delay Differential Equation with Reflection (DDER)

In this section, we define what we mean by a solution of a delay differential equation with reflection and remark on its relation to the one-dimensional Skorokhod problem and delay differential equations with discontinuous right hand sides. Throughout this section, we fix a delay $\tau \in (0, \infty)$ and a continuous function $f : \mathcal{C}^+_{[-\tau,0]} \to \mathbb{R}$.

Definition 2.1. A *solution of the delay differential equation with reflection (DDER) associated with* f is a continuous function $x \in \mathcal{C}^+_{[-\tau,\infty)}$ satisfying (1.1), where $y \in \mathcal{C}^+_{[0,\infty)}$ is a continuous and non-decreasing function such that $y(0) = 0$ and $\int_0^t x(s)dy(s) = 0$ for all $t \geq 0$.

Remark 2.1. Note that the requirement $\int_0^t x(s)dy(s) = 0$ for all $t \geq 0$ can be interpreted as meaning that $y(\cdot)$ can only have a point of increase at time $t > 0$ if $x(t) = 0$.

Remark 2.2. Given a solution x of DDER, (1.1) can be rewritten for $t \geq 0$ as

$$x(t) = z(t) + y(t), \tag{2.1}$$

$$z(t) = x(0) + \int_0^t f(x_s)ds. \tag{2.2}$$

It follows that $(\tilde{x}, y)$, where $\tilde{x}$ denotes the restriction of x to the interval $[0, \infty)$, is a solution of the one-dimensional Skorokhod problem for z (see Appendix A). It is a well-known fact in the theory of the one-dimensional Skorokhod problem that given z, y is uniquely defined by

$$y(t) = \sup_{0 \leq s \leq t} z^-(s), \ t \geq 0. \tag{2.3}$$

In the notation of Appendix A,

$$(\tilde{x}, y) = (\Phi, \Psi)(z). \tag{2.4}$$

Remark 2.3. It will be assumed throughout the paper that given any $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, there exists a unique solution x of DDER with initial condition $x_0 = \varphi$. We do not prescribe any further conditions than continuity on f. Under this assumption, given $x \in \mathcal{C}^+_{[-\tau,\infty)}$, z in (2.2) is well-defined; however, usually additional assumptions are required to guarantee existence and uniqueness of solutions. For example, if f satisfies a local Lipschitz condition as in Lemma 2.1 below and a condition for non-explosion of solutions in finite time is imposed, then existence and uniqueness holds. Our assumptions (see Assumptions 3.1 and 3.2) for the existence of periodic solutions will ensure the existence of a unique solution x of DDER given $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, which is bounded for all time.

The following lemma provides sufficient conditions for continuity in the initial condition of solutions of DDER. The requirement of an a priori bound on solutions of DDER is not as strong a condition as it may appear since a negative feedback condition that we will impose for the existence of slowly oscillating periodic solutions (see Assumption 3.2) ensures that such a bound exists.

Lemma 2.1. *Suppose that there exist $C > 0$ and $K_{f,C} > 0$ such that*

$$|f(\varphi) - f(\varphi^\dagger)| \leq K_{f,C} \|\varphi - \varphi^\dagger\|_{[-\tau,0]}, \tag{2.5}$$

for all $\varphi, \varphi^\dagger \in \mathcal{C}^+_{[-\tau,0]}$ satisfying $\|\varphi\|_{[-\tau,0]} \leq C$ and $\|\varphi^\dagger\|_{[-\tau,0]} \leq C$. If x and $x^\dagger$ are solutions of DDER bounded by C on $[-\tau, t]$ for some $t \geq 0$, then

$$\|x - x^\dagger\|_{[-\tau,s]} \leq 2\exp(2K_{f,C}s)\|x - x^\dagger\|_{[-\tau,0]}, \ 0 \leq s \leq t. \tag{2.6}$$

Proof. Fix $t \geq 0$. Suppose x and $x^\dagger$ are solutions of DDER bounded by C on $[-\tau, t]$. Define z as in (2.2) and define $z^\dagger$ as in (2.2), but with $x^\dagger$ and $z^\dagger$ in place of x and z, respectively. Then by (2.2) and (2.5),

$$|z(r) - z^\dagger(r)| \le |x(0) - x^\dagger(0)| + K_{f,C} \int_0^r \|x_u - x_u^\dagger\|_{[-\tau,0]} du, \ 0 \le r \le t.$$

Fix $s \in [0,t]$. By taking the supremum over $r \in [0,s]$, we obtain a uniform bound on $\mathcal{C}_{[0,s]}$,

$$\|z - z^\dagger\|_{[0,s]} \le |x(0) - x^\dagger(0)| + K_{f,C} \int_0^s \|x - x^\dagger\|_{[-\tau,r]} dr.$$

By (2.4), the Lipschitz continuity of the Skorokhod map (see Proposition A.1) and an extension of the uniform bound to the interval $[-\tau, s]$, we have

$$\|x - x^\dagger\|_{[-\tau,s]} \le 2\|x - x^\dagger\|_{[-\tau,0]} + 2K_{f,C} \int_0^s \|x - x^\dagger\|_{[-\tau,r]} dr.$$

A simple application of Gronwall's inequality yields (2.6). □

Recall that if a function $x : [0,\infty) \to \mathbb{R}$ is *absolutely continuous*, then it is almost everywhere differentiable and there exists a Lebesgue integrable function $u : [0,\infty) \to \mathbb{R}$ such that

$$x(t) = x(0) + \int_0^t u(s) ds, \ t \ge 0,$$

and at almost every $t > 0$, x is differentiable and $\frac{dx(t)}{dt} = u(t)$.

Lemma 2.2. *Suppose that x is a solution of DDER. Then on $[0,\infty)$, x is locally Lipschitz continuous and so is absolutely continuous. For the almost every $t \in (0,\infty)$ at which x is differentiable, we have*

$$\frac{dx(t)}{dt} = \begin{cases} f(x_t) & \text{if } x(t) > 0, \\ 0 & \text{if } x(t) = 0. \end{cases} \tag{2.7}$$

Furthermore, x is continuously differentiable at all $t > 0$ such that $x(t) > 0$.

Proof. By the fundamental theorem of calculus and the continuity of $t \to f(x_t)$, z defined in (2.2) is locally Lipschitz on $[0,\infty)$ and continuously differentiable on $(0,\infty)$ with $\frac{dz(t)}{dt} = f(x_t)$ at each $t > 0$. By (2.4) and Proposition A.2, x inherits the local Lipschitz property from z on $[0,\infty)$ and so is absolutely continuous there. Consider a time $t > 0$ where $x(t) > 0$. Then y is constant in an open interval about t and so y is differentiable in a neighborhood of t with derivative $\frac{dy(t)}{dt} = 0$. By (2.1), x is continuously differentiable at t with derivative $\frac{dx(t)}{dt} = f(x_t)$. Now consider $t > 0$ where x is differentiable and $x(t) = 0$. By considering derivatives from the left and the right and using the fact that $x(s) \geq 0$ for all $s \geq 0$, we see that $\frac{dx(t)}{dt} = 0$. □

Delay differential equations with discontinuous right hand side are often used in engineering models (see e.g., [23, 24, 25, 26]) to account for state constraints. A consequence of the discontinuous right hand side is that solutions do not have to be continuously differentiable. Consider, for example,

$$\frac{dx(t)}{dt} = \begin{cases} f(x_t), & \text{if } x(t) > 0, \\ f(x_t)^+, & \text{if } x(t) = 0. \end{cases} \tag{2.8}$$

We consider solution of (2.8) to be an absolutely continuous function $x \in \mathcal{C}^+_{[-\tau,\infty)}$ satisfying (2.8) at the almost every $t \in (0, \infty)$ where x is differentiable. In the following lemma we show that given $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, there exists a unique solution of (2.8) with initial condition φ and it coincides with the unique solution of DDER with initial condition φ (Recall that we have assumed the existence of a unique solution of DDER with initial condition φ.)

Lemma 2.3. *Suppose x is a solution of* (2.8)*, then x is a solution of DDER. Conversely, if x is a solution of DDER, then x is a solution of* (2.8)*. By assumption, for each $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, there exists a unique solution x of DDER with initial condition φ; therefore there exists a unique solution of* (2.8) *with initial condition φ.*

Proof. Suppose x is a solution of (2.8). Then x satisfies

$$x(t) = x(0) + \int_0^t 1_{\{x(s)>0\}} f(x_s)ds + \int_0^t 1_{\{x(s)=0\}} f(x_s)^+ ds, \; t \geq 0. \tag{2.9}$$

We can rewrite (2.9) as

$$x(t) = x(0) + \int_0^t f(x_s)ds + \int_0^t 1_{\{x(s)=0\}} f(x_s)^- ds, \; t \geq 0.$$

If we define $y \in \mathcal{C}^+_{[0,\infty)}$ by $y(t) = \int_0^t 1_{\{x(s)=0\}} f(x_s)^- ds$ for all $t \geq 0$, then y is nondecreasing, $y(0) = 0$, and $\int_0^t x(s)dy(s) = 0$ for all $t \geq 0$. Therefore x is a solution of DDER associated with f.

Now suppose x is a solution of DDER. Let z and y be defined as in (2.2) and (2.3), respectively. By Lemma 2.2, x is an absolutely continuous function satisfying (2.7) for the almost every $t > 0$ at which x is differentiable. Clearly, (2.8) holds at such $t > 0$ if $x(t) > 0$. On the other hand, consider such $t > 0$ at which $x(t) = 0$. Then $\frac{dx(t)}{dt} = 0$ and by the fundamental theorem of calculus and the continuity of f, z is a continuously differentiable function and therefore y is differentiable at t with derivative given by

$$\frac{dy(t)}{dt} = \frac{dx(t)}{dt} - \frac{dz(t)}{dt} = -f(x_t) \geq 0,$$

where the inequality follows since y is a non-decreasing function. Thus, for $t > 0$ such that x is differentiable and $x(t) = 0$, we have $\frac{dy(t)}{dt} = f(x_t)^-$ and so

$$\frac{dx(t)}{dt} = \frac{dz(t)}{dt} + \frac{dy(t)}{dt} = f(x_t) + f(x_t)^- = f(x_t)^+.$$

Hence x is a solution of (2.8). □

Remark 2.4. Note that if x is a solution of DDER and $t > 0$ such that $x(t) = 0$ and $x(\cdot)$ is differentiable at t, it follows from Lemmas 2.2 and 2.3 that

$$\frac{dx(t)}{dt} = f(x_t)^+ = 0. \tag{2.10}$$

This chapter is based on the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

Chapter 3

Main Results

In this section we present our main results on sufficient conditions for the existence, uniqueness and stability of slowly oscillating periodic solutions to the DDER.

3.1 Slowly Oscillating Periodic Solutions (SOPS)

We will be assuming that there is a positive equilibrium point for the DDER which is defined as follows:

Definition 3.1. A point $L > 0$ is an *equilibrium point* of DDER if $x \equiv L$ on $[-\tau, \infty)$ is a solution of DDER.

A solution x of DDER that oscillates about an equilibrium point L and such that the times at which x is at the equilibrium point are separated by more than they delay τ is called *slowly oscillating*. Throughout this paper, we consider periodic solutions with this property.

Definition 3.2. A solution x of DDER is called a *periodic solution with period* $p > 0$ if

$$x(t + p) = x(t) \text{ for all } t \geq -\tau. \tag{3.1}$$

Suppose $L > 0$ is an equilibrium point of DDER. Then a periodic solution x^* of DDER is called a *slowly oscillating periodic solution (SOPS)* if there exist $q_0 \geq -\tau$,

$q_1 > q_0 + \tau$, and $q_2 > q_1 + \tau$ such that (3.1) holds with $p = q_2 - q_0$, and

$$\begin{aligned} x^*(q_0) &= L, \\ x^*(t) &> L, \ q_0 < t < q_1, \\ 0 \le x^*(t) &< L, \ q_1 < t < q_2. \end{aligned} \tag{3.2}$$

See Figure 5.3 for an example of a SOPS of DDER when $q_0 = -\tau$ and f is of the form exhibited in (3.9) and satisfies Assumptions 3.3 and 3.4 below. Throughout the remainder of this paper we will use x^* to denote a SOPS of DDER.

3.2 Existence of SOPS

To establish the existence of a SOPS, we assume that f is solely a function of the current and delayed states of the system:

$$f(\varphi) = g(\varphi(0), \varphi(-\tau)), \text{ for all } \varphi \in \mathcal{C}^+_{[-\tau,0]}, \tag{3.3}$$

where $g : \mathbb{R}^2_+ \to \mathbb{R}$ is a continuous function that satisfies two sets of assumptions. The first set of assumptions is used to establish the existence of an equilibrium point and to specify regularity properties of g.

Assumption 3.1. The function $g : \mathbb{R}^2_+ \to \mathbb{R}$ is locally Lipschitz continuous, there is an $L > 0$ such that $g(L, L) = 0$, g is differentiable at (L, L) and

$$A \equiv -\partial_1 g(L, L) \ge 0, \ B \equiv -\partial_2 g(L, L) > 0, \tag{3.4}$$

satisfy $B > A \ge 0$. Here $\partial_i g$ denotes the first partial derivative with respect to the i^{th} argument of g, for $i = 1, 2$.

The condition (3.4) imposes a negative feedback condition on the local linearization about the equilibrium; for this linearization, the condition $B > A$ is known to be necessary for the equilibrium solution to be unstable. The following is a global negative feedback type of condition.

Assumption 3.2. For all $r, s \in \mathbb{R}_+$,

(i) $(g(r,s) - g(r,L))(s - L) < 0$ if $s \neq L$, and

(ii) $(g(r,s) - g(L,s))(r - L) \leq 0$ if $r \neq L$.

Remark 3.1. Assumptions 3.1 and 3.2 imply that if $r \geq L$ and $s > L$, then $g(r,s) < g(r,L) \leq g(L,L) = 0$; similarly, if $r \leq L$ and $s < L$, then $g(r,s) > g(r,L) \geq g(L,L) = 0$. Hence Assumption 3.2 can be thought of as imposing a global negative feedback condition. Also, by (i) and (ii), $g(r,r)(r - L) \leq g(L,r)(r - L) < 0$ for $r \neq L$, and so $g(r,r) = 0$ if and only if $r = L$; this ensures that L is a unique equilibrium point of DDER.

Previous results [2] on the existence of SOPS in the unconstrained setting typically require a third set of conditions bounding $g(L,\cdot)$ and providing linear growth conditions on g in both components to prevent unbounded oscillations. The presence of the lower boundary in (1.1) prevents unbounded oscillations and a version of the third condition in [2] is instead a consequence of Assumptions 3.1 and 3.2, as follows.

Lemma 3.1. *Under Assumptions 3.1 and 3.2, there exists $G = G(g) > 0$ such that $g(L,s) \leq G$ for all $s \in \mathbb{R}_+$. Additionally, there exist positive constants $\kappa_1 = \kappa_1(g)$ and $\kappa_2 = \kappa_2(g)$ such that*

$$|g(r,s)| \leq \kappa_1|r - L| + \kappa_2|s - L|, \; 0 \leq r, s \leq L + \tau G. \tag{3.5}$$

Proof. The continuity of g implies that $g(L,\cdot)$ is bounded on compact sets. Therefore, there exists $G > 0$ such that $g(L,s) \leq G$ for all $s \in [0,L]$. For $s > L$, using the fact that $g(L,L) = 0$ and part (i) of Assumption 3.2, we have that $g(L,s) < 0$. Thus, $g(L,s) \leq G$ for all $s \in \mathbb{R}_+$. Additionally, since g is continuous on $\mathbb{R}^2_+$, locally Lipschitz continuous at (L,L) and $g(L,L) = 0$, we have the linear growth bounds in (3.5). □

Under Assumption 3.1, consider the unconstrained linear delay differential equation obtained by linearizing g about its equilibrium point L and centering about the origin:

$$\frac{du(t)}{dt} = -Au(t) - Bu(t - \tau). \tag{3.6}$$

Equation (3.6) has characteristic equation

$$\lambda + A + Be^{-\lambda\tau} = 0. \tag{3.7}$$

Let θ_0 be the unique solution in $[\pi/2, \pi)$ to $\cos\theta_0 = -A/B$ and define

$$\tau_0 = \frac{\theta_0}{\sqrt{B^2 - A^2}}. \tag{3.8}$$

If $\tau > \tau_0$, the characteristic equation (3.7) will have a solution λ with positive real part, from which it follows there exist solutions of the linear delay differential equation (3.6) that exhibit unbounded oscillations (see [10], Chapter 7). Using this we show that $x \equiv L$ is an unstable equilibrium solution for DDER and then prove the following result on the existence of a non-constant (oscillating) periodic solution.

Theorem 3.1. *Under Assumptions 3.1 and 3.2, if τ_0 is given by* (3.8)*, then for any $\tau > \tau_0$, there exists a SOPS of the DDER* (1.1)*.*

The proof of Theorem 3.1 is given in Section 4. Our proof is similar to proofs of the existence of SOPS for unconstrained one-dimensional non-linear delay differential equations (see e.g., [2, 9, 12, 18]). The main difference in our work is the presence of the lower boundary with the associated function y controlling the dynamics in (1.1).

3.3 Uniqueness and Stability of SOPS

To establish uniqueness and stability of SOPS, we will now impose more restrictive conditions on f; in particular, we only allow f to depend on the delayed state:

$$f(\varphi) = h(\varphi(-\tau)), \text{ for all } \varphi \in \mathcal{C}^+_{[-\tau,0]}, \tag{3.9}$$

where h is a continuous function that satisfies two sets of assumptions. The first set of assumptions imply Assumptions 3.1 and 3.2 used in proving the existence of a SOPS. It includes further assumptions on the differentiability of h and its asymptotic behavior at infinity.

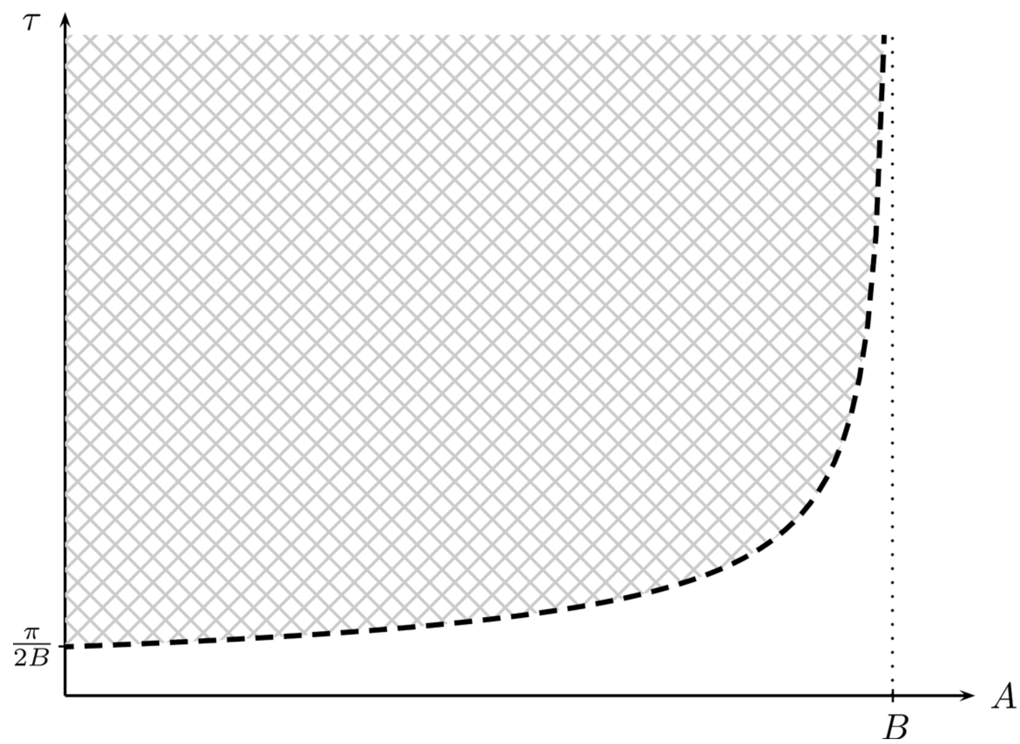

Figure 3.1: For a fixed parameter $B > 0$, the equilibrium solution $x \equiv L$ is unstable for parameters τ and A in the shaded region, which is given by $\{(A, \tau) : A \in [0, B) \text{ and } \tau > \tau_0\}$, where τ_0 is as in (3.8).

Assumption 3.3. The function $h : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable on $\mathbb{R}_+$, there are positive constants α, β such that $\lim_{s\to\infty} h(s) = -\alpha$, $h(0) = \beta$, and there is an $L > 0$ such that $h(L) = 0$, $h'(L) < 0$ and $(s - L)h(s) < 0$ for all $L \neq s \in \mathbb{R}_+$.

Lemma 3.2. *Under Assumption 3.3,* $H \equiv \sup\{|h(s)| : s \in \mathbb{R}_+\} < \infty$. *If x is a solution of DDER associated with h, then x is uniformly Lipschitz continuous with Lipschitz constant H:*

$$|x(t) - x(s)| \leq H|t - s|, \ 0 \leq s, t < \infty. \tag{3.10}$$

Proof. Since h is continuous on $\mathbb{R}_+$ and has a finite limit at infinity, h is uniformly bounded. By Lemma 2.2, if x is a solution of DDER, then it is absolutely continuous on $[0,\infty)$ and for the almost every $t \in (0,\infty)$ at which x is differentiable,

$$\frac{dx(t)}{dt} = \begin{cases} h(x(t-\tau)) & \text{if } x(t) > 0, \\ 0 & \text{if } x(t) = 0. \end{cases}$$

It follows that its almost everywhere in $[0,\infty)$ defined derivative is bounded by H and (3.10) holds. □

On setting $g(r,s) = h(s)$ for $r, s \geq 0$, Assumption 3.3 implies that g satisfies Assumptions 3.1 and 3.2 (with $A = 0$ and $B = -h'(L)$). Define

$$\tau_0 = -\frac{\pi}{2h'(L)} > 0. \tag{3.11}$$

Then Theorem 3.1 ensures that for any $\tau > \tau_0$ there exists a SOPS of DDER. In order to prove the uniqueness and stability of a SOPS, we assume that $h'(s)$ converges zero sufficiently fast as $s \to \infty$. Recall that $L^1(\mathbb{R}_+)$ denotes the space of Lebesgue measurable functions from $\mathbb{R}_+$ into $\mathbb{R}$ with finite L^1-norm.

Assumption 3.4. The function $h : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable on $\mathbb{R}_+$, its derivative h' is in $L^1(\mathbb{R}_+)$ and $m \equiv \sup\{|sh'(s)| : s \in \mathbb{R}_+\} < \infty$.

Lemma 3.3. *Under Assumption 3.4,* $K_h \equiv \sup\{|h'(s)| : s \in \mathbb{R}_+\} < \infty$ *and so* h *is uniformly Lipschitz continuous with Lipschitz constant* K_h*:*

$$|h(s) - h(r)| = \left| \int_r^s h'(u)du \right| \leq K_h|s-r|, \; 0 \leq r, s < \infty. \tag{3.12}$$

Proof. Since h' is continuous and $h'(s) \to 0$ as $s \to \infty$, h' is uniformly bounded on $\mathbb{R}_+$ and (3.12) follows. □

We say a SOPS x^* of DDER is unique (up to time translation) if given another SOPS $x^\dagger$ of DDER, there exists $t_0 \geq 0$ such that $x^*(t) = x^\dagger(t + t_0)$ for all $t \geq 0$. The following is our main result on the uniqueness and stability of a SOPS.

Theorem 3.2. *Under Assumptions 3.3 and 3.4, if* τ_0 *is given by* (3.11)*, then there exists* $\tau^* \geq \tau_0$ *such that for any* $\tau > \tau^*$*, there exists a SOPS* x^* *of DDER with delay* τ *and it is unique (up to time translation). Furthermore, the SOPS satisfies the following property, which we call (local) uniform exponential asymptotic stability:*

there are positive constants ε, γ, K_γ *and* K_ρ *such that for any member* x^* *of the family of equivalent (up to time translation) SOPS and for* p *equal to the period of* x^*, *if* $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ *satisfies* $\|\varphi - x^*_\sigma\|_{[-\tau,0]} < \varepsilon$ *for some* $\sigma \in [0,p)$, *then there is a* $\rho \in (-p,p)$ *that satisfies*

$$|\rho| \leq K_\rho \|\varphi - x^*_\sigma\|_{[-\tau,0]}, \tag{3.13}$$

and is such that

$$\|x_t - x^*_{t+p+\sigma+\rho}\|_{[-\tau,0]} \leq K_\gamma e^{-\gamma t} \|\varphi - x^*_\sigma\|_{[-\tau,0]}, \; t \geq 0, \tag{3.14}$$

where x *denotes the unique solution of DDER with initial condition* φ.

The proof of Theorem 3.2 is given in Section 5. Our proof adapts an approach used in [30, 31] to prove uniform exponential asymptotic stability and uniqueness of unconstrained slowly oscillating periodic solutions. The proof relies on the construction of an approximate Poincaré map associated with a SOPS. The construction requires a linear variational equation relative to constrained solutions. Difficulties arise because solutions to the linear variational equation in the constrained setting can be discontinuous.

3.4 Examples

We illustrate our results with some simple examples of deterministic models of biochemical reactions and Internet rate control.

Example 3.1. Fix $\tau > 0$ and $A, B, C, D, E > 0$. We consider a simple biochemical model for the production and degradation of a protein X. In the model, protein X is produced by components external to the system at rate A and each protein molecule degrades at rate B, which is represented by the following reactions:

$$\emptyset \xrightarrow{A} X, \qquad X \xrightarrow{B} \emptyset,$$

where $\emptyset$ denotes "nothing" (or a quantity external to the system). Furthermore, X is a transcription factor activating the production of a protein Y, which, after production, quickly eliminates a molecule of protein X if it is available or otherwise

Y rapidly degrades. The production process for protein Y is a multistage process, including lengthy transcription and translation stages, which leads to a delay in its production. The production of protein Y and the subsequent degradation of a molecule of X by a molecule of protein Y can be represented by the following reactions:

$$X \overset{C}{\Rightarrow} X + Y, \qquad X + Y \overset{D}{\to} \emptyset, \qquad Y \overset{E}{\to} \emptyset,$$

where the double arrow indicates a delayed reaction. Both D and E are very large constants with D considerably larger than E. As a simplification, we assume that after a molecule of Y is produced, it eliminates a molecule of protein X instantaneously if such a molecule of X is available or otherwise the molecule of Y degrades instantaneously. With this simplification, a deterministic model for the concentration of protein X at time t is given by the DDER associated with

$$f(\varphi) = C - A\varphi(0) - B\varphi(-\tau),\ \varphi \in \mathcal{C}^+_{[-\tau,0]}.$$

If $B > A$, f clearly satisfies Assumptions 3.1 and 3.2 with equilibrium point $L = \frac{C}{A+B}$. Let τ_0 be defined as in (3.8). If $\tau > \tau_0$, then by Theorem 3.1 there exists a SOPS of DDER.

In [4], Bratsun et al. analyzed a similar deterministic biochemical reaction model with delayed dynamics and linear f, but without the non-negativity constraint. If $\tau > \tau_0$, then an initial condition corresponding to a SOPS in the constrained setting will correspond to a solution with unbounded oscillations as in Figure 3.2.

Example 3.2. Fix $\tau > 0$ and $\alpha, \beta, \gamma, C_0, R_0 > 0$. We consider a simple model for a biochemical reaction system in which the quantity of a repressor protein is affected by three factors: production, enzymatic degradation and dilution. The protein is self-regulating in that it represses its own production. It also has a lengthy transcription and translation time. A simple model was proposed by Mather et al. [15] where the deterministic dynamics of the system are described by the following delay differential equation:

$$\frac{dx(t)}{dt} = \frac{\alpha C_0^2}{(C_0 + x(t-\tau))^2} - \frac{\gamma x(t)}{R_0 + x(t)} - \beta x(t),\ t \geq 0,$$

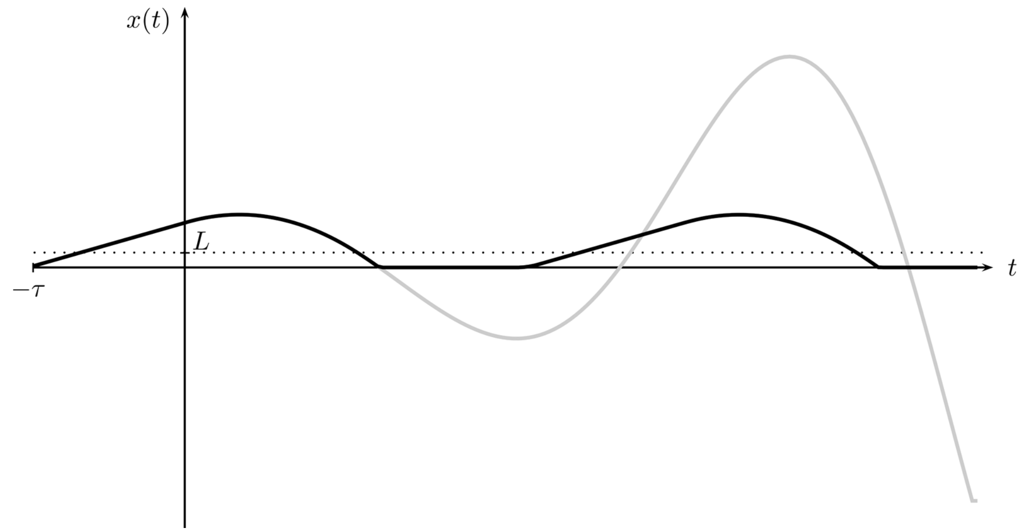

Figure 3.2: A SOPS of DDER (in black) and a solution of an unconstrained delay differential equation (in gray) with identical initial conditions and identical linear drift functions as described in Example 3.1.

where $x(t)$ represents the concentration of the repressor protein at time t and takes values in the non-negative real numbers. The first term on the right is the protein production rate with delayed negative feedback and the second and third terms represent the effects of enzymatic degradation and dilution, respectively. In the above model, the non-negativity of the protein concentration is ensured by the form of the delay differential equation: if $x(t) = 0$, then $\frac{dx(t)}{dt} > 0$. In [15], R_0 and β are very small and the authors consider the limiting case of the deterministic system where $R_0 = 0$ and $\beta = 0$. However, in this formal limit, the delay differential equation loses the inherent non-negativity of its solutions, and the equation must be modified at the boundary, i.e., when $x(t) = 0$. One way to account for the non-negativity constraint is to consider solutions of our DDER associated with

$$f(\varphi) = \frac{\alpha C_0^2}{(C_0 + \varphi(-\tau))^2} - \gamma, \; \varphi \in \mathcal{C}^+_{[-\tau,0]}.$$

If $\alpha > \gamma$, then f satisfies Assumptions 3.3 and 3.4 with equilibrium point $L = C_0(\sqrt{\alpha/\gamma} - 1)$. Then by Theorem 3.1, if $\tau > \frac{\pi}{4} C_0 \sqrt{\alpha/\gamma^3}$, there exists a slowly oscillating periodic solution of the DDER. Furthermore, by Theorem 3.2, there exists $\tau^* \geq \frac{\pi}{4} C_0 \sqrt{\alpha/\gamma^3}$ such that if $\tau > \tau^*$, then any SOPS of the DDER is unique and uniformly exponentially asymptotically stable.

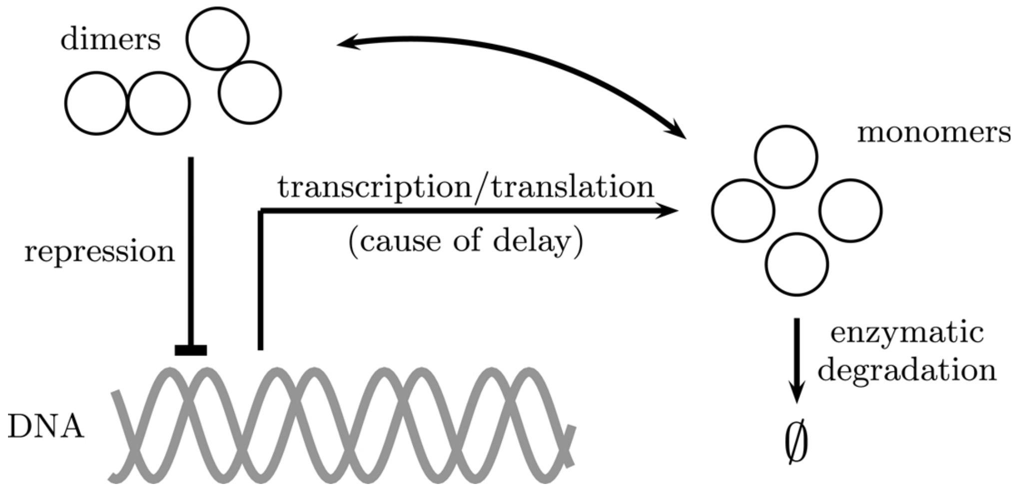

Figure 3.3: A depiction of the simple negative feedback circuit described in Example 3.2. In our model $x(t)$ denotes the concentration of the protein monomers and the squared term in the denominator of the production term arises because the dimerized form of the protein represses transcription.

Example 3.3. Deterministic delay differential equations have been used as approximate (fluid) models for Internet rate control, where the finiteness of transmission speeds leads to a delay in the dynamics. Here we consider the one-dimensional case of a model introduced by Paganini, Doyle and Low [22], and studied by Paganini and Wang [23], Peet and Lall [26], Papachristadolou [24], and Papachristadolou, Doyle and Low [25]. The dynamics of the model are described by (2.8), where

$$f(\varphi) = \frac{e^{-\alpha\varphi(-\tau)}}{c} - 1, \ \varphi \in \mathcal{C}^+_{[-\tau,0]}, \tag{3.15}$$

$\tau > 0$ is the fixed delay, $0 < c < 1$ is the capacity of a singe link, and $\alpha > 0$ is a fixed constant. By Lemma 2.3, it follows that solutions of (2.8) are in one-to-one correspondence with solutions of the DDER where f is as in (3.15). Since f satisfies Assumptions 3.3 and 3.4 with equilibrium point $L = -\ln(c)/\alpha$, by Theorem 3.1, if $\tau > \pi/(2\alpha)$, then there exists a SOPS of the DDER. Furthermore, by Theorem 3.2, there is a $\tau^* \geq \pi/(2\alpha)$ such that for each $\tau > \tau^*$, there exists a unique SOPS, which is uniformly exponentially asymptotically stable. Internet rate control protocols are typically designed to prevent such oscillatory behavior. It is noted in [22] that as the delay τ increases, solutions to (2.8) with f as in (3.15) do exhibit oscillatory behavior. To counteract this problem, α is often designed so as to depend on the delay τ, e.g., in [22] it is observed that if $\alpha(\tau) = \beta/\tau$ for $\beta \in (0, \pi/2)$, then the equilibrium solution is asymptotically stable for all $\tau > 0$.

This chapter is based on the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

Chapter 4

Existence of Slowly Oscillating Periodic Solutions

In this section we prove Theorem 3.1 which provides sufficient conditions for the existence of slowly oscillating periodic solutions (SOPS) of DDER. The proof follows an argument similar to one used for unconstrained systems [2, 9, 18], which utilizes a form of Browder's fixed point theorem. New difficulties need to be addressed in our context because of the lower boundary constraint and the associated discontinuous dynamics. Throughout this section, we assume that f is of the form exhibited in (3.3) and that Assumptions 3.1 and 3.2 hold.

4.1 Browder's Fixed Point Theorem

Definition 4.1. Let X be a topological space, $f : X \to X$ a continuous function and $x_0 \in X$ be a fixed point of f, so that $f(x_0) = x_0$. Then x_0 is an *ejective fixed point* if there exists an open neighborhood U of x_0 such that for every $x \in U \setminus \{x_0\}$, there exists $n = n(x) \in \mathbb{N}$ such that the n^{th} iterate of f, $f^n(x)$, is not in U.

The following is a version of Browder's fixed point theorem [6], proved by Nussbaum (a special case of Corollary 1.1 in [18]), which provides sufficient conditions for the existence of non-ejective fixed points.

Theorem 4.1. *Let K be a closed, bounded, convex, infinite-dimensional subset of a Banach space. Suppose that $f : K \to K$ is a continuous, compact function. Then f has a fixed point in K that is not ejective.*

Briefly, our proof of Theorem 3.1 proceeds as follows. We first perform a spatial shift and rescale time in (1.1) so that the equilibrium solution is at the origin and the delay interval $[-\tau, 0]$ is normalized to $[-1, 0]$. We show that it suffices to prove existence of a corresponding slowly oscillating periodic solution for this normalized equation. We denote solutions of the normalized equation with a hat: $\hat{x}$. Existence will be proved by finding a suitable set $\widetilde{\mathcal{K}}$ in the Banach space $\mathcal{C}_{[-1,0]}$ and proving that if $\hat{x}$ is a solution of the normalized equation such that $\hat{x}_0$ in $\widetilde{\mathcal{K}}$, then there exists a time $t > 0$ such that $\hat{x}_t \in \widetilde{\mathcal{K}}$. A function Λ on $\widetilde{\mathcal{K}}$ will map $\hat{x}_0 \in \widetilde{\mathcal{K}}$ to $\hat{x}_t$ at the first time $t > 0$ that the solution $\hat{x}$ starting from $\hat{x}_0$ reenters $\widetilde{\mathcal{K}}$. An element of $\widetilde{\mathcal{K}}$ that is mapped by Λ to itself corresponds to a periodic solution (which may be constant). Under Assumptions 3.1 and 3.2, for $\tau > \tau_0$, the unique constant solution will be an ejective fixed point of the map Λ and Browder's fixed point theorem will imply the existence of a non-ejective fixed point, which will correspond to a slowly oscillating periodic solution.

4.2 Normalized Solutions

It will be convenient to work with normalized solutions of the DDER (1.1), obtained from a solution x of DDER by subtracting off L and rescaling time so that the delay is of length one. The normalized solutions will satisfy a normalized version of (1.1). We work with this normalized equation here in the proof of existence, as well as in Section 5 in the proof of stability and uniqueness. There is no loss of generality in this as there is a one-to-one correspondence between solutions of the normalized equation and those of the original DDER, as we will show below in Lemma 4.1.

We first need some definitions. Recall that g is assumed to satisfy Assumptions 3.1 and 3.2. Let $\hat{g} : [-L, \infty)^2 \to \mathbb{R}$ be the function defined by:

$$\hat{g}(r, s) = g(r + L, s + L), \; r, s \in [-L, \infty). \tag{4.1}$$

Then $\hat{g}$ inherits the following properties from g: the function $\hat{g}$ is locally Lipschitz continuous, $\hat{g}(0,0) = 0$, $\hat{g}$ is differentiable at $(0,0)$ with

$$-\partial_1 \hat{g}(0,0) = A \geq 0, \ -\partial_2 \hat{g}(0,0) = B > 0,$$

where $\partial_i \hat{g}$ denotes the first partial derivative with respect to the i^{th} argument of $\hat{g}$, for $i = 1, 2$, and $B > A \geq 0$ are defined in Assumption 3.1. By Assumption 3.2, $\hat{g}$ satisfies the following inequalities for $r, s \in [-L, \infty)$:

$$\hat{g}(r,s) > \hat{g}(r,0) \text{ if } -L \leq s < 0, \tag{4.2}$$

$$\hat{g}(r,s) < \hat{g}(r,0) \text{ if } s > 0, \tag{4.3}$$

$$\hat{g}(r,s) \geq \hat{g}(0,s) \text{ if } -L \leq r \leq 0, \tag{4.4}$$

$$\hat{g}(r,s) \leq \hat{g}(0,s) \text{ if } r \geq 0. \tag{4.5}$$

By (4.2) and (4.4), if $r \leq 0$ and $s < 0$, then $\hat{g}(r,s) > 0$ and similarly, by (4.3) and (4.5), if $r \geq 0$ and $s > 0$, then $\hat{g}(r,s) < 0$. Finally, for $G = G(g)$, $\kappa_1 = \kappa_1(g)$ and $\kappa_2 = \kappa_2(g)$ as in Lemma 3.1, we have

$$\hat{g}(0,s) \leq G, \ s \geq -L, \tag{4.6}$$

and

$$|\hat{g}(r,s)| \leq \kappa_1 |r| + \kappa_2 |s|, \ -L \leq r, s \leq \tau G. \tag{4.7}$$

We can now define a solution of a normalized delay differential equation with reflection associated with $\hat{g}$.

Definition 4.2. A continuous function $\hat{x} \in \mathcal{C}_{[-1,\infty)}$ is a *solution of a normalized delay differential equation with reflection (DDER^n) associated with* $\hat{g}$ if $\hat{x}(t) \geq -L$ for all $t \geq -1$ and satisfies

$$\hat{x}(t) = \hat{x}(0) + \tau \int_0^t \hat{g}(\hat{x}(s), \hat{x}(s-1)) ds + \hat{y}(t), \ t \geq 0, \tag{4.8}$$

where $\hat{y} \in \mathcal{C}^+_{[0,\infty)}$ is a continuous and non-decreasing function such that $\hat{y}(0) = 0$, and $\int_0^t (\hat{x}(s) + L) d\hat{y}(s) = 0$ for all $t \geq 0$.

For the remainder of this section, we use $\hat{x}$ to denote a solution of DDER$^{\text{n}}$ associated with the $\hat{g}$ defined in (4.1). From the conditions on $\hat{g}$, the point 0 is the unique equilibrium point for $\mathrm{DDER^n}$, i.e., $\hat{x} \equiv 0$ is the only constant solution of $\mathrm{DDER^n}$. We define a slowly oscillating periodic solution ($\mathrm{SOPS^n}$) of $\mathrm{DDER^n}$.

Definition 4.3. A solution $\hat{x}$ of $\mathrm{DDER^n}$ is called *periodic* if there exists $\hat{p} > 0$ such that

$$\hat{x}(t + \hat{p}) = \hat{x}(t), \text{ for all } t \geq -1. \tag{4.9}$$

A periodic solution $\hat{x}^*$ of $\mathrm{DDER^n}$ is called a *slowly oscillating periodic solution ($SOPS^n$)* if there exists $\hat{q}_0 \geq -1$, $\hat{q}_1 > \hat{q}_0 + 1$, and $\hat{q}_2 > \hat{q}_1 + 1$ such that (4.9) holds with $\hat{p} = \hat{q}_2 - \hat{q}_0$, and

$$\begin{aligned} \hat{x}^*(\hat{q}_0) &= 0, \\ \hat{x}^*(t) &> 0 \text{ for } \hat{q}_0 < t < \hat{q}_1, \\ -L \leq \hat{x}^*(t) &< 0 \text{ for } \hat{q}_1 < t < \hat{q}_2. \end{aligned} \tag{4.10}$$

See Figure 5.1 for an example of a $\mathrm{SOPS^n}$ of $\mathrm{DDER^n}$ when $\hat{q}_0 = -1$ and $\hat{g}$ is of the form $\hat{g}(r, s) = \hat{h}(s)$ where $\hat{h}$ is as in Section 5.1. Throughout the remainder of this paper we will use $\hat{x}^*$ to denote a $\mathrm{SOPS^n}$ of $\mathrm{DDER^n}$. In the following lemma we show that there is a one-to-one correspondence between solutions of DDER and solutions of $\mathrm{DDER^n}$ as well as between SOPS and $\mathrm{SOPS^n}$.

Lemma 4.1. *If x is a solution of DDER associated with g and $\hat{x} \in \mathcal{C}_{[-1,\infty)}$ is defined by*

$$\hat{x}(t) = x(\tau t) - L, \; t \geq -1, \tag{4.11}$$

then $\hat{x}$ is a solution of $DDER^n$ associated with $\hat{g}$. Furthermore, if x is a SOPS, then $\hat{x}$ is a $SOPS^n$. Conversely, if $\hat{x}$ is a solution of $DDER^n$ associated with $\hat{g}$ and $x \in \mathcal{C}^+_{[-\tau,\infty)}$ is defined by

$$x(t) = \hat{x}(\tau^{-1} t) + L, \; t \geq -\tau, \tag{4.12}$$

then x is a solution of DDER associated with g. Furthermore, if $\hat{x}$ is a $SOPS^n$, then x is a SOPS.

Proof. Suppose x is a solution of DDER. By subtracting L from both sides, scaling time by τ, and performing a change of variables in the integration, we see that x

satisfies

$$x(\tau t) - L = x(0) - L + \tau \int_0^t \hat{g}(x(\tau s) - L, x(\tau s - \tau) - L)ds + y(\tau t),\ t \geq 0.$$

Define $\hat{x} \in \mathcal{C}_{[-1,\infty)}$ as in (4.11) and $\hat{y} \in \mathcal{C}^+_{[0,\infty)}$ by $\hat{y}(t) = y(\tau t)$ for $t \geq 0$. Then $\hat{x}(t) \geq -L$ for all $t \geq -1$, and $\hat{x}$ satisfies

$$\hat{x}(t) = \hat{x}(0) + \tau \int_0^t \hat{g}(\hat{x}(s), \hat{x}(s-1))ds + \hat{y}(t),\ t \geq 0,$$

where $\hat{y} \in \mathcal{C}^+_{[0,\infty)}$ is non-decreasing, $\hat{y}(0) = 0$ and $\int_0^t 1_{(-L,\infty)}(\hat{x}(s))d\hat{y}(s) = 0$ for all $t \geq 0$. Therefore $\hat{x}$ is a solution of $\mathrm{DDER}^{\mathrm{n}}$. Furthermore, if x is a SOPS with period p and q_0, q_1, and q_2 are as in (3.2), then $\hat{x}$ is a $\mathrm{SOPS}^{\mathrm{n}}$ with period $\hat{p} = \tau^{-1}p$ and $\hat{q}_0 = \tau^{-1}q_0$, $\hat{q}_1 = \tau^{-1}q_1$, and $\hat{q}_2 = \tau^{-1}q_2$. Now suppose $\hat{x}$ is a solution of $\mathrm{DDER}^{\mathrm{n}}$. If we define $x \in \mathcal{C}^+_{[-\tau,\infty)}$ as in (4.12) and $y \in \mathcal{C}^+_{[0,\infty)}$ by $y(t) = \hat{y}(\tau^{-1}t)$, then by reversing the above steps, we have that x is a solution of DDER and if $\hat{x}$ is a $\mathrm{SOPS}^{\mathrm{n}}$, then x is a SOPS. □

By the unique correspondence between solutions x of DDER and $\hat{x}$ of $\mathrm{DDER}^{\mathrm{n}}$ described in Lemma 4.1, we have the following results for solutions $\hat{x}$ of $\mathrm{DDER}^{\mathrm{n}}$.

Lemma 4.2. *Suppose that $\hat{x}$ is a solution of $DDER^{n}$. Then, on $[0,\infty)$, $\hat{x}$ is locally Lipschitz continuous and so is absolutely continuous. For the almost every $t \in (0,\infty)$ that $\hat{x}(\cdot)$ is differentiable,*

$$\frac{d\hat{x}(t)}{dt} = \begin{cases} \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)), & \text{if } \hat{x}(t) > -L, \\ 0, & \text{if } \hat{x}(t) = -L. \end{cases} \tag{4.13}$$

Furthermore, $\hat{x}$ is continuously differentiable at all $t > 0$ for which $\hat{x}(t) > -L$.

Proof. This follows immediately from Lemmas 4.1 and 2.2. □

Remark 4.1. By Lemmas 4.1 and 2.3, we have that if $\hat{x}$ is a solution of $\mathrm{DDER}^{\mathrm{n}}$ and $\hat{x}(\cdot)$ is differentiable at $t > 0$, then

$$\frac{d\hat{x}(t)}{dt} = \begin{cases} \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)), & \text{if } \hat{x}(t) > -L, \\ \tau\hat{g}(\hat{x}(t), \hat{x}(t-1))^+, & \text{if } \hat{x}(t) = -L. \end{cases} \tag{4.14}$$

4.3 Slowly Oscillating Solutions

In this section we prove that solutions of DDER$^{\mathrm{n}}$ with initial condition in a certain subset of $\mathcal{C}_{[-1,0]}$ are slowly oscillating. Throughout this section, let $G = G(g)$, $\kappa_1 = \kappa_1(g)$, and $\kappa_2 = \kappa_2(g)$ be as in Lemma 3.1 so that (4.7) holds. Define

$$\mathcal{K} = \left\{ \hat{\varphi} \in \mathcal{C}^+_{[-1,0]} : \hat{\varphi}(-1) = 0 \right\} \tag{4.15}$$

$$\widehat{\mathcal{K}} = \{ \hat{\varphi} \in \mathcal{K} : \exp(\tau\kappa_1 \cdot)\hat{\varphi}(\cdot) \text{ is non-decreasing on } [-1,0] \} \tag{4.16}$$

$$\widetilde{\mathcal{K}} = \left\{ \hat{\varphi} \in \widehat{\mathcal{K}} : \|\hat{\varphi}\|_{[-1,0]} \le \tau G \right\}. \tag{4.17}$$

Then $\widetilde{\mathcal{K}}$ is a closed, convex, bounded, infinite-dimensional subset of the Banach space $\mathcal{C}_{[-1,0]}$ endowed with the supremum norm $\|\cdot\|_{[-1,0]}$. The zero element of $\widetilde{\mathcal{K}}$ is the function $\hat{\varphi} \in \mathcal{C}_{[-1,0]}$ such that $\hat{\varphi} \equiv 0$ on $[-1,0]$. In this section we show that the trajectory of a solution $\hat{x}$ of DDER$^{\mathrm{n}}$ with initial condition $\hat{x}_0$ in $\widetilde{\mathcal{K}}$ reenters $\widetilde{\mathcal{K}}$ at some time after time zero.

The following lemma gives a lower bound on the magnitude of $\hat{g}$ in a certain set near the equilibrium point.

Lemma 4.3. *For each $\eta \in (0,1)$, there exists $\delta \in (0,L)$ such that*

$$|\hat{g}(r,s)| \ge \eta|Ar + Bs| \text{ for all } (r,s) \in \mathbb{B}_\delta, \tag{4.18}$$

where $\mathbb{B}_\delta := \{(r,s) \in \mathbb{R}^2 : rs \ge 0 \text{ and } |r|,|s| \le \delta\}$.

Remark 4.2. Lemma 4.3 is similar to the converse statement of Lemma 3 in [2] with the main difference being that here we allow A to equal zero.

Proof. Fix $\eta \in (0,1)$. Suppose that there does not exist $\delta \in (0,L)$ for which (4.18) holds. Let $\{\delta_n\}_{n=1}^\infty$ be a sequence in $(0,L)$ such that δ_n goes to zero as $n \to \infty$. Then there exists a sequence $\{(r_n,s_n)\}_{n=1}^\infty$ such that for each n, $(r_n,s_n) \in \mathbb{B}_{\delta_n}$ and

$$|\hat{g}(r_n,s_n)| < \eta|Ar_n + Bs_n|. \tag{4.19}$$

Note that (4.19) ensures that either $A > 0$ or $A = 0$ and $s_n \neq 0$ for all n. We first treat the case where $A > 0$. By the definition of $\mathbb{B}_{\delta_n}$, for each n, r_n and s_n are both non-negative or both non-positive. It follows that we can take a subsequence, also denoted $\{(r_n, s_n)\}_{n=1}^{\infty}$, such that (i) r_n and s_n are both non-negative for all n or both non-positive for all n, (ii) for each n, at least one of r_n, s_n is non-zero, and (iii) $(r_n, s_n) \to (0,0)$ as $n \to \infty$. We consider the case where r_n, s_n are both non-negative for each n, with the case that they are both non-positive being similar. Dividing both sides of (4.19) by $A|r_n| + B|s_n|$, we have for all n sufficiently large that

$$1 + \frac{o\left(|(r_n, s_n)|\right)}{A|r_n| + B|s_n|} < \eta, \tag{4.20}$$

where we have approximated $\hat{g}$ at the origin using its first partial derivatives. Note that the mapping $(r, s) \to A|r| + B|s|$ defines a norm on $\mathbb{R}^2$ so taking limits as $n \to \infty$ on both sides in (4.20) yields the contradiction $1 \leq \eta$, which proves the lemma in the case $A > 0$.

We now consider the case where $A = 0$. As before, we can take a subsequence, also denoted $\{(r_n, s_n)\}_{n=1}^{\infty}$, such that r_n and s_n are both non-negative for all n or both non-positive for all n, $s_n \neq 0$ for all n since $A = 0$, and $(r_n, s_n) \to (0,0)$ as $n \to \infty$. Again, we treat the case where r_n, s_n are both non-negative for all n, with the case that they are both non-positive being similar. Equation (4.19) becomes

$$-\hat{g}(r_n, s_n) < \eta B s_n. \tag{4.21}$$

Since $r_n \geq 0$ for each n, the inequality (4.5) implies that $-\hat{g}(0, s_n) \leq -\hat{g}(r_n, s_n)$ for all n and when combined with (4.21), we obtain that

$$-\hat{g}(0, s_n) < \eta B s_n. \tag{4.22}$$

Substituting the first order approximation $-\hat{g}(0, s_n) = Bs_n + o(|s_n|)$ into (4.22) and dividing by $B|s_n|$ on either side, we have

$$1 + \frac{o(|s_n|)}{B|s_n|} < \eta. \tag{4.23}$$

Taking limits as $n \to \infty$ on both sides in (4.23), we arrive at the contradiction $1 \leq \eta$, which proves the lemma in the case $A = 0$. □

To prove the existence of SOPS$^{\mathrm{n}}$ of DDER$^{\mathrm{n}}$, we first show that solutions of DDER$^{\mathrm{n}}$ with initial conditions in $\widetilde{\mathcal{K}}$ are slowly oscillating. The next lemma is an adaptation of analogous results in the unconstrained setting, which are detailed (for g satisfying various assumptions) in Lemma 2.3 of [18], Lemma 6 of [9] and Lemma 4 of [2]. The main difference in the following lemma is the presence of the lower reflective boundary.

Lemma 4.4. *Suppose $\tau > 1/B$. Let $\hat{\varphi} \in \widetilde{\mathcal{K}}$ such that $\hat{\varphi} \not\equiv 0$ and let $\hat{x} \in \mathcal{C}_{[-1,\infty)}$ denote the unique solution of DDER^n with initial condition $\hat{\varphi}$. Then there is a positive constant Q depending only on $\hat{g}$ and L, and countably many points $0 < \hat{q}_1 < \hat{q}_2 < \cdots$ such that*

(i) $\hat{x}(\hat{q}_k) = 0$ *for* $k = 1, 2, \ldots,$

(ii) $0 < \hat{q}_1 < Q$,
$1 < \hat{q}_{k+1} - \hat{q}_k < 1 + Q$ *for* $k = 1, 2, \ldots,$

(iii) $\hat{x}(t) > 0$ *for* $t \in (0, \hat{q}_1)$,
$\hat{x}(t) < 0$ *for* $t \in (\hat{q}_{2k-1}, \hat{q}_{2k})$ *for* $k = 1, 2, \ldots,$
$\hat{x}(t) > 0$ *for* $t \in (\hat{q}_{2k}, \hat{q}_{2k+1})$ *for* $k = 1, 2, \ldots,$

(iv) The function $\exp(\tau\kappa_1 \cdot)\hat{x}(\cdot)$ *is non-increasing on the intervals* $(\hat{q}_{2k-1}, \hat{q}_{2k-1}+1)$ *and non-decreasing on the intervals* $(\hat{q}_{2k}, \hat{q}_{2k} + 1)$, $k = 1, 2, \ldots,$ *and*

(v) $\hat{x}(t) \leq \tau G$ *for all* $t \geq -1$.

Remark 4.3. We call $\hat{q}_1, \hat{q}_2, \ldots,$ the zeros of $\hat{x}$. Note that if $\tau > \tau_0$, where τ_0 is defined in (3.8), then τ satisfies the condition in Lemma 4.4.

Proof. Fix $\tau > 1/B$ and $\hat{\varphi} \in \widetilde{\mathcal{K}}$ such that $\hat{\varphi} \not\equiv 0$. Let $\hat{x}$ denote the unique solution of DDER$^{\mathrm{n}}$ with initial condition $\hat{\varphi}$. By Lemma 4.2, $\hat{x}$ is absolutely continuous on $[0, \infty)$ and therefore can be recovered by integrating its almost everywhere defined derivative. Additionally, at $t > 0$ such that $\hat{x}(t) > -L$, $\hat{x}$ is continuously differentiable with derivative $\frac{d\hat{x}(t)}{dt} = \tau\hat{h}(\hat{x}(t), \hat{x}(t-1))$ and at $t > 0$ such that $\hat{x}(t) = -L$ and $\hat{x}$ is differentiable, its derivative satisfies $\frac{d\hat{x}(t)}{dt} = 0$.

Since $\tau > 1/B$, there exists $\eta \in (0, 1)$ such that $\eta\tau > 1/B$, and by Lemma 4.3 there exists $\delta_1 \in (0, L)$ such that

$$|\hat{g}(r, s)| \geq \eta|Ar + Bs| \text{ whenever } (r, s) \in \mathbb{B}_{\delta_1},$$

where $\mathbb{B}_{\delta_1}$ is as in Lemma 4.3. By choosing a smaller δ_1 if necessary, we can assume $\delta_1 \le \tau G$. Since $\hat{\varphi} \not\equiv 0$ and $\hat{\varphi}(t) \ge 0$ for all $t \in [-1, 0]$, there exists $t_0 \in [-1, 0]$ such that $\hat{x}(t_0) > 0$. Therefore $\hat{x}(0) > 0$ since $\exp(\tau\kappa_1 \cdot)\hat{x}(\cdot)$ is non-decreasing on $[-1, 0]$. By (4.3) and (4.5), for $t \ge 0$ prior to the first time after zero that $\hat{x}$ reaches zero, we have $\hat{g}(\hat{x}(t), \hat{x}(t-1)) \le 0$ for $t \ge 0$ and since $\hat{x}(0) \le \tau G$, by (4.13), $\hat{x}(t) \le \tau G$ for such t. We next prove that $\hat{x}(\cdot)$ will become negative. Let

$$t_1 = \inf\{t \ge 0 : \hat{x}(t) \le \delta_1\},$$

where by convention we define $t_1 = \infty$ if the infimum is taken over the empty set. We first show that t_1 is finite. Suppose $t_1 > 1$. Then $\hat{x}(t) > \delta_1$ for $t \in [0, 1]$. By (4.3) and (4.5),

$$-d_1 \equiv \max\{\hat{g}(r, s) : \delta_1 \le r, s \le \tau G\} < 0.$$

For $t \in [1, t_1)$, where t_1 is possibly infinite, we have $\hat{x}(t), \hat{x}(t-1) \ge \delta_1$, and so

$$\frac{d\hat{x}(t)}{dt} = \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)) \le -\tau d_1.$$

Therefore $\hat{x}(t) \le \tau G - \tau d_1(t-1)$ for $t \in [1, t_1)$ and it follows that t_1 is finite with

$$t_1 \le 1 + \frac{\tau G - \delta_1}{\tau d_1}.$$

Define $\hat{q}_1 = \inf\{t \ge t_1 : \hat{x}(t) \le 0\}$. We will prove that $\hat{q}_1 < t_1 + 2$. For a contradiction, suppose $\hat{q}_1 \ge t_1 + 2$. Then, for $t \in [t_1 + 1, t_1 + 2]$, we have $\delta_1 \ge \hat{x}(t-1) \ge \hat{x}(t_1+1)$ since $\hat{x}$ is decreasing on the interval, and by (4.13) and Lemma 4.3, we have

$$\begin{aligned}\frac{d\hat{x}(t)}{dt} &= \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)) \\ &\le -\eta\tau(A\hat{x}(t) + B\hat{x}(t-1)) \\ &\le -\eta\tau B\hat{x}(t-1) \\ &\le -\eta\tau B\hat{x}(t_1 + 1),\end{aligned}$$

and so

$$\hat{x}(t) \le \hat{x}(t_1 + 1) \cdot [1 - (t - t_1 - 1)\eta\tau B].$$

Since $\eta\tau > 1/B$, the right hand-side of the inequality is negative for $t = t_1 + 2$. This contradicts the assumption that $\hat{q}_1 \geq t_1 + 2$. It follows that

$$\hat{q}_1 < t_1 + 2 < 3 + \frac{\tau G - \delta_1}{\tau d_1} < 3 + \frac{G}{d_1}. \tag{4.24}$$

If $\hat{q}_1 \geq 1$, then by (4.13), (4.3) and the fact that $\hat{x}(\hat{q}_1 - 1) > 0$,

$$\begin{aligned}\left.\frac{dx(t)}{dt}\right|_{t=\hat{q}_1} &= \tau\hat{g}(\hat{x}(\hat{q}_1), \hat{x}(\hat{q}_1 - 1)) \\ &= \tau\hat{g}(0, \hat{x}(\hat{q}_1 - 1)) < 0,\end{aligned}$$

To show this inequality also holds for $\hat{q}_1 < 1$, we argue by contradiction. Suppose that $\hat{q}_1 < 1$ and $\frac{d\hat{x}(t)}{dt}|_{t=\hat{q}_1} = \tau\hat{g}(0, \hat{x}(\hat{q}_1 - 1)) = 0$ (the derivative at $t = \hat{q}_1$ must be non-positive as having a positive derivative there would contradict the definition of $\hat{q}_1$). Then by (4.2)–(4.3), $\hat{x}(\hat{q}_1 - 1) = \hat{x}_0(\hat{q}_1 - 1) = 0$. Since $\exp(\tau\kappa_1 \cdot)\hat{x}_0(\cdot)$ is non-decreasing, $\hat{x}_0(-1) = 0$ and $\hat{x}_0(t) \geq 0$ for all $t \in [-1, 0]$, it follows that $\hat{x}_0(t) = 0$ for all $t \in [-1, \hat{q}_1 - 1]$. Combining this with (4.5) and (4.7), we have, for $t \in [0, \hat{q}_1]$,

$$0 \geq \hat{g}(\hat{x}(t), \hat{x}(t-1)) = \hat{g}(\hat{x}(t), 0) \geq -\kappa_1|\hat{x}(t)| = -\kappa_1\hat{x}(t).$$

Thus, $\frac{d\hat{x}(t)}{dt} \geq -\tau\kappa_1\hat{x}(t)$ for $t \in [0, \hat{q}_1]$. It follows that

$$\hat{x}(\hat{q}_1) \geq \exp(-\tau\kappa_1\hat{q}_1)\hat{x}(0) > 0,$$

which contradicts the definition of $\hat{q}_1$. Hence $\frac{d\hat{x}(t)}{dt}|_{t=\hat{q}_1} < 0$.

Since $\frac{d\hat{x}(t)}{dt}|_{t=\hat{q}_1} < 0$, $\hat{x}$ will be negative for an interval after $\hat{q}_1$. Indeed, we show that $\hat{x}$ stays negative throughout the interval $(\hat{q}_1, \hat{q}_1 + 1)$. Note that

$$\left.\frac{d\hat{x}(t)}{dt}\right|_{t=\hat{q}_1} = \tau\hat{g}(0, \hat{x}(\hat{q}_1 - 1)) < 0$$

and the negative feedback condition on $\hat{g}$ imply that $\hat{x}(\hat{q}_1 - 1) > 0$. Then by the definition of $\hat{q}_1$ and the fact that $\exp(\kappa_1 \cdot)\hat{x}(\cdot)$ is non-decreasing on $[-1, 0]$, it follows that $\hat{x}(t-1) > 0$ for all $t \in [\hat{q}_1, \hat{q}_1 + 1)$. Let $\hat{q}_2 = \inf\{t > \hat{q}_1 : \hat{x}(t) = 0\}$ and suppose $\hat{q}_2 \in (\hat{q}_1, \hat{q}_1 + 1)$. Then $\hat{q}_2 - 1 \in (\hat{q}_1 - 1, \hat{q}_1)$, so that $\hat{x}(\hat{q}_2 - 1) > 0$ and

$$\left.\frac{d\hat{x}(t)}{dt}\right|_{t=\hat{q}_2} = \tau\hat{g}(0, \hat{x}(\hat{q}_2 - 1)) < 0.$$

Since $\hat{x}$ is continuously differentiable in a neighborhood of $\hat{q}_2$, $\hat{x}$ would be strictly decreasing in a neighborhood of $\hat{q}_2$, which contradicts the definition of $\hat{q}_2$. This contradiction proves that $\hat{x}$ is strictly negative on $(\hat{q}_1, \hat{q}_1 + 1)$.

For $t \in [\hat{q}_1, \hat{q}_1 + 1]$ such that $\hat{x}(t) > -L$, we have $\hat{x}(\cdot)$ is differentiable at t and

$$\begin{aligned}\frac{d}{dt}(\exp(\tau\kappa_1 t)\cdot\hat{x}(t)) &= \exp(\tau\kappa_1 t)\left(\tau\kappa_1\hat{x}(t) + \frac{d\hat{x}(t)}{dt}\right)\\ &= \exp(\tau\kappa_1 t)(\tau\kappa_1\hat{x}(t) + \tau\hat{g}(\hat{x}(t), \hat{x}(t-1))\\ &\leq \exp(\tau\kappa_1 t)(\tau\kappa_1\hat{x}(t) + \tau\hat{g}(\hat{x}(t), 0))\\ &\leq \tau\kappa_1\exp(\tau\kappa_1 t)(\hat{x}(t) + |\hat{x}(t)|) = 0,\end{aligned}$$

where we have used (4.13), (4.3) and (4.7) in the above. For $t \in [\hat{q}_1, \hat{q}_1 + 1]$ such that $\hat{x}(t) = -L$ and $\hat{x}$ is differentiable at t, we have by (4.13) that

$$\frac{d}{dt}(\exp(\tau\kappa_1 t)\cdot\hat{x}(t)) = -\tau\kappa_1 L\exp(\tau\kappa_1 t) < 0.$$

Then by the absolutely continuity of $\hat{x}$, $\exp(\tau\kappa_1\cdot)\hat{x}(\cdot)$ is non-increasing on $[\hat{q}_1, \hat{q}_1+1]$. Since $\hat{x}(t) < 0$ for some $t \in [\hat{q}_1, \hat{q}_1 + 1]$, it follows that $\hat{x}(\hat{q}_1 + 1) < 0$.

From the above, we have that $\hat{q}_2 > \hat{q}_1 + 1$ and since $\hat{x}(t), \hat{x}(t-1) \in [-L, 0)$ for $t \in (\hat{q}_1 + 1, \hat{q}_2)$, it follows from (4.2) and (4.4) that at times $t \in (\hat{q}_1 + 1, \hat{q}_2)$ we have $\hat{g}(\hat{x}(t), \hat{x}(t-1)) > 0$ and so for any such t at which $\hat{x}$ is differentiable, it follows from (4.14) that its derivative satisfies $\frac{d\hat{x}(t)}{dt} = \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)) > 0$. We now show that $\hat{q}_2$ is finite. Choose a fixed $\delta_2 \in (0, \delta_1)$. Let

$$t_2 = \inf\{t \geq \hat{q}_1 + 1 : \hat{x}(t) \geq -\delta_2\},$$

We first show that t_2 is finite. Suppose $t_2 > \hat{q}_1 + 2$. Then $\hat{x}(t) \leq -\delta_2$ for $t \in [\hat{q}_1 + 1, t_2]$, and so by (4.2) and (4.4), $\hat{g}(\hat{x}(t), \hat{x}(t-1)) > 0$ for $t \in [\hat{q}_1 + 2, t_2]$. By (4.2) and (4.4),

$$d_2 \equiv \min\{\hat{g}(r, s) : -L \leq r, s \leq -\delta_2\} > 0.$$

For $t \in [\hat{q}_1 + 2, t_2]$, we have $\hat{x}(t), \hat{x}(t-1) \leq -\delta_2$, and so at anytime t in this interval where $\hat{x}$ is differentiable, by (4.14),

$$\frac{d\hat{x}(t)}{dt} = \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)) \geq \tau d_2.$$

Since $\hat{x}$ is bounded below by $-L$, $\hat{x}(t) \geq -L + \tau d_2(t - \hat{q}_1 - 2)$ for $t \in [\hat{q}_1 + 2, t_2]$. It follows that t_2 is finite and

$$t_2 \leq \hat{q}_1 + 2 + \frac{L - \delta_2}{\tau d_2}.$$

Next we will prove that $\hat{q}_2 < t_2 + 2$. For a contradiction, suppose $\hat{q}_2 \geq t_2 + 2$. Since $t_2 \geq \hat{q}_1 + 1$, we have $\hat{x}(t), \hat{x}(t-1) < 0$ and therefore $\hat{g}(\hat{x}(t), \hat{x}(t-1)) > 0$ for $t \in (t_2, \hat{q}_2)$. It follows from (4.14) that $\hat{x}$ is strictly increasing on $(t_2, \hat{q}_2)$ and for $t \in [t_2 + 1, t_2 + 2]$, we have $-L < -\delta_2 \leq \hat{x}(t-1) \leq \hat{x}(t_2 + 1)$. Therefore, as long as $t \in [t_2 + 1, t_2 + 2]$, $\hat{x}$ is differentiable and by Lemma 4.3,

$$\begin{aligned}
\frac{d\hat{x}(t)}{dt} &= \tau \hat{g}(\hat{x}(t), \hat{x}(t-1)) \\
&\geq -\eta\tau(A\hat{x}(t) + B\hat{x}(t-1)) \\
&\geq -\eta\tau B\hat{x}(t-1) \\
&\geq -\eta\tau B\hat{x}(t_2 + 1),
\end{aligned}$$

and so

$$\hat{x}(t) \geq \hat{x}(t_2 + 1) \cdot [1 - (t - t_2 - 1)\eta\tau B].$$

Since $\eta\tau > 1/B$, the right hand side of the last inequality is positive for $t = t_2 + 2$. This contradicts the assumption that $\hat{q}_2 \geq t_2 + 2$ by continuity of $\hat{x}$. It follows that

$$\hat{q}_2 < t_2 + 2 < \hat{q}_1 + 4 + \frac{L - \delta_2}{\tau d_2} < 7 + \frac{G}{d_1} + \frac{LB}{d_2}, \tag{4.25}$$

and $\hat{q}_2 - \hat{q}_1 < 4 + LB/d_2$.

Since $\hat{x}(t)$ is negative for $t \in (\hat{q}_1, \hat{q}_2)$ and $\hat{q}_2 > \hat{q}_1 + 1$, we have $\hat{x}(\hat{q}_2 - 1) < 0$, and so by (4.13) and (4.2), $\frac{d\hat{x}(t)}{dt}|_{t=\hat{q}_2} = \tau\hat{g}(0, \hat{x}(\hat{q}_2 - 1)) > 0$. A similar argument to that used before can be used to prove that $\hat{x}(t)$ is strictly positive for $t \in (\hat{q}_2, \hat{q}_2 + 1)$ and therefore is differentiable on this interval. By (4.13), (4.5) and (4.6), we have $\frac{d\hat{x}(t)}{dt} = \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)) \leq \tau\hat{g}(0, \hat{x}(t-1)) \leq \tau G$ for $t \in [\hat{q}_2, \hat{q}_2 + 1]$, and therefore $\hat{x}(t) \leq \tau G$ on $[\hat{q}_2, \hat{q}_2 + 1]$. For $t \in [\hat{q}_2, \hat{q}_2 + 1]$, $\hat{x}(t) \in [0, \tau G]$ and $\hat{x}(t-1) \in [-L, 0]$, so

$$\frac{d}{dt}(\exp(\tau\kappa_1 t)\cdot\hat{x}(t)) = \exp(\tau\kappa_1 t)\left(\tau\kappa_1\hat{x}(t) + \frac{d\hat{x}(t)}{dt}\right)$$
$$= \exp(\tau\kappa_1 t)(\tau\kappa_1\hat{x}(t) + \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)))$$
$$\geq \exp(\tau\kappa_1 t)(\tau\kappa_1\hat{x}(t) + \tau\hat{g}(\hat{x}(t), 0))$$
$$\geq \exp(\tau\kappa_1 t)(\tau\kappa_1\hat{x}(t) - \tau\kappa_1|\hat{x}(t)|) = 0,$$

where we have used (4.13), (4.2) and (4.7) in the above.

The function $\hat{x}_{\hat{q}_2+1} \equiv \{\hat{x}(\hat{q}_2+1+t), -1 \leq t \leq 0\}$ satisfies $\hat{x}_{\hat{q}_2+1} \not\equiv 0$ and $\hat{x}_{\hat{q}_2+1} \in \widetilde{\mathcal{K}}$. Thus $\hat{x}_{\hat{q}_2+1}$ satisfies the conditions originally imposed on $\hat{\varphi}$ and by shifting the time origin, the preceding argument can be repeated countably many times to prove (i)-(iv). By (4.24)–(4.25), (ii) holds for $Q = 3+\max(Gd_1^{-1}, LBd_2^{-1})$. To prove (v), we note that by assumption $\hat{x}(t) \leq \tau G$ for $t \in [-1, 0]$, $\hat{x}$ is non-increasing on $[0, \hat{q}_1]$, $\hat{x}(t)$ is strictly negative for $t \in (\hat{q}_1, \hat{q}_2)$ and we have shown $\hat{x}(t) \leq \tau G$ for $t \in [\hat{q}_2, \hat{q}_2+1]$. Therefore $\hat{x}(t) \leq \tau G$ for $t \in [-1, \hat{q}_2+1]$ and the argument can be repeated to complete the proof of (v). □

Consider the function $\Lambda : \widetilde{\mathcal{K}} \to \mathcal{C}_{[-1,0]}$ defined for $\hat{\varphi} \in \widetilde{\mathcal{K}}$ by $\Lambda(\hat{\varphi}) = \hat{\varphi}$ when $\hat{\varphi} \equiv 0$ and when $\hat{\varphi} \not\equiv 0$,

$$\Lambda(\hat{\varphi}) = \hat{x}_{\hat{q}_2+1}, \tag{4.26}$$

where $\hat{x}$ is the solution of DDER with initial condition $\hat{\varphi}$ and $\hat{q}_2$ is the second zero of $\hat{x}$, as in Lemma 4.4. The following two lemmas are used to prove the continuity of Λ.

Lemma 4.5. *There exists a constant $K_{\hat{g},\tau G} > 0$ such that if $\hat{\varphi}, \hat{\varphi}^\dagger \in \widetilde{\mathcal{K}}$, then*

$$\|\hat{x} - \hat{x}^\dagger\|_{[-1,t]} \leq 2\exp(2K_{\hat{g},\tau G}t)\|\hat{\varphi} - \hat{\varphi}^\dagger\|_{[-1,0]},\ t \geq 0, \tag{4.27}$$

where $\hat{x}$ and $\hat{x}^\dagger$ denote the unique solution of DDER^n with respective initial condition $\hat{\varphi}$ and $\hat{\varphi}^\dagger$.

Proof. Let $\hat{\varphi}, \hat{\varphi}^\dagger \in \widetilde{\mathcal{K}}$. By part (v) of Lemma 4.4 and the lower boundary $-L$, if $\hat{x}$ and $\hat{x}^\dagger$ are solutions of DDER^n with respective initial conditions $\hat{\varphi}$ and $\hat{\varphi}^\dagger$, then $|\hat{x}(t)|$ and $|\hat{x}^\dagger(t)|$ are bounded by $L \vee \tau G$ for all $t \geq -1$. It then follows from (4.12) that the corresponding solutions x and $x^\dagger$ of DDER are bounded by $L+\tau G$ for all $t \geq -\tau$. Since g is locally Lipschitz continuous, there exists a positive constant $K_{g,L+\tau G}$ such that g is uniformly Lipschitz continuous with Lipschitz constant $K_{g,L+\tau G}$ on the set $[0, L+\tau G] \times [0, L+\tau G]$. Therefore, by Lemma 2.1,

$$\|x - x^\dagger\|_{[-\tau,\tau t]} \leq 2\exp(2K_{g,L+\tau G}\tau t)\|x - x^\dagger\|_{[-\tau,0]},\ t \geq 0. \tag{4.28}$$

Then (4.27) follows from (4.11) and (4.28) with $K_{\hat{g},\tau G} = \tau K_{g,L+\tau G}$. □

Lemma 4.6. *The function $\hat{\varphi} \to \hat{q}_2$, where $\hat{q}_2$ is the second zero of $\hat{x}$ as defined in Lemma 4.4, is continuous as a function from $\widetilde{\mathcal{K}} \setminus \{0\}$ into $[0,\infty)$.*

Proof. Fix $0 \not\equiv \hat{\varphi} \in \widetilde{\mathcal{K}}$, let $\hat{x}$ denote the unique solution of DDER$^{\rm n}$ with initial condition $\hat{\varphi}$ and let $\hat{q}_2$ be defined as in (4.4). Choose $0 < \eta < \min(\hat{q}_1, \frac{1}{4})$. By part (iii) of Lemma 4.4 we see that $\hat{q}_2$ is bounded by $1+2Q$, which only depends on $\hat{g}$ and L. By our choice of η and the definition of $\hat{q}_k$, we have that $0 < \hat{q}_1 - \eta < \hat{q}_1 + \eta < \hat{q}_2 - \eta$ and

$$\begin{aligned}
\hat{x}(t) > 0 &\text{ for all } t \in (\hat{q}_1 - \eta, \hat{q}_1),\\
\hat{x}(t) < 0 &\text{ for all } t \in (\hat{q}_1, \hat{q}_1 + \eta),\\
\hat{x}(t) < 0 &\text{ for all } t \in (\hat{q}_2 - \eta, \hat{q}_2),\\
\hat{x}(t) > 0 &\text{ for all } t \in (\hat{q}_2, \hat{q}_2 + \eta).
\end{aligned}$$

Finally, choose $\delta > 0$ satisfying

$$\delta < \inf\{|\hat{x}(t)| : t \in [0, \hat{q}_1 - \eta] \cup [\hat{q}_1 + \eta, \hat{q}_2 - \eta] \cup \{\hat{q}_2 + \eta\}\} \tag{4.29}$$

and $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{\delta}{2\exp(2K_{\hat{g},\tau G}(1 + 2Q + \eta))}, \tag{4.30}$$

where $K_{\hat{g},\tau G}$ is as in Lemma 4.5. By (4.27), if $\hat{\varphi}^\dagger \in \widetilde{\mathcal{K}}$ and $\|\hat{\varphi} - \hat{\varphi}^\dagger\|_{[-1,0]} < \varepsilon$, then $\|\hat{x} - \hat{x}^\dagger\|_{[-1,\hat{q}_2+\eta]} < \delta$, where $\hat{x}^\dagger$ denotes the unique solution of DDER$^{\rm n}$ with initial condition $\hat{\varphi}^\dagger$. It then follows from (4.29) and the continuity of $\hat{x}^\dagger$ that $\hat{x}^\dagger(t) = 0$ for some $t \in (\hat{q}_1 - \eta, \hat{q}_1 + \eta)$ and also for some $t \in (\hat{q}_2 - \eta, \hat{q}_2 + \eta)$. Then by (4.29) and the fact that the zeros of $\hat{x}^\dagger$ must be separated by at least one, we have $\hat{q}_1^\dagger \in (\hat{q}_1 - \eta, \hat{q}_1 + \eta)$ and $\hat{q}_2^\dagger \in (\hat{q}_2 - \eta, \hat{q}_2 + \eta)$, where $q_1^\dagger$ and $q_2^\dagger$ are the zeros of $\hat{x}^\dagger$, proving the desired continuity result. □

Lemma 4.7. *Λ is a continuous and compact function mapping $\widetilde{\mathcal{K}}$ into $\widetilde{\mathcal{K}}$.*

Proof. The fact that Λ maps $\widetilde{\mathcal{K}}$ into itself follows from the definition of $\hat{q}_2$ and parts (iv) and (v) of Lemma 4.4. The continuity of Λ at $\hat{\varphi} \equiv 0$ follows from part (iii) of Lemma 4.4 and (4.27). The continuity of Λ at $\hat{\varphi} \not\equiv 0$ follows from the continuity of the map $\hat{\varphi} \to \hat{q}_2$ defined in Lemma 4.6, the continuity of $\hat{x}$, part (iii) of Lemma 4.4, (4.27), and the triangle inequality:

$$\|\Lambda(\hat{\varphi}) - \Lambda(\hat{\varphi}^\dagger)\|_{[-1,0]} \leq \|\hat{x}_{\hat{q}_2+1} - \hat{x}_{\hat{q}_2^\dagger+1}\|_{[-1,0]} + \|\hat{x}_{\hat{q}_2^\dagger+1} - \hat{x}^\dagger_{\hat{q}_2^\dagger+1}\|_{[-1,0]},$$

where $\hat{x}$ and $\hat{x}^\dagger$ denote the unique solutions of DDER$^{\mathrm{n}}$ with respective initial conditions $\hat{x}_0 = \hat{\varphi}$ and $\hat{x}_0^\dagger = \hat{\varphi}^\dagger$, and $\hat{q}_2$ and $\hat{q}_2^\dagger$ denote the second zeros, defined in Lemma 4.4, of $\hat{x}$ and $\hat{x}^\dagger$, respectively. The compact property of Λ follows from the Arzelà-Ascoli theorem since $\hat{x}(\cdot)$ is bounded and differentiable on $[\hat{q}_2, \hat{q}_2 + 1]$ with bounded derivative:

$$\begin{aligned}\left|\frac{d\hat{x}(t)}{dt}\right| &= |\tau\hat{g}(\hat{x}(t), \hat{x}(t-1))| \\ &\leq \tau \cdot \sup\{|\hat{g}(r,s)| : (r,s) \in [0, \tau G] \times [-L, 0]\} < \infty,\end{aligned}$$

for all $t \in [\hat{q}_1, \hat{q}_2 + 1]$, where we have used the fact that $\hat{g}$ is continuous on this compact set. □

4.4 Ejective Equilibrium Solution

In order to prove the existence of a SOPS, it remains to show that the zero solution of DDER$^{\mathrm{n}}$ is an ejective fixed point for Λ. It will then follow from Theorem 4.1 that there exists another fixed point that is non-ejective which will correspond to a SOPS$^{\mathrm{n}}$. Since the ejective property of the equilibrium solution is related to its local stability, we consider the approximation of DDER$^{\mathrm{n}}$ by the following unconstrained linear delay differential equation

$$\frac{du(t)}{dt} = -\tau A u(t) - \tau B u(t-1). \tag{4.31}$$

The linear delay differential equation (4.31) has characteristic equation

$$\lambda + \tau A + \tau B e^{-\lambda} = 0. \tag{4.32}$$

The following proposition is a standard result from stability theory for linear delay differential equations.

Lemma 4.8. *Let $B > A \geq 0$ and $\tau > \tau_0$, where τ_0 is given by* (3.8)*. Then the characteristic equation* (4.32) *has a solution $\lambda = \mu + i\nu$ satisfying $\mu > 0$ and $\pi/2 < \nu < \pi$.*

Proof. The lemma is a well-known result for linear delay differential equations and a complete proof can be found in the appendix of [12]. □

Under the conditions of Lemma 4.8, there exist solutions of the linear delay differential equation (4.31) that exhibit unbounded oscillations; namely, if $\lambda = \mu + i\nu$ is a solution of (4.32) such that $\mu > 0$ and $\nu < \pi$, then the function $u(t) = e^{\mu t}\cos(\nu t)$, $t \geq -1$, is a solution of (4.31) with unbounded oscillations. Therefore, we anticipate that the equilibrium solution $\hat{x} \equiv L$ of (4.8) will be locally unstable if $\tau > \tau_0$. Indeed, we have the following lemma.

Lemma 4.9. *Let τ_0 be given by* (3.8)*. If $\tau > \tau_0$ and Λ is defined by* (4.26)*, then $\hat{\varphi} \equiv 0$ is an ejective fixed point of Λ.*

Before proceeding with the proof of this lemma, we introduce a key lemma which states that a solution of DDER$^{\text{n}}$ with nonzero initial condition in $\widetilde{\mathcal{K}}$ will eventually leave a certain neighborhood of the origin.

Lemma 4.10. *There exists $\gamma > 0$ such that for each $0 \not\equiv \hat{\varphi} \in \widetilde{\mathcal{K}}$, the unique solution $\hat{x}$ of DDER^{n} with initial condition $\hat{\varphi}$ satisfies*

$$\sup_{t \geq \hat{q}_1} |\hat{x}(t)| \geq \gamma, \tag{4.33}$$

where $\hat{q}_1$ is the first zero of $\hat{x}$ defined in Lemma 4.4.

Remark 4.4. The proofs of Lemmas 4.9 and 4.10 are adaptations of the proofs for Lemmas 2.6, 2.7 and 2.8 in [18], Lemmas 10 and 11 in [9] and Lemma 6 in [2], and are given here for completeness. In [18], the author notes the basic idea behind the proofs appears to be due to Wright (see Theorem 4 of [29]).

Proof. Define the positive constant T by

$$T = \min\left\{\frac{1}{2}, \frac{\exp(-\tau\kappa_1)}{4\tau(\kappa_1+\kappa_2)}\right\}.$$

By Lemma 4.8, the characteristic equation (4.32) has a solution $\lambda = \mu + i\nu$ with $\mu > 0$ and $\pi/2 < \nu < \pi$. Define the positive constant c by

$$c = \inf\{\sin(\nu t) : 1 - T \leq t \leq 1\}$$

and choose $\varepsilon > 0$ such that

$$\varepsilon < \frac{\mu}{4} c \cdot TB \exp[-(\mu + \tau\kappa_1)].$$

Define the function $P : [-L, \infty)^2 \to \mathbb{R}$ by

$$P(r,s) = \hat{g}(r,s) + Ar + Bs, \text{ for } r, s \geq -L. \tag{4.34}$$

Since $\hat{g}$ is differentiable at the origin with first partial derivatives $-A$ and $-B$, respectively, the quotient $|P(r,s)|/|(r,s)|$ approaches zero as $0 \neq |(r,s)| \to 0$. Choose a positive $\gamma \in (0, L)$ such that

$$|P(r,s)| \leq \varepsilon|(r,s)| \text{ when } |r| \leq \gamma \text{ and } |s| \leq \gamma.$$

Let $0 \not\equiv \hat{\varphi} \in \mathcal{K}$ and $\hat{x}$ be the solution of DDER$^{\text{n}}$ with initial condition $\hat{\varphi}$. Suppose (4.33) is false, then

$$\sup_{t \geq \hat{q}_1} |\hat{x}(t)| = \delta < \gamma. \tag{4.35}$$

It follows that we can choose $n \geq 1$ and $\sigma \in [\hat{q}_n, \hat{q}_{n+1}]$ such that $\hat{x}(\sigma) = \sup\{|\hat{x}(t)| : \hat{q}_n \leq t \leq \hat{q}_{n+1}\}$ and $|\hat{x}(\sigma)| > \delta/2$. By assumption, $\hat{x}(t) > -\gamma > -L$ for all $t \geq \hat{q}_1$, and therefore, by Lemma 4.2, $\hat{x}(\cdot)$ is differentiable at all $t \geq \hat{q}_1$ with $\frac{d\hat{x}(t)}{dt} = \tau\hat{g}(\hat{x}(t), \hat{x}(t-1))$. We treat the case where n is even and σ is a maximum, with the case that n is odd and σ is a minimum being similar.

Since $\hat{x}(\cdot)$ is bounded on $[-1, \infty)$ by part (v) of Lemma 4.4, and $\mu > 0$ by assumption, we can integrate by parts to obtain the following identity

$$\begin{aligned}
&\int_{\hat{q}_n+1+T}^{\infty} \frac{d\hat{x}(t)}{dt} \exp(-\lambda t)dt \\
&\quad = -\hat{x}(\hat{q}_n + 1 + T)\exp[-\lambda(\hat{q}_n + 1 + T)] + \lambda \int_{\hat{q}_n+1+T}^{\infty} \hat{x}(t)\exp(-\lambda t)dt.
\end{aligned} \tag{4.36}$$

Using (4.13) and (4.34), we can write the left hand side of (4.36) as

$$\int_{\hat{q}_n+1+T}^{\infty} \frac{d\hat{x}(t)}{dt} \exp(-\lambda t)dt = \tau \int_{\hat{q}_n+1+T}^{\infty} P(\hat{x}(t), \hat{x}(t-1)) \exp(-\lambda t)dt \\ - \tau A \int_{\hat{q}_n+1+T}^{\infty} \hat{x}(t) \exp(-\lambda t)dt - \tau B \exp(-\lambda) \int_{\hat{q}_n+T}^{\infty} \hat{x}(t) \exp(-\lambda t)dt. \quad (4.37)$$

Combining (4.36)–(4.37), multiplying both sides by $\exp[\lambda(\hat{q}_n+1+T)]$ and, in light of the characteristic equation (4.32), substituting $\tau A+\lambda = -\tau B \exp(-\lambda)$, we have

$$\tau \int_{\hat{q}_n+1+T}^{\infty} P(\hat{x}(t), \hat{x}(t-1)) \exp[-\lambda(t-\hat{q}_n-1-T)]dt \\ = -\hat{x}(\hat{q}_n+1+T) + \tau B \int_{\hat{q}_n+T}^{\hat{q}_n+1+T} \hat{x}(t) \exp[-\lambda(t-\hat{q}_n-T)]dt. \quad (4.38)$$

To reach a contradiction, we note that the left hand side of (4.38) is bounded above by

$$\tau\varepsilon\delta \int_{\hat{q}_n+1+T}^{\infty} |\exp[-\lambda(t-\hat{q}_n-1-T)]|dt \leq \tau\varepsilon\delta \int_0^{\infty} \exp(-\mu t)dt \quad (4.39) \\ \leq \tau\varepsilon\delta\mu^{-1}.$$

Next, we need to find a lower bound of the absolute value of the right hand side of (4.38). To do so, we first show that $\hat{x}(t)$ remains non-negative for $t \in [\hat{q}_n, \hat{q}_n + 1+T]$. By Lemma 4.4, the function $\exp(\tau\kappa_1\cdot)\hat{x}(\cdot)$ is non-decreasing on the interval $[\hat{q}_n, \hat{q}_n+1]$, hence

$$\hat{x}(\hat{q}_n+1) \geq \hat{x}(\sigma) \exp[-\tau\kappa_1(\hat{q}_n+1-\sigma)] \geq \frac{\delta}{2} \exp(-\tau\kappa_1).$$

Using this inequality, we can bound $\hat{x}(t)$ for $t \in [\hat{q}_n+1, \hat{q}_n+1+T]$:

$$\begin{aligned} \hat{x}(t) &= \hat{x}(\hat{q}_n+1) + \tau \int_{\hat{q}_n+1}^{t} \hat{g}(\hat{x}(s), \hat{x}(s-1))ds \quad (4.40) \\ &\geq \frac{\delta}{2} \exp(-\kappa_1\tau) - \tau\delta(\kappa_1+\kappa_2)(t-\hat{q}_n-1) \\ &\geq \delta\left(\frac{1}{2}\exp(-\kappa_1\tau) - \tau(\kappa_1+\kappa_2)T\right) \\ &\geq \frac{\delta}{4}\exp(-\kappa_1\tau), \end{aligned}$$

where we have used (4.7) and the definition of T. The absolute value of the right hand side of (4.38) is bounded below by the absolute value of the imaginary part, which is equal to

$$\tau B \int_{\hat{q}_n+T}^{\hat{q}_n+1+T} \hat{x}(t)\exp[-\mu(t-\hat{q}_n-T)]\cdot\sin[\nu(t-\hat{q}_n-T)]dt$$
$$\geq \tau B \int_{\hat{q}_n+1}^{\hat{q}_n+1+T} \hat{x}(t)\exp[-\mu(t-\hat{q}_n-T)]\cdot\sin[\nu(t-\hat{q}_n-T)]dt, \quad (4.41)$$

where the non-negativity of the integrand follows since $\nu \in (\pi/2, \pi)$, $\hat{x}(t)$ is non-negative for $t \in [\hat{q}_n + T, \hat{q}_n + 1 + T]$, and $T \leq 1/2$. Using (4.40), we have that (4.41) is bounded below by

$$\frac{\delta}{4}\tau B \exp(-\kappa_1\tau) \int_{\hat{q}_n+1}^{\hat{q}_n+1+T} \exp[-\mu(t-\hat{q}_n-T)]\cdot\sin[\nu(t-\hat{q}_n-T)]dt$$
$$\geq \frac{\delta}{4}\tau c \cdot TB\exp[-(\kappa_1\tau+\mu)].$$

Combining with (4.39), we have the following inequality

$$\frac{1}{4}c\cdot TB\exp[-(\tau\kappa_1+\mu)] \leq \varepsilon\mu^{-1},$$

which contradicts our choice of ε. □

Proof of Lemma 4.9. Consider $\hat{\varphi} \in \widetilde{\mathcal{K}}$ such that $\hat{\varphi} \not\equiv 0$ and let $\hat{x}$ denote the solution of DDER$^{\mathrm{n}}$ with initial condition given by $\hat{\varphi}$. For each $n \in \mathbb{N}$, let $\sigma_n \in [\hat{q}_n, \hat{q}_{n+1}]$ such that

$$|\hat{x}(\sigma_n)| = \sup_{t\in[\hat{q}_n,\hat{q}_{n+1}]} |\hat{x}(t)|.$$

By the definition of $\hat{q}_n$, for $t \in [\hat{q}_n + 1, \hat{q}_{n+1}]$, $\hat{x}(t), \hat{x}(t-1) \geq 0$ if n is even and $\hat{x}(t), \hat{x}(t-1) \leq 0$ if n is odd. It follows from (4.13) and (4.2)–(4.5) that $\hat{x}$ is non-increasing on $[\hat{q}_n + 1, \hat{q}_{n+1}]$ if n is even and non-decreasing on $[\hat{q}_n + 1, \hat{q}_{n+1}]$ if n is odd. Therefore we may choose $\sigma_n \in [\hat{q}_n, \hat{q}_n + 1]$. By Lemma 4.10, there exists $\gamma > 0$, not depending on $\hat{\varphi}$, such that

$$|\hat{x}(\sigma_n)| \geq \gamma$$

for some positive integer n. Let $\eta > 0$ such that

$$\eta < \frac{\gamma}{\tau\kappa_2} \wedge \gamma. \quad (4.42)$$

To prove ejectivity, we show that

$$\|\hat{x}_{\hat{q}_{2n+1}}\|_{[-1,0]} = \hat{x}(\sigma_{2n}) \geq \eta \tag{4.43}$$

for some $n \in \mathbb{N}$. Suppose that $\hat{x}(\sigma_{2n}) < \eta$ for all n. Let $t_\gamma^n = \inf\{t \geq \hat{q}_{2n+1} : \hat{x}(t) = -\gamma\}$. Since $\gamma < L$, for $n \geq 1$ and $t \in [\hat{q}_{2n+1}, t_\gamma^n]$, $\hat{x}(\cdot)$ is differentiable at t and

$$\begin{aligned}\frac{d\hat{x}(t)}{dt} &= \tau\hat{g}(\hat{x}(t), \hat{x}(t-1)) \\ &\geq \tau\hat{g}(0, \hat{x}(t-1)) \\ &\geq -\tau\kappa_2|\hat{x}(t-1)| = -\tau\kappa_2\hat{x}(t-1) \\ &\geq -\tau\kappa_2\eta.\end{aligned}$$

where we have used the fact that $\hat{x}(t) \leq 0$ and $\hat{x}(t-1) \geq 0$, as well as equations (4.13), (4.4), (4.7) and (4.42). By (4.42), $t_\gamma^n > \hat{q}_{2n+1} + 1$ and so $\hat{x}(t) > -\gamma$ for all $t \in [\hat{q}_{2n+1}, \hat{q}_{2n+1} + 1]$. It follows that

$$\hat{x}(\sigma_{2n+1}) > -\gamma$$

for all n and therefore $|\hat{x}(\sigma_n)| < \gamma$ for all n. This contradicts (4.33), so (4.43) holds for some n. Since $\Lambda^n(\hat{\varphi}) = \hat{x}_{\hat{q}_{2n+1}}$ for each n, the desired ejectivity property follows and the proof of the lemma is completed. □

4.5 Proof of Existence

Proof of Theorem 3.1. By Browder's fixed point theorem, the mapping $\Lambda : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}$ has a non-ejective fixed point. By Lemma 4.9, the constant function $\hat{\varphi} \equiv 0$ is an ejective fixed point of Λ and so there must be another fixed point $0 \not\equiv \hat{x}_0 \in \widetilde{\mathcal{K}}$. Let $\hat{x}$ denote the associated solution of DDER$^{\mathrm{n}}$. By the uniqueness of solutions and time homogeneity of DDER$^{\mathrm{n}}$, $\hat{x}$ is periodic with period $\hat{p} = \hat{q}_2 + 1$ and by Lemma 4.4, $\hat{x}$ is a SOPS$^{\mathrm{n}}$. Lastly, by Lemma 4.1, the associated solution x of DDER, which is defined via (4.12), is a SOPS. □

This chapter is based on the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

Chapter 5

Uniqueness and Stability of Slowly Oscillating Periodic Solutions

In this section we prove Theorem 3.2 which provides sufficient conditions for the uniqueness and uniform exponential asymptotic stability of slowly oscillating periodic solutions (SOPS) of the delay differential equation with reflection (DDER) (1.1). Both the proof of uniqueness of SOPS and the proof of uniform exponential asymptotic stability of SOPS are inspired by arguments used to prove similar results in the unconstrained setting [30, 31]. In these papers the author considered a Poincaré map associated with a slowly oscillating periodic solution in the unconstrained setting and showed that the Floquet multipliers associated with the slowly oscillating periodic solution are bounded by the spectral norm of the derivative of the Poincaré map evaluated at the initial condition of the slowly oscillating periodic solution. After providing conditions for the spectral norm to be less than one, including long delay interval lengths, the author employed existing theory relating Floquet multipliers to the stability of periodic solutions of delay differential equations in the unconstrained setting to show that any slowly oscillating periodic solution is uniformly exponentially asymptotically stable. Then using theorems for fixed point indices, the author proved there must be exactly one slowly oscillating periodic solution.

Here, in the constrained setting, we consider an approximate Poincaré map in a neighborhood of the initial condition of a SOPS. We show that it is continuously Fréchet differentiable in this neighborhood and provide conditions for its derivative evaluated at the initial condition of the SOPS to have operator norm less than one, which is a sufficient condition for the SOPS to be uniformly exponentially asymptotically stable. We then use theorems for fixed point indices to prove their must be exactly one slowly oscillating periodic solution. This is a slightly different approach because we are proving a stronger condition that the operator norm of the approximate Poincaré map is less than one rather than proving the spectral norm is less than one. The main reason for this is that the estimates we use to prove that the spectral is less than one also prove that the operator norm is less than one, so we would not obtain a stronger result by considering the spectral norm.

To construct our approximate Poincaré map, new difficulties that arise because of the lower boundary constraint need to be overcome. In particular, a new form of the linear variational equation needs to be developed and analyzed. In Appendix C such an equation is derived for functions f that are more general than the functions considered in this section. Throughout this section we assume that f is of the form exhibited in (3.9) and that h satisfies Assumptions 3.3 and 3.4, although we note that the results in Section 5.2 only require that h satisfy Assumption 3.3.

5.1 Normalized Solutions

As was done in Section 4.2, we normalize solutions of the DDER (1.1) by subtracting off L and rescaling time so that the normalized delay interval is of length one.

Let $\hat{h} : [-L, \infty) \to \mathbb{R}$ be the normalized function defined by:

$$\hat{h}(s) = h(s + L), \ s \in [-L, \infty). \tag{5.1}$$

The function $\hat{h}$ inherits the following properties from h. By Assumption 3.3, the function $\hat{h}$ is bounded by $H \equiv \|h\|_{[0,\infty)} = \|\hat{h}\|_{[-L,\infty)} < \infty$ and is continuously differentiable on $[-L,\infty)$,

$$s\hat{h}(s) < 0 \text{ for all } s \neq 0, \tag{5.2}$$

there exist positive constants $\alpha, \beta > 0$ such that $\hat{h}(-L) = \beta$ and

$$\lim_{s\to\infty} \hat{h}(s) = -\alpha < 0. \tag{5.3}$$

By Assumption 3.4, the derivative of $\hat{h}$, denoted by $\hat{h}' : [-L,\infty) \to \mathbb{R}$, satisfies

$$\|\hat{h}'\|_{L^1([-L,\infty))} \equiv \int_{-L}^{\infty} |\hat{h}'(s)|ds < \infty, \tag{5.4}$$

and

$$s\hat{h}'(s) \to 0 \text{ as } s \to \infty, \tag{5.5}$$

and is uniformly bounded by $K_h \equiv \|h'\|_{[0,\infty)} = \|\hat{h}'\|_{[-L,\infty)}$ so that $\hat{h}$ satisfies

$$|\hat{h}(s) - \hat{h}(r)| \leq K_h|s-r|, \ -L \leq r,s < \infty. \tag{5.6}$$

On setting $g(r,s) = h(s)$ for all $r,s \geq 0$, Assumption 3.3 implies that Assumptions 3.1 and 3.2 hold for g, so Lemma 3.1 and Theorem 3.1 hold. It follows that $\hat{g}(r,s) = \hat{h}(s)$ for all $r,s \geq -L$, so we can define a solution of DDER$^{\mathrm{n}}$ and a SOPS$^{\mathrm{n}}$ of DDER$^{\mathrm{n}}$ as in Definitions 4.2 and 4.3, respectively, and Lemmas 4.1–4.10 hold.

Given a solution $\hat{x}$ of DDER$^{\mathrm{n}}$, (4.8) can be rewritten for $t \geq 0$ as

$$\hat{x}(t) = \hat{z}(t) + \hat{y}(t), \tag{5.7}$$

$$\hat{z}(t) = \hat{x}(0) + \tau \int_0^t \hat{h}(\hat{x}(s-1))ds. \tag{5.8}$$

Adding L to either side of (5.7), we obtain

$$(\hat{x}(t) + L) = (\hat{z}(t) + L) + \hat{y}(t), \ t \geq 0,$$

where $\hat{x}(t)+L \geq 0$ and $\int_0^t (\hat{x}(s)+L)d\hat{y}(s) = 0$ for all $t \geq 0$. It follows that $(\hat{x}+L, \hat{y})$, where $\hat{x}$ here is restricted to the interval $[0,\infty)$, is a solution of the one-dimensional Skorokhod problem for $\hat{z} + L$ (see Appendix A) and by Proposition A.3,

$$\hat{y}(t) = \sup_{0\leq s\leq t} (\hat{z}(s) + L)^-, \ t \geq 0. \tag{5.9}$$

For the following, recall the definition of a SOPS$^{\text{n}}$ of DDER$^{\text{n}}$ from Definition 4.3 and note that we can use $G = \sup\{h(s) : s \in \mathbb{R}_+\} \in (0, \infty)$ for the G in Lemma 3.1.

Lemma 5.1. *Under Assumption 3.3, let $\hat{x}^*$ be a SOPS^n of DDER^n such that $\hat{q}_0 = -1$. Then $\hat{x}^*$ is bounded above by τG and satisfies:*

(i) $\hat{x}^*$ is positive on $(-1, 0]$, continuously differentiable on $[-1, 0]$ and is increasing on $(-1, 0)$;

(ii) $\hat{x}^*$ is positive on $[0, \hat{q}_1)$, continuously differentiable on $[0, \hat{q}_1]$ and is decreasing on $(0, \hat{q}_1]$;

(iii) $\hat{x}^*$ is negative on $(\hat{q}_1, \hat{q}_1 + 1]$ and is non-increasing on $[\hat{q}_1, \hat{q}_1 + 1]$;

(iv) $\hat{x}^*$ is negative on $[\hat{q}_1 + 1, \hat{q}_2)$ and is continuously differentiable and increasing on $(\hat{q}_1 + 1, \hat{q}_2]$.

Furthermore, $\hat{z}^*$ is negative on $(\hat{q}_1, \hat{q}_1 + 1]$ and is decreasing on $(\hat{q}_1, \hat{q}_1 + 1)$ and $\hat{x}^*$, $\hat{z}^*$ satisfy

$$\hat{x}^*(t) = \hat{z}^*(t), \qquad \textit{for } t \in [0, \hat{q}_1], \tag{5.10}$$

$$\hat{x}^*(t) = \max(\hat{z}^*(t), -L), \qquad \textit{for } t \in (\hat{q}_1, \hat{q}_1 + 1], \tag{5.11}$$

$$\hat{x}^*(t) = \hat{x}^*(\hat{q}_1 + 1) + (\hat{z}^*(t) - \hat{z}^*(\hat{q}_1 + 1)), \qquad \textit{for } t \in (\hat{q}_1 + 1, \hat{p}]. \tag{5.12}$$

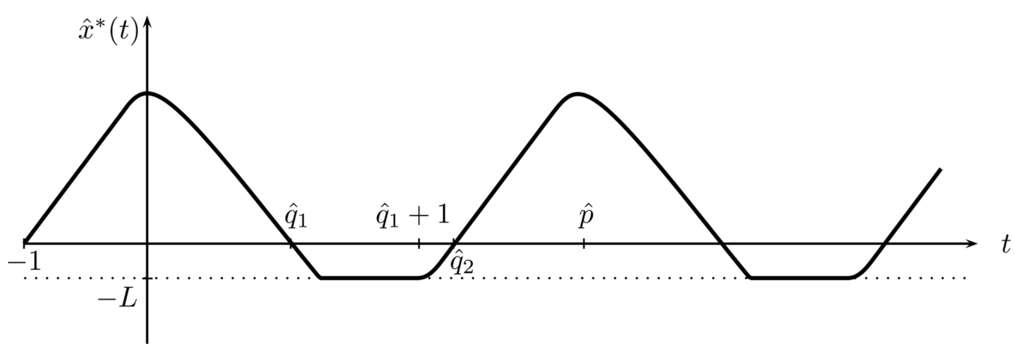

Figure 5.1: An example of a SOPS$^{\text{n}}$ as described in Lemma 5.1.

Proof. Proof of (ii): By (4.10), $\hat{x}^*(t) > 0$ for all $t \in [0, \hat{q}_1)$ and $\hat{x}^*(t) > -L$ for all $t \in [0, \hat{q}_1]$ and so $\hat{y}^*$ is zero there. Then by (5.7), $\hat{x}^*(t) = \hat{z}^*(t)$ for all $t \in [0, \hat{q}_1]$ and it follows from (5.8), (5.2) and (4.10) that $\hat{x}^*$ is differentiable on $[0, \hat{q}_1]$ with

$$\frac{d\hat{x}^*(t)}{dt} = \frac{d\hat{z}^*(t)}{dt} = \tau\hat{h}(\hat{x}^*(t-1)) < 0, \ t \in (0, \hat{q}_1],$$

so $\hat{x}^*$ is decreasing on $(0, \hat{q}_1]$.

Proof of (iii): By (4.10), $\hat{x}^*$ is negative on $(\hat{q}_1, \hat{q}_1 + 1]$ and since $\hat{x}^* = \hat{z}^* - \hat{y}^* \geq \hat{z}^*$, $\hat{z}^*$ is also negative there. It follows from (5.8), (5.2) and (4.10) that $\hat{z}^*$ is differentiable on $[\hat{q}_1, \hat{q}_1 + 1)$ with derivative

$$\frac{d\hat{z}^*(t)}{dt} = \tau\hat{h}(\hat{x}^*(t-1)) < 0, \ t \in [\hat{q}_1, \hat{q}_1 + 1). \tag{5.13}$$

By (ii) and (5.13), $\hat{z}^*$ is decreasing on $(0, \hat{q}_1 + 1)$, so by (5.9),

$$\hat{y}^*(t) = \sup_{0 \leq s \leq t} (\hat{z}^*(s) + L)^- = (\hat{z}^*(t) + L)^-, \ t \in [0, \hat{q}_1 + 1]. \tag{5.14}$$

Then by (5.7) and (5.14), we have

$$\hat{x}^*(t) = \hat{z}^*(t) + (\hat{z}^*(t) + L)^- = \max(\hat{z}^*(t), -L), \ t \in [\hat{q}_1, \hat{q}_1 + 1], \tag{5.15}$$

and since $\hat{z}^*$ is decreasing on $(\hat{q}_1, \hat{q}_1 + 1)$, $\hat{x}^*$ is non-increasing there.

Proof of (iv) and (i): By (4.10), $\hat{x}^*(t) < 0$ for all $t \in (\hat{q}_1+1, \hat{q}_2)$ and $\hat{x}^*(t) > 0$ for all $t \in (-1, 0)$. Then by (5.8), (5.2), (4.10) and the fact that $\hat{p} = \hat{q}_2 + 1$, we have

$$\frac{d\hat{z}^*(t)}{dt} = \tau\hat{h}(\hat{x}^*(t-1)) > 0, \ t \in (\hat{q}_1 + 1, \hat{p}).$$

Therefore $-\hat{z}^*$ is decreasing on $(\hat{q}_1 + 1, \hat{p})$ so by (5.9) and (5.14), we have, using the continuity of $\hat{y}^*$ to obtain the value at $\hat{p}$, for $t \in [\hat{q}_1 + 1, \hat{p}]$,

$$\hat{y}^*(t) = \sup_{0 \leq s \leq t} (\hat{z}^*(s) + L)^- = (\hat{z}^*(\hat{q}_1 + 1) + L)^- = \hat{y}^*(\hat{q}_1 + 1). \tag{5.16}$$

By (5.7) and (5.15)–(5.16), for $t \in [\hat{q}_1 + 1, \hat{p}]$,

$$\begin{aligned} \hat{x}^*(t) &= \hat{z}^*(t) + (\hat{z}^*(\hat{q}_1 + 1) + L)^- \\ &= \max(\hat{z}^*(\hat{q}_1 + 1), -L) + (\hat{z}^*(t) - \hat{z}^*(\hat{q}_1 + 1)) \\ &= \hat{x}^*(\hat{q}_1 + 1) + (\hat{z}^*(t) - \hat{z}^*(\hat{q}_1 + 1)). \end{aligned}$$

Since $\hat{z}^*$ is continuously differentiable on $(\hat{q}_n + 1, p]$, $\hat{x}^*$ is also continuously differentiable with $\frac{d\hat{x}^*}{dt} = \frac{d\hat{z}^*}{dt}$ there. Since $\hat{z}^*$ is increasing on $(\hat{q}_1 + 1, \hat{p})$, $\hat{x}^*$ is also increasing on $(\hat{q}_1 + 1, \hat{p})$. Then by periodicity, $\hat{x}^*$ is differentiable and increasing on $(-1, 0)$ with derivative bounded by $\sup_{s\in[-L,\infty)} \tau\hat{h}(s) = \sup_{s\in[0,\infty)} \tau h(s) = \tau G$. Lastly, the fact that $\hat{x}^*(t) \le \tau G$ for all $t \ge -1$ follows because of the properties of $\hat{x}^*$ described in (i)-(iv), the bound on the derivative of $\hat{x}^*$ on the interval $(-1, 0)$ and the periodicity of $\hat{x}^*$. □

5.2 Convergence of Scaled SOPS$^{\mathrm{n}}$

In this section we prove the convergence of a family of scaled SOPS$^{\mathrm{n}}$ as the delay τ goes to infinity. The results in this section only require that h satisfy Assumption 3.3.

Define $\tau_0 > 0$ as in (3.11). By Theorem 3.1, for each $\tau > \tau_0$ there exists a SOPS of DDER with delay τ, which we denote by x^τ, and real numbers $q_0^\tau \ge -\tau$, $q_1^\tau > q_0^\tau + \tau$, $q_2^\tau > q_1^\tau + \tau$, $p^\tau = q_2^\tau + \tau$ such that x^τ satisfies (3.1)–(3.2), but with q_0^τ, q_1^τ, q_2^τ and p^τ in place of q_0, q_1, q_2 and p, respectively. Furthermore, since DDER is time homogeneous, by performing a time shift on x^τ, we can assume that $q_0^\tau = -\tau$. By Lemma 4.1, for each SOPS x^τ, there is an associated SOPS$^{\mathrm{n}}$, denoted $\hat{x}^\tau$, satisfying (4.9)–(4.10) with zeros $\hat{q}_0^\tau = \tau^{-1}q_0^\tau = -1$, $\hat{q}_1^\tau = \tau^{-1}q_1^\tau$, $\hat{q}_2^\tau = \tau^{-1}q_2^\tau$ and period $\hat{p}^\tau = \tau^{-1}p^\tau$. For each $\tau > \tau_0$, define the scaled functions $\bar{x}^\tau \in \mathcal{C}_{[-1,\infty)}$, $\bar{y}^\tau \in \mathcal{C}^+_{[0,\infty)}$ and $\bar{z}^\tau \in \mathcal{C}_{[0,\infty)}$ by

$$\bar{x}^\tau(t) = \tau^{-1}\hat{x}^\tau(t),\ t \ge -1, \tag{5.17}$$

$$\bar{y}^\tau(t) = \tau^{-1}\hat{y}^\tau(t),\ t \ge 0, \tag{5.18}$$

$$\bar{z}^\tau(t) = \tau^{-1}\hat{z}^\tau(t),\ t \ge 0. \tag{5.19}$$

By (5.7)–(5.9) and (5.17)–(5.19), $\bar{x}^\tau$, $\bar{y}^\tau$ and $\bar{z}^\tau$ satisfy, for $t \ge 0$,

$$\bar{x}^\tau(t) = \bar{z}^\tau(t) + \bar{y}^\tau(t) \tag{5.20}$$

$$\bar{z}^\tau(t) = \bar{x}^\tau(0) + \int_0^t \hat{h}(\tau\bar{x}^\tau(s-1))ds, \tag{5.21}$$

where

$$\bar{y}^\tau(t) = \sup_{0 \le s \le t} (\bar{z}^\tau(s) + \tau^{-1}L)^-. \tag{5.22}$$

Note that if we add $\tau^{-1}L$ to either side of (5.20), we obtain $(\bar{x}^\tau(t) + \tau^{-1}L) = (\bar{z}^\tau(t) + \tau^{-1}L) + \bar{y}^\tau(t) \ge 0$ and $\int_0^t (\bar{x}^\tau(s) + \tau^{-1}L) d\bar{y}^\tau(s) = 0$ for all $t \ge 0$. Therefore $(\bar{x}^\tau + \tau^{-1}L, \bar{y}^\tau)$, where here $\bar{x}^\tau$ is restricted to the interval $[0, \infty)$, is a solution of the one-dimensional Skorokhod problem for $\bar{z}^\tau + \tau^{-1}L$ (see Appendix A) and so $(\bar{x}^\tau + \tau^{-1}L, \bar{y}^\tau) = (\Phi, \Psi)(\bar{z}^\tau + \tau^{-1}L)$.

The following three lemmas are used to prove that $\bar{x}^\tau$ converges to a non-trivial function in $\mathcal{C}_{[-1,\infty)}$ as $\tau \to \infty$. The proof of Lemma 5.2 is relegated to Appendix D as it is similar to the proof of Theorem 12 in [30]. The proof in [30] relies on estimates for slowly oscillating periodic solutions of an unconstrained delay differential equation and we provide analogous estimates for a SOPS$^{\text{n}}$ in Appendix D. The main difference is that the estimates for our SOPS$^{\text{n}}$ need to take account of the lower boundary constraint.

Lemma 5.2. *There exists $\tau^\dagger \ge \tau_0$ and $\gamma > 0$ such that if $\tau > \tau^\dagger$, then $\bar{x}^\tau$ satisfies*

$$\|\bar{x}^\tau\|_{[-1,\infty)} = \|\bar{x}^\tau\|_{[0,\hat{p}^\tau]} \ge \gamma. \tag{5.23}$$

Proof. See Appendix D. □

For the following recall that by Assumption 3.3, $H \equiv \|h\|_{[0,\infty)} = \|\hat{h}\|_{[-L,\infty)} < \infty$.

Lemma 5.3. *For each $\tau > \tau_0$, $\bar{x}^\tau$, $\bar{y}^\tau$ and $\bar{z}^\tau$ satisfy, for $0 \le s < t < \infty$,*

$$|\bar{x}^\tau(t) - \bar{x}^\tau(s)| \le H|t-s|, \tag{5.24}$$

$$|\bar{y}^\tau(t) - \bar{y}^\tau(s)| \le H|t-s|, \tag{5.25}$$

$$|\bar{z}^\tau(t) - \bar{z}^\tau(s)| \le H|t-s|. \tag{5.26}$$

Proof. By (5.21), for $0 \le s \le t < \infty$, we have

$$|\bar{z}^\tau(t) - \bar{z}^\tau(s)| \le \int_s^t |\hat{h}(\tau\bar{x}^\tau(u-1))| du \le H|t-s|.$$

It follows that $\operatorname{Osc}(\bar{z}^\tau, [s,t]) \le H|t-s|$. Recall that $(\bar{x}^\tau + \tau^{-1}L, \bar{y}^\tau) = (\Phi, \Psi)(\bar{z}^\tau + \tau^{-1}L)$. Then by Proposition A.2, for $0 \le s \le t < \infty$,

$$
\begin{aligned}
|\bar{x}^\tau(t) - \bar{x}^\tau(s)| &\le H|t-s|, \\
|\bar{y}^\tau(t) - \bar{y}^\tau(s)| &\le H|t-s|.
\end{aligned}
$$

□

Lemma 5.4. *The families* $\{\hat{q}_1^\tau : \tau > \tau_0\}$ *and* $\{\hat{q}_2^\tau : \tau > \tau_0\}$ *are uniformly bounded in* $[0,\infty)$.

Proof. Since $\hat{q}_1^\tau + 1 < \hat{q}_2^\tau$ for each $\tau > \tau_0$, it suffices to show that $\{\hat{q}_2^\tau : \tau > \tau_0\}$ is uniformly bounded. By Lemma 5.1(i), for each $\tau > \tau_0$, $\hat{x}^\tau(-1) = 0$, $\hat{x}^\tau$ is increasing on $[-1,0]$ and is bounded by τG there. Therefore $\hat{x}_0 \in \widetilde{\mathcal{K}}$, where $\widetilde{\mathcal{K}}$ is defined as in (4.15)–(4.17), and we can apply Lemma 4.4(ii) to obtain that there exists a positive constant Q, which does depend on τ, such that $\hat{q}_2^\tau$ is bounded by $1 + 2Q$. □

Let $\bar{q} = \alpha^{-1}\beta$. Define $\bar{x} \in \mathcal{C}^+_{[-1,\infty)}$ to be a periodic function with period $\bar{q} + 2$ satisfying

$$
\bar{x}(t) = \begin{cases} \beta(t+1) & \text{for } t \in [-1,0], \\ \beta - \alpha t & \text{for } t \in [0,\bar{q}], \\ 0 & \text{for } t \in [\bar{q}, \bar{q}+1]. \end{cases} \tag{5.27}
$$

Define $\bar{z} \in \mathcal{C}_{[0,\infty)}$ and $\bar{y} \in \mathcal{C}^+_{[0,\infty)}$ by

$$
\bar{z}(t) = \bar{x}(0) + \int_0^t \bar{h}(\bar{x}(s-1))ds, \ t \ge 0, \tag{5.28}
$$

$$
\bar{y}(t) = \sup_{0 \le s \le t} (\bar{z}(s))^-, \ t \ge 0, \tag{5.29}
$$

where

$$
\bar{h}(s) = \begin{cases} -\alpha, & \text{if } s > 0, \\ \beta, & \text{if } s = 0. \end{cases} \tag{5.30}
$$

Note that $(\bar{x}, \bar{y}) = (\Phi, \Psi)(\bar{z})$, where $\bar{x}$ is restricted to the interval $[0,\infty)$. Therefore $(\bar{x}, \bar{y})$ is the unique solution of the one-dimensional Skorokhod problem for $\bar{z}$.

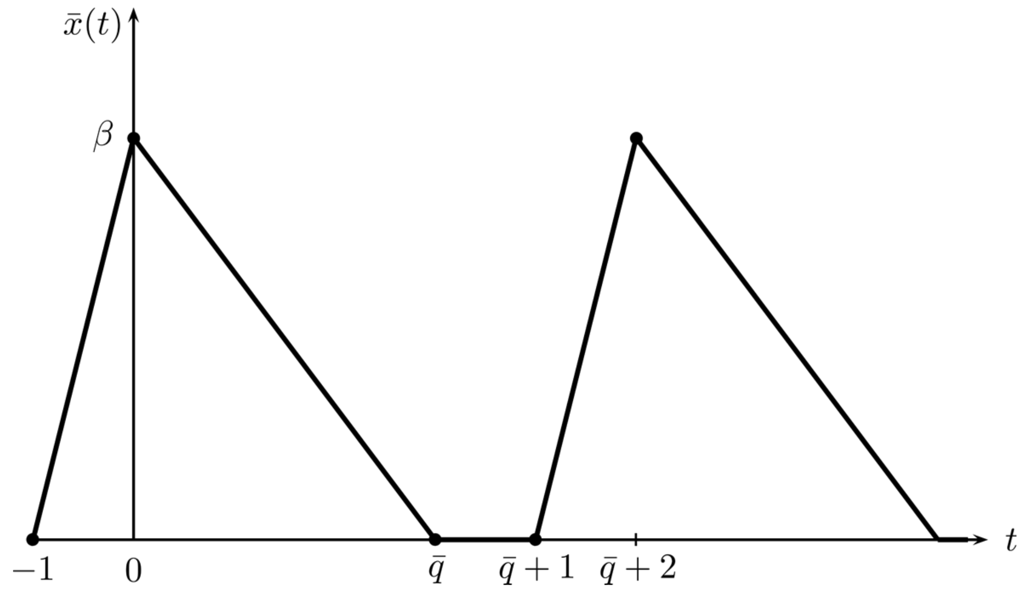

Figure 5.2: An example of $\bar{x}$ as described in (5.27).

Theorem 5.1. *The family* $\{(\bar{x}^\tau, \bar{y}^\tau, \bar{z}^\tau, \hat{q}_1^\tau, \hat{q}_2^\tau, \hat{p}^\tau) : \tau > \tau_0\}$ *converges to* $(\bar{x}, \bar{y}, \bar{z}, \bar{q}, \bar{q} +1, \bar{q}+2)$ *in* $\mathcal{C}_{[-1,\infty)} \times \mathcal{C}_{[0,\infty)} \times \mathcal{C}_{[0,\infty)} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ *as* $\tau \to \infty$*.*

Proof. Fix a sequence $\{\tau_n\}_{n=1}^\infty$ in (τ_0, ∞) such that $\tau_n \to \infty$ as $n \to \infty$. By (5.24), the periodicity of $\bar{x}^\tau$ and the fact that $\bar{x}^\tau(-1) = 0$ for all $\tau > \tau_0$, the family $\{\bar{x}^\tau : \tau > \tau_0\}$ is uniformly bounded and uniformly Lipschitz continuous on compact intervals in $[-1, \infty)$. Similarly, by (5.25)–(5.26) and the fact that $\bar{y}^\tau(0) = 0$, $\bar{z}^\tau(0) = \bar{x}^\tau(0)$ and $\bar{x}^\tau(0)$ is uniformly bounded as τ varies, the families $\{\bar{y}^\tau : \tau > \tau_0\}$ and $\{\bar{z}^\tau : \tau > \tau_0\}$ are uniformly bounded and uniformly Lipschitz continuous on compact intervals in $[0, \infty)$. Therefore, by the theorem of Arzelà and Ascoli, there exist nested subsequences $\{n_{1,j}\}_{j=1}^\infty \supset \{n_{2,j}\}_{j=1}^\infty \supset \cdots$ such that for each $k \in \mathbb{N}$, $x^{\tau_{n_{k,j}}}$, $y^{\tau_{n_{k,j}}}$ and $z^{\tau_{n_{k,j}}}$ converge to continuous functions x^k, y^k and z^k uniformly on the intervals $[-1, k]$, $[0, k]$ and $[0, k]$, respectively, as $j \to \infty$. By taking the diagonal subsequence $\{n_{j,j}\}_{j=1}^\infty$, we see that there exist continuous functions $x^0 \in \mathcal{C}_{[-1,\infty)}$, $y^0 \in \mathcal{C}_{[0,\infty)}$ and $z^0 \in \mathcal{C}_{[0,\infty)}$ that agree with x^k, y^k and z^k on $[-1, k]$, $[0, k]$ and $[0, k]$, respectively, for each $k \in \mathbb{N}$, and are such that $x^{\tau_{n_{j,j}}}$, $y^{\tau_{n_{j,j}}}$ and $z^{\tau_{n_{j,j}}}$ converge to x^0, y^0 and z^0 uniformly on compact intervals in $[-1, \infty)$, $[0, \infty)$ and $[0, \infty)$, respectively, as $j \to \infty$. For notational purposes, henceforth, we will simply use $\{\tau_n\}_{n=1}^{\infty}$ to denote the subsequence $\{\tau_{n_{j,j}}\}_{j=1}^{\infty}$.

By Lemma 5.4, $\{(\hat{q}_1^{\tau}, \hat{q}_2^{\tau}) : \tau > \tau_0\}$ is uniformly bounded in $\mathbb{R}_+^2$ and hence relatively compact. Therefore, by taking a further subsequence if necessary, we have that (i) $\bar{x}^{\tau_n}$ converges to a non-trivial continuous function x^0 in $\mathcal{C}_{[-1,\infty)}$ as $n \to \infty$; (ii) $\bar{y}^{\tau_n}$ and $\bar{z}^{\tau_n}$ converge in $\mathcal{C}_{[0,\infty)}$ to continuous functions y^0 and z^0, respectively, as $n \to \infty$; and (iii) $\hat{q}_1^{\tau_n}$ and $\hat{q}_2^{\tau_n}$ converge to non-negative real numbers q_1^0 and q_2^0, respectively, as $n \to \infty$. Additionally, since $\hat{q}_1^{\tau_n}+1 < \hat{q}_2^{\tau_n}$ and $\hat{p}^{\tau_n} = \hat{q}_2^{\tau_n}+1$ for each $n \in \mathbb{N}$, then $q_1^0 + 1 \leq q_2^0$ and $p^0 := \lim_{n\to\infty} \hat{p}^{\tau_n} = q_2^0 + 1$. Furthermore, since for each $n \geq 1$ we have $\hat{x}^{\tau_n}(t) = \hat{x}^{\tau_n}(t+\hat{p}^{\tau_n})$ for all $t \geq -1$, on taking $n \to \infty$, we obtain $x^0(t) = x^0(t + p^0)$ for all $t \geq -1$, i.e., x^0 is periodic with period p^0. It follows from the convergence of $\bar{x}^{\tau_n}$ to x^0, the periodicity and boundedness of $\{\hat{p}^{\tau_n} : \tau > \tau_0\}$, and Lemma 5.2 that

$$\|x^0\|_{[-1,\infty)} \geq \gamma > 0. \tag{5.31}$$

For each $n \geq 1$, $\bar{x}^{\tau_n}(t) = \tau_n^{-1}\hat{x}^{\tau_n}(t) \geq -\tau_n^{-1}L$ for all $t \geq -1$. Taking $n \to \infty$, we have that $x^0(t) \geq 0$ for all $t \geq -1$. By Lemma 5.1(iii)–(iv), for each $n \geq 1$, $\bar{x}^{\tau_n}(t) \leq 0$ for all $t \in [\hat{q}_1^{\tau_n}, \hat{q}_2^{\tau_n}]$. Combining this with the previous statement and the fact that $(\hat{q}_1^{\tau_n}, \hat{q}_2^{\tau_n}) \to (q_1^0, q_2^0)$ as $n \to \infty$, we have that $x^0(t) = 0$ for all $t \in [q_1^0, q_2^0]$. It follows from this and (5.31) that x^0 is non-trivial, i.e., not identically constant. By Lemma 5.1(i)–(ii), for each $n \geq 1$, $\bar{x}^{\tau_n}$ is positive on $(-1, \hat{q}_1^{\tau_n})$, increasing on $[-1, 0]$ and decreasing on $[0, \hat{q}_1^{\tau_n}]$. It follows that x^0 is non-decreasing on $[-1, 0]$ and non-increasing on $[0, q_1^0]$. Combining the above we see that there exist $t_1 \in [-1, 0)$ and $t_2 \in (0, q_1^0]$ such that

$$\begin{aligned}
&x^0(t) = 0, \ t \in [-1, t_1], \\
&x^0(t) > 0, \ t \in (t_1, t_2), \\
&x^0(t) = 0, \ t \in [t_2, q_1^0] \cup [q_1^0, q_2^0],
\end{aligned} \tag{5.32}$$

By (5.10), $\bar{z}^{\tau_n}(t) = \bar{x}^{\tau_n}(t)$ for all $t \in [0, \hat{q}_1^{\tau_n}]$. Taking $n \to \infty$, we have that $z^0(t) = x^0(t)$ for all $t \in [0, q_1^0]$.

We first show that $t_2 - t_1 > 1$. Suppose for a proof by contradiction that

$t_2 - t_1 \leq 1$. By (5.21),

$$\bar{z}^{\tau_n}(t) = \bar{x}^{\tau_n}(t_1+1) + \int_{t_1+1}^{t} \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds, \; t \in [t_1+1, t_2+1]. \tag{5.33}$$

Since $\hat{h}$ is bounded (see Assumption 3.3), we can pass to the limit as $n \to \infty$ in (5.33) to obtain

$$z^0(t) = -\alpha(t - t_1 - 1), \; t \in [t_1+1, t_1+1]. \tag{5.34}$$

Here we have used the facts that $\bar{x}^{\tau_n}(\cdot)$ converges pointwise on $[t_1, t_2+1]$ to $x^0(\cdot)$, $x^0(t_1+1) = 0$ since $t_2 < t_1 + 1 \leq q_2^0$, $x^0(t) > 0$ for all $t \in (t_1, t_2)$, $\tau_n \to \infty$ as $n \to \infty$ and $\hat{h}(s) \to -\alpha$ as $s \to \infty$. Using that $\bar{z}^{\tau_n}(t) = \tau_n^{-1}\hat{z}^{\tau_n}(t) \to z^0(t) < 0$ for all $t \in (t_1+1, t_2+1]$ and $\tau_n \to \infty$ as $n \to \infty$, we have that

$$\hat{z}^{\tau_n}(t) = \tau_n \bar{z}^{\tau_n}(t) \to -\infty \text{ as } n \to \infty, \; t \in (t_1+1, t_2+1]. \tag{5.35}$$

If $t_1 + 1 < q_1^0$, then for each $t \in (t_1+1, t_2+1)$ we have that $t \in (t_1+1, \hat{q}_1^{\tau_n})$ and $\hat{z}^{\tau_n}(t) < 0$ for all n sufficiently large, which contradicts (5.10) since $\hat{z}^{\tau_n}(t) = \hat{x}^{\tau_n}(t) > 0$ for all $t \in (-1, \hat{q}_1^{\tau_n})$. Thus, we must have $t_1 + 1 \geq q_1^0$. Then, since $t_2 \leq q_1^0$, we must have that $[t_1+1, t_2+1] \subset [q_1^0, q_1^0+1]$. For each $t \in (t_1+1, t_2+1)$, we can choose n sufficiently large enough so that $t \in (\hat{q}_1^{\tau_n}, \hat{q}_1^{\tau_n}+1)$, $\hat{z}^{\tau_n}(t) \leq -L$ and by (5.11), $\hat{x}^{\tau_n}(t) = -L$. Hence, $\lim_{n\to\infty} \tau_n \bar{x}^{\tau_n}(t) = \lim_{n\to\infty} \hat{x}^{\tau_n}(t) = -L$ for all $t \in (t_1+1, t_2+1)$. By (5.20), for $t \in [t_1+2, t_2+2]$,

$$\begin{aligned}
\bar{x}^{\tau_n}(t) &= \bar{z}^{\tau_n}(t) + \bar{y}^{\tau_n}(t) \\
&= \bar{x}^{\tau_n}(t_1+2) + (\bar{z}^{\tau_n}(t) - \bar{z}^{\tau_n}(a+2)) + (\bar{y}^{\tau_n}(t) - \bar{y}^{\tau_n}(t_1+2)) \\
&\geq \bar{x}^{\tau_n}(t_1+2) + \int_{t_1+2}^{t} \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds \\
&\geq -\tau_n^{-1}L + \int_{t_1+2}^{t} \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds,
\end{aligned}$$

where we have used that $\bar{y}^{\tau_n}(\cdot)$ is non-decreasing. Since $\hat{h}$ is bounded, we can pass to the limit as $n \to \infty$ to obtain

$$x^0(t) \geq \beta(t - t_1 - 2), \; t \in [t_1+2, t_2+2],$$

Here we have used the fact that $\lim_{n\to\infty} \tau_n \bar{x}^{\tau_n}(t) = -L$ for all $t \in (t_1+1, t_2+1)$, the continuity of $\hat{h}$ and that $\hat{h}(-L) = \beta$. It follows that $x^0(t) > 0$ for all $t \in (t_1+2, t_2+2)$ and $x^0(t_2+2) \geq \beta(t_2-t_1) > 0$. Since x^0 is continuous, x^0 is positive for some $t > t_2+2$ and hence positive on an interval of length greater than t_2-t_1, which contradicts (5.32) and the periodicity of x^0. With the contradiction thus obtained, we must have $t_2 - t_1 > 1$.

By (5.20),

$$\bar{z}^{\tau_n}(t) = \bar{z}^{\tau_n}(t_2) + \int_{t_2}^{t} \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds, \; t \in [t_2, t_2+1]. \tag{5.36}$$

Taking limits as $n \to \infty$ in (5.36), we have

$$\bar{z}^0(t) = -\alpha(t-t_2), \; t \in [t_2, t_2+1]. \tag{5.37}$$

Here we have used the facts that $z^0(t_2) = x^0(t_2) = 0$, that $\bar{x}^{\tau_n}(\cdot)$ converges pointwise to $x^0(\cdot)$ on $(t_2-1, t_2] \subset (t_1, t_2]$ and $x^0(t) > 0$ for all $t \in (t_1, t_2)$, that $\tau_n \to \infty$ as $n \to \infty$ that $\hat{h}$ is bounded and that $\lim_{s\to\infty} \hat{h}(s) = -\alpha$. By (5.37), for each $t \in (t_2, t_2+1]$, $z^0(t) < 0$ and so $\bar{z}^{\tau_n}(t) < 0$ and $\hat{q}_1^{\tau_n} < t$ for all n sufficiently large. Since this holds for each $t \in (t_2, t_2+1]$, it follows that $q_1^0 \leq t_2$. Combining this with the fact that $t_2 \leq q_1^0$ as in (5.32), we have that $t_2 = q_1^0$. Furthermore, since $\bar{z}^{\tau_n} \to z^0$ uniformly on compact intervals and $(\hat{q}_1^{\tau_n}, \hat{q}_1^{\tau_n}+1) \to (q_1^0, q_1^0+1)$ as $n \to \infty$, we have that if $t \in (t_2, t_2+1)$, then $t \in (\hat{q}_1^{\tau_n}, \hat{q}_1^{\tau_n}+1)$ and $\hat{z}^{\tau_n}(t) = \tau_n \bar{z}^{\tau_n}(t) \leq -L$ for all n sufficiently large. Then for such n, by (5.11), $\hat{x}^{\tau_n}(t) = \tau_n \bar{x}^{\tau_n}(t) = -L$. Hence

$$\lim_{n\to\infty} \tau_n \bar{x}^{\tau_n}(t) = -L, \; t \in (q_1^0, q_1^0+1) = (t_2, t_2+1). \tag{5.38}$$

For each $t \in [q_1^0+1, q_1^0+2]$, we have

$$\begin{aligned} \bar{x}^{\tau_n}(t) &= \bar{z}^{\tau_n}(t) + \bar{y}^{\tau_n}(t) \qquad (5.39) \\ &= \bar{x}^{\tau_n}(q_1^0+1) + (\bar{z}^{\tau_n}(t) - \bar{z}^{\tau_n}(q_1^0+1)) + (\bar{y}^{\tau_n}(t) - \bar{y}^{\tau_n}(q_1^0+1)) \\ &\geq \bar{x}^{\tau_n}(q_1^0+1) + \int_{q_1^0+1}^{t} \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds. \end{aligned}$$

Here we have used the fact that $\bar{y}^{\tau_n}(\cdot)$ is non-decreasing. Taking limits as $n \to \infty$, we have that

$$x^0(t) \geq \beta(t - q_1^0 - 1), \; t \in [q_1^0+1, q_1^0+2],$$

where we have used the fact that $x^0(\hat{q}_1^0+1) =$, (5.38) and the continuity and boundedness of $\hat{h}$, as well as the fact that $\hat{h}(-L) = \beta$. It follows that for each $t \in (q_1^0+1, q_1^0+2)$, $x^0(t) > 0$, so for all n sufficiently large we have $\bar{x}^{\tau_n}(t) > 0$ and $\hat{q}_2^{\tau_n} < t$. Thus $q_2^0 = \lim_{n\to\infty} \hat{q}_2^{\tau_n} \leq q_1^0 + 1$. Combining this with the fact that $q_1^0 + 1 \leq q_2^0$ (since $\hat{q}_1^{\tau_n} + 1 \leq \hat{q}_2^{\tau_n}$ for all n), we have $q_2^0 = q_1^0 + 1$. Then we have $p^0 = q_2^0 + 1 = q_1^0 + 2$. Using that $\bar{x}^{\tau_n}$ converges to x^0 uniformly on compact intervals in $[-1,\infty)$ and x^0 is positive on $(\hat{q}_2, p^0) = (q_1^0+1, q_1^0+2)$, we have that for each closed interval I contained in (q_1^0+1, q_1^0+2), $\bar{x}^{\tau_n}$ is positive on I for all n sufficiently large and so $\bar{y}^{\tau_n}$ is constant on I for all such n. Since this holds for all closed intervals I in (q_1^0+1, q_1^0+2), y^0 must be constant on (q_1^0+1, q_1^0+2). Then taking limits as $n \to \infty$ in (5.39), we have $x^0(t) = \beta(t - q_1^0 - 1)$ for all $t \in [q_1^0+1, q_1^0+2]$. Finally, since x^0 is periodic with period p^0, $x^0(t) = \beta(t+1)$ for all $t \in [-1, 0]$ and so $t_1 = -1$.

For $t \in [0, q_1^0]$,

$$\bar{z}^{\tau_n}(t) = \bar{x}^{\tau_n}(0) + \int_0^t \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds.$$

Taking $n \to \infty$, we have

$$x^0(t) = \beta - \alpha t,\ t \in [0, q_1^0].$$

Here we have used the fact that $z^0(t) = x^0(t)$ for all $t \in [0, q_1^0]$ and $\lim_{n\to\infty} \bar{x}^{\tau_n}(t) = x^0(t) > 0$ for all $t \in (-1, q_1^0) = (t_1, t_2)$, as well as the boundedness of $\hat{h}$ and the fact that $\lim_{s\to\infty} \hat{h}(s) = -\alpha$. Furthermore, since $x^0(q_1^0) = 0$, we have that $q_1^0 = \alpha^{-1}\beta$.

Using the fact that $\lim_{n\to\infty} \bar{x}^{\tau_n}(t) = x^0(t) > 0$ for all $t \in (-1, q_1^0)$ and the fact that $\lim_{s\to\infty} \hat{h}(s) = -\alpha$, we have that $\lim_{n\to\infty} \hat{h}(\tau_n \bar{x}^{\tau_n}(t)) = \bar{h}(x^0(t)) = -\alpha$ for all $t \in (-1, q_1^0)$. By (5.38), the continuity of $\hat{h}$ and the fact that $x^0(t) = 0$ on $[q_1^0, q_1^0+1]$, we have that $\lim_{n\to\infty} \hat{h}(\tau_n \bar{x}^{\tau_n}(t)) = \bar{h}(x^0(t)) = \beta$ for all $t \in (q_1^0, q_1^0+1)$. Hence, $\lim_{n\to\infty} \hat{h}(\tau_n \bar{x}^{\tau_n}(t)) = \bar{h}(x^0(t))$ at all $t \neq -1, q_1^0, q_1^0+1$ in $[-1, q_1^0+1]$. Since x^0 is periodic with period $p^0 = q_1^0+2$, we can repeat this argument countably many times to obtain that $\lim_{n\to\infty} \hat{h}(\tau_n \bar{x}^{\tau_n}(t)) = \bar{h}(x^0(t))$ for all but countably many t in $[-1,\infty)$. Thus, by bounded convergence, we can take limits as $n \to \infty$ in

$$\bar{z}^{\tau_n}(t) = \bar{x}^{\tau_n}(0) + \int_0^t \hat{h}(\tau_n \bar{x}^{\tau_n}(s-1))ds,\ t \geq 0,$$

to obtain

$$z^0(t) = x^0(0) + \int_0^t \bar{h}(x^0(s-1))ds, \ t \geq 0.$$

Finally, since $\bar{z}^{\tau_n} \to z^0$ uniformly on compact intervals in $[0,\infty)$,

$$y^0(t) = \lim_{n\to\infty} \bar{y}^{\tau_n}(t) = \lim_{n\to\infty} \sup_{0\leq s\leq t} (\bar{z}^{\tau_n}(s) + \tau_n^{-1}L)^- = \sup_{0\leq s\leq t} (z^0(s))^-.$$

We have shown that $(x^0, y^0, z^0, q_1^0, q_2^0, p^0) = (\bar{x}, \bar{y}, \bar{z}, \bar{q}, \bar{q}+1, \bar{q}+2)$. Since $\{\tau_n\}_{n=1}^\infty$ was an arbitrary increasing sequence in (τ_0,∞) with $\tau_n \to \infty$ as $n\to\infty$, it follows that the family $\{(\bar{x}^\tau, \bar{y}^\tau, \bar{z}^\tau, \hat{q}_1^\tau, \hat{q}_2^\tau, \hat{p}^\tau) : \tau > \tau_0\}$ converges to $(\bar{x}, \bar{y}, \bar{z}, \bar{q}, \bar{q}+1, \bar{q}+2)$ as $\tau\to\infty$. If not, there necessarily exists a sequence $\{\tau_n\}_{n=1}^\infty$, an $\varepsilon > 0$, a compact interval I_x in $[-1,\infty)$ and compact intervals I_y, I_z in $[0,\infty)$ such that $\tau_n \to \infty$ as $n\to\infty$ and one of the following holds for all n: $\|\bar{x}^{\tau_n} - \bar{x}\|_{I_x} > \varepsilon$, $\|\bar{y}^{\tau_n} - \bar{y}\|_{I_y} > \varepsilon$, $\|\bar{z}^{\tau_n} - \bar{z}\|_{I_z} > \varepsilon$, $|\hat{q}_1^{\tau_n} - \bar{q}| > \varepsilon$, $|\hat{q}_2^{\tau_n} - \bar{q} - 1| > \varepsilon$ or $|\hat{p}^{\tau_n} - \bar{q} - 2| > \varepsilon$. However, we have just shown that there must exists a subsequence $\{n_k\}_{k=1}^\infty$ such that $(\bar{x}^{\tau_{n_j}}, \bar{y}^{\tau_{n_j}}, \bar{z}^{\tau_{n_j}}, \hat{q}_1^{\tau_{n_j}}, \hat{q}_2^{\tau_{n_j}}, \hat{p}^{\tau_{n_j}})$ converges to $(\bar{x}, \bar{y}, \bar{z}, \bar{q}, \bar{q}+1, \bar{q}+2)$ as $j\to\infty$, a contradiction. Hence, we must have that $(\bar{x}^\tau, \bar{y}^\tau, \bar{z}^\tau, \hat{q}_1^\tau, \hat{q}_2^\tau, \hat{p}^\tau)$ converges to $(\bar{x}, \bar{y}, \bar{z}, \bar{q}, \bar{q}+1, \bar{q}+2)$ as $\tau\to\infty$, completing the proof. □

In the following corollaries, we use Theorem 5.1 to further describe the family $\{\hat{x}^\tau : \tau > \tau_0\}$ for τ sufficiently large.

Corollary 5.1. *There exists $\tau_1 \geq \tau_0$, such that (i) for each $\tau > \tau_1$, there exists a unique $\hat{\ell}_1^\tau \in (\hat{q}_1^\tau, \hat{q}_1^\tau + 1)$ satisfying*

$$\begin{aligned} 0 > \hat{x}^\tau(t) > -L \ &\textit{for all } t \in (\hat{q}_1^\tau, \hat{\ell}_1^\tau) \\ \hat{x}^\tau(t) = -L \ &\textit{for all } t \in [\hat{\ell}_1^\tau, \hat{q}_1^\tau + 1], \end{aligned}$$

and (ii) $\hat{\ell}_1^\tau \to \bar{q}$ as $\tau\to\infty$.

Proof. Fix $\delta \in (0,1)$. By Theorem 5.1 and (5.28), $\hat{q}_1^\tau \to \bar{q}$ and $\bar{z}^\tau \to \bar{z}$ as $\tau\to\infty$ and $\bar{z}(t) = -\alpha(t-\bar{q})$ for all $t\in[\bar{q},\bar{q}+1]$. Hence, there exists $\tau^\delta \geq \tau_0$ such that $\bar{q}+\delta \in (\hat{q}_1^\tau, \hat{q}_1^\tau+1)$ and $\hat{z}^\tau(\bar{q}+\delta) = \tau\bar{z}^\tau(\bar{q}+\delta) \leq -L$ for all $\tau \geq \tau^\delta$. By Lemma 5.1, $\hat{z}^\tau(\hat{q}_1^\tau) = 0$ and $\hat{z}^\tau$ is decreasing on $[\hat{q}_1^\tau, \hat{q}_1^\tau+1]$, so by the continuity of $\hat{z}^\tau$, for each $\tau \geq \tau^\delta$, there exists a unique $\hat{\ell}_1^\tau \in (\hat{q}_1^\tau, \hat{q}_1^\tau + 1]$ such that

$$\begin{aligned} \hat{z}^\tau(t) &> -L \text{ for all } t \in [\hat{q}_1^\tau, \hat{\ell}_1^\tau), \\ \hat{z}^\tau(\hat{\ell}_1^\tau) &= -L, \\ \hat{z}^\tau(t) &< -L \text{ for all } t \in (\hat{\ell}_1^\tau, \hat{q}_1^\tau + 1]. \end{aligned}$$

Moreover, $\hat{\ell}_1^\tau \in (\hat{q}_1^\tau, \bar{q}+\delta)$. Note that nominally the family $\{\hat{\ell}_1^\tau : \tau \geq \tau^\delta\}$ depends on δ. However, because of the uniqueness, the families for two different values of δ will agree on the range of τ that is common to both. Then by (5.11), $\hat{x}^\tau(t) = \hat{z}^\tau(t) > -L$ for all $t \in (\hat{q}_1^\tau, \hat{\ell}_1^\tau)$ and $\hat{x}^\tau(t) = \max(\hat{z}^\tau(t), -L) = -L$ for all $t \in [\hat{\ell}_1^\tau, \hat{q}_1^\tau + 1]$. Let $\tau_1 = \inf\{\tau^\delta : \delta \in (0,1)\}$. If $\tau > \tau_1$, then $\tau \geq \tau^{\delta_0}$ for some $\delta_0 \in (0,1)$ and the first part of the lemma holds with the sequence $\{\hat{\ell}_1^\tau : \tau \geq \tau^{\delta_0}\}$ associated with δ_0. Furthermore, using the uniqueness property mentioned above, it follows that $\hat{\ell}_1^\tau \in (\hat{q}_1^\tau, \bar{q}+\delta)$ for all $\tau \geq \tau^\delta \vee \tau^{\delta_0}$, for any $\delta \in (0,1)$, and so $\hat{\ell}_1^\tau \to \bar{q}$ as $\tau \to \infty$. □

Corollary 5.2. *There exists $\tau_2 > \tau_1$ such that if $\tau \geq \tau_2$, then $0 < \hat{\ell}_1^\tau - \hat{q}_1^\tau < \frac{2L}{\alpha\tau}$ and $0 < \hat{q}_2^\tau - \hat{q}_1^\tau - 1 < (\frac{2}{\alpha} + \frac{1}{\beta})\frac{L}{\tau}$.*

Proof. Let $\delta = \frac{1}{4}\min(\bar{q}, 1) > 0$ and $d = \frac{1}{2}\min(\bar{x}(-1+\delta), \bar{x}(\bar{q}-\delta)) > 0$. By Theorem 5.1, Corollary 5.1 and (5.3), there exists $\tau_2 > \tau_1$ such that if $\tau \geq \tau_2$, then the following hold:

(i) $\bar{q} - \delta < \hat{q}_1^\tau < \hat{\ell}_1^\tau < \bar{q} + \delta < \bar{q} + 1 - \delta < \hat{q}_1^\tau + 1 < \hat{q}_2^\tau < \bar{q} + 1 + \delta$,

(ii) $\hat{x}^\tau(t) \geq \tau d$ for all $t \in [-1+\delta, \bar{q}-\delta]$, and

(iii) $\hat{h}(s) \leq -\alpha/2$ for all $s \geq \tau d$.

Combining the above, we have $\hat{h}(\hat{x}^\tau(t-1)) \leq -\alpha/2$ for all $t \in [\hat{q}_1^\tau, \hat{\ell}_1^\tau]$ and $\tau \geq \tau_2$. Fix $\tau \geq \tau_2$. By Lemma 4.2, $\hat{x}^\tau$ is differentiable with $\frac{d\hat{x}^\tau(t)}{dt} = \tau\hat{h}(\hat{x}^\tau(t-1)) \leq -\alpha\tau/2$ for all $t \in (\hat{q}_1^\tau, \hat{\ell}_1^\tau)$. Thus,

$$\hat{\ell}_1^\tau - \hat{q}_1^\tau \leq \frac{\hat{x}^\tau(\hat{q}_1^\tau) - \hat{x}^\tau(\hat{\ell}_1^\tau)}{\alpha\tau/2} \leq \frac{2L}{\alpha\tau}, \tag{5.40}$$

and by definition, $\hat{\ell}_1^\tau > \hat{q}_1^\tau$. Now consider $\hat{q}_2^\tau - \hat{q}_1^\tau - 1$. If $\hat{q}_2^\tau \leq \hat{\ell}_1^\tau + 1$, then $\hat{q}_2^\tau - \hat{q}_1^\tau - 1 \leq \hat{\ell}_1^\tau - \hat{q}_1^\tau$ and the desired inequality holds. Suppose on the other hand that $\hat{q}_2^\tau > \hat{\ell}_1^\tau + 1$. Then for $t \in [\hat{\ell}_1^\tau + 1, \hat{q}_2^\tau] \subset [\hat{\ell}_1^\tau + 1, \hat{q}_1^\tau + 2]$, we have $\hat{h}(\hat{x}^\tau(t-1)) = \hat{h}(-L) = \beta$. Since $\hat{x}^\tau(\cdot) > -L$ on $(\hat{q}_1^\tau + 1, \hat{q}_2^\tau)$ and $\hat{q}_2^\tau - 2 < \bar{q} + \delta < \hat{q}_1^\tau + 1$, by Lemma 4.2, $\hat{x}^\tau$ is differentiable with $\frac{d\hat{x}^\tau(t)}{dt} = \tau\hat{h}(\hat{x}^\tau(t-1)) = \tau\beta$ for all $t \in (\hat{\ell}_1^\tau + 1, \hat{q}_2^\tau)$. Thus,

$$\begin{aligned} |\hat{q}_2^\tau - \hat{q}_1^\tau - 1| &\leq |\hat{q}_2^\tau - \hat{\ell}_1^\tau - 1| + |\hat{\ell}_1^\tau - \hat{q}_1^\tau| \\ &\leq \frac{\hat{x}^\tau(\hat{q}_2^\tau) - \hat{x}^\tau(\ell_1^\tau + 1)}{\beta\tau} + \frac{2L}{\alpha\tau} \\ &\leq \frac{L}{\beta\tau} + \frac{2L}{\alpha\tau}. \end{aligned} \tag{5.41}$$

□

5.3 Solutions of DDER near a SOPS

In this section we study solutions of DDER whose initial conditions are in a neighborhood of the initial condition of a SOPS. Throughout this section we fix a delay $\tau \geq \tau_2$, where τ_2 is defined as in Corollary 5.2, and consider the associated SOPS x^τ from the family $\{x^\tau : \tau > \tau_0\}$. Since we are fixing the delay τ, we will drop the superscript τ notation and use x^* to denote the SOPS x^τ and use $x, x^\dagger$ to denote solutions of DDER. We similarly use $\hat{x}^*$ to denote the associated SOPS^n $\hat{x}^\tau$, so that Corollaries 5.1 and 5.2 hold with $\hat{x}^*$ in place of $\hat{x}^\tau$.

Lemma 5.5. *There exists $\ell_1 \in (q_1, q_1 + \tau)$ such that x^* satisfies*

(i) $x^*(t) > L$ *for all* $t \in (-\tau, 0]$, *is increasing on* $[-\tau, 0)$ *and is continuously differentiable on* $(-\tau, 0]$;

(ii) $x^*(t) > L$ *for all* $t \in [0, q_1)$, *is decreasing on* $(0, q_1]$ *and is continuously differentiable on* $[0, q_1]$;

(iii) $0 < x^*(t) < L$ *for all* $t \in (q_1, \ell_1)$, *is decreasing on* $[q_1, \ell_1)$ *and is continuously differentiable on* $[q_1, \ell_1)$;

(iv) $x^*(t) = 0$ *for all* $t \in [\ell_1, q_1 + \tau]$;

(v) $0 < x^*(t) < L$ *for all* $t \in (q_1 + \tau, q_2)$*, is increasing on* $(q_1 + \tau, q_2]$ *and is continuously differentiable on* $(q_1 + \tau, q_2]$*.*

Furthermore,

$$0 < \ell_1 - q_1 < \frac{2L}{\alpha}, \tag{5.42}$$

and

$$0 < q_2 - q_1 - \tau < \left(\frac{2}{\alpha} + \frac{1}{\beta}\right) L. \tag{5.43}$$

Moreover, z^* *is positive and decreasing on* (q_1, ℓ_1) *and negative and decreasing on* $(\ell_1, q_1 + \tau)$ *and*

$$x^*(t) = z^*(t), \qquad \text{for } t \in [0, \ell_1], \tag{5.44}$$

$$x^*(t) = x^*(q_1 + \tau) + (z^*(t) - z^*(q_1 + \tau)), \qquad \text{for } t \in [q_1 + \tau, p]. \tag{5.45}$$

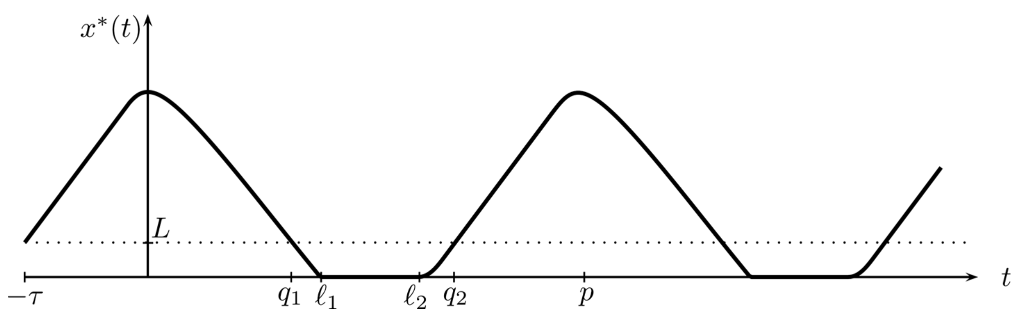

Figure 5.3: An example of a SOPS as described in Lemma 5.5, where $\ell_2 = q_1 + \tau$.

Proof. By Lemmas 4.1, 5.1 (parts (i),(ii) and (iv)) and (5.10)–(5.12), we have that (i), (ii), (v) and (5.44)–(5.45) hold. By Lemmas 4.1 and 5.1(iii) and Corollary 5.1, we have that (iii) and (iv) hold. By Lemma 4.1 and Corollary 5.2, (5.42)–(5.45) hold. □

Lemma 5.6. *Suppose* x*,* $x^\dagger$ *are two solutions of DDER. The function* z *is defined as in (2.2) and let* $z^\dagger$ *be defined as in (2.2) but with* $x^\dagger$*,* $z^\dagger$ *in place of* x*,* z*, respectively. Then for each* $t \geq 0$*,*

$$\|x - x^\dagger\|_{[-\tau,t]} \leq 2\exp(2K_h t)\|x - x^\dagger\|_{[-\tau,0]}, \tag{5.46}$$

and

$$\|z - z^{\dagger}\|_{[0,t]} \leq (1 + K_h\tau)\exp(2K_h t)\|x - x^{\dagger}\|_{[-\tau,0]}, \tag{5.47}$$

where K_h is the Lipschitz constant for h.

Proof. Fix $t \geq 0$. Since h is uniformly Lipschitz continuous with Lipschitz constant K_h, we can apply Lemma 2.1, with $f(\varphi) = h(\varphi(-\tau))$, to obtain (5.46). By (2.2) and (3.12), for $s \in [0,t]$,

$$\begin{aligned}
|z(s) - z^{\dagger}(s)| &\leq |x(0) - x^{\dagger}(0)| + K_h \int_{-\tau}^{0} |x(u) - x^{\dagger}(u)| du \\
&\quad + K_h \int_{0}^{s-\tau} |x(u) - x^{\dagger}(u)| du \\
&\leq (1 + K_h\tau)\|x - x^{\dagger}\|_{[-\tau,0]} + K_h \int_0^s \|x - x^{\dagger}\|_{[0,u]} du \\
&\leq (1 + K_h\tau)\|x - x^{\dagger}\|_{[-\tau,0]} + 2K_h \int_0^s \|z - z^{\dagger}\|_{[0,u]} du.
\end{aligned}$$

The last inequality follows from (2.4) and the Lipschitz continuity of the Skorokhod map Φ (see Appendix A). Taking the supremum over $s \in [0,t]$ and applying Gronwall's inequality yields (5.47). □

Lemma 5.7. *Suppose $\eta_0 > 0$ satisfies $\eta_0 < \min\left\{\frac{\tau}{2}, \frac{q_1}{2}, \frac{\ell_1 - q_1}{2}, \frac{q_1+\tau-\ell_1}{2}, \frac{q_2-q_1-\tau}{2}\right\}$. There exists $\varepsilon_0 > 0$ such that if x is a solution of DDER satisfying $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$, then there exist points $q_1^x \in (q_1 - \eta_0, q_1 + \eta_0)$, $\ell_1^x \in (\ell_1 - \eta_0, \ell_1 + \eta_0)$ and $q_2^x \in (q_2 - \eta_0, q_2 + \eta_0)$ such that x satisfies*

(i) $x(t) > L$ *for all* $t \in [-\tau + \eta_0, \eta_0]$;

(ii) $x(t) > L$ *for all* $t \in [\eta_0, q_1^x)$ *and is decreasing and continuously differentiable on* $[\eta_0, q_1^x]$;

(iii) $0 < x(t) < L$ *for all* $t \in (q_1^x, \ell_1^x)$, *is decreasing and continuously differentiable on* $[q_1^x, \ell_1^x)$;

(iv) $x(t) = 0$ *for all* $t \in [\ell_1^x, q_1^x + \tau]$;

(v) $0 < x(t) < L$ *for all* $t \in (q_1^x + \tau, q_2^x)$ *and is increasing and continuously differentiable on* $t \in (q_1^x + \tau, q_2^x]$;

(vi) $x(t) > L$ *for all* $t \in (q_2^x, q_2^x+\tau)$, *is increasing on* $[q_2^x, q_2^x+\tau)$ *and is continuously differentiable on* $[q_2^x, q_2^x+\tau]$;

(vii) $x(t) > L$ *for all* $t \in [q_2^x+\tau, p+\eta_0]$, *is decreasing on* $(q_2^x+\tau, p+\eta_0]$ *and is continuously differentiable on* $[q_2^x+\tau, p+\eta_0]$.

Furthermore, $h(x(t-\tau)) < 0$ *for all* $t \in [\eta_0, q_1^x+\tau)$, z *is decreasing on* $[\eta_0, q_1^x+\tau)$ *and*

$$\begin{aligned} &\overline{-z}(t) < 0 && \text{for all } t \in [0, \ell_1^x), \\ &\overline{-z}(t) > 0,\ \mathbb{S}_{-z}(t) = \{t\} && \text{for all } t \in (\ell_1^x, q_1^x+\tau], \\ &\overline{-z}(t) > 0,\ \mathbb{S}_{-z}(t) = \{q_1^x+\tau\} && \text{for all } t \in (q_1^x+\tau, p+\eta_0], \end{aligned}$$

where $\mathbb{S}_{-z}(t) = \{s \in [0,t] : -z(s) = \overline{-z}(t)\}$.

Proof. Fix η_0 as in the statement of the lemma. Let $\delta = \delta(\eta_0) > 0$ be given by

$$\begin{aligned} \delta = (x^*(-\tau+\eta_0) - L) &\wedge (x^*(q_1-\eta_0) - L) \wedge x^*(\ell_1-\eta_0) \\ &\wedge (L - x^*(q_1+\eta_0)) \wedge (-z^*(\ell_1+\eta_0)) \wedge x^*(q_1+\tau+\eta_0) \\ &\wedge (L - x^*(q_2-\eta_0)) \wedge (x^*(q_2+\eta_0) - L) \wedge (x^*(p+\eta_0) - L), \end{aligned} \tag{5.48}$$

and define $\varepsilon_0 = \varepsilon_0(\eta_0) > 0$ by

$$\varepsilon_0 = \frac{\delta}{\max(2, 1+K_h\tau)\exp(2K_h(p+\eta_0))}. \tag{5.49}$$

Suppose that x is a solution of DDER satisfying $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$. Then by (5.46)–(5.47) and (5.49),

$$\|x - x^*\|_{[-\tau,p+\eta_0]} < \delta, \tag{5.50}$$

and

$$\|z - z^*\|_{[0,p+\eta_0]} < \delta. \tag{5.51}$$

We first note that by Lemma 5.5, (5.48) and (5.50)–(5.51), we have the following bounds on x and z. Since $x^*(-\tau) = x^*(q_1) = L$ and x^* is increasing on $[-\tau, 0]$ and decreasing $[0, q_1]$, we have, for $t \in [-\tau+\eta_0, q_1-\eta_0]$,

$$\begin{aligned} x(t) &\geq x^*(t) - \|x - x^*\|_{[-\tau,p+\eta_0]} \\ &> L + \min\{x^*(-\tau+\eta_0) - L, x^*(q_1-\eta_0) - L\} - \delta, \end{aligned}$$

so by (5.48),

$$x(t) > L,\ t \in [-\tau + \eta_0, q_1 - \eta_0]. \tag{5.52}$$

Since $x^*(\ell_1) = 0$ and x is non-increasing on $[0, \ell_1]$, we have, for $t \in [q_1 - \eta_0, \ell_1 - \eta_0]$,

$$x(t) \geq x^*(t) - \|x - x^*\|_{[-\tau, p+\eta_0]} > x^*(\ell_1 - \eta_0) - \delta,$$

so by (5.48),

$$x(t) > 0,\ t \in [q_1 - \eta_0, \ell_1 - \eta_0]. \tag{5.53}$$

Since $x^*(q_1) = x^*(q_2) = L$, $x^*(\ell_1) = 0$ and x^* is decreasing on (q_1, ℓ_1), constant on $[\ell_1, q_1 + \tau]$ and increasing on $(q_1 + \tau, q_2)$, we have, for $t \in [q_1 + \eta_0, q_2 - \eta_0]$,

$$\begin{aligned} x(t) &\leq x^*(t) + \|x - x^*\|_{[-\tau, p+\eta_0]} \\ &< L - \min\{L - x^*(q_1 + \eta_0), L - x^*(q_2 - \eta_0)\} + \delta, \end{aligned}$$

so by (5.48),

$$x(t) < L,\ t \in [q_1 + \eta_0, q_2 - \eta_0]. \tag{5.54}$$

Since $z^*(\ell_1) = 0$, and z^* is decreasing on $(\ell_1, q_1 + \tau)$, we have, for $t \in [\ell_1 + \eta_0, q_1 + \tau - \eta_0]$,

$$z(t) \leq z^*(t) + \|z - z^*\|_{[0, p+\eta_0]} < z^*(\ell_1 + \eta_0) + \delta,$$

so by (5.48),

$$z(t) < 0,\ t \in [\ell_1 + \eta_0, q_1 + \tau - \eta_0]. \tag{5.55}$$

Since $x^*(q_1 + \tau) = 0$ and x^* is increasing on $(q_1 + \tau, p)$, we have, for $t \in [q_1 + \tau + \eta_0, p]$,

$$x(t) \geq x^*(t) - \|x - x^*\|_{[-\tau, p+\eta_0]} > x^*(q_1 + \tau + \eta_0) - \delta,$$

so by (5.48),

$$x(t) > 0,\ t \in [q_1 + \tau + \eta_0, p]. \tag{5.56}$$

Since $x^*(q_2) = L$, $x^*(p + \eta_0) > L$, x^* is increasing on (q_2, p) and x^* is decreasing on $(p, p + \eta_0)$, we have, for $t \in [q_2 + \eta_0, p + \eta_0]$,

$$\begin{aligned} x(t) &\geq x^*(t) - \|x - x^*\|_{[-\tau, p+\eta_0]} \\ &> L + \min\{x^*(q_2 + \eta_0) - L, x^*(p + \eta_0) - L\} - \delta, \end{aligned}$$

so by (5.48),

$$x(t) > L, \ t \in [q_2 + \eta_0, p + \eta_0]. \tag{5.57}$$

Proof of (i): By (5.52) and the fact that $\eta_0 < \frac{q_1}{2}$, we have that $x(t) > L$ for all $t \in [-\tau + \eta_0, \eta_0]$.

Proof of (ii): By (5.52) and the fact that $\eta_0 < \frac{\tau}{2}$, we have that $x(t-\tau) > L$ for all $t \in [\eta_0, q_1 + \eta_0]$. It follows from the negative feedback condition on h, Lemma 2.2 and (5.52)–(5.53) that x is differentiable and decreasing on the interval $[\eta_0, q_1 + \eta_0]$. Then by (5.52)–(5.54), there exists $q_1^x \in (q_1 - \eta_0, q_1 + \eta_0)$ such that $x(t) > L$ for all $t \in [q_1 - \eta_0, q_1^x)$ and $0 < x(t) < L$ for all $t \in (q_1^x, q_1 + \eta_0]$.

Proof of (iii) and (iv): By (5.52) and the definition of q_1^x, we have that $x(t-\tau) > L$ for all $t \in [\eta_0, q_1^x + \tau)$. It follows from the negative feedback condition on h and (2.2) that z is differentiable and decreasing on the interval. By (5.52)–(5.53), $x(t) > 0$ for all $t \in [0, \ell_1 - \eta_0]$ and so $y(t) = 0$ and $z(t) = x(t) > 0$ on the interval. Combining this with (5.55), we have that there exists $\ell_1^x \in (\ell_1 - \eta_0, \ell_1 + \eta_0)$ such that $x(t) = z(t) > 0$ for all $t \in [q_1^x, \ell_1^x)$ and $z(t) < 0$ for all $t \in (\ell_1^x, q_1^x + \tau]$. By Lemma 2.2, the negative feedback condition on h and the fact that $x(t-\tau) > L$ for all $t \in (q_1^x, \ell_1^x) \subset [\eta_0, q_1^x + \tau)$, x is differentiable and decreasing on (q_1^x, ℓ_1^x). Furthermore, $\overline{-z}(t) < 0$ and $x(t) > 0$ for all $t \in [0, \ell_1^x)$ and since $-z$ is increasing on $[\ell_1^x, q_1^x + \tau]$, it follows that $\overline{-z}(t) = -z(t)$ and $x(t) = 0$ for all $t \in [\ell_1^x, q_1^x + \tau]$.

Proof of (v): By the definition of q_1^x, (iii)–(iv) and (5.54), we have that $x(t-\tau) < L$ for all $t \in (q_1^x + \tau, q_2 - \eta_0 + \tau)$. It follows from (2.2) and the negative feedback condition on h that z is differentiable and increasing on the interval. Therefore using the results from (iii)–(iv) above $\overline{-z}(t) = -z(q_1^x + \tau)$, $x(t) = z(t) - z(q_1^x + \tau) > 0$ for all $t \in (q_1^x + \tau, q_2 - \eta_0 + \tau)$ and so x is differentiable and increasing on the interval. Then by (5.54) and (5.57), there exists $q_2^x \in (q_2 - \eta_0, q_2 + \eta_0)$ such that $x(t) < L$ for all $t \in (q_1^x + \tau, q_2^x)$ and $x(t) > L$ for all $t \in (q_2^x, q_2 + \eta_0)$.

Proof of (vi): By the definitions of q_1^x and q_2^x and the fact that $q_2^x > q_1^x + \tau$, we have that $x(t-\tau) < L$ for all $t \in (q_2^x, q_2^x + \tau)$. It follows from the negative feedback condition on h, Lemma 2.2 and (5.56) that $x(t) > L$ and is differentiable and increasing at all $t \in (q_2^x, q_2^x + \tau)$.

Proof of (vii): By the definition of q_2^x and (5.57), $x(t-\tau) > L$ for all

$t \in (q_2^x+\tau, p+\eta_0]$. It follows from the negative feedback condition on h, Lemma 2.2 and (5.57) that $x(t) > L$ and is differentiable and decreasing at all $t \in (q_2^x+\tau, p+\eta_0]$.

Since $x(t) > 0$ for all $t \in [0, \ell_1^x)$, $y(t) = 0$ there and by (2.1), $z(t) = x(t) > 0$ for all $t \in [0, \ell_1^x)$. By the proof of (iv), z is decreasing on $[\ell_1^x, q_1^x + \tau]$, so $\overline{-z}(t) = -z(t) > 0$ and $\mathbb{S}_{-z}(t) = \{t\}$ for all $t \in [\ell_1^x, q_1^x + \tau]$. Then since $x(t) > 0$ for all $t \in (q_1^x + \tau, p + \eta_0]$ it follows from (2.1) and (2.3) that $-z(t) = \overline{-z}(t) - x(t) < \overline{-z}(t) = -z(q_1^x + \tau)$ there. $\square$

For a solution x of DDER with $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$, define

$$\varepsilon_1 = \varepsilon_1(x) = \varepsilon_0 - \|x - x^*\|_{[-\tau,0]} > 0, \tag{5.58}$$

and

$$\eta_1 = \eta_1(x) = \eta_0 - \max\{|q_1^x - q_1|, |\ell_1^x - \ell_1|, |q_2^x - q_2|\} > 0. \tag{5.59}$$

By the definition of η_0 in Lemma 5.7 and (5.59), we have

$$(q_1^x - \eta_1, q_1^x + \eta_1) \subset (q_1 - \eta_0, q_1 + \eta_0), \tag{5.60}$$

$$(\ell_1^x - \eta_1, \ell_1^x + \eta_1) \subset (\ell_1 - \eta_0, \ell_1 + \eta_0), \tag{5.61}$$

$$(q_2^x - \eta_1, q_2^x + \eta_1) \subset (q_2 - \eta_0, q_2 + \eta_0). \tag{5.62}$$

Suppose that $x^\dagger$ is another solution of DDER satisfying $\|x - x^\dagger\|_{[-\tau,0]} < \varepsilon_1$. Then by (5.58), we have

$$\|x^\dagger - x^*\|_{[-\tau,0]} \le \|x^\dagger - x\|_{[-\tau,0]} + \|x - x^*\|_{[-\tau,0]} < \varepsilon_0, \tag{5.63}$$

and Lemma 5.7 holds with $x^\dagger$ in place of x. The following lemma ensures that $|q_1^x - q_1^{x^\dagger}|$, $|\ell_1^x - \ell_1^{x^\dagger}|$ and $|q_2^x - q_2^{x^\dagger}|$ converge to zero as $x^\dagger$ converges to x in $\mathcal{C}^+_{[-\tau,0]}$.

Lemma 5.8. *Suppose x is a solution of DDER satisfying $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$ and define $\varepsilon_1 = \varepsilon_1(x) > 0$ and $\eta_1 = \eta_1(x) > 0$ as in* (5.58)–(5.59). *For each $\eta \in (0, \eta_1)$, there exists $\varepsilon \in (0, \varepsilon_1)$ such that if $x^\dagger$ is solution of DDER satisfying $\|x - x^\dagger\|_{[-\tau,0]} < \varepsilon$, then $q_1^{x^\dagger} \in (q_1^x - \eta, q_1^x + \eta)$, $\ell_1^{x^\dagger} \in (\ell_1^x - \eta, \ell_1^x + \eta)$, $q_2^{x^\dagger} \in (q_2^x - \eta, q_2^x + \eta)$.*

Proof. Fix a solution x of DDER satisfying $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$. Let $\eta \in (0, \eta_1)$ and define $\delta > 0$ by

$$\delta = (x(q_1^x - \eta) - L) \wedge (L - x(q_1^x + \eta)) \wedge z(\ell_1^x - \eta) \wedge (-z(\ell_1^x + \eta)) \wedge (L - x(q_2^x - \eta)) \wedge (x(q_2^x + \eta) - L). \tag{5.64}$$

The strict positivity of δ follows from Lemma 5.7. Choose $\varepsilon \in (0, \varepsilon_1)$ satisfying

$$\varepsilon < \frac{\delta}{\max(2, 1 + K_h\tau)\exp(2K_h(p + \eta_0))}. \tag{5.65}$$

Suppose $x^\dagger$ is a solution of DDER satisfying $\|x^\dagger - x\|_{[-\tau,0]} < \varepsilon$. If we define $z^\dagger$ as in (2.2), but with $x^\dagger$ and $z^\dagger$ in place of x and z, respectively, then by (5.63), Lemma 5.7 holds with $x^\dagger$ and $z^\dagger$ in place of x and z, respectively. By (5.46)–(5.47) and (5.65), we have

$$\|x - x^\dagger\|_{[-\tau,p+\eta_0]} < \delta \tag{5.66}$$

and

$$\|z - z^\dagger\|_{[0,p+\eta_0]} < \delta. \tag{5.67}$$

It follows from (5.64) and (5.66)–(5.67) and that

$$x^\dagger(q_1^x - \eta) > L > x^\dagger(q_1^x + \eta), \tag{5.68}$$

$$z^\dagger(\ell_1^x - \eta) > 0 > z^\dagger(\ell_1^x + \eta), \tag{5.69}$$

$$x^\dagger(q_2^x - \eta) < L < x^\dagger(q_2^x + \eta). \tag{5.70}$$

By Lemma 5.7, (5.60)–(5.62), (5.68)–(5.70) and the continuity of $x^\dagger$ and $z^\dagger$, we have that $q_1^{x^\dagger} \in (q_1^x - \eta, q_1^x + \eta)$, $\ell_1^{x^\dagger} \in (\ell_1^x - \eta, \ell_1^x + \eta)$ and $q_2^{x^\dagger} \in (q_2^x - \eta, q_2^x + \eta)$. $\square$

5.4 Linear Variational Equation (LVE)

In this section we consider solutions of a linear variational equation (LVE) relative to solutions of DDER with initial conditions in a small neighborhood of the initial condition of a SOPS. Linear variational equations have been well studied in the unconstrained setting, but require a first principles approach in the constrained setting. Indeed, solutions of LVE in the constrained setting can exhibit considerably different behavior than in the unconstrained setting. In particular, solutions of LVE in the constrained setting can be discontinuous. A general definition and properties of a solution of LVE relative to a solution of DDER are presented in Appendix C. The treatment in the Appendix describes solutions of LVE relative to solutions of DDER associated with a function f in (1.1) that is more general than we need for the proof of stability and uniqueness of SOPS.

Throughout this section we assume that f is of the form (3.9) and satisfies Assumptions 3.3 and 3.4. We fix $\tau \geq \tau_2$, where τ_2 is as in Corollary 5.2, and let x^* denote the SOPS x^τ so that Lemmas 5.5–5.8 hold. We briefly summarize important definitions and properties from Appendix C regarding solutions of LVE relative to a solution x of DDER satisfying $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$, where $\varepsilon_0 > 0$ is as in Lemma 5.7. The following definition is a version of Definition C.1 specific to this section. Recall that $\mathcal{D}_{[-\tau,\infty)}$ denotes the set of functions from $[-\tau, \infty)$ to $\mathbb{R}$ with finite left and right limits at each $t > -\tau$ and a finite right limit at $-\tau$.

Definition 5.1. Suppose x is a solution of DDER satisfying $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$. A function $v \in \mathcal{D}_{[-\tau,\infty)}$ is a solution of LVE relative to x if $v(t) \geq 0$ at all $t \geq -\tau$ such that $x(t) = 0$ and v satisfies

$$v(t) = \partial_w \Phi(z)(t), \; t \geq 0, \tag{5.71}$$

where Φ denotes the Skorokhod map given by (A.1)–(A.2), $z \in \mathcal{C}_{[0,\infty)}$ is defined in (2.2), $w \in \mathcal{C}_{[0,\infty)}$ is defined by

$$w(t) = v(0) + \int_0^t h'(x(s-\tau))v(s-\tau)ds, \; t \geq 0, \tag{5.72}$$

and the directional derivative of Φ at z in the direction w is denoted by $\partial_w \Phi(z)$, is well defined as an element of $\mathcal{D}_{[0,\infty)}$ by Theorem B.1 and is given by

$$\partial_w \Phi(z)(t) = w(t) + R(-z, -w)(t), \tag{5.73}$$

where

$$R(-z,-w)(t)=\begin{cases}\sup_{s\in\mathbb{S}_{-z}(t)}(-w(s)) & \text{if } \overline{-z}(t)>0,\\ \sup_{s\in\mathbb{S}_{-z}(t)}(-w(s))\vee 0 & \text{if } \overline{-z}(t)=0,\\ 0 & \text{if } \overline{-z}(t)<0,\end{cases}\tag{5.74}$$

and

$$\mathbb{S}_{-z}(t)=\{s\in[0,t]:-z(s)=\overline{-z}(t)\}.\tag{5.75}$$

A solution v of LVE relative to x and with initial condition $v_0\in\mathcal{C}_{[-\tau,0]}$ can be thought of as the direction that x is perturbed in when its initial condition x_0 is perturbed in the direction v_0. In general, the element v_0 is constrained by the fact that the initial condition of a solution of DDER cannot be perturbed in the negative direction at points that it is at the boundary. However, in the case that $\|x-x^*\|_{[-\tau,0]}<\varepsilon_0$, it follows from Lemma 5.7 that the initial condition x_0 is strictly above the lower boundary, so we can take v_0 to be any element of $\mathcal{C}_{[-\tau,0]}$. For each $\varepsilon>0$ sufficiently small so that $x_0^\varepsilon\equiv x_0+\varepsilon v_0\in\mathcal{C}^+_{[-\tau,0]}$, let x^ε denote the unique solution of DDER with initial condition x_0^ε and define $v^\varepsilon\in\mathcal{C}_{[-\tau,\infty)}$ by

$$v^\varepsilon(t)=\frac{x^\varepsilon(t)-x(t)}{\varepsilon},\ t\geq-\tau.\tag{5.76}$$

Also, if z is defined as in (2.2) and z^ε is defined as in (2.2), but with x^ε and z^ε in place of x and z, respectively, then we can define $w^\varepsilon\in\mathcal{C}_{[0,\infty)}$ by

$$w^\varepsilon(t)=\frac{z^\varepsilon(t)-z(t)}{\varepsilon},\ t\geq 0.\tag{5.77}$$

The following proposition is a version of Theorem C.1 specific to this section. Recall that the family $\{u^\varepsilon:\varepsilon>0\}$ in $\mathcal{C}_{[-\tau,\infty)}$ converges to $u\in\mathcal{D}_{[-\tau,\infty)}$ uniformly on compact intervals of continuity as $\varepsilon\to0$ provided that for each compact interval I in $[-\tau,\infty)$ such that u is continuous on I, u^ε converges to u uniformly on I as $\varepsilon\to0$.

Proposition 5.1. *Suppose x is a solution of DDER satisfying $\|x-x^*\|_{[-\tau,0]}<\varepsilon_0$ and $\psi\in\mathcal{C}_{[-\tau,0]}$. Then there exists a unique solution v of LVE relative to x with initial condition ψ and v is Borel measurable. Furthermore, if v^ε and w^ε are defined is* (5.76)–(5.77)*, then v^ε converges to v pointwise and uniformly on compact* *intervals of continuity in* $[-\tau,\infty)$ *as* $\varepsilon \to 0$ *and* w^ε *converges to* w *uniformly on compact intervals in* $[0,\infty)$ *as* $\varepsilon \to 0$.

In the following lemma we further describe solutions of LVE.

Lemma 5.9. *Suppose* x *is a solution of DDER satisfying* $\|x-x^*\|_{[-\tau,0]} < \varepsilon_0$. *If* $v \in \mathcal{D}_{[-\tau,\infty)}$ *is a solution of LVE relative to* x, *then* v *satisfies*

$$v(t) = \begin{cases} v(0) + \int_0^t h'(x(s-\tau))v(s-\tau)ds, & t \in [0,\ell_1^x), \\ \left(v(0) + \int_0^{\ell_1^x} h'(x(s-\tau))v(s-\tau)ds\right)^+, & t = \ell_1^x, \\ 0, & t \in (\ell_1^x, q_1^x+\tau], \\ \int_{q_1^x+\tau}^t h'(x(s-\tau))v(s-\tau)ds, & t \in (q_1^x+\tau, p+\eta_0]. \end{cases} \tag{5.78}$$

If $v^\dagger \in \mathcal{D}_{[-\tau,\infty)}$ *also satisfies* (5.78) *and* $v^\dagger(t) = v(t)$ *at almost every* $t \in [-\tau,0]$, *then* $v^\dagger(t) = v(t)$ *for all* $t \in [0,p+\eta_0]$.

Furthermore, if v *is a solution of LVE relative to* x^*, *then* v *satisfies*

$$v(kp+t) = \begin{cases} v(kp) + \int_{kp}^{kp+t} h'(x^*(s-\tau))v(s-\tau)ds, & t \in [0,\ell_1), \\ \left(v(kp) + \int_{kp}^{kp+\ell_1} h'(x^*(s-\tau))v(s-\tau)ds\right)^+, & t = \ell_1, \\ 0, & t \in (\ell_1, q_1+\tau], \\ \int_{kp+q_1+\tau}^{kp+t} h'(x^*(s-\tau))v(s-\tau)ds, & t \in (q_1+\tau, p], \end{cases} \tag{5.79}$$

for each $k \in \mathbb{N}_0$. *If* $v^\dagger \in \mathcal{D}_{[-\tau,\infty)}$ *also satisfies* (5.79) *and* $v^\dagger(t) = v(t)$ *at almost every* $t \in [-\tau,0]$, *then* $v^\dagger(t) = v(t)$ *for all* $t \geq 0$.

Proof. First consider the case that $\|x-x^*\|_{[-\tau,0]} < \varepsilon_0$ and v is a solution of LVE relative to x. It follows from Lemma 5.7 and parts (i), (ii) and (iv) of Lemma C.4 (with $\partial_{x_s} f(x_s) = h'(x(s-\tau))$) that v satisfies (5.78). To prove uniqueness, suppose that $v^\dagger \in \mathcal{D}_{[-\tau,\infty)}$ also satisfies (5.78) and $v^\dagger(t) = v(t)$ at almost every $t \in [-\tau,0]$. By (3.12), for $t \in [0,\tau \wedge \ell_1^x)$,

$$|v^\dagger(t) - v(t)| \leq K_h \int_0^t |v^\dagger(s-\tau) - v(s-\tau)|ds = 0.$$

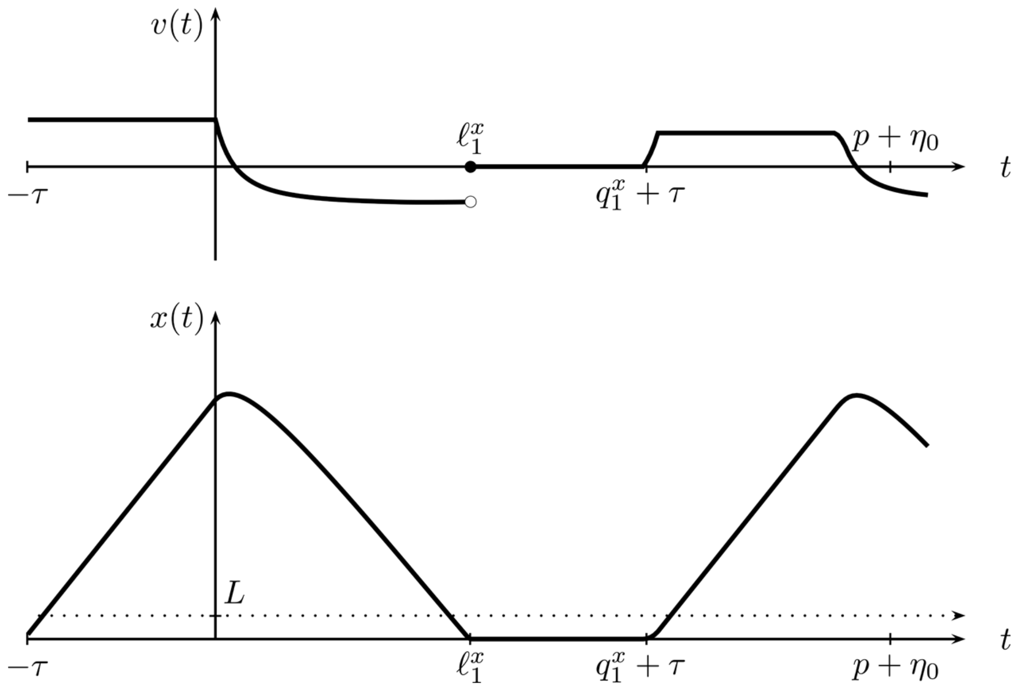

Figure 5.4: An example of a solution of LVE (on the top) relative to a solution of DDER (on the bottom). Here $h(x(t-\tau)) = \frac{\alpha C_0^2}{(C_0+x(t-\tau))^2} - \gamma$, where $\alpha > \gamma > 0$ and $C_0 > 0$ as in Example 3.2.

If $\ell_1^x > \tau$, we can iterate this argument to obtain $v^\dagger(t) = v(t)$ for all $t \in [-\tau, \ell_1^x)$. At $t = \ell_1^x$, we have that $v^\dagger(\ell_1^x) = \left(v^\dagger(\ell_1^x-)\right)^+ = (v(\ell_1^x-))^+ = v(\ell_1^x)$. By (5.78), $v^\dagger(t) = v(t) = 0$ for all $t \in (\ell_1^x, q_1^x+\tau]$. Since $v^\dagger(q_1^x+\tau+t) = v(q_1^x+\tau+t)$ for all $t \in [-\tau, 0]$, we can apply a similar argument as we did on the interval $[0, \ell_1^x)$ to obtain that $v^\dagger(t) = v(t)$ for all $t \in [q_1^x+\tau, p+\eta_0]$.

Now consider the case that v is a solution of LVE relative to x^*. It follows from Lemma 5.5, the periodicity of x^* and parts (i), (ii) and (iv) of Lemma C.4 that v satisfies (5.79) for $k = 0, 1, \ldots$. Suppose that $v^\dagger$ also satisfies (5.79) and $v^\dagger(t) = v(t)$ at almost every $t \in [-\tau, 0]$. By the same argument as above, we have that $v^\dagger(t) = v(t)$ for all $t \in [0, p]$. By iterating this argument for $k = 1, 2, \ldots$, we see that $v^\dagger(t) = v(t)$ for all $t \geq 0$. □

Lemma 5.10. *Suppose x is a solution of DDER satisfying $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$. If $a, b \in \mathbb{R}$ and $v, v^\dagger \in \mathcal{D}_{[-\tau,\infty)}$ are solutions of LVE relative to x and if $v^\ddagger$ is the* *unique solution of LVE relative to* x *with initial condition* $v_0^\ddagger := av_0 + bv_0^\dagger$*, then*

$$v^\ddagger(t) = av(t) + bv^\dagger(t) \text{ for all } t \in [-\tau, p+\eta_0] \setminus \{\ell_1^x\}. \tag{5.80}$$

Proof. Suppose that $a, b \in \mathbb{R}$ and $v, v^\dagger \in \mathcal{D}_{[-\tau,\infty)}$ are solutions of LVE relative to x. Let $v^\ddagger$ denote the unique solution of LVE relative to x satisfying $v_0^\ddagger = av_0 + bv_0^\dagger$. By (5.78), we have, for $t \in [0, \ell_1^x)$,

$$(av + bv^\dagger)(t) = (av + bv^\dagger)(0) + \int_0^t h'(x(s-\tau))((av + bv^\dagger)(s-\tau))ds.$$

By (5.78) and the fact that $v^\ddagger(t) = av(t) + bv^\dagger(t)$ for all $t \in [-\tau, 0]$, we have, for $t \in [0, \tau]$,

$$\begin{aligned}|v^\ddagger(t) - (av + bv^\dagger)(t)| &\le \int_0^t |h'(x(s-\tau))(v^\ddagger(s-\tau) - (av + bv^\dagger)(s-\tau))|ds \\ &= 0.\end{aligned}$$

We can iterate this argument for $t \in [\tau, \ell_1^x)$ to obtain that $v^\ddagger(t) = av(t) + bv^\dagger(t)$ for all $t \in [0, \ell_1^x)$. By (5.78), we have $v^\ddagger(t) = av(t) + bv^\dagger(t) = 0$ for all $t \in (\ell_1^x, q_1^x + \tau]$. Again by (5.78), we have, for $t \in [q_1^x + \tau, p + \eta_0]$,

$$(av + bv^\dagger)(t) = \int_{q_1^x+\tau}^t h'(x(s-\tau))((av + bv^\dagger)(s-\tau))ds.$$

By (5.78) and the fact that $v^\ddagger(t) = av(t) + bv^\dagger(t)$ at all but one $t \in [q_1^x, q_1^x + \tau]$, we have, for $t \in [q_1^x + \tau, q_1^x + 2\tau]$,

$$\begin{aligned}|v^\ddagger(t) - (av + bv^\dagger)(t)| &\le \int_{q_1^x+\tau}^t |h'(x(s-\tau))(v^\ddagger(s-\tau) - (av + bv^\dagger)(s-\tau))|ds \\ &= 0.\end{aligned}$$

Again iterating this argument for $t \in [q_1^x + 2\tau, p + \eta_0]$, we have that $v^\ddagger(t) = av(t) + bv^\dagger(t)$ for all $t \in [q_1^x + 2\tau, p + \eta_0]$ and the conclusion of the lemma follows. □

For a solution x of DDER, define $\dot{x} : [0, \infty) \to \mathbb{R}$ by

$$\dot{x}(t) = \begin{cases} h(x(t-\tau)), & \text{if } x(t) > 0, \\ 0, & \text{if } x(t) = 0. \end{cases} \tag{5.81}$$

By Lemma 2.2, $\dot{x}$ is equal to the derivative of x at the almost every $t \in [0, \infty)$ that x is differentiable. By Lemma 5.7, if x satisfies $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$, then $\dot{x}$ satisfies

$$\dot{x}(t) = \begin{cases} h(x(t-\tau)), & \text{for all } t \in [0, \ell_1^x), \\ 0, & \text{for all } t \in [\ell_1^x, q_1^x + \tau], \\ h(x(t-\tau)), & \text{for all } t \in (q_1^x + \tau, p + \eta_0]. \end{cases} \tag{5.82}$$

Since h and x are continuous and $q_1^x + \tau < p - \tau - \eta_0$, where η_0 is defined as in Lemma 5.7, we have that $\dot{x}_t \in \mathcal{C}_{[-\tau,0]}$ for all $t \in (p - \eta_0, p + \eta_0)$. For a SOPS x^*, we can uniquely define $\dot{x}^*$ on $[-\tau, \infty)$ by requiring that $\dot{x}^*$ satisfy $\dot{x}^*(p+t) = \dot{x}^*(t)$ for all $t \geq -\tau$.

Lemma 5.11. *The function $\dot{x}^*$ is a solution of LVE relative to x^*.*

Proof. To show that $\dot{x}^* \in \mathcal{D}_{[-\tau,\infty)}$, it suffices to show that $\dot{x}^*$ has finite left and right limits at each $t \in [0, p]$ (finite right limit at 0 and finite right limit at p). Since $x^*(\cdot)$ is positive on $[0, \ell_1)$ and on $(q_1 + \tau, p]$, we have that $\dot{x}^*(t) = h(x^*(t-\tau))$ for all $t \in [0, \ell_1) \cup (q_1 + \tau, p]$. By the continuity of h and x^*, $\dot{x}^*$ is continuous there. For $t \in [\ell_1, q_1 + \tau]$, $x^*(t) = 0$, so $\dot{x}^*(t) = 0$ there. It follows that $\dot{x}^*$ has finite left and right limits at each $t \in [0, p]$, so $\dot{x}^* \in \mathcal{D}_{[-\tau,\infty)}$. By Proposition 5.1, since $\dot{x}_0^* \in \mathcal{C}_{[-\tau,0]}$, there exists a unique solution of LVE with initial condition $\dot{x}_0^*$. Then by Lemma 5.9, it suffices to show that $\dot{x}^*$ satisfies (5.79) for all $t \geq 0$. Since x^* is absolutely continuous with its almost everywhere defined derivative equal to $\dot{x}^*$, we have $x^*(t_2) = x^*(t_1) + \int_{t_1}^{t_2} \dot{x}^*(s)ds$ for all $t_1, t_2 \in [-\tau, \infty)$.

For $t \in [0, \ell_1)$, $x^*(t) > 0$ so by (5.81), $\dot{x}^*(t) = h(x^*(t-\tau))$. Since h is continuously differentiable and x^* is positive and therefore continuously differentiable on $[-\tau, \ell_1 - \tau]$, it follows from the fundamental theorem of calculus and chain rule that

$$\dot{x}^*(t) = \dot{x}^*(0) + \int_0^t h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds, \ t \in [0, \ell_1). \tag{5.83}$$

By (5.83), the fact that $\dot{x}^*(t) = h(x^*(t-\tau))$ for all $t \in [0, \ell_1)$, the continuity of h and x^*, the negative feedback condition on h and the fact that $x^*(\ell_1 - \tau) > L$, we

have

$$\left(\dot{x}^*(0) + \int_0^{\ell_1} h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds\right)^+ = \left(\lim_{t\uparrow\ell_1} \dot{x}^*(t)\right)^+$$
$$= \left(\lim_{t\uparrow\ell_1} h(x^*(t-\tau))\right)^+$$
$$= [h(x^*(\ell_1-\tau))]^+$$
$$= 0.$$

By (5.81) and the fact that $x^*(\ell_1) = 0$, we also have $\dot{x}^*(\ell_1) = 0$. For $t \in (\ell_1, q_1+\tau]$, $x^*(t) = 0$, so by (5.81), $\dot{x}^*(t) = 0$. For $t \in (q_1+\tau, \ell_1+\tau)$, $x^*(t) > 0$, so by (5.81), $\dot{x}^*(t) = h(x^*(t-\tau))$. Since h is continuously differentiable, x^* is continuously differentiable on $[q_1, \ell_1)$ and $h(x^*(q_1)) = h(L) = 0$, it follows from the fundamental theorem of calculus and chain rule that for $t \in [q_1+\tau, \ell_1+\tau)$,

$$\begin{aligned}\dot{x}^*(t) &= h(x^*(t-\tau)) \\ &= h(x^*(q_1)) + \int_{q_1+\tau}^{t} h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds \\ &= \int_{q_1+\tau}^{t} h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds.\end{aligned} \tag{5.84}$$

At $t = \ell_1+\tau$, $x^*(\ell_1+\tau) > 0$, so by (5.81), $\dot{x}^*(\ell_1+\tau) = h(x^*(\ell_1))$. By the continuity of h and x^* and by (5.84), we have

$$\begin{aligned}\dot{x}^*(\ell_1+\tau) = \lim_{t\uparrow\ell_1} h(x^*(t)) &= \lim_{t\uparrow\ell_1+\tau} h(x^*(t-\tau)) \\ &= \lim_{t\uparrow\ell_1+\tau} \int_{q_1+\tau}^{t} h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds \\ &= \int_{q_1+\tau}^{\ell_1+\tau} h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds.\end{aligned} \tag{5.85}$$

For $t \in (\ell_1+\tau, q_1+2\tau]$, $x^*(t) > 0$, so by (5.81), $\dot{x}^*(t) = h(x^*(t-\tau))$. Since $x^*(t-\tau) = 0$ for all $t \in [\ell_1+\tau, q_1+2\tau]$, $\dot{x}^*(t) = h(x^*(t-\tau)) = h(0)$ for all $t \in [\ell_1+\tau, q_1+2\tau]$. By the continuity of h and x^*, (5.85) and the fact that

$\dot{x}^*(t-\tau) = 0$ for all $t \in [\ell_1+\tau, q_1+2\tau]$, we have, for $t \in [\ell_1+\tau, q_1+2\tau]$,

$$\begin{aligned} \dot{x}^*(t) &= \dot{x}^*(\ell_1+\tau) \\ &= \int_{q_1+\tau}^{\ell_1+\tau} h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds \\ &= \int_{q_1+\tau}^{t} h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds. \end{aligned} \tag{5.86}$$

For $t \in [q_1+2\tau, p]$, $x^*(t) > 0$, so by (5.81), $\dot{x}^*(t) = h(x^*(t-\tau))$. Since h is continuously differentiable and x^* is continuously differentiable on $[q_1+\tau, p-\tau]$, it follows from the fundamental theorem of calculus and (5.86) that for $t \in [q_1+2\tau, p]$,

$$\begin{aligned} \dot{x}^*(t) &= h(x^*(t-\tau)) \\ &= h(x^*(q_1+\tau)) + \int_{q_1+2\tau}^{t} h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds \\ &= \int_{q_1+\tau}^{t} h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds \end{aligned}$$

Therefore $\dot{x}^*$ satisfies (5.79) for $k=0$. Since both x^* and $\dot{x}^*$ are periodic with period p, we can repeat the previous argument to observe that $\dot{x}^*$ also satisfies (5.79) for all $k = 1, 2, \ldots$, which concludes the proof of the lemma. □

Lemma 5.12. *Let v be a solution of LVE relative to x^* and define $\hat{v} \in \mathcal{D}_{[-1,\infty)}$ by $\hat{v}(t) = v(\tau t)$ for all $t \geq -1$. Then $\hat{v}$ satisfies*

$$\hat{v}(t) = \begin{cases} \hat{v}(0) + \tau \int_0^t \hat{h}'(\hat{x}^*(s-1))\hat{v}(s-1)ds, & t \in [0, \hat{\ell}_1), \\ \left(\hat{v}(0) + \tau \int_0^t \hat{h}'(\hat{x}^*(s-1))\hat{v}(s-1)ds\right)^+, & t = \hat{\ell}_1, \\ 0, & t \in (\hat{\ell}_1, \hat{q}_1+1], \\ \tau \int_{\hat{q}_1+1}^t \hat{h}'(\hat{x}^*(s-1))\hat{v}(s-1)ds, & t \in [\hat{q}_1+1, \hat{p}+\hat{\eta}_0]. \end{cases} \tag{5.87}$$

where $\hat{h}$ is as in (5.1), $\hat{x}^*$ is as in Lemma 4.1, $\hat{\ell}_1 = \tau^{-1}\ell_1$, $\hat{q}_1 = \tau^{-1}q_1$, $\hat{p} = \tau^{-1}p$ and $\hat{\eta}_0 = \tau^{-1}\eta_0$.

Proof. If v is a solution of LVE relative to x^*, then by (5.78),

$$v(\tau t) = \begin{cases} v(0) + \int_0^{\tau t} h'(x^*(s-\tau))v(s-\tau)ds, & \tau t \in [0, \ell_1), \\ \left(v(0) + \int_0^{\ell_1} h'(x^*(s-\tau))v(s-\tau)ds\right)^+, & \tau t = \ell_1, \\ 0, & \tau t \in (\ell_1, q_1+\tau], \\ \int_{q_1+\tau}^{\tau t} h'(x^*(s-\tau))v(s-\tau)ds, & \tau t \in [q_1+\tau, p+\eta_0], \end{cases}$$

By substituting expressions in terms of $\hat{v}$, $\hat{h}$, $\hat{x}^*$, $\hat{q}_1$, $\hat{\ell}_1$, $\hat{p}$ and $\hat{\eta}_0$, for v, h, x^*, q_1, ℓ_1, p and η_0, we obtain (5.87). □

Lemma 5.13. *There exist a constant $M_h > 0$, which does not depend on τ, and a constant $\tau_3 \geq \tau_2$ such that if $\tau \geq \tau_3$ and v is a solution of LVE relative to x^*, then*

$$\|v\|_{[-\tau,p]} \leq M_h \|v\|_{[-\tau,0]}. \tag{5.88}$$

Proof. Let $\delta = \frac{1}{4}\min(\bar{q}, 1)$. By Theorem 5.1, Corollary 5.1 and (5.3), there exist constants $\tau_3 \geq \tau_2$ and $d > 0$ such that if the fixed $\tau \geq \tau_3$, then the following hold:

(i) $\bar{q} - \delta < \hat{q}_1 < \hat{\ell}_1 < \bar{q} + \delta < \bar{q} + 1 - \delta < \hat{q}_2 < \bar{q} + 1 + \delta$,

(ii) $\hat{x}^*(t) > d$ for all $t \in [-1+\delta, \bar{q}-\delta]$, and

(iii) $\hat{x}^*(t) = -L$ and $\hat{h}(\hat{x}^*(t)) = \beta$ for all $t \in [\bar{q}+\delta, \hat{q}_1+1]$.

Suppose $\tau \geq \tau_3$ and let v be a solution of LVE relative to x^*. If we define $\hat{x}^*$ as in Lemma 4.1 and $\hat{v}$ as in Lemma 5.12, then $\hat{x}^*$ is a SOPS^{n} that satisfies (i)-(iii) and $\|v\|_{[-\tau,\tau t]} = \|\hat{v}\|_{[-1,t]}$ for all $t \geq 0$. Thus, letting $\hat{p} = \tau^{-1}p$, we have that (5.88) is equivalent to

$$\|\hat{v}\|_{[-1,\hat{p}]} \leq M_h \|\hat{v}\|_{[-1,0]}. \tag{5.89}$$

By (5.87), for $t \in [0, \delta]$,

$$\begin{aligned} |\hat{v}(t)| &\leq |\hat{v}(0)| + \tau \int_0^t |h'(\hat{x}^*(s-1))\hat{v}(s-1)|ds \\ &\leq \|\hat{v}\|_{[-1,0]} \left(1 + \tau \int_{\hat{q}_2}^{\hat{q}_2+t} |\hat{h}'(\hat{x}^*(s))|ds\right). \end{aligned} \tag{5.90}$$

Here we have used the fact that $\hat{x}^*(s-1) = \hat{x}^*(\hat{q}_2 + s)$ for all $s \geq 0$. If $\hat{\ell}_1 + 1 \leq \hat{q}_2$, then by (i) and the definition of δ, $[\hat{q}_2, \hat{q}_2 + \delta] \subset [\hat{\ell}_1 + 1, \hat{q}_1 + 2]$. Since $\hat{x}^*(t) > -L$ and $\hat{x}^*(t-1) = -L$ for all $t \in [\hat{q}_2, \hat{q}_2 + \delta]$, it follows from Lemma 4.2 that $\hat{x}^*$ is differentiable on the interval with derivative

$$\frac{d\hat{x}^*(t)}{dt} = \tau \hat{h}(\hat{x}^*(t-1)) = \tau \hat{h}(-L) = \tau\beta, \; t \in [\hat{q}_2, \hat{q}_2 + \delta]. \tag{5.91}$$

Hence $\hat{x}^*(\hat{q}_2 + t) = \tau\beta t$ for all $t \in [0, \delta]$. By (5.90), for $t \in [0, \delta]$,

$$\begin{aligned} |\hat{v}(t)| &\leq \|\hat{v}\|_{[-1,0]} \left(1 + \tau \int_0^t |\hat{h}'(\tau\beta s)| ds\right) \\ &\leq \|\hat{v}\|_{[-1,0]} \left(1 + \frac{1}{\beta} \int_0^{\tau\beta t} |\hat{h}'(u)| du\right) \\ &\leq \|\hat{v}\|_{[-1,0]} \left(1 + \frac{1}{\beta} \|\hat{h}'\|_{L^1([-L,\infty))}\right), \end{aligned} \tag{5.92}$$

where

$$\|h'\|_{L^1([-L,\infty))} = \int_{-L}^{\infty} |\hat{h}'(s)| ds < \infty$$

is finite by Assumption 3.4. The second inequality follows from performing the substitution $u = \tau\beta s$. Alternatively, if $\hat{q}_2 \leq \hat{\ell}_1 + 1$, then by (5.6), (5.90)–(5.91), Corollary 5.2 and (iii), we have, for $t \in [0, \delta]$,

$$\begin{aligned} |\hat{v}(t)| &\leq \|\hat{v}\|_{[-1,0]} \left(1 + \tau \int_{\hat{q}_2}^{\hat{\ell}_1+1} |\hat{h}'(\hat{x}^*(s))| ds + \tau \int_{\hat{\ell}_1+1}^{\hat{q}_2+\delta} |\hat{h}'(\hat{x}^*(s))| ds\right) \\ &\leq \|\hat{v}\|_{[-1,0]} \left(1 + \tau K_h |\hat{\ell}_1 - \hat{q}_1| + \tau \int_0^{\delta} |\hat{h}'(\hat{x}^*(\hat{\ell}_1 + 1) + \tau\beta s)| ds\right) \\ &\leq \|\hat{v}\|_{[-1,0]} \left(1 + \frac{2LK_h}{\alpha} + \frac{1}{\beta} \int_{\hat{x}^*(\hat{\ell}_1+1)}^{\hat{x}^*(\hat{\ell}_1+1)+\tau\beta\delta} |\hat{h}'(u)| du\right) \\ &\leq \|\hat{v}\|_{[-1,0]} \left(1 + \frac{2LK_h}{\alpha} + \frac{1}{\beta} \|\hat{h}'\|_{L^1([-L,\infty))}\right). \end{aligned} \tag{5.93}$$

Here the third inequality follows from performing the substitution $u = \hat{x}^*(\hat{\ell}_1 + 1) + \tau\beta s$. Taking the supremum over t in the interval $[-1, \delta]$, using (5.92)–(5.93), we have the following bound, which does not depend on the relation between $\hat{\ell}_1 + 1$ and $\hat{q}_2$,

$$\|\hat{v}\|_{[-1,\delta]} \leq \left(1 + \frac{2LK_h}{\alpha} + \frac{1}{\beta} \|\hat{h}\|_{L^1([-L,\infty))}\right) \|\hat{v}\|_{[-1,0]}. \tag{5.94}$$

By (i) and the definition of δ, $[-1+\delta, \hat{\ell}_1 - 1] \subset [-1+\delta, \bar{q}-\delta]$ and so the property in (ii) holds on the interval. If $\hat{\ell}_1 \leq \delta + 1$, then by (5.87), we have, for $t \in [\delta, \hat{\ell}_1]$,

$$
\begin{aligned}
|\hat{v}(t)| &\leq |\hat{v}(\delta)| + \tau \int_{\delta}^{t} |h'(\hat{x}^*(s-1))\hat{v}(s-1)| ds \\
&\leq |\hat{v}(\delta)| + \left(\tau \int_{-1+\delta}^{t-1} \frac{|h'(\hat{x}^*(s))\hat{x}^*(s)|}{|\hat{x}^*(s)|} |\hat{v}(s)| ds \right) \\
&\leq |\hat{v}(\delta)| + \frac{m}{d} \int_{-1+\delta}^{\delta} |\hat{v}(s)| ds \\
&\leq \|\hat{v}\|_{[-1+\delta,\delta]} \left(1 + \frac{m}{d}\right),
\end{aligned}
$$

where $m = \sup_{s\in[-L,\infty)} |s\hat{h}'(s)|$ is finite by Assumption 3.4. Alternatively, if $\hat{\ell}_1 > \delta + 1$, then by (5.87) and Gronwall's inequality, we have, for $t \in [\delta, \hat{\ell}_1]$,

$$
\begin{aligned}
|\hat{v}(t)| &\leq |\hat{v}(\delta)| + \tau \int_{\delta}^{t} |h'(\hat{x}^*(s-1))\hat{v}(s-1)| ds \\
&\leq |\hat{v}(\delta)| + \left(\tau \int_{-1+\delta}^{t-1} \frac{|h'(\hat{x}^*(s))\hat{x}^*(s)|}{|\hat{x}^*(s)|} |\hat{v}(s)| ds \right) \\
&\leq |\hat{v}(\delta)| + \frac{m}{d} \int_{-1+\delta}^{(t-1)\wedge\delta} |\hat{v}(s)| ds + \frac{m}{d} \int_{\delta}^{(t-1)\vee\delta} |\hat{v}(s)| ds \\
&\leq \|\hat{v}\|_{[-1+\delta,\delta]} \left(1 + \frac{m}{d}\right) \exp\left(\frac{m}{d}(t-1-\delta)^+\right).
\end{aligned}
$$

By (5.87), $\hat{v}(t) = 0$ for all $t \in (\hat{\ell}_1, \hat{q}_1 + 1]$. Taking the supremum over t in the interval $[-1, \hat{q}_1 + 1]$, we have

$$
\|\hat{v}\|_{[-1,\hat{q}_1+1]} \leq \left(1 + \frac{m}{d}\right) \exp\left(\frac{m}{d}(\hat{\ell}_1 - 1 - \delta)^+\right) \|\hat{v}\|_{[-1,\delta]}. \tag{5.95}
$$

By (5.87) and Corollary 5.2, we have, for $t \in [\hat{q}_1 + 1, \hat{\ell}_1 + 1]$,

$$
\begin{aligned}
|\hat{v}(t)| &\leq \tau \int_{\hat{q}_1+1}^{t} |\hat{h}'(x(s-1))\hat{v}(s-1)| ds \\
&\leq \tau K_h \|\hat{v}\|_{[\hat{q}_1,\hat{\ell}_1]} |\hat{\ell}_1 - \hat{q}_1| \\
&\leq \frac{2LK_h}{\alpha} \|\hat{v}\|_{[\hat{q}_1,\hat{\ell}_1]}.
\end{aligned}
$$

For $t \in (\hat{\ell}_1 + 1, \hat{q}_1 + 2]$, $\hat{v}(t-1) = 0$, so by (5.87),

$$
|\hat{v}(t)| = |\hat{v}(\hat{\ell}_1 + 1)| \leq \frac{2LK_h}{\alpha} \|\hat{v}\|_{[\hat{q}_1,\hat{\ell}_1]}.
$$

Taking the supremum over t in the interval $[-1, \hat{q}_1+2]$, we have

$$\|\hat{v}\|_{[-1,\hat{q}_1+2]} \le \frac{2LK_h}{\alpha}\|\hat{v}\|_{[-1,\hat{\ell}_1]}. \tag{5.96}$$

By (5.87) and Corollary 5.2, we have, for $t \in [\hat{q}_1+2, (\hat{\ell}_1+2)\wedge\hat{p}]$,

$$\begin{aligned}|\hat{v}(t)| &\le |\hat{v}(\hat{q}_1+2)| + \tau\int_{\hat{q}_1+2}^{(\hat{\ell}_1+2)\wedge\hat{p}} |h'(\hat{x}(s-1))\hat{v}(s-1)|ds \\ &\le |\hat{v}(\hat{q}_1+2)| + \tau K_h\|\hat{v}\|_{[\hat{q}_1+1,\hat{\ell}_1+1]}|\hat{\ell}_1-\hat{q}_1| \\ &\le |\hat{v}(\hat{q}_1+2)| + \frac{2LK_h}{\alpha}\|\hat{v}\|_{[\hat{q}_1+1,\hat{\ell}_1+1]}.\end{aligned}$$

Taking the supremum over t in the interval $[-1, (\hat{\ell}_1+2)\wedge\hat{p}]$, we have

$$\|\hat{v}\|_{[-1,(\hat{\ell}_1+2)\wedge\hat{p}]} \le \left(1+\frac{2LK_h}{\alpha}\right)\|\hat{v}\|_{[-1,\hat{q}_1+2]}. \tag{5.97}$$

If $\hat{\ell}_1+2 < \hat{p}$, then consider the interval $[\hat{\ell}_1+2, \hat{p}]$. By (i)–(ii), if $t \in [\hat{\ell}_1+2, \hat{p}]$, then $t-1 \in [\hat{\ell}_1+1, \hat{q}_2] \subset [\hat{\ell}_1+1, \hat{q}_1+2]$. By Lemma 4.2, $\hat{x}^*$ is differentiable on $[\hat{\ell}_1+1, \hat{q}_1+2]$ with $\frac{d\hat{x}^*(t)}{dt} = \tau h(\hat{x}^*(t-1)) = \tau\beta$. By (5.87), we have, for $t \in [\hat{\ell}_1+2, \hat{p}]$,

$$\begin{aligned}|\hat{v}(t)| &\le |\hat{v}(\hat{\ell}_1+2)| + \tau\int_{\hat{\ell}_1+2}^{t} |\hat{h}'(\hat{x}^*(s-1))\hat{v}(s-1)|ds \\ &\le |\hat{v}(\hat{\ell}_1+2)| + \tau\|\hat{v}\|_{[\hat{\ell}_1+1,t-1]}\int_0^{(t-\hat{\ell}_1-2)^+} |\hat{h}'(\hat{x}^*(\hat{\ell}_1+1)+\tau\beta s)|ds \\ &\le |\hat{v}(\hat{\ell}_1+2)| + \frac{1}{\beta}\|\hat{v}\|_{[\hat{\ell}_1+1,t-1]}\int_{\hat{x}^*(\hat{\ell}_1+1)}^{\hat{x}^*(\hat{\ell}_1+1)+\tau\beta(\hat{p}-\hat{\ell}_1-2)} |\hat{h}'(u)|du \\ &\le |\hat{v}(\hat{\ell}_1+2)| + \frac{1}{\beta}\|h'\|_{L^1([-L,\infty))}\|\hat{v}\|_{[\hat{\ell}_1+1,\hat{q}_2]}.\end{aligned}$$

The third inequality follows by performing the substitution $u = \hat{x}^*(\hat{\ell}_1+1)+\tau\beta s$ and for the fourth inequality we have used that $\hat{q}_2 = \hat{p}-1$. Taking the supremum over t in the interval $[-1,\hat{p}]$ and using (5.94)–(5.97), we have

$$\|\hat{v}\|_{[-1,\hat{p}]} \le \left(1+\frac{1}{\beta}\|\hat{h}\|_{L^1([-L,\infty))}\right)\|\hat{v}\|_{[-1,\hat{q}_2\vee(\hat{\ell}_1+2)]} \le M_h\|\hat{v}\|_{[-1,0]},$$

where

$$\begin{aligned}M_h = &\left(1+\frac{1}{\beta}\|\hat{h}\|_{L^1([-L,\infty))}\right)\left(1+\frac{2LK_h}{\alpha}\right)\frac{2LK_h}{\alpha}\left(1+\frac{m}{d}\right) \\ &\cdot\exp\left(\frac{m}{d}(\hat{\ell}_1-1-\delta)^+\right)\left(1+\frac{2LK_h}{\alpha}+\frac{1}{\beta}\|\hat{h}\|_{L^1([-L,\infty))}\right)\|\hat{v}\|_{[-1,0]}.\end{aligned}$$

□

Lemma 5.14. *Suppose x is a solution of DDER satisfying $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$. For each $\delta > 0$, there exists $\varepsilon > 0$ such that whenever $x^\dagger$ is a solution of DDER satisfying $\|x - x^\dagger\|_{[-\tau,0]} < \varepsilon$, and $\psi \in \mathcal{C}_{[-\tau,0]}$ satisfies $\|\psi\|_{[-\tau,0]} \leq 1$, then*

$$\|v^x - v^{x^\dagger}\|_{[p-\tau-\eta_0, p+\eta_0]} < \delta, \tag{5.98}$$

where v^x and $v^{x^\dagger}$ denote the unique solutions of LVE relative to x and $x^\dagger$, respectively, with initial conditions $v_0^x = v_0^{x^\dagger} = \psi$.

Proof. Fix $\delta > 0$. Define $\varepsilon_1 = \varepsilon_1(x) > 0$ and $\eta_1 = \eta_1(x) > 0$ as in (5.58)–(5.59). Let

$$M_1 = 2(p + \eta_0)\exp(2K_h(p + \eta_0)) + 2K_h(\ell_1 + \eta_0)^2 \exp(3K_h(\ell_1 + \eta_0)), \tag{5.99}$$

$$M_2 = 2K_h \exp(2K_h(p + \eta_0)). \tag{5.100}$$

For a solution $x^\dagger$ of DDER and $t \geq 0$, define

$$d_{h'}(x, x^\dagger, t) = \sup_{s \in [0,t]} |h'(x(s-\tau)) - h'(x^\dagger(s-\tau))|. \tag{5.101}$$

Note that $d_{h'}(x, x^\dagger, \cdot)$ is an increasing function and by (5.46) and the continuity of h', we have that for each fixed $t \geq 0$, $d_{h'}(x, x^\dagger, t) \to 0$ as $\|x - x^\dagger\|_{[-\tau,0]} \to 0$.

Choose $\eta \in (0, \eta_1)$ such that

$$\eta < \frac{\delta}{18M_2 \exp(K_h(q_2 - q_1 + 2\eta_0))}. \tag{5.102}$$

Given η, we can choose $\varepsilon \in (0, \varepsilon_1)$ such that the conclusion of Lemma 5.8 holds and

$$d_{h'}(x, x^\dagger, q_2) < \frac{\delta}{4M_1 \exp(K_h(q_2 - q_1 + 2\eta_0))} \tag{5.103}$$

whenever $\|x^\dagger - x\|_{[-\tau,0]} < \varepsilon$.

Define z as in (2.2). Let $\psi \in \mathcal{C}_{[-\tau,0]}$ such that $\|\psi\|_{[-\tau,0]} \leq 1$ and let $v^x, v^{x^\dagger}$ be defined as in the lemma. By Definition 5.1, we have that v^x satisfies (5.71)–(5.75), but with v^x and w^x in place of v and w, respectively. Similarly, $v^{x^\dagger}$ satisfies (5.71)–(5.75), but with $x^\dagger, z^\dagger, v^{x^\dagger}$ and $w^{x^\dagger}$ in place of x, z, v and w, respectively. By (C.8) (with $v^\dagger \equiv 0$ and v^x, $v^{x^\dagger}$ in place of v), we have that v^x, $v^{x^\dagger}$ are bounded on compact intervals by

$$\|v^x\|_{[-\tau,t]} \leq 2\exp(2K_h t) \text{ and } \|v^{x^\dagger}\|_{[-\tau,t]} \leq 2\exp(2K_h t) \text{ for all } t \geq 0. \tag{5.104}$$

Here we have used the fact that $\|v^x\|_{[-\tau,0]} = \|v^{x^\dagger}\|_{[-\tau,0]} = \|\psi\|_{[-\tau,0]} \le 1$.

We first consider $|v^x(t) - v^{x^\dagger}(t)|$ when $t \in [0, \ell_1^x - \eta]$. By Lemmas 5.7–5.8, $\ell_1^{x^\dagger} > \ell_1^x - \eta$ and so $\overline{-z}(t) < 0$ and $\overline{-z^\dagger}(t) < 0$ for all $t \in [0, \ell_1^x - \eta]$. It follows from (5.74) that $R(-z, -w^x)(t) = R(-z^\dagger, -w^{x^\dagger})(t) = 0$ for all $t \in [0, \ell_1^x - \eta]$. By (5.71)–(5.72), (5.101), (5.104), (3.12) and the fact that $v_0^x = v_0^{x^\dagger} = \psi$, we have, for $t \in [0, \ell_1^x - \eta]$,

$$\begin{aligned}
|v^x(t) - v^{x^\dagger}(t)| &= |w^x(t) - w^{x^\dagger}(t)| \\
&\le \int_0^t |h'(x(s-\tau))v^x(s-\tau) - h'(x^\dagger(s-\tau))v^{x^\dagger}(s-\tau)|ds \\
&\le \int_0^t |h'(x(s-\tau)) - h'(x^\dagger(s-\tau))| \cdot |v^x(s-\tau)|ds \\
&\qquad + \int_\tau^t |h'(x^\dagger(s-\tau))| \cdot |v^x(s-\tau) - v^{x^\dagger}(s-\tau)|ds \\
&\le d_{h'}(x, x^\dagger, t)2t\exp(2K_h t) + K_h \int_0^t |v^x(s) - v^{x^\dagger}(s)|ds.
\end{aligned}$$

Applying Gronwall's inequality, we obtain

$$|v^x(t) - v^{x^\dagger}(t)| \le d_{h'}(x, x^\dagger, t)2t\exp(3K_h t),\ t \in [0, \ell_1^x - \eta]. \tag{5.105}$$

Second, we consider $|v^x(t) - v^{x^\dagger}(t)|$ when $t \in [\ell_1^x - \eta, \ell_1^x + \eta]$. By (5.104), we have,

$$|v^x(t) - v^{x^\dagger}(t)| \le \|v^x\|_{[0,t]} + \|v^{x^\dagger}\|_{[0,t]} \le 4\exp(2K_h(\ell_1^x + \eta)). \tag{5.106}$$

Third, we consider $|v^x(t) - v^{x^\dagger}(t)|$ when $t \in [\ell_1^x + \eta, q_1^x + \tau - \eta]$. By Lemmas 5.7–5.8, $\ell_1^{x^\dagger} < \ell_1^x + \eta < q_1^x + \tau - \eta < q_1^{x^\dagger} + \tau$ and so we have that $\mathbb{S}_{-z}(t) = \mathbb{S}_{-z^\dagger}(t) = \{t\}$ and $\overline{-z}(t), \overline{-z^\dagger}(t) > 0$ for all $t \in [\ell_1^x + \eta, q_1^x + \tau - \eta]$. Then by (5.74), $R(-z, -w^x) = -w^x(t)$ and $R(-z^\dagger, -w^{x^\dagger}) = -w^{x^\dagger}(t)$, so by (5.71)–(5.73), $v^x(t) = v^{x^\dagger}(t) = 0$ for all $t \in [\ell_1^x + \eta, q_1^x + \tau - \eta]$.

Fourth, we consider $|v^x(t) - v^{x^\dagger}(t)|$ when $t \in [q_1^x + \tau - \eta, q_1^x + \tau + \eta]$. By (5.104), we have

$$|v^x(t) - v^{x^\dagger}(t)| \le \|v^x\|_{[0,t]} + \|v^{x^\dagger}\|_{[0,t]} \le 4\exp(2K_h(q_1^x + \tau + \eta)), \tag{5.107}$$

Fifth, we consider $|v^x(t) - v^{x^\dagger}(t)|$ when $t \in [q_1^x + \tau + \eta, p + \eta_0]$. By (5.71)–(5.73),

$$|v^x(t) - v^{x^\dagger}(t)| \le |w^x(t) - w^{x^\dagger}(t)| + |R(-z, -w^x)(t) - R(-z^\dagger, -w^{x^\dagger})(t)|. \tag{5.108}$$

By (5.72), the first term on the right hand side of (5.108) satisfies

$$
\begin{aligned}
|w^x(t) - w^{x^\dagger}(t)| &\leq \int_0^t |h'(x(s-\tau)) - h'(x^\dagger(s-\tau))| \cdot |v^x(s-\tau)| ds \\
&\quad + \int_0^t |h'(x^\dagger(s-\tau))| \cdot |v^x(s-\tau) - v^{x^\dagger}(s-\tau)| ds \\
&\leq d_{h'}(x, x^\dagger, t) 2t \exp(2K_h t) + K_h \int_0^{\ell_1^x - \eta} |v^x(s) - v^{x^\dagger}(s)| ds \\
&\quad + K_h \int_{\ell_1^x - \eta}^{\ell_1^x + \eta} |v^x(s) - v^{x^\dagger}(s)| ds \\
&\quad + K_h \int_{q_1^x + \tau - \eta}^{q_1^x + \tau + \eta} |v^x(s) - v^{x^\dagger}(s)| ds \\
&\quad + K_h \int_{q_1^x + \tau + \eta}^{t} |v^x(s) - v^{x^\dagger}(s)| ds \\
&\leq d_{h'}(x, x^\dagger, t) 2t \exp(2K_h t) \\
&\quad + d_{h'}(x, x^\dagger, \ell_1^x - \eta) 2K_h (\ell_1^x - \eta)^2 \exp(3K_h(\ell_1^x - \eta)) \\
&\quad + 8K_h \eta \exp(2K_h(\ell_1^x + \eta)) + 8K_h \eta \exp(2K_h(q_1^x + \tau + \eta)) \\
&\quad + K_h \int_{q_1^x + \tau + \eta}^{t} |v^x(s) - v^{x^\dagger}(s)| ds \\
&\leq M_1 d_{h'}(x, x^\dagger, t) + 8M_2 \eta \\
&\quad + K_h \int_{q_1^x + \tau + \eta}^{t} |v^x(s) - v^{x^\dagger}(s)| ds.
\end{aligned} \tag{5.109}
$$

The second inequality follows from (5.104), (3.12) and because $v_0^x = v_0^{x^\dagger} = \psi$ and $v^x(s) = v^{x^\dagger}(s) = 0$ for all $s \in [\ell_1^x + \eta, q_1^x + \tau - \eta]$. The third inequality follows from (5.105)–(5.107). The constants M_1 and M_2 are defined in (5.99)–(5.100). By

(5.74), the second term on the right hand side of (5.108) satisfies

$$\begin{aligned}|R(-z,-w^x)(t)-R(-z^\dagger,-w^{x^\dagger})(t)| &\le |R(-z,-w^x)(t)-R(-z,-w^{x^\dagger})(t)| \qquad (5.110)\\ &\quad + |R(-z,-w^{x^\dagger})(t)-R(-z^\dagger,-w^{x^\dagger})(t)|\\ &\le |w^x(q_1^x+\tau)-w^{x^\dagger}(q_1^x+\tau)|\\ &\quad + |w^{x^\dagger}(q_1^x+\tau)-w^{x^\dagger}(q_1^{x^\dagger}+\tau)|\\ &\le |w^x(q_1^x+\tau)-w^{x^\dagger}(q_1^x+\tau)|\\ &\quad + \eta K_h\|v^{x^\dagger}\|_{[-\tau,q_1+\eta_0]}\\ &\le |w^x(q_1^x+\tau)-w^{x^\dagger}(q_1^x+\tau)| + M_2\eta.\end{aligned}$$

The second inequality follows from (5.74) and Lemma 5.7. The third inequality follows from (5.72) and because $|q_1^x - q_1^{x^\dagger}| < \eta$. The fourth inequality follows from (5.104) and the definition of M_2. By (5.72),

$$\begin{aligned}|w^x(q_1^x+\tau)-w^{x^\dagger}(q_1^x+\tau)| &\le \int_0^{q_1^x+\tau} |h'(x(s-\tau))-h'(x^\dagger(s-\tau))|\cdot|v^x(s-\tau)|ds \qquad (5.111)\\ &\quad + \int_0^{q_1^x+\tau} |h'(x^\dagger(s-\tau))||v^x(s-\tau)-v^{x^\dagger}(s-\tau)|ds\\ &\le d_{h'}(x,x^\dagger,q_1^x+\tau)2(q_1^x+\tau)\exp(2K_h(q_1^x+\tau))\\ &\quad + K_h\int_0^{q_1^x+\tau} |v^x(s-\tau)-v^{x^\dagger}(s-\tau)|ds\\ &\le d_{h'}(x,x^\dagger,q_1^x+\tau)2(q_1^x+\tau)\exp(2K_h(q_1^x+\tau))\\ &\quad + d_{h'}(x,x^\dagger,q_1^x+\tau)2K_h(q_1^x+\tau)^2\exp(3K_h(q_1^x+\tau))\\ &\le M_1 d_{h'}(x,x^\dagger,q_1^x+\tau).\end{aligned}$$

The second inequality follows from (5.101), (5.104), (3.12) and uses the fact that $v_0^x = v_0^{x^\dagger} = \psi$. The third inequality follows from (5.105) and because $q_1^x < \ell_1^x - \eta$. By (5.108)–(5.111), for $t \in [q_1^x+\tau+\eta, p+\eta_0]$,

$$|v^x(t)-v^{x^\dagger}(t)| \le 2M_1 d_{h'}(x,x^\dagger,q_1^x+\tau) + 9M_2\eta + K_h\int_{q_1^x+\tau+\eta}^t |v^x(s)-v^{x^\dagger}(s)|ds.$$

Applying Gronwall's inequality, we obtain

$$|v^x(t)-v^{x^\dagger}(t)| \le (2M_1 d_{h'}(x,x^\dagger,q_1^x+\tau)+9M_2\eta)\exp(K_h(t-q_1^x-\tau-\eta)), \quad (5.112)$$

for all $t \in [q_1^x + \tau + \eta, p + \eta_0]$. By Lemma 5.8, we have that $q_1 + \tau < q_1^x + \tau + \eta < q_1 + \tau + \eta_0 < q_2 - \eta_0 = p - \tau - \eta_0$. Then by (5.112) and (5.102)–(5.103), we have

$$\begin{aligned}\|v^x - v^{x^\dagger}\|_{[p-\tau-\eta_0, p+\eta_0]} &\leq \|v^x - v^{x^\dagger}\|_{[q_1^x+\tau+\eta, p+\eta_0]} \\ &\leq (2M_1 d_{h'}(x, x^\dagger, q_2) + 9M_2\eta)\exp(K_h(q_2 - q_1 + 2\eta_0)) \\ &< \delta.\end{aligned}$$

□

5.5 Semiflow and Approximate Poincaré Map

In this section we define a semiflow and an approximate Poincaré map that will be used to prove the uniform exponential asymptotic stability of a SOPS. Throughout this section we fix a $\tau \geq \tau_3$, where τ_3 is defined as in Lemma 5.13, and we use x^* to denote the associated SOPS x^τ defined in Section 5.2.

Define the semiflow $\Sigma : \mathbb{R}_+ \times \mathcal{C}^+_{[-\tau,0]} \to \mathcal{C}^+_{[-\tau,0]}$ by

$$\Sigma(t, \varphi) = x_t, \tag{5.113}$$

where x is the unique solution of DDER with initial condition φ. Note that

$$\Sigma(t, \Sigma(s, \varphi)) = \Sigma(t, x_s) = x_{s+t} = \Sigma(s + t, \varphi), \tag{5.114}$$

by time-homogeneity of DDER and uniqueness of solutions to DDER.

Lemma 5.15. *The semiflow $\Sigma : \mathbb{R}_+ \times \mathcal{C}^+_{[-\tau,0]} \to \mathcal{C}^+_{[-\tau,0]}$ is continuous.*

Proof. Suppose that (t, φ) and $(t^\dagger, \varphi^\dagger)$ are in $\mathbb{R}_+ \times \mathcal{C}^+_{[-\tau,0]}$. Let x and $x^\dagger$ denote the unique solutions of DDER with respective initial conditions φ and $\varphi^\dagger$. By (5.46), for $t \geq 0$,

$$\|\Sigma(t, \varphi) - \Sigma(t, \varphi^\dagger)\|_{[-\tau,0]} \leq \|x - x^\dagger\|_{[-\tau,t]} \leq 2\exp(2K_h t)\|\varphi - \varphi^\dagger\|_{[-\tau,0]}. \tag{5.115}$$

By (3.10) with $x^\dagger$ in place of x, for $t, t^\dagger \geq 0$,

$$\begin{aligned}\|\Sigma(t, \varphi^\dagger) - \Sigma(t^\dagger, \varphi^\dagger)\|_{[-\tau,0]} &= \sup_{u\in[-\tau,0]} |x^\dagger(t + u) - x^\dagger(t^\dagger + u)| \\ &\leq H|t - t^\dagger|.\end{aligned} \tag{5.116}$$

Joint continuity then follows from (5.115)–(5.116) and the triangle inequality. □

Recall $\eta_0 > 0$ and $\varepsilon_0 > 0$ as introduced in Lemma 5.7. Define the neighborhood $\mathcal{U}$ of (p, x_0^*) in the Banach space $\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$ (with norm $\|(t,\varphi)\| = |t|\vee\|\varphi\|_{[-\tau,0]}$) by

$$\mathcal{U} = \left\{(t,\varphi) \in \mathbb{R}\times\mathcal{C}_{[-\tau,0]} : |t-p| < \eta_0 \text{ and } \|\varphi - x_0^*\|_{[-\tau,0]} < \varepsilon_0\right\}. \tag{5.117}$$

By the definition of η_0 and ε_0, we have $\mathcal{U} \subset \mathbb{R}_+ \times \mathcal{C}^+_{[-\tau,0]}$.

Lemma 5.16. *The semiflow Σ is continuously Fréchet differentiable on $\mathcal{U}$ and for each $(t,\varphi)\in\mathcal{U}$, the derivative $D\Sigma(t,\varphi) \in \mathcal{L}(\mathbb{R}\times\mathcal{C}_{[-\tau,0]}, \mathcal{C}_{[-\tau,0]})$ at (t,φ) is given by*

$$D\Sigma(t,\varphi)(s,\psi) = s\dot{x}_t + v_t, \quad \textit{for all } (s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}. \tag{5.118}$$

Here x denotes the unique solution of DDER with initial condition φ, $\dot{x}$ is defined as in (5.81) *and v denotes the unique solution of LVE relative to x with initial condition ψ.*

Remark 5.1. Note that if $(t,\varphi)\in\mathcal{U}$ and x is the unique solution of DDER with initial condition φ, then by Lemma 5.7 and the definition of $\mathcal{U}$, $x_t(s) > 0$ for all $s\in[-\tau,0]$. Thus, by (5.81), $\dot{x}_t = \{h(x(t+s-\tau)) : -\tau\le s\le 0\} \in \mathcal{C}_{[-\tau,0]}$.

Proof. For each $(t,\varphi)\in\mathcal{U}$, define the operator $F(t,\varphi):\mathbb{R}\times\mathcal{C}_{[-\tau,0]}\to\mathcal{C}_{[-\tau,0]}$ by

$$F(t,\varphi)(s,\psi) = s\dot{x}_t + v_t,\ (s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}. \tag{5.119}$$

We first show that for each $(t,\varphi)\in\mathcal{U}$, $F(t,\varphi)$ is a bounded linear operator from $\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$ into $\mathcal{C}_{[-\tau,0]}$, i.e., $F(t,\varphi)\in\mathcal{L}(\mathbb{R}\times\mathcal{C}_{[-\tau,0]},\mathcal{C}_{[-\tau,0]})$. Fix $(t,\varphi)\in\mathcal{U}$. To show that $F(t,\varphi)$ is linear, let $a,b\in\mathbb{R}$, $s,s^\dagger\in\mathbb{R}$ and $\psi,\psi^\dagger\in\mathcal{C}_{[-\tau,0]}$. Let x denote the unique solution of DDER with initial condition φ, and let v and $v^\dagger$ denote the unique solutions of LVE relative to x with respective initial conditions ψ and $\psi^\dagger$. Let $v^\ddagger$ denote the unique solution of LVE relative to x with initial condition $a\psi + b\psi^\dagger$. By Lemma 5.10 and the fact that $\|x - x^*\|_{[-\tau,0]} < \varepsilon_0$ and $\ell_1^x \notin [t-\tau,t]$, we have $v_t^\ddagger = av_t + bv_t^\dagger$. Hence,

$$\begin{aligned} F(t,\varphi)(as+bs^\dagger, a\psi+b\psi^\dagger) &= (as+bs^\dagger)\dot{x}_t + v_t^\ddagger \\ &= a(s\dot{x}+v_t) + b(s^\dagger\dot{x}+v_t^\dagger) \\ &= aF(t,\varphi)(s,\psi) + bF(t,\varphi)(s^\dagger,\psi^\dagger). \end{aligned}$$

To show that $F(t,\varphi)$ is a bounded linear operator, consider $(s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$ such that $|s|\vee\|\psi\|_{[-\tau,0]}=1$. By (3.10) and (C.8) with $v^\dagger\equiv 0$,

$$\|F(t,\varphi)(s,\psi)\|_{[-\tau,0]}\leq\|\dot{x}_t\|_{[-\tau,0]}+\|v_t\|_{[-\tau,0]}\leq H+2\exp(2K_ht).$$

Since the bound is only subject to the constraint $|s|\vee\|\psi\|_{[-\tau,0]}=1$, $F(t,\varphi)$ is a bounded linear operator.

We now show that the function $(t,\varphi)\to F(t,\varphi)$ is continuous as a mapping from $\mathcal{U}$ into $\mathcal{L}(\mathbb{R}\times,\mathcal{C}_{[-\tau,0]},\mathcal{C}_{[-\tau,0]})$. Fix $(t,\varphi)\in\mathcal{U}$ and $\delta>0$. Let x denote the unique solution of DDER with initial condition φ. By Lemma 5.14, we can choose $\varepsilon^\dagger>0$ such if $x^\dagger$ is another solution of DDER and satisfies $\|x-x^\dagger\|_{[-\tau,0]}<\varepsilon^\dagger$ and $\psi\in\mathcal{C}_{[-\tau,0]}$ satisfies $\|\psi\|_{[-\tau,0]}\leq 1$, then $\|v^x-v^{x^\dagger}\|_{[p-\tau-\eta_0,p+\eta_0]}<\frac{\delta}{4}$, where v^x and $v^{x^\dagger}$ denote the unique solutions of LVE relative to x and $x^\dagger$, respectively, and both with initial condition ψ. By choosing a possibly smaller $\varepsilon^\dagger>0$, we can assume that $\varepsilon^\dagger$ satisfies

$$\varepsilon^\dagger\leq\frac{1}{2K_h[H+\exp(2K_h(p+\eta_0))]}. \tag{5.120}$$

Now choose $(t^\dagger,\varphi^\dagger)\in\mathcal{U}$ satisfying $|t^\dagger-t|\vee\|\varphi-\varphi^\dagger\|_{[-\tau,0]}<\varepsilon^\dagger$. Let $x^\dagger$ denote the unique solution of DDER with initial condition $\varphi^\dagger$. Consider $(s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$ such that $|s|\vee\|\psi\|_{[-\tau,0]}=1$. Define $\dot{x}$ as in (5.81). Let v^x be the unique solution of LVE relative to x with initial condition ψ and let $v^{x^\dagger}$ be the unique solution of LVE relative to $x^\dagger$ with initial condition ψ. By the triangle inequality

$$\begin{aligned}&\|F(t,\varphi)(s,\psi)-F(t^\dagger,\varphi^\dagger)(s,\psi)\|_{[-\tau,0]}\\&\quad\leq\|F(t,\varphi)(s,\psi)-F(t^\dagger,\varphi)(s,\psi)\|_{[-\tau,0]}\\&\qquad+\|F(t^\dagger,\varphi)(s,\psi)-F(t^\dagger,\varphi^\dagger)(s,\psi)\|_{[-\tau,0]}.\end{aligned} \tag{5.121}$$

For the first term on the right hand side of (5.121), by (5.81) and (5.78),

$$\begin{aligned}\|F(t,\varphi)(s,\psi)-F(t^{\dagger},\varphi)(s,\psi)\|_{[-\tau,0]} &\leq \|\dot{x}_t-\dot{x}_{t^{\dagger}}\|_{[-\tau,0]}+\|v^x_t-v^x_{t^{\dagger}}\|_{[-\tau,0]} \\ &\leq \sup_{u\in[-2\tau,-\tau]}|h(x(t+u))-h(x(t^{\dagger}+u))| \\ &\quad+\sup_{u\in[-2\tau,-\tau]}\left|\int_{t^{\dagger}+u}^{t+u}h'(x(r))v^x(r)dr\right| \\ &\leq K_h\sup_{u\in[-2\tau,-\tau]}|x(t+u)-x(t^{\dagger}+u)| \\ &\quad+K_h\|v^x\|_{[-\tau,t\vee t^{\dagger}]}|t-t^{\dagger}| \\ &\leq K_h[H+\exp(2K_h(p+\eta_0))]|t-t^{\dagger}| \\ &< \frac{\delta}{2}.\end{aligned} \tag{5.122}$$

The third inequality follows from (3.12). The fourth inequality follows (3.10) and (C.8) with $v^{\dagger}\equiv 0$. The final equality follows from our choice of $\varepsilon^{\dagger}$. For the second term on the right of (5.121), we have

$$\begin{aligned}&\|F(t^{\dagger},\varphi)(s,\psi)-F(t^{\dagger},\varphi^{\dagger})(s,\psi)\|_{[-\tau,0]} \\ &\quad\leq \|\dot{x}_{t^{\dagger}}-\dot{x}^{\dagger}_{t^{\dagger}}\|_{[-\tau,0]}+\|v^x_{t^{\dagger}}-v^{x^{\dagger}}_{t^{\dagger}}\|_{[-\tau,0]}.\end{aligned} \tag{5.123}$$

For the first term on the right hand side of (5.123), we have

$$\begin{aligned}\|\dot{x}_{t^{\dagger}}-\dot{x}^{\dagger}_{t^{\dagger}}\|_{[-\tau,0]} &\leq \sup_{u\in[-\tau,0]}|h(x(t^{\dagger}-\tau+u))-h(x^{\dagger}(t^{\dagger}-\tau+u))| \\ &\leq K_h\sup_{u\in[-\tau,0]}|x(t^{\dagger}-\tau+u))-x^{\dagger}(t^{\dagger}-\tau+u)| \\ &\leq K_h\|x-x^{\dagger}\|_{[-\tau,t^{\dagger}-\tau]} \\ &\leq 2K_h\exp(2K_h(p+\eta_0))\|\varphi-\varphi^{\dagger}\|_{[-\tau,0]} \\ &< \frac{\delta}{4}.\end{aligned} \tag{5.124}$$

The second and fourth inequalities follow from (3.12) and (5.46), respectively. The final inequality follows from (5.120). For the second term on the right hand side, we have

$$\|v^x_{t^{\dagger}}-v^{x^{\dagger}}_{t^{\dagger}}\|_{[-\tau,0]}\leq\|v^x-v^{x^{\dagger}}\|_{[p-\tau-\eta_0,p+\eta_0]}<\frac{\delta}{4}. \tag{5.125}$$

Combining (5.121)–(5.125), we see that if $|t - t^\dagger| \vee \|\varphi - \varphi^\dagger\|_{[-\tau,0]} < \varepsilon^\dagger$, then $\|F(t,\varphi)(s,\psi) - F(t^\dagger,\varphi^\dagger)(s,\psi)\|_{[-\tau,0]} < \delta$. Since this bound was obtained only subject to the restriction that $|s| \vee \|\psi\|_{[-\tau,0]} = 1$, we have

$$\|F(t,\varphi) - F(t^\dagger,\varphi^\dagger)\| < \delta$$

where $\|\cdot\|$ denotes the operator norm on $\mathcal{L}(\mathbb{R} \times \mathcal{C}_{[-\tau,0]}, \mathcal{C}_{[-\tau,0]})$. It follows that the function $(t,\varphi) \to F(t,\varphi)$ is continuous at each $(t,\varphi) \in \mathcal{U}$ and therefore continuous on $\mathcal{U}$.

We now show that for each $(t,\varphi) \in \mathcal{U}$ and $(s,\psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}$,

$$F(t,\varphi)(s,\psi) = \partial_{(s,\psi)}\Sigma(t,\varphi) \equiv \lim_{\varepsilon\to 0} \frac{\Sigma(t+\varepsilon s, \varphi+\varepsilon\psi) - \Sigma(t,\varphi)}{\varepsilon}, \tag{5.126}$$

where the convergence is taken to be uniform on $[-\tau,0]$. Fix $(t,\varphi) \in \mathcal{U}$ and $(s,\psi) \in \mathbb{R}\times\mathcal{C}_{[-\tau,0]}$. Let x denote the unique solution of DDER with initial condition φ. Define $\dot{x}$ as in (5.81) and let v be the unique solution of LVE relative to x with initial condition ψ. For $\varepsilon > 0$ sufficiently small so that $(t+\varepsilon s, \varphi+\varepsilon\psi) \in \mathcal{U}$, let x^ε denote the unique solution of DDER with initial condition $x_0^\varepsilon = \varphi + \varepsilon\psi$ and define $v^\varepsilon = \varepsilon^{-1}(x^\varepsilon - x) \in \mathcal{C}_{[-\tau,\infty)}$. By the triangle inequality, for such $\varepsilon > 0$,

$$\begin{aligned} &\left\| \frac{\Sigma(t+\varepsilon s,\varphi) - \Sigma(t,\varphi)}{\varepsilon} - F(t,\varphi)(s,\psi) \right\|_{[-\tau,0]} \\ &\quad = \left\| \frac{x^\varepsilon_{t+\varepsilon s} - x_t}{\varepsilon} - s\dot{x}_t - v_t \right\|_{[-\tau,0]} \\ &\quad \le \varepsilon^{-1} \left\| x^\varepsilon_{t+\varepsilon s} - x^\varepsilon_t - \varepsilon s \dot{x}_t \right\|_{[-\tau,0]} + \left\| v^\varepsilon_t - v_t \right\|_{[-\tau,0]} . \end{aligned} \tag{5.127}$$

For the first term on the right hand side of the inequality in (5.127), we have

$$
\begin{aligned}
\left\|x_{t+\varepsilon s}^{\varepsilon}-x_{t}^{\varepsilon}-\varepsilon s \dot{x}_{t}\right\|_{[-\tau,0]} &= \sup_{u\in[-\tau,0]} \int_{0}^{\varepsilon s} |\dot{x}^{\varepsilon}(t+u+r)-\dot{x}(t+u)|dr \\
&= \sup_{u\in[-\tau,0]} \int_{0}^{\varepsilon s} |h(x^{\varepsilon}(t+u-\tau+r))-h(x(t+u-\tau))|dr \\
&\leq K_h \sup_{u\in[-\tau,0]} \int_{0}^{\varepsilon s} |x^{\varepsilon}(t+u-\tau+r)-x(t+u-\tau)|dr \\
&\leq K_h \sup_{u\in[-\tau,0]} \int_{0}^{\varepsilon s} |x^{\varepsilon}(t+u-\tau+r)-x^{\varepsilon}(t+u-\tau)|dr \\
&\quad + K_h \sup_{u\in[-\tau,0]} \int_{0}^{\varepsilon s} |x^{\varepsilon}(t+u-\tau)-x(t+u-\tau)|dr \\
&\leq K_h H \int_{0}^{\varepsilon s} r dr \\
&\quad + \sup_{u\in[-\tau,0]} K_h \varepsilon s |x^{\varepsilon}(t+u-\tau)-x(t+u-\tau)| \\
&\leq \frac{1}{2} K_h H \varepsilon^2 s^2 + 2K_h \varepsilon^2 s \exp(2K_h t).
\end{aligned}
$$

The second equality follows from (5.81) and the fact that (by our choice of $\varepsilon > 0$) x and x^{ε} are positive in the $|\varepsilon s|$ neighborhood of $[t-\tau, t]$. The first and third inequalities follow from (3.12) and (3.10), respectively. The final inequality follows from (5.46), but with x^{ε} in place of $x^{\dagger}$.

It follows that the first term on the right hand side of the inequality in (5.127) converges to zero as $\varepsilon \to 0$. For the second term on the right hand side of the inequality in (5.127), by (5.78) and the fact that $[t-\tau, t] \subset [p-\tau-\eta_0, p+\eta_0] \subset [q_1^x+\tau, p+\eta_0]$, v is continuous on $[t-\tau, t]$ and so by Proposition 5.1, $\|v_t^{\varepsilon} - v_t\|_{[-\tau,0]}$ converges to zero as $\varepsilon \to 0$. Then by (5.126)–(5.127) and because the convergence holds for each $(t,\varphi) \in \mathcal{U}$ and $(s,\psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}$, $F(t,\varphi)(s,\psi) = \partial_{(s,\psi)}\Sigma(t,\varphi)$ for all $(t,\varphi) \in \mathcal{U}$ and $(s,\psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}$.

We have shown that $(t,\varphi) \to F(t,\varphi)$ is a continuous function from $\mathcal{U}$ into $\mathcal{L}(\mathbb{R} \times \mathcal{C}_{[-\tau,0]}, \mathcal{C}_{[-\tau,0]})$ and that $\partial_{(s,\psi)}\Sigma(t,\varphi)$ exists and is equal to $F(t,\varphi)(s,\psi)$ at each $(t,\varphi) \in \mathcal{U}$ and $(s,\psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}$. Then by Proposition F.1, Σ is continuously Fréchet differentiable on $\mathcal{U}$ and $D\Sigma = F$. $\square$

Define the function $Z : \mathbb{R}_+ \times \mathcal{C}^+_{[-\tau,0]} \to \mathbb{R}$ by

$$Z(t,\varphi) = z(t), \tag{5.128}$$

where if x is the unique solution of DDER with initial condition φ, then z is defined as in (2.2). Recall $\eta_0 > 0$ and $\varepsilon_0 > 0$ as introduced in Lemma 5.7. Define the neighborhood $\mathcal{V}$ of (ℓ_1, x_0^*) in the Banach space $\mathbb{R} \times \mathcal{C}_{[-\tau,0]}$ by

$$\mathcal{V} = \left\{(t,\varphi) \in \mathbb{R}_+ \times \mathcal{C}^+_{[-\tau,0]} : |t - \ell_1| < \eta_0 \text{ and } \|\varphi - x_0^*\|_{[-\tau,0]} < \varepsilon_0\right\}.$$

Note that by the definition of $\eta_0 > 0$ and $\varepsilon_0 > 0$ in Lemma 5.7, $\mathcal{V} \subset \mathbb{R}_+ \times \mathcal{C}^+_{[-\tau,0]}$.

Lemma 5.17. *The function Z is continuously Fréchet differentiable on $\mathcal{V}$ and for each $(t,\varphi) \in \mathcal{V}$, the derivative $DZ(t,\varphi) \in \mathcal{L}(\mathbb{R} \times \mathcal{C}_{[-\tau,0]}, \mathbb{R})$ at (t,φ) is given by*

$$DZ(t,\varphi)(s,\psi) = s\dot{z}(t) + w(t), \text{ for all } (s,\psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}. \tag{5.129}$$

Here $\dot{z} \in \mathcal{C}_{[0,\infty)}$ is defined by $\dot{z}(t) = h(x(t-\tau))$ for all $t \geq 0$, and if v is the unique solution of LVE relative x with initial condition ψ, then w is defined by

$$w(t) = v(0) + \int_0^t h'(x(s-\tau))v(s-\tau)ds, \ t \geq 0. \tag{5.130}$$

Proof. For each $(t,\varphi) \in \mathcal{V}$, define the operator $G : \mathbb{R} \times \mathcal{C}_{[-\tau,0]} \to \mathbb{R}$ by

$$G(t,\varphi)(s,\psi) = s\dot{z}(t) + w(t), \ (s,\psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}. \tag{5.131}$$

We first show that for each $(t,\varphi) \in \mathcal{V}$, $G(t,\varphi)$ is a bounded linear operator from $\mathbb{R} \times \mathcal{C}_{[-\tau,0]}$ into $\mathbb{R}$, i.e., $G(t,\varphi) \in \mathcal{L}(\mathbb{R} \times \mathcal{C}_{[-\tau,0]}, \mathbb{R})$. Fix $(t,\varphi) \in \mathcal{V}$. To show that $G(t,\varphi)$ is linear, let $a, b \in \mathbb{R}$, $s, s^\dagger \in \mathbb{R}$ and $\psi, \psi^\dagger \in \mathcal{C}_{[-\tau,0]}$. Let x denote the unique solution of DDER with initial condition φ and define z as in (2.2). Let v and $v^\dagger$ denote the unique solutions of LVE relative to x with respective initial conditions ψ and $\psi^\dagger$. Define w as in (5.72) and define $w^\dagger$ as in (5.72), but with $v^\dagger$ and $w^\dagger$ in place of v and w, respectively. Let $v^\ddagger$ denote the unique solution of LVE relative to x with initial condition $a\psi + b\psi^\dagger$ and define $w^\ddagger$ as in (5.72), but with $v^\ddagger$ and $w^\ddagger$ in place of v and w, respectively. By Lemma 5.10 and the fact that

$\ell_1 + \eta_0 - \tau < \ell_1^x$, we have, for $t \in [0, \ell_1 + \eta_0]$,

$$\begin{aligned} w^\ddagger(t) &= v^\ddagger(0) + \int_0^t h'(x(s-\tau))v^\ddagger(s-\tau)ds \\ &= av(0) + bv^\dagger(0) + \int_0^t h'(x(s-\tau))(av(s-\tau) + bv^\dagger(s-\tau))ds \\ &= aw(t) + bw^\dagger(t), \end{aligned}$$

from which we obtain that $G(t, \varphi)$ is linear:

$$\begin{aligned} G(t,\varphi)(as + bs^\dagger, a\psi + b\psi^\dagger) &= (as + bs^\dagger)\dot{z}(t) + (aw(t) + bw^\dagger(t)) \\ &= a(s\dot{z}(t) + w(t)) + b(s^\dagger \dot{z}(t) + w^\dagger(t)) \\ &= aG(t,\varphi)(s,\psi) + bG(t,\varphi)(s^\dagger, \psi^\dagger). \end{aligned}$$

To show that $G(t, \varphi)$ is a bounded linear operator, consider $(s, \psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}$ such that $|s| \vee \|\psi\|_{[-\tau,0]} = 1$. Then by (5.72), (3.12) and (C.8) with $v^\dagger \equiv 0$,

$$\begin{aligned} |w(t)| &\leq |v(0)| + \int_0^t |h'(x(s-\tau))||v(s-\tau)|ds \\ &\leq 1 + K_h \|v\|_{[-\tau, \ell_1+\eta_0]}(\ell_1 + \eta_0) \\ &\leq 1 + 2K_h \exp(2K_h(\ell_1 + \eta_0))(\ell_1 + \eta_0). \end{aligned}$$

It follows that

$$|G(t,\varphi)(s,\psi)| \leq |\dot{z}(t)| + |w(t)| \leq H + 1 + 2K_h \exp(2K_h(\ell_1 + \eta_0))(\ell_1 + \eta_0).$$

Since the bound was obtained only subject to the constraint $|s| \vee \|\psi\|_{[-\tau,0]} = 1$, $G(t, \varphi)$ is a bounded linear operator.

We now show that the function $(t, \varphi) \to G(t, \varphi)$ is continuous as a mapping from $\mathcal{V}$ into $\mathcal{L}(\mathbb{R}, \mathcal{C}_{[-\tau,0]}, \mathbb{R})$. Fix $(t, \varphi) \in \mathcal{V}$ and $\delta > 0$. By (5.46) and the continuity of h', we can choose a possibly smaller $\varepsilon^\ddagger > 0$ such that if $\|\varphi - \varphi^\dagger\|_{[-\tau,0]} < \varepsilon^\ddagger$, then

$$\sup_{u \in [-\tau, t^\dagger - \tau]} |h'(x(u)) - h'(x^\dagger(u))| < \frac{\delta}{8 \exp(3K_h(\ell_1 + \eta_0))(\ell_1 + \eta_0)}.$$

By choosing a possibly smaller $\varepsilon^\ddagger > 0$, we have that $\varepsilon^\ddagger$ satisfies

$$\varepsilon^\ddagger \leq \frac{1}{K_h(H + 2\exp(2K_h(\ell_1 + \eta_0))}. \tag{5.132}$$

Suppose $(t^\dagger, \varphi^\dagger) \in \mathcal{V}$ satisfies $|t - t^\dagger| \vee \|\varphi - \varphi^\dagger\|_{[-\tau,0]} < \varepsilon^\ddagger$. Let x and $x^\dagger$ denote the unique solutions of DDER with respective initial conditions φ and $\varphi^\dagger$. Define z as in (2.2) and define $z^\dagger$ as in (2.2), but with $x^\dagger$ and $z^\dagger$ in place of x and z, respectively. Consider $(s, \psi) \in \mathbb{R} \times \mathcal{C}_{[-\tau,0]}$ such that $|s| \vee \|\psi\|_{[-\tau,0]} = 1$. Let v^x be the unique solution of LVE relative to x with initial condition ψ and let $v^{x^\dagger}$ be the unique solution of LVE relative to $x^\dagger$ with initial condition ψ. Define w^x as in (5.72), but with v^x and w^x in place of v and w, respectively and define $w^{x^\dagger}$ as in (5.72), but with $v^{x^\dagger}$ and $w^{x^\dagger}$ in place of v and w, respectively. By the triangle inequality,

$$\begin{aligned} |G(t,\varphi)(s,\psi) - G(t^\dagger,\varphi^\dagger)(s,\psi)| \le\ & |G(t,\varphi)(s,\psi) - G(t^\dagger,\varphi)(s,\psi)| \\ & + |G(t^\dagger,\varphi)(s,\psi) - G(t^\dagger,\varphi^\dagger)(s,\psi)|. \end{aligned} \tag{5.133}$$

For the first term on the right hand side of (5.133), we have

$$\begin{aligned} |G(t,\varphi)(s,\psi) - G(t^\dagger,\varphi)(s,\psi)| &\le |\dot{z}(t) - \dot{z}(t^\dagger)| + |w(t) - w(t^\dagger)| \\ &\le |h(x(t-\tau)) - h(x(t^\dagger - \tau))| \\ &\quad + \left| \int_{t^\dagger}^{t} h'(x(u-\tau))v(u-\tau)du \right| \\ &\le K_h |x(t-\tau) - x(t^\dagger - \tau)| \\ &\quad + K_h \|v\|_{[-\tau, t\wedge t^\dagger]} |t - t^\dagger| \\ &\le K_h(H + 2\exp(2K_h(\ell_1 + \eta_0))|t - t^\dagger| \\ &< \frac{\delta}{2}. \end{aligned} \tag{5.134}$$

The third inequality follows from (3.12). The fourth inequality follows from (3.10) and (C.8) with $v^\dagger \equiv 0$. The final inequality follows from our bound on $\varepsilon^\ddagger$ in (5.132). For the second term on the right hand side of (5.133), we have

$$|G(t^\dagger,\varphi)(s,\psi) - G(t^\dagger,\varphi^\dagger)(s,\psi)| \le |\dot{z}(t^\dagger) - \dot{z}^\dagger(t^\dagger)| + |w^x(t^\dagger) - w^{x^\dagger}(t^\dagger)|. \tag{5.135}$$

For the first term on the right hand side of (5.135), we have

$$\begin{aligned}|\dot{z}(t^\dagger) - \dot{z}^\dagger(t^\dagger)| &\leq |h(x(t^\dagger - \tau)) - h(x^\dagger(t^\dagger - \tau))| \\ &\leq K_h \|x - x^\dagger\|_{[-\tau, \ell_1^x \wedge \ell_1^{x^\dagger}]} \\ &\leq 2K_h \exp(2K_h(\ell_1 + \eta_0))\|\varphi - \varphi^\dagger\|_{[-\tau,0]} \\ &< \frac{\delta}{4}.\end{aligned} \tag{5.136}$$

The second inequality follows from (3.12). The third inequality follows because of (5.46) and because, by Lemma 5.7, $\ell_1^x, \ell_1^{x^\dagger} \in (\ell_1 - \eta_0, \ell_1 + \eta_0)$. For the second term on the right hand side of (5.135), we have, for $r \in [0, t^\dagger]$,

$$\begin{aligned}|w^x(r) - w^{x^\dagger}(r)| &\leq \int_{-\tau}^{r-\tau} |h'(x(u))v^x(u) - h'(x^\dagger(u))v^{x^\dagger}(u)|du \\ &\leq \int_0^{r-\tau} |h'(x(u))| \cdot |v^x(u) - v^{x^\dagger}(u)|du \\ &\quad + \int_{-\tau}^{r-\tau} |h'(x(u)) - h'(x^\dagger(u))| \cdot |v^{x^\dagger}(u)|du \\ &\leq K_h \int_0^{r-\tau} |w^x(u) - w^{x^\dagger}(u)|du \\ &\quad + \|v^{x^\dagger}\|_{[-\tau,\ell_1]} \int_{-\tau}^{r-\tau} |h'(x(u)) - h'(x^\dagger(u))|du \\ &\leq K_h \int_0^r |w^x(u) - w^{x^\dagger}(u)|du \\ &\quad + 2\exp(2K_h\ell_1)(\ell_1 + \eta_0) \\ &\qquad \cdot \sup_{u \in [-\tau, t^\dagger - \tau]} |h'(x(u)) - h'(x^\dagger(u))|.\end{aligned} \tag{5.137}$$

The second inequality follows because $v^x(u) = v^{x^\dagger}(u) = \psi(u)$ for all $u \in [-\tau, 0]$. The third inequality follows from (3.12) and the fact that, by (5.72) and (5.78), $w^x(u) = v^x(u)$ for all $u \in [0, \ell_1^x)$, $w^{x^\dagger}(u) = v^{x^\dagger}(u)$ for all $u \in [0, \ell_1^{x^\dagger})$, and by Lemma 5.7, $\ell_1^x, \ell_1^{x^\dagger} \notin [0, \ell_1 + \eta_0 - \tau]$. The fourth inequality follows from (C.8), but with $v^{x^\dagger}$ in place of v and with $v^\dagger \equiv 0$. Applying Gronwall's inequality and our choice of $\varepsilon^{\ddagger}$ yields

$$|w^x(t^{\dagger}) - w^{x^{\dagger}}(t^{\dagger})| \leq 2\exp(3K_h(\ell_1+\eta_0))(\ell_1+\eta_0) \cdot \sup_{s\in[-\tau,\ell_1-\tau+\eta_0]} |h'(x(s)) - h'(x^{\dagger}(s))| < \frac{\delta}{4}. \tag{5.138}$$

It then follows from (5.133)–(5.138) that if $|t-t^{\dagger}| \vee \|\varphi-\varphi^{\dagger}\|_{[-\tau,0]} < \varepsilon^{\ddagger}$, then $|G(t,\varphi)(s,\psi) - G(t^{\dagger},\varphi^{\dagger})(s,\psi)| < \delta$. Since this bound was obtained only subject to the restriction that $|s| \vee \|\psi\|_{[-\tau,0]} = 1$, we have

$$\|G(t,\varphi) - G(t^{\dagger},\varphi^{\dagger})\| < \delta,$$

where $\|\cdot\|$ denotes the operator norm on $\mathcal{L}(\mathbb{R}\times\mathcal{C}_{[-\tau,0]},\mathbb{R})$. It follows that the function $(t,\varphi)\to G(t,\varphi)$ is continuous at each $(t,\varphi)\in\mathcal{V}$ and therefore continuous on $\mathcal{V}$.

We now show that for each $(t,\varphi)\in\mathcal{V}$ and $(s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$,

$$G(t,\varphi)(s,\psi) = \partial_{(s,\psi)}Z(t,\varphi) \equiv \lim_{\varepsilon\to 0}\frac{Z(t+\varepsilon s,\varphi+\varepsilon\psi) - Z(t,\varphi)}{\varepsilon}. \tag{5.139}$$

Fix $(t,\varphi)\in\mathcal{V}$ and $(s,\psi)\in\mathbb{R}\times\mathcal{C}_{[-\tau,0]}$. Let x be the unique solution of DDER with initial condition φ, z be defined as in (2.2), v be the unique solution of LVE relative to x, and w be defined as in (5.72). For $\varepsilon>0$ sufficiently small that $(t+\varepsilon s,\varphi+\varepsilon\psi)\in\mathcal{V}$, let x^{ε} denote the unique solution of DDER with initial condition $\varphi+\varepsilon\psi$ and define z^{ε} as in (2.2), but with x^{ε} and z^{ε} in place of x and z, respectively, and define $w^{\varepsilon} = \varepsilon^{-1}(z^{\varepsilon}-z)\in\mathcal{C}_{[-\tau,0]}$. Then by the triangle inequality,

$$\begin{aligned}\left|\frac{Z(t+\varepsilon s,\varphi+\varepsilon\psi) - Z(t,\varphi)}{\varepsilon} - s\dot{z}(t) - w(t)\right| \\ = \left|\frac{z^{\varepsilon}(t+\varepsilon s) - z^{\varepsilon}}{\varepsilon} - s\dot{z}(t) - w(t)\right| \\ \leq \varepsilon^{-1}|z^{\varepsilon}(t+\varepsilon s) - z^{\varepsilon}(t) - \varepsilon s\dot{z}(t)| + |w^{\varepsilon}(t) - w(t)|.\end{aligned} \tag{5.140}$$

For the first term on the right hand side of the inequality in (5.140), we have

$$\begin{aligned}|z^\varepsilon(t+\varepsilon s) - z^\varepsilon(t) - \varepsilon s\dot z(t)| &= \left|\int_0^{\varepsilon s} h(x^\varepsilon(t+u-\tau)) - h(x(t-\tau))du\right| \\ &\le K_h \int_0^{\varepsilon s} |x^\varepsilon(t+u-\tau) - x(t-\tau)|du \\ &\le K_h \int_0^{\varepsilon s} |x^\varepsilon(t+u-\tau) - x^\varepsilon(t-\tau)|du \\ &\quad + K_h\varepsilon s|x^\varepsilon(t-\tau) - x(t-\tau)| \\ &\le K_h H \int_0^{\varepsilon s} udu \\ &\quad + K_h\varepsilon s\|x^\varepsilon - x\|_{[-\tau,\ell_1+\eta_0]} \\ &\le \frac{1}{2}K_h H\varepsilon^2 s^2 + 2K_h\varepsilon^2 s\exp(2K_h(\ell_1+\eta_0)).\end{aligned}$$

The first inequality follows from (3.12). The third inequality follows from (3.10). The final inequality follows from (5.46) with x^ε in place of $x^\dagger$. It follows that the first term on the right hand side of the inequality in (5.140) converges to zero as $\varepsilon \to 0$. Then by Proposition 5.1, second term on the right hand side of the inequality in (5.140) converges to zero as $\varepsilon \to 0$. Hence $G(t,\varphi)(s,\psi) = \partial_{(s,\psi)}Z(t,\varphi)$ for all $(t,\varphi) \in \mathcal{V}$ and $(s,\psi) \in \mathbb{R}\times\mathcal{C}_{[-\tau,0]}$.

We have shown that $(t,\varphi) \to Z(t,\varphi)$ is a continuous function from $\mathcal{V}$ into $\mathcal{L}(\mathbb{R}\times\mathcal{C}_{[-\tau,0]},\mathbb{R})$ and that $\partial_{(s,\psi)}Z(t,\varphi)$ exists and is equal to $G(t,\varphi)(s,\psi)$ at each $(t,\varphi) \in \mathcal{V}$ and $(s,\psi) \in \mathbb{R}\times\mathcal{C}_{[-\tau,0]}$. Then by Proposition F.1, Z is continuously Fréchet differentiable on $\mathcal{V}$ and $DZ = G$. $\square$

Lemma 5.18. *There exists a neighborhood $\mathcal{W}$ of x_0^* in $\mathcal{C}^+_{[-\tau,0]}$ and a continuously Fréchet differentiable function $\Delta : \mathcal{W} \to \mathbb{R}$ such that $\Delta(x_0^*) = 0$, $|\Delta(\varphi)| < \eta_0$ and*

$$Z(\ell_1 + \Delta(\varphi), \varphi) = 0 \text{ for all } \varphi \in \mathcal{W}.$$

It follows that $\Delta(\varphi) = \ell_1^x - \ell_1$, where ℓ_1^x is as in Lemma 5.7. Furthermore, at x_0^*, $D\Delta(x_0^*)$ is given by

$$D\Delta(x_0^*)\psi = -\frac{w(\ell_1)}{\dot z^*(\ell_1)} \text{ for all } \psi \in \mathcal{C}_{[-\tau,0]}, \tag{5.141}$$

where w is given by (5.72) with x replaced by x^* and v is the solution of LVE relative to x^* with initial condition ψ.

Proof. By Lemma 5.17, Z is continuously Fréchet differentiable on $\mathcal{V}$, which is an open neighborhood of (ℓ_1, x_0^*) in the Banach space $\mathbb{R} \times \mathcal{C}_{[-\tau,0]}$. We also have that $Z(\ell_1, x_0^*) = z^*(\ell_1) = 0$ and $D_1 Z(\ell_1, x_0^*) = \dot{z}^*(\ell_1) \neq 0$, where $D_1 Z(\ell_1, x_0^*) \equiv (DZ(\ell_1, x_0^*))(1,0)$. By Proposition F.2, there exists a neighborhood $\mathcal{W}$ of x_0^* and a unique continuous function $\Delta^\ell : \mathcal{W} \to \mathbb{R}$ such that $\Delta^\ell(x_0^*) = \ell_1$, $(\Delta^\ell(\varphi), \varphi) \in \mathcal{V}$ and $Z(\Delta^\ell(\varphi), \varphi) = 0$ for all $\varphi \in \mathcal{W}$. Moreover, Δ^ℓ is continuously Fréchet differentiable with derivative at x_0^* given by

$$D\Delta^\ell(x_0^*)\psi = -[D_1 Z(\ell_1, x_0^*)]^{-1} D_2 Z(\ell_1, x_0^*)\psi = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}$$

If we define $\Delta : \mathcal{W} \to \mathbb{R}$ by $\Delta(\varphi) = \Delta^\ell(\varphi) - \ell_1$ for each $\varphi \in \mathcal{W}$, Δ is continuously Fréchet differentiable, $\Delta(x_0^*) = 0$ and $(\ell_1 + \Delta(\varphi), \varphi) \in \mathcal{V}$, so by the definition of $\mathcal{V}$, $|\Delta(\varphi)| < \eta_0$. To see that $\Delta(\varphi) = \ell_1^x$, note that if $\varphi \in \mathcal{W}$, then $\|\varphi - x_0^*\|_{[-\tau,0]} < \varepsilon_0$, so by Lemma 5.7 there exists a unique $\ell_1^x \in (\ell_1 - \eta_0, \ell_1 + \eta_0)$ such that $z(\ell_1^x) = 0$. Since $|\Delta(\varphi)| < \eta_0$ and $z(\ell_1 + \Delta(\varphi)) = Z(\ell_1 + \Delta(\varphi)) = 0$, it follows that $\Delta(\varphi) = \ell_1^x - \ell_1$. □

We can now define an approximate *Poincaré map* $\Gamma : \mathcal{W} \to \mathcal{C}^+_{[-\tau,0]}$ by

$$\Gamma(\varphi) = \Sigma(p + \Delta(\varphi), \varphi). \tag{5.142}$$

Since Σ is continuously Fréchet differentiable on $\mathcal{U}$, Δ is continuously Fréchet differentiable on $\mathcal{W}$ and $(\Delta(\varphi), \varphi) \in \mathcal{U}$ for all $\varphi \in \mathcal{W}$, it follows that Γ is also continuously Fréchet differentiable on $\mathcal{W}$. Note that x_0^* is a fixed point of Γ and

$$D\Gamma(x_0^*) = D_1\Sigma(p, x_0^*) D\Delta(x_0^*) + D_2\Sigma(p, x_0^*). \tag{5.143}$$

By Lemma 5.16 and (5.141),

$$D\Gamma(x_0^*)\psi = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}\dot{x}_0^* + v_p, \text{ for all } \psi \in \mathcal{C}_{[-\tau,0]}. \tag{5.144}$$

Here v denotes the unique solution of LVE relative to x^* with initial condition ψ and w is defined as in (5.72), but with x^* in place of x.

5.6 Proof of Stability

In this section we prove that if the delay τ is sufficiently large, then any SOPS, x^*, of DDER with delay τ is uniformly exponentially asymptotically stable. In Section 5.7, we prove that such a SOPS (with τ fixed but sufficiently large) x^* is unique (up to time translation), which will complete the proof of Theorem 3.2.

Lemma 5.19. *Let $\delta > 0$ and $\{x^\tau : \tau > \tau_0\}$ denote the family of SOPS defined in Section 5.2. Then there exists $\tau^\delta \geq \tau_3$ such that if $\tau > \tau^\delta$ and x^* denotes the SOPS x^τ, then $\|D\Gamma(x_0^*)\| \leq \delta$.*

Proof. Fix $\delta > 0$. Let $\hat{x}^*$ be the associated SOPS$^{\mathrm{n}}$ defined by (4.11). By Theorem 5.1, Corollary 5.1(ii) and (5.27), there exist constants $d > 0$ and $\tau^\delta \geq \tau_3$ such that if the fixed delay satisfies $\tau \geq \tau^\delta$, then $-1 < \hat{q}_1 - 1 < \hat{\ell}_1 - 1 < \bar{q} < \hat{q}_2 < \hat{q}_1 + 2$, $\hat{x}^*(t) > \tau d$ for all $t \in [\hat{q}_1 - 1, \hat{\ell}_1 - 1]$, and $\hat{h}(\tau d) \leq -\alpha/2$. Then by (4.12), $-\tau < q_1 - \tau < \ell_1 - \tau < \tau\bar{q} < q_2 < q_1 + 2\tau$, $x^*(t) > L + \tau d$ for all $t \in [q_1 - \tau, \ell_1 - \tau]$ and $h(L + \tau d) \leq -\alpha/2$. By choosing a possibly larger τ^δ, we have that τ^δ satisfies

$$\tau^\delta \geq \left(\frac{2mM_h L^2(1 + K_h)(2M_h H + \alpha)(2\beta + \alpha)}{\alpha^3 \beta d} \right) \delta^{-1}. \tag{5.145}$$

Suppose that $\psi \in \mathcal{C}_{[-\tau,0]}$ satisfies $\|\psi\|_{[-\tau,0]} = 1$. By (5.144), we have

$$D\Gamma(x_0^*)\psi = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}\dot{x}_0^* + v_p, \tag{5.146}$$

where $\dot{x}^*$ is defined via (5.81) and $\dot{x}_0^* \in \mathcal{C}_{[-\tau,0]}$ since x^* is positive on $[-\tau, 0]$, v denotes the unique solution of LVE relative to x^* with initial condition ψ and w is defined as in (5.72) with x^* in place of x there.

Let $\xi \in \mathcal{D}_{[-\tau,\infty)}$ denote the unique solution of LVE relative to x^* with initial condition $\xi_0 = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}\dot{x}_0^* + \psi \in \mathcal{C}_{[-\tau,0]}$. Note that ξ_0 satisfies

$$\|\xi_0\|_{[-\tau,0]} \leq \frac{|w(\ell_1)|}{|\dot{z}^*(\ell_1)|}\|\dot{x}_0^*\|_{[-\tau,0]} + \|\psi\|_{[-1,0]} \leq 2\alpha^{-1}M_h H + 1. \tag{5.147}$$

Here we have used that $x(\ell_1 - \tau) \geq L + \tau d$ and so $\dot{z}(\ell_1) = h(x(\ell_1 - \tau)) \leq -\alpha/2$; we have used that $\dot{x}$ is bounded by $H \equiv \|h\|_{[0,\infty)}$; and we have used (5.79) and (5.72) to deduce that $w(t) = v(t)$ for all $t \in [0, \ell_1)$ and so by the continuity of w,

$|w(\ell_1)| = \lim_{t\uparrow\ell_1} |w(t)| = \lim_{t\uparrow\ell_1} |v(t)| \le M_h$. By Lemma 5.11, $\dot{x}^*$ is a solution of LVE relative to x^*. By Lemma 5.10,with $a = 1$, $b = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}$ and $\dot{x}^*$ in place of $v^\dagger$, we have

$$\xi(t) = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}\dot{x}^*(t) + v(t),\ t \in [-\tau, p] \setminus \{\ell_1\}. \tag{5.148}$$

Since $\ell_1 \notin [p-\tau, p]$, by (5.146) and the periodicity of x^*,

$$\xi_p = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}\dot{x}^*_0 + v_p = D\Gamma(x^*_0)\psi,$$

By (5.79) and (5.148), for $t \in [0, \ell_1)$,

$$\begin{aligned}
\xi(t) &= \xi(0) + \int_0^t h'(x^*(s-\tau))\xi(s-\tau)ds \\
&= -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}\left(\dot{x}^*(0) + \int_0^t h'(x^*(s-\tau))\dot{x}^*(s-\tau)ds\right) \\
&\quad + v(0) + \int_0^t h'(x^*(s-\tau))v(s-\tau)ds \\
&= -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}h(x^*(t-\tau)) + w(t).
\end{aligned}$$

Here we have used that h is continuously differentiable, x is continuously differentiable on $[0, \ell_1)$ with derivative equal to $\dot{x}^*$ there, $\dot{x}^*(0) = h(x^*(-\tau))$, the fundamental theorem of calculus, the chain rule and (5.72). Taking the limit as $t \uparrow \ell_1$, we have

$$\lim_{t\uparrow\ell_1} \xi(t) = -\frac{w(\ell_1)}{\dot{z}^*(\ell_1)}h(x^*(\ell_1-\tau)) + w(\ell_1) = 0, \tag{5.149}$$

where we have used that $\dot{z}^*(\ell_1) = h(x^*(\ell_1 - \tau))$. By (5.79), for $q_1 \le s \le t < \ell_1$,

$$\xi(t) - \xi(s) = \int_t^s h'(x^*(u-\tau))\xi(u-\tau)du.$$

Then taking the limit as $t \uparrow \ell_1$ and using (5.149), we have, for $s \in [q_1, \ell_1)$,

$$\begin{aligned}
|\xi(s)| &\le \int_s^{\ell_1} |h'(x^*(u-\tau))\xi(u-\tau)|du \qquad (5.150)\\
&\le \|\xi\|_{[-\tau,\ell_1]} \int_s^{\ell_1} \frac{|h'(x^*(u-\tau))x^*(u-\tau)|}{|x^*(u-\tau)|}du \\
&\le \frac{M_h|\ell_1 - q_1|m}{\tau d}\|\xi_0\|_{[-\tau,0]}.
\end{aligned}$$

The final inequality follows from Lemma 5.13, Assumption 3.4 and the fact that $x^*(u) > L + \tau d$ for all $u \in [q_1 - \tau, \ell_1 - \tau]$.

By (5.79), $\xi(t) = 0$ for all $t \in (\ell_1, q_1 + \tau]$ so by (5.149), ξ is continuous at ℓ_1. It then follows from (5.150) that

$$\|\xi\|_{[q_1,q_1+\tau]} \leq \frac{M_h|\ell_1 - q_1|m}{\tau d}\|\xi_0\|_{[-\tau,0]}. \tag{5.151}$$

By (5.79) and (3.12), for $t \in [q_1 + \tau, \ell_1 + \tau]$,

$$|\xi(t)| \leq \int_{q_1+\tau}^{\ell_1+\tau} |h'(x^*(s-\tau))\xi(s-\tau)|ds \leq K_h|\ell_1 - q_1|\|\xi\|_{[q_1,q_1+\tau]},$$

For $t \in [\ell_1 + \tau, q_1 + 2\tau]$, $\xi(t-\tau) = 0$, so by (5.79), $\xi(t) = \xi(\ell_1 + \tau)$ and hence

$$\|\xi\|_{[q_1+\tau,q_1+2\tau]} \leq K_h|\ell_1 - q_1|\|\xi\|_{[q_1,q_1+\tau]}. \tag{5.152}$$

By our choice of τ^δ at the beginning of the proof, $q_2 < q_1 + 2\tau$. Then by (5.79) and (5.42)–(5.43), for $t \in [q_1 + 2\tau, p] = [q_1 + 2\tau, q_2 + \tau]$,

$$\begin{aligned} |\xi(t)| &\leq |\xi(q_1 + 2\tau)| + \int_{q_1+2\tau}^{p} |h'(x^*(s-\tau))\xi(s-\tau)|ds \\ &\leq (1 + K_h)|q_2 - q_1 - \tau|\|\xi\|_{[q_1+\tau,q_1+2\tau]}. \end{aligned}$$

Then by (5.145), (5.147), (5.151)–(5.152) and (5.42)–(5.43),

$$\|\xi_p\|_{[-\tau,0]} = \left(\frac{2mM_hL^2(1+K_h)(2M_hH+\alpha)(2\beta+\alpha)}{\tau\alpha^3\beta d}\right) \leq \delta. \tag{5.153}$$

Since (5.153) holds for all $\psi \in \mathcal{C}_{[-\tau,0]}$ satisfying $\|\psi\|_{[-\tau,0]} = 1$, the conclusion of the lemma follows. □

Corollary 5.3. *Suppose that $\{x^\tau : \tau > \tau_0\}$ is a family of SOPS such that for each $\tau > \tau_0$, x^τ is a SOPS of DDER with delay τ and $q_0 = -\tau$. Then for each $\delta > 0$, there exists $\tau^\delta \geq \tau_0$ such that if $\tau > \tau^\delta$, then*

(i) the semiflow Σ, defined in (5.113) *but with x^τ in place of x^*, is continuously Fréchet differentiable in a neighborhood of (p^τ, x_0^τ), where p^τ is the period of x^τ;*

(ii) the function Z, defined in (5.128) *but with x^τ in place of x^*, is continuously Fréchet differentiable in a neighborhood of (ℓ_1^τ, x_0^τ), where ℓ_1^τ is defined as in Lemma 5.5, but with x^τ and ℓ_1^τ in place of x^* and ℓ_1;*

(iii) the functions Δ and Γ are well defined in a neighborhood $\mathcal{W}$ of x_0^τ as in Lemma 5.18 and (5.142)*, respectively, but with x^τ in place of x^*, and Δ and Γ are continuously Fréchet differentiable in $\mathcal{W}$;*

(iv) the Fréchet derivative $D\Gamma$ satisfies $\|D\Gamma(x_0^\tau)\| \leq \delta$.

Proof. This is an immediate consequence of Lemmas 5.16–5.19 and the fact that the family $\{x^\tau : \tau > \tau_0\}$ of SOPS we chose in Section 5.2 was only subject to the constraint that for each $\tau > \tau_0$, x^τ denoted a SOPS of DDER with delay τ and such that $q_0 = -\tau$. □

Theorem 5.2. *For each $\delta > 0$ there exists $\bar{\tau}^\delta \geq \tau_0$ such that if $\tau > \bar{\tau}^\delta$ and x^* is a SOPS of DDER with delay τ such that $q_0 = -\tau$, then*

(i) the semiflow Σ, defined in (5.113)*, is continuously Fréchet differentiable in a neighborhood of (p, x_0^*);*

(ii) the function Z, defined in (5.128)*, is continuously Fréchet differentiable in a neighborhood of (ℓ_1, x_0^*);*

(iii) the functions Δ and Γ are well defined in a neighborhood $\mathcal{W}$ of x_0^* as in Lemma 5.18 and (5.142)*, respectively, and Δ and Γ are continuously Fréchet differentiable in $\mathcal{W}$;*

(iv) the Fréchet derivative $D\Gamma$ satisfies $\|D\Gamma(x_0^*)\| \leq \delta$.

Proof. We give a proof by contradiction. Fix $\delta > 0$. Suppose that such a $\bar{\tau}^\delta \geq \tau_0$ does not exist. Then there exists a sequence of delays $\{\tau_n\}_{n=1}^\infty$ such that $\tau_n > \tau_0$ for all n and $\tau_n \to \infty$ as $n \to \infty$, and an associated sequence $\{x^n\}_{n=1}^\infty$ of SOPS where for each n, x^n is a SOPS of DDER with delay τ_n and $q_0 = -\tau_n$, and for each n at least one of (i)–(iv) does not hold with x^n in place of x^*. To obtain a contradiction, embed the sequence $\{x^n\}_{n=1}^\infty$ into a family of SOPS $\{x^\tau : \tau > \tau_0\}$

such that (a) for each $\tau > \tau_0$, x^τ denotes a SOPS of DDER with delay τ and such that $q_0 = -\tau$; and (b) for each $n \in \mathbb{N}$, $x^{\tau_n} = x^n$. Then by Corollary 5.3 there exists $\tau^\delta > \tau_0$ such that (i)–(iv) hold for all $\tau > \tau^\delta$. Then there exists $n \in \mathbb{N}$ such that $\tau_n > \tau^\delta$ and so (i)–(iv) hold with x^n in place of x^*. This contradicts our assumption, proving the theorem. □

Theorem 5.3. *Fix $\delta_0 \in (0,1)$ and $\tau > \bar{\tau}^{\delta_0}$. Let x^* be a SOPS of DDER with delay τ such that $q_0 = -\tau$. Then there exist constants $\varepsilon > 0$ and $K_\rho > 0$ such that given any*

$$0 < \gamma < \frac{|\log \delta_0|}{p}, \tag{5.154}$$

there exists $K_\gamma > 0$ such that if $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ satisfies $\|\varphi - x^*_\sigma\|_{[-\tau,0]} \leq \varepsilon$ for some $\sigma \in [0,p)$, then there exists $\rho \in (-p,p)$ satisfying

$$|\rho| \leq K_\rho \|\varphi - x^*_\sigma\|_{[-\tau,0]} \tag{5.155}$$

and such that

$$\|x_t - x^*_{t+\sigma+\rho+p}\|_{[-\tau,0]} \leq K_\gamma e^{-\gamma t} \|\varphi - x^*_\sigma\|_{[-\tau,0]} \text{ for all } t \geq 0, \tag{5.156}$$

where x is the unique solution of DDER with delay τ and initial condition φ.

Remark 5.2. Note that since σ is non-negative and $|\rho| < p$, it follows that $t+\sigma+\rho+p \geq 0$ for all $t \geq 0$ and so $x^*_{t+\sigma+\rho+p}$ in (5.156) is well defined for all $t \geq 0$. When $\sigma + \rho \geq 0$ $x^*_{t+\sigma+\rho}$ is well-defined for all $t \geq 0$, so by the periodicity of x^*, we can replace $x^*_{t+\sigma+\rho+p}$ with $x^*_{t+\sigma+\rho}$.

Proof. In this proof, all solutions of DDER are with the fixed delay $\tau > \bar{\tau}^{\delta_0}$. Note that by (3.10), for any solution x of DDER, for all $\tau \leq s < t < \infty$,

$$\|x_t - x_s\|_{[-\tau,0]} = \sup_{u \in [-\tau,0]} |x(t+u) - x(s+u)| \leq H|t-s|. \tag{5.157}$$

For the SOPS x^*, by periodicity and (5.157), for all $0 \leq s < t < \infty$,

$$\|x^*_t - x^*_s\|_{[-\tau,0]} = \|x^*_{t+p} - x^*_{s+p}\|_{[-\tau,0]} \leq H|t-s|. \tag{5.158}$$

We break the proof into three parts:

(a) First, we show that for each γ satisfying (5.154), there exist positive constants $\varepsilon_1(\gamma)$, $\widetilde{K}_\rho(\gamma)$ and $K_1(\gamma)$ such that if $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ satisfies $\|\varphi - x_0^*\|_{[-\tau,0]} \le \varepsilon_1(\gamma)$, then there exists $\rho \in (-p,p)$ such that (5.155) holds with $\widetilde{K}_\rho(\gamma)$ in place of K_ρ and (5.156) holds with $\sigma = 0$ and $K_1(\gamma)$ in place of K_γ.

(b) Second, we show there exist positive constants ε_2 and $\overline{K}_\rho$ such that for each γ satisfying (5.154), there exists $K_5(\gamma) > 0$ such that if $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ satisfies $\|\varphi - x_0^*\|_{[-\tau,0]} \le \varepsilon_2$, then there exists $\rho \in (-p,p)$ such that (5.155) holds with $\overline{K}_\rho$ in place of K_ρ and (5.156) holds with $\sigma = 0$ and $K_5(\gamma)$ in place of K_γ.

(c) Lastly, we prove the statement of the theorem.

Proof of part (a): suppose $\gamma > 0$ satisfies (5.154). Let $\delta = e^{-\gamma p}$. Then $\delta \in (\delta_0, 1)$. By parts (iii) and (iv) of Theorem 5.2, Γ is continuously Fréchet differentiable in a neighborhood $\mathcal{W}$ of x_0^* and $\|D\Gamma(x_0^*)\| \le \delta_0 < \delta < 1$, there exists a possibly smaller neighborhood $\mathcal{W}^\delta$ of x_0^* such that

$$\|\Gamma(\varphi) - x_0^*\|_{[-\tau,0]} \le \delta \|\varphi - x_0^*\|_{[-\tau,0]} \text{ for all } \varphi \in \mathcal{W}^\delta. \tag{5.159}$$

By part (iii) of Theorem 5.2, Δ is also continuously Fréchet differentiable in $\mathcal{W}$ and $\Delta(x_0^*) = 0$, so we can choose $\varepsilon_1(\gamma), K_2(\gamma) > 0$ such that

$$\mathcal{W}(\varepsilon_1(\gamma)) = \left\{\varphi \in \mathcal{C}^+_{[-\tau,0]} : \|\varphi - x_0^*\|_{[-\tau,0]} < \varepsilon_1(\gamma)\right\} \subseteq \mathcal{W}^\delta$$

and

$$|\Delta(\varphi)| \le K_2(\gamma)\|\varphi - x_0^*\|_{[-\tau,0]} \text{ for all } \varphi \in \mathcal{W}(\varepsilon_1(\gamma)). \tag{5.160}$$

By choosing a possibly smaller $\varepsilon_1(\gamma) > 0$ such that $\varepsilon_1(\gamma)K_2(\gamma) < \tau$, (5.160) ensures that

$$|\Delta(\varphi)| < \tau \text{ for all } \varphi \in \mathcal{W}(\varepsilon_1(\gamma)). \tag{5.161}$$

Given $\varphi \in \mathcal{W}(\varepsilon_1(\gamma))$, since $\delta < 1$ we can iterate (5.159) to obtain that $\Gamma^k(\varphi) \in \mathcal{W}(\varepsilon_1(\gamma))$ for each $k \in \mathbb{N}$ and

$$\|\Gamma^k(\varphi) - x_0^*\|_{[-\tau,0]} \le \delta^k \|\varphi - x_0^*\|_{[-\tau,0]} = e^{-\gamma k p}\|\varphi - x_0^*\|_{[-\tau,0]}. \tag{5.162}$$

For each $k \in \mathbb{N}$, define

$$\Delta(\varphi, k) = \sum_{j=0}^{k-1} \Delta\left(\Gamma^j(\varphi)\right), \tag{5.163}$$

$$t_k = kp + \Delta(\varphi, k), \tag{5.164}$$

where $\Gamma^0(\varphi) = \varphi$. Let x denote the unique solution of DDER with initial condition φ. We will show by induction that $\Gamma^k(\varphi) = x_{t_k}$ for all $k \in \mathbb{N}$. By the definition of Γ in (5.142), $\Gamma(\varphi) = x_{t_1}$. Now suppose that $\Gamma^k(\varphi) = x_{t_k}$ for some $k \geq 1$. Then

$$\begin{aligned}
\Gamma^{k+1}(\varphi) &= \Gamma\left(\Gamma^k(\varphi)\right) = \Sigma\left(p + \Delta(\Gamma^k(\varphi)), x_{t_k}\right) \\
&= \Sigma\left(p + \Delta(\Gamma^k(\varphi)), \Sigma(t_k, \varphi)\right) \\
&= \Sigma\left(t_k + p + \Delta(\Gamma^k(\varphi)), \varphi\right) \\
&= x_{t_{k+1}}.
\end{aligned}$$

where the third equality uses the semiflow property of Σ. Therefore, by the induction principle, $\Gamma^k(\varphi) = x_{t_k}$ for each $k \in \mathbb{N}$. Define $t_0 = 0$ and note that by (5.161), $|\Delta(\varphi)| < \tau < \frac{p}{2}$, so $t_k < t_{k+1}$ for each $k \in \mathbb{N}_0$. By (5.160) and (5.162), for each $k \in \mathbb{N}$ we have

$$\left|\Delta\left(\Gamma^k(\varphi)\right)\right| \leq K_2(\gamma) e^{-\gamma kp} \|\varphi - x_0^*\|_{[-\tau,0]}. \tag{5.165}$$

Define

$$\rho = -\lim_{k\to\infty} \Delta(\varphi, k), \tag{5.166}$$

where the convergence follows from (5.163) and (5.165). By (5.163) and (5.165)–(5.166), we have for each $k \in \mathbb{N}$,

$$|\Delta(\varphi, k)| \leq \sum_{j=0}^{\infty} |\Delta(\Gamma^j(\varphi))| \leq \widetilde{K}_\rho(\gamma) \|\varphi - x_0^*\|_{[-\tau,0]}, \tag{5.167}$$

$$|\rho| \leq \sum_{j=0}^{\infty} |\Delta(\Gamma^j(\varphi))| \leq \widetilde{K}_\rho(\gamma) \|\varphi - x_0^*\|_{[-\tau,0]}, \tag{5.168}$$

and

$$|\rho + \Delta(\varphi, k)| \leq \sum_{j=k}^{\infty} |\Delta(\Gamma^j(\varphi))| \leq \widetilde{K}_\rho(\gamma) e^{-\gamma kp} \|\varphi - x_0^*\|_{[-\tau,0]}, \tag{5.169}$$

where

$$\widetilde{K}_\rho(\gamma) = K_2(\gamma) \sum_{j=0}^{\infty} e^{-\gamma jp} = \frac{K_2(\gamma)}{1 - e^{-\gamma p}}. \tag{5.170}$$

By choosing a possibly smaller $\varepsilon_1(\gamma) > 0$ such that $\varepsilon_1(\gamma)\widetilde{K}_\rho(\gamma) < p$, it follows from (5.178) that $\rho \in (-p, p)$.

Now let $I_k = [t_k, t_{k+1}]$ for each $k \in \mathbb{N}_0$. By (5.164), (5.167) and the fact that $\varepsilon_1(\gamma)\widetilde{K}_\rho(\gamma) < p$, $t_k \to \infty$ as $n \to \infty$, so $\bigcup_{k=0}^{\infty} I_k$ covers $[0, \infty)$. By (5.163)–(5.165) and (5.167), we have that for $\varphi \in \mathcal{W}(\varepsilon_1(\gamma))$ and $k \in \mathbb{N}_0$,

$$t_{k+1} - t_k = p + \Delta(\Gamma^k(\varphi)) \leq p + \widetilde{K}_\rho(\gamma)\varepsilon_1(\gamma). \tag{5.171}$$

By (5.46), for $K_3(\gamma) = 2\exp[2K_h(p + \widetilde{K}_\rho(\gamma)\varepsilon_1(\gamma))]$, we have

$$\|x_t - x_t^*\|_{[-\tau,0]} \leq K_3(\gamma)\|\varphi - x_0^*\|_{[-\tau,0]} \tag{5.172}$$

for all $\varphi \in \mathcal{W}(\varepsilon_1(\gamma))$ and $0 \leq t \leq p + \widetilde{K}_\rho(\gamma)\varepsilon_1(\gamma)$. It follows from (5.162), (5.171)–(5.172), the fact that $\Gamma^k(\varphi) = x_{t_k}$ and the semiflow property of Σ that for $t \in I_k$

$$\begin{aligned} \|x_t - x_{t-t_k}^*\|_{[-\tau,0]} &= \|\Sigma(t - t_k, x_{t_k}) - \Sigma(t - t_k, x_0^*)\|_{[-\tau,0]} \\ &\leq K_3(\gamma)\|\Gamma^k(\varphi) - x_0^*\|_{[-\tau,0]} \\ &\leq K_3(\gamma)e^{-\gamma kp}\|\varphi - x_0^*\|_{[-\tau,0]}. \end{aligned}$$

Also, by (5.158), (5.164) and (5.169), for all $t \in I_k$,

$$\begin{aligned} \|x_{t-t_k}^* - x_{t+p+\rho}^*\|_{[-\tau,0]} &= \|x_{t-t_k+(k+1)p}^* - x_{t+p+\rho}^*\|_{[-\tau,0]} \\ &\leq H|\rho - kp + t_k| \\ &\leq H|\rho + \Delta(\varphi, k)| \\ &\leq H\widetilde{K}_\rho(\gamma)e^{-\gamma kp}\|\varphi - x_0^*\|_{[-\tau,0]}. \end{aligned}$$

Combining the previous two inequalities yields

$$\|x_t - x_{t+p+\rho}^*\|_{[-\tau,0]} \leq K_4(\gamma)e^{-\gamma kp}\|\varphi - x_0^*\|_{[-\tau,0]} \text{ for all } t \in I_k, \tag{5.173}$$

where

$$K_4(\gamma) = K_3(\gamma) + H\widetilde{K}_\rho(\gamma). \tag{5.174}$$

Further, by (5.164) and (5.167), we have

$$t_{k+1} - kp = p + \Delta(\varphi, k+1) \leq p + \widetilde{K}_\rho(\gamma)\varepsilon_1(\gamma). \tag{5.175}$$

Therefore, by (5.173) and (5.175), for all $t \in I_k$,

$$\begin{aligned} \|x_t - x^*_{t+p+\rho}\|_{[-\tau,0]} &\leq K_4(\gamma)e^{-\gamma kp}e^{\gamma(t_{k+1}-t)}\|\varphi - x^*_0\|_{[-\tau,0]} \\ &\leq K_1(\gamma)e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]}, \end{aligned} \tag{5.176}$$

where $K_1(\gamma) = K_4(\gamma)e^{\gamma(p+\widetilde{K}_\rho(\gamma)\varepsilon_1(\gamma))}$. From (5.168) and (5.176), we see that part (a) holds.

We now prove part (b): fix $\bar{\gamma} > 0$ satisfying (5.154) and set

$$\varepsilon_2 = \varepsilon_1(\bar{\gamma}),\ \overline{K}_\rho = \widetilde{K}_\rho(\bar{\gamma}).$$

By part (a), if $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ satisfies $\|\varphi - x^*_0\|_{[-\tau,0]} \leq \varepsilon_2$ and x denotes the unique solution of DDER with initial condition φ, then for $t \geq 0$,

$$\|x_t - x^*_{t+p+\rho}\|_{[-\tau,0]} \leq K_1(\bar{\gamma})e^{-\bar{\gamma}t}\|\varphi - x^*_0\|_{[-\tau,0]}, \tag{5.177}$$

where $\rho \in (-p, p)$ satisfies

$$|\rho| \leq \overline{K}_\rho\|\varphi - x^*_0\|_{[-\tau,0]}. \tag{5.178}$$

Now take any $\gamma > 0$ satisfying (5.154) and set

$$T(\gamma) = \max\left\{0, \frac{1}{\bar{\gamma}}\log\left(\frac{K_1(\bar{\gamma})\varepsilon_2}{\varepsilon_1(\gamma)}\right)\right\}. \tag{5.179}$$

By (5.177) and (5.179), if $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ satisfies $\|\varphi - x^*_0\|_{[-\tau,0]} \leq \varepsilon_2$ and x denotes the unique solution of DDER with initial condition φ, then for $t \geq T(\gamma)$,

$$\|x_t - x^*_{t+p+\rho}\|_{[-\tau,0]} \leq K_1(\bar{\gamma})e^{-\bar{\gamma}t}\|\varphi - x^*_0\|_{[-\tau,0]} \leq \varepsilon_1(\gamma). \tag{5.180}$$

Let

$$t_\gamma = n_\gamma p - \rho, \tag{5.181}$$

where $n_\gamma = \min\{k \in \mathbb{N} : kp \geq T(\gamma) + \varepsilon_2\overline{K}_\rho\}$. Note that by (5.179) and the fact that $\rho \in (-p, p)$,

$$0 \leq t_\gamma \leq T(\gamma) + \varepsilon_2\overline{K}_\rho + p + |\rho| \leq \frac{1}{\bar{\gamma}}\log\left(\frac{K_1(\bar{\gamma})\varepsilon_2}{\varepsilon_1(\gamma)}\right) + \varepsilon_2\overline{K}_\rho + 2p. \tag{5.182}$$

Then, by (5.178), we have that $t_\gamma \geq n_\gamma p - \varepsilon_2 \overline{K}_\rho \geq T(\gamma)$. Therefore, by (5.180)–(5.181) and the periodicity of x^*, we have

$$\|x_{t_\gamma} - x_0^*\|_{[-\tau,0]} = \|x_{t_\gamma} - x^*_{t_\gamma+p+\rho}\|_{[-\tau,0]} \leq \varepsilon_1(\gamma). \tag{5.183}$$

Define $x^\gamma \in \mathcal{C}^+_{[-\tau,\infty)}$ by $x^\gamma(t) = x(t_\gamma + t)$ for all $t \geq -\tau$, so that x^γ is a solution of DDER with initial condition $x_0^\gamma = x_{t_\gamma}$. By (5.183), $\|x_0^\gamma - x_0^*\|_{[-\tau,0]} \leq \varepsilon_1(\gamma)$, so by part (a), there exists $\tilde{\rho} \in (-p, p)$ such that for all $t \geq 0$,

$$\|x_t^\gamma - x^*_{t+p+\tilde{\rho}}\|_{[-\tau,0]} \leq K_1(\gamma) e^{-\gamma t} \|x_0^\gamma - x_0^*\|_{[-\tau,0]}. \tag{5.184}$$

From the definition of x^γ and (5.180)–(5.184), for $t \geq 0$,

$$\begin{aligned} \|x_{t+t_\gamma} - x^*_{t+p+\tilde{\rho}}\|_{[-\tau,0]} &= \|x_t^\gamma - x^*_{t+p+\tilde{\rho}}\|_{[-\tau,0]} \\ &\leq K_1(\gamma) e^{-\gamma t} \|x_0^\gamma - x_0^*\|_{[-\tau,0]} \\ &\leq K_1(\gamma) e^{-\gamma t} \|x_{t_\gamma} - x^*_{t_\gamma+p+\rho}\|_{[-\tau,0]} \\ &\leq K_1(\gamma) e^{-\gamma t} K_1(\bar{\gamma}) e^{-\bar{\gamma} t_\gamma} \|\varphi - x_0^*\|_{[-\tau,0]}. \end{aligned} \tag{5.185}$$

By (5.180)–(5.183), (5.185) and the fact that $\|\varphi - x_0^*\|_{[-\tau,0]} < \varepsilon_2$, for $t \geq 0$,

$$\begin{aligned} \|x_t^* - x^*_{t+p+\tilde{\rho}}\|_{[-\tau,0]} &= \|x^*_{t+t_\gamma+\rho} - x^*_{t+p+\tilde{\rho}}\| \\ &\leq \|x^*_{t+t_\gamma+\rho} - x_{t+t_\gamma}\|_{[-\tau,0]} + \|x_{t+t_\gamma} - x^*_{t+p+\tilde{\rho}}\|_{[-\tau,0]} \\ &\leq K_1(\bar{\gamma}) e^{-\bar{\gamma} t} \|\varphi - x_0^*\|_{[-\tau,0]} \\ &\qquad + K_1(\gamma) K_1(\bar{\gamma}) e^{-\bar{\gamma} t_\gamma} e^{-\gamma t} \|\varphi - x_0^*\|_{[-\tau,0]} \\ &\to 0 \text{ as } t \to \infty. \end{aligned} \tag{5.186}$$

It follows that $x^*(t) = x^*(t + p + \tilde{\rho})$ for all $t \geq 0$. If not, then there exists $t_0 > 0$ such that $|x^*(t_0) - x^*(t_0 + p + \tilde{\rho})| > 0$. By the periodicity of x^*, it follows that $|x^*(t_0 + np) - x^*(t_0 + (n+1)p + \tilde{\rho})| > 0$ for all $n \geq 1$, which contradicts (5.186). Hence, $x^*(t) = x^*(t + p + \tilde{\rho})$ for all $t \geq 0$, so x^* is periodic with period $p + \tilde{\rho}$. Since p is the minimal period of x^*, $\tilde{\rho}$ is an integer multiple of p. Then using that $|\tilde{\rho}| < p$, $\tilde{\rho} = 0$. By (5.185), with $\tilde{\rho} = 0$ and t replaced by $t - t_\gamma$, we have, for $t \geq t_\gamma$,

$$\begin{aligned} \|x_t - x^*_{t+p-t_\gamma}\|_{[-\tau,0]} &\leq K_1(\gamma) K_1(\bar{\gamma}) e^{-\gamma(t-t_\gamma) - \bar{\gamma} t_\gamma} \|\varphi - x_0^*\|_{[-\tau,0]} \\ &\leq K_1(\gamma) K_1(\bar{\gamma}) e^{t_\gamma(\gamma - \bar{\gamma})} e^{-\gamma t} \|\varphi - x_0^*\|_{[-\tau,0]} \\ &\leq K_6(\gamma) e^{-\gamma t} \|\varphi - x_0^*\|_{[-\tau,0]}, \end{aligned} \tag{5.187}$$

where $K_6(\gamma) = K_1(\gamma)K_1(\bar{\gamma})e^{t_\gamma(\gamma-\bar{\gamma})}$. For $0 \le t \le t_\gamma$, by (5.177) and (5.182), we have

$$\begin{aligned} \|x_t - x^*_{t+p+\rho}\|_{[-\tau,0]} &\le K_1(\bar{\gamma})e^{-\bar{\gamma}t}\|\varphi - x^*_0\|_{[-\tau,0]} \\ &\le K_1(\bar{\gamma})e^{\gamma t_\gamma}e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]} \\ &\le K_1(\bar{\gamma})\exp(\gamma(\varepsilon_2\overline{K}_\rho + 2p))\left(\frac{K_1(\bar{\gamma})\varepsilon_2}{\varepsilon_1(\gamma)}\right)^{\gamma/\bar{\gamma}} e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]} \\ &\le K_7(\gamma)e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]}, \end{aligned} \tag{5.188}$$

where $K_7(\gamma) = K_1(\bar{\gamma})\exp(\gamma(\varepsilon_2\overline{K}_\rho + 2p))\left(\frac{K_1(\bar{\gamma})\varepsilon_2}{\varepsilon_1(\gamma)}\right)^{\gamma/\bar{\gamma}}$. Combining (5.187)–(5.188) and letting $K_5(\gamma) = \max(K_6(\gamma), K_7(\gamma))$ we have, for $t \ge 0$,

$$\|x_t - x^*_{t+p+\rho}\|_{[-\tau,0]} \le K_5(\gamma)e^{-\gamma t}\|\varphi - x^*_0\|_{[-\tau,0]}. \tag{5.189}$$

It follows from (5.178) and (5.189) that part (b) holds.

We now complete the proof. Suppose $\sigma \in [0,p)$ and $\varphi \in \mathcal{C}^+_{[-\tau,0]}$. Let x denote the unique solution of DDER with initial condition φ. By (5.46), with $x^*(\sigma + \cdot)$ in place of $x^\dagger(\cdot)$, for $\sigma \in [0,p)$ and $0 \le t \le p$,

$$\|x_t - x^*_{t+\sigma}\|_{[-\tau,0]} \le 2\exp(2K_h p)\|\varphi - x^*_\sigma\|_{[-\tau,0]}. \tag{5.190}$$

Choose $\varepsilon > 0$ satisfying

$$\varepsilon \le \frac{\varepsilon_2}{2\exp(2K_h p)}. \tag{5.191}$$

Fix $\gamma > 0$ satisfying (5.154) and suppose $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ satisfies $\|\varphi - x^*_\sigma\|_{[-\tau,0]} \le \varepsilon$ for some $\sigma \in [0,p)$. By (5.190), for $0 \le t \le p - \sigma$,

$$\begin{aligned} \|x_t - x^*_{t+\sigma}\|_{[-\tau,0]} &\le 2\exp(2K_h p)\|\varphi - x^*_\sigma\|_{[-\tau,0]} \\ &\le 2\exp(2K_h p)e^{\gamma p}e^{-\gamma t}\|\varphi - x^*_\sigma\|_{[-\tau,0]} \end{aligned} \tag{5.192}$$

By (5.190)–(5.191), we have

$$\|x_{p-\sigma} - x^*_0\|_{[-\tau,0]} \le 2\exp(2K_h p)\|\varphi - x^*_\sigma\|_{[-\tau,0]} \le \varepsilon_2. \tag{5.193}$$

Then by part (b) and (5.193), there exists $K_5(\gamma) > 0$ and $\rho \in (-p,p)$ such that for $t \ge 0$,

$$\begin{aligned} \|x_{t+p-\sigma} - x^*_{t+p+\rho}\|_{[-\tau,0]} &\le K_5(\gamma)e^{-\gamma t}\|x_{p-\sigma} - x^*_0\|_{[-\tau,0]} \\ &\le 2K_5(\gamma)\exp(2K_h p)e^{-\gamma t}\|\varphi - x^*_\sigma\|_{[-\tau,0]} \end{aligned} \tag{5.194}$$

and

$$|\rho| \le \overline{K}_\rho \|x_{p-\sigma} - x_0^*\|_{[-\tau,0]} \le 2\overline{K}_\rho \exp(2K_h p)\|\varphi - x_\sigma^*\|_{[-\tau,0]}. \tag{5.195}$$

Set $K_\rho = 2\overline{K}_\rho \exp(2K_h p)$ so that (5.155) holds. By (5.194) and the periodicity of x^*, for $t \ge p - \sigma$,

$$\|x_t - x^*_{t+p+\sigma+\rho}\|_{[-\tau,0]} \le 2K_5(\gamma)\exp(2K_h p)e^{-\gamma t}\|\varphi - x_\sigma^*\|_{[-\tau,0]}. \tag{5.196}$$

By (5.192), (5.158) and (5.195), we have for $0 \le t \le p - \sigma$ that

$$\begin{aligned}\|x_t - x^*_{t+p+\sigma+\rho}\|_{[-\tau,0]} &\le \|x_t - x^*_{t+\sigma}\|_{[-\tau,0]} + \|x^*_{t+p+\sigma} - x^*_{t+p+\sigma+\rho}\|_{[-\tau,0]} \\ &\le 2\exp(2K_h p)e^{\gamma p}e^{-\gamma t}\|\varphi - x_\sigma^*\|_{[-\tau,0]} \\ &\quad + 2H\overline{K}_\rho \exp(2K_h p)e^{\gamma p}e^{-\gamma t}\|\varphi - x_\sigma^*\|_{[-\tau,0]}.\end{aligned} \tag{5.197}$$

Upon setting $K_\gamma = 2\exp(2K_h p)\max(K_5(\gamma), e^{\gamma p}(1 + H\overline{K}_\rho))$, we see from (5.196)–(5.197) that (5.156) holds, which completes the proof. □

5.7 Proof of Uniqueness

In this section we show that if the delay τ is sufficiently large, then any SOPS x^* of DDER with delay τ, is unique up to time translation, which will allow us to complete the proof of Theorem 3.2. The main tool which we use to prove the uniqueness of a SOPS is the *fixed point index*. For an in-depth discussion of the fixed point index and its properties, see [20]. We briefly summarize some important definitions and properties regarding the special case of the fixed point index used here. Suppose that X is a Banach space, K is a closed, convex subset of X and U is a relatively open subset of K. Assume that $f : K \to K$ is a continuous, compact map and $S = \{x \in U : f(x) = x\}$ is compact (possibly empty). Then there is defined an integer $\iota_K(f, U)$ called the fixed point index of f on U. If $\iota_K(f, U) \ne 0$, then f has a fixed point in U. If $U = K$, then since f is continuous and compact and K is closed, $S = \{x \in K : f(x) = x\}$ is a compact set and so $\iota_K(f, K)$ is well defined. The following proposition is a special case of Corollary 3 in [17].

Proposition 5.2. *Suppose that K is bounded. Then $\iota_K(f, K) = 1$.*

The following property is known as the additivity property of the fixed point index.

Proposition 5.3. *Suppose that U_1 and U_2 are disjoint subsets of U that are relatively open in K and such that $S \subset U_1 \cup U_2$, then $\iota_K(f, U_j)$ is defined for $j = 1, 2$ and*

$$\iota_K(f, U) = \iota_K(f, U_1) + \iota_K(f, U_2).$$

For the following, recall the definition of an ejective fixed point from Definition 4.1.

Proposition 5.4. *Suppose K is bounded and infinite-dimensional. If $x_0 \in K$ is an ejective fixed point of f and U is a relatively open neighborhood of x_0 in K such that the closure of U does not contain another fixed point of f, then $\iota_K(f, U) = 0$.*

Proof. See Corollary 1.1 in [18]. □

Definition 5.2. Suppose $x_0 \in K$ is a fixed point of f. Then x_0 is an *attractive fixed point* if there exists a relatively open neighborhood U of x_0 in K such that if V is any relatively open neighborhood of x_0 in K, there exists $n_0 = n_0(V) \in \mathbb{N}$ such that $f^n(x) \in V$ for all $n \geq n_0$ and $x \in U$.

Proposition 5.5. *Suppose that $x_0 \in K$ is an attractive fixed point of f. If V is a relatively open neighborhood of x_0 in K such that x_0 is the only fixed point of f in V, then $\iota_K(f, V) = 1$.*

Proof. See Theorem 3.5 in [20] □

For the following, recall the definitions of $\widetilde{\mathcal{K}}$ and $\Lambda : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}$ from (4.15)–(4.17) and (4.26), respectively.

Lemma 5.20. *Fix $\delta_0 \in (0, 1)$ and let $\tau > \bar{\tau}^{\delta_0}$ be as in Theorem 5.2. Suppose that x^* is a SOPS of DDER with delay τ such that $q_0 = -\tau$. Let $\hat{x}^*$ denote the associated solution of DDER^n defined via (4.11). Then $\hat{x}^*_0 \in \widetilde{\mathcal{K}}$ and $\hat{x}^*_0$ is an attractive fixed point of Λ.*

Proof. By Lemma 5.1 and the periodicity of $\hat{x}^*$, $\hat{x}_0^* \in \widetilde{\mathcal{K}}$ and $\Lambda(\hat{x}_0^*) = \hat{x}_0^*$. To show that $\hat{x}_0^*$ is an attractive fixed point of Λ, we need to find an $\varepsilon^* > 0$ such that for each $\delta^* > 0$, there exists $n_0 = n_0(\delta^*) \in \mathbb{N}$ such that if $\hat{\varphi} \in \widetilde{\mathcal{K}}$ satisfies $\|\hat{\varphi} - \hat{x}_0^*\|_{[-1,0]} < \varepsilon^*$, then $\|\Lambda^n(\hat{\varphi}) - \hat{x}_0^*\|_{[-\tau,0]} < \delta^*$ for all $n \geq n_0$.

Let ε, γ, K_γ and K_ρ be positive constants as in Theorem 5.3. Recall that if $\hat{\varphi} \in \widetilde{\mathcal{K}}$, then by Lemma 4.4, the unique solution $\hat{x}$ of DDER$^{\text{n}}$ with initial condition $\hat{\varphi}$ is slowly oscillating, i.e., there are $0 < \hat{q}_1^{\hat{x}} < \hat{q}_2^{\hat{x}} < \cdots$ (called the zeros of $\hat{x}$) such that

(i) $\hat{x}(\hat{q}_n^{\hat{x}}) = 0$ for $n = 1, 2, \dots,$

(ii) $\hat{q}_1^{\hat{x}} > 0$ and $\hat{q}_{n+1}^{\hat{x}} - \hat{q}_n^{\hat{x}} > 1$ for $n = 1, 2, \dots,$

(iii) $\hat{x}(t) > 0$ for $t \in [0, \hat{q}_1^{\hat{x}})$,
$\hat{x}(t) < 0$ for $t \in (\hat{q}_{2n-1}^{\hat{x}}, \hat{q}_{2n}^{\hat{x}})$ for $n = 1, 2, \dots,$
$\hat{x}(t) > 0$ for $t \in (\hat{q}_{2n}^{\hat{x}}, \hat{q}_{2n+1}^{\hat{x}})$ for $n = 1, 2, \dots.$

We will show by induction that $\Lambda^n(\hat{\varphi}) = \hat{x}_{\hat{q}_{2n}^{\hat{x}}+1}$ for all $n \in \mathbb{N}$. By definition $\Lambda(\hat{\varphi}) = \hat{x}_{\hat{q}_2^{\hat{x}}+1}$. Suppose that $\Lambda^n(\hat{\varphi}) = \hat{x}_{\hat{q}_{2n}^{\hat{x}}+1}$ for some $n \in \mathbb{N}$. Define

$$\begin{aligned}\hat{q}_1^n &= \inf\{t \geq 0 : \hat{x}(\hat{q}_{2n}^{\hat{x}} + 1 + t) = 0\}\\ &= \inf\{t \geq \hat{q}_{2n+1}^{\hat{x}} + 1 : \hat{x}(t) = 0\} - \hat{q}_{2n}^{\hat{x}} - 1\\ &= \hat{q}_{2n+1}^{\hat{x}} - \hat{q}_{2n}^{\hat{x}} - 1,\end{aligned}$$

$$\begin{aligned}\hat{q}_2^n &= \inf\{t > \hat{q}_1^n : \hat{x}(\hat{q}_{2n}^{\hat{x}} + 1 + t) = 0\}\\ &= \inf\{t > \hat{q}_{2n+1}^{\hat{x}} : \hat{x}(t) = 0\} - \hat{q}_{2n}^{\hat{x}} - 1\\ &= \hat{q}_{2(n+1)}^{\hat{x}} - \hat{q}_{2n}^{\hat{x}} - 1.\end{aligned}$$

Then $\hat{q}_1^n$ and $\hat{q}_2^n$ denote the first and second zeros of the unique solution of DDER with initial condition $\hat{x}_{\hat{q}_{2n}^{\hat{x}}+1} \in \widetilde{\mathcal{K}}$. Then by the definition of Λ, the induction hypothesis and the semiflow property of Σ,

$$\begin{aligned}\Lambda^{n+1}(\hat{\varphi}) &= \Lambda(\Lambda^n(\hat{\varphi})) = \Lambda(\hat{x}_{\hat{q}_{2n}^{\hat{x}}+1})\\ &= \Sigma(\hat{q}_2^n + 1, \hat{x}_{\hat{q}_{2n}^{\hat{x}}+1}) = \Sigma(\hat{q}_{2n}^{\hat{x}} + 1 + \hat{q}_2^n + 1, \hat{\varphi})\\ &= \hat{x}_{\hat{q}_{2(n+1)}^{\hat{x}}+1}.\end{aligned}$$

Therefore, by the induction principle, $\Lambda^n(\hat{\varphi}) = \hat{x}_{\hat{q}^{\hat{x}}_{2n}+1}$ for all $n \in \mathbb{N}$. If x is the solution of DDER associated with $\hat{x}$ via (4.12) and $0 < q_1^x < q_2^x < \cdots$ are given by $q_n^x = \tau \hat{q}_n^{\hat{x}}$ for each $n \in \mathbb{N}$, then these are the times in $[0, \infty)$ where $x(\cdot)$ is equal to L (called the "zeros" of x) and for each $n \in \mathbb{N}$,

$$\|\Lambda^n(\hat{\varphi}) - \hat{x}_0^*\|_{[-1,0]} = \|x_{q_{2n}^x+\tau} - x_0^*\|_{[-\tau,0]}. \tag{5.198}$$

Let $-\tau = q_0 < q_1 < q_2 < \cdots$ denote the zeros of x^*. Let $0 < \eta < \frac{1}{2}\min\{\tau, q_1\}$. Then $0 < q_1 - \eta < q_1 + \eta < q_2 - \eta$ and

$$x^*(t) > L \text{ for all } t \in [-\tau + \eta, q_1 - \eta], \tag{5.199}$$

$$x^*(t) < L \text{ for all } t \in [q_1 + \eta, q_2 - \eta]. \tag{5.200}$$

By (5.199)–(5.200) and the continuity of x^*, we can choose $d > 0$ satisfying

$$d < \min\{|x^*(t) - L| : t \in [-\tau + \eta, q_1 - \eta] \cup [q_1 + \eta, q_2 - \eta]\}. \tag{5.201}$$

Let $\varepsilon > 0$ and K_ρ be as in the statement of Theorem 5.3. Fix γ such that (5.154) holds and let K_γ be the associated constant from Theorem 5.3. Choose $\varepsilon^* \in (0, \varepsilon)$ satisfying

$$\varepsilon^* < \min\left(\frac{d}{K_\gamma + HK_\rho}, \frac{\eta}{2K_\rho}\right). \tag{5.202}$$

Then if $\|\varphi - x_0^*\|_{[-\tau,0]} < \varepsilon^*$, there exists $\rho \in (-p, p)$ satisfying (5.155) such that (5.156) holds with $\sigma = 0$ and ε^* in place of ε. It then follows from (3.10), (5.156), (5.201)–(5.202) and the periodicity of x^* that for each $n \in \mathbb{N}_0$ and all $t \in [q_{2n} + \eta, q_{2n+1} - \eta]$, the solution x of DDER with delay τ and initial condition φ satisfies

$$\begin{aligned} x(t) &\geq x^*(t) - |x^*(t) - x^*(t+\rho)| - |x^*(t+\rho) - x(t)| \\ &> L + d - (HK_\rho + K_\gamma e^{-\gamma t})\varepsilon^* > L. \end{aligned}$$

Similarly, for each $n \in \mathbb{N}_0$ and all $t \in [q_{2n+1} + \eta, q_{2n+2} - \eta]$,

$$\begin{aligned} x(t) &\leq x^*(t) + |x^*(t) - x^*(t+\rho)| + |x^*(t+\rho) - x(t)| \\ &< L - d + (HK_\rho + K_\gamma e^{-\gamma t})\varepsilon^* < L. \end{aligned}$$

Since x is continuous, its zeros are separated by at least τ, $\eta < \frac{\tau}{2}$, $q_1^x > 0$, $x(t) > L$ for all $t \in [-\tau + \eta, q_1 - \eta]$ and $x(q_1 + \eta) < L$, we have $q_1^x \in (q_1 - \eta, q_1 + \eta)$ and x

does not have another zero in $(q_1-\eta, q_1+\eta)$. By iterating this argument, we have that $q_n^x \in (q_n-\eta, q_n+\eta)$ for each $n \in \mathbb{N}$.

Now fix $\delta^* > 0$. Choose $\eta^* \in (0, \frac{\eta}{2})$ such that $\eta^* < \frac{\delta^*}{2H}$. Let $d^* = \min\{|x^*(q_2-\eta^*)-L|, |x^*(q_2+\eta^*)-L|\}$. Since $q_{2n} \to \infty$ as $n \to \infty$, there exists $n_0 = n_0(\delta^*) \in \mathbb{N}$ such that for all $n \geq n_0$,

$$q_{2n} - \eta^* - |\rho| > \max\left\{\frac{1}{\gamma}\log\left(\frac{2K_\gamma\varepsilon^*}{d^* \wedge \delta^*}\right), 0\right\}.$$

Then by (5.156) with $\sigma = 0$ and the periodicity of x^*, for $t \geq q_{2n_0} - \eta^* - \rho$,

$$\|x_t - x^*_{t+\rho}\|_{[-\tau,0]} \leq K_\gamma \varepsilon^* e^{-\gamma(q_{2n_0}-\eta^*-\rho)} < \frac{d^*}{2}.$$

Hence for $n \geq n_0$,

$$x(q_{2n}-\eta^*-\rho) \leq x^*(q_{2n}-\eta^*) + |x^*(q_{2n}-\eta^*) - x(q_{2n}-\eta^*-\rho)| < L$$

and

$$x(q_{2n}+\eta^*-\rho) \geq x^*(q_{2n}+\eta^*) - |x^*(q_{2n}+\eta^*) - x(q_{2n}+\eta^*-\rho)| > L.$$

It follows that there exists $t_n \in (q_{2n}-\rho-\eta^*, q_{2n}-\rho+\eta^*)$ such that $x(t_n) = L$. By (5.155) and (5.202), $|\rho| < \frac{\eta}{2}$. This combined with our choice of η^* implies that $t_n \in (q_{2n}-\eta, q_{2n}+\eta)$ and since q_{2n}^x is the unique zero of x in the open interval $(q_{2n}-\eta, q_{2n}+\eta)$, $q_{2n}^x = t_n \in (q_{2n}-\rho-\eta^*, q_{2n}-\rho+\eta^*)$, and so $|q_{2n}^x - \rho - q_{2n}| < \eta^*$. Thus, by (5.156), the periodicity of x^*, (3.10), the definition of η^* and our choice of n_0, for all $n \geq n_0$,

$$\begin{aligned}\|x_{q_{2n}^x+\tau} - x_0^*\|_{[-\tau,0]} &\leq \|x_{q_{2n}^x+\tau} - x^*_{q_{2n}^x+\tau+p+\rho}\|_{[-\tau,0]} + \|x^*_{q_{2n}^x+\tau+\rho} - x^*_{q_{2n}+\tau}\|_{[-\tau,0]} \\ &\leq K_\gamma e^{-\gamma(q_{2n}^x+\tau)}\|x_0 - x_0^*\|_{[-\tau,0]} + H|q_{2n}^x - q_{2n} - \rho| \\ &< \delta^*.\end{aligned}$$

By (5.198), for all $n \geq n_0$,

$$\|\Lambda^n(\hat{\varphi}) - \hat{x}_0^*\|_{[-1,0]} = \|x_{q_{2n}^x+\tau} - x_0^*\|_{[-\tau,0]} < \delta^*,$$

which completes the proof that $\hat{x}_0^*$ is an attractive fixed point of Λ. □

Theorem 5.4. *Fix $\delta_0 \in (0,1)$ and let $\tau > \bar{\tau}^{\delta_0}$. Then there exists a unique SOPS x^* of DDER with delay τ such that $q_0 = -\tau$.*

Proof. Fix $\tau > \bar{\tau}^{\delta_0}$. Recall that if we let $g(r,s) = h(s)$ for all $(r,s) \in \mathbb{R}^2_+$, then g satisfies Assumptions 3.1 and 3.2. Define $\widetilde{\mathcal{K}}$ and Λ as in (4.15)–(4.17) and (4.26). Recall that DDER$^{\mathrm{n}}$ has a unique constant solution $\hat{x} \equiv 0$. By Lemma 4.9, the constant function $\hat{\varphi} \equiv 0$ is an ejective fixed point of Λ. If $\hat{\varphi}$ is a non-constant fixed point of Λ and $\hat{x}$ is the unique solution of DDER$^{\mathrm{n}}$ with initial condition $\hat{\varphi}$, then $\hat{x}$ is periodic and by Lemma 4.4, $\hat{x}$ is a SOPS$^{\mathrm{n}}$ such that $\hat{q}_0 = -1$. Conversely, if $\hat{x}^*$ is a SOPS$^{\mathrm{n}}$ such that $\hat{q}_0 = -1$, then by Lemma 5.1, $\hat{x}^*_0 \in \widetilde{\mathcal{K}}$ and is a non-constant fixed point of Λ. Furthermore, since $\hat{x}^*$ is the normalized version of a SOPS x^* satisfying $q_0 = -\tau$, by Lemma 5.20, $\hat{x}^*_0$ is an attractive fixed point of Λ. It follows that there is a one-to-one correspondence between non-constant fixed points of Λ and SOPS$^{\mathrm{n}}$, $\hat{x}^*$, such that $\hat{q}_0 = -1$, and furthermore, all non-constant fixed points of Λ are attractive fixed points.

Recall that $\widetilde{\mathcal{K}}$ is a closed, bounded, convex, infinite-dimensional subset of a Banach space and $\Lambda : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}$ is continuous and compact by Lemma 4.7. By Proposition 5.2, the fixed point index of Λ on $\widetilde{\mathcal{K}}$ is defined and $\iota_{\widetilde{\mathcal{K}}}(\Lambda, \widetilde{\mathcal{K}}) = 1$. Let $\mathcal{S} = \{\hat{\varphi} \in \widetilde{\mathcal{K}} : \Lambda(\hat{\varphi}) = \hat{\varphi}\}$, the set of fixed points of Λ. Since Λ is continuous and compact, $\mathcal{S}$ is compact. By Lemma 4.9, the unique constant fixed point of Λ, $\hat{\varphi} \equiv 0$, is ejective, and so there exists a neighborhood $\mathcal{U}$ of $\hat{\varphi} \equiv 0$ that does not contain another fixed point of Λ in its closure. By the paragraph above, non-constant fixed points of Λ are attractive, and so for each non-constant fixed point $\hat{\varphi}$ of Λ, there exists a neighborhood $\mathcal{V}_{\hat{\varphi}}$ of $\hat{\varphi}$ that does not contain another fixed point of Λ. Since $\mathcal{S}$ is compact and each point in $\mathcal{S}$ is a fixed point that is contained in an open set that does not contain another fixed point, it follows that $\mathcal{S}$ is a finite set. By the additivity property described in Proposition 5.3 and the fact that $\mathcal{S} \subset \mathcal{U} \cup \left(\bigcup_{\hat{\varphi} \in \mathcal{S}: \hat{\varphi} \not\equiv 0} \mathcal{V}_{\hat{\varphi}} \right) \subset \widetilde{\mathcal{K}}$,

$$1 = \iota_{\widetilde{\mathcal{K}}}(\Lambda, \widetilde{\mathcal{K}}) = \iota_{\widetilde{\mathcal{K}}}(\Lambda, \mathcal{U}) + \sum_{\hat{\varphi} \in \mathcal{S}: \hat{\varphi} \not\equiv 0} \iota_{\widetilde{\mathcal{K}}}(\Lambda, \mathcal{V}_{\hat{\varphi}}) = \sum_{\hat{\varphi} \in \mathcal{S}: \hat{\varphi} \not\equiv 0} 1,$$

where the last equality follows by Propositions 5.4 and 5.5. Therefore $\mathcal{S}$ contains exactly one point besides $\hat{\varphi} \equiv 0$ and so Λ has exactly one non-constant fixed point. By the one-to-one correspondence between non-constant fixed points of Λ and $\mathrm{SOPS^n}$ with $\hat{q}_0 = -1$, there is a unique $\mathrm{SOPS^n}$ of $\mathrm{DDER^n}$ with $\hat{q}_0 = -1$ and hence by Lemma 4.1, there is a unique SOPS of DDER with $q_0 = -\tau$. $\square$

Proof of Theorem 3.2. Choose $\delta_0 \in (0,1)$. Define $\bar{\tau}^{\delta_0} \geq \tau_0$ as in Theorem 5.2 and set $\tau^* = \bar{\tau}^{\delta_0}$. Fix a $\tau > \tau^*$. By Theorem 5.4 there exists a unique SOPS x^* of DDER such that $q_0 = -\tau$. Suppose $x^\dagger$ is also a SOPS of DDER as defined in Definition 3.2, but with $q_0^\dagger, q_1^\dagger$ and $q_2^\dagger$ in place of q_0, q_1 and q_2. Define $\tilde{x}^\dagger \in \mathcal{C}^+_{[-\tau,\infty)}$ by $\tilde{x}^\dagger(t) = x^\dagger(q_0^\dagger + \tau + t)$ for all $t \geq -\tau$. Then $\tilde{x}^\dagger$ is a SOPS of DDER such that its associated value of q_0 is $-\tau$. By Theorem 5.4, $x^* = \tilde{x}^\dagger$ and so $x^*(t) = x^\dagger(q_0^\dagger + \tau + t)$ for all $t \geq -\tau$. Hence x^* is the unique (up to time translation) SOPS of DDER with delay τ. Since $\tau > \bar{\tau}^{\delta_0}$, we can choose positive constants ε, γ, K_γ and K_ρ as in the statement Theorem 5.3. Let $x^\dagger$ be a member from the family of equivalent (up to time translation) SOPS, i.e., there exists $t_0 \geq 0$ such that $x^*(t) = x^\dagger(t + t_0)$ for all $t \geq -\tau$. Suppose that $\varphi \in \mathcal{C}^+_{[-\tau,0]}$ satisfies $\|\varphi - x^\dagger_\sigma\|_{[-\tau,0]} < \varepsilon$ for some $\sigma \in [0,p)$. Let $n = \min\{k \in \mathbb{N} : \sigma + kp - t_0 \geq 0\}$. Then $\sigma^\dagger \equiv \sigma + np - t_0 \in [0,p)$ and $\|\varphi - x^*_{\sigma^\dagger}\|_{[-\tau,0]} = \|\varphi - x^\dagger_\sigma\|_{[-\tau,0]} < \varepsilon$. By Theorem 5.3, we have that there is a $\rho \in (-p,p)$ satisfying

$$|\rho| \leq K_\rho \|\varphi - x^*_{\sigma^\dagger}\|_{[-\tau,0]} = K_\rho \|\varphi - x^\dagger_\sigma\|_{[-\tau,0]}$$

and such that, for $t \geq 0$,

$$\begin{aligned}\|x_t - x^\dagger_{t+p+\sigma+\rho}\|_{[-\tau,0]} &= \|x_t - x^*_{t+p+\sigma^\dagger+\rho}\|_{[-\tau,0]} \\ &\leq K_\gamma e^{-\gamma t} \|\varphi - x^*_{\sigma^\dagger}\|_{[-\tau,0]} = K_\gamma e^{-\gamma t} \|\varphi - x^\dagger_\sigma\|_{[-\tau,0]},\end{aligned}$$

where x denotes the unique solution of DDER with initial condition φ. $\square$

This chapter is based on the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

Appendix A

One-dimensional Skorokhod Problem

Define the *one-dimensional Skorokhod map* $(\Phi, \Psi) : \mathcal{C}_{[0,\infty)} \to \mathcal{C}^+_{[0,\infty)} \times \mathcal{C}^+_{[0,\infty)}$ by

$$\Phi(z)(t) = z(t) + \Psi(z)(t),\ t \geq 0, \tag{A.1}$$

$$\Psi(z)(t) = \sup_{0 \leq s \leq t} z^-(s),\ t \geq 0. \tag{A.2}$$

Here we note some well known properties of the one-dimensional Skorokhod map.

Proposition A.1. *For $z, z^\dagger \in \mathcal{C}_{[0,\infty)}$ and $t \geq 0$,*

$$\|\Phi(z) - \Phi(z^\dagger)\|_{[0,t]} \leq 2\|z - z^\dagger\|_{[0,t]},$$
$$\|\Psi(z) - \Psi(z^\dagger)\|_{[0,t]} \leq \|z - z^\dagger\|_{[0,t]}.$$

It follows that the map $(\Phi, \Psi) : \mathcal{C}_{[0,\infty)} \to \mathcal{C}^+_{[0,\infty)} \times \mathcal{C}^+_{[0,\infty)}$ is continuous (Recall that $\mathcal{C}_{[0,\infty)}$ is endowed with the topology of uniform convergence on compact intervals.)

Proposition A.2. *For $z \in \mathcal{C}_{[0,\infty)}$,*

$$Osc(\Phi(z), [t_1, t_2]) \leq Osc(z, [t_1, t_2]),$$
$$Osc(\Psi(z), [t_1, t_2]) \leq Osc(z, [t_1, t_2]),$$

for each $0 \leq t_1 \leq t_2 < \infty$, where for any $u \in \mathcal{C}_{[0,\infty)}$,

$$Osc(u, [t_1, t_2]) = \sup_{t_1 \leq s < t \leq t_2} |u(t) - u(s)|.$$

To ensure that solutions of (1.1) remain non-negative, we have employed the well known (one-dimensional) Skorokhod problem constraining a continuous function to be non-negative.

Definition A.1. Let $z \in \mathcal{C}_{[0,\infty)}$ satisfy $z(0) \geq 0$. A pair $(x, y) \in \mathcal{C}^+_{[0,\infty)} \times \mathcal{C}^+_{[0,\infty)}$ is a solution of the *(one-dimensional) Skorokhod problem for* z if the following hold:

(i) $x(t) = z(t) + y(t),\ t \geq 0,$

(ii) $x(t) \geq 0,\ t \geq 0,$

(iii) y satisfies the following:

- (a) $y(0) = 0,$
- (b) y is non-decreasing,
- (c) $\int_0^t x(s)dy(s) = 0,\ t \geq 0.$

Remark A.1. Here y is the *reflection* or *regulator* term that prevents $x(t)$ from taking negative values. Condition (iii)(c) ensures that $y(\cdot)$ does not increase in an interval $[s, t]$ where $x(\cdot)$ is positive.

Proposition A.3. *Suppose* $z \in \mathcal{C}_{[0,\infty)}$ *satisfies* $z(0) \geq 0$. *Then there exists a unique solution* $(x, y) \in \mathcal{C}^+_{[0,\infty)} \times \mathcal{C}^+_{[0,\infty)}$ *of the Skorokhod problem for* z*, given by*

$$(x, y) = (\Phi, \Psi)(z).$$

Proof. See Section 8.2 of [7]. □

This appendix is a formulation of known results and based on a similar formulation of these results contained in the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

Appendix B

Derivative of the One-dimensional Skorokhod Map

Throughout this section we use the following notation. Fix an interval I in $\mathbb{R}$. Given a family $\{u^\varepsilon : \varepsilon > 0\}$ in $\mathcal{C}_I$ that converges pointwise to $u \in \mathcal{D}_I$ as $\varepsilon \to 0$, we say that u^ε converges to u *uniformly on compact intervals of continuity (u.o.c.c.)* provided that for any compact interval $J \subset I$ such that u is continuous on J, u^ε converges to u uniformly on J as $\varepsilon \to 0$. For $z \in \mathcal{C}_{[0,\infty)}$ and $t \geq 0$, consider the set of times in the interval $[0,t]$ that the function z is coincident with its upper envelope function at time t, i.e.,

$$\mathbb{S}_z(t) = \{s \in [0,t] : z(s) = \bar{z}(t)\}, \text{ where } \bar{z}(t) = \max_{0 \leq s \leq t} z(s). \tag{B.1}$$

For $z, w \in \mathcal{C}_{[0,\infty)}$ and $t \geq 0$, define

$$R(z,w)(t) = \begin{cases} 0 & \text{if } \bar{z}(t) < 0, \\ S(z,w)(t) \vee 0 & \text{if } \bar{z}(t) = 0, \\ S(z,w)(t) & \text{if } \bar{z}(t) > 0, \end{cases} \tag{B.2}$$

where

$$S(z,w)(t) = \sup_{s \in \mathbb{S}_z(t)} w(s). \tag{B.3}$$

Let $(\Phi, \Psi) : \mathcal{C}_{[0,\infty)} \to \mathcal{C}^+_{[0,\infty)} \times \mathcal{C}^+_{[0,\infty)}$ be the one-dimensional Skorokhod map defined in (A.1)–(A.2). For $z, w \in \mathcal{C}_{[0,\infty)}$, define $\partial^\varepsilon_w \Phi(z), \partial^\varepsilon_w \Psi(z) \in \mathcal{C}_{[0,\infty)}$ by

$$\partial^\varepsilon_w \Phi(z) = \frac{\Phi(z+\varepsilon w) - \Phi(z)}{\varepsilon} = w + \partial^\varepsilon_w \Psi(z), \tag{B.4}$$

$$\partial^\varepsilon_w \Psi(z) = \frac{\Psi(z+\varepsilon w) - \Psi(z)}{\varepsilon}. \tag{B.5}$$

In the following we prove that if z, w and $\{w^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ are in $\mathcal{C}_{[0,\infty)}$ such that $w^\varepsilon \to w$ uniformly on compact intervals in $[0, \infty)$, then $\partial^\varepsilon_{w^\varepsilon} \Phi(z)$ and $\partial^\varepsilon_{w^\varepsilon} \Psi(z)$ converge pointwise as $\varepsilon \to 0$ and we denote the limits by $\partial_w \Phi(z)$ and $\partial_w \Psi(z)$, respectively. We refer to $\partial_w \Phi(z)$ and $\partial_w \Psi(z)$ as the *directional derivatives of* Φ *and* Ψ, respectively, *in the direction* w *at* z. The existence of these limits (and a bit more) is given by Theorem B.1. The theorem follows from Theorem 9.5.3 in [28] and a proof is provided here for completeness.

Theorem B.1. *Let z, w and $\{w^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ be in $\mathcal{C}_{[0,\infty)}$ such that $w^\varepsilon \to w$ uniformly on compact intervals in $[0, \infty)$ as $\varepsilon \to 0$. If $\partial^\varepsilon_{w^\varepsilon} \Phi(z)$ and $\partial^\varepsilon_{w^\varepsilon} \Psi(z)$ are defined via* (B.4)–(B.5) *for $\varepsilon \in (0, \varepsilon^*]$, then as $\varepsilon \to 0$,*

$$\partial^\varepsilon_{w^\varepsilon} \Phi(z) \to \partial_w \Phi(z) \text{ pointwise and u.o.c.c.},$$
$$\partial^\varepsilon_{w^\varepsilon} \Psi(z) \to \partial_w \Psi(z) \text{ pointwise and u.o.c.c.},$$

where $\partial_w \Phi(z)$ and $\partial_w \Psi(z)$ are given by

$$\partial_w \Phi(z) = w + \partial_w \Psi(z), \tag{B.6}$$

$$\partial_w \Psi(z) = R(-z, -w). \tag{B.7}$$

Further, $\partial_w \Phi(z)$ and $\partial_w \Psi(z)$ are both in $\mathcal{D}_{[0,\infty)}$.

Before proving the theorem, we introduce the following lemma which is similar to Lemma 5.2 in [13], Theorem 9.4.3 in [28] and Theorem 3.2 in [14]. The proof of the lemma is adapted from the proofs in [13, 14, 28] and provided here for completeness.

Lemma B.1. *Let $z, w, \{w^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ be in $\mathcal{C}_{[0,\infty)}$ such that $w^\varepsilon \to w$ uniformly on compact intervals of $[0, \infty)$ as $\varepsilon \to 0$. Then as $\varepsilon \to 0$,*

$$\frac{\overline{z + \varepsilon w^\varepsilon} \vee 0 - \overline{z} \vee 0}{\varepsilon} \to R(z, w) \text{ pointwise and u.o.c.c.}$$

Proof. Fix $t \geq 0$. We first prove that

$$\lim_{\varepsilon \to 0} \left\{ \overline{\varepsilon^{-1} z + w^{\varepsilon}}(t) - \overline{\varepsilon^{-1} z}(t) \right\} = S(z, w)(t). \tag{B.8}$$

For each $\varepsilon \in (0, \varepsilon^*)$, choose $s_\varepsilon \in [0, t]$ such that

$$(\varepsilon^{-1} z + w^{\varepsilon})(s_\varepsilon) = \overline{\varepsilon^{-1} z + w^{\varepsilon}}(t).$$

Since $w^\varepsilon \to w$ u.o.c. and w is continuous and therefore bounded on $[0, t]$, there exists $\varepsilon_0 \in (0, \varepsilon^*]$ such that $\sup_{\varepsilon \in (0, \varepsilon_0]} \|w^\varepsilon\|_{[0,t]}$ is finite. It follows that

$$\lim_{\varepsilon \to 0} z(s_\varepsilon) = \lim_{\varepsilon \to 0} \{ \overline{z + \varepsilon w^{\varepsilon}}(t) - \varepsilon w^{\varepsilon}(s_\varepsilon) \} = \bar{z}(t). \tag{B.9}$$

Now we have

$$\overline{\varepsilon^{-1} z + w^{\varepsilon}}(t) - \overline{\varepsilon^{-1} z}(t) = w^{\varepsilon}(s_\varepsilon) + \varepsilon^{-1} \left[z(s_\varepsilon) - \bar{z}(t) \right] \leq w^{\varepsilon}(s_\varepsilon),$$

and therefore

$$\limsup_{\varepsilon \to 0} \left\{ \overline{\varepsilon^{-1} z + w^{\varepsilon}}(t) - \overline{\varepsilon^{-1} z}(t) \right\} \leq \limsup_{\varepsilon \to 0} w^{\varepsilon}(s_\varepsilon). \tag{B.10}$$

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence in $(0, \varepsilon_0]$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$\lim_{n \to \infty} w^{\varepsilon_n}(s_{\varepsilon_n}) = \limsup_{\varepsilon \to 0} w^{\varepsilon}(s_\varepsilon).$$

Since $\{s_{\varepsilon_n}\}_{n=1}^{\infty}$ is uniformly bounded, we can assume (by taking a further subsequence if necessary) that there exists $s_0 \in [0, t]$ such that $\lim_{n \to \infty} s_{\varepsilon_n} = s_0$. By (B.9), $z(s_0) = \bar{z}(t)$, so $s_0 \in \mathbb{S}_z(t)$. Thus,

$$\limsup_{\varepsilon \to 0} w^{\varepsilon}(s_\varepsilon) = \lim_{n \to \infty} w^{\varepsilon_n}(s_{\varepsilon_n}) = w(s_0) \leq \sup_{s \in \mathbb{S}_z(t)} w(s).$$

Combining with (B.10) this yields

$$\limsup_{\varepsilon \to 0} \left\{ \overline{\varepsilon^{-1} z + w^{\varepsilon}}(t) - \overline{\varepsilon^{-1} z}(t) \right\} \leq \sup_{s \in \mathbb{S}_z(t)} w(s). \tag{B.11}$$

To establish the limit we need to show the reverse inequality

$$\liminf_{\varepsilon \to 0} \left\{ \overline{\varepsilon^{-1} z + w^{\varepsilon}}(t) - \overline{\varepsilon^{-1} z}(t) \right\} \geq \sup_{s \in \mathbb{S}_z(t)} w(s).$$

Suppose $s \in \mathbb{S}_z(t)$, then

$$\overline{\varepsilon^{-1}z + w^\varepsilon}(t) - \overline{\varepsilon^{-1}z}(t) \geq (\varepsilon^{-1}z + w^\varepsilon)(s) - \varepsilon^{-1}\bar{z}(t) = w^\varepsilon(s).$$

Since $w^\varepsilon \to w$ uniformly on compact intervals of $[0, \infty)$ as $\varepsilon \to 0$, we have

$$\liminf_{\varepsilon \to 0} \left\{ \overline{\varepsilon^{-1}z + w^\varepsilon}(t) - \overline{\varepsilon^{-1}z}(t) \right\} \geq w(s).$$

Taking supremums over $s \in \mathbb{S}_z(t)$ yields

$$\liminf_{\varepsilon \to 0} \left\{ \overline{\varepsilon^{-1}z + w^\varepsilon}(t) - \overline{\varepsilon^{-1}z}(t) \right\} \geq \sup_{s \in \mathbb{S}_z(t)} w(s).$$

The above inequality along with (B.11) establishes the pointwise limit

$$\lim_{\varepsilon \to 0} \left\{ \overline{\varepsilon^{-1}z + w^\varepsilon}(t) - \overline{\varepsilon^{-1}z}(t) \right\} = \sup_{s \in \mathbb{S}_z(t)} w(s) = S(z, w)(t).$$

We now treat the three cases: $\bar{z}(t) > 0$, $\bar{z}(t) = 0$ and $\bar{z} < 0$. Suppose $\bar{z}(t) > 0$. Since $\sup_{\varepsilon \in (0, \varepsilon_0]} \|w^\varepsilon\|_{[0,t]} < \infty$, $\overline{z + \varepsilon w^\varepsilon}(t) > 0$ for all $\varepsilon > 0$ sufficiently small. For such ε we have

$$\overline{\varepsilon^{-1}z + w^\varepsilon}(t) \vee 0 - \overline{\varepsilon^{-1}z}(t) \vee 0 = \overline{\varepsilon^{-1}z + w^\varepsilon}(t) - \overline{\varepsilon^{-1}z}(t).$$

The above equality along with (B.8) establishes the pointwise limit

$$\lim_{\varepsilon \to 0} \left\{ \overline{\varepsilon^{-1}z + w^\varepsilon}(t) \vee 0 - \overline{\varepsilon^{-1}z}(t) \vee 0 \right\} = S(z, w)(t) = R(z, w)(t).$$

Suppose $\bar{z}(t) = 0$. Then,

$$\overline{\varepsilon^{-1}z + w^\varepsilon}(t) \vee 0 - \overline{\varepsilon^{-1}z}(t) \vee 0 = \left\{ \overline{\varepsilon^{-1}z + w^\varepsilon}(t) - \overline{\varepsilon^{-1}z}(t) \right\} \vee 0.$$

The above equality along with (B.8) establishes the pointwise limit

$$\lim_{\varepsilon \to 0} \left\{ \overline{\varepsilon^{-1}z + w^\varepsilon}(t) \vee 0 - \overline{\varepsilon^{-1}z}(t) \vee 0 \right\} = S(z, w)(t) \vee 0 = R(z, w)(t).$$

Suppose $\bar{z}(t) < 0$. Since $\sup_{\varepsilon \in (0, \varepsilon_0]} \|w^\varepsilon\|_{[0,t]} < \infty$, $\overline{\varepsilon^{-1}z + w^\varepsilon}(t) < 0$ for all $\varepsilon > 0$ sufficiently small. Hence, for such ε,

$$\overline{\varepsilon^{-1}z + w^\varepsilon}(t) \vee 0 - \overline{\varepsilon^{-1}z}(t) \vee 0 = 0,$$

and so

$$\lim_{\varepsilon\to 0}\left\{\overline{\varepsilon^{-1}z+w^{\varepsilon}}(t)\vee 0-\overline{\varepsilon^{-1}z}(t)\right\}=0=R(z,w)(t).$$

This proves pointwise convergence.

Note that for $z,w\in\mathcal{C}_{[0,\infty)}$,

$$\overline{z}(t)-\overline{w}(t)\leq\overline{z-w}(t),\ \text{for each } t\geq 0. \tag{B.12}$$

If $t=0$, the result is trivial. For $t>0$, let $s\in[0,t]$ be such that $z(s)=\overline{z}(t)$, then

$$\overline{z}(t)-\overline{w}(t)=z(s)-\overline{w}(t)\leq z(s)-w(s)\leq\overline{z-w}(t).$$

We now show the convergence is u.o.c.c. By (B.12),

$$\overline{\varepsilon_1^{-1}z+w}-\overline{\varepsilon_2^{-1}z+w}\leq\overline{(\varepsilon_1^{-1}-\varepsilon_2^{-1})z}.$$

If $\varepsilon_1\leq\varepsilon_2$, then it follows that

$$\overline{\varepsilon_1^{-1}z+w}-\overline{\varepsilon_1^{-1}z}\leq\overline{\varepsilon_2^{-1}z+w}-\overline{\varepsilon_2^{-1}z}.$$

Therefore $\left\{\overline{\varepsilon^{-1}z+w}:0<\varepsilon\leq\varepsilon^*\right\}$ is a monotone decreasing sequence as $\varepsilon\to 0$. Convergence of monotone decreasing continuous functions to a continuous limit must be u.o.c., so $\overline{\varepsilon^{-1}z+w}-\overline{\varepsilon^{-1}z}\to S(z,w)$ u.o.c.c. By (B.12),

$$-\overline{w-w^{\varepsilon}}\leq\overline{\varepsilon^{-1}z+w^{\varepsilon}}-\overline{\varepsilon^{-1}z+w}\leq\overline{w^{\varepsilon}-w},$$

and since $\overline{w^{\varepsilon}-w}\to 0$ u.o.c., it follows that

$$\begin{aligned}\overline{\varepsilon^{-1}z+w^{\varepsilon}}-\overline{\varepsilon^{-1}z}&=\left(\overline{\varepsilon^{-1}z+w^{\varepsilon}}-\overline{\varepsilon^{-1}z+w}\right)+\left(\overline{\varepsilon^{-1}z+w}-\overline{\varepsilon^{-1}z}\right)\\&\to S(z,w)\ \text{u.o.c.c.}\end{aligned}$$

□

Proof of Theorem B.1. By (A.2), (B.5) and Lemma B.1, we have

$$\begin{aligned}\partial^{\varepsilon}_{w^{\varepsilon}}\Psi(z)&=\frac{\Psi(z+\varepsilon w^{\varepsilon})-\Psi(z)}{\varepsilon}\\&=\frac{\overline{-z-\varepsilon w^{\varepsilon}}\vee 0-\overline{-z}\vee 0}{\varepsilon}\\&\to R(-z,-w)\ \text{pointwise and u.o.c.c. as } \varepsilon\to 0.\end{aligned}$$

The convergence of $\partial^{\varepsilon}_{w^{\varepsilon}}\Phi(z)$ follows from (B.4). The fact that $\partial_w\Phi(z)$ and $\partial_w\Psi(z)$ lie in $\mathcal{D}_{[0,\infty)}$ follows from Theorem 1.1 in [14]. □

Lemma B.2. *For $z, w, w^\dagger \in \mathcal{C}_{[0,\infty)}$ and $t \geq 0$,*

$$\|\partial_w \Phi(z) - \partial_{w^\dagger} \Phi(z)\|_{[0,t]} \leq 2\|w - w^\dagger\|_{[0,t]}, \tag{B.13}$$

$$\|\partial_w \Psi(z) - \partial_{w^\dagger} \Psi(z)\|_{[0,t]} \leq \|w - w^\dagger\|_{[0,t]}. \tag{B.14}$$

Proof. The Lipschitz condition for $\partial_\cdot \Phi(z)$ follows from (B.6) and (B.14). The Lipschitz condition from $\partial_\cdot \Psi(z)$ follows from

$$\begin{aligned}
\|\partial_w \Psi(z) - \partial_{w^\dagger} \Psi(z)\|_{[0,t]} &\leq \|R(-z, -w) - R(-z, -w^\dagger)\|_{[0,t]} \\
&\leq \|S(-z, -w) - S(-z, -w^\dagger)\|_{[0,t]} \\
&\leq \|w - w^\dagger\|_{[0,t]},
\end{aligned}$$

where the second inequality follows from the fact that for any $r, s \in \mathbb{R}$,

$$|r \vee 0 - s \vee 0| \leq |r - s|,$$

and the third inequality follows from the fact that for any $t \geq 0$,

$$\left| \sup_{s \in \mathbb{S}_{-z}(t)} (-w(s)) - \sup_{s \in \mathbb{S}_{-z}(t)} (-w^\dagger(s)) \right| \leq \sup_{s \in \mathbb{S}_{-z}(t)} \left| w(s) - w^\dagger(s) \right| \leq \|w - w^\dagger\|_{[0,t]}.$$

□

This appendix is a formulation of known results and based on a similar formulation of these results in the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

Appendix C

Linear Variational Equation (LVE)

In this section, we introduce a linear variational equation relative to a solution x of DDER. We prove that a solution of the linear variational equation relative to x can be represented as a pointwise limit of the difference between x and solutions of DDER with perturbed initial conditions. As we will see, solutions of our linear variational equation differ considerably from those of the analogous equation in the unconstrained setting. In particular, the lower boundary constraint in the DDER can result in discontinuous solutions of the linear variational equation.

Recall that $\mathcal{D}_{[-\tau,0]}$ is the space of functions from the interval $[-\tau, 0]$ to $\mathbb{R}$ that have finite left and right limits at each $t \in (-\tau, 0)$ and finite right limits at $-\tau$ and finite left limits at 0. For each $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, define $\mathcal{D}^\varphi$ to be the directions in $\mathcal{D}_{[-\tau,0]}$ that we allow φ to be perturbed:

$$\mathcal{D}^\varphi = \{\psi \in \mathcal{D}_{[-\tau,0]} : \psi(s) \geq 0 \text{ if } \varphi(s) = 0,\ s \in [-\tau, 0]\}. \tag{C.1}$$

For $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, let $\mathcal{C}^\varphi = \{\psi \in \mathcal{C}_{[-\tau,0]} : \varphi + \varepsilon\psi \in \mathcal{C}^+_{[-\tau,0]}$ for all ε sufficiently small$\}$, the directions in $\mathcal{C}_{[-\tau,0]}$ in which we allow φ to be perturbed. To ensure that the linear variational equation is well-defined, we assume that the function f in (1.1) satisfies the following regularity properties.

Assumption C.1. The function $f : \mathcal{C}^+_{[-\tau,0]} \to \mathbb{R}$ satisfies the following uniform

Lipschitz continuity property:

$$|f(\varphi) - f(\varphi^\dagger)| \leq K_f \|\varphi - \varphi^\dagger\|_{[-\tau,0]} \tag{C.2}$$

for all $\varphi, \varphi^\dagger \in \mathcal{C}^+_{[-\tau,0]}$ and some fixed finite positive constant K_f.

Assumption C.2. At each $\varphi \in \mathcal{C}^+_{[-\tau,0]}$, for each $\psi \in \mathcal{D}^\varphi$, there is a unique *derivative of f in the direction ψ*, denoted $\partial_\psi f(\varphi)$, that satisfies the following:

(i) Whenever $\{\psi_n\}_{n=1}^\infty$ is a uniformly bounded sequence in $\mathcal{C}_{[-\tau,0]}$ that converges *pointwise* to $\psi \in \mathcal{D}^\varphi$ as $n \to \infty$ and $\{\varepsilon_n\}_{n=1}^\infty$ is a sequence of positive real numbers such that $\varepsilon_n \to 0$ as $n \to \infty$ and $\varphi + \varepsilon_n \psi_n \in \mathcal{C}^+_{[-\tau,0]}$ for each n, we have that

$$\partial_\psi f(\varphi) = \lim_{n\to\infty} \frac{f(\varphi + \varepsilon_n \psi_n) - f(\varphi)}{\varepsilon_n}.$$

(ii) If $r, s \in \mathbb{R}$ and $\psi, \psi^\dagger \in \mathcal{D}^\varphi$ such that $r\psi + s\psi^\dagger \in \mathcal{D}^\varphi$, then

$$\partial_{r\psi + s\psi^\dagger} f(\varphi) = r\partial_\psi f(\varphi) + s\partial_{\psi^\dagger} f(\varphi).$$

(iii) For all $\psi, \psi^\dagger \in \mathcal{D}^\varphi$

$$|\partial_\psi f(\varphi) - \partial_{\psi^\dagger} f(\varphi)| \leq K_f \|\psi - \psi^\dagger\|_{[-\tau,0]},$$

where K_f is as in (C.2).

Lemma C.1. *Let $f : \mathcal{C}^+_{[-\tau,0]} \to \mathbb{R}$ be given by*

$$f(\varphi) = \int_{[-\tau,0]} \zeta(\varphi(s)) d\mu(s) \text{ for all } \varphi \in \mathcal{C}^+_{[-\tau,0]},$$

where $\zeta : \mathbb{R}_+ \to \mathbb{R}$ is uniformly Lipschitz continuous (with Lipschitz constant K_ζ) and continuously differentiable on $\mathbb{R}_+$ and μ is a finite measure on the interval $[-\tau, 0]$. Then f satisfies Assumptions C.1 and C.2 with $K_f = K_\zeta \mu([-\tau, 0])$ and

$$\partial_\psi f(\varphi) = \int_{[-\tau,0]} \zeta'(\varphi(s))\psi(s) d\mu(s),$$

for all $\psi \in \mathcal{D}^\varphi$, where $\zeta' : \mathbb{R}_+ \to \mathbb{R}$ denotes the first derivative of ζ.

Proof. To prove that Assumption C.1 holds, suppose that $\varphi, \varphi^\dagger \in \mathcal{C}^+_{[-\tau,0]}$. Then

$$\begin{aligned}|f(\varphi) - f(\varphi^\dagger)| &\le \int_{[-\tau,0]} |\zeta(\varphi(s)) - \zeta(\varphi^\dagger(s))| d\mu(s) \\ &\le K_\zeta \int_{[-\tau,0]} |\varphi(s) - \varphi^\dagger(s)| d\mu(s) \\ &\le K_\zeta \|\varphi - \varphi^\dagger\|_{[-\tau,0]} \mu([-\tau, 0]).\end{aligned}$$

To prove that Assumption C.2 holds, suppose that $\{\psi_n\}_{n=1}^\infty$ is a bounded sequence in $\mathcal{C}_{[-\tau,0]}$ that converges pointwise to $\psi \in \mathcal{D}^\varphi$ as $n \to \infty$ and $\{\varepsilon_n\}_{n=1}^\infty$ is a sequence of positive real numbers such that $\varepsilon_n \to 0$ as $n \to \infty$ and $\varphi + \varepsilon_n \psi_n \in \mathcal{C}^+_{[-\tau,0]}$ for each n. For each $s \in [-\tau, 0]$

$$\lim_{n\to\infty} \frac{\zeta(\varphi(s) + \varepsilon_n \psi_n(s)) - \zeta(\varphi(s))}{\varepsilon_n} = \zeta'(\varphi(s))\psi(s).$$

Let $m = \sup_n \|\psi_n\|_{[-\tau,0]} < \infty$. Then for each n,

$$\frac{|\zeta(\varphi(s) + \varepsilon_n \psi_n(s)) - \zeta(\varphi(s))|}{\varepsilon_n} \le K_\zeta m.$$

Therefore, by bounded convergence,

$$\lim_{n\to\infty} \frac{f(\varphi + \varepsilon_n \psi_n) - f(\varphi)}{\varepsilon_n} = \int_{[-\tau,0]} \zeta'(\varphi(s))\psi(s) d\mu(s).$$

Part (ii) of the assumption follows because the integral is linear in ψ. Part (iii) then follows from the fact that ζ' is bounded by K_ζ and so

$$|\partial_\psi f(\varphi) - \partial_{\psi^\dagger} f(\varphi)| \le K_\zeta \|\psi - \psi^\dagger\|_{[-\tau,0]} \mu([-\tau, 0]).$$

□

Example C.1. Let $f : \mathcal{C}^+_{[-\tau,0]} \to \mathbb{R}$ be given by

$$f(\varphi) = h(\varphi(-\tau)) \text{ for all } \varphi \in \mathcal{C}^+_{[-\tau,0]},$$

where $h : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable with uniformly bounded derivative on $\mathbb{R}_+$. Then by Lemma C.1, with $\zeta = h$ and μ equal to the point mass at $s = -\tau$, f satisfies Assumptions C.1 and C.2 with

$$\partial_\psi f(\varphi) = h'(\varphi(-\tau))\psi(-\tau),$$

for all $\psi \in \mathcal{D}^\varphi$, where $h' : \mathbb{R}_+ \to \mathbb{R}$ denotes the first derivative of h.

Throughout the remainder of this section, we assume that f satisfies Assumptions C.1 and C.2 and we fix a solution x of DDER and define z as in (2.2).

Definition C.1. A function $v \in \mathcal{D}_{[-\tau,\infty)}$ is a solution of the *linear variational equation (LVE) relative to* x if for each $s \geq 0$, $v_s \in \mathcal{D}^{x_s}$, the function $s \to \partial_{v_s} f(x_s)$ is measurable and integrable on each compact set in $[0,\infty)$, and v satisfies

$$v(t) = \partial_w \Phi(z)(t), \ t \geq 0, \tag{C.3}$$

where Φ denotes the Skorokhod map given by (A.1)–(A.2), $z \in \mathcal{C}_{[0,\infty)}$ is defined in (2.2), $w \in \mathcal{C}_{[0,\infty)}$ is defined by

$$w(t) = v(0) + \int_0^t \partial_{v_s} f(x_s) ds, \ t \geq 0. \tag{C.4}$$

and the directional derivative of Φ at z in the direction w is denoted by $\partial_w \Phi(z)$ and is well defined as an element of $\mathcal{D}_{[0,\infty)}$ by Theorem B.1.

Suppose $\psi \in \mathcal{C}^{x_0}$. Then there exists $\varepsilon^* > 0$ such that $x_0 + \varepsilon\psi \in \mathcal{C}^+_{[-\tau,0]}$ for all $\varepsilon \in (0,\varepsilon^*]$. For each $\varepsilon \in (0,\varepsilon^*]$ there exists a unique solution x^ε of DDER satisfying $x_0^\varepsilon = x_0 + \varepsilon\psi$. Define $v^\varepsilon \in \mathcal{C}_{[-\tau,\infty)}$ by

$$v^\varepsilon(t) = \frac{x^\varepsilon(t) - x(t)}{\varepsilon}, \ t \geq -\tau. \tag{C.5}$$

Furthermore, for each $\varepsilon \in (0,\varepsilon^*]$ define $z^\varepsilon \in \mathcal{C}_{[0,\infty)}$ as in (2.2) but with x and z replaced with x^ε and z^ε, respectively, and define $w^\varepsilon \in \mathcal{C}_{[0,\infty)}$ by

$$w^\varepsilon(t) = \frac{z^\varepsilon(t) - z(t)}{\varepsilon} = \psi(0) + \int_0^t \frac{f(x_s + \varepsilon v_s^\varepsilon) - f(x_s)}{\varepsilon} ds, \ t \geq 0. \tag{C.6}$$

Recall that a family $\{u^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ in $\mathcal{D}_{[0,\infty)}$ converges to $u \in \mathcal{D}_{[0,\infty)}$ uniformly on compact intervals of continuity (u.o.c.c.) as $\varepsilon \to 0$ if for each compact interval I contained in $[0,\infty)$ on which u is continuous, u^ε converges to u uniformly on I as $\varepsilon \to 0$. We have the following theorem on the existence and uniqueness of a solution of LVE given an appropriate initial condition as well as the pointwise and u.o.c.c. convergence of v^ε to v as $\varepsilon \to 0$.

Theorem C.1. *Suppose $\psi \in \mathcal{C}^{x_0}$. Then there exists a unique solution v of LVE relative to x with initial condition ψ and v is a Borel measurable function. Furthermore, $v^\varepsilon \to v$ pointwise and uniformly on compact intervals of continuity in $[-\tau,\infty)$ as $\varepsilon \to 0$ and $w^\varepsilon \to w$ uniformly on compact intervals in $[0,\infty)$ as $\varepsilon \to 0$, where w is defined by (5.72) and for each $\varepsilon \in (0,\varepsilon^*]$, v^ε and w^ε are defined by (C.5) and (C.6), respectively.*

In preparation for proving Theorem C.1, we prove the following lemmas.

Lemma C.2. *Suppose that v^ε is defined as in (C.5), then*

$$\|v^\varepsilon\|_{[-\tau,t]} \leq 2\|\psi\|_{[-\tau,0]}\exp(2K_f t),\ t \geq 0. \tag{C.7}$$

Proof. Fix $t \geq 0$. By (C.6) and (C.2), for each $\varepsilon \in (0,\varepsilon^*]$ and all $s \in [0,t]$ we have

$$\begin{aligned} |w^\varepsilon(s)| &\leq |\psi(0)| + \int_0^s \frac{|f(x_r+\varepsilon v_r^\varepsilon) - f(x_r)|}{\varepsilon}dr \\ &\leq \|\psi\|_{[-\tau,0]} + K_f\int_0^s \|v^\varepsilon\|_{[-\tau,r]}dr. \end{aligned}$$

By taking the supremum over s in the interval $[0,t]$, using (2.4) and applying the Lipschitz continuity of the Skorokhod map (see Proposition A.1) we have

$$\begin{aligned} \|v^\varepsilon\|_{[0,t]} &= \varepsilon^{-1}\|x^\varepsilon - x\|_{[0,t]} \\ &\leq 2\varepsilon^{-1}\|z^\varepsilon - z\|_{[0,t]} = 2\|w^\varepsilon\|_{[0,t]} \\ &\leq 2\|\psi\|_{[-\tau,0]} + 2K_f\int_0^t \|v^\varepsilon\|_{[-\tau,s]}ds. \end{aligned}$$

We can easily extend the supremum norm on the left to the interval $[-\tau,t]$ and then apply Gronwall's inequality to complete the proof. □

Lemma C.3. *Suppose $v, v^\dagger$ are solutions of LVE relative to x. Then we have*

$$\|v - v^\dagger\|_{[-\tau,t]} \leq \widetilde{K}_f(t)\|v - v^\dagger\|_{[-\tau,0]},\ t \geq 0, \tag{C.8}$$

where $\widetilde{K}_f(t) = 2\exp(2K_f t)$.

Proof. Suppose v and $v^\dagger$ are solutions of the linear variational equation relative to x. Let $w \in \mathcal{C}_{[0,\infty)}$ be given by (5.72) and $w^\dagger \in \mathcal{C}_{[0,\infty)}$ also be defined as in (5.72), but with $v^\dagger$ instead of v. By definition, the restrictions $\tilde{v}, \tilde{v}'$ of $v, v^\dagger$ (respectively) to $[0, \infty)$ satisfy

$$\tilde{v} = \partial_w \Phi(z), \ \tilde{v}' = \partial_{w^\dagger} \Phi(z),$$

where z is defined as in (2.2). Fix $t \geq 0$. By (5.72) and Assumption C.2(iii), for $s \in [0, t]$, we have

$$\begin{aligned} |w(s) - w'(s)| &\leq |v(0) - v^\dagger(0)| + \int_0^t |\partial_{v_r} f(x_r) - \partial_{v_r^\dagger} f(x_r)| dr \\ &\leq |v(0) - v^\dagger(0)| + K_f \int_0^t \|v - v^\dagger\|_{[-\tau, r]} dr. \end{aligned}$$

By taking the supremum over s in the interval $[0, t]$ and using the Lipschitz continuity of $D.\Phi(x)$ (see Appendix B), we have

$$\begin{aligned} \|v - v^\dagger\|_{[0,t]} = \|\partial_w \Phi(z) - \partial_{w^\dagger} \Phi(z)\|_{[0,t]} &\leq 2\|w - w^\dagger\|_{[0,t]} \\ &\leq 2\|v - v^\dagger\|_{[-\tau, 0]} + 2K_f \int_0^t \|v - v^\dagger\|_{[-\tau, r]} dr. \end{aligned}$$

The supremum norm on the left can clearly be extended to the interval $[-\tau, t]$ after which a simple application of Gronwall's inequality yields (C.8). □

Proof of Theorem C.1. We first establish uniqueness, suppose that v and $v^\dagger$ are solutions of LVE such that $v_0 = v_0^\dagger = \psi$. By (C.8),

$$\|v - v^\dagger\|_{[-\tau, t]} \leq \widetilde{K}_f(t) \|\psi - \psi\|_{[-\tau, 0]} = 0,$$

for all $t \geq 0$ and so $v = v^\dagger$.

We now establish existence. First, we prove that the family $\{w^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ is relatively compact in $\mathcal{C}_{[0,\infty)}$. Fix $t \geq 0$. For $0 \leq t_1 \leq t_2 \leq t$, by (C.2) and Lemma C.2 we have

$$\begin{aligned} |w^\varepsilon(t_2) - w^\varepsilon(t_1)| &\leq \int_{t_1}^{t_2} \frac{|f(x_s + \varepsilon v_s^\varepsilon) - f(x_s)|}{\varepsilon_n} ds \\ &\leq K_f \int_{t_1}^{t_2} \|v^\varepsilon\|_{[-\tau, s]} ds \\ &\leq 2K_f \|\psi\|_{[-\tau, 0]} \exp(2K_f t) |t_2 - t_1|. \end{aligned}$$

Hence $\{w^\varepsilon : 0 < \varepsilon \le \varepsilon^*\}$ is uniformly bounded and uniformly Lipschitz continuous on each interval $[0,t]$. Since this holds for each $t \ge 0$, by the Arzelà-Ascoli theorem and a diagonal sequence argument, $\{w^\varepsilon : 0 < \varepsilon \le \varepsilon^*\}$ is relatively compact in $\mathcal{C}_{[0,\infty)}$.

It follows that for any sequence $\{\varepsilon_n\}_{n=1}^\infty$ in $(0,\varepsilon^*]$ such that $\varepsilon_n \to 0$ as $n \to \infty$, there exists a subsequence, also denoted $\{\varepsilon_n\}_{n=1}^\infty$, and a $w \in \mathcal{C}_{[0,\infty)}$ such that w^{ε_n} converges to w uniformly on compact intervals in $[0,\infty)$ as $n \to \infty$. By extending $\{w^{\varepsilon_n}\}_{n=1}^\infty$ to a family $\{\hat{w}^\varepsilon : 0 < \varepsilon \le \varepsilon^*\}$ that converges to w as $\varepsilon \to 0$, where $\hat{w}^\varepsilon = w^{\varepsilon_n}$ when $\varepsilon = \varepsilon_n$, and applying Theorem B.1 in Appendix B, we have for each $t \ge 0$,

$$\begin{aligned}\lim_{n\to\infty} v^{\varepsilon_n}(t) &= \lim_{n\to\infty} \frac{x^{\varepsilon_n}(t) - x(t)}{\varepsilon_n} \\ &= \lim_{n\to\infty} \frac{\Phi(z + \varepsilon_n w^{\varepsilon_n})(t) - \Phi(z)(t)}{\varepsilon_n} \\ &= \partial_w \Phi(z)(t),\end{aligned}$$

where $\partial_w \Phi(z)$ is defined as in (B.6) and the convergence is pointwise at all times and uniform on compact intervals in $[0,\infty)$ on which $\partial_w \Phi(z)(\cdot)$ is continuous. Define $v \in \mathcal{D}_{[-\tau,\infty)}$ by $v(t) = \psi(t)$ for $t \in [-\tau, 0]$ and by $v(t) = \partial_w \Phi(z)(t)$ for $t \ge 0$ (Note that v is well-defined at zero because $\partial_w \Phi(z)(0) = w(0) = \psi(0)$.) Then $v^{\varepsilon_n} \to v$ pointwise on $[-\tau,\infty)$ as $n \to \infty$.

For each $s \ge 0$, $x_s^{\varepsilon_n} = x_s + \varepsilon_n v_s^{\varepsilon_n} \in \mathcal{C}^+_{[-\tau,0]}$ for all n, so it follows from Assumption C.2(i) that

$$\lim_{n\to\infty} \frac{f(x_s + \varepsilon_n v_s^{\varepsilon_n}) - f(x_s)}{\varepsilon_n} = \partial_{v_s} f(x_s). \tag{C.9}$$

By (C.2) and Lemma C.2, for each n we have

$$\left| \frac{f(x_s + \varepsilon_n v_s^{\varepsilon_n}) - f(x_s)}{\varepsilon_n} \right| \le K_f \|v^{\varepsilon_n}\|_{[-\tau,s]} \le 2K_f \|\psi\|_{[-\tau,0]} \exp(2K_f s). \tag{C.10}$$

Since the function $s \to \partial_{v_s} f(x_s)$ is the pointwise limit of a sequence of measurable functions, it is also measurable. Furthermore, by (C.9)–(C.10), the function is bounded and hence integrable on compact sets in $[0,\infty)$. It then follows by bounded convergence that w satisfies

$$w(t) = \psi(0) + \lim_{n\to\infty} \int_0^t \frac{f(x_s + \varepsilon_n v_s^{\varepsilon_n}) - f(x_s)}{\varepsilon_n} ds = \psi(0) + \int_0^t \partial_{v_s} f(x_s) ds,$$

for all $t \geq 0$. This establishes the existence of a solution v of LVE relative to x with initial condition ψ and that each sequence along which $\varepsilon \to 0$ in $\{v^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ has a subsequence that converges pointwise on $[0, \infty)$ and uniformly on compact intervals of continuity to a solution of LVE relative to x. By the uniqueness of solutions, it follows that this limit is always v and since v is the pointwise limit of continuous functions, v is Borel measurable.

We show that the family $\{v^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ converges to v pointwise and uniformly on compact intervals of continuity as $\varepsilon \to 0$. Suppose not, then there exists a subsequence $\{v^{\varepsilon_n}\}_{n=1}^\infty$ and $\eta > 0$ such that one of the following holds: (i) there exists $s > 0$ such that $|v^{\varepsilon_n}(s) - v(s)| > \eta$ for all $n \geq 1$ or (ii) there exists a compact interval $I \subset \mathbb{R}_+$ such that v is continuous on I and $\|v^{\varepsilon_n} - v\|_I > \eta$ for all $n \geq 1$. However, as we have shown above, there must exist a further subsequence $\{v^{\varepsilon_{n_k}}\}_{k=1}^\infty$ such that $|v^{\varepsilon_{n_k}}(s) - v(s)| \to 0$ and $\|v^{\varepsilon_{n_k}} - v\|_I \to 0$ as $k \to \infty$, a contradiction. Therefore the family $\{v^\varepsilon : 0 < \varepsilon \leq \varepsilon^*\}$ converges to v pointwise and uniformly on compact intervals of continuity as $\varepsilon \to 0$. □

In the following lemma we further describe solutions of LVE relative to a solution x of DDER.

Lemma C.4. *Suppose v is a solution of LVE relative to x. For $0 \leq t_1 < t_2$:*

(i) If $x(t) > 0$ for $t \in (t_1, t_2)$, then

$$v(t) = v(t_1) + \int_{t_1}^{t} \partial_{v_s} f(x_s) ds, \ t \in [t_1, t_2).$$

(ii) If $x(t_2) = 0$ and $x(t) > 0$ for $t \in (t_1, t_2)$, then

$$v(t_2) = \left(v(t_1) + \int_{t_1}^{t_2} \partial_{v_s} f(x_s) ds \right)^+ .$$

(iii) If $x(t) = 0$ for $t \in [t_1, t_2]$ and $f(x_t) = 0$ for all $t \in [t_1, t_2]$, then

$$v(t) = v(t_1) + \int_{t_1}^{t} \partial_{v_s} f(x_s) ds + \sup_{r \in [t_1, t]} \left(-v(t_1) - \int_{t_1}^{r} \partial_{v_s} f(x_s) ds \right) \vee 0,$$

for all $t \in [t_1, t_2]$.

(iv) If $x(t) = 0$ for $t \in (t_1, t_2]$ and $f(x_t) < 0$ for all $t \in (t_1, t_2)$, then

$$v(t) = 0, \ t \in (t_1, t_2].$$

Proof. By Appendix B, we have $v = w + R(-z, -w)$ where R, z and w are defined by (B.2), (2.2) and (5.72), respectively. For $t_1 \leq t$, we have

$$\begin{aligned} w(t) &= w(t_1) + \int_{t_1}^{t} \partial_{v_s} f(x_s) ds \\ &= v(t_1) + \int_{t_1}^{t} \partial_{v_s} f(x_s) ds - R(-z, -w)(t_1) \end{aligned} \tag{C.11}$$

Proof of (i): Suppose $x(t) > 0$ for $t \in (t_1, t_2)$. Then $-z(t) < \overline{-z}(t) \vee 0$ for all $t \in (t_1, t_2)$. Given $t \in (t_1, t_2)$, if $\overline{-z}(t) < 0$, then $\overline{-z}(t_1) < 0$ and $R(-z, -w)(t) = R(-z, -w)(t_1) = 0$. Therefore, by (C.11),

$$v(t) = w(t) + R(-z, -w)(t) = v(t_1) + \int_{t_1}^{t} \partial_{v_s} f(x_s) ds.$$

If $\overline{-z}(t) \geq 0$, then $-z(s) < \overline{-z}(t)$ for all $s \in (t_1, t]$ (since $x(s) > 0$ for all $s \in (t_1, t]$), so $\mathbb{S}_{-z}(t) = \mathbb{S}_{-z}(t_1)$. Therefore $R(-z, -w)(t) = R(-z, -w)(t_1)$ and by (C.11),

$$\begin{aligned} v(t) &= w(t) + R(-z, -w)(t) \\ &= v(t_1) + \int_{t_1}^{t} \partial_{v_s} f(x_s) ds - R(-z, -w)(t_1) + R(-z, -w)(t) \\ &= v(t_1) + \int_{t_1}^{t} \partial_{v_s} f(x_s) ds. \end{aligned}$$

Proof of (ii): Suppose $x(t) > 0$ for $t \in (t_1, t_2)$ and $x(t_2) = 0$. Then for $t \in (t_1, t_2)$, as for (i), $-z(t) < \overline{-z}(t) \vee 0 \leq \overline{-z}(t_2) \vee 0$. At $t = t_2$, $x(t_2) = 0$ implies $-z(t_2) = \overline{-z}(t_2) \geq 0$. Therefore $\mathbb{S}_{-z}(t_2) = \mathbb{S}_{-z}(t_1) \cup \{t_2\}$ and by (C.11) (with t_2 in place of t),

$$\begin{aligned} v(t_2) &= w(t_2) + R(-z, -w)(t_2) \\ &= w(t_2) + R(-z, -w)(t_1) \vee (-w(t_2)) \\ &= \left(v(t_1) + \int_{t_1}^{t_2} \partial_{v_s} f(x_s) ds \right)^{+}. \end{aligned}$$

Proof of (iii): Suppose $x(t) = 0$ on $[t_1, t_2]$ and $f(x_t) = 0$ for all $t \in [t_1, t_2]$. Then $-z(t) = \overline{-z}(t) \geq 0$ for all $t \in [t_1, t_2]$ and z and is constant on $[t_1, t_2]$. It follows that $\mathbb{S}_{-z}(t) = \mathbb{S}_{-z}(t_1) \cup [t_1, t]$ for all $t \in [t_1, t_2]$. Then by (C.11), for $t \in [t_1, t_2]$,

$$\begin{aligned} v(t) &= w(t) + R(-z, -w)(t) \\ &= w(t) + R(-z, -w)(t_1) \vee \sup_{s \in [t_1, t]} (-w(s)) \\ &= v(t_1) + \int_{t_1}^{t} \partial_{v_s} f(x_s) ds + \left(\sup_{s \in [t_1, t]} (-w(s)) - R(-z, -w)(t_1) \right) \vee 0 \\ &= v(t_1) + \int_{t_1}^{t} \partial_{v_s} f(x_s) ds + \sup_{s \in [t_1, t]} \left(-v(t_1) - \int_{t_1}^{s} \partial_{v_r} f(x_r) dr \right) \vee 0. \end{aligned}$$

Proof of (iv): Suppose $x(t) = 0$ on $(t_1, t_2]$ and $f(x_t) < 0$ for all $t \in (t_1, t_2)$. Then $-z(t) = \overline{-z}(t) \geq 0$ for all $t \in (t_1, t_2]$ and since $-z$ is strictly increasing on (t_1, t_2), we have $\mathbb{S}_{-z}(t) = \{t\}$ and $-z(t) > 0$ for $t \in (t_1, t_2]$. Thus for $t \in (t_1, t_2]$,

$$v(t) = w(t) + (-w(t)) = 0.$$

□

This appendix is based on the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

Appendix D

Proof of Lemma 5.2

In this section we prove Lemma 5.2, which provides a lower bound on $\|\bar{x}^\tau\|_{[-1,\infty)}$ for sufficiently large delays τ, where $\{\bar{x}^\tau : \tau > \tau_0\}$ is the family of scaled SOPS$^{\mathrm{n}}$ defined in Section 5.2. We first need the following propositions, which follow directly from Lemmas 1 and 3 of [19] and Lemmas 5 and 6 of [30] once we extend our function $\hat{h}$ to be defined on the whole real line. For this, we define the function $\tilde{h} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{h}(s) = \begin{cases} \hat{h}(-L), & \text{if } s < -L, \\ \hat{h}(s), & \text{if } s \geq -L, \end{cases}$$

Then $-\tilde{h}$ satisfies **H1** and **H2** in [19] and so Lemmas 1 and 3 hold for the function $-\tilde{h}$. The function $-\tilde{h}$ also satisfies the conditions in **H1** of [30], except for the condition that $-\tilde{h}$ be a smooth function. Instead $-\tilde{h}$ is continuously differentiable on $(-L, \infty)$ and smooth on $(-\infty, -L)$. However, as we will show, Lemmas 5 and 6 in [30] still hold for the function $-\tilde{h}$.

Proposition D.1. *There exists a constant* $C_h \geq 1$ *such that*

$$\hat{h}(r_1) \geq C_h \hat{h}(r_2) \text{ for all } 0 \leq r_1 \leq r_2 \tag{D.1}$$

and

$$\hat{h}(s_1) \leq C_h \hat{h}(s_2) \text{ for all } -L \leq s_2 \leq s_1 \leq 0. \tag{D.2}$$

If $\hat{h}$ *is non-increasing, then we can take* $C_h = 1$.

Proof. As stated in Remark 1 of [19], this is a straightforward consequence of the facts that $\tilde{h}$ is a continuous function, $s\tilde{h}(s) < 0$ for all $s \neq 0$, $\tilde{h}'(0)$ exists and $\tilde{h}'(0) < 0$, and $\tilde{h}$ has finite, non-zero limits at $\pm\infty$. □

Proposition D.2. *Define the function $\zeta_h : [-L, \infty) \to \mathbb{R}$ by*

$$\zeta_h(r) = \begin{cases} \frac{1}{r}\int_0^r \hat{h}(s)ds, & \text{if } r \neq 0, \\ 0, & \text{if } r = 0. \end{cases} \tag{D.3}$$

Then ζ_h satisfies the following properties:

(i) ζ_h *is continuous on* $[-L, \infty)$;

(ii) $\lim_{r\to\infty} \zeta_h(r) = \lim_{s\to\infty} \hat{h}(s) = -\alpha$;

(iii) *there is a constant* $d_h > 0$ *such that*

$$|\zeta_h(r)| \geq d_h|r|, \quad \text{for } |r| \leq 1,$$
$$|\zeta_h(r)| \geq d_h, \quad \text{for } |r| \geq 1;$$

(iv)

$$\zeta_h(r_1) \geq C_h^2\zeta_h(r_2), \quad \text{for } 0 \leq r_1 \leq r_2,$$
$$C_h^2\zeta_h(s_2) \geq \zeta_h(s_1), \quad \text{for } -L \leq s_2 \leq s_1 \leq 0;$$

(v) *Then for* $0 \leq r_1 \leq r_2$ *or* $-L \leq r_2 \leq r_1 \leq 0$,

$$\left|\int_{r_1}^{r_2} \hat{h}(u)du\right| \geq C_h^{-2}|r_2 - r_1||\zeta_h(r_2)|.$$

Here C_h is as in Lemma D.1.

Proof. Properties (i)–(iv) follow from parts (1)–(4) of Lemma 1 in [19]. From equation (2) in [19], we see that for $0 \leq r_1 \leq r_2$ or $-L \leq r_2 \leq r_1 \leq 0$,

$$\left|r_2^{-1}\int_{r_1}^{r_2} \tilde{h}(u)du\right| \geq \frac{|r_2 - r_1||\zeta_h(r_2)|}{|r_2 - r_1| + C_h^2|r_1|}.$$

In the case that $0 \leq r_1 \leq r_2$, property (v) follows from the fact that

$$\frac{r_2}{r_2 - r_1 + C_h^2 r_1} \geq \frac{1}{C_h^2}.$$

The case $-L \leq r_2 \leq r_1 \leq 0$ follows similarly. □

Proposition D.3. *Let* $-\infty < t_1 < t_2 < \infty$ *and* $u : [t_1, t_2] \to \mathbb{R}$ *be a continuously differentiable function such that* $u(t_1) \geq 0$ *and* $u(t_2) \geq 0$. *Assume that there exists a constant* $C \geq 1$ *such that* $\frac{du(t)}{dt}|_{t=s_2} \leq C\left(\frac{du(t)}{dt}|_{t=s_1}\right)$ *for all* $t_1 \leq s_1 \leq s_2 \leq t_2$. *Then*

$$u(t) \geq \frac{(t-t_1)u(t_2) + C(t_2-t)u(t_1)}{(t-t_1) + C(t_2-t)} \text{ for all } t_1 \leq t \leq t_2. \tag{D.4}$$

Alternatively, suppose there exists $C \geq 1$ *such that* $C\left(\frac{du(t)}{dt}|_{t=s_2}\right) \leq \frac{du(t)}{dt}|_{t=s_1}$ *for all* $t_1 \leq s_1 \leq s_2 \leq t_2$. *Then*

$$u(t) \geq \frac{C(t-t_1)u(t_2) + (t_2-t)u(t_1)}{C(t-t_1) + (t_2-t)} \text{ for all } t_1 \leq t \leq t_2. \tag{D.5}$$

Proof. This follows from an application of the mean value theorem; see Lemma 3 in [19]. Note that [19] assumes that $u(t_1) > 0$, but the proof still applies when $u(t_1) = 0$. □

Remark D.1. Suppose $u : [t_1, t_2] \to \mathbb{R}$ is continuously differentiable with $u(t_1) \leq 0$ and $u(t_2) \leq 0$ and there exists $C \geq 1$ such that $\frac{du(t)}{dt}|_{t=s_2} \geq C\left(\frac{du(t)}{dt}|_{t=s_1}\right)$ for all $t_1 \leq s_1 \leq s_2 \leq t_2$. Then by considering the function $-u$ in place of u in the above lemma, we have

$$u(t) \leq \frac{(t-t_1)u(t_2) + C(t_2-t)u(t_1)}{(t-t_1) + C(t_2-t)} \text{ for all } t_1 \leq t \leq t_2. \tag{D.6}$$

Alternatively, suppose there exists $C \geq 1$ such that $C\left(\frac{du(t)}{dt}|_{t=s_2}\right) \geq \frac{du(t)}{dt}|_{t=s_1}$ for all $t_1 \leq s_1 \leq s_2 \leq t_2$, then

$$u(t) \leq \frac{C(t-t_1)u(t_2) + (t_2-t)u(t_1)}{C(t-t_1) + (t_2-t)} \text{ for all } t_1 \leq t \leq t_2. \tag{D.7}$$

Proposition D.4. *For* $M > m > 0$, *there exist two constants* $\gamma > 0$ *and* $\tau' > 0$ *such that, if* $m \leq r \leq M$, $|s| \geq \tau|\hat{h}(rs)|$ *and* $\tau \geq \tau'$, *then*

$$|s| > \gamma\tau.$$

Proof. The proposition follows from applying the proof of Lemma 5 in [30] to $-\tilde{h}$ with τ in place of ε^{-1} and τ' in place of ε_0^{-1} and considering the restriction of $\tilde{h}$ to the interval $[-L, \infty)$. The main difference is that the function f in [30] is assumed to be smooth. However, the proof of Lemma 5 only requires that f is continuous, $f(0) = 0$, $f'(0)$ exists and is non-zero, $sf(s) > 0$ for all $s \neq 0$ and f has finite, non-zero limits at $\pm\infty$. Since $-\tilde{h}$ satisfies these properties, the conclusion of Lemma 5 holds for $\tilde{h}$ as well. □

Proposition D.5. *For each $M > 0$, there exists $C_M \geq 1$ such that*

$$C_M^{-1}\hat{h}(s) \geq \hat{h}(r) \geq C_M\hat{h}(s) \tag{D.8}$$

whenever $r, s > 0$ with $M^{-1} \leq (r/s) \leq M$, and

$$C_M^{-1}\hat{h}(s) \leq \hat{h}(r) \leq C_M\hat{h}(s) \tag{D.9}$$

whenever $r, s < 0$ with $M^{-1} \leq (r/s) \leq M$.

Proof. The proposition follows from applying the proof of Lemma 6 in [30] to $-\tilde{h}$ and considering the restriction of $\tilde{h}$ to the interval $[-L, \infty)$. The main difference is that the function f in [30] is assumed to be smooth. However, the proof of Lemma 6 only requires that f satisfy the properties stated in the proof of Proposition D.4. Since $-\tilde{h}$ also satisfies these properties, the conclusion of Lemma 6 hold for $-\tilde{h}$. □

Suppose that $\hat{x}^*$ is a SOPS$^{\mathrm{n}}$ of DDER$^{\mathrm{n}}$ such that $\hat{q}_0 = -1$. In Lemmas D.1, D.2 and D.3 below, we provide bounds on $\hat{x}^*$ that are dependent on the distance between its zeros. The lemmas and proofs are adapted from Lemmas 8, 10 and 11 in [30]. The main difference is that the SOPS$^{\mathrm{n}}$ $\hat{x}^*$ is bounded below by $-L$. For notational convenience, we define $\hat{q}_{2,1} = \hat{q}_2 - (\hat{q}_1 + 1)$.

Lemma D.1. *If $\hat{q}_1 \leq 1$, then $\hat{x}^*$ satisfies*

$$\hat{x}^*(t) \geq \frac{\hat{x}^*(0)}{C_h\hat{q}_1}(\hat{q}_1 - t), \qquad t \in [0, \hat{q}_1], \tag{D.10}$$

$$\hat{x}^*(t) \leq \max\left(-\frac{\hat{x}^*(0)}{C_h^2\hat{q}_1}(t - \hat{q}_1), -L\right), \qquad t \in [\hat{q}_1, 1], \tag{D.11}$$

$$\hat{x}^*(t) \leq \max\left(-\frac{\hat{x}^*(0)}{C_h^2\hat{q}_1}(1 - \hat{q}_1) - \frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(0)}{C_h}\right)\right|(t-1), -L\right), \quad t \in [1, \hat{q}_1 + 1]. \tag{D.12}$$

If $\hat{q}_{2,1} \leq 1$, then $\hat{x}^*$ satisfies

$$\hat{x}^*(t) \leq -\frac{|\hat{x}^*(\hat{q}_1+1)|}{C_h \hat{q}_{2,1}}(\hat{q}_2 - t), \qquad t \in [\hat{q}_1+1, \hat{q}_2], \tag{D.13}$$

$$\hat{x}^*(t) \geq \frac{|\hat{x}^*(\hat{q}_1+1)|}{C_h^2 \hat{q}_{2,1}}(t - \hat{q}_2), \qquad t \in [\hat{q}_2, \hat{q}_1+2], \tag{D.14}$$

$$\hat{x}^*(t) \geq \frac{|\hat{x}^*(\hat{q}_1+1)|}{C_h^2 \hat{q}_{2,1}}(1 - \hat{q}_{2,1}) + \frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(\hat{q}_1+1)}{C_h}\right)\right|(t - \hat{q}_1 - 2), \qquad t \in [\hat{q}_1+2, \hat{q}_2+1]. \tag{D.15}$$

Lemma D.2. *If $1 \leq \hat{q}_1 \leq 3/2$, then*

$$\hat{x}^*(t) \leq \max\left(-\frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(0)}{C_h}(2-\hat{q}_1)\right)\right|(t-\hat{q}_1), -L\right), \quad t \in [\hat{q}_1, 2], \tag{D.16}$$

$$\hat{x}^*(t) \leq \max\left(-\frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(0)}{C_h}(2-\hat{q}_1)\right)\right|(2-\hat{q}_1), -L\right), \quad t \in [2, \hat{q}_1+1]. \tag{D.17}$$

If $1 \leq \hat{q}_{2,1} \leq 3/2$, then

$$\hat{x}^*(t) \geq \frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(\hat{q}_1+1)}{C_h}(2-\hat{q}_{2,1})\right)\right|(t-\hat{q}_2), \qquad t \in [\hat{q}_2, \hat{q}_1+3] \tag{D.18}$$

$$\hat{x}^*(t) \geq \frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(\hat{q}_1+1)}{C_h}(2-\hat{q}_{2,1})\right)\right|(2-\hat{q}_{2,1}), \qquad t \in [\hat{q}_1+3, \hat{q}_2+1]. \tag{D.19}$$

Lemma D.3. *There exists $\tau'' > 0$ and $\delta'' > 0$ such that if $\tau \geq \tau''$, then*

(i) $\hat{q}_{2,1} < 3/2$ *and*

(ii) *if $\hat{q}_1 \geq 3/2$, then*

$$\hat{x}^*(t) \geq \tau\delta'', \; t \in [0, \hat{q}_1 - 1]. \tag{D.20}$$

Proof of Lemma D.1. Suppose that $\hat{q}_1 \leq 1$. By parts (i) and (ii) of Lemma 5.1, $\hat{x}^*$ is positive on $(-1, \hat{q}_1)$, continuously differentiable on $[-1, \hat{q}_1]$, increasing on $(-1, 0)$ and decreasing on $(0, \hat{q}_1]$. Then for $0 \leq t_1 \leq t_2 \leq \hat{q}_1$, $\hat{x}^*(t_1 - 1) \leq \hat{x}^*(t_2 - 1)$ and so by (4.13) and (D.1),

$$\left.\frac{d\hat{x}^*(t)}{dt}\right|_{t=t_1} = \tau\hat{h}(\hat{x}^*(t_1-1)) \geq C_h\tau\hat{h}(\hat{x}^*(t_2-1)) = C_h \cdot \left.\frac{d\hat{x}^*(t)}{dt}\right|_{t=t_2}.$$

Applying (D.5) to $\hat{x}^*$ on the interval $[0, \hat{q}_1]$ and with C_h in place of C, we obtain

$$\hat{x}^*(t) \geq \frac{C_h t\hat{x}^*(\hat{q}_1) + (\hat{q}_1 - t)\hat{x}^*(0)}{C_h t + (\hat{q}_1 - t)} \geq \frac{(\hat{q}_1 - t)\hat{x}^*(0)}{C_h \hat{q}_1}, \; t \in [0, \hat{q}_1]. \tag{D.21}$$

By (5.8), (D.1), (5.10) and the facts that $\hat{q}_1 \le 1$, $\hat{x}^*(\hat{q}_1) = 0$ and $\hat{x}^*$ is non-negative and non-decreasing on $[-1, 0]$, we have

$$\hat{z}^*(t) = \tau \int_{\hat{q}_1}^{t} \hat{h}(\hat{x}^*(s-1))ds \le \frac{\tau \hat{h}(\hat{x}^*(\hat{q}_1 - 1))(t - \hat{q}_1)}{C_h}, \ t \in [\hat{q}_1, 1]. \tag{D.22}$$

By (5.8) and (5.10) and since $\hat{q}_1 \le 1$, $\hat{x}^*$ is non-negative and non-decreasing on $[-1, 0]$, and using (D.1), we have

$$-\hat{x}^*(0) = \tau \int_{0}^{\hat{q}_1} \hat{h}(\hat{x}^*(s-1))ds \ge \tau C_h \hat{h}(\hat{x}^*(\hat{q}_1 - 1))\hat{q}_1. \tag{D.23}$$

Combining (D.22)–(D.23) yields

$$\hat{z}^*(t) \le -\frac{\hat{x}^*(0)}{C_h^2 \hat{q}_1}(t - \hat{q}_1), \ t \in [\hat{q}_1, 1], \tag{D.24}$$

and so by (5.11),

$$\hat{x}^*(t) \le \max\left(-\frac{\hat{x}^*(0)}{C_h^2 \hat{q}_1}(t - \hat{q}_1), -L\right), \ t \in [\hat{q}_1, 1]. \tag{D.25}$$

Now by (5.8), (D.1), (D.21) and part (v) of Proposition D.2, for $t \in [1, \hat{q}_1 + 1]$,

$$\begin{aligned}
\hat{z}^*(t) - \hat{z}^*(1) &= \tau \int_{1}^{t} \hat{h}(\hat{x}^*(s-1))ds \\
&\le \frac{\tau}{C_h} \int_{1}^{t} \hat{h}\left(\frac{\hat{x}^*(0)}{C_h \hat{q}_1}(\hat{q}_1 + 1 - s)\right) ds \\
&\le \frac{\tau \hat{q}_1}{\hat{x}^*(0)} \int_{(C_h \hat{q}_1)^{-1}(\hat{q}_1 - t + 1)\hat{x}^*(0)}^{C_h^{-1}\hat{x}^*(0)} \hat{h}(s)ds \\
&\le -\frac{\tau \hat{q}_1}{\hat{x}^*(0)} C^{-2} \left| \frac{\hat{x}^*(0)}{C_h} - \frac{(\hat{q}_1 - t + 1)\hat{x}^*(0)}{C_h \hat{q}_1} \right| \left| \zeta_h \left(\frac{\hat{x}^*(0)}{C_h}\right) \right| \\
&\le -\frac{\tau(t-1)}{C_h^3} \left| \zeta_h \left(\frac{\hat{x}^*(0)}{C_h}\right) \right|.
\end{aligned}$$

After combining the above with (D.24), it follows from (5.11) that, for $t \in [1, \hat{q}_1 + 1]$,

$$\hat{x}^*(t) \le \max\left(-\frac{\hat{x}^*(0)(1 - \hat{q}_1)}{C_h^2 \hat{q}_1} - \frac{\tau(t-1)}{C_h^3} \left| \zeta_h \left(\frac{\hat{x}^*(0)}{C_h}\right) \right|, -L\right). \tag{D.26}$$

Now suppose that $\hat{q}_{2,1} = \hat{q}_2 - (\hat{q}_1 + 1) \le 1$. By part (iii) of Lemma 5.1, $\hat{x}^*$ is non-positive and non-increasing on $(\hat{q}_1, \hat{q}_1 + 1]$ and continuously differentiable on $(\hat{q}_1 + 1, \hat{q}_2]$. Then by (4.13) and (D.2), for $\hat{q}_1 + 1 < t_1 \le t_2 \le \hat{q}_2$,

$$\left.\frac{d\hat{x}^*(t)}{dt}\right|_{t=t_1} = \tau \hat{h}(\hat{x}^*(t_1 - 1)) \le C_h \tau \hat{h}(\hat{x}^*(t_2 - 1)) = C_h \cdot \left.\frac{d\hat{x}^*(t)}{dt}\right|_{t=t_2}.$$

For each $t_0 \in (\hat{q}_1 + 1, \hat{q}_2]$, $\hat{x}^*$ is continuously differentiable on $[t_0, \hat{q}_2]$, so by (D.7) with $t_1 = t_0$, $t_2 = \hat{q}_2$ and $C = C_h$,

$$\hat{x}^*(t) \leq \frac{C_h(t - t_0)\hat{x}^*(\hat{q}_2) + (\hat{q}_2 - t)\hat{x}^*(t_0)}{C_h(t - t_0) + (\hat{q}_2 - t)}, \ t \in [t_0, \hat{q}_2].$$

Since this holds for each $\hat{q}_1 + 1 < t_0 \leq t \leq \hat{q}_2$, by the continuity of $\hat{x}^*$, we have, for $\hat{q}_1 + 1 \leq t \leq \hat{q}_2$,

$$\begin{aligned}\hat{x}^*(t) &\leq \frac{C_h(t - (\hat{q}_1 + 1))\hat{x}^*(\hat{q}_2) + (\hat{q}_2 - t)\hat{x}^*(\hat{q}_1 + 1)}{C_h(t - (\hat{q}_1 + 1)) + (\hat{q}_2 - t)} \\ &\leq \frac{\hat{x}^*(\hat{q}_1 + 1)(\hat{q}_2 - t)}{C_h \hat{q}_{2,1}},\end{aligned} \tag{D.27}$$

where we used the fact that $C_h \geq 1$ for the last inequality. By (4.13), the fact that $\hat{x}^*$ is positive on $[\hat{q}_2, \hat{q}_1 + 2]$, and non-positive and non-increasing on $[\hat{q}_2 - 1, \hat{q}_1 + 1] \subset [\hat{q}_1, \hat{q}_1 + 1]$, part (iv) of Lemma 5.1, and (D.2),

$$\hat{x}^*(t) = \tau \int_{\hat{q}_2}^{t} \hat{h}(\hat{x}^*(s - 1))ds \geq \frac{\tau \hat{h}(\hat{x}^*(\hat{q}_2 - 1))(t - \hat{q}_2)}{C_h}, \ t \in [\hat{q}_2, \hat{q}_1 + 2]. \tag{D.28}$$

By (4.13) and the fact that $\hat{x}^*(t) > -L$ for all $t \in (\hat{q}_1 + 1, \hat{q}_2)$, part (iii) of Lemma 5.1, the fact that $\hat{x}^*$ is non-positive and non-increasing on $[\hat{q}_1, \hat{q}_2 - 1] \subset [\hat{q}_1, \hat{q}_1 + 1]$ and (D.1), we have

$$-\hat{x}^*(\hat{q}_1 + 1) = \tau \int_{\hat{q}_1 + 1}^{\hat{q}_2} \hat{h}(\hat{x}^*(s - 1))ds \leq C_h \tau \hat{h}(\hat{x}^*(\hat{q}_2 - 1))\hat{q}_{2,1}. \tag{D.29}$$

Combining inequalities (D.28)–(D.29) yields

$$\hat{x}^*(t) \geq -\frac{\hat{x}^*(\hat{q}_1 + 1)(t - \hat{q}_2)}{C_h^2 \hat{q}_{2,1}} \geq 0, \ t \in [\hat{q}_2, \hat{q}_1 + 2]. \tag{D.30}$$

Now by (D.2), (D.27) and part (v) of Proposition D.2, for $t \in [\hat{q}_1 + 2, \hat{q}_2 + 1]$, we

have

$$\begin{aligned}
\hat{x}^*(t) - \hat{x}^*(\hat{q}_1+2) &= \tau \int_{\hat{q}_1+2}^{t} \hat{h}(\hat{x}^*(s-1))ds \\
&\geq \frac{\tau}{C_h} \int_{\hat{q}_1+2}^{t} \hat{h}\left(\frac{\hat{x}^*(\hat{q}_1+1)(\hat{q}_2-s+1)}{C_h\hat{q}_{2,1}}\right) ds \\
&\geq \frac{\tau\hat{q}_{2,1}}{|\hat{x}^*(\hat{q}_1+1)|} \int_{(C_h\hat{q}_{2,1})^{-1}\hat{x}^*(\hat{q}_1+1)(\hat{q}_2-t+1)}^{C_h^{-1}\hat{x}^*(\hat{q}_1+1)} \hat{h}(u)du \\
&\geq \frac{\tau\hat{q}_{2,1}}{|\hat{x}^*(\hat{q}_1+1)|} C_h^{-2} \left|\frac{\hat{x}^*(\hat{q}_1+1)}{C_h} - \frac{(\hat{q}_2-t+1)\hat{x}^*(\hat{q}_1+1)}{C_h\hat{q}_{2,1}}\right| \\
&\quad \cdot \left|\zeta_h\left(\frac{\hat{x}^*(\hat{q}_1+1)}{C_h}\right)\right| \\
&\geq \frac{\tau(t-(\hat{q}_1+2))}{C_h^3} \left|\zeta_h\left(\frac{\hat{x}^*(\hat{q}_1+1)}{C_h}\right)\right|.
\end{aligned}$$

Combining with (D.30), we have that, for $t \in [\hat{q}_1+2, \hat{q}_2+1]$,

$$\hat{x}^*(t) \geq \frac{|\hat{x}^*(\hat{q}_1+1)|(1-\hat{q}_{2,1})}{C_h^2\hat{q}_{2,1}} + \frac{\tau(t-(\hat{q}_1+2))}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(\hat{q}_1+1)}{C_h}\right)\right|. \tag{D.31}$$

The conclusion then follows from (D.21), (D.25)–(D.27) and (D.30)–(D.31). □

Proof of Lemma D.2. Suppose that $1 \leq \hat{q}_1 \leq 3/2$. By part (i) of Lemma 5.1, $\hat{x}^*$ is non-negative and non-decreasing on $[-1, 0]$, so by (D.1), for $0 \leq t_1 \leq t_2 \leq 1$,

$$C_h \cdot \left.\frac{d\hat{x}^*(t)}{dt}\right|_{t=t_2} = C_h\tau\hat{h}(\hat{x}^*(t_2-1)) \leq \tau\hat{h}(\hat{x}^*(t_1-1)) = \left.\frac{d\hat{x}^*(t)}{dt}\right|_{t=t_1}.$$

By part (ii) of Lemma 5.1 and the fact that $\hat{q}_1 \geq 1$, $\hat{x}^*$ is non-negative and continuously differentiable on $[0, 1]$, so we can apply (D.5) to $\hat{x}^*$ on the interval with C_h in place of C to obtain

$$\hat{x}^*(t) \geq \frac{C_h t\hat{x}^*(1) + (1-t)\hat{x}^*(0)}{C_h t + (1-t)} \geq \frac{\hat{x}^*(0)}{C_h}(1-t),\ t \in [0, 1]. \tag{D.32}$$

By (5.10), $\hat{z}^*(\hat{q}_1) = \hat{x}^*(\hat{q}_1) = 0$. Then by (5.8), (D.1), (D.32) and part (v) of

Proposition D.2, for $\hat{q}_1 \le t \le 2$,

$$\begin{aligned}\hat{z}^*(t) &= \tau \int_{\hat{q}_1}^{t} \hat{h}(\hat{x}^*(s-1))ds \\ &\le \frac{\tau}{C_h}\int_{\hat{q}_1}^{t} \hat{h}\left(\frac{\hat{x}^*(0)}{C_h}(2-s)\right)ds \\ &\le \frac{\tau}{\hat{x}^*(0)}\int_{C_h^{-1}\hat{x}^*(0)(2-t)}^{C_h^{-1}\hat{x}^*(0)(2-\hat{q}_1)} \hat{h}(u)du \\ &\le -\frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(0)}{C_h}(2-\hat{q}_1)\right)\right|(t-\hat{q}_1).\end{aligned} \tag{D.33}$$

Since $\hat{q}_1 \ge 1$, $\hat{x}^*$ is non-negative on $[\hat{q}_1 - 1, \hat{q}_1]$, and then by (5.8) and the negative feedback condition on $\hat{h}$, $\hat{z}^*$ is non-increasing on $[\hat{q}_1, \hat{q}_1 + 1]$. It follows that

$$\hat{z}^*(t) \le \hat{z}^*(2) \le -\frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(0)}{C_h}(2-\hat{q}_1)\right)\right|(2-\hat{q}_1),\ t \in [2, \hat{q}_1+1]. \tag{D.34}$$

By (5.11) and (D.33)–(D.34), we have

$$\hat{x}^*(t) \le \max\left(-\frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(0)}{C_h}(2-\hat{q}_1)\right)\right|(t-\hat{q}_1), -L\right), \quad t \in [\hat{q}_1, 2], \tag{D.35}$$

$$\hat{x}^*(t) \le \max\left(-\frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(0)}{C_h}(2-\hat{q}_1)\right)\right|(2-\hat{q}_1), -L\right), \quad t \in [2, \hat{q}_1+1]. \tag{D.36}$$

Now suppose that $1 \le \hat{q}_{2,1} \le 3/2$. By part (iii) of Lemma 5.1, $\hat{x}^*$ is non-positive and non-increasing on $[\hat{q}_1, \hat{q}_1 + 1]$. Then by (4.13) and (D.1), for $\hat{q}_1 + 1 < t_1 \le t_2 \le \hat{q}_1 + 2$,

$$C_h \cdot \left.\frac{d\hat{x}^*(t)}{dt}\right|_{t=t_2} = C_h\tau\hat{h}(\hat{x}^*(t_2-1)) \ge \tau\hat{h}(\hat{x}^*(t_1-1)) = \left.\frac{d\hat{x}^*(t)}{dt}\right|_{t=t_1}.$$

By part (iv) of Lemma 5.1 and the fact that $\hat{q}_{2,1} = \hat{q}_2 - (\hat{q}_1 + 1) \ge 1$, $\hat{x}^*$ is non-positive on $[\hat{q}_1 + 1, \hat{q}_1 + 2]$ and continuously differentiable on $(\hat{q}_1 + 1, \hat{q}_1 + 2]$. Then by (D.7), for $\hat{q}_1 + 1 < t_0 \le t \le \hat{q}_1 + 2$,

$$\hat{x}^*(t) \le \frac{C_h(t-t_0)\hat{x}^*(\hat{q}_1+2) + (\hat{q}_1+2-t)\hat{x}^*(t_0)}{C_h(t-t_0) + (\hat{q}_1+2-t)}.$$

Since $\hat{x}^*$ is continuous, we have, for $\hat{q}_1 + 1 \le t \le \hat{q}_1 + 2$,

$$\begin{aligned}\hat{x}^*(t) &\le \frac{C_h(t-(\hat{q}_1+1))\hat{x}^*(\hat{q}_1+2) + (\hat{q}_1+2-t)\hat{x}^*(\hat{q}_1+1)}{C_h(t-(\hat{q}_1+1)) + (\hat{q}_1+2-t)} \\ &\le \frac{\hat{x}^*(\hat{q}_1+1)}{C_h}(\hat{q}_1+2-t).\end{aligned} \tag{D.37}$$

Note that $\hat{q}_2 = \hat{q}_1 + 1 + \hat{q}_{2,1} \le \hat{q}_1 + 5/2$. By (4.13) and the fact that $\hat{x}^*(t) > -L$ for all $t \in [\hat{q}_2, \hat{q}_1 + 3]$, (D.2), (D.37) and part (v) of Proposition D.2, for $t \in [\hat{q}_2, \hat{q}_1 + 3]$,

$$\begin{aligned} \hat{x}^*(t) &= \tau \int_{\hat{q}_2}^{t} \hat{h}(\hat{x}^*(s-1))ds \\ &\ge \frac{\tau}{C_h} \int_{\hat{q}_2}^{t} \hat{h}\left(\frac{\hat{x}^*(\hat{q}_1+1)}{C_h}(\hat{q}_1 + 3 - s)\right) ds \\ &\ge \frac{\tau}{\hat{x}^*(\hat{q}_1+1)} \int_{C_h^{-1}\hat{x}^*(\hat{q}_1+1)(\hat{q}_1+3-t)}^{C_h^{-1}\hat{x}^*(\hat{q}_1+1)(2-\hat{q}_{2,1})} \hat{h}(u)du \\ &\ge \frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(\hat{q}_1+1)}{C_h}(2-\hat{q}_{2,1})\right)\right|(t-\hat{q}_2). \end{aligned} \tag{D.38}$$

By the periodicity of $\hat{x}^*$, part (i) of Lemma 5.1 and the fact that $\hat{q}_{2,1} = \hat{q}_2 - (\hat{q}_1 + 1) \ge 1$, $\hat{x}^*$ is non-decreasing on $[\hat{q}_2, \hat{q}_1 + 3]$, so by (D.38), for $t \in [\hat{q}_1 + 3, \hat{q}_2 + 1]$,

$$\hat{x}^*(t) \ge \hat{x}^*(\hat{q}_1 + 2) \ge \frac{\tau}{C_h^3}\left|\zeta_h\left(\frac{\hat{x}^*(\hat{q}_1+1)}{C_h}(2-\hat{q}_{2,1})\right)\right|(2-\hat{q}_{2,1}) \tag{D.39}$$

The lemma then follows from (D.35)–(D.36) and (D.38)–(D.39). □

Proof of Lemma D.3. Fix $\delta \in (0, 1/2)$. Let $r = \delta/C_h$ and choose positive constants m and M satisfying $0 < m < r < M$. Let $\gamma > 0$ and $\tau'' > 0$ be such that $\tau'' > \frac{LC_h}{\delta\gamma}$ and the conclusion of Proposition D.4 holds with τ'' in place of τ' there. Let $\delta'' = \frac{\delta\gamma}{C_h}$. In the following, fix $\tau \ge \tau''$.

First suppose that $\hat{q}_1 \ge 3/2$. By part (ii) of Lemma 5.1, $\hat{x}^*$ is continuously differentiable on $[\hat{q}_1 - \delta, \hat{q}_1]$ and $\hat{x}^*$ is positive and decreasing on $[\hat{q}_1 - \delta - 1, \hat{q}_1 - 1]$. Then by Lemma 4.2, the fact that $\hat{x}^*(\hat{q}_1) = 0$ and (D.1), we have

$$\begin{aligned} \hat{x}^*(\hat{q}_1 - \delta) &= -\tau \int_{\hat{q}_1-\delta}^{\hat{q}_1} \hat{h}(\hat{x}^*(s-1))ds \\ &\ge -\frac{\tau}{C_h} \int_{\hat{q}_1-\delta}^{\hat{q}_1} \hat{h}(\hat{x}^*(\hat{q}_1-1))ds \\ &\ge -\frac{\tau\delta}{C_h}\hat{h}(\hat{x}^*(\hat{q}_1-1)). \end{aligned}$$

Then by applying the conclusion of Proposition D.4 with $s = \frac{\hat{x}^*(\hat{q}_1-1)C_h}{\delta}$, we have

$$\hat{x}^*(\hat{q}_1 - 1) \ge \frac{\tau\delta\gamma}{C_h} = \tau\delta''. \tag{D.40}$$

Then (D.20) follows from the fact that $\hat{x}^*$ is decreasing on $[0, \hat{q}_1 - 1]$.

Now suppose instead that $\hat{q}_{2,1} = \hat{q}_2 - (\hat{q}_1 + 1) \geq 3/2$. We will obtain a contradiction. Since $\hat{q}_2 \geq \hat{q}_1 + 5/2$ and $\hat{q}_2 - 1 - \delta > \hat{q}_1 + 1$, by part (iv) of Lemma 5.1, $\hat{x}^*$ is continuously differentiable on $[\hat{q}_2 - \delta, \hat{q}_2]$ and $\hat{x}^*$ is negative and increasing on $[\hat{q}_2 - \delta - 1, \hat{q}_2 - 1]$. Then by (D.2), we have

$$\begin{aligned}\hat{x}^*(\hat{q}_2 - \delta) &= -\tau \int_{\hat{q}_2 - \delta}^{\hat{q}_2} \hat{h}(\hat{x}^*(s-1))ds \\ &\leq -\frac{\tau}{C_h} \int_{\hat{q}_2 - \delta}^{\hat{q}_2} \hat{h}(\hat{x}^*(\hat{q}_2 - 1))ds \\ &\leq -\frac{\tau\delta}{C_h} \hat{h}(\hat{x}^*(\hat{q}_2 - 1)).\end{aligned}$$

It follows from the conclusion of Proposition D.4, with $s = \frac{\hat{x}^*(\hat{q}_2 - 1)C_h}{\delta}$, that

$$\hat{x}^*(\hat{q}_2 - 1) \leq -\frac{\tau\delta\gamma}{C_h}. \tag{D.41}$$

However, this implies that $\hat{x}^*(\hat{q}_2 - 1) < -L$, a contradiction and so we must have $\hat{q}_{2,1} < 3/2$. □

Lemma D.4. *There exists $\gamma > 0$ and $\tau^\dagger \geq \tau_0$ such that if $\tau > \tau^\dagger$ and $\hat{x}^*$ is a SOPS^n, then*

$$\|\hat{x}^*\|_{[-1,\infty)} = \|\hat{x}^*\|_{[0,\hat{p}]} \geq \tau\gamma. \tag{D.42}$$

Proof. Let $\tau'' > 0$ and $\delta'' > 0$ be as in the statement of Lemma D.3. Define $\tau^\dagger > 0$ by

$$\tau^\dagger = \max\left\{\tau'', \frac{4C_h^{10}(L \vee 1)^2}{d_h(L \wedge 1)}\right\} \tag{D.43}$$

and $\gamma > 0$ by

$$\gamma = \min\left\{\delta'', \frac{(L \wedge 1)d_h}{4C_h^4}, \frac{\beta}{4(L \vee 1)C_h^4}\right\}, \tag{D.44}$$

where $C_h \geq 1$ and $d_h > 0$ are the constants from Propositions D.1 and D.2. Fix $\tau > \tau^\dagger$. Then by part (i) of Lemma D.3, $\hat{q}_{2,1} < 3/2$. We consider the following remaining cases:

(i) $\hat{q}_1 \geq 3/2$,

(ii) $1 \leq \hat{q}_1 \leq 3/2$ and $1 \leq \hat{q}_{2,1} < 3/2$,

(iii) $1 \leq \hat{q}_1 \leq 3/2$ and $1/2 \leq \hat{q}_{2,1} \leq 1$,

(iv) $1 \leq \hat{q}_1 \leq 3/2$ and $0 < \hat{q}_{2,1} \leq 1/2$,

(v) $1/2 \leq \hat{q}_1 \leq 1$ and $1 \leq \hat{q}_{2,1} < 3/2$,

(vi) $0 < \hat{q}_1 \leq 1/2$ and $1 \leq \hat{q}_{2,1} < 3/2$,

(vii) $0 < \hat{q}_1 < \frac{1}{4(L\vee 1)C_h^4}$ and $0 < \hat{q}_{2,1} \leq 1/2$,

(viii) $0 < \hat{q}_1 < \frac{1}{4(L\vee 1)C_h^4}$ and $1/2 \leq \hat{q}_{2,1} \leq 1$,

(ix) $\frac{1}{4(L\vee 1)C_h^4} \leq \hat{q}_1 \leq 1$ and $1/2 \leq \hat{q}_{2,1} \leq 1$,

(x) $\frac{1}{4(L\vee 1)C_h^4} \leq \hat{q}_1 \leq 1$ and $0 < \hat{q}_{2,1} \leq 1/2$.

(i) Suppose $\hat{q}_1 \geq 3/2$. By Lemma D.3, we have $\|\hat{x}^*\|_{[-1,\infty)} \geq \tau\delta'' \geq \tau\gamma$.

(ii) Suppose $1 \leq \hat{q}_1 \leq 3/2$ and $1 \leq \hat{q}_{2,1} < 3/2$. By (D.19), the periodicity of $\hat{x}^*$, (D.17), the facts that $\hat{p} = \hat{q}_2 + 1$, $2 - \hat{q}_{2,1} \geq 1/2$ and $2 - \hat{q}_1 \geq 1/2$, and part (iii) of Proposition D.2, we have

$$\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3}\left(\frac{|\hat{x}^*(\hat{q}_1+1)|}{2C_h} \wedge 1\right), \tag{D.45}$$

$$|\hat{x}^*(\hat{q}_1+1)| \geq \min\left(\frac{\tau d_h}{2C_h^3}\left(\frac{\hat{x}^*(0)}{2C_h} \wedge 1\right), L\right). \tag{D.46}$$

First, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{2C_h} \notin (-1,1)$. By (D.45), $\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3} \geq \tau\gamma$.

Second, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{2C_h}$ and $\frac{\hat{x}^*(0)}{2C_h}$ are in $(-1,1)$. Combining (D.45)–(D.46), we obtain

$$\hat{x}^*(0) \geq \frac{\tau d_h}{4C_h^4}\min\left(\frac{\tau d_h}{4C_h^4}\hat{x}^*(0), L\right). \tag{D.47}$$

We show that $\min\left(\frac{\tau d_h}{4C_h^4}\hat{x}^*(0), L\right) = L$. For a proof by contradiction, suppose that instead $\min\left(\frac{\tau d_h}{4C_h^4}\hat{x}^*(0), L\right) = \frac{\tau d_h}{4C_h^4}\hat{x}^*(0) < L$. Rearranging (D.47), we obtain $\tau \leq \frac{4C_h^4}{d_h}$, which contradicts our choice of τ. Hence $\min\left(\frac{\tau d_h}{4C_h^4}\hat{x}^*(0), L\right) = L$ and by substituting back into (D.47), we have $\hat{x}^*(0) \geq \frac{\tau d_h L}{4C_h^4} \geq \tau\gamma$.

Third, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{2C_h} \in (-1,1)$ and $\frac{\hat{x}^*(0)}{2C_h} \notin (-1,1)$. By (D.46) and our choice of τ, $|\hat{x}^*(\hat{q}_1+1)| \geq \min\left(\frac{\tau d_h}{2C_h^3}, L\right) = L$. It follows from (D.45) that $\hat{x}^*(0) \geq \frac{\tau d_h L}{4C_h^4} \geq \tau\gamma$.

(iii) Suppose $1 \leq \hat{q}_1 \leq 3/2$ and $1/2 \leq \hat{q}_{2,1} \leq 1$. By (D.15), the periodicity of $\hat{x}^*$, the fact that $\hat{q}_2 - (\hat{q}_1 + 1) = \hat{q}_{2,1} \geq 1/2$, (D.17), the fact that $2 - \hat{q}_1 \geq 1/2$, and part (iii) of Proposition D.2, we have

$$\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3}\left(\frac{|\hat{x}^*(\hat{q}_1+1)|}{C_h} \wedge 1\right) \tag{D.48}$$

$$|\hat{x}^*(\hat{q}_1+1)| \geq \min\left(\frac{\tau d_h}{2C_h^3}\left(\frac{\hat{x}^*(0)}{2C_h} \wedge 1\right), L\right). \tag{D.49}$$

First, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{C_h} \notin (-1,1)$. Then by (D.48), $\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3} \geq \tau\gamma$.

Second, consider the case that $\frac{\hat{x}^*(0)}{2C_h} \notin (-1,1)$. Then by (D.49) and our choice of τ, $|\hat{x}^*(\hat{q}_1+1)| \geq L$. Combining this with (D.48), we have $\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3}\left(\frac{L}{C_h} \wedge 1\right) = \frac{\tau d_h}{2C_h^4}(L \wedge C_h) \geq \frac{\tau d_h}{2C_h^4}(L \wedge 1) \geq \tau\gamma$.

Third, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{C_h} \in (-1,1)$ and $\frac{\hat{x}^*(0)}{2C_h} \in (-1,1)$. By (D.48)–(D.49),

$$\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^4}\min\left(\frac{\tau d_h}{4C_h^4}\hat{x}^*(0), L\right). \tag{D.50}$$

We show that $\min\left(\frac{\tau d_h}{4C_h^4}\hat{x}^*(0), L\right) = L$. For a proof by contradiction, suppose that instead $\min\left(\frac{\tau d_h}{4C_h^4}\hat{x}^*(0), L\right) = \frac{\tau d_h}{4C_h^4}\hat{x}^*(0) < L$. Rearranging (D.50), we obtain $\tau \leq \frac{\sqrt{8}C_h^4}{d_h}$, which contradictions our choice of τ. Hence $\min\left(\frac{\tau d_h}{4C_h^4}\hat{x}^*(0), L\right) = L$ and by substituting back into (D.50), we have $\hat{x}^*(0) \geq \frac{\tau d_h L}{2C_h^4} \geq \tau\gamma$.

(iv) Suppose $1 \leq \hat{q}_1 \leq 3/2$ and $0 < \hat{q}_{2,1} \leq 1/2$. By (D.15)–(D.17), the periodicity of $\hat{x}^*$, the facts that $\frac{1}{\hat{q}_{2,1}} \geq 2$ and $2 - \hat{q}_1 \geq 1/2$, and part (iii) of Proposition D.2, we have

$$\hat{x}^*(0) \geq \frac{|\hat{x}^*(\hat{q}_1+1)|}{C_h^2} \tag{D.51}$$

$$|\hat{x}^*(t)| \geq \min\left(\frac{\tau d_h}{C_h^3}\left(\frac{\hat{x}^*(0)}{2C_h} \wedge 1\right)(t-\hat{q}_1), L\right),\ t \in [\hat{q}_1, 2] \tag{D.52}$$

$$|\hat{x}^*(t)| \geq \min\left(\frac{\tau d_h}{C_h^3}\left(\frac{\hat{x}^*(0)}{2C_h} \wedge 1\right)(2-\hat{q}_1), L\right),\ t \in [2, \hat{q}_1+1]. \tag{D.53}$$

First, consider the case that $\frac{\hat{x}^*(0)}{2C_h} \in (-1,1)$. By (D.51)–(D.53),

$$|\hat{x}^*(t)| \geq \min\left(\frac{\tau d_h}{4C_h^6}|\hat{x}^*(\hat{q}_1+1)|, L\right),\ t \in [\hat{q}_1+1/2, \hat{q}_1+1]. \tag{D.54}$$

We show that $\min\left(\frac{\tau d_h}{4C_h^6}|\hat{x}^*(\hat{q}_1+1)|, L\right) = L$. For a proof by contradiction, suppose that instead $\min\left(\frac{\tau d_h}{4C_h^6}|\hat{x}^*(\hat{q}_1+1)|, L\right) = \frac{\tau d_h}{4C_h^6}|\hat{x}^*(\hat{q}_1+1)| < L$. By considering (D.54) with $t = \hat{q}_1 + 1$ and rearranging, we obtain $\tau \leq \frac{4C_h^6}{d_h}$, which contradicts our choice of τ. Hence $\min\left(\frac{\tau d_h}{4C_h^6}|\hat{x}^*(\hat{q}_1+1)|, L\right) = L$ and by substituting back into (D.54), we have $|\hat{x}^*(t)| \geq L$ for all $t \in [\hat{q}_1+1/2, \hat{q}_1+1]$.

Second, consider the case that $\frac{\hat{x}^*(0)}{2C_h} \notin (-1,1)$. By (D.52)–(D.53),

$$|\hat{x}^*(t)| \geq \min\left(\frac{\tau d_h}{4C_h^3}, L\right) = L,\ t \in [\hat{q}_1+1/2, \hat{q}_1+1],$$

where the last equality follows our choice of τ.

It follows that $|\hat{x}^*(t)| \geq L$ for all $t \in [\hat{q}_1+1/2, \hat{q}_1+1]$ (independent of whether $\frac{\hat{x}^*(0)}{2C_h}$ is in $(-1,1)$ or not). By part (iii) of Lemma 5.1, $-L \leq \hat{x}^*(t) < 0$ on the interval, and so $\hat{x}^*(t) = -L$ for all $t \in [\hat{q}_1+1/2, \hat{q}_1+1]$. Then by the periodicity of $\hat{x}^*$, the fact that $\hat{x}^*$ is non-decreasing on $[\hat{q}_1, +1, \hat{q}_2+1]$, (5.12), (5.8) and the fact that $\hat{q}_2 = \hat{q}_1 + 1 + \hat{q}_{2,1} \leq \hat{q}_1 + 3/2$, we have

$$\begin{aligned}
\hat{x}^*(0) = \hat{x}^*(\hat{q}_2+1) &\geq \hat{x}^*(\hat{q}_1+2) \\
&\geq \hat{x}^*(\hat{q}_1+3/2) + \tau\int_{\hat{q}_1+3/2}^{\hat{q}_1+2} \hat{h}(-L)ds \\
&\geq \frac{\beta\tau}{2} \geq \tau\gamma.
\end{aligned}$$

(v) Suppose $1/2 \leq \hat{q}_1 \leq 1$ and $1 \leq \hat{q}_{2,1} \leq 3/2$. By (D.19), the periodicity of $\hat{x}^*$, the fact that $2 - \hat{q}_{2,1} \geq 1/2$, (D.12) and part (iii) of Proposition D.2, we have

$$\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3}\left(\frac{|\hat{x}^*(\hat{q}_1+1)|}{2C_h} \wedge 1\right) \tag{D.55}$$

$$|\hat{x}^*(\hat{q}_1+1)| \geq \min\left(\frac{\tau d_h}{2C_h^3}\left(\frac{\hat{x}^*(0)}{C_h} \wedge 1\right), L\right) \tag{D.56}$$

First, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{2C_h} \notin (-1,1)$. By (D.55), we have $\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3} \geq \tau\gamma$.

Second, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{2C_h} \in (-1,1)$ and $\frac{\hat{x}^*(0)}{C_h} \in (-1,1)$. Combining (D.55)–(D.56), we obtain

$$\hat{x}^*(0) \geq \frac{\tau d_h}{4C_h^4} \min\left(\frac{\tau d_h}{2C_h^4}\hat{x}^*(0), L\right). \tag{D.57}$$

We show that $\min\left(\frac{\tau d_h}{2C_h^4}\hat{x}^*(0), L\right) = L$. For a proof by contradiction, suppose that instead $\min\left(\frac{\tau d_h}{2C_h^4}\hat{x}^*(0), L\right) = \frac{\tau d_h}{2C_h^4}\hat{x}^*(0) < L$. Then rearranging (D.57), we obtain $\tau \leq \frac{\sqrt{8}C_h^4}{d_h}$, which contradicts our choice of τ. Hence $\min\left(\frac{\tau d_h}{2C_h^4}\hat{x}^*(0), L\right) = L$ and after substituting back into (D.57), we have $\hat{x}^*(0) \geq \frac{\tau d_h L}{4C_h^4} \geq \tau\gamma$.

Third, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{2C_h} \in (-1,1)$ and $\frac{\hat{x}^*(0)}{C_h} \notin (-1,1)$. By (D.55)–(D.56) and our choice of τ,

$$\hat{x}^*(0) \geq \frac{\tau d_h}{4C_h^4} \min\left(\frac{\tau d_h}{2C_h^3}, L\right) \geq \frac{\tau d_h L}{4C_h^4} \geq \tau\gamma.$$

(vi) Suppose $0 < \hat{q}_1 \leq 1/2$ and $1 \leq \hat{q}_{2,1} \leq 3/2$. By (D.19), the facts that $\frac{1}{\hat{q}_1} \geq 2$ and $2 - \hat{q}_{2,1} \geq 1/2$, (D.12) and part (iii) of Proposition D.2, we have

$$\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3}\left(\frac{|\hat{x}^*(\hat{q}_1+1)|}{2C_h} \wedge 1\right) \tag{D.58}$$

$$|\hat{x}^*(\hat{q}_1+1)| \geq \min\left(\frac{\hat{x}^*(0)}{C_h^2}, L\right). \tag{D.59}$$

First, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{2C_h} \notin (-1,1)$. Then by (D.58), $\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3} \geq \tau\gamma$.

Second, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{2C_h} \in (-1,1)$. Combining (D.58)–(D.59), we have

$$\hat{x}^*(0) \geq \frac{\tau d_h}{4C_h^4} \min\left(\frac{\hat{x}^*(0)}{C_h^2}, L\right). \tag{D.60}$$

We show that $\min\left(\frac{\hat{x}^*(0)}{C_h^2}, L\right) = L$. For a proof by contradiction, suppose that instead $\min\left(\frac{\hat{x}^*(0)}{C_h^2}, L\right) = \frac{\hat{x}^*(0)}{C_h^2} < L$. Then rearranging (D.60), we obtain $\tau \leq \frac{4C_h^6}{d_h}$, which contradicts our choice of τ. Hence $\min\left(\frac{\hat{x}^*(0)}{C_h^2}, L\right) = L$ and by substituting back into (D.60), we have $\hat{x}^*(0) \geq \frac{\tau d_h L}{4C_h^4} \geq \tau\gamma$.

(vii) Suppose $0 < \hat{q}_1 < \frac{1}{4(L \vee 1)C_h^4}$ and $0 < \hat{q}_{2,1} \leq 1/2$. By (D.11)–(D.12), (D.15), the periodicity of $\hat{x}^*$ and the fact that $\frac{1-\hat{q}_{2,1}}{\hat{q}_{2,1}} \geq 1$, we have

$$|\hat{x}^*(t)| \geq \min\left(\frac{\hat{x}^*(0)}{C_h^2 \hat{q}_1}(t \wedge 1 - \hat{q}_1), L\right), \qquad \hat{q}_1 \leq t \leq \hat{q}_1 + 1, \tag{D.61}$$

$$\hat{x}^*(0) \geq \frac{|\hat{x}^*(\hat{q}_1 + 1)|}{C_h^2}. \tag{D.62}$$

Combining (D.61)–(D.62) yields

$$|\hat{x}^*(t)| \geq \min\left(\frac{|\hat{x}^*(\hat{q}_1 + 1)|}{C_h^4 \hat{q}_1}(t \wedge 1 - \hat{q}_1), L\right), \ \hat{q}_1 \leq t \leq \hat{q}_1 + 1. \tag{D.63}$$

Since $\hat{q}_1 < \frac{1}{4}$, we have

$$\frac{|\hat{x}^*(\hat{q}_1 + 1)|}{C_h^4 \hat{q}_1}(1 - \hat{q}_1) > 3|\hat{x}^*(\hat{q}_1 + 1)|.$$

Substituting this inequality into (D.63) when $t = \hat{q}_1 + 1$, we obtain $|\hat{x}^*(\hat{q}_1 + 1)| \geq \min(3|\hat{x}^*(\hat{q}_1 + 1)|, L)$. Since $|\hat{x}^*(\hat{q}_1 + 1)| \neq 0$, we must have $|\hat{x}^*(\hat{q}_1 + 1)| = L$. In light of this and the fact that $\hat{q}_1 < \frac{1}{4(L \vee 1)C_h^4}$, (D.63) implies that

$$|\hat{x}^*(t)| \geq \min(\frac{L}{2C_h^4 \hat{q}_1}, L) \geq L \text{ for all } \hat{q}_1 + 1/2 \leq t \leq \hat{q}_1 + 1.$$

By part (iii) of Lemma 5.1, $-L \leq \hat{x}^*(t) \leq 0$ for all $\hat{q}_1 + 1/2 \leq t \leq \hat{q}_1 + 1$, so $\hat{x}^*(t) = -L$ on the interval. Then by (5.12), (5.8) and the fact that $\hat{q}_2 \leq \hat{q}_1 + 3/2$, we have

$$\hat{x}^*(\hat{q}_1 + 2) = \hat{x}^*(\hat{q}_1 + 3/2) + \tau \int_{\hat{q}_1 + 3/2}^{\hat{q}_1 + 2} \hat{h}(\hat{x}^*(s - \tau))ds \geq \frac{\beta}{2}\tau \geq \tau\gamma.$$

(viii) Suppose $0 < \hat{q}_1 < \frac{1}{4(L \vee 1)C_h^4}$ and $1/2 \leq \hat{q}_{2,1} \leq 1$. By (D.12), the fact that $\frac{1-\hat{q}_1}{\hat{q}_1} > 3$, (D.15), the periodicity of $\hat{x}^*$, and part (iii) of Proposition D.2, we have

$$|\hat{x}^*(\hat{q}_1 + 1)| \geq \min\left(\frac{3\hat{x}^*(0)}{C_h^2}, L\right), \tag{D.64}$$

$$\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3}\left(\frac{|\hat{x}^*(\hat{q}_1 + 1)|}{C_h} \wedge 1\right). \tag{D.65}$$

First, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{C_h} \notin (-1, 1)$. Then by (D.65), $\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3} \geq \tau\gamma$.

Second, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{C_h} \in (-1,1)$. By (D.64)–(D.65),

$$\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^4} \min\left(\frac{3\hat{x}^*(0)}{C_h^2}, L\right). \tag{D.66}$$

We show that $\min\left(\frac{3\hat{x}^*(0)}{C_h^2}, L\right) = L$. For a proof by contradiction, suppose that instead $\min\left(\frac{3\hat{x}^*(0)}{C_h^2}, L\right) = \frac{3\hat{x}^*(0)}{C_h^2} < L$. Then rearranging (D.66), we have $\tau \leq \frac{2C_h^6}{3d_h}$, which contradicts our choice of τ. Hence $\min\left(\frac{3\hat{x}^*(0)}{C_h^2}, L\right) = L$ and by substituting back into (D.66), we have $\hat{x}^*(0) \geq \frac{\tau d_h L}{2C_h^4} \geq \tau\gamma$.

(ix) Suppose $\frac{1}{4(L\vee 1)C_h^4} \leq \hat{q}_1 \leq 1$ and $1/2 \leq \hat{q}_{2,1} \leq 1$. By (D.15), (D.12), the periodicity of $\hat{x}^*$, part (iii) of Proposition D.2, and the lower bound on $\hat{q}_1$, we have

$$\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3}\left(\frac{|\hat{x}^*(\hat{q}_1+1)|}{C_h} \wedge 1\right) \tag{D.67}$$

$$|\hat{x}^*(\hat{q}_1+1)| \geq \min\left(\frac{\tau d_h}{4(L\vee 1)C_h^7}\left(\frac{\hat{x}^*(0)}{C_h} \wedge 1\right), L\right). \tag{D.68}$$

First, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{C_h} \notin (-1,1)$. Then by (D.67), $\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^3} \geq \tau\gamma$.

Second, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{C_h} \in (-1,1)$ and $\frac{\hat{x}^*(0)}{C_h} \notin (-1,1)$. By (D.68) and our choice of τ, $|\hat{x}^*(\hat{q}_1+1)| \geq \min\left(\frac{\tau d_h}{4(L\vee 1)C_h^7}, L\right) = L$. Then by (D.67), $\hat{x}^*(0) \geq \frac{\tau d_h L}{2C_h^4} \geq \tau\gamma$.

Third, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{C_h} \in (-1,1)$ and $\frac{\hat{x}^*(0)}{C_h} \in (-1,1)$. Then by (D.67)–(D.68),

$$\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^4} \min\left(\frac{\tau d_h}{4(L\vee 1)C_h^8}\hat{x}^*(0), L\right). \tag{D.69}$$

We show that $\min\left(\frac{\tau d_h}{4(L\vee 1)C_h^8}\hat{x}^*(0), L\right) = L$. For a proof by contradiction, suppose that instead $\min\left(\frac{\tau d_h}{4(L\vee 1)C_h^8}\hat{x}^*(0), L\right) = \frac{\tau d_h}{4(L\vee 1)C_h^8}\hat{x}^*(0) < L$. Then rearranging (D.69), we obtain $\tau \leq \frac{\sqrt{8(L\vee 1)}C_h^6}{d_h}$, which contradicts our choice of τ. It follows that $\min\left(\frac{\tau d_h}{4(L\vee 1)C_h^8}\hat{x}^*(0), L\right) = L$ and by substituting back into (D.69), we have $\hat{x}^*(0) \geq \frac{\tau d_h}{2C_h^4} \geq \tau\gamma$.

(x) Suppose $\frac{1}{4(L\vee 1)C_h^4} \leq \hat{q}_1 \leq 1$ and $0 < \hat{q}_{2,1} \leq 1/2$. By (D.15), the periodicity of $\hat{x}^*$, the fact that $\frac{1-\hat{q}_{2,1}}{\hat{q}_{2,1}} \geq 1$, (D.12) and part (iii) of Proposition D.2, we

have

$$\hat{x}^*(0) \geq \frac{|\hat{x}^*(\hat{q}_1+1)|}{C_h^2},$$
$$|\hat{x}^*(t)| \geq \min\left(\frac{\tau d_h}{C_h^3}\left(\frac{\hat{x}^*(0)}{C_h} \wedge 1\right)(t-1), L\right),\ t \in [1, \hat{q}_1+1].$$

Combining the above inequalities, we have

$$|\hat{x}^*(t)| \geq \min\left(\frac{\tau d_h}{C_h^3}\left(\frac{|\hat{x}^*(\hat{q}_1+1)|}{C_h^3} \wedge 1\right)(t-1), L\right),\ t \in [1, \hat{q}_1+1]. \tag{D.70}$$

If we set $t = \hat{q}_1 + 1$ in (D.70) and use the fact that $\hat{q}_1 \geq \frac{1}{4(L\vee 1)C_h^4}$, then we obtain

$$|\hat{x}^*(\hat{q}_1+1)| \geq \min\left(\frac{\tau d_h}{4(L\vee 1)C_h^7}\left(\frac{|\hat{x}^*(\hat{q}_1+1)|}{C_h^3} \wedge 1\right), L\right). \tag{D.71}$$

First, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{C_h^3} \notin (-1,1)$. By (D.71) and our choice of τ,

$$|\hat{x}^*(\hat{q}_1+1)| \geq \min\left(\frac{\tau d_h}{4(L\vee 1)C_h^7}, L\right) = L.$$

Second, consider the case that $\frac{\hat{x}^*(\hat{q}_1+1)}{C_h^3} \in (-1,1)$. By (D.71),

$$|\hat{x}^*(\hat{q}_1+1)| \geq \min\left(\frac{\tau d_h}{4(L\vee 1)C_h^{10}}|\hat{x}^*(\hat{q}_1+1)|, L\right). \tag{D.72}$$

We show that $\min\left(\frac{\tau d_h}{4(L\vee 1)C_h^{10}}|\hat{x}^*(\hat{q}_1+1)|, L\right) = L$. For a proof by contradiction, suppose that instead $\min\left(\frac{\tau d_h}{4(L\vee 1)C_h^{10}}|\hat{x}^*(\hat{q}_1+1)|, L\right) = \frac{\tau d_h}{4(L\vee 1)C_h^{10}}|\hat{x}^*(\hat{q}_1+1)| < L$. Rearranging (D.72) and using the fact that $\hat{x}^*(\hat{q}_1+1) \neq 0$ yields $\tau \leq \frac{4(L\vee 1)C_h^{10}}{d_h}$, which contradicts our choice of τ. Hence, $\min\left(\frac{\tau d_h}{4(L\vee 1)C_h^{10}}|\hat{x}^*(\hat{q}_1+1)|, L\right) = L$ and so $|\hat{x}^*(\hat{q}_1+1)| \geq L$.

Thus, $|\hat{x}(\hat{q}_1+1)| \geq L$ in either case and because $-L \leq \hat{x}^*(\hat{q}_1+1) \leq 0$, we must have $\hat{x}^*(\hat{q}_1+1) = -L$. Then by (D.70), the fact that $\hat{q}_1 \geq \frac{1}{4(L\vee 1)C_h^4}$ and our choice of τ, for $t \in [1+\hat{q}_1/2, 1+\hat{q}_1]$,

$$|\hat{x}^*(t)| \geq \min\left(\frac{\tau d_h}{4(L\vee 1)C_h^7}\left(\frac{L}{C_h^3} \wedge 1\right), L\right) \geq \min\left(\frac{\tau d_h}{4C_h^{10}}\frac{L\wedge 1}{L\vee 1}, L\right) = L.$$

By Lemma 5.1, since $\hat{q}_1 \leq 1$, $-L \leq \hat{x}^*(t) \leq 0$ for all $t \in [1+\hat{q}_1/2, 1+\hat{q}_1] \subset [\hat{q}_1, \hat{q}_1+1]$, so $\hat{x}^*(t) = -L$ on the interval. Then by (5.8), (5.12) and the fact that

$\hat{q}_2 = \hat{q}_1 + 1 + \hat{q}_{2,1} \leq \hat{q}_1/2 + 2$, we have

$$\hat{x}^*(\hat{q}_1 + 2) \geq \hat{x}^*\left(\frac{\hat{q}_1}{2} + 2\right) + \tau \int_{\frac{\hat{q}_1}{2}+2}^{\hat{q}_1+2} \hat{h}(-L)ds \geq \frac{\tau\beta}{8(L \vee 1)C_h^4} \geq \tau\gamma.$$

Thus for $\tau > \tau^\dagger$, we have $\|\hat{x}^*\|_{[-1,\infty)} \geq \tau\gamma$ in each case, which completes the proof of the lemma. □

Proof of Lemma 5.2. Since $\bar{x}^\tau = \tau^{-1}\hat{x}^\tau$ and $\hat{x}^\tau$ is a SOPS$^{\mathrm{n}}$, by Lemma D.4, there exists $\tau^\dagger > 0$ and $\gamma > 0$ such that $\|\bar{x}^\tau\|_{[-1,\infty)} = \tau^{-1}\|\hat{x}^\tau\|_{[-1,\infty)} \geq \gamma$ for all $\tau \geq \tau^\dagger$. □

This appendix is based on the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

Appendix E

The Space $\mathcal{D}$

Fix $-\infty < a < b < \infty$. Let $\mathcal{D}_{[a,b]}$ denote the space of real valued functions on $[a, b]$ with finite left and right limits at each time t in (a, b), finite right limits at a and finite left limits at b.

Lemma E.1. *The space $\mathcal{D}_{[a,b]}$ is a Banach space under the uniform norm $\|\cdot\|_{[a,b]}$.*

Remark E.1. Note that $\mathcal{D}_{[a,b]}$ is not a *separable* Banach space under the uniform norm (see e.g., [3]).

Proof. It is simple to check that $\mathcal{D}_{[a,b]}$ is a vector space under $\|\cdot\|_{[a,b]}$, so in order to prove that $\mathcal{D}_{[a,b]}$ is a Banach space, it is left to show that $\mathcal{D}_{[a,b]}$ is complete. Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{D}_{[a,b]}$. Then for each $t \in [a, b]$, the sequence $\{u_n(t)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$, so we can define the real-valued function u on $[a, b]$ by

$$u(t) = \lim_{n\to\infty} u_n(t),\ t \in [a, b].$$

Since $\{u_n\}_{n=1}^{\infty}$ is Cauchy in $\mathcal{D}_{[a,b]}$, for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\|u_n - u\|_{[a,b]} < \varepsilon$ for all $n \geq n_\varepsilon$.

We now prove that $u \in \mathcal{D}_{[a,b]}$. Fix $t \in (0, 1]$. For each $n \in \mathbb{N}$, u_n has a finite left limit at t, which we denote by $u_n(t-)$. Since $|u_n(t-)-u_m(t-)| \leq \|u_n - u_m\|_{[a,b]}$ for all $n, m \in \mathbb{N}$, it follows that $u(t-) \equiv \lim_{n\to\infty} u_n(t-)$ exists. Fix $\varepsilon > 0$ and choose $n_\varepsilon \in \mathbb{N}$ such that $\|u - u_n\|_{[a,b]} < \varepsilon$ for all $n \geq n_\varepsilon$. Now fix $\delta > 0$ such that for all $s \in (t - \delta, t)$, $|u_n(t-) - u_n(s)| < \varepsilon$. Then for all $s \in (t - \tau, t)$,

$$|u(t-) - u(s)| \leq |u(t-) - u_n(t-)| + |u_n(t-) - u_n(s)| + |u_n(s) - u(s)| < 3\varepsilon.$$

Therefore $\lim_{s\uparrow t} u(s) = u(t-) < \infty$. Since $t \in (a,b]$ was arbitrary, u has a finite left limit at each $t \in (a,b]$. Following a similar approach on $[a,b)$, we can show that u has a finite right limit at each $t \in [a,b)$. Hence $u \in \mathcal{D}_{[a,b]}$. □

This appendix is based on the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

Appendix F

Some Functional Analysis Results

Here we review some results extending the concepts of differential calculus to the infinite-dimensional Banach space setting. The following results and more can be found in Chapter 4 of [32]. Let X, Y and Z denote Banach spaces. Let U be an open subset of X and let V be an open subset of $X \times Y$. Recall that $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X to Y.

Definition F.1. A function $f : U \to Y$ is *Fréchet differentiable at* $x_0 \in U$ if there exists $L \in \mathcal{L}(X,Y)$ such that

$$f(x_0 + h) - f(x_0) - Lh = o\,(h) \ \text{ as } h \to 0.$$

We denote L by $Df(x_0)$ if it exists. If f is Fréchet differentiable at all $x \in U$, then f is *Fréchet differentiable on* U. If $x \to Df(x)$ is continuous as a function from $U \to \mathcal{L}(X,Y)$, then f is *continuously Fréchet differentiable on* U.

Definition F.2. A function $f : U \to Y$ is *differentiable at* $x_0 \in U$ *in the direction* $h \in X$ if

$$\partial_h f(x_0) \equiv \lim_{\varepsilon \to 0} \frac{f(x_0 + \varepsilon h) - f(x_0)}{\varepsilon}$$

exists, where the convergence is taken to be in the Banach space Y. We call $\partial_h f(x_0)$ the directional derivative of f at x_0 in the direction h.

Proposition F.1. *Let $f : U \to Y$ be a continuous function. Suppose that $\partial_h f(x)$ exists for all $x \in U$ and $h \in X$, and there exists a continuous function*

$L : U \to \mathcal{L}(X,Y)$ such that $L(x)h = \partial_h f(x)$ for all $x \in U$ and $h \in X$. Then f is continuously Fréchet differentiable on U with derivative $Df = L$.

The following proposition is a version of the implicit function theorem for the general Banach space setting. For a Fréchet differentiable function $g : V \to Z$, define $(D_1 g(x_0, y_0))(\cdot) \equiv (Dg(x_0, y_0))(\cdot, 0) : X \to Z$ and $(D_2 g(x_0, y_0))(\cdot) \equiv (Dg(x_0, y_0))(0, \cdot) : Y \to Z$.

Proposition F.2. *Suppose $(x_0, y_0) \in V$ and $g : V \to Z$ is continuously Fréchet differentiable on V, $g(x_0, y_0) = 0$ and the inverse operator $[D_2 g(x_0, y_0)]^{-1}$ exists and is in $\mathcal{L}(Z, Y)$. Then there is an open neighborhood W of x_0 in X and a unique continuous function $u : W \to Y$ such that $u(x_0) = y_0$, $(x, u(x)) \in V$ and $g(x, u(x)) = 0$ for all $x \in W$. Moreover, W can be chosen such that u is continuously Fréchet differentiable on W and*

$$Du(x) = -[D_2 g(x, u(x))]^{-1} D_1 g(x, u(x)) \text{ for all } x \in W,$$

where $x \to [D_2 g(x, u(x))]^{-1}$ is well-defined and continuous as a function from W into $\mathcal{L}(Z, Y)$.

This appendix is a formulation of known results based on a similar formulation of these results in the paper "Existence, Uniqueness and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints" written jointly with Ruth J. Williams and currently in preparation.

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